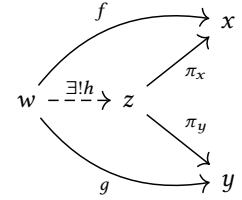


Category Theory from Scratch: Products Worksheet

Mathcamp 2025 - Della Hendrickson & Riley Shahar

We have now defined a *product*:

Definition 1. Let x and y be objects in a category C , and let z be an object with morphisms $\pi_x : z \rightarrow x$ and $\pi_y : z \rightarrow y$. We say that (z, π_x, π_y) is a *product* of x and y if, for any object w with morphisms $f : w \rightarrow x$ and $g : w \rightarrow y$, there exists a unique morphism $h : w \rightarrow z$ such that $h \circ \pi_x = f$ and $h \circ \pi_y = g$.

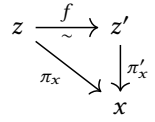


Our goal in this worksheet is to work out some properties of this definition.

Problem 1. Describe what a product is in as many categories as you can.

Problem 2. In this exercise, we will prove that products are unique up to unique isomorphism.

(a) Prove that, if (z, π_x, π_y) and (z', π'_x, π'_y) are products of x and y , then there is an isomorphism $f : z \rightarrow z'$ such that $\pi'_x = \pi_x \circ f$ and $\pi'_y = \pi_y \circ f$. We are therefore justified in writing z or z' as $x \times y$.



(b) Prove that if f and g are isomorphisms satisfying this property, then $f = g$.

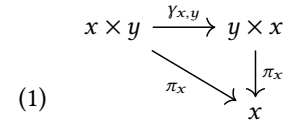
Problem 3 (optional). In this exercise, we will prove that the product is commutative.

(a) Prove that, for objects x and y , there is an isomorphism

$$\gamma_{x,y} : x \times y \xrightarrow{\sim} y \times x.$$

(b) Prove that $\gamma_{x,y}$ commutes with the projections, in that

$$\pi_x \circ \gamma_{x,y} = \pi_x \quad \text{and} \quad \pi_y \circ \gamma_{x,y} = \pi_y$$



Warning: the π_x on the left is a different map from the π_x on the right! The former is projection from $y \times x$, while the latter is projection from $x \times y$.

(c) Prove that $\gamma_{x,y}$ is the unique isomorphism satisfying Eq. (1).

Problem 4 (optional). In this exercise, we will prove that the product is associative. We are thus justified in writing either $(x \times y) \times z$ and $x \times (y \times z)$ as $x \times y \times z$.

(a) Prove that, for objects x , y , and z , there is an isomorphism

$$\alpha_{x,y,z} : (x \times y) \times z \xrightarrow{\sim} x \times (y \times z).$$

(b) Prove that $\alpha_{x,y,z}$ commutes with the projections, in that

$$\pi_x \circ \alpha_{x,y,z} = \pi_x \circ \pi_{x \times y}, \quad \pi_y \circ \pi_{y \times z} \circ \alpha_{x,y,z} = \pi_y \circ \pi_{x \times y}, \quad \text{and} \quad \pi_z \circ \pi_{y \times z} \circ \alpha_{x,y,z} = \pi_z. \quad (2)$$

(c) Prove that $\gamma_{x,y}$ is the unique isomorphism satisfying Eq. (2).

Problem 5 (optional). Show that, if $f : x \rightarrow x'$ and $g : y \rightarrow y'$ are morphisms, then there is an induced morphism

$$f \times g : x \times y \rightarrow x' \times y'.$$