

# Category Theory from Scratch: Sums Worksheet

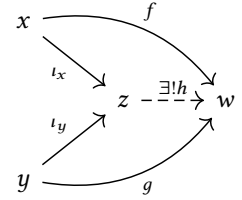
Mathcamp 2025 - Della Hendrickson & Riley Shahar

We have now defined a *sum*:

**Definition 1.** Let  $x$  and  $y$  be objects in a category  $C$ , and let  $z$  be an object with morphisms  $\iota_x : x \rightarrow z$  and  $\iota_y : y \rightarrow z$ . We say  $(z, \iota_x, \iota_y)$  is a *sum* of  $x$  and  $y$  if, for any object  $w$  with morphisms  $f : x \rightarrow w$  and  $g : y \rightarrow w$ , there exists a unique morphism  $h : z \rightarrow w$  such that  $h \circ \iota_x = f$  and  $h \circ \iota_y = g$ .

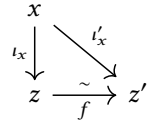
Our goal in this worksheet is to work out some properties of this definition.

**Problem 1.** Describe what a sum is in as many categories as you can.



**Problem 2.** In this exercise, we will prove that sums are unique up to unique isomorphism.

(a) Prove that, if  $(z, \iota_x, \iota_y)$  and  $(z', \iota'_x, \iota'_y)$  are sums of  $x$  and  $y$ , then there is an isomorphism  $f : z \rightarrow z'$  such that  $\iota'_x = f \circ \iota_x$  and  $\iota'_y = f \circ \iota_y$ . We are therefore justified in writing  $z$  or  $z'$  as  $x + y$ .



(b) Prove that if  $f$  and  $g$  are isomorphisms satisfying this property, then  $f = g$ .

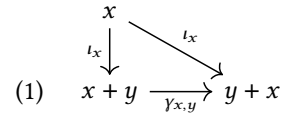
**Problem 3 (optional).** In this exercise, we will prove that the sum is commutative.

(a) Prove that, for objects  $x$  and  $y$ , there is an isomorphism

$$\gamma_{x,y} : x + y \xrightarrow{\sim} y + x.$$

(b) Prove that  $\gamma_{x,y}$  commutes with the inclusions, in that

$$\gamma_{x,y} \circ \iota_x = \iota_x \quad \text{and} \quad \gamma_{x,y} \circ \iota_y = \iota_y.$$



Warning: the  $\iota_x$  on the left is a different map from the  $\iota_x$  on the right! The former is inclusion from  $y + x$ , while the latter is inclusion from  $x + y$ .

(c) Prove that  $\gamma_{x,y}$  is the unique isomorphism satisfying Eq. (1).

**Problem 4 (optional).** In this exercise, we will prove that the sum is associative. We are thus justified in writing either  $(x + y) + z$  and  $x + (y + z)$  as  $x + y + z$ .

(a) Prove that, for objects  $x$ ,  $y$ , and  $z$ , there is an isomorphism

$$\alpha_{x,y,z} : (x + y) + z \xrightarrow{\sim} x + (y + z).$$

(b) Prove that  $\alpha_{x,y,z}$  commutes with the inclusions, in that

$$\alpha_{x,y,z} \circ \iota_{x+y} \circ \iota_x = \iota_x, \quad \alpha_{x,y,z} \circ \iota_{x+y} \circ \iota_y = \iota_{y+z} \circ \iota_y, \quad \text{and} \quad \alpha_{x,y,z} \circ \iota_z = \iota_{y+z} \circ \iota_z. \quad (2)$$

(c) Prove that  $\gamma_{x,y}$  is the unique isomorphism satisfying Eq. (2).

**Problem 5 (optional).** Show that, if  $f : x \rightarrow x'$  and  $g : y \rightarrow y'$  are morphisms, then there is an induced morphism

$$f + g : x + y \rightarrow x' + y'.$$