Category Theory from Scratch: Terminal Objects Worksheet

Mathcamp 2025 - Della Hendrickson & Riley Shahar

We have now defined a *terminal object*:

Definition 1. A *terminal object* in a category C is an object $*$ such that, for any object x, there is a unique morphism $!_x : x \to *$.

Our goal in this worksheet is to work out some properties of this definition.

Problem 1. Describe what a terminal object is in as many categories as you can.

Problem 2. In this exercise, we will prove that terminal objects are unique up to unique isomorphism.

  (a) Prove that, if $*$ and $*'$ are terminal objects in C, then there is an isomorphism $f : * \to *'$.

  (b) Prove that if f and g are isomorphisms $* \to *'$, then $f = g$.

Problem 3. Given a category C and an object x, define the *over category* category C/x as follows:

  - The objects of C/x are pairs (y, f), where y is an object in C and $f : y \to x$ is a morphism in C.
  - The morphisms between (y, f) and (z, g) in C/x are the morphisms $h : y \to z$ in C so that $g \circ h = f$.

Define composition and the identities, and prove that C/x is a category.
What is the terminal object in C/x?

Problem 4 (optional). Prove that there is no terminal object in the category of fields. (Hint: think about the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$.)

Problem 5 (optional). Let P be a preordered set, which has an associated category $|P|$, according to the first day's homework. What is a terminal object in $|P|$?

Problem 6 (optional). Let M be a monoid, which has an associated category BM, according to the first day's homework. For which monoids M does BM have a terminal object?