

Saturated Transfer Systems: Homework II

Mathcamp 2025

Problem 1 (recommended). Describe $c(R)$ for all seven saturated transfer systems R on $\underline{2} \times \underline{2}$. Check that our computations of $|c^{-1}(A)|$ are correct in this case.

Problem 2 (recommended). Use the recurrence to give explicit closed forms for $s(m, n)$ for $n = 1, 2, 3, 4$.

Problem 3 (optional). Prove the following facts about the Stirling numbers of the second kind:

- (a) $\left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1$ for $n \geq 0$,
- (b) $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 1$ for $n \geq 1$,
- (c) $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = 0$ for $k > n$,
- (d) $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = 1$ for $n \geq 1$,
- (e) $\left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\} = 2^{n-1} - 1$ for $n \geq 2$,
- (f) $\left\{ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\} = \binom{n}{2}$ for $n \geq 2$,
- (g) $\left\{ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\}$ $0 < k < n$,
- (h) $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \sum_{r_1 + \dots + r_k = n-k} \prod_{i=1}^k i^{r_i}$, where the sum is over all nonnegative ordered partitions of $n-k$ (hint: what is i^{r_i} counting?), and
- (i) $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$ (hint: use inclusion-exclusion.)