Lecture 11: Frequency Domain Analysis

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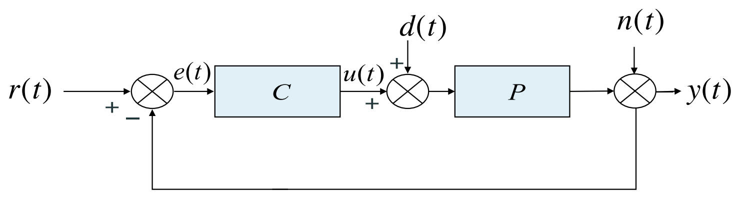
The University of Sydney

Previous Lecture – Transfer functions and Block Diagrams



- **If a system is stable**, then in steady state the response due to initial conditions will have vanished and we simply have

$$y(t) = G(s)u(t).$$

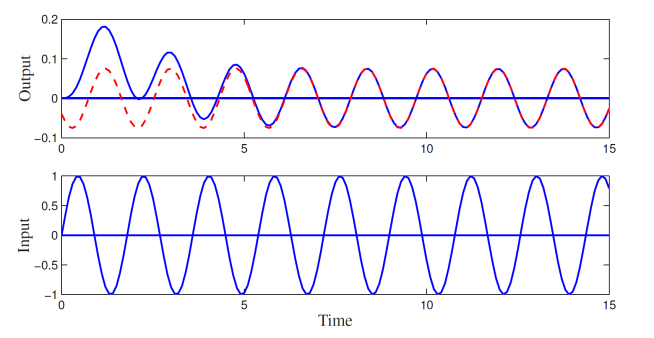
- Block diagrams allow us to examine **interconnections of systems**.
- A block diagram may include a number of input-output pairs, and therefore **several transfer functions**.
- These can be found using block diagram algebra.
- For example, we have

$$r(t) \to e(t): \qquad G_{er}(s) = \frac{1}{1 + P(s)C(s)}.$$

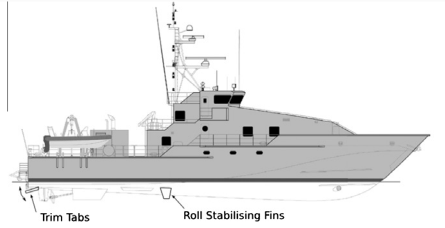
Previous Lecture – Frequency Response of LTI Systems

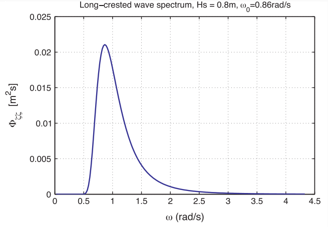
A sine wave input to an LTI system with transfer function $G(s)$ forces a sine wave output with the **same frequency**.
The output is **amplified** by $|G(j\omega)|$ and **phase-shifted** by $\phi(\omega)$.

$$u(t) = A\sin(\omega t) \longrightarrow y(t) = |G(j\omega)| A \sin(\omega t + \phi(\omega)).$$



Example – Ship Roll Control





- Roll motion due to waves can affect the performance of ships.
- Whilst ocean waves are random in time and space, the sea surface elevation can be modelled by its power spectral density.
- Typically, the wave energy is centered around 0.5-2 rad/s, i.e., an oscillation period of about 3-12 seconds.

Frequency Response of Ship Roll

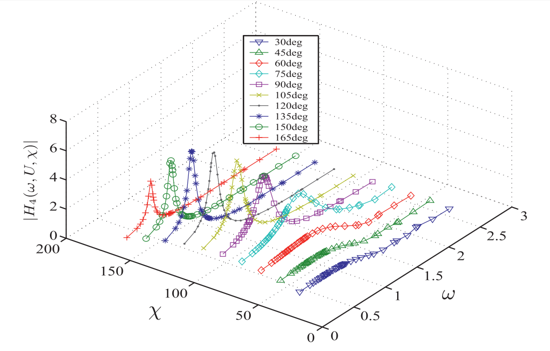


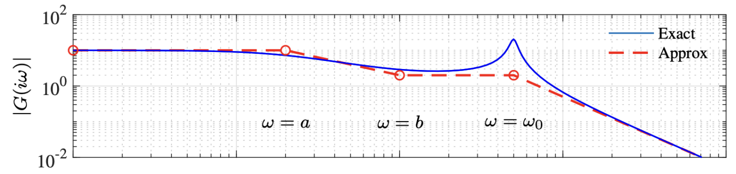
Figure 1: Magnitude response of ship roll ($\chi = 180^\circ$ is head-on into waves)

- When the vessel moves, the frequency experienced depends not only on the speed of the craft, but also on the enounter angle χ.
- Feedback control systems (e.g., using fins or the rudder) are used to reduce ship roll motion, see
  T. Perez, "Ship roll damping control", *Annual Rev in Control*, 2012.

Previous Lecture – Bode Plots

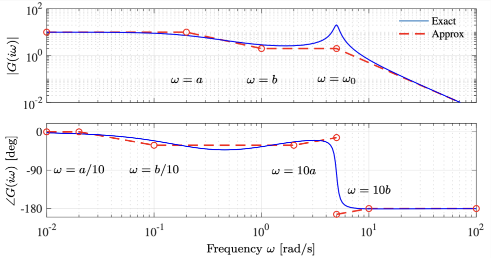
A Bode plot visualises the range of sinusoidal responses for all frequencies. It contains two subplots:

1. A plot of **magnitude**: $20\log_{10}|G(j\omega)|$ vs frequency $\log_{10}\omega$.
2. A plot of **phase angle** $\angle G(j\omega)$ vs frequency $\log_{10}\omega$.



- Bode plots of can be built from plots of basic terms.
- This can be done by hand using approximations, or using computers.
- We may be able to **identify a transfer function** from an empirical Bode plot, e.g., using the Matlab System Identification Toolbox.

Today's complement: Bode's gain-phase Relation

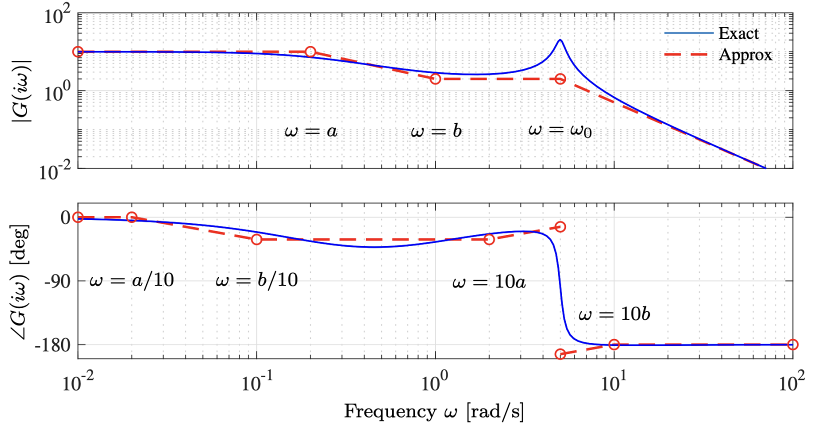


Bode discovered that if a system has **no poles or zeros in the right-half-plane** (called a "minimum phase system"), then the gain curve completely determines the phase curve (and vice versa):[1]

$$\angle G(j\omega_0) = \frac{\pi}{2}\int_0^\infty f(\omega)\frac{d\log|G(j\omega)|}{d\log\omega}\frac{d\omega}{\omega} \approx \frac{\pi}{2}\frac{d\log|G(j\omega)|}{d\log\omega}\bigg|_{\omega=\omega_0}.$$

[1] The weighting kernel $f(\omega) = \frac{2}{\pi^2}\log|\frac{\omega+\omega_0}{\omega-\omega_0}|$, $\int_0^\infty \big(f(\omega)/\omega\big)d\omega = 1$ is approximately shaped like a Dirac delta.

Bode's gain-phase Relation



The phase curve for a **minimum phase system** is thus a weighted average of the derivative of the gain curve, on a Bode plot. Each $\pm$ 20 dB/dec slope roughly corresponds to a $\pm 90^\circ$ phase shift.

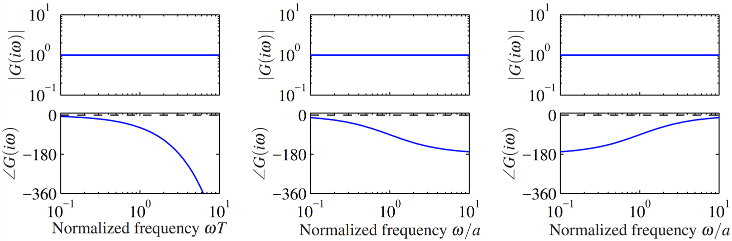
Non-minimum Phase Systems

Systems with poles and/or zeros in the right-half plane (RHP) or with time-delays can have the **same gain curve**, but a **different phase curve**.

Example. (1) The transfer function $G(s) = 1$ has gain 1 (0dB) and phase zero for all frequencies ω.

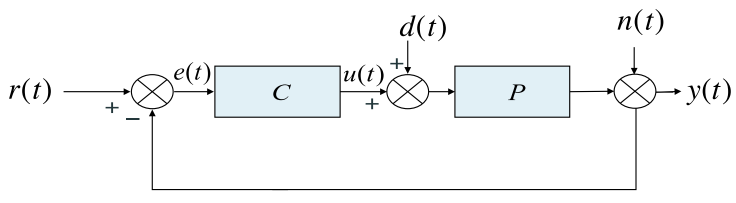
(2) The following transfer functions also have $|G(j\omega)| = 1$ for all frequencies, but increased (non-minimum) phase lag ($\tau, a > 0$):

$$G(s) = e^{-\tau s}, \qquad G(s) = \frac{a-s}{s+a}, \qquad G(s) = \frac{s+a}{s-a},$$



(a) Time delay (b) RHP zero (c) RHP pole

Remainder of today's lecture: Frequency Domain Analysis



- We have seen that a transfer function $G(s)$ describes **transmission of complex exponential signals**, e.g., sinusoidal oscillations.
- Interestingly, a transfer function tells us much more than just that.
- We will next learn how $G(j\omega)$ can be used to tell us:
    1. Whether a system will be **stable** when placed in a feedback loop.
    2. How robust the feedback system is to **model uncertainties**.

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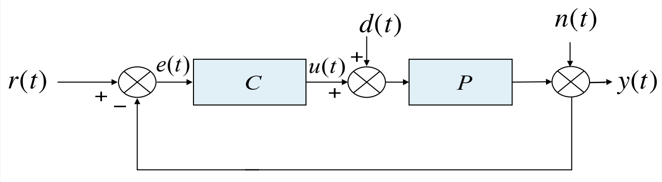
Closed-Loops and the Nyquist Criterion

Open-loop Transfer Function



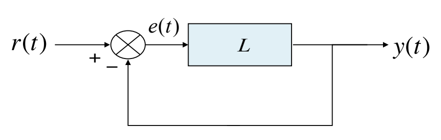
To simplify notation, we combine controller and plant model into the **open-loop transfer function**

$$L(s) = P(s)C(s).$$

The open-loop transfer function is a rational transfer function and we can write

$$L(s) = \frac{n(s)}{d(s)}.$$

Open-loop Transfer Function



Using block- or regular algebra, we obtain the transfer functions:

$$G_{yr}(s) = \frac{L(s)}{1+L(s)} = \frac{\frac{n(s)}{d(s)}}{1+\frac{n(s)}{d(s)}} = \frac{n(s)}{n(s)+d(s)}$$

$$G_{er}(s) = \frac{1}{1+L(s)} = \frac{1}{1+\frac{n(s)}{d(s)}} = \frac{d(s)}{n(s)+d(s)}$$

These transfer functions have **different zeros**, but the **same poles**:

s such that $n(s)+d(s) = 0$ or, equivalently, s such that $L(s) = -1$.

Closed-Loop Stability from the Open-loop Transfer Function

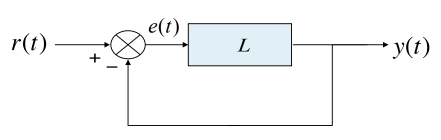
Closed loop poles are the complex values s, such that $L(s) = -1$.

- Is it possible to say anything about **closed-loop stability** from the **open-loop frequency response** $L(j\omega)$, i.e., before closing the loop?
- The key closed-loop stability question is: Are any of the closed loop poles in the right-half of the complex plane?
- If we know $L(j\omega)$ as a function of ω, does this tell us anything about the locations of the closed loop poles?

Surprisingly, a plot of $L(j\omega)$ allows us to precisely state how many closed loop poles are located in the right half plane!

It provides the information, without having to calculate the pole locations exactly.

Nyquist's Intuition

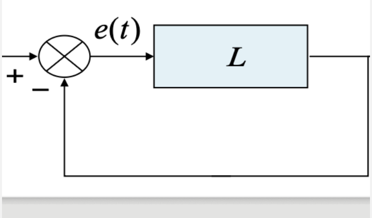


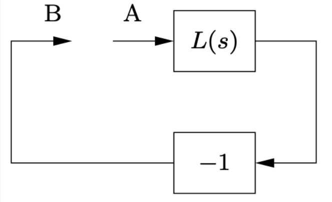
- The basic idea is to recall that a sustained oscillation (neither growing nor decaying) lies the **boundary between stability and instability**.
- **Under which conditions can such oscilations happen?**
- For an oscillatory reference signal $r(t) = e^{j\omega_0 t}$, in steady state we have:

$$e(t) = G_{er}(j\omega_0)r(t) = \frac{1}{1 + L(j\omega_0)}r(t).$$

- Could there be a sustained oscillation in $e(t)$ at a frequency ω_0 even if $r(t) = 0$?

To study this, let's break the feedback loop!

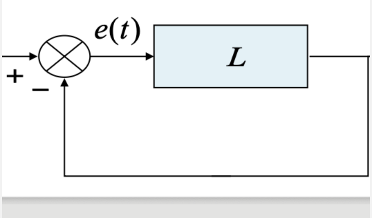


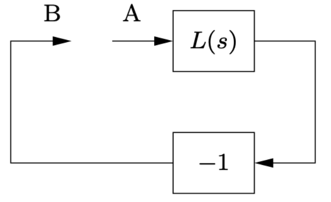


- Assume that a sinusoid of frequency ω_0 is injected at point A.
- It seems reasonable that an oscillation can be maintained if the signal at B is the **same as the injected signal**, because we can then disconnect the injected signal and directly connect A to B.
- Therefore, a condition for maintaining an oscillation is:

$$L(j\omega_0) = -1$$

- This condition implies that the frequency response goes through the value -1, which is called the **critical point**.
- Let ω_c represent a frequency at which $\angle L(j\omega_c) = 180°$.



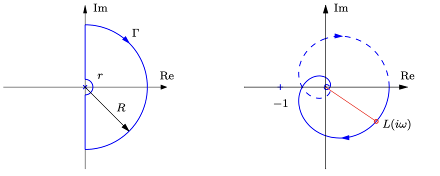


- Let ω_c represent a frequency at which $\angle L(j\omega_c) = 180°$.
- We can further reason that the system is stable if $|L(j\omega_c)| < 1$, since the signal at point B will have smaller amplitude than the injected signal.
- We note that the closed-loop transfer function $r \to y$ is

$$G_{yr}(s) = \frac{L(s)}{1 + L(s)}$$

  Thus, if $L(j\omega) = -1$, then there is a closed-loop pole on the imaginary axis, i.e. on the boundary between stability and instability.

The Nyquist Plot



(a) Nyquist contour (b) Nyquist plot

$$L(s) = 1.4\frac{e^{-s}}{(s+1)^2}$$

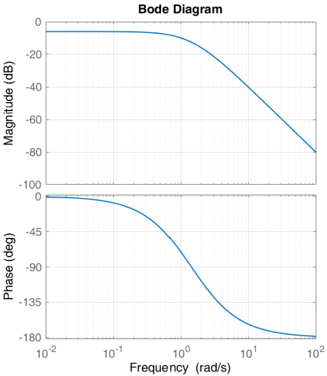
- A Nyquist plot (or "polar plot") is a **single** plot, which is obtained by tracing s around the Nyquist contour.
- If $L(s)$ goes to zero as s gets large (the usual case), then the portion of the contour Γ ("at infinity") maps to the origin.[2]
- **Plot $L(j\omega)$ on the complex plane, for all ω from $-\infty$ to ∞.**
- The portion of the plot corresponding to $\omega < 0$, shown in dashed lines, is the mirror image of the portion with $\omega > 0$.

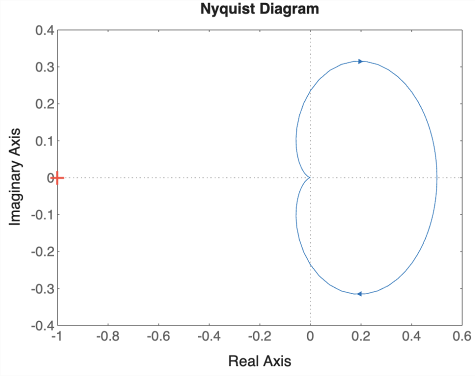
[2] There are some technicalities for systems that have poles on the imaginary axis.

Example

Matlab command `nyquist` can be used to produce a Nyquist plot:

$$L(s) = \frac{1}{(s+1)(s+2)}, \quad \texttt{L = tf([1],[1 3 2]), nyquist(L)}$$





Nyquist's Theorem (stable case)

In many cases, the open-loop system $L(s)$ is stable, apart from possible integrators. The aim is to make sure that feedback does not destabilise it.

In such cases, we can use this simple version of the Nyquist criterion:[3]

Theorem
Suppose that $L(s)$ has no poles in the right-half-plane, except possibly at the origin. Then the closed-loop system

$$G_{yr}(s) = \frac{L(s)}{1 + L(s)}$$

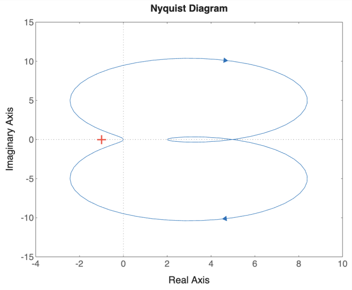
is stable, if and only if the Nyquist plot of $L(s)$ has no net (clockwise) encirclements of the critical point $s = -1$.

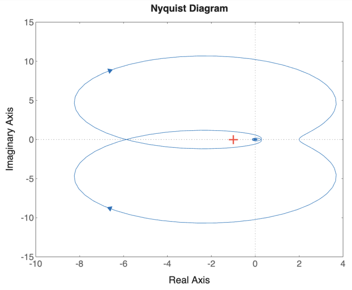
[3] A formal proof requires results from the theory of complex variables, more specifically, the principle of variation of the argument.

Example – Effect of a time-delay

Consider the two (stable) open-loop transfer functions[4]

$$L(s) = \frac{s+2}{s^2+0.2s+1}, \quad L_1(s) = e^{-s}\frac{s+2}{s^2+0.2s+1}.$$





- The Nyquist plot of $L(s)$ (left) does not encircle the critical point.
- However, the plot of $L_1(s)$ (right) encircles $s = -1$ once.
- **The delay destabilises this particular closed loop system!**

[4]Use Matlab commands: `s = tf("s"); L1 = exp(-s)*tf([1 2],[1 0.2 1])`

General Nyquist Theorem

Theorem
Consider the open-loop transfer function $L(s)$ and the closed-loop transfer function

$$G_{yr}(s) = \frac{L(s)}{1 + L(s)}.$$

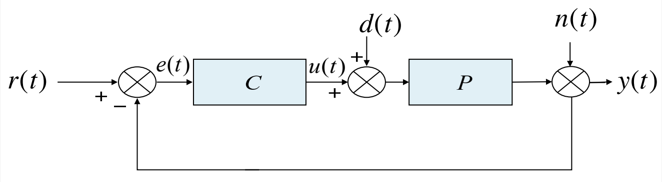
Suppose that:

1. *$L(s)$ has P poles in the right-half-plane.*
2. *For the Nyquist plot of $L(s)$, the* **net number of clockwise** *encirclement of the point $s = -1$ is N.*

Then $G_{yr}(s)$ has $N + P$ poles in the right half-plane.

In other words, for stability the Nyquist plot should have $N = -P$ clockwise encirclements, i.e., exactly P anti-clockwise encirclements.

Encirclement Equivalence



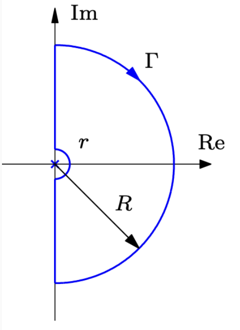
- Suppose we have designed a controller, up to a gain K: $C(s) = K\bar{C}(s)$.
- We would like find the range of gains K for which the closed loop system is stable.
- Direct calculations give

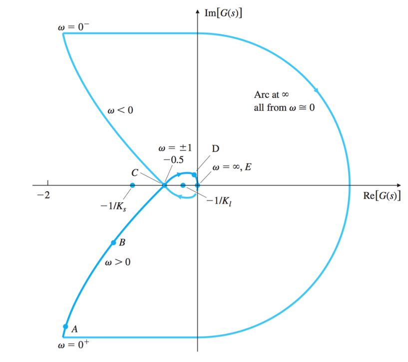
$$L(s) = P(s)C(s) = K\bar{L}(s), \quad \text{where} \quad \bar{L}(s) = P(s)\bar{C}(s).$$

- Every encirclement of -1 for L is an encirclement of $-1/K$ for $\bar{L}$.
- Thus, we only need to consider $\bar{L}(s)$ and count the number of encirclements of $-1/K$.

Example – Applying Nyquist Theorem to Controller Design

$$P(s) = \frac{1}{(s+1)^2}, \quad C(s) = \frac{K}{s} \quad \Longrightarrow \quad L(s) = K\bar{L}(s), \quad \bar{L}(s) = \frac{1}{s(s+1)^2}.$$



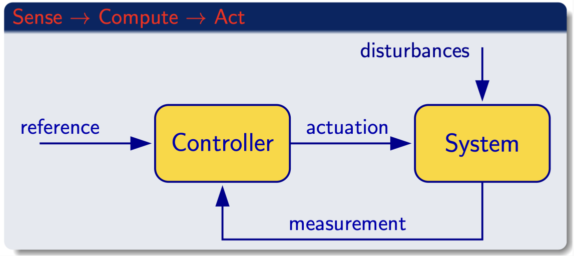


If $K >$ is large (K_l in the plot), then there are $N = 2$ encirclements and we have an unstable loop.

For small $K > 0$ (K_s in the plot), we have $N = 0$ and a stable loop.

Robustness Margins

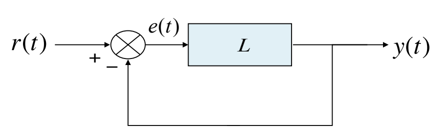
Model Uncertainty and Variability



An important property of feedback systems is their ability to **be robust to variability and model uncertainty**:

1. A single model is used for many systems with small variations (e.g. a large production run of manufactured products),
2. The true system is nonlinear,
3. The true system is higher-order (even infinite order),
4. The true system is time-varying,
5. An accurate model is unavailable or too expensive to obtain.

Stability Margins



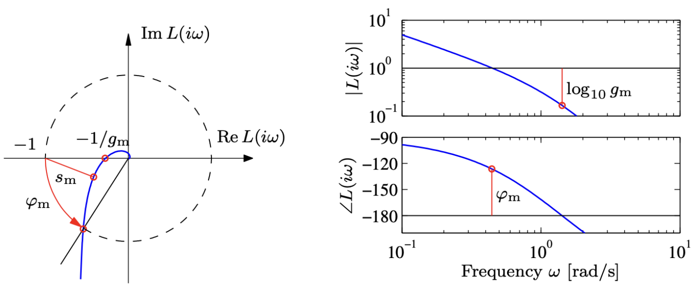
If there is model uncertainty, then the true (but **unknown**) open-loop transfer function, say $L_{true}(j\omega)$, is different to its model $L_{model}(j\omega)$.

1. Changes in **gain** are common if the effect of an actuator on the system is uncertain (e.g., aircraft at different speeds/altitudes).
2. **Phase** uncertainties are common due to filtering of signals, time-delay in computation, and higher-order unmodelled dynamics.

Stability margins alow us to assess how bad the model can be before there is instability.

Assuming that the nominal closed-loop system is stable, stability margins quantify **distance to instability**.

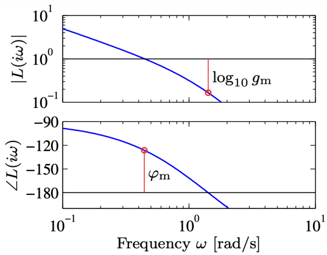
Stability Margins



- In a Nyquist plot, the question is: How much can the curve change before it changes the number of encirclements of $s = -1$?
- In a Bode plot, the critical point of $L(j\omega) = -1$ corresponds to gain $|L(j\omega)| = 1$, i.e., 0dB, and phase[5] $\angle L(j\omega) = -180°$.

[5] Or in general $(180 \pm 360k)°$ for some integer k, but $-180°$ is the most common since most physical systems have phase lag

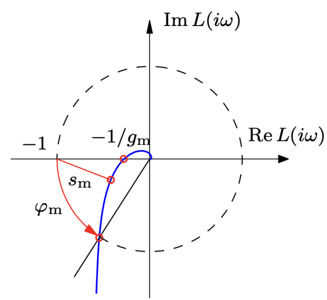
Gain and Phase Margins from Bode Plots



The critical points are the **0dB gain point**, and the $-180°$ **phase point**.

1. **Gain Margin g_m:** How much can the open loop gain be changed (usually increased) before instability?
   Look at frequencies where phase is already at $-180° \pm 360k$, and check the "gap" between the gain and 0dB.
2. **Phase Margin φ_m:** How much can the phase be changed (usually increased) before instability?
   Look at where the gain is already 0dB, and see how much the phase would need to change (usually more lag) to reach $-180° \pm 360k$.

Stability Margins from Nyquist Plots



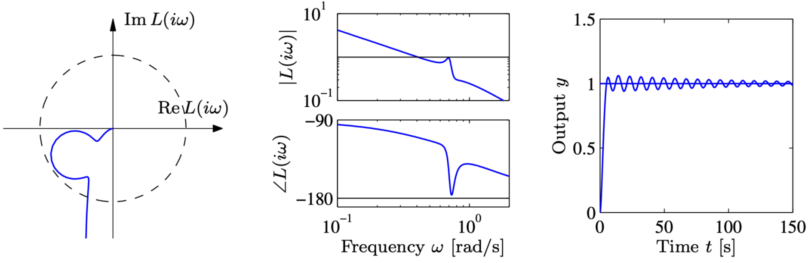
From a Nyquist plot we can analyse “distance to instability”, while **at the same time** checking closed-loop stability based on Nyquist’s theorem.

1. **Gain Margin** g_m**:** How much can the Nyquist plot be scaled before the number of encirclements changes?
2. **Phase Margin** φ_m**:** How much would the Nyquist plot have to be rotated before the number of encirclements changes?
3. **Stability Margin** s_m**:** Shortest distance to the critical point $s = -1$.

Example – Good gain and phase margins, but poor robustness

Consider the open-loop transfer function

$$L(s) = \frac{0.38(s^2 + 0.1s + 0.55)}{s(s+1)(s^2 + 0.06s + 0.5)}.$$



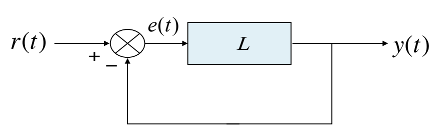
(a) Nyquist plot (b) Bode plot (c) Step response

This system has good gain and phase margins, but a poor stability margin!

Summary and Outlook

Closed-Loop Stability from the Open-loop Transfer Function

Open-loop transfer function: $L(s) = P(s)C(s)$

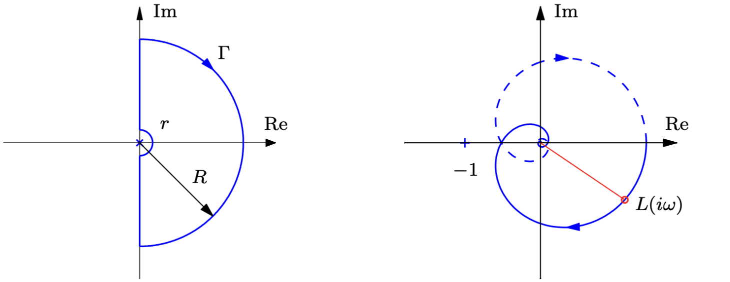


- (Some) closed-loop transfer functions are given by:

$$G_{yr}(s) = \frac{L(s)}{1 + L(s)}, \quad G_{er}(s) = \frac{1}{1 + L(s)}.$$

- The closed loop poles are the values s, such that $L(s) = -1$.
- A plot of the **open-loop** frequency response $L(j\omega)$ allows us to precisely state how many **closed-loop** poles are located in the right half plane!

Nyquist Plots



(a) Nyquist contour (b) Nyquist plot

- Plot $L(j\omega)$ on the complex plane, for all ω from $-\infty$ to ∞.
- Matlab command `nyquist` can be used to produce a Nyquist plot:

$$L(s) = 1.4\frac{e^{-s}}{(s+1)^2}$$

```
s = tf("s"); L = 1.4*exp(-s)*tf([1],[1 2 1]), nyquist(L)
```

Nyquist Theorem

Theorem
Consider the open-loop transfer function $L(s)$ and the closed-loop transfer function

$$G_{yr}(s) = \frac{L(s)}{1 + L(s)}.$$

Suppose that:

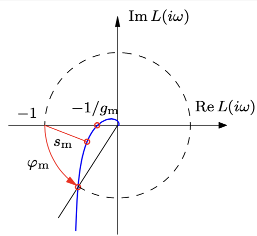
1. $L(s)$ *has* P *poles in the right-half-plane.*
2. *For the Nyquist plot of* $L(s)$*, the net number of clockwise encirclement of the point* $s = -1$ *is* N*.*

Then $G_{yr}(s)$ *has* $N + P$ *poles in the right half-plane.*

In other words, for stability the Nyquist plot should have $N = -P$ clockwise encirclements, i.e., exactly P anti-clockwise encirclements.

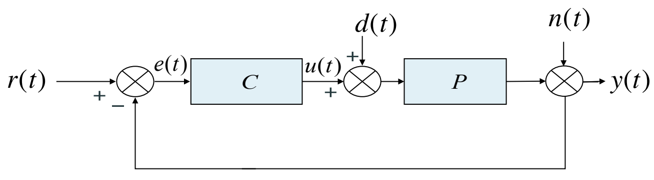
Robustness Margins

- An important property of feedback systems is their ability to **be robust to variability and model uncertainty**.
- If there is model uncertainty, then the true (but **unknown**) open-loop transfer function is different to its model.
- Assuming that the nominal closed-loop system (using the model $L(s)$) is stable, stability margins quantify **distance to instability**.
- Commonly used **Stability margins** can be read out from Bode or Nyquist plots:



1. Gain Margin: g_m
2. Phase Margin: φ_m
3. Stability Margin: s_m

Outlook: PID Control and Frequency Domain Design



In Lecture 10, we raised the following questions, some of which we have now worked on:

1. How do **interconnected systems** behave?
2. What is the response to **time-varying** references and disturbances?
3. What about measurement **noise**?
4. For an n^{th} order system, the controller is also n^{th} order. Is this really necessary? Would a **simpler** controller (e.g., PID) suffice?
5. What happens if the model is **inexact**?

Next week, we will tackle the remaining questions by going further into **PID control** and **frequency domain design methods**.