Lecture Notes 02

First-order Dynamical Systems

These notes present the mathematical essentials for this week's material. They are *complementary* to the material that will be presented in the lecture and tutorial. Material marked ”advanced” will not be assessed in the final exam.

Preliminaries: Real Exponentials

Let $a \in \mathbb{R}$ be a real number and then we define a function

$$y(t) = e^{at}$$

which maps every real number $t \in \mathbb{R}$ to a positive value $y \in \mathbb{R}^+$. Here $\mathbb{R}^+$ represents the set of all positive numbers.

> There holds
> $$\dot{y} = ae^{at} = ay(t),$$
> a key property that makes exponential functions extremely useful in solving linear differential equations.

We are also familiar with the following properties.

- When $a = 0$, $y(t) \equiv 1$ is a constant function;
- When $a < 0$, $\lim_{t\to+\infty} y(t) = 0$;
- When $a > 0$, $\lim_{t\to+\infty} y(t) = +\infty$.

Further, the following understandings will be useful.

(i) For any two real numbers a and b, there holds $e^{at} \cdot e^{bt} = e^{(a+b)t}$ for all $t \in \mathbb{R}$.

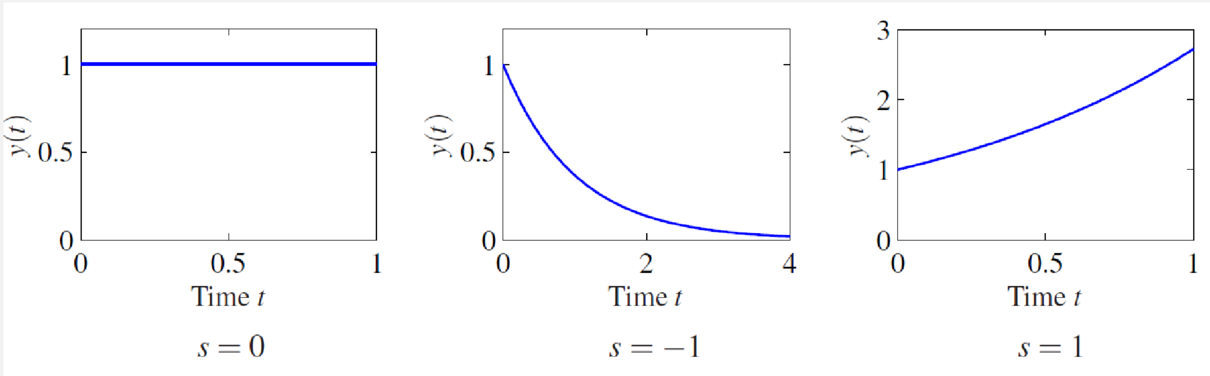
(ii) The Taylor series of $y(t)$ at the origin

$$\sum_{k=0}^{\infty} \frac{(at)^k}{k!} = 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots$$

converges to a finite number for any $t \in \mathbb{R}$, and the limit is exactly e^{at}. This means

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}, \quad t \in \mathbb{R}. \tag{1}$$

Figure 1: Real Exponentials e^{st}



Advanced: Differentiating Exponential Functions from Taylor Series

Let us differentiate the k'th term ($k \geq 1$) in the Taylor series of the exponential function e^{at}:

$$\frac{d}{dt}\frac{(at)^k}{k!} = \frac{a^k}{k!}\frac{d}{dt}t^k = \frac{a^k}{k!}kt^{k-1} = a\frac{(at)^{k-1}}{(k-1)!}.$$

Consequently,

$$\sum_{k=0}^{\infty}\frac{d}{dt}\frac{(at)^k}{k!} = a\sum_{k=1}^{\infty}\frac{(at)^{k-1}}{(k-1)!}.$$

From (1) we can see

$$\frac{d}{dt}e^{at} = \sum_{k=0}^{\infty}\frac{d}{dt}\frac{(at)^k}{k!}, \quad a\sum_{k=1}^{\infty}\frac{(at)^{k-1}}{(k-1)!} = ae^{at}.$$

In other words, we just proved (!)

$$\frac{d}{dt}e^{at} = ae^{at}.$$

1 First-order Dynamical Systems

We study first-order differential equations of the form

$$\dot{x} = f(x, t) \tag{2}$$

where x is a scaler system. The system is called *autonomous* if function $f(x, t)$ does not depend on t. Given an equation of any type, our primary concern would be whether such equation can possibly admits any solution, and if so how many solutions there are. In general, differential equations can be quite strange and difficult in that regard.

Example 1. Consider the differential equation about a scalar signal x:

$$\dot{x} = 1 + x^2.$$

We can easily verify

$$x(t) = \tan(t)$$

is indeed a solution of the differential equation, for the time interval $(-\pi/2, \pi/2)$. The signal however cannot be extended passing the point $\pi/2$. This is to say, solutions may not exist for all time for differential equations.

Example 2. Consider the equation

$$\dot{x} = \sqrt{x}. \tag{3}$$

First of all we can verify that the signal

$$x(t) = \begin{cases} 0, & t \leq t_0 \\ \frac{1}{4}(t - t_0)^2, & t > t_0 \end{cases}$$

satisfies the ODE for all t for whatever choice of t_0. Therefore a differential equation can have an infinite number of solutions. This is in fact the case for many differential equations.

From this general form, we can for example verify that

$$x_1(t) \equiv 0, \quad t \in \mathbb{R}$$

and

$$x_2(t) = t^2/4, \quad t \in \mathbb{R}$$

are both solutions of ODE whose values pass through the zero with $t = 0$. The first solution $x_1(t)$ remains a constant all the time. This is a bit wield because supposing (3) describes the motion of some real-world physical system $x(t)$, the trajectory $x(t)$ should be unique looking ahead of time if we start from $x(0) = 0$! Thoughts along this line pointed to the notion of well-posedness of ODEs as models of physical systems.

1.1 External Input and Equilibria

The system (2) can be subject to external input even it is a simple autonomous system, resulting in

$$\dot{x} = f(x, u). \tag{4}$$

This is the state-space equation of dynamical systems with u being a constant or time-varying input signal. Suppose $(x_e, u_e) \in \mathbb{R}^2$ is a pair of points for the state and input such that

$$f(x_e, u_e) = 0.$$

In this case (x_e, u_e) is an *equilibrium* of the differential equation (4), which implies

$$x(t) \equiv x_e, \ t \in \mathbb{R}$$

is always a solution[1] of (4). The equation can of course have multiple equilibria.

2 First-order Linear Dynamical Systems

Let us now investigate the following differential equation

$$\dot{x}(t) + ax(t) = u(t) \tag{5}$$

describing the dynamics of a system response $x(t)$ over time, where $a \in \mathbb{R}$ is a constant and $u(t)$ is the system input. This differential equation is called to be linear time invariant because only linear combinations of $x(t)$ and its derivatives are involved in the equation, and the coefficients are all constants.

2.1 Homogeneous Solutions

Let $u(t) \equiv 0$. Then the equation (5) is reduced to

$$\dot{x}(t) + ax(t) = 0. \tag{6}$$

The differential equation (6) is called a homogeneous equation, and we can easily verify that

$$c_1 x_1(t) + c_2 x_2(t)$$

is also a solution of (6) for all c_1, c_2 given any two solutions $x_1(t)$ and $x_2(t)$ of the differential equation (6). This means all solutions of (6) form a linear space $\mathcal{H}$. We call any solution in $\mathcal{H}$ a homogeneous solution of (5). If $e^{\lambda t}$ is a homogeneous solution, then it satisfies

$$\lambda e^{\lambda t} + a e^{\lambda t} = 0.$$

Hence $\lambda = -a$. We can proceed to verify that any function in the form of ce^{-at} with $c \in \mathbb{R}$ satisfies the equation (6), and ce^{-at} is called a general solution.

[1] Have another look at Example 2!

Next, we want to solve the equation (6) with a given initial condition $x(0) = x_0$. Such a problem is called an initial-value problem in which we derive solutions to differential equations with fixed boundary conditions. Because the system is linear, from the theory of ordinary differential equations the initial-value problem has a unique solution which extends to the entire time horizon. We can see if we let $c = x_0$ in a general solution,

$$x(t) = e^{-at}x_0$$

will be a particular solution with $x(0) = x_0$. This further assures that

$$\mathcal{H} = \left\{ce^{-at},\ c \in \mathbb{R}\right\}$$

since every solution must have certain initial value.

2.2 Stability

The system (5) is called to be *stable* if all its homogeneous solutions tend to zero as time grows to infinity. It is obvious from the form of $\mathcal{H}$ that the system is stable if and only if a is a positive number. The system is called *marginally stable* if $a = 0$, in which case $\mathcal{H}$ contains all constant functions. The system is *unstable* if $a < 0$, in which case the homogeneous solution $x(t)$ grows to infinity as long as $x(0) \neq 0$.

2.3 Step Response

One simple yet important class of input functions is step functions: $u(t) = m\mathbf{1}(t)$ with

$$\mathbf{1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

where m is a constant. The function $\mathbf{1}(t)$ is called a unit step function. The physical meaning of such an input is obvious in that we are giving the system a constant input starting from the initial time $t = 0$. The resulting response of $x(t)$ for $t \geq 0$ is called the step response. It is worth mentioning that now that $u(t)$ is only continuous from the right-hand side at $t = 0$, the response $x(t)$ should be understood to satisfy the differential equation (5) for $t > 0$, but only satisfy the equation from the right-hand side of $t = 0$.

Given initial value $x(0) = x_0$. Let us do a trial solution $x(t) = c_1e^{-at} + c_2$ for the differential equation

$$\dot{x}(t) + ax(t) = m,\ t \geq 0. \tag{7}$$

From the initial value we know $c_1 + c_2 = x_0$. From the equation (7) we have

$$(-ac_1 + ac_1)e^{-at} + ac_2 = m.$$

Thus we obtain $c_1 = x_0 - m/a$ and $c_2 = m/a$. We conclude that the response of the system (5) to input function $u(t) = m\mathbf{1}(t)$ is

$$x(t) = \left(x_0 - \frac{m}{a}\right)e^{-at} + \frac{m}{a}, t \geq 0.$$

2.4 Exponential Inputs

2.4.1 General Solutions

Let us now consider the solution of (5) with $u(t)$ being an exponential function $u(t) = Ae^{st}$ with $s \in \mathbb{R}$. Let's do a trial solution $x(t) = Be^{st}$ assuming $u = Ae^{st}$. From the equation (5) we obtain

$$sBe^{st} + aBe^{st} = Ae^{st}.$$

Hence

$$B = \frac{A}{s+a} = G(s)A$$

with $G(s) = \frac{1}{s+a}$. $G(s)$ is called the *transfer function* and relates the output to the input. Therefore, we learned that

Fact. $x(t) = \frac{A}{s+a}e^{st}$ is a solution of (5) with $u = Ae^{st}$.

Now select $x_h(t) \in \mathcal{H}$ as an arbitrary homogeneous solution of (5). We can verify straightforwardly that

$$x(t) = \frac{A}{s+a}e^{st} + x_h(t)$$

is a solution of (5)! Therefore, indeed the equation (5) has many solutions with the input function $u = Ae^{st}$. In fact, given $u = Ae^{st}$, all solutions of (5) form the set

$$\mathcal{S} := \Big\{ \frac{A}{s+a}e^{st} + x_h(t) : x_h \in \mathcal{H} \Big\}.$$

2.4.2 Particular Solutions

As discussed above, any signal in the set $\mathcal{H}$ is a solution (5) with input $u = Ae^{st}$. Now we assume the solution that we are interested in starts from initial value $x(0) = x_0$. From

$$x(t) = \frac{A}{s+a}e^{st} + ce^{-at}$$

we obtain

$$x_0 = \frac{A}{s+a} + c$$

and therefore $c = x_0 - \frac{A}{s+a}$. Consequently, of all solutions in the set $\mathcal{S}$ there is a unique one

$$\begin{aligned} x(t) &= (x_0 - \frac{A}{s+a})e^{-at} + \frac{A}{s+a}e^{st} \\ &= e^{-at}x_0 + \int_0^t e^{-a(t-\tau)} Ae^{s\tau} d\tau \\ &= e^{-at}x_0 + \int_0^t e^{-a(t-\tau)} u(\tau) d\tau \end{aligned} \tag{8}$$

corresponding to the initial condition $x(0) = x_0$. We can write $x(t) = x_1(t) + x_2(t)$ with

$$x_1(t) = e^{-at}x_0, \quad x_2(t) = \int_0^t e^{-a(t-\tau)} u(\tau) d\tau.$$

The part $x_1(t)$ is from the initial value x_0, while the part $x_2(t)$ is generated from the input $u(t)$.

2.4.3 Steady-state Responses

When the system is stable, the system response $x(t)$ will be close to

$$x_{ss}(t) := \frac{A}{s+a}e^{st} = G(s)u(t)$$

as t grows to infinity. Here $x_{ss}(t)$ is termed the *steady-state response*. From the third equality of (8) we know that having stability, the steady state response with input $c_1u_1(t)+c_2u_2(t)$ will be $c_1y_{ss}(t)+c_2z_{ss}(t)$, where $y_{ss}(t)$ and $z_{ss}(t)$ are the steady-state responses under u_1 and u_2, respectively.

Exercise 1 *Can you extend this material to sinusoidal inputs? (Hint: consider the complex exponential.)*

3 Existence and Uniqueness (advanced)

3.1 Cauchy Solutions

Consider the following differential equation subject to initial conditions

$$\dot{x} = f(x,t) \tag{9}$$

$$x(t_0) = x_0 \tag{10}$$

where $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a function, $t_0 \in \mathbb{R}$ is a given time and $x_0 \in \mathbb{R}$. Finding solutions to (9) and (10) is called a Cauchy problem. If $x(t)$ is a function defined over the interval $(t_0 - \delta, t_0 + \delta)$ satisfying (9) and (10) for any $t \in (t_0 - \delta, t_0 + \delta)$, then we call $x(t)$ a local Cauchy solution.

Theorem 2 (Peano Existence Theorem) *Suppose f is continuous in a neighborhood of (x_0, t_0) in $\mathbb{R}^2$. Then the equations (9) and (10) always admit local solutions defined over $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$.*

Further, we call f to be locally Lipschitz continuous at x_0 and uniformly with respect to t if we can find a neighborhood $\mathcal{D}$ of (x_0, t_0) in $\mathbb{R}^2$ such that there exists $L > 0$ with

$$|f(y,t) - f(z,t)| \leq L|y-z|$$

for all $(y,t) \in \mathcal{D}$ and $(z,t) \in \mathcal{D}$. With this Lipschitz continuity, we can go beyond the Peano Existence Theorem and conclude that the equations (9) and (10) admit unique local solutions.

Theorem 3 (Picard Lindelöf Existence Theorem) *Further, we call f to be locally Lipschitz continuous at x_0 and uniformly with respect to t if we can find a neighborhood $\mathcal{D}$ of (x_0, t_0) in $\mathbb{R}^2$ such that there exists $L > 0$ with*

$$|f(y,t) - f(z,t)| \leq L|y-z|$$

for all $(y,t) \in \mathcal{D}$ and $(z,t) \in \mathcal{D}$. With this Lipschitz continuity, we can go beyond the Peano Existence Theorem and conclude that the equations (9) and (10) admit unique local solutions.

Exercise 4 *Can you verify that linear systems always satisfy these conditions?*