First and Second Order Systems

1. [10 Marks] Consider the system

$$\dot{x} + ax = u.$$

   (a) [2 Marks] Let $u = 0$ for all time, and consider $x(t)$. If $x(0) = 1$ and $x(2) = e^{-8}$, what is the value of a?

   (b) [4 Marks] Now let a be the value found in Part (a). If $u(t) = \mathbf{1}(t)$ is the unit step function and $x(0) = 0$, derive the response of $x(t)$.

   (c) [4 Marks] For $a = 6$, what is the value of u, so that $x(t)$ approaches 6 as t tends to infinity for any initial value $x(0)$? **Explain your reasoning.**

2. [15 Marks] Consider the following second-order system

$$\ddot{x} + 3\dot{x} + 2x = u.$$

   (a) [2 Marks] What are the poles of the system?

   (b) [3 Marks] What is the meaning that the system be stable in terms of system response $x(t)$? Is the system stable or not?

   (c) [10 Marks] Let $r(t)$ be a constant reference. Design a PD controller

$$u(t) = K_p(r - x) - K_d\dot{x}$$

   so that the system response to a step input has a settling time around 2 sec and an overshoot of about 5%. **Show every step of working**.

State Space Control Design

3. [10 Marks] Let us reconsider the second-order system

$$\ddot{x} + 3\dot{x} + 2x = u.$$

   (a) [3 Mark] Rewrite the dynamics in state-space form.

   (b) [3 Mark] Is the system reachable? **Show your working**.

(d) [4 Marks] Suppose the system output is given by

$$y(t) = \dot{x}.$$

Is the system observable? **Show your working**.

4. [20 Marks] Consider the system given by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$.

(a) [4 Marks] What is the meaning of the system being observable?

(b) [4 Marks] What is the purpose of an observer?

(c) [4 Marks] Consider an observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

where $L \in \mathbb{R}^{n\times 1}$. What is the condition that L should satisfy for the observer to work?

(d) [8 Marks] Now, let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [0\ 1]. \tag{1}$$

Design an observer with observer gain $L = [l_1, l_2]$ so that the poles of $A - LC$ are exactly -1 and -2. **Show every step of working**.

Frequency-Domain Control Design

5. [15 Marks] Consider the unity feedback system shown in Figure 1.

The Nyquist diagram for the plant

$$G(s) = \frac{s^2 - 5s - 4}{2(s^2 + 5s + 6)}$$

(with gain $K = 1$) is given in Figure 2.

(a) [4 marks] Is the closed loop system stable for $K = 1$? **Explain your answer.**

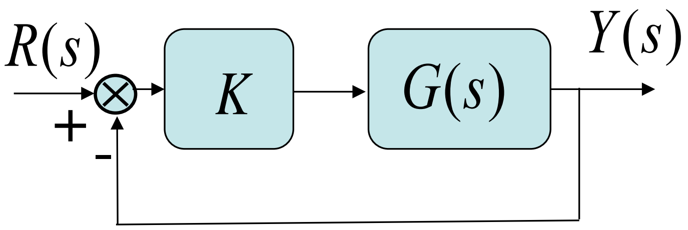

Figure 1: Unity feedback system for Question 6.

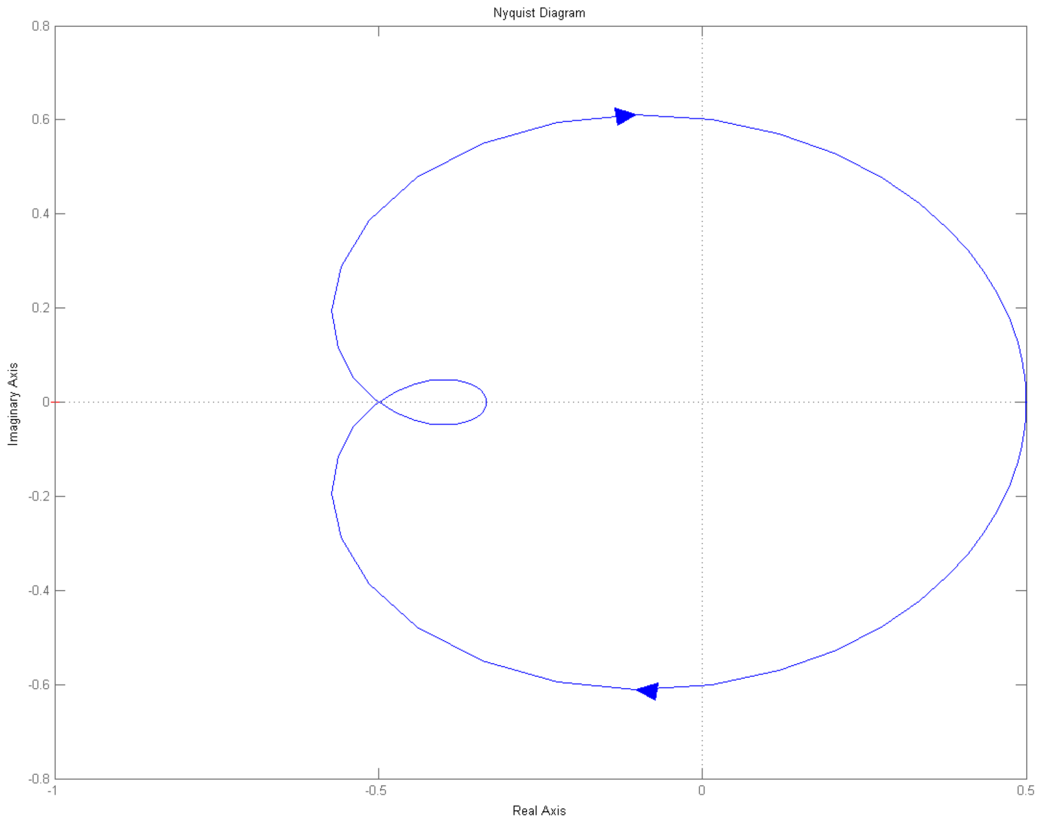

Figure 2: Nyquist plot for Question 5.

(b) [3 marks] What is the gain margin of the system with $K = 1$?

(c) [8 marks] The Nyquist plot goes through the point $(-0.5, 0)$. Compute the value of s in $G(s)$ that corresponds to that point. **Show all working.**

6. [10 Marks] Consider a plant with transfer function

$$P(s) = \frac{1}{s(s+1)}.$$

(a) [6 Marks] For a PD control $C(s) = s + 3$, compute the sensitivity and complementary sensitivity functions, respectively.

(b) [4 Marks] Explain the significance of the sensitivity function with respect to system uncertainty.

Real-world Dynamics and Control

7. [20 Marks] In *vivo*, e.g., within a single living cell, various types of proteins are interacting with each other, which are in the meantime continuously synthesized by genes through mRNA. A full description of such a system would involve thousands of signals, which is difficult to achieve. However, we can simplify our world by focusing on the relationship between one particular type of protein and one particular type of mRNA.

Let $x_1(t)$ be the concentration of the mRNA, and $x_2(t)$ be the concentration of the protein, respectively. The interaction between the two **states** is described by

$$\dot{x}_1 = p_1 u - k_1 x_1 \tag{2}$$

$$\dot{x}_2 = p_2 x_1 - k_2 x_2 \tag{3}$$

where p_1, p_2 and k_1, k_2 are parameters for the production rates and degradation rates, and u is our **input**.

(a) [4 Marks] The system (2)-(3) can be written as a state-space model in the form of $\dot{x} = Ax + Bu$ with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{4}$$

Write down A and B.

(b) [3 Marks] Suppose we have sensors that can measure the real-time concentration of the protein $x_2(t)$. Let us call $x_2(t)$ the system output $y(t)$. That is, $y(t) = x_2(t)$. Represent $y(t)$ as $y(t) = Cx(t)$ using state-space model.

(c) [9 Marks] Now, we hope to design a controller under which the system output $y(t)$ (that is, $x_2(t)$) can track a constant reference signal r. Note that we only have $y(t)$ known for the controller. What would be your approach for the design of the controller? **Describe each step of your design**.

(d) [4 Marks] Explain possible *uncertainties* for the equation (2)-(3) considering the real world scenario as described from the beginning.

END OF EXAM.