First and Second Order Systems

1. [10 Marks] Consider the system

$$\dot{x} + ax = u.$$

   (a) [2 Marks] Let $u = 0$ for all time, and consider $x(t)$. If $x(0) = 3$ and $x(2) = e^{-6}$, what is the value of a?

   (b) [4 Marks] Now let a be the value found in Part (a). If $u(t) = \mathbf{1}(t)$ is the unit step function and $x(0) = 0$, derive the response of $x(t)$.

   (c) [4 Marks] For $a = 6$, what is the value of u, so that $x(t)$ approaches 3 as t tends to infinity for any initial value $x(0)$?

2. [10 Marks] Consider the following second-order system

$$\ddot{x} + 4\dot{x} + 3x = u.$$

   (a) [2 Marks] What are the poles of the system?

   (b) [3 Marks] What is the meaning that the system be stable in terms of system response $x(t)$? Is the system stable or not?

   (c) [5 Marks] Let $r(t)$ be a constant reference. Design a PD controller

$$u(t) = K_p(r - x) - K_d\dot{x}$$

   so that the system response to a step input has a settling time around 2 sec and an overshoot of about 5%. Show all working.

3. [10 Marks] The dynamics of a plane pendulum subject to some external force is described by

$$\ddot{x} + \frac{g}{\ell}\sin x = u(t),$$

   where x is the angular displacement, ℓ is the length of its link, and g is the standard gravity.

   (a) [2 Marks] What is the equilibrium of the pendulum with $u(t) = F$ being a constant?

   (b) [3 Marks] Write down the linearized equation for small angular displacement.

   (c) [5 Marks] Derive the response $x(t) = X\sin(\omega t)$ subject to $u(t) = \sin(\omega t)$ from the linearized equation.

State Space Control Design

4. [12 Marks] Let us reconsider the second-order system

$$\ddot{x} + 4\dot{x} + 3x = u.$$

   (a) [2 Mark] Rewrite the dynamics in state-space form.

   (b) [5 Mark] Is the system reachable? Show your working.

   (d) [5 Marks] Suppose the system output is given by

$$y(t) = \dot{x}.$$

   Is the system observable? Show your working.

5. [13 Marks] Consider the system given by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$.

   (a) [4 Marks] What is the meaning of the system being observable?

   (b) [4 Marks] What is the purpose of an observer?

   (c) [5 Marks] Consider an observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

   where $L \in \mathbb{R}^{n\times 1}$. What is the condition that L should satisfy for the observer to work? Explain your reasoning.

Frequency-Domain Control Design

6. [15 Marks] Consider the unity feedback system shown in Figure 1.

The Nyquist diagram for the plant

$$G(s) = \frac{s^2 - 5s - 4}{2(s^2 + 5s + 6)}$$

(with gain $K = 1$) is given in Figure 2.

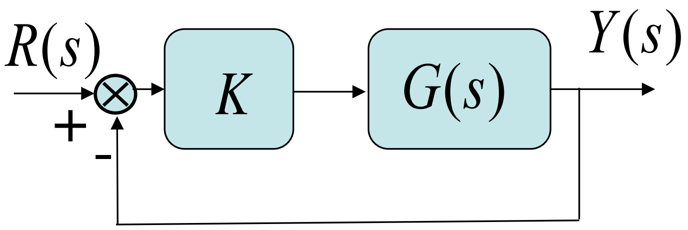

Figure 1: Unity feedback system for Question 6.

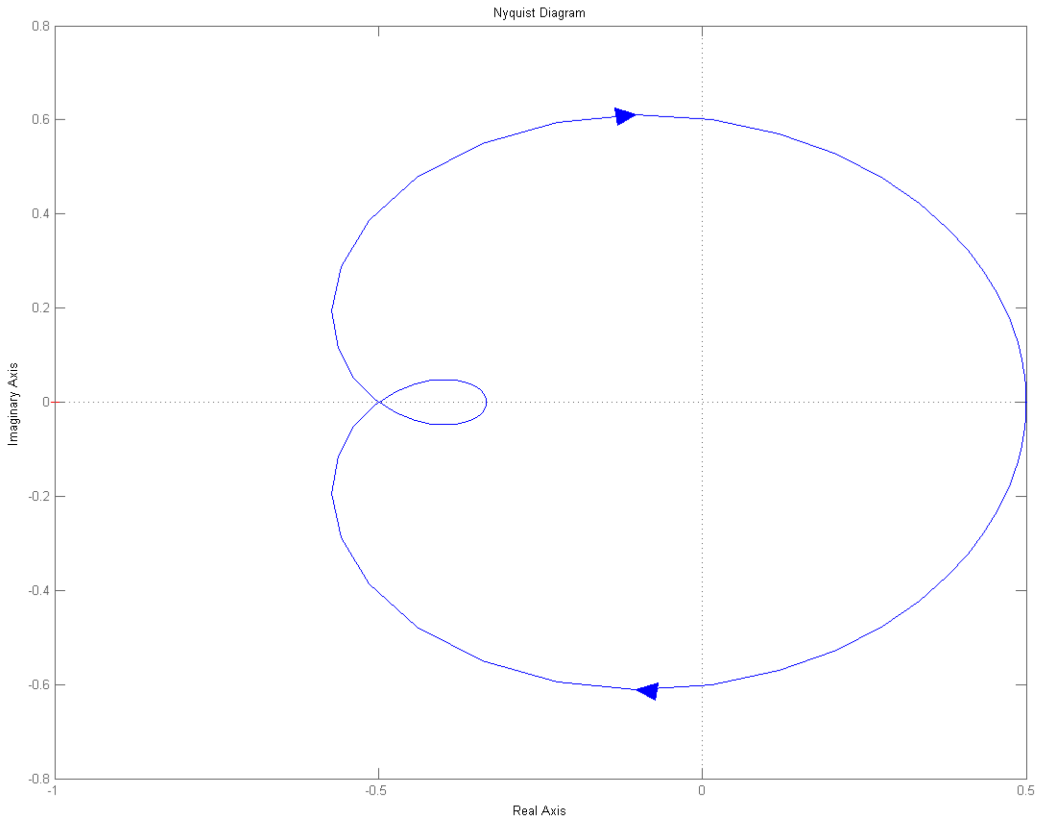

Figure 2: Nyquist plot for Question 6.

(a) [4 marks] Is the closed loop system stable for $K = 1$? *Explain your answer.*

(b) [4 marks] What is the definition of gain margin? Particularly, what is the gain margin of the system with $K = 1$? *Explain your answer.*

(c) [7 marks] The Nyquist plot goes through the point $(-0.5, 0)$. Find the value of s in $G(s)$ that corresponds to that point.

7. [15 Marks] Consider a plant with transfer function

$$P(s) = \frac{1}{s(s+1)}.$$

(a) [6 Marks] For a PD control $C(s) = s + 2$, compute the sensitivity and complementary sensitivity functions, respectively. Show the working and the results.

(b) [4 Marks] Explain the significance of the sensitivity function with respect to system uncertainty.

(c) [5 Marks] Explain how the sensitivity function and complementary sensitivity function impose limitations to the performance of feedback controllers in the presence of disturbances and noises.

Real-world Dynamics and Control

8. [15 Marks] In *vivo*, e.g., within a single living cell, various types of proteins are interacting with each other, which are in the meantime continuously synthesized by genes through mRNA. A full description of such a system would involve thousands of signals, which is difficult to achieve. However, we can simplify our world by focusing on the relationship between one particular type of protein and one particular type of mRNA.

Let $x_1(t)$ be the concentration of the mRNA, and $x_2(t)$ be the concentration of the protein, respectively. The interaction between the two **states** is described by

$$\dot{x}_1 = p_1 u - k_1 x_1 \tag{1}$$

$$\dot{x}_2 = p_2 x_1 - k_2 x_2 \tag{2}$$

where p_1, p_2 and k_1, k_2 are parameters for the production rates and degradation rates, and u is our **input**.

(a) [2 Marks] Let us for the moment assume x_1 is a constant, i.e., $x_1(t) = a$ for all t. Thus, we can just look at the Eq. (2), which becomes

$$\dot{x}_2 = -k_2 x_2 + p_2 a. \tag{3}$$

Should k_2 be a positive or negative number for a real-world cell? Show your reasoning.

(b) [3 Marks] The system (1)-(2) can be written as a state-space model in the form of $\dot{x} = Ax + Bu$ with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{4}$$

Write down A and B.

(c) [2 Marks] Suppose we have sensors that can measure the real-time concentration of the protein $x_2(t)$. Let us call $x_2(t)$ the system output $y(t)$. That is, $y(t) = x_2(t)$. Represent $y(t)$ as $y(t) = Cx(t)$ using state-space model.

(d) [5 Marks] Now, we hope to design a controller under which the system output $y(t)$ (that is, $x_2(t)$) can track a constant reference signal r. Note that we only have $y(t)$ known for the controller. What would be your approach for the design of the controller? Describe each step of your design.

(c) [3 Marks] Explain possible *uncertainties* for the equation (1)-(2) considering the real world scenario as described from the beginning.

END OF EXAM.