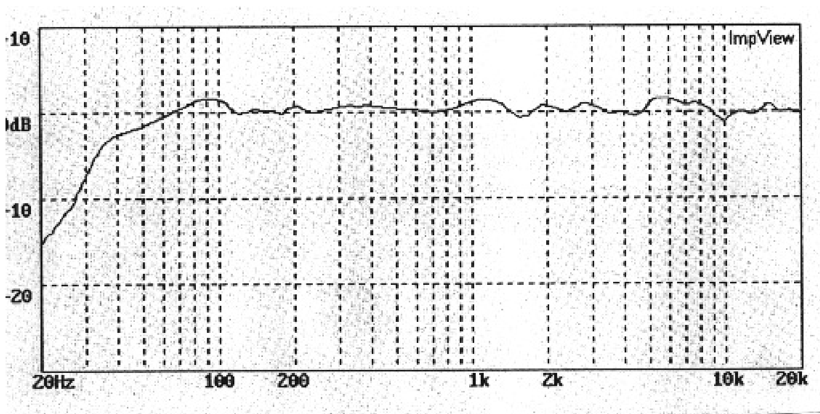
Lecture Notes 09
Bode Diagrams

Further Readings. Chapter 9. Åström and Murray, *Feedback Systems*, Second Edition.

Frequency response is something that can be directly measured in the laboratory, with a sinusoidal source of variable frequency and an oscilloscope. The idea is to measure the magnitude and phase of the output relative to the input at a range of frequency values.



[Magnitude frequency response of a loudspeaker]

1 Transfer Functions: Review

For an LTI dynamical system with input $u(t)$ and output $y(t)$, we have come to the understanding that if the input signal $u(t)$ is some exponential function, i.e., $u(t) = Ue^{st}$ with s being a complex number, the output $y(t)$ will have **a** solution given by

$$y_*(t) = G(s)e^{st}$$

where $G(s)$ is the so-called transfer function. For now we have established two equivalent representations of LTI dynamical systems, and we can drive the $G(s)$ from each of them:

(i) ODE Form. The system is described by a high-order differential equation written as

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + ... + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + ... + b_m u.$$

Because $\frac{d^k u}{dt^k} = s^k U e^{st}$ and $\frac{d^k y}{dt^k} = s^k G(s) U e^{st}$, the transfer function is

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{s^n + a_1 s^{n-1} + ... + a_n}.$$

A few common examples are:

– Integrator $\dot{y} = u$: $G(s) = \frac{1}{s}$.

– Differentiator $y = \dot{u}$: $G(s) = s$.

– First-order system $\dot{y} + ay = bu$: $G(s) = \frac{b}{s+a}$

– Second-order system: $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u$: $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$.

(ii) State-space Form. For state space models

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

we derived

$$G(s) = C(sI - A)^{-1}B + D.$$

This came from the introduction of $x(t) = X(s)Ue^{st}$, so

$$\dot{x}(t) = sX(s)Ue^{st} = AX(s)Ue^{st} + BUe^{st}$$
$$sX(s) = AX(s) + B$$
$$X(s) = (sI - A)^{-1}B$$

and $y(t) = (CX(s) + D)Ue^{st} = (C(sI - A)^{-1}B + D)u(t)$.

The importance of this particular solution given by the transfer function lies in the fact that if the system is stable, then such a solution will be the steady-state solution, i.e., all solutions regardless of their corresponding initial conditions will eventually converge to this steady-state response.

Steady-state Response

For s stable LTI sytem with sinusoid inputs $u(t) = e^{j\omega t}$, the *steady state* output is

$$y_{ss}(t) = G(j\omega)u(t).$$

Here $G(s)$ is the transfer function of the system.

In view of this steady-state exponential response, we can further derive the frequency response for the input $u(t) = A\cos\omega t$.

Frequency Response

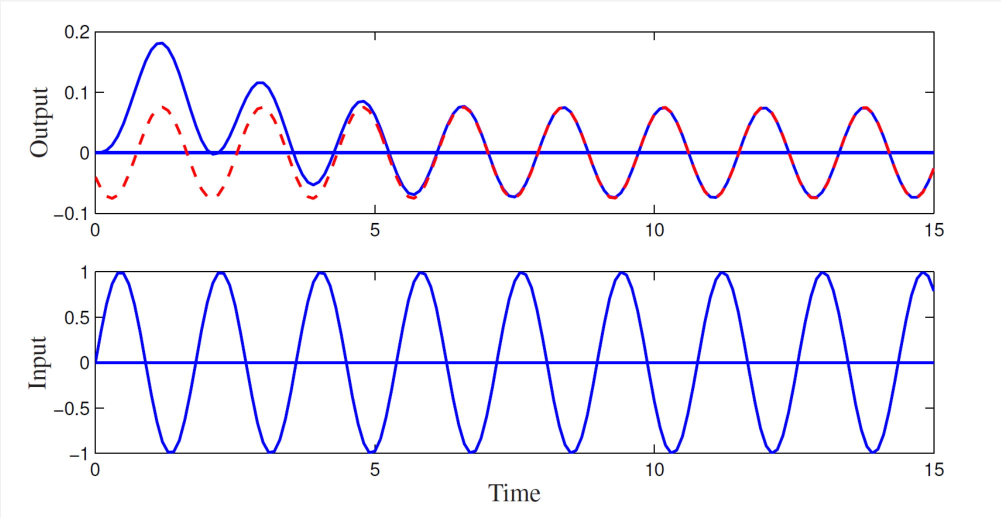
If $u(t) = A\cos\omega t$ we can represent this as

$$u(t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right).$$

Then we can compute the steady state response as

$$y(t) = |G(j\omega)|A\cos(\omega t + \phi(\omega))$$

where $\phi(\omega) = \angle G(j\omega)$.



Therefore, $G(j\omega)$ is kind of important for us to understand the dynamic behavior of the system. In fact, in practice, for most of the time we just call $G(s)$ the system, without worrying too much about how the original representation of the system equations.

2 Bode Diagrams

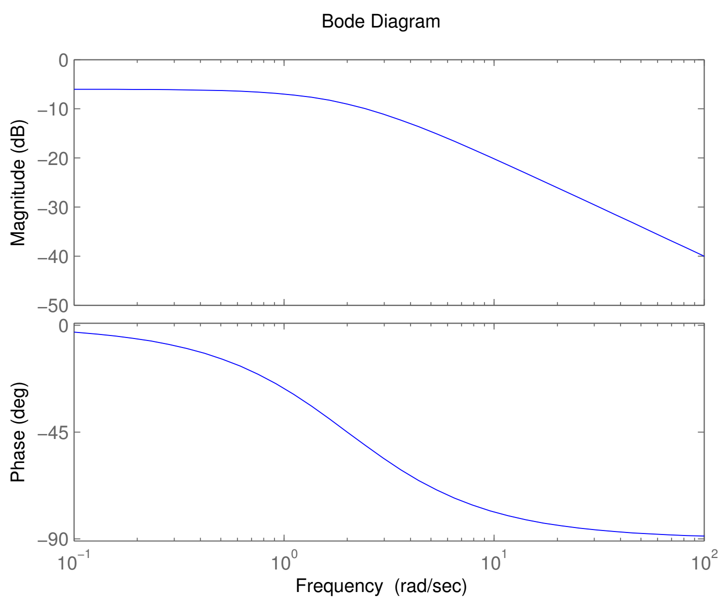
Now that $G(j\omega)$ fully describes the input-output relations in the time domain at the steady state for sinusoid input functions, we want to take a further look at $G(j\omega)$. We notice that **both** $|G(j\omega)|$ **and** $\angle G(j\omega)$ **are functions of** ω**.** By plotting these two functions in logarithmic scales, we obtain the Bode diagrams.

2.1 Definition

A Bode diagram is simply a graph of the transfer function $G(j\omega)$, respectively by magnitude and phase, plotted for $0 < \omega < +\infty$ where

- ω is expressed in a logarithmic scale,
- Magnitude is plotted in decibels (dB), i.e. $20\log_{10}|G(j\omega)|$,
- $\angle G(j\omega)$ is expressed in degrees.

Example. $H(s) = \frac{1}{s+2}$. The Bode diagram is:



This is easy in matlab:

```
g=tf(1, [1 2])
bode(g)
```

2.2 Bode Diagram Characteristics

We will be able to establish some very intuitive characteristics of Bode diagrams. This can be done entirely based on the properties of complex numbers of logarithms.

Complex Numbers

Recall that for complex numbers in polar form, e.g. $v = M_1 e^{j\phi_1}, u = M_2 e^{j\phi_2}, w = M_3 e^{j\phi_3}$ then

$$\begin{aligned}\frac{uv}{w} &= \frac{M_1 M_2 e^{j\phi_1} e^{j\phi_2}}{M_3 e^{j\phi_1}} \\ &= \frac{M_1 M_2}{M_3} e^{j(\phi_1+\phi_2-\phi_3)}\end{aligned}$$

I.e. magnitudes multiply/divide, but angles add/subtract.

Logarithms

Recall that for logarithms

$$\log(ab/c) = \log a + \log b - \log c.$$

Consider a system with gain k, zeros $-a, -b$, and poles $-c, -d$:

$$G(s) = \frac{k(s+a)(s+b)}{(s+c)(s+d)}$$

The magnitude is:

$$|G(s)| = \frac{|k| \cdot |(s+a)| \cdot |(s+b)|}{|(s+c)| \cdot |(s+d)|}$$

Let us take logs:

$$\log|G(s)| = \log|k| + \log|s+a| + \log|s+b| - \log|s+c| - \log|s+d|.$$

For angles we have

$$\angle G(s) = \angle k + \angle(s+a) + \angle(s+b) - \angle(s+c) - \angle(s+d).$$

This is pretty remarkable because now **if we can understand each lower order term of a transfer function, the Bode plots for the entire TF can be obtained by linear combinations.** It turns out to be convenient for plotting if we normalise each term

$$G(s) = \frac{kab}{cd} \frac{(\frac{s}{a}+1)(\frac{s}{b}+1)}{(\frac{s}{c}+1)(\frac{s}{d}+1)}$$

The log magnitude is then

$$\log|G(s)| = \log\left|\frac{kab}{cd}\right| + \log|\tfrac{s}{a}+1| + \log|\tfrac{s}{b}+1| - \log|\tfrac{s}{c}+1| - \log|\tfrac{s}{d}+1|$$

and for angles:

$$\angle G(s) = \angle(kab/(cd)) + \angle(\tfrac{s}{a}+1) + \angle(\tfrac{s}{b}+1) - \angle(\tfrac{s}{c}+1) - \angle(\tfrac{s}{d}+1).$$

Note that if $s = j\omega = 0$, then $G(0) = kab/cd$. This is called the zero-frequency gain, or DC gain (DC for "direct current"), and is the steady-state response to a constant input. Also, if $s = j\omega = \infty$, then $G(j\infty) = k$ which is known as the high-frequency gain.

Exercise. For a transfer function $G(s)$, verify that its high-frequency gain is a nonzero constant only if the numbers of zeros and poles are the same.

2.3 Constants, Integrators, Differentiators

For individual term in a transfer function being a constant, an integrator, or a differentiator, the corresponding characteristics of Bode diagrams are very easy to derive.

(i) The simplest is a constant gain k. This is obviously independent of s and always has magnitude $|k|$ and phase of either 0 (if $k > 0$) or 180 if ($k < 0$). This means, the magnitude plot is a straight line with zero slope.

(ii) A differentiator is a term s. When $s = j\omega$

$$|s| = |j\omega| = \omega. \quad 20\log|s| = 20\log\omega. \quad \angle s = \angle j\omega = 90^\circ.$$

This means, the magnitude plot is a straight line with 20 db/decade slope.

(iii) An integrator is a term $\frac{1}{s}$. When $s = j\omega$,

$$\left|\frac{1}{s}\right| = \left|\frac{1}{j\omega}\right| = \frac{1}{\omega}. \quad 20\log\left|\frac{1}{s}\right| = 20\log\left|\frac{1}{j\omega}\right| = -20\log\omega.$$

$$\angle\left|\frac{1}{s}\right| = \angle\left|\frac{1}{j\omega}\right| = -\angle j\omega = -90^\circ.$$

This means, the magnitude plot is a straight line with -20 db/decade slope.

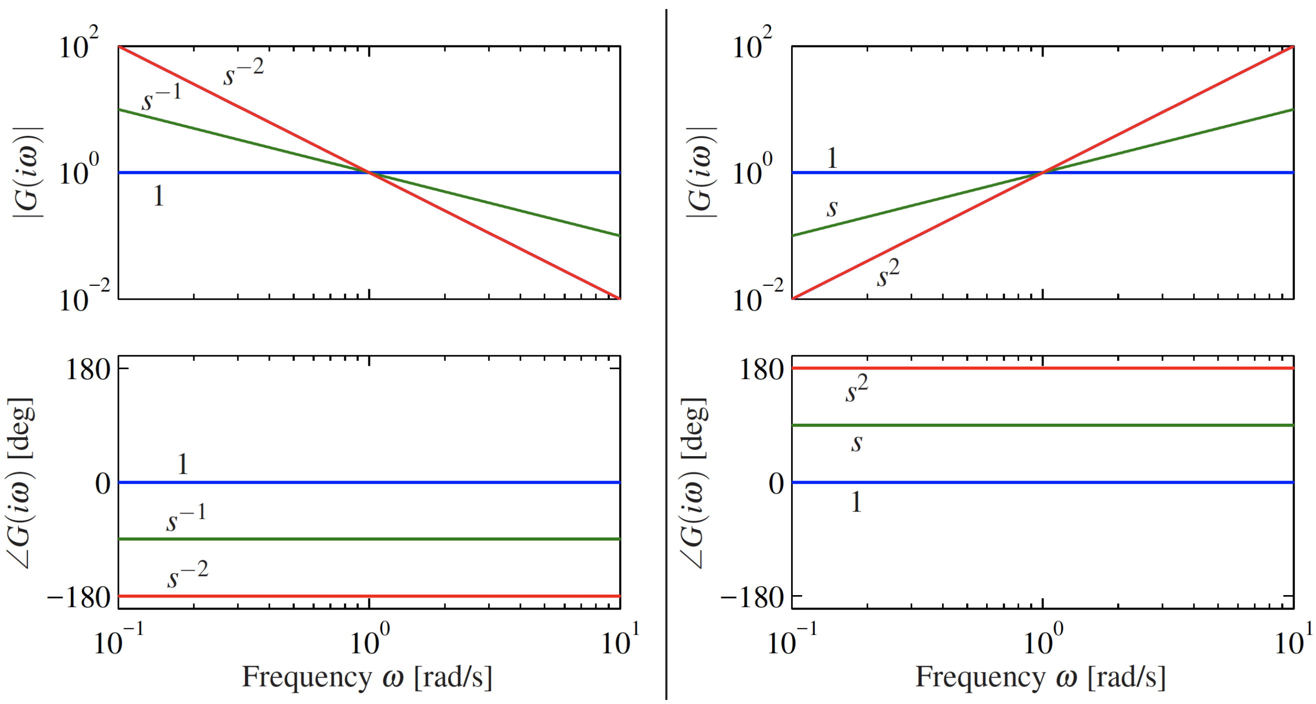


Figure 9.12: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope k. The phase curves for the transfer functions are constants, with phase equal to $90^\circ \times k$

2.4 First-order and Second-order Components

Now, for normalized first-order and second-order components, we can establish their characteristics by approximations. Such approximations eventually go back to constants, integrators, and differentiators at low or high frequencies.

First-order Component:

(i) Consider a normalised term $\frac{s}{a} + 1$. Note that this corresponds to a zero at $-a$. Then by direct computation we have

a) $|\frac{j\omega}{a} + 1| = \sqrt{1 + \omega^2/a^2}$

b) $20 \log \sqrt{1 + \omega^2/a^2} = 10 \log(1 + \omega^2/a^2)$

c) $\angle(\frac{j\omega}{a} + 1) = \arctan(\omega/a)$

which are fairly complex. However, we can make the following approximations:

- When $\omega \ll a$, $\frac{j\omega}{a} + 1 \approx 1$. As a result, the magnitude $10 \log(1 + \omega^2/a^2) \approx 10 \log 1 = 0$, and the angle $\arctan(\omega/a) \approx 0$.
- When $\omega = a$, $\frac{j\omega}{a} + 1 = 1 + \omega$. As a result, $10 \log(1 + \omega^2/a^2) = 10 \log 2 \approx 3dB$, and $\arctan(\omega/a) = 45°$.
- When $\omega \gg a$, $\frac{j\omega}{a} + 1 \approx \frac{j\omega}{a}$. Hence $10 \log(1 + \omega^2/a^2) \approx 10 \log(\omega^2/a^2) = 20 \log(\omega/a)$ and $\arctan(\omega/a) \approx 90°$.

(ii) For a term $\frac{1}{\frac{s}{a}+1}$, magnitudes are just reciprocal, and angles have opposite sign.

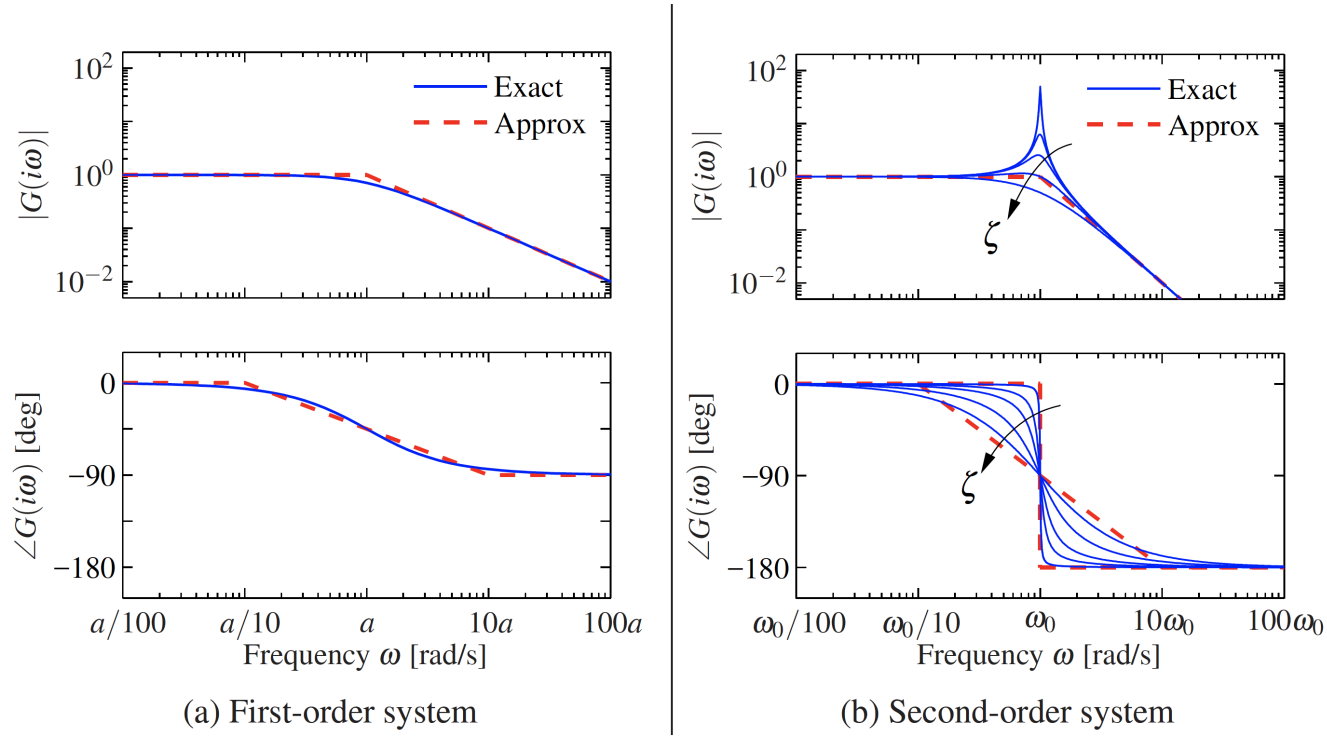
Second-order Component:

(i) Second-order overdamped or critically-damped systems can be factored into first-order terms with real zeros/poles.

(ii) For underdamped terms, i.e. complex zeros/poles, it is more convenient to keep the second order term, normalised by natural frequency:

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{1}{(\frac{j\omega}{\omega_n})^2 + 2\zeta j \frac{\omega}{\omega_n} + 1}$$
$$= \frac{1}{(1 - (\frac{\omega}{\omega_n})^2) + j2\zeta\frac{\omega}{\omega_n}}$$

- When $\omega \ll \omega_n$, $20 \log |G(j\omega)| \approx 20 \log 1 = 0$. $\angle G(j\omega) \approx 0$.
- When $\omega = \omega_n$, $20 \log |G(j\omega)| = 20 \log |\frac{1}{j2\zeta}| = 20 \log \frac{1}{2\zeta}$. $\angle G(j\omega) = \angle \frac{1}{j2\zeta} = -90°$.
- When $\omega \gg \omega_n$, $20 \log |G(j\omega)| \approx 20 \log \frac{\omega_n^2}{\omega^2} = -40 \log \frac{\omega}{\omega_n}$. $\angle G(j\omega) = \angle \frac{-\omega_n^2}{\omega^2} \approx -180°$.



(a) First-order system

(b) Second-order system

2.5 Sketching Bode Diagram

Let us reconsider a system with gain k, zeros $-a, -b$, and poles $-c, -d$:

$$G(s) = \frac{k(s+a)(s+b)}{(s+c)(s+d)}.$$

When we normalise each term we get

$$G(s) = \frac{kab}{cd} \frac{(\frac{s}{a}+1)(\frac{s}{b}+1)}{(\frac{s}{c}+1)(\frac{s}{d}+1)}$$

Now using the above characteristics and approximations for each term, we can get a straight line sketch for the overall Bode diagram by the following procedure.

S1. Find the low frequency gain $G(0) = \frac{kab}{cd}$. The Bode magnitude plot will start from $20 \log |\frac{kab}{cd}|$ db at low frequencies.

S2. Find the four frequencies, a, b, c, d. They are the so-called separation frequencies, and at logarithms scales they correspond to $\log a, \log b, \log c, \log d$, respectively. Suppose $a < b < c < d$.

S3. We carry out the sketch from low to high frequencies.

- Before the frequency a, it is a straight line with zero slope as $G(j\omega)$ is a constant approximately.
- Extending the above line we fist meet the frequency a. As a is from a zero, the slope of the line should add by 20 db/decade from a.
- Extending 20 db/decade-slope line from a until we meet b. As b is from a zero again, add another 20 db/decade slope.

- Extending the now 40 db/decade-slope line from b until we meet c. As c is from a pole, reduce the slope by 20 db/decade. So the slope becomes 20 db/decade again.
- Extending the now 20 db/decade-slope line from c until we meet d. As d is from a pole again, reduce the slope further by 20 db/decade. So the slope becomes 0 db/decade. Extend the plot beyond d and that is a sketch of the Bode plot for $G(s)$!

Exercise. Complete the sketching procedure by yourself as described above.

3 Bode' gain-phase relationship

> For any stable minimum phase system (i.e. no RHP poles and no RHP zeros), the phase of $G(j\omega)$ is uniquely related to the magnitude of $G(j\omega)$.

When the *slope* of $|G(j\omega)|$ is approximately constant over a decade or so (on a log-log plot), we have

$$\angle G(j\omega) \approx n \times 90^\circ,$$

where n is the slope in units of decade of amplitude per decade of frequency.