Lecture Notes 03

Feedback Control of First-order Systems

Let us revisit the linear dynamical system

$$\dot{x}(t) + ax(t) = bu(t) \tag{1}$$

where again we will focus on the case $b = 1$, i.e.

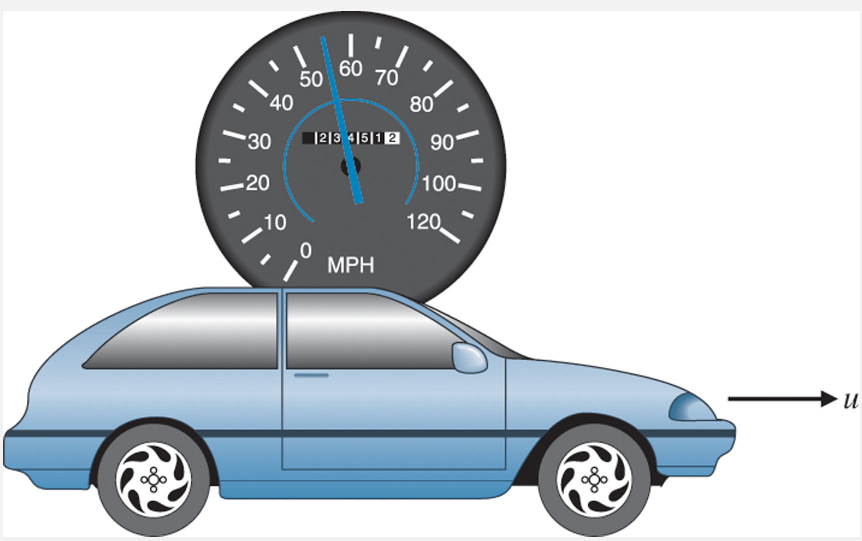
$$\dot{x}(t) + ax(t) = u(t) \tag{2}$$

describing the dynamics of a system response $x(t)$ over time, where $a \in \mathbb{R}$ is a constant and $u(t)$ is the system input. We consider the objective of making the response of the system (2) track a given reference r. That is, the objective is that $x(t) = r$ or that $|x(t) - r|$ is small.

Example 1. Consider a car moving along a straight line. We use y to denote the position of the car, and therefore $\dot{y}$ and $\ddot{y}$ are the velocity and acceleration of the car, respectively. Let u be the force applied to the car due to engine. We assume the drag force is proportional to the car's speed with a proportionality constant k_d (at high speeds a more accurate model may be quadratic). From mechanics, the equation of motion is $F = m\ddot{y}$, where

$$F = u - k_d\dot{y},$$

Figure 1: Cruise control model



This gives the following differential equation

$$\ddot{y} + \frac{k_d}{m}\dot{y} = \frac{u}{m}$$

Denote the velocity as $v = \dot{y}$. Then one obtains a first order linear differential equation

$$\dot{v} + \underbrace{\frac{k_d}{m}}_{a} v = \underbrace{\frac{1}{m}}_{b} u,$$

where the input is the applied force u and the output is the velocity v.

The aim is to maintain a set speed. In control engineering, this is called a *regulation* problem. Regulation is one of the most important purposes of control, going back to even before the Watt used a flyball governor to regulate the speed of steam engines due to varying loads and other disturbances. Feedback control systems were among the key technological innovations responsible for the industrial revolution. $\square$

1 Open-loop and Feedback Control

Open-loop Control. We just let $u(t) = r\mathbf{1}(t)$. In other words, $u(t)$ is a signal that is prescribed at $t = 0$ independent with $x(t)$. Then from Section 2.3 of Week 2's notes we have

$$x(t) = \left(x_0 - \frac{r}{a}\right) e^{-at} + \frac{r}{a}, \; t \geq 0. \tag{3}$$

There will be two cases.

- When the system is unstable, i.e., $-a > 0$, the response $x(t)$ diverges.
- When the system is stable, e^{-at} will decay to zero as time grows. Then there holds $\lim_{t\to\infty} x(t) = x(\infty) = r/a$. There will be a steady-state error

$$e(\infty) = r - x(\infty) = r \cdot \frac{a-1}{a}.$$

This steady-state error can be overcome by putting a signal amplifier before the controller. Instead of $u(t) = r\mathbf{1}(t)$ we use $u(t) = K_r r\mathbf{1}(t)$. With $K_r = a$ we have $x(\infty) = r$ and thus $e(\infty) = 0$. This K_r is called a feedforward gain. This approach appears to achieve zero error, but the disadvantage is that it requires *perfect* knowledge of a.

Feedback Control. At each time t we measure $e(t) = r - x(t)$ and let

$$u(t) = K_P e(t) = K_P(r - x(t)). \tag{4}$$

Here $K_P > 0$ is the feedback gain. The controller is called a feedback controller since the information of $x(t)$ is fed back to the input. Plugging (4) into (2) we obtain the closed loop system

$$\dot{x} + (K_P + a)x = K_P r.$$

Note that this is a **new first-order linear system** of the form (1), but with input r and output x. The new parameters are $a_{CL} = K_P + a$, $b_{CL} = K_P$.

The closed-loop system has the step response

$$x(t) = \left(x_0 - \frac{K_P r}{K_P + a}\right) e^{-(a+K_P)t} + \frac{K_P r}{K_P + a}. \tag{5}$$

We have the following points.

- We can always select K_P greater than $-a$, so that the closed-loop system will be stable even the original system is unstable. This is called feedback stabilization.
- When the closed-loop system has been stabilized, we have $x(\infty) = K_P r/(K_P + a)$ and the resulting steady-state error is
$$e(\infty) = r - x(\infty) = r \cdot \frac{a}{K_P + a}.$$
When we increase K_P, the steady-state error will decrease.

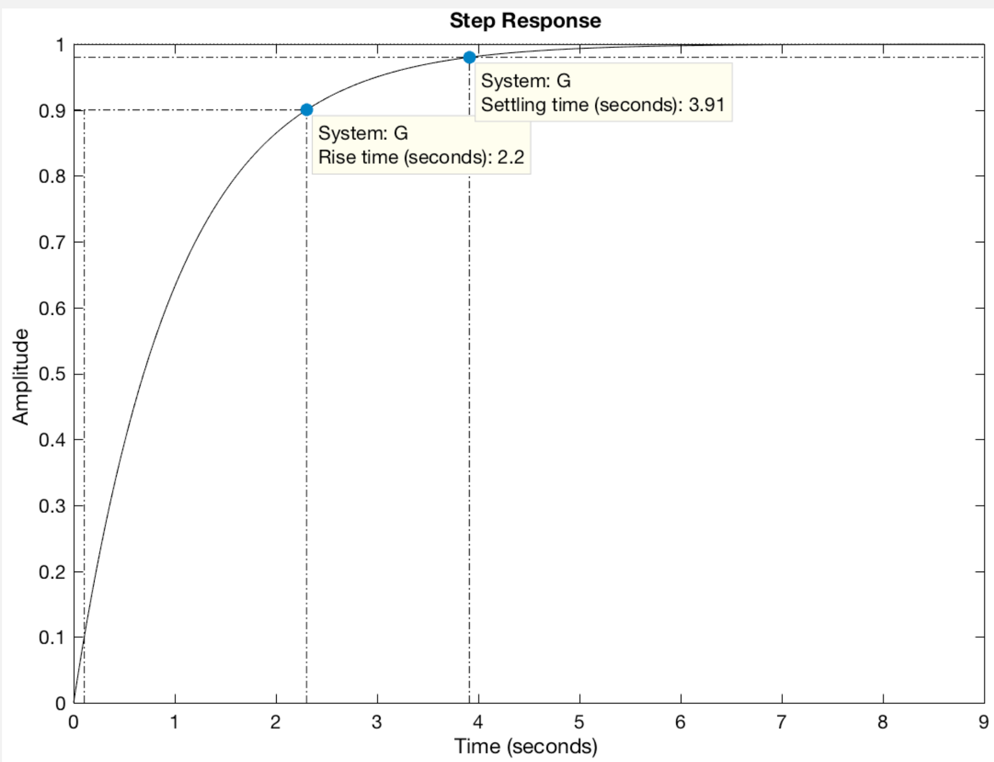
Note that we can achieve arbitrarily small steady-state error *regardless* of whether we have accurate knowledge of a by choosing large K_P.

2 Step Response Plots

The time solutions we presented for (3) and (5) are examples of *step responses*.

A common way of analysing systems is looking at the *step response plot*. An example is shown below for the system
$$\dot{x} + x = u.$$

Figure 2: Step Response Plots



This can be be produced in matlab with the following code:

```
G = ss(-1,1,1,0);
figure;step(G)
```

Type `doc ss` in the Matlab workspace to understand the usage of this function.

This is a plot of $x(t)$ vs t for a unit step input. Three important characteristics can be read from this graph:

- **Steady-state response (final value)**: in this figure, the system converges to $x(\infty) = 1$.
- **Settling time**: the time it takes for the system response to reach a 2% band around its steady state and *stay there*. In this case, it takes about 3.91 seconds.
- **Rise time**: the time taken to move from 10% to 90% of the steady state. In this case it is about 2.2 seconds.

Exercise 1 *Show that for first order systems of the form* (2)*, we have the following estimates:*

- *Settling time* $\approx \frac{3.91}{a}$, *often simplified to* $\frac{4}{a}$.
- *Rise time* $\approx \frac{2.2}{a}$

Solutions. Let us consider a unit step input $\mathbf{1}(t)$, i.e., $u(t) \equiv 1$ for all $t \geq 0$. Then (2) becomes

$$\dot{x}(t) + ax(t) = 1$$

Let the initial value be $x(0) = 0$. Then the overall system response is (see (3)) $x(t) = -\frac{1}{a}e^{-at} + \frac{1}{a}$. We can now establish each of the three characteristics.

- The steady state is $\frac{1}{a}$ because $-\frac{1}{a}e^{-at}$ vanaishes when time tends to infinity.
- The settling time, denoted t_s, should satisfy from the definition that $|x(t_s) - \frac{1}{a}| = 0.02\frac{1}{a}$. This leads to $e^{-at_s} = 0.02$ after subsituting the $x(t)$. We get $t_s = \frac{\ln 50}{a} \approx \frac{3.91}{a}$.
- The rise time, denoted t_r, satisfies $t_r = t_2 - t_1$, where t_1 and t_2 are the 10% and 90% steady state values, respectively. For t_1, we have $|x(t_1) - \frac{1}{a}| = 0.1\frac{1}{a}$, and then simiarly $t_1 = \frac{\ln 10}{a} \approx \frac{2.3}{a}$. We can also get $t_2 = \frac{\ln(10/9)}{a} \approx \frac{0.1}{a}$. Combining these two we get rise time $t_r \approx \frac{2.2}{a}$.

Now, these calculations are based on a unit step input. However, with a general step input $u(t) = r\mathbf{1}(t)$, the response will scale with a factor r. The above calculation can be repeated and we will arrive at the same formula for the rise time and settling time. Thus, those two rule-of-thumb estimates continue to apply.

3 Disturbance Attenuation

When implementing the control signal in the actuator, there often some "external disturbance" d, e.g. a wind gust affecting an aeroplane, so that the real input affecting the system is $u(t) + d$. By simple calculation we know that subject to such disturbance, open-loop control leads to

$$x(\infty) = \frac{r}{a} + \frac{d}{a}$$

while closed-loop control leads to

$$x(\infty) = \frac{K_P r}{K_P + a} + \frac{d}{K_P + a}.$$

Notice that as K_P gets larger, the second term gets smaller, while the first term gets closer to r.

Therefore, the reference tracking is improved and the influence of the disturbance is attenuated by the feedback controller, and this effect is increased with larger K_P

4 Measurement errors

Unfortunately, most sensors have some error, or "noise", compared to the real physical quantity. Hence, the real feedback system might be represented as

$$\dot{x}(t) + ax(t) = \underbrace{K_P(r - (x(t) + n(t)))}_{u(t)} + d$$

or, grouping terms

$$\dot{x}(t) + (a + K_P)x(t) = K_P(r(t) - n(t)) + d$$

Exercise 2 *What is the effect of constant noise on $x(\infty)$, and how does it relate to K_P?*

5 Frequency Response

Another commonly-used "test input" is *frequency response*. This can be thought of as a test of how fast a control system can respond to changes in the input, as an alternative to settling time. Though, as we will see later, the frequency response tells use much deeper information about a system, it's response, robustness, and other properties.

Suppose we have the following input

$$u = \cos(\omega t)$$

Here ω is the frequency in radians per second. The higher ω is the faster the input oscillates up and down.

What we observe in experiments is that the output seems to be an oscillation with the same frequency, but different magnitude and phase offset. So let's guess a response

$$x(t) = M\cos(\omega t + \phi) \tag{6}$$

where ϕ is a fixed phase offset. Note that by angle-sum identities

$$x(t) = M\cos(\omega t)\cos\phi - M\sin(\omega t)\sin\phi.$$

Using shorthand $p = -M\sin\phi, q = M\cos\phi$ and substituting into (2), we get:

$$\omega p\cos(\omega t) - \omega q\sin(\omega t) + ap\sin(\omega t) + aq\cos(\omega t) = \cos(\omega t)$$

This can only hold for all t if coefficients on sin and cos match:

$$\omega p + aq = 1, -\omega q + ap = 0$$

Making the substitution $p = \frac{\omega}{a} q$ we have

$$\frac{\omega^2 q}{a} + aq = 1$$

Which has the solution

$$q = \frac{a}{\omega^2 + a^2}$$

and substituting back into $p = \frac{w}{a} q$

$$p = \frac{\omega}{\omega^2 + a^2}.$$

So

$$x(t) = \frac{\omega}{\omega^2 + a^2} \sin(\omega t) + \frac{a}{\omega^2 + a^2} \cos(\omega t)$$

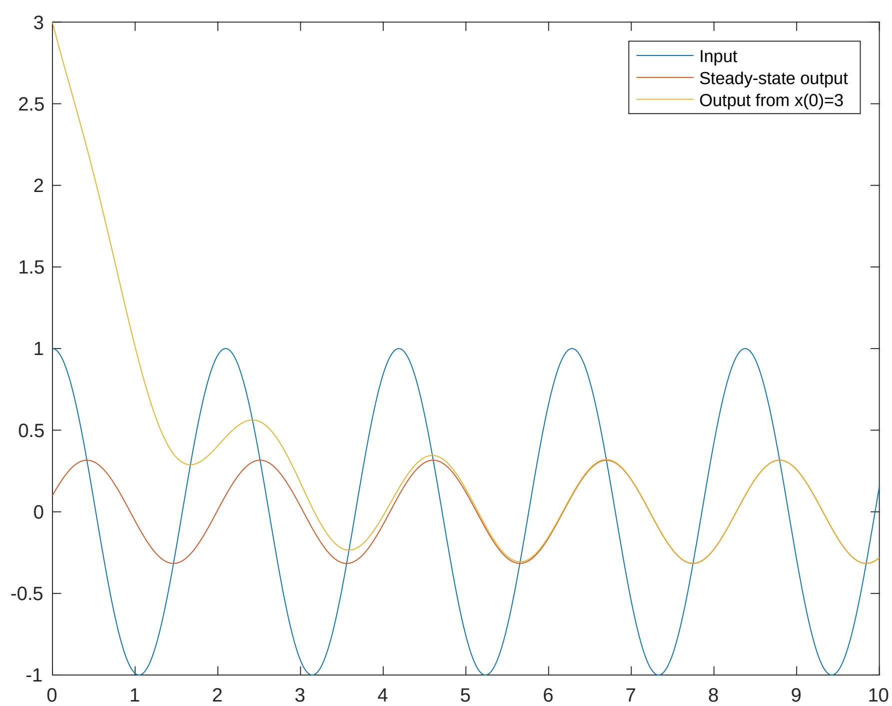
or equivalently (6) with $M = 1/\sqrt{\omega^2 + a^2)}$, $\phi = \arctan(-p/q) = \arctan(-\omega/a)$.

Remark 3 *A step input $u(t) = \mathbf{1}(t)$ can be thought of as a cosine with zero frequency: $u(t) = \cos(0*t) = 1$. So we can recover the steady-state response to a step from the frequency response!*

5.1 Example

For the system (2) with $a = 1$, and the input $u(t) = \cos(3t)$ we $M = 0.31, \phi = -1.24$ radians. The response is illustrated in the figure below, as well as another response with initial condition $x(0) = 3$.

Figure 3: Response to Sinusoidal Input



Exercise 4 *What is the general formula for response to a sinusoidal input with known initial condition? (Hint: think about homogeneous solutions).*

5.2 Complex exponential

The same result can be arrived at via the complex exponential. Suppose $u = e^{st}$, $s \in \mathbb{C}$ and guess that $x = G(s)e^{st}$ for some *complex* number $G(s) \in \mathbb{C}$. Substituting into (2) we get $(sG + a)e^{st} = e^{st}$ so

$$G(s) = \frac{1}{s+a}$$

Take $s = j\omega$ and recall that

$$u(t) = \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Exercise 5 *Verify that you can recover the same answers as above using this formulation of* $\cos(\omega t)$.

6 Integral Control for a Static Plant

Suppose $y = f(u)$ is some unknown scalar function, except that it is *monotone increasing:* (i.e. always has an uphill slope): $\frac{df}{du} > 0$.

How do we choose an input u so that $y = r$ for some desired reference r?

Here the "plant" has no dynamics, it is just a static function. But we can introduce dynamics into the *controller*.

Let's set

$$u(t) = k_I \int_0^t (r(\tau) - y(\tau))d\tau$$

i.e.

$$\dot{u} = r - y$$

so

$$\dot{y} = \underbrace{\tfrac{df}{du}}_{>0}(r - y).$$

Note that since $\frac{df}{du} > 0$, if $y < r$, the y increases, and if $y > r$ then y decreases, and the only equilibrium $\dot{y} = 0$ is when $r = y$.

This is an example of *integral control*, which is used to provide *zero error* even when the true system dynamics are not fully known. We will see much more of this in the future, including for cases when the system has dynamics.