Lecture Notes 04

Second and Higher-order Linear Systems

Preliminaries: Complex Exponentials

Complex Numbers

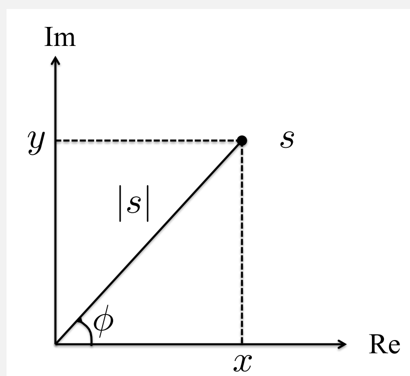
A complex number can be represented as

$$s = x + jy$$

where x is the real part, y is the imaginary part, and[1] $j = \sqrt{-1}$. Using x and y as two coordinates we can associate every complex number with a unique point in a two-dimensional space, a space often referred to as the complex plane. In control engineering the complex plane is often termed the s-plane since s is often used to denote a complex number. The conjugate of s is $\bar{s} = x - jy$, and it is easy to verify that $s\bar{s} = x^2 + y^2$. The Euler equation[2] suggests that

$$e^{x+jy} = e^{x} \left(\cos y + j \sin y \right). \tag{1}$$

Figure 1: Complex number s



Define the *magnitude* or a.k.a. *modulus* or *amplitude* of s as $|s| = \sqrt{x^2 + y^2}$, and the *angle* a.k.a. *argument* or *phase* of s, denoted $\angle(s)$, as the angle between the positive direction of the real axis and the vector pointing from the origin to s, measured in radians $\angle(s) = \tan^{-1}(y/x)$. Of course $\angle(s)$ is the same as $\angle(s) + 2\pi$. Then letting $\phi = \angle(s)$, we have from the Euler equation that

$$s = x + jy = |s| e^{i\phi}.$$

[1] You might be more familiar with the notation $i = \sqrt{-1}$, but control engineering is heavily influenced by electrical engineering, in which i usually refers to current, so we use j instead

[2] Particularly, $e^{i\pi} + 1 = 0$ is regarded as the most beautiful mathematical formula where five magic numbers 0, 1, e, π, and i meet each other in such an elegant way.

The form $|s|e^{i\phi}$ is termed the polar form of s. The magnitude and angle of two complex numbers $s_1, s_2 \in \mathbb{C}$ satisfy the following rules.

(i) $|s_1 s_2| = |s_1| \cdot |s_2|$ and $\left|\frac{s_1}{s_2}\right| = \frac{|s_1|}{|s_2|}$;

(ii) $\angle(s_1 s_2) = \angle(s_1) + \angle(s_2)$ and $\angle(\frac{s_1}{s_2}) = \angle(s_1) - \angle(s_2)$.

Complex Exponentials

We can now generalize the exponential functions with complex exponents. Let $s \in \mathbb{C}$. From the equation (1) we see

$$y(t) = e^{st}$$

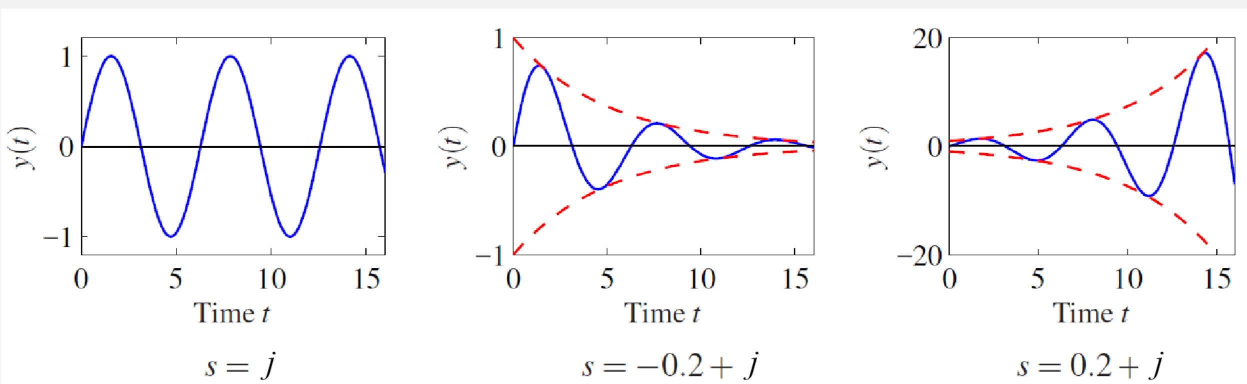
is a function that maps a real number $t \in \mathbb{R}$ to a complex number $e^{st} \in \mathbb{C}$. With $s = -\sigma + j\omega$, we can further write

$$y(t) = e^{st} = e^{-\sigma t}\big(\cos(\omega t) + i\sin(\omega t)\big).$$

The real and imaginary parts of $y(t)$ are $e^{-\sigma t}\cos(\omega t)$ and $e^{-\sigma t}\sin(\omega t)$, respectively.

If you think about this, it should be clear that the resulting signals look like oscillations (the sinusoidal part), multiplied by an "envelope" $e^{\sigma t}$ that exponentially grows or shrinks, or stays constant, depending on the sign of σ. This is illustrated below.

Figure 2: The Imaginary Part of Complex Exponentials e^{st}



The following properties are useful for understanding complex exponentials.

- There still holds
$$\frac{d}{dt}e^{st} = se^{st}.$$
Here note that the function e^{st} consists of a real part and an imaginary part, and therefore the derivative is taken to each of the two parts as real functions, respectively.

- For any two complex numbers s_1 and s_2, there holds
$$e^{s_1 t} \cdot e^{s_2 t} = e^{(s_1+s_2)t}, \; t \in \mathbb{R}.$$

- The complex Taylor series $\sum_{k=0}^{\infty} \frac{(st)^k}{k!}$ continues to converge for every $s \in \mathbb{C}$ and $t \in \mathbb{R}$, leading to
$$e^{st} = \sum_{k=0}^{\infty} \frac{(st)^k}{k!}, \quad t \in \mathbb{R}.$$

1 Second-Order Linear Differential Equations

The equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = cu(t) \tag{2}$$

with a, b, c being constants is a second-order linear differential equation. For most of this lecture we will take, for simplicity, $c = 1$.

Example 1 *Consider the spring-mass-damper system as displayed in Figure 1. Let $q \in \mathbb{R}$ denote the position of the mass m with respect to the rest position where the spring is at the equilibrium. The spring force is proportional to the displacement of the mass; the damper generates a friction $c(\dot{q})$ depending on the velocity of the mass. By Newton's second law of motion $F = ma$ there holds*

$$m\ddot{q} = -c(\dot{q}) - k_0 q$$

which yields a second order differential equation

$$m\ddot{q} + c(\dot{q}) + k_0 q = 0.$$

When the damper is linear in the sense that $c(\dot{q}) = c_0\dot{q}$, the equation becomes linear. Finally, with an external force u the equation of motion is

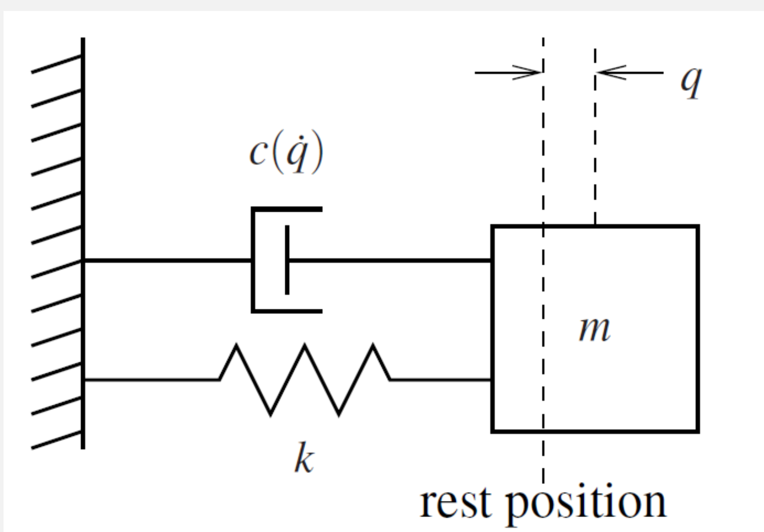
$$m\ddot{q} + c_0\dot{q} + k_0 q = u.$$

or

$$\ddot{q} + \frac{c_0}{m}\dot{q} + \frac{k_0}{m}q = \frac{1}{m}u.$$

which has the form (2).

Figure 3: Spring-Mass-Damper System



1.1 Homogenous Solutions

Once again, if $x(t)$ is a *particular* solution of (2), then so is $x(t)+x_h(t)$ where x_h satisfies the homogeneous equation

$$\ddot{x}_h(t) + a\dot{x}_h(t) + bx_h(t) = 0. \tag{3}$$

This follows from direct substitution of $x(t) + x_h(t)$ into (2) and linearity of the equations. If solutions of the homogeneous equation converge to zero, then it means that solutions from all initial conditions will converge to the same steady-state solution, i.e. the system is stable.

Let us try a solution $x_h(t) = e^{\lambda t}$ for the homogenous equation then we obtain

$$\lambda^2 e^{\lambda t} + a\lambda e^{\lambda t} + be^{\lambda t} = 0,$$

and since $e^{\lambda t}$ is never zero, this yields

$$\lambda^2 + a\lambda + b = 0.$$

This equation is the so-called characteristic equation of the differential equation (2) and (3). To find the possible values of λ we find the roots of this polynomial, which are obtained from the familiar quadratic formula

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

This equation might yield real or complex roots, depending on the sign of the term inside the square-root. The roots of the characteristic equation are called the *poles* of the system.

- **Two distinct real roots** $\lambda_1, \lambda_2 \in \mathbb{R}$ $(a^2 - 4b > 0)$.

  In this case, the general solution of (3) is

  $$x_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},\ C_1, C_2 \in \mathbb{R}.$$

- **Two equal roots** $\lambda_1 = \lambda_2 = \lambda_* \in \mathbb{R}$ $(a^2 - 4b = 0)$.

  The general solution is

  $$x_h(t) = C_1 e^{\lambda_* t} + C_2 t e^{\lambda_* t},\ C_1, C_2 \in \mathbb{R}.$$

  In particular, when $\lambda_* = 0$ we have a $\ddot{x} = 0$ and so velocity is constant, and solutions have the form

  $$x_h(t) = C_1 + C_2 t$$

  for arbitrary C_1, C_2.

- **Two distinct complex roots** $\lambda_1, \lambda_2 \in \mathbb{C}$ $(a^2 - 4b < 0)$.

  In this case, the general solution of (3) can still be formally written as

  $$x_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},\ C_1, C_2 \in \mathbb{C}.$$

  Here note that the above signals would take complex values. They are understood as solutions of (3) in the sense that both the real and imaginary parts of the signals are solutions of (3). Alternatively, we can write $\lambda_1 = -\sigma + j\omega_d$ and $\lambda_2 = -\sigma - j\omega_d$. Then the general solution of (3) can also be represented as

  $$x_h(t) = C_3 e^{-\sigma t}\cos(\omega_d t) + C_4 e^{-\sigma t}\sin(\omega_d t),\ C_3, C_4 \in \mathbb{R}. \tag{4}$$

1.2 Stability

The system (2) is called *stable* if and only if all its homogenous solutions tend to zero as time tends to infinity; *unstable* if there are homogenous solutions that tend to infinity; *marginally stable* if all of the homogenous solutions are bounded and there are the ones that do not converge to zero.

From the above discussion on the general homogenous solutions we know that the system (2) is stable if the roots of the characteristic equation have negative real parts; unstable if the roots have positive real parts or they are two repeated zero roots; and marginally stable if the two roots are purely imaginary.

Exercise 2 *Show that the system is stable if and only if a and b are both strictly positive.*

2 Responses to Exponential Inputs

Now let $u(t) = Ae^{st}$ and consider the corresponding solutions of the equation (2). Here we allow s to be a complex number and again, a complex signal is understood as a solution when both its real and imaginary parts are solutions of the differential equation.

2.0.1 A Special Solution

We do a trial solution with

$$x(t) = Be^{st}.$$

Then $\dot{x}(t) = Bse^{st}$ and $\ddot{x}(t) = Bs^2e^{st}$. Plugging these functions to the equation (2), we obtain

$$Bs^2e^{st} + aBse^{st} + bBe^{st} = Ae^{st}.$$

The above equation holds when

$$B = \frac{1}{s^2 + as + b}A = G(s)A.$$

Here

$$G(s) = \frac{1}{s^2 + as + b}$$

is the so-called transfer function of the system (2) since it relates the input to the output. This is to say,

$$x(t) = G(s)Ae^{st}$$

is ***a*** solution of the equation (2) with $u(t) = Ae^{st}$.

2.0.2 The General Solution

Let us now consider the equation (2) with

(i) $u(t) = Ae^{st}$

(ii) $x(0) = x_0$, $\dot{x}(0) = v_0$.

This defines an initial-value problem and the solution exists for all time and is unique since the differential equation is linear. Since the combination of any homogenous solution and the special solution $x(t) = G(s)Ae^{st}$ under $u(t) = Ae^{st}$ will be a solution as well, we consider

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + G(s)Ae^{st}$$

assuming two distinct roots for the characteristic equation. The conditions $x(0) = x_0$, $\dot{x}(0) = v_0$ lead to

$$\begin{aligned} x(0) &= C_1 + C_2 + G(s)A = x_0 \\ \dot{x}(0) &= C_1\lambda_1 + C_2\lambda_2 + sG(s)A = v_0. \end{aligned}$$

For particular values of s, x_0 and v_0, this is a linear system of equations in the unknowns C_1, C_2, which can be solved for.

2.1 Step Reponse

The step response of the system is the response of the system with a step input $u(t) = m\mathbf{1}(t)$. Recall that this can also be written as $u(t) = me^{0t}$, i.e. an exponential input with $s = 0$. The system (2) under step input becomes

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = m.$$

In this case with $s = 0$, we have

$$\begin{aligned} x(0) &= C_1 + C_2 + G(0)A = x_0 \\ \dot{x}(0) &= C_1\lambda_1 + C_2\lambda_2 = v_0 \end{aligned}$$

This is a linear equation about C_1 and C_2. With λ_1 and λ_2 being distinct, noting that $G(0) = \frac{1}{b}$, we can obtain the following solutions:

$$\begin{aligned} C_1 &= \frac{\lambda_2(x_0 - \frac{A}{b}) - v_0}{\lambda_2 - \lambda_1} \\ C_2 &= \frac{\lambda_1(x_0 - \frac{A}{b}) - v_0}{\lambda_1 - \lambda_2}, \end{aligned}$$

When the system is stable, for any initial value $x(0) = x_0$ and $\dot{x}(0) = v_0$, the response of the system subject to input $u(t) = Ae^{st}$ will become more and more close to $G(s)Ae^{st}$ as time increases! We call $x_{ss}(t) = G(s)Ae^{st}$ the steady state response.

Exercise 3 *What goes wrong in the above working if the roots are not distinct, i.e. $\lambda_1 = \lambda_2$? Try with the trial solution from Section 1.1:*

$$x(t) = C_1 e^{\lambda_* t} + C_2 t e^{\lambda_* t}.$$

2.2 Distinct Real Poles

From the above general form we have, with $x(0) = 0$ and $\dot{x}(0) = 0$:

$$x(t) = \frac{m}{b}\left(1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_1 - \lambda_2}e^{\lambda_2 t}\right)$$

Note that as long as the system is stable, the limiting value is m/b and the response converges exponentially. If one of the λ is much smaller in magnitude than the other, this part of the response will tend to dominate since convergence to zero of this part will be much slower.

2.3 Complex Poles

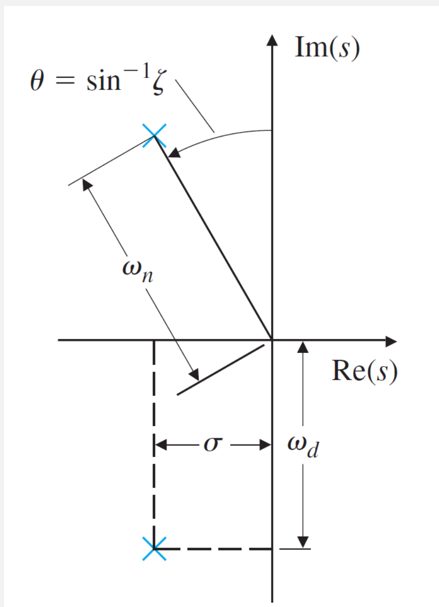
Let us now assume the characteristic equation admits two stable complex poles $\lambda_1 = -\sigma + j\omega_d$ and $\lambda_2 = -\sigma - j\omega_d$. Here we assume for convenience that $\sigma > 0$ and $\omega_d > 0$.

The following notation is common in control engineering:

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2} = \sqrt{b},\ \zeta = \sigma/\omega_n.$$

Note that this can be thought of as a change from cartesian form to polar form for the pole locations, this is illustrated in the figure below

Figure 4: Two complex poles.



The system equation becomes

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = u(t).$$

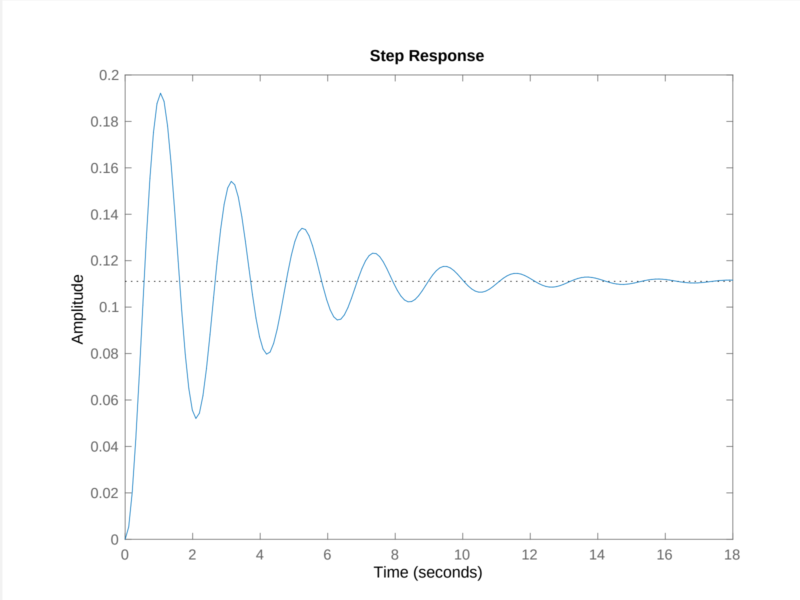
Here ω_n is the *undamped natural frequency* and ζ is the *damping ratio*. Using the general solution form (4) and subject to zero initial values $x(0) = \dot{x}(0) = 0$ we can find the step response is

$$x(t) = \left(1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \beta)\right)\frac{m}{\omega_n^2},$$

where $\beta = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$. This implies $\lim_{t\to\infty} x(t) = x(\infty) = \frac{m}{\omega_n^2}$, but there is an oscillation as the convergence happens.

In this case, we can have overshoot in the step response, and it is particularly large if ζ is very small. An example is shown in the figure below, corresponding to $\zeta = 0.1, \omega_n = 3$. We will investigate this further next week.

Figure 5: Step response example showing overshoot.



2.4 Frequency Response

We would like to investigate the steady-state response with input $u(t) = A\cos(\omega t)$. From Euler's relation we know that

$$A\cos(\omega t) = \frac{A}{2}(e^{j\omega t} + e^{-j\omega t}).$$

Assume the system is stable. We consider two cases for exponential inputs $u(t) = e^{st}$, respectively.

- Let $u_1(t) = e^{j\omega t}$. From the above analysis we know that

  $$x_{ss}(t) = G(j\omega)Ae^{j\omega t} = AMe^{j(\omega t+\varphi)}.$$

  where we have rewritten $G(j\omega)$ as $G(j\omega) = Me^{j\varphi}$ with $M = |G(j\omega)|$ and $\varphi = \angle G(j\omega)$.

- Let $u_2(t) = e^{-j\omega t}$. Noting $G(-j\omega)$ is the complex conjugate of $G(j\omega)$, we have

  $$x_{ss}(t) = G(-j\omega)Ae^{-j\omega t} = AMe^{-j(\omega t+\varphi)}.$$

Now that the system is linear, with $u(t) = A\cos(\omega t)$, the steady-state response is

$$x_{ss}(t) = AMe^{j(\omega t+\varphi)} + AMe^{-j(\omega t+\varphi)} = AM\cos(\omega t + \varphi).$$

This is to say, the steady-state response will have the same frequency, but the amplitude has been amplified by a factor $M = |G(j\omega)|$ and the phase will be shifted by an angle $\phi = \angle G(j\omega)$.

Plots of $|G(j\omega)|$ and $\angle G(j\omega)$ vs ω are very useful in control engineering and are called *Bode plots* or *Bode diagrams*, named after Hendrick Bode who made many major contributions to control theory and telecommunications.

For second-order systems with complex conjugate poles we have

$$G(j\omega) = \frac{1}{-\omega^2 + 2\zeta\omega_n\omega j + \omega_n^2}$$

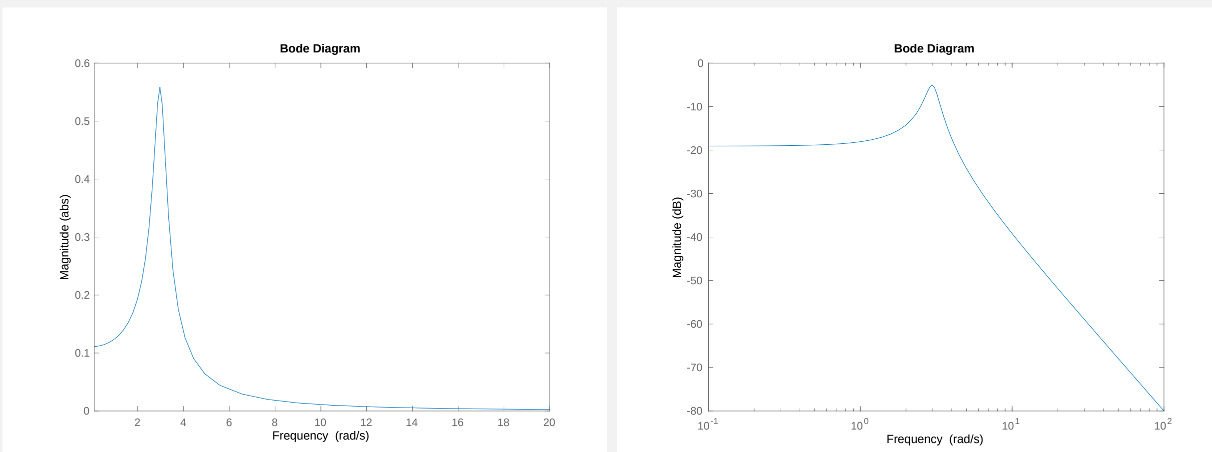
Let us note a few things

- If ω is zero, then $G(j\omega) = \frac{1}{\omega_n^2} > 0$.
- If ω is extremely large, then the magnitude $|G(j\omega)|$ is very small and its sign is now negative.
- If the damping ratio ζ is small, and the frequency ω is close to the natural frequency ω_n, the denominator becomes very small and we get *resonance*.

The magnitude response $|G(j\omega)|$ for the special case $\omega_n = 3$ and $\zeta = 0.1$ is shown in the Figure 5. Notice that the response starts at $1/\omega_n^2 = 1/9 \approx 0.111$, then gets very large when $\omega \approx \omega_n = 3$, this is resonance, and then dies away to zero.

It is common to present these frequency response on a *log scale*, for reasons that we will explore later. This is shown in the right hand side of the figure.

Figure 6: Frequency Response Magnitude: Linear and Log Scales



3 Zeros and High-Order Differential Equations

Let us begin by discussing a common example:

Example 4 *Suppose we have a first order system dynamics:*

$$\dot{y}(t) + ay(t) = u(t)$$

and a proportional + integral (PI) controller:

$$u(t) = K_P(r(t) - y(t)) + K_I \int_0^t (r(\tau) - y(\tau))d\tau.$$

Differentiating the above equations gives:

$$\begin{aligned} \ddot{y}(t) + a\dot{y}(t) &= \dot{u}(t) \\ \dot{u}(t) &= K_P(\dot{r}(t) - \dot{y}(t)) + K_I(r(t) - y(t)) \end{aligned}$$

and combining we get

$$\ddot{y} + a\dot{y} = K_P(\dot{r} - \dot{y}) + K_I(r - y)$$

and rearranging:

$$\ddot{y} + (a + K_P)\dot{y} + K_I y = K_P\dot{r} + K_I r$$

and supposing the system is stable and $r(t) = e^{st}$ *we get the steady-state solution* $y(t) = G(s)e^{st}$ *where*

$$G(s) = \frac{K_P s + K_I}{s^2 + (a + K_P)s + K_I}$$

This example is almost in the form discussed above, except it also has a numerator polynomial $K_P s + K_I$. This is quite a common situation. The root of this polynomial $s = -K_I/K_P$ is called a *zero* of the system. Can you guess why?

Consider the n^{th} order differential equation relating input u to output y:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1\dot{y} + a_0 y = b_n u^{(n)} + b_{n-1}u^{(n-1)} + ... + b_1\dot{u} + b_0 u \tag{5}$$

Note that if we have a coefficient on $y^{(n)}$ we can always make the coefficient one by dividing through.

Once again, we can consider the homogeneous equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1\dot{y} + a_0 y = 0$$

and the solutions of the are obtained from the roots of the characteristic polynomial, i.e. λ for which

$$\lambda^n + a_{n-1}\lambda^{(n-1)} + ... + a_1\lambda + a_0 = 0$$

In this case, there is a bit more subtle behaviour with repeated roots, which we will discuss in Lecture 6.

Particular solutions to exponential inputs $u(t) = e^{st}$ are given by $G(s)e^{st}$ where $G(s)$ is the transfer function:

$$G(s) = \frac{b_n s^n + b_{n-1}s^{n-1} ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_s + a_0}.$$

The roots of the denominator polynomial of $G(s)$ are called *poles* and the roots of the numerator polynomial are called *zeros.*

The system is stable if all poles have strictly negative real parts. The zeros of the system do not affect stability (they don't show up in the homogeneous equation) but they do have subtle effects on the input-output response. We will discuss this further in future lectures.