Lecture Notes 07

Linear Systems: State Feedback

Further Readings. Chapter 7. Åström and Murray, *Feedback Systems*, Second Edition.

1 Feedback Control under State-space Model

Now, we have learned that the state-space model for linear dynamical system is described by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

where $x \in \mathbb{R}^n$ is the *state*, $u \in \mathbb{R}^p$ is the *input*, $y \in \mathbb{R}^q$ is the *output*. Then $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, and $D \in \mathbb{R}^{q\times p}$. The system is *single-input single-output* (SISO) if both u and y are scalars, which will be the standing assumption for our system for the reminder of the course. Note that for SISO systems B is just a column vector while C is a row vector both with n dimensions.

We summarize a few important points that are foundational to the idea of feedback control under this state-space model.

(i) The system is stable if and only if all solutions to $\dot{x} = Ax$ converge to zero as time tends to infinity. The condition for stability is that *all poles, i.e., eigenvalues of the matrix A, have negative real parts*. Stability means that with a constant or oscillating input u, the responses (that is, $x(t)$) of the system (1) and therefore the system output $y(t)$, can possibly go to a steady state, which is our primary concern for many control design problems (see Design Project 1).

(ii) For the system (1)-(2), we can assume a feedback controller taking the form of

$$u = -Kx \tag{3}$$

where u must be a scaler, and x is an n-dimensional column vector. Therefore, K must be an n-dimensional row vector. This is a feedback controller because u is depending on x for all time. The controller can be realized in real world if we can measure the system state $x(t)$ for all time.

(iii) Now, substituting (3) to (1), we obtain the closed loop system equation

$$\dot{x} = (A - BK)x. \tag{4}$$

Note that B is $n \times 1$ column vector, and K is $1 \times n$ row vector, which lead to BK being an $n \times n$ matrix[1]. For the system (5), it is stable if the eigenvalues of $A - BK$ all have negative real parts.

[1] As a result, the rank of the matrix BK must be one if $BK \neq 0$. You can verify this by the definition of matrix rank

(iv) The eigenvalue of $A-BK$ can certainly be different than those of A. Therefore, the stability and step specifications are altered from the open loop system with the state feedback controller $u=-Kx$! If we hope the system output $y(t)$ to track a reference signal $r(t)$, a natural controller would then be $u=-Kx+k_r r$ so that

$$\dot{x}=(A-BK)x+Bk_r r \tag{5}$$

where k_r is a design parameter as a feedforward gain.

Example 1. Let us revisit the simple example discussed in LN06 on heating/cooling of a 3-room building:

$$\dot{T}_1(t)=r_{12}(T_2(t)-T_1(t))+r_{13}(T_3(t)-T_1(t)) \tag{6}$$
$$\dot{T}_2(t)=r_{21}(T_1(t)-T_2(t))+r_{23}(T_3(t)-T_2(t))+u_a(t) \tag{7}$$
$$\dot{T}_3(t)=r_{31}(T_3(t)-T_1(t))+r_{32}(T_2(t)-T_3(t)). \tag{8}$$

The system can be written in the form (1) with

$$A=\begin{bmatrix}-r_{12}-r_{13} & r_{12} & r_{13}\\ r_{21} & -r_{21}-r_{23} & r_{23}\\ r_{31} & r_{32} & -r_{31}-r_{32}\end{bmatrix}, B=\begin{bmatrix}0\\1\\0\end{bmatrix},$$

and if there were temperature sensors in rooms 1 and 2, then we could have a vector measurement of the form (2) with

$$C=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, D=0.$$

When we have sensors measuring the temperature of the three rooms, a state-feedback controller can be described by

$$u(t)=-[k_1\ k_2\ k_3]\begin{bmatrix}T_1(t)\\T_2(t)\\T_3(t)\end{bmatrix}=-k_1T_1(t)-k_2T_2(t)-k_3T_3(t)$$

where $K=[k_1\ k_2\ k_3]$.

Although the idea of state-feedback becomes clear at this point, there are a few questions for this to be feasible.

- Q1. To what extent the eigenvalues can be altered by K from A to $A-BK$?
- Q2. For a pair (A,B), if there is indeed K so that $A-BK$ can have desired poles, then how to find a suitable K?
- Q3. When there are many K under which the controller $u=-Kx$ all gives reasonable closed-loop response, where one is the best? Or, how to say one K is better than another K?

2 Pole Placement for Linear Systems

2.1 Reachability Canonical Form

To answer Q1, it turns out the following particular form of (A, B) leads to immediate answers. A system is in reachable (or controllable) canonical (RC) form if

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0. \end{bmatrix} \tag{9}$$

This pair of A, B appears to be funny coming from nowhere. However, from matrix algebra it can be shown that

$$\det(sI - A) = s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n. \tag{10}$$

This is quite interesting because in general for a matrix A, it is difficult to find the coefficients of $\det(sI-A)$ from the entries of A.

Now we aim to design a state-feedback controller $u = -Kx$ with

$$K = \begin{bmatrix} k_1 & k_2 & ... & k_n. \end{bmatrix}$$

Again direct calculation with matrix algebra shows

$$A - BK = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 & \cdots & -a_n - k_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and again, this is in the same form but now[2]:

$$\det(sI - (A - BK)) = s^n + p_1 s^{n-1} + ... + p_{n-1}s + p_n$$

with $p_i = a_i + k_i$. We can use this to design K.

To summarise, for a system (1) that is already in RC form, we can place closed-loop poles, i.e., eigenvalues of $A - BK$, wherever we want by the following procedure:

1. Choose n desired pole (eigenvalue) locations $\lambda_1, \ldots, \lambda_n$ for $A - BK$.
2. Compute desired characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + ... + p_{n-1}s + p_n = (s - \lambda_1)(s - \lambda_2)...(s - \lambda_n)$$

[2]Note that the above A and $A - BK$ are no different in terms of their formats.

3. Set $k_i = p_i - a_i$.

4. Apply control $u = -Kx$.

As we can see, after the procedure, $\det(sI - (A - BK))$ is forced to be $s^n + p_1 s^{n-1} + ... + p_{n-1}s + p_n$, which exactly gives us the poles $\lambda_1, \ldots, \lambda_n$.

Obviously a general system (1) hardly has the form (9). We need to find a way to transform the system (1) to (9).

2.2 Reachability

Let us first be clear about the term *reachability*. In some other textbooks, reachability is also called *controllability*.

Definition 1 *The system (1) is reachable if*

$$\operatorname{rank}\begin{bmatrix}B & AB & A^2B & \ldots & A^{n-1}B\end{bmatrix} = n. \tag{11}$$

Note here each A^kB is a column vector for $k = 0, 1, \ldots, n-1$ for SISO systems, which implies the matrix $W = \begin{bmatrix}B & AB & A^2B & \ldots & A^{n-1}B\end{bmatrix}$ is an $n \times n$ square matrix.

Reachability defined from the rank condition has geometric indication, which was actually the origin of this concept. An alternative definition for reachability is

Definition 2 *The system (1) is reachable if any final state $x(T)$ can be reached from any initial state $x(0)$ in finite time T by choice of control $u(t)$ on $t \in [0, T]$.*

Definition 1 and Definition 2 can be shown to be **equivalent**. Later, it was found that the two definitions are further equivalent to the third one described below.

Definition 3 *The system (1) is reachable if For any desired set of Closed-Loop eigenvalues (poles), there exists a gain matrix K such that $A - BK$ has those eigenvalues.*

Now we can see, this third definition addresses Q1. Q2 can then be answered with the help of the RC form, for which the matrix W is the key.

For a system already in reachable canonical form (9), the reachability matrix has particularly simple form:

$$\tilde{W} = \begin{bmatrix} 1 & \star & \star & \cdots & \star \\ 0 & 1 & \star & \cdots & \star \\ 0 & 0 & 1 & \cdots & \star \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{12}$$

where each $\star$ is some possibly non-zero value depending on $a_1, ..., a_n$. Such a matrix $\tilde{W}$ is *always* full rank, since no column can be written as a linear combination of other columns. Therefore, a system of the RC form is always reachable.

Recall if $z = Tx$ with T invertible then

$$\begin{aligned}\dot{z} &= TAx + TBu &(13)\\ &= TAT^{-1}z + TBu &(14)\\ &=: \tilde{A}z + \tilde{B}u &(15)\end{aligned}$$

Then, if we can find T so that $\tilde{A}, \tilde{B}$ are in the reachability canonical form, we can design state-feedback controller under $\tilde{A}, \tilde{B}$ using the above RC procedure, which will give us the controller for the original system under A, B.

Now let a system described in (1) be reachable. Note $W = [B \ AB \ \ldots \ A^{n-1}B]$. And so under coordinate change $z = Tx$, the resulting $\tilde{A}, \tilde{B}$ satisfy

$$\begin{aligned}\tilde{W} &= \begin{bmatrix}\tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B}\end{bmatrix}\\ &= \begin{bmatrix}TB & TAT^{-1}TB & (TAT^{-1})^2TB & \cdots & (TAT^{-1})^{n-1}B\end{bmatrix}\\ &= T\begin{bmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix}\\ &= TW\end{aligned}$$

So if we choose $\tilde{W}$ from the canonical form and use

$$T = \tilde{W}W^{-1}, \tag{16}$$

then $\tilde{A}, \tilde{B}$ are in the reachability canonical form! We only need to note that $\tilde{A}$ and $\tilde{B}$ can be written down directly if we know the poles. This solves our Q2.

General Pole Placement Procedure

A general procedure for placing the poles of the closed-loop system by state feedback is as follows. For general form $\dot{x} = Ax + Bu$, with A, B reachable:

1. Compute characteristic polynomial

   $$\det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

   to get reachable canonical form $z = \tilde{A}z + \tilde{B}u$.

2. From desired poles, compute desired characteristic polynomial:

   $$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n = (s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n)$$

3. Compute gain matrix $u = -\tilde{K}z$ using coefficient matching, i.e., $k_i = p_i - a_i$.

4. Compute $W, \tilde{W}$ from A, B and $\tilde{A}, \tilde{B}$, respective. Then compute $T = \tilde{W}W^{-1}$.

5. Compute $u = -Kx$ with $K = \tilde{K}T$, so that $\tilde{K}z = \tilde{K}Tx = Kx$.

3 Steady State

3.1 Feedforward Gain Selection

Now, suppose we have completed the pole-placement procedure to find a state feedback gain K for the system (1)-(2) with $D = 0$. As mentioned above we can now consider

$$\dot{x} = (A - BK)x + Bk_r r \tag{17}$$

where k_r is a feedforward gain and r is a reference for output y. If the closed-loop system is stable, with a constant step input r, steady-state will be achieved for both x and y. We can compute the steady state and output. First

$$0 = (A - BK)x_{ss} + Bk_r r$$

and therefore $x_{ss} = -(A - BK)^{-1}Bk_r r$. The steady-state output will be $(D = 0)$

$$y_{ss} = -C(A - BK)^{-1}Bk_r r.$$

Now, we can set

$$k_r = \frac{-1}{C(A - BK)^{-1}B}$$

to get $y = r$, i.e., zero steady-sate error.

But this will only work perfectly with precise knowledge of A, B, C.

3.2 Adding an Integrator

We have learned in the previous discussions (particular, Design Project 1) that integrator is very useful in dealing with steady-state error. If we add

$$z = \int_0^t (y(\tau) - r(\tau))d\tau$$

Then

$$\dot{z} = Cx - r.$$

This integrator adds a state to the original system (1)-(2):

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}}_{A_a} \underbrace{\begin{bmatrix} x \\ z \end{bmatrix}}_{x_a} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_a} u + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B_r} r$$

Design a controller for the augmented system A_a, B_a, and we get[3]

$$u = -K_a x_a = [-K \;\; -K_I] \begin{bmatrix} x \\ z \end{bmatrix} = -Kx - K_I z \tag{18}$$

[3] A PI controller!

If the closed-loop system is stable, i.e. $A_a - B_a K_a$ has all its eigenvalues in the left half plane, then the system reaches the steady-state

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A - BK & -BK_I \\ C & 0 \end{bmatrix} \begin{bmatrix} x_{ss} \\ z_{ss} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

From the second row, we can read off $0 = Cx_{ss} - r$ i.e. $y = r$.