# First and Second Order Systems

1. [10 Marks] Consider the system

$$\dot{x} + ax = u.$$

   (a) [2 Marks] Let $u = 0$ for all time, and consider $x(t)$. If $x(0) = 1$ and $x(2) = e^{-4}$, what is the value of $a$?

   (b) [4 Marks] Now let $a$ be the value found in Part (a). If $u(t) = \mathbf{1}(t)$ is the unit step function and $x(0) = 0$, derive the response of $x(t)$.

   (c) [4 Marks] For $a = 3$, what is the value of $u$, so that $x(t)$ approaches 2 as $t$ tends to infinity for any initial value $x(0)$?

2. [10 Marks] Consider the following second-order system

$$\ddot{x} + 5\dot{x} + 6x = u.$$

   (a) [2 Marks] What are the poles of the system?

   (b) [2 Marks] What is the meaning that the system be stable in terms of system response $x(t)$? Is the system stable or not?

   (c) [6 Marks] Design a rate-feedback PD controller

$$u(t) = K_p(r - x) - K_d\dot{x}$$

   so that the system response to a step input has a settling time around 2 sec and an overshoot of about 5%. Show all working.

3. [10 Marks] The dynamics of a plane pendulum subject to some external force is described by

$$\ddot{x} + \frac{g}{\ell}\sin x = u(t),$$

   where $x$ is the angular displacement, $\ell$ is the length of its link, and $g$ is the standard gravity.

   (a) [2 Marks] What is the equilibrium of the pendulum with $u(t) = F$ being a constant?

   (b) [2 Marks] Write down the linearized equation for small angular displacement.

   (c) [6 Marks] Derive the harmonic response $x(t) = X\sin(\omega t)$ subject to $u(t) = F\sin(\omega t)$ from the linearized equation.

# State Space Control Design

4. [15 Marks] Let us reconsider the second-order system

$$\ddot{x} + 5\dot{x} + 6x = u.$$

   (a) [2 Marks] Rewrite the dynamics in state-space form.

   (b) [2 Marks] Is the system reachable?

   (c) [2 Marks] Compute the poles from the state-space form.

   (d) [4 Marks] Suppose the system output is given by

$$y(t) = \dot{x}.$$

   Is the system observable?

   (e) [5 Marks] Design a state feedback controller so that the closed-loop system has poles placed at $1 + j$ and $1 - j$.

5. [10 Marks] Consider the system given by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, and $C \in \mathbb{R}^{1 \times n}$.

   (a) [3 Marks] What is the meaning of the system being observable?

   (b) [3 Marks] What is the purpose of an observer?

   (c) [4 Marks] Consider an observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

   where $L \in \mathbb{R}^{n \times 1}$. What is the condition that $L$ should satisfy for the observer to work? Explain your reasoning.

6. [15 Marks] Consider the following two single-input single-output subsystems:

$$S_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 \end{cases} \qquad S_2 : \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 \end{cases}$$

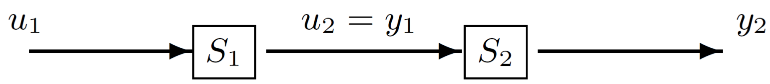


Figure 1: Block diagram of the system $S$.

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2 & 1 \end{bmatrix},$$
$$A_2 = -2, \qquad B_2 = 1, \qquad C_2 = 1.$$

The two subsystems are cascaded forming an overall system $S$, as shown in Figure 1.

(a) [3 Marks] Verify reachability (controllability) and observability of the subsystems $S_1$ and $S_2$, respectively.

(b) [4 Marks] Denote

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Write the state-space equation of the overall system $S$ with input $u_1$, output $y_2$, and state $x$.

(c) [4 Marks] Verify reachability (controllability) and observability of the system $S$.

(d) [4 Marks] Calculate the transfer function from $u_1$ to $y_2$ of the system $S$.

# Frequency-Domain Control Design

7. [10 Marks] The Bode diagram of a transfer function $H(s)$ is shown in Figure 2.

(a) [2 Marks] Estimate the number of zeros and poles for $H(s)$ from the Bode gain plot. Explain your reasoning.

(b) [4 Marks] Estimate the transfer function $H(s)$ from both the gain and phase plots. Explain your reasoning.

(c) [4 Marks] Plot the steady-state response of the system with an input signal $u(t) = \sin(2t)$. Clearly show key features.

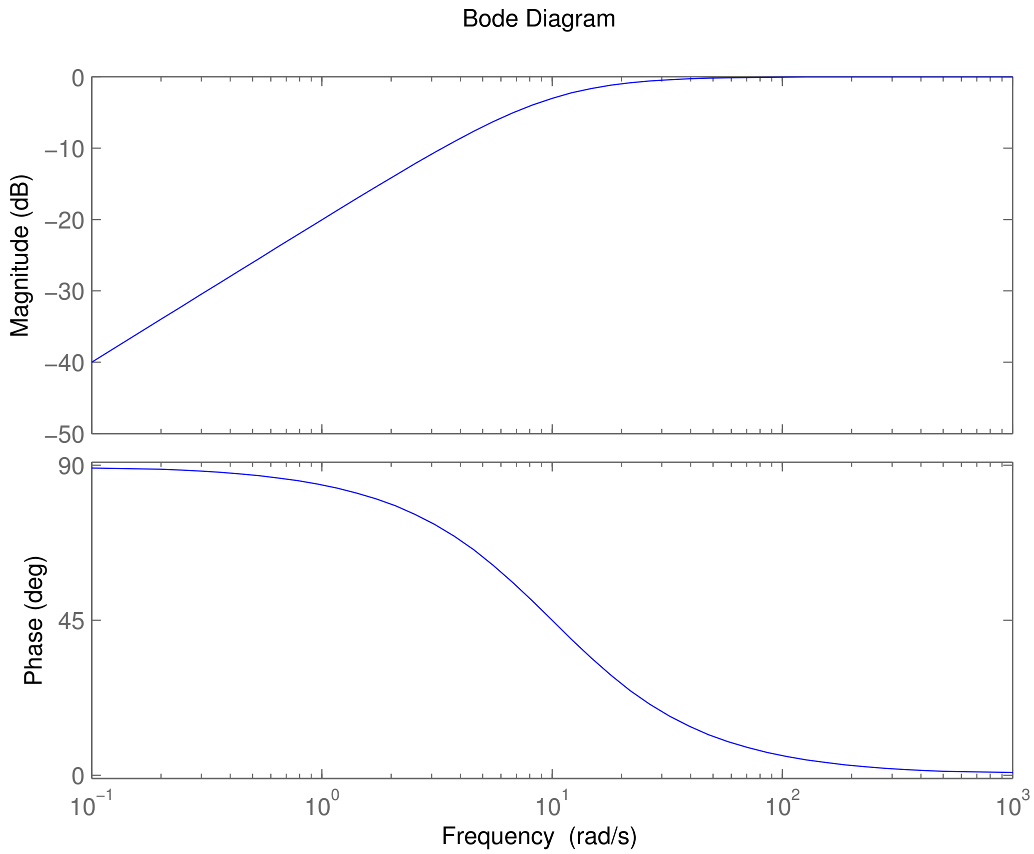


Figure 2: The Bode diagram for Problem 7.

8. [10 Marks] Consider a plant with the transfer function

$$G(s) = \frac{s+6}{s(s+10)}$$

placed in a unity feedback loop with a PID controller $C(s)$ of the form

$$C(s) = K\frac{(s+z_1)(s+z_2)}{s}$$

where $K, z_1, z_2 > 0$.

(a) [4 Marks] Find the condition of $z_1$ and $z_2$ under which the closed loop system is second order.

(b) [6 Marks] Design a $C(s)$ so that the closed loop system is second order with poles at $s = -3 \pm 4j$. Show all working.

9. [10 Marks] Consider a plant with transfer function

$$P(s) = \frac{1}{s(s+1)}.$$

   (a) [4 Marks] For a PD control $C(s) = 3s + 2$, what are the sensitivity and complementary sensitivity functions?

   (b) [2 Marks] Explain the significance of sensitivity functions in terms of plant uncertainty.

   (c) [4 Marks] Explain the idea of loop shaping for frequency-domain design of feedback controllers.

END OF EXAM.