Lecture 2: First-order dynamical systems

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Some useful advice!

1- Get comfortable with math! The math used in this course is not hard!

*Use math as a tool to better understand and design control systems.

2- Review your math notes from past years and contents of AMME2000 and AMME2500.

3- Use AI to help you learn the concepts and understand the maths.

4- It's ok to feel confused at the start! You need some iterations and more exposure:

* Read extra contents on Canvas and watch the uploaded videos

* Attend tutorials and lab sessions to see the examples and applications of the contents

* Ask your tutor or post your questions on Ed Discussion on Canvas

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The real exponential

"The most important function in mathematics"

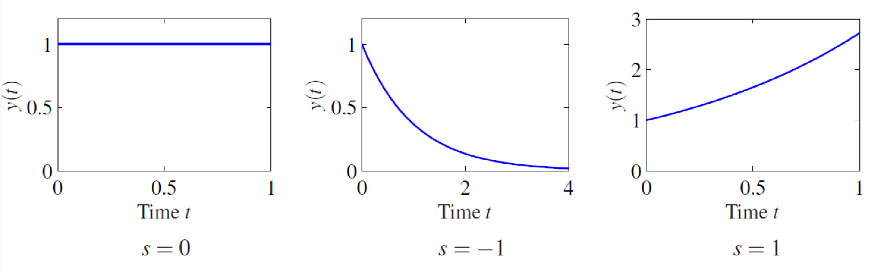
- Walter Rudin, 1987

$$y(t) = e^{at}$$

- $\dot{y}(t) = ae^{at} = ay(t)$
  note: $\dot{y}(t)$ can be also written as dy/dt (shows how fast y is growing or shrinking as time increases)
- $y(t) > 0$ for all t
- When $a = 0$, $y(t) \equiv 1$ is a constant function;
- When $a < 0$, $\lim_{t\to+\infty} y(t) = 0$;
- When $a > 0$, $\lim_{t\to+\infty} y(t) = +\infty$.

Real exponential functions

$y(t) = e^{st}$



Physical systems and differential equations

Differential equations

What are "differential equations"?

Equations of functions (signals) and their derivatives.

- Many fundamental laws of physics can be written as relationships between signals $x(t)$ and their time derivatives $\dot{x}(t)$, e.g.

$$\dot{x} = f(x,t), \text{ or } h(\dot{x}, x, t) = 0$$

Meaning: At each moment, the rate of change of x depends directly on the current value of x and on time t

- Often there is some *external input* (forcing)

$$\dot{x}(t) = f(x(t), u(t))$$

e.g. Car's velocity $\dot{x}(t)$ is affected by an external force $u(t)$ like breaking etc.

Differential equations from engineering systems

- Newton's second law (F=ma):

$$m\ddot{x}(t) = \sum_i F_i(t)$$

  where forces $F_i(t)$ are often functions of x or $\dot{x}$ (springs, friction, gravity).

- Electrical inductance and capacitance:

$$L\frac{di(t)}{dt} = v(t), \quad C\frac{dv(t)}{dt} = i(t)$$

- Newton's law of cooling:

$$C\frac{dT(t)}{dt} = hA(T(t) - T_{env})$$

Example 1: Friction and Aerodynamic Drag

A vehicle moving along a straight line. From the Newton's 2nd law of motion we have:

$$M\ddot{x}(t) + f(\dot{x}(t)) = u(t)$$

- $x(t)$ - displacement (position)
- $u(t)$ - external force (e.g., engine force, braking force)
- $f(\dot{x}(t))$ - represents the resistive forces (e.g., friction, air drag)
- M - mass (constant)

Example 2: DC Motor

From Newton's Second Law for Rotational Motion:

$$J\ddot{\theta}(t) + b\dot{\theta}(t) = K_1 i(t), \quad Ri(t) = v(t) - K_2\dot{\theta}(t).$$

- $\theta(t)$ - angular displacement
- $i(t)$ - armature current
- $v(t)$ - voltage
- J, b, R, K_1, K_2 - physical parameters (constant)

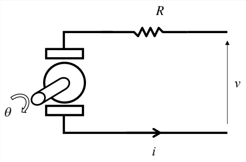


Figure 1: Voltage-controlled DC Motor circuit diagram and photo

Example 3: Water Flow

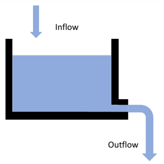


Figure 2: Water flowing in and out of a tank

Describe how the water level in the tank changes over time based on the inflow and outflow rates: Let the tank have constant area A. Let $h(t)$ be the liquid level, so $Ah(t)$ is the volume of water in the tank.

$$\text{Conservation of mass: } A\dot{h}(t) = q_i(t) - q_o(t)$$

$$\text{Simplifying Bernoulli's equation: } \frac{v^2}{2} = gh(t)$$

$$A\dot{h}(t) + B\sqrt{h(t)} = q_i(t)$$

Solutions of differential equations I: Equilibrium

We saw various forms of differential equations. What about the solution?

EX: What is a solution to a first-order differential equation $\dot{x} = f(x)$?

- Very simple example $\dot{x} = x + 1$.
- A solution is a signal $x(t)$ so that $\dot{x}(t) = x(t) + 1$ for all t in an interval of interest.
- OK $x(t) \equiv -1$ seems to be a solution! Substitute $x(t) \equiv -1$ into the differential equation, and indeed there is an identity. So it is a solution. It is a special solution called equilibrium.
- Any point x_e that satisfies $f(x_e) = 0$ is an equilibrium.

A differential equation in general may have many solutions.

Example

$$\dot{x} = x^3 - x = -x(1-x)(1+x)$$

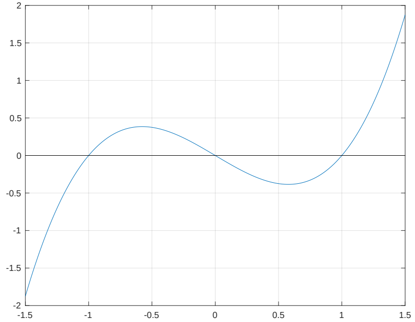


Figure 3: Cubic nonlinearity

To satisfy the ODE: 3 solutions or roots or equilibrium at x= -1, 0, and 1

Solutions of differential equations II: Stability

What is equilibrium x_e ? An equilibrium is the "happy place" of the system—a state where everything is balanced so that the system has no reason to change over time.

What does stability mean? If a system starts **close to** the equilibrium, it should **stay close** or **move toward**. It should not move away (diverge).

Why this is important? 1- It means small errors in starting conditions don't cause big problems. 2- The system will behave predictably even if we don't know the exact initial state.

Example: Imagine a ball in a bowl:

1-If you push it slightly, it returns to the bottom $\rightarrow$ **Stable equilibrium.**
2-If the ball were on top of a hill, a small push would make it roll away $\rightarrow$ **Unstable equilibrium.**

Stability of equilibrium 0: Starts slightly close to 0 and quickly goes back to (converge) to 0 again

$$\dot{x} = x^3 - x = -x(1-x)(1+x)$$

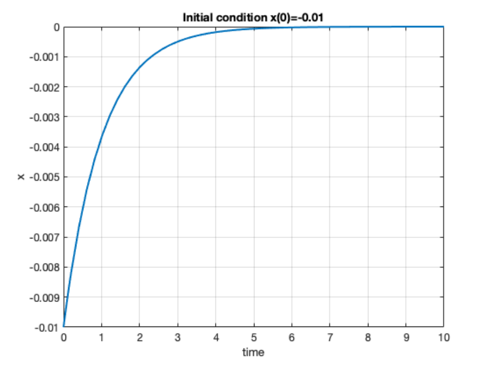


Figure 4: Starting from a bit left to 0.

Stability of equilibrium 0:

$$\dot{x} = x^3 - x = -x(1-x)(1+x)$$

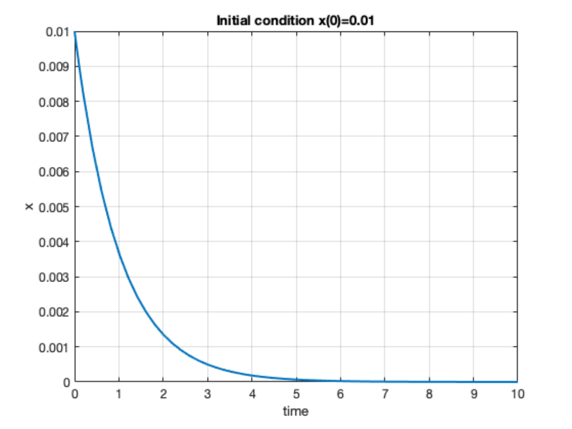


Figure 5: Starting from a bit right to 0.

Solutions of differential equations II: Stability

Instability of equilibrium -1: Starts slightly close to -1 and quickly moves away (diverge) from -1 (unstable)

$$\dot{x} = x^3 - x = -x(1-x)(1+x)$$

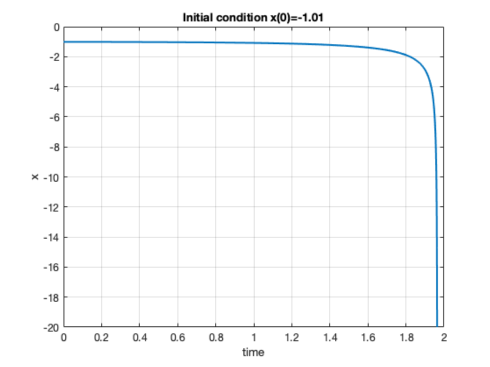


Figure 6: Starting from a bit left to -1.

Solutions of differential equations II: Stability

Instability of equilibrium -1:

$$\dot{x} = x^3 - x = -x(1-x)(1+x)$$

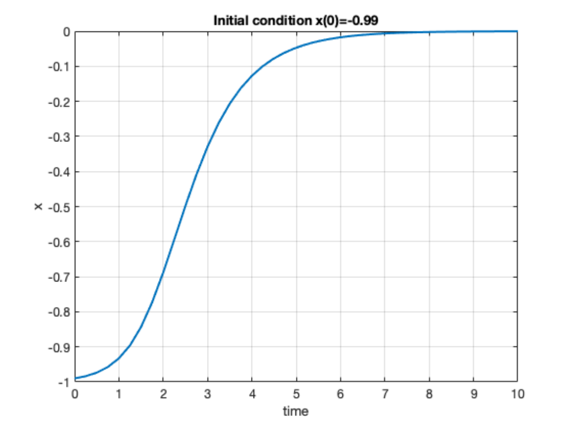


Figure 7: Starting from a bit right to -1.

Solutions of differential equations III: Presence of input

Consider the following differential equation describing a dynamical system with an external input:

$$\dot{x} = f(x, u).$$

- A solution is still $x(t)$ defined over a time interval. The $u(t)$ is a known signal: $\dot{x} = x + u$.
- An equilibrium is now a pair of points (x_e, u_e) satisfying

$$f(x_e, u_e) = 0.$$

Q1: What this equilibrium means in the physical sense? The system settles into a steady state where all forces or influences balance out.

Q2: Can we find an equilibrium for the system $\dot{x} = x + u$?

First-order linear systems and exponential response

First order linear systems

Let us now investigate the following differential equation

$$\dot{x}(t) + ax(t) = bu(t) \tag{1}$$

and for this lecture take $b = 1$.

This form is called a *linear time-invariant* (LTI) system because it is a linear relationship between $\dot{x}, x, u$, and the coefficients a, b don't change with time.

Note: For some examples it is more natural to use the form

$$\dot{x}(t) = ax(t) + bu(t)$$

this is completely equivalent, except the sign of a has changed. We will stick with the first form for this lecture.

A nice property

Linear time invariant (LTI) systems **always** have unique solutions that exist for all time for all given initial conditions.

Understanding LTI Systems

A Linear Time-Invariant (LTI) system can be described by a simple first-order equation:

$$\dot{x}(t) + ax(t) = bu(t) \tag{2}$$

Where:

- $\dot{x}(t)$: Rate of change of the system's state (e.g., water level).
- $x(t)$: The current state of the system (e.g., water level in the tank).
- $u(t)$: The input signal (e.g., water flow into the tank).
- a: System parameter (e.g., leak rate).
- b: Input gain (e.g., efficiency).

These systems are called **time-invariant** because a and b do not change with time.

Real-Life Example: Water Tank

Imagine a tank with:

- An **inflow** of water through a faucet $u(t) = 5$ L/min.
- A **leak** that drains water at 10% of the current water level per minute.

The LTI equation becomes:

$$\dot{x}(t) + 0.1x(t) = 5 \tag{3}$$

The tank's water level $x(t)$ will stabilize when:

$$\dot{x}(t) = 0 \Rightarrow 0.1x(t) = 5 \tag{4}$$

$$\therefore x(t) = 50 \text{ liters} \tag{5}$$

This illustrates how the system reaches an equilibrium when inflow equals outflow. What about tank level at different time intervals (e.g. 5min, 10min, etc)? Need a right LTI solution (response equation)!

Solutions for LTI Systems - Overview

An LTI system **responds** differently depending on the input $u(t)$. Throughout this course, we will explore how different types of inputs affect the system's behavior:

- **No Input (Homogeneous Solution):** $u(t) = 0$
  - The system evolves based solely on initial conditions.
  - Solution: $x(t) = Ce^{-at}$
- **Step Input:** $u(t) = m$
  - The system reacts to a constant input.
  - Solution: $x(t) = [x(0) - \frac{m}{a}]e^{-at} + \frac{m}{a}$
- **Exponential Input:** $u(t) = Ae^{st}$
  - The system's response involves both transient and steady-state behavior.
  - Solution: $x(t) = \frac{A}{s+a}e^{st} + x_h(t)$

Key Insight: LTI systems can respond predictably to different inputs, helping us design controllers for desired outcomes.

Solving First-Order LTI Systems I: Homogeneous Solutions

To begin, let $u(t) \equiv 0$:

First-order homogeneous differential equation:

$$\dot{x}(t) + ax(t) = 0$$

Real-world examples:

- A cooling object **naturally** losing heat (with no input like fan or AC)
- A car slowing down when releasing the gas pedal (e.g. no break)
- A capacitor discharging in an electrical circuit

Assumed solution:

$$x(t) = Ce^{\lambda t}$$

where C is a constant and λ is to be determined. (see Lecture Notes 02)

Solving First-Order LTI Systems I: Homogeneous Solutions

Finding λ: Substituting $x(t) = Ce^{\lambda t}$ into the equation, we get:

$$C\lambda e^{\lambda t} + aCe^{\lambda t} = 0$$

Factoring out $Ce^{\lambda t}$, we obtain: $\lambda = -a$

Thus, the general solution is:

$$x(t) = Ce^{-at}$$

System Behavior:

- If $a > 0 \rightarrow$ **Stable system** (exponential decay to zero).
- If $a < 0 \rightarrow$ **Unstable system** (exponential growth).
- If $a = 0 \rightarrow$ **System remains constant**.

Key Takeaway: Homogeneous solutions describe natural decay or growth in systems. Understanding stability helps in **control system design**!

Solving 1st order LTI systems II: Step Response

What is a Step Response?

- A step response describes how a system reacts when a sudden, constant input is applied.
- It shows how the system **adjusts to a new steady state**.

Real-Life Examples:

- **Room Heating:** Turning on a heater – temperature gradually increases.
- **Filling a Tank:** Water level rises steadily when a valve is opened.
- **Cruise Control:** Car speed slowly reaches the target speed.

Solving 1st order LTI systems II: Step Response

Equation of Step Response:

$$\dot{x}(t) + ax(t) = m$$

Solution: (see Lecture Notes 02 for more info)

$$x(t) = \left(x_0 - \frac{m}{a}\right) e^{-at} + \frac{m}{a}, \quad t \geq 0$$

Use this solution for the water tank example to calculate tank level as a function of time (e.g. 5min, 10min, etc)

System Behavior:

- **Fast response** (a large) $\rightarrow$ System quickly reaches the final value.
- **Slow response** (a small) $\rightarrow$ System takes longer to adjust.
- System always **smoothly transitions** (never instant).

Key Takeaway: Step responses describe how systems adjust to sudden changes, helping us design fast, stable control systems!

Solving 1st order LTI systems III: Exponential Inputs

- Let us now consider a solution for $\dot{x} + ax = u$ where $u(t)$ is an exponential function: $u(t) = Ae^{st}$ with $s \in \mathbb{R}$. *A* is the input amplitude and *s* determines the growth or decay rate.
- Let's guess a trial solution $x(t) = Be^{st}$:

$$sBe^{st} + aBe^{st} = Ae^{st}.$$

  So $B = \frac{A}{s+a}$.
- Define $G(s) = \frac{B}{A} = \frac{1}{s+a}$. Notice that $x(t) = G(s)u(t)$.
- $G(s)$ is called the *transfer function* and relates the output to the input. We will have a lot more to say about transfer functions in later lectures!
- Infinitely many other solutions (with different initial conditions) are given by $x(t) = \frac{A}{s+a}e^{st} + x_h(t)$ where $x_h(t)$ is *any* homogeneous solution.

Some reflections:

- We looked at solving first-order LTI systems for 3 cases: 1- no input, 2- step input, and 3- exponential input.
- These 3 systems are important to learn and use. They will come up later in this course.
- No one can learn these by looking at them or listening to someone talking about them, you need to practice yourself! Spend some time to read the lecture notes/extra contents and try to solve the equations yourself.
- Read the supplementary materials on Canvas (e.g. videos, book chapter, etc.)
- Attend your tutorial/lab sessions and **review the worked examples** to learn how these equations are used and applied to solve real-world problems.

Challenges of nonlinear dynamics

Real-world example: Friction

Want to know friction as a function of time? Experiment or model?

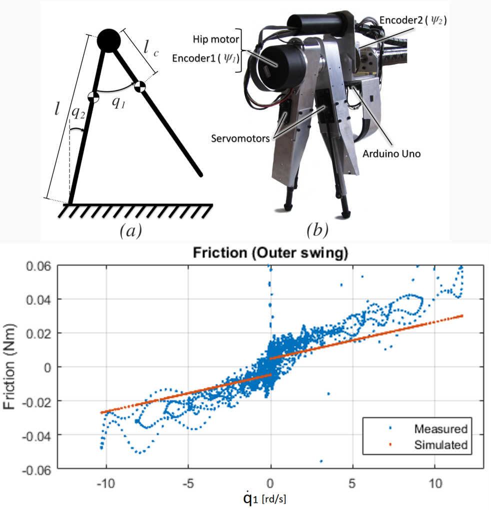


Figure 8: Real experimental data on friction vs. angular velocity

Challenge 1: Non-Existence (Finite Escape)

Consider the differential equation for a scalar signal x:

$$\dot{x} = 1 + x^2.$$

We can easily verify

$$x(t) = \tan(t)$$

is indeed a solution of the differential equation, for the time interval $(-\pi/2, \pi/2)$, but blows up to infinity at the boundaries of this interval.

Hint: Tan(t)=sin(t)/cos(t)

Real-life example: Imagine driving toward a wall. As you get closer, the distance decreases, but the required braking force increases exponentially. Similarly, the tangent function grows infinitely near the boundaries.

Challenge 2: Non-Uniqueness (multiple possible solutions)

Suppose we have a system that behaves like this equation:

$$\dot{x} = \sqrt{|x|}. \tag{6}$$

Here are possible solutions for any t_0 (arbitrary starting time):

$$x(t) = \begin{cases} 0, & t \leq t_0 \\ \frac{1}{4}(t - t_0)^2, & t > t_0 \end{cases}$$

Imagine a car with no engine on a flat hill.

If you give it a tiny push, it starts rolling slowly at first, then speeds up. But if you don't push, it can sit still indefinitely. This is like the equation waiting for an external push (or a starting point t_0) before changing.

Challenge 3: Lack of Closed-Form Solution

- We have seen a few cases in which we were able to verify explicit formulas for the (or a) solution.
- For the vast majority of nonlinear ODEs (especially higher order), there is no known formula for the solution.
- In such cases, we can use numerical simulation (e.g. Simulink) or local analysis via **LINEARISATION (makes life easier!)**

During Week 3 LAB, your tutor will help you build a Simulink block on your laptop for simulating first order systems.No prior knowledge is required, just show up!

Linearisation of nonlinear dynamical systems

Simplifying Nonlinear Systems Using Taylor Series

Nonlinear systems are hard to analyse. Linearising helps us approximate them near equilibrium for easier control and analysis.

What is Linearisation?

- Many real-world systems are **nonlinear**.
- Solving nonlinear equations is often too complex.
- **Linearisation approximates a nonlinear system with a simple linear equation near an equilibrium point**.

Real-World Example:

- Imagine a hilly road
- **Zoomed out**: The road is curved (nonlinear).
- **Zoomed in**: A small section looks straight (linear).
- We approximate the curve using a straight-line equation!

Example: Non-linear cubic function

$$f(x) = x^3 - x = x(1-x)(1+x)$$

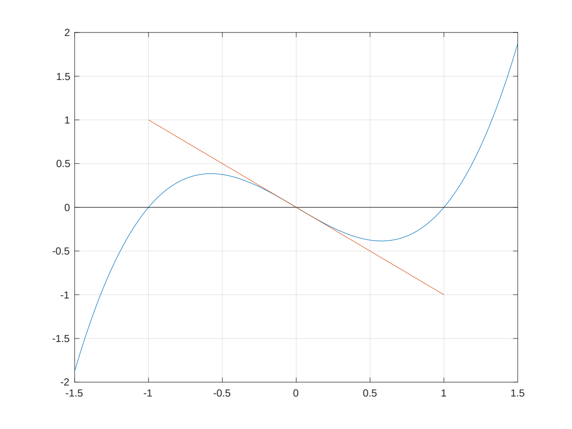


Figure 9: Cubic nonlinearity

Taylor Expansion

$$\dot{x} \approx f(x_e, u_e) + \frac{\partial f}{\partial x}(x - x_e) + \frac{\partial f}{\partial u}(u - u_e)$$

Explanation of the Terms:

- $f(x_e, u_e)$: The function value at equilibrium.
- $\frac{\partial f}{\partial x}(x - x_e)$: The effect of small changes in x.
- $\frac{\partial f}{\partial u}(u - u_e)$: The effect of small changes in u.

*Memorise this equation!

Steps to Linearise a Function using Taylor Expansion

Step 1: Find the Equilibrium Point

- An equilibrium point satisfies:

$$f(x_e, u_e) = 0$$

- At equilibrium, the system remains at rest.

Example:

$$\dot{x} = f(x, u) = x^3 - x + u$$

Solving for equilibrium:

$$x^3 - x + u = 0$$

If $x_e = 1$ and $u_e = 0$, then:

$$1^3 - 1 + 0 = 0$$

So, $(x_e, u_e) = (1, 0)$ is an equilibrium point.

Step 2: Compute Partial Derivatives

Find the first-order derivatives (first with respect to x):

$$\frac{\partial f}{\partial x} = 3x^2 - 1$$

Evaluating at $x_e = 1$:

$$\left.\frac{\partial f}{\partial x}\right|_{x_e} = 3(1)^2 - 1 = 2$$

Find the first-order derivatives (then with respect to u):

$$\frac{\partial f}{\partial u} = 1$$

These values tell us how the system changes near the equilibrium.

Step 3: Write the Linearised Equation

Using Taylor Expansion:

$$\dot{x} \approx f(x_e, u_e) + \frac{\partial f}{\partial x}(x - x_e) + \frac{\partial f}{\partial u}(u - u_e)$$

Since $f(x_e, u_e) = 0$, we get:

$$\dot{x} = 2(x - 1) + (u - 0)$$

Introducing small deviations:

$$\delta_x = x - x_e, \quad \delta_u = u - u_e$$

Final Linearised Equation:

$$\dot{\delta_x} = 2\delta_x + \delta_u$$

This is a **simple linear equation** that approximates the nonlinear system near equilibrium.

Real-life Example: Linearising a Rocket Launch

Figure 10: Rocket Launch Example

Real-life Example: Linearising a Rocket Launch

- Rocket motion follows **complex nonlinear equations**.
- We need to **simplify** the system to control altitude and thrust efficiently.
- Taylor expansion helps us **approximate** the system near equilibrium.

Newton's Second Law for the Rocket:

$$m\ddot{h} = T - mg$$

where:

- $h =$ altitude (m)
- $m =$ mass (kg) - Assume 500 kg
- $T =$ thrust (N)
- $g =$ gravity 9.81 m/s^2

Linearizing a Rocket Equation Using Taylor Expansion (1/2)

Step 1: Equilibrium Thrust

At equilibrium (hover), thrust balances weight:

$$T_e = m\,g.$$

The rocket's vertical velocity at hover is $x_e = 0$ (no climb or descent).

Step 2: Taylor Expansion Setup

We define

$$x(t) = \dot{h}(t), \quad u(t) = T(t), \quad f(x,u) = \frac{u}{m} - g.$$

Then our system is

$$\dot{x} = f(x,u).$$

The Taylor expansion of $f(x,u)$ around (x_e, u_e) is:

$$\dot{x} \approx f(x_e, u_e) + \frac{\partial f}{\partial x}(x - x_e) + \frac{\partial f}{\partial u}(u - u_e)$$

Linearizing a Rocket Equation Using Taylor Expansion (2/2)

Step 3: Evaluate Terms at $(x_e, u_e) = (0, m\,g)$

$$f(x_e, u_e) = \frac{m\,g}{m} - g = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial u} = \frac{1}{m}.$$

$$\dot{x} \;\approx\; 0 \;+\; 0\,(x - 0) \;+\; \frac{1}{m}\,\big(u - m\,g\big) \;\Longrightarrow\; \dot{x} = \frac{1}{m}\,\big(u - m\,g\big).$$

Since $u = T$, we get:

$$\dot{x} = \dot{v} = \frac{T - m\,g}{m}.$$

Numerical Example:

If $m = 500$ kg, $g = 9.8$ m/s^2, then $T_e = 4900$ N.
A small increase $\Delta T = 100$ N yields:

$$\dot{x} = \frac{(4900 + 100) - 4900}{500} = 0.2 \text{ m/s}^2.$$

Why This Is Useful?

- This means a small thrust increase makes the rocket accelerate **0.2 m/s**^{2} faster!

Real-World Impact:

- NASA and SpaceX may use similar approximations for rocket launch/landing control.
- Allows real-time altitude adjustments during launch and re-entry.
- Without this, rocket stability and autopilot would be much harder!

Taylor expansion helps rockets fly smarter!