Lecture Notes 08

State Estimation and Output Feedback

Further Readings. Chapter 8. Åström and Murray, *Feedback Systems*, Second Edition.

1 From State feedback to Output feedback

So for now, we have learned that for the linear dynamical system described by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

where $x \in \mathbb{R}^n$ is the *state*, $u \in \mathbb{R}^p$ is the *input*, $y \in \mathbb{R}^q$ is the *output*, we can carry out feedback control by feedback the system state $x(t)$ back to the input. This gives us the so-called state-feedback controller in the form of

$$u = -Kx \tag{3}$$

and the closed loop system equation $\dot{x} = (A - BK)x$. We we further hope the system output $y(t)$ to track a reference signal $r(t)$, a modified controller would then be

$$u = -Kx + k_r r \tag{4}$$

where k_r is a design parameter as a feedforward gain.

For a real-world system, to implement the controller (3) and (4), however, we would need to measure the system state $x(t)$ at any time t. Therefore, physically it means we need to have a sensor for each dimension of the state signal $x(t)$. Sometimes we afford to do that, but often don't.

Note that $y(t)$ is the system output, a quantity that we suppose to care for practical reasons, and therefore, for such systems $y(t)$ is equipped with sensor from the beginning for the purpose of monitoring the system performance. On the other hand, we simply note that $y(t)$ is often a signal with a lower dimension compared to $x(t)$, even a scalar! Therefore, to monitor $y(t)$ in real time, would also be inherently easier compared to monitoring $x(t)$. A natural idea would be: Maybe we can design a controller u that depends on y only, which becomes a feedback controller depending only on output? This is the origin of the idea for developing output feedback.

Along that thought, we have the following two questions:

1. (Yes or No?) Can we design a control system that chooses u based on measurements of y?
2. (How?) Can we *estimate* the state from y, and if so could we use this state estimate in place of the true state?

If we have clear answers to these questions, a road map in building output feedback framework becomes a two-step process: (1) Build an estimator $\hat{x}(t)$ that estimates $x(t)$ from $y(t)$; (2) Replace $x(t)$ by $\hat{x}(t)$ in a state-feedback controller. Then we hope we have a guarantee that this intuitive approach would work. This estimator, is what so-called an observer.

2 The Observer and Observability

In this section, let us definitively settle the first step: Can we and how do we establish an estimator $\hat{x}(t)$ that estimates $x(t)$ from $y(t)$? We start from a simple example illustrating that the existence of such an estimator is not always a certainty.

2.1 An Example

Suppose there is a point-mass moving in a straight line

$$m\ddot{q} = 0.$$

So q is position, $\dot{q}$ is velocity, and $\ddot{q}$ is acceleration. Now, could we estimate $\dot{q}$ from q? Or vice versa? We have the following straightforward observations.

- We can get the exact $\dot{q}(t)$ from $q(t)$ by differentiating. But a noise with small magnitude but high rate of change (frequency) will significantly upset the estimation.
- We can never obtain $q(t)$ with certainty from $\dot{q}(t)$ because $q(0)$ is not known:

$$q(t) = q(0) + \int_0^t \dot{q}(s)ds$$

We can take a state-space perspective to explain where this difference comes from. The system has a state-space representation: $\dot{x} = Ax$ with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Let $y(t) = Cx(t)$ be the system output (measurement). Then

- Knowing $q(t)$ is to say $C = [1 \;\; 0]$;
- Knowing $\dot{q}(t)$ is to say $C = [0 \;\; 1]$.

So this difference might arise from the different forms of the C vector for the respective forms of output.

2.2 The Observer

A common structure for an observer, also called a "state estimator", is the following "predictor + corrector" form

$$\frac{d}{dt}\hat{x}(t) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{Model prediction}} + \underbrace{L(y(t) - C\hat{x}(t))}_{\text{Measurement correction}} \tag{5}$$

This is the so-called Luenberger observer named after its discoverer Luenberger from Stanford. The idea is that the gain L is chosen to balance between:

1. predictions based on the model and the current estimate $\hat{x}(t)$
2. new information arriving in the measurement $y(t)$.

This $\hat{x}(t)$ attempts to estimate, and thus to get close to $x(t)$. Translating this to mathematical terms we need to have $\hat{x}(t) - x(t)$ converge to zero as time increases.

We define $\tilde{x} = x - \hat{x}$, and collect the dynamics of x and $\hat{x}$:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{\hat{x}} &= A\hat{x}(t) + Bu(t) + L(y - C\hat{x}) \end{aligned}$$

Noting $y = Cx$, we have

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= Ax + Bu - (A\hat{x} + Bu + L(Cx - C\hat{x})) \\ &= A(x - \hat{x}) + \cancel{Bu} - \cancel{Bu} - LC(x - \hat{x}) \\ &= (A - LC)\tilde{x} \end{aligned}$$

Therefore, we obtain $\tilde{x}(t) = e^{(A-LC)t}\tilde{x}(0)$, which goes to zero only if eigenvalues of $A - LC$ have negative real parts. The key turns out to be the existence of L so that $A - LC$ becomes stable. If such an L exists, we have an observer in the form of (5) with proven capability of making state estimations. Such an L turns out also to be necessary for finding *any* state estimator, as explained in the following observability theory.

2.3 The Observability

Let us throw out the full concept of observability all together in the table below. As we easily notice, this what we call observability theory (1) establishes that the existence of an observer relies on the system being *observable* i.e. it is conditional; (2) seems to be very similar as the reachability theory established for state feedback.

Observability Theory

For system described by (1)-(2), the following conditions are equivalent:

- (State computability) The state $x(T)$ at any finite time $T > 0$ can be computed from measurements of $y(t)$ and $u(t)$ on $t \in [0, T]$. Note that computing $x(T)$ is equivalent to computing $x(0)$.
- (Pole placement) For any desired set of eigenvalues, there exists a gain matrix L such that $A - LC$ has those eigenvalues.
- (Observability matrix) The observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank.

We term the system (1)-(2) to be **observable** if one of the three conditions holds.

Instead of proving this observability theory in mathematical terms, we give some insights from the notion of duality. We have seen that state-feedback design amounted to choosing K to set the eigenvalues of $(A - BK)$, which is possible only if the system was *reachable*. Now setting the eigenvalues of $(A - LC)$ by choosing L is a very similar problem. In fact, the only difference is the order of known B, C and unknown/design K, L terms. However, the eigenvalues are not changed by transpose, so

$$\mathrm{eig}(A - LC) = \mathrm{eig}\left(A^T - C^T L^T\right) = \mathrm{eig}(\tilde{A} - \tilde{B}\tilde{K})$$

with $\tilde{A} = A^T$, $\tilde{B} = C^T$, $\tilde{K} = L^T$. Consequently, observer design for system[1] (A, C) is identical to controller design for system[2] (A^T, C^T). This relationship is called Duality.

The magic is matrix transpose!

Duality For the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

its dual system is

$$\dot{x} = A^T x + C^T u$$
$$y = B^T x + D^T u.$$

State-feedback control design for one is observer design for the other, and vice versa.

[1]System (A, C) here means $\dot{x} = Ax, y = Cx$.

[2]System (A^T, C^T) here means $\dot{x} = A^T x + C^T u$.

The insights we gain from this duality are threefold:

(i) The theory of observability is tightly connected to reachability.

(ii) If we design a state-feedback controller $u = -Kx$ for $\dot{x} = A^T x + C^T u$ (which we already know how to do by coefficient matching with the help of reachability canonical form), we will obtain an observer in the form of (5) for the system (1)-(2), with $L = K^T$.

(iii) The step (ii) is possible only if the system (1)-(2) is reachable i.e. the observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank.

We have now established a clear understanding of the problem for estimating x from y! The observable canonical for is also useful to know, but as shown in the point (ii) above, we don't need to specifically work on it for the design of observers.

Observable Canonical Form

By duality transform, a system is in *observable canonical form* if

$$\dot{x} = Ax, \quad y = Cx \tag{6}$$

with

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \ddots & \vdots \\ \vdots & 0 & 0 & \ddots & 1 \\ -a_n & 0 & 0 & \ddots & 0 \end{bmatrix} \tag{7}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{8}$$

It can be shown that

$$\det(sI - A) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n. \tag{9}$$

2.4 Effects of Noise and Disturbance

We will often have disturbances and measurement noise

$$\dot{x} = Ax + d$$
$$y = Cx + n.$$

As a result, the actual observer in use becomes

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + L(y - C\hat{x}) \\ &= (A - LC)\hat{x} + L(Cx + n).\end{aligned}$$

Now with $\tilde{x} = x - \hat{x}$, we have

$$\begin{aligned}\dot{\tilde{x}} &= Ax + d - ((A - LC)\hat{x} + L(Cx + n)) \\ &= (A - LC)\tilde{x} + d - Ln.\end{aligned}$$

Generally larger gains L correspond to faster observer convergence, less effect from disturbance, but measurement noise will have worse effect. This can not be very obvious when A, L, C are matrices.

Example. Let all signals be scalars, i.e. $A = a < 0$ and $C = 1$ (so $y = x + n$), and $L = l \geq 0$. Then

$$\dot{\tilde{x}} = (a - l)\tilde{x} + d - ln.$$

In the absence of d and n, the error decays as $\tilde{x}(t) = e^{(a-l)t}\tilde{x}(0)$.

- Let d be a constant and $n = 0$. At equilibrium $\dot{\tilde{x}}_e = 0$
  $$\frac{\tilde{x}_e}{d} = \frac{1}{l - a}$$
  i.e. as l increases, the error due to disturbances decreases towards zero.
- Let n be a constant and $d = 0$. At equilibrium $\dot{\tilde{x}}_e = 0$
  $$\frac{\tilde{x}_e}{n} = \frac{-l}{l - a}, \tag{10}$$
  as l increases, the error due to measurement noise increases (in magnitude) from $\tilde{x}_e = 0$ at $l = 0$ to $\tilde{x}_e \approx -n$ at large l (as $-l/(l - a) \approx -1$).

3 Output Feedback

3.1 The Idea

So far we have established an observer (5) for the system (1)-(2), that produces $\hat{x}(t)$ estimating $x(t)$ from $y(t)$ (and of course $u(t)$). So we might just replace x by $\hat{x}$ in the state-feedback controller (4), to establish

$$u = -K\hat{x} + k_r r \tag{11}$$

and hope this works. Before we show it is indeed working, we emphasize that the controller (11) depends only on $\hat{x}$. While $\hat{x}$ depends only on y, the controller (11) in turn depends only on $y(t)$ i.e. it is an output feedback.

To show this idea is actually feasible, we need to study the closed-loop dynamics. We collect all dynamical equations of the system:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ u &= -K\hat{x} + k_r r\end{aligned}$$

The closed-loop dynamics can be written (via some complicated but elementary matrix algebra) as the following complete form

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A-LC-BK \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ Bk_r \end{bmatrix} r$$

This is nice as a clear linear system is established. But the matrix in red representing the closed-loop dynamics is complicated, so in this form it's a bit hard to see what's going on. It is better to work with[3] $\tilde{x} = x - \hat{x}$ for which we have $\dot{\tilde{x}} = (A - LC)\tilde{x}$. Now integrating feedback in the system (noting $\hat{x} = x - \tilde{x}$) leads to

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax - BK\hat{x} + Bk_r r \\ &= (A - BK)x + BK\tilde{x} + Bk_r r. \end{aligned}$$

The closed-loop system can be written as

$$\frac{d}{dt}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

for which the corresponding characteristic polynomial is

$$\det(sI - (A - BK)) \times \det(sI - (A - LC)).$$

The poles (eigenvalues) of the closed-loop system are just the eigenvalues of $A - BK$ and the eigenvalues of $A - LC$! This is called the **separation principle**.

Note that, at this point, we have established everything we need for (i) designing an output-feedback controller and (ii) ensuring the closed-loop system is stable. Let us have a brief summary. For the system (1)-(2), we can

- S1 Design a state-feedback gain K for $\dot{x} = Ax + Bu$;
- S2 Design an observer with observer gain L by designing a state-feedback for the dual system $\dot{x} = A^T x + C^T u$;
- S3 Combine the state-feedback and the observer and obtain an output-feedback controller;
- S4 The overall closed-loop system is stable if $A - BK$ and $A - LC$ are both stable.

3.2 The Procedure

Finally, how should we choose the poles for the closed-loop system, which now has $2n$ poles? We just use the same method as before[4]:

1. Choose second-order dominant poles based on desired transient response;
2. Place poles 3 to $2n$, **including observer poles** so they are "less dominant":
   (a) Settling time 5-10 times faster (real part σ of poles 5-10 times further left).
   (b) Low overshoot (high damping factor ζ, i.e., close to the real axis).

[3]This is a change of state space.

[4]Observer gains can also be chosen based on statistical properties of noise and disturbance (Kalman filter)