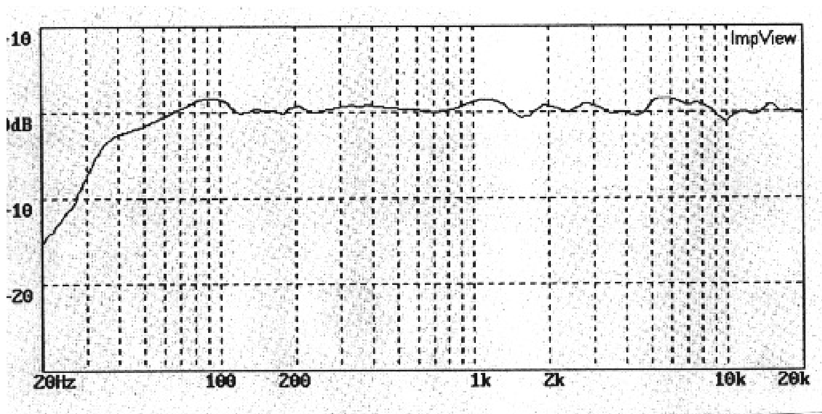
Lecture Notes 10
Frequency Response Analysis

Further Readings. Chapter 10. Åström and Murray, *Feedback Systems*, Second Edition.

Frequency response is something that can be directly measured in the laboratory, with a sinusoidal source of variable frequency and an oscilloscope. The idea is to measure the magnitude and phase of the output relative to the input at a range of frequency values.



[Magnitude frequency response of a loudspeaker]

In these notes: Nyquist test and Stability margins

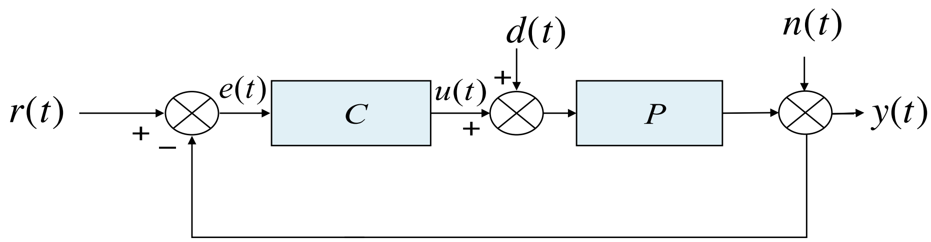
- The Nyquist stability criterion relates the open loop frequency response to the number of closed loop poles in the RHP.
- Is stability guaranteed in the presence of uncertainty and disturbances? This is a question of robustness. The Stability margins can be used to determine how far a stable system is from becoming unstable.

Stability margins may be defined using the Bode or Nyquist plot. The gain and phase margins provide measures of robustness.

1 Nyquist Stability Criterion

1.1 Motivation

Let us consider a standard reference-tracking control loop



To simplify notation, we can combine controller and plant into $L(s) = P(s)C(s)$, which is referred to as a **loop transfer function**. Then we study the loop:

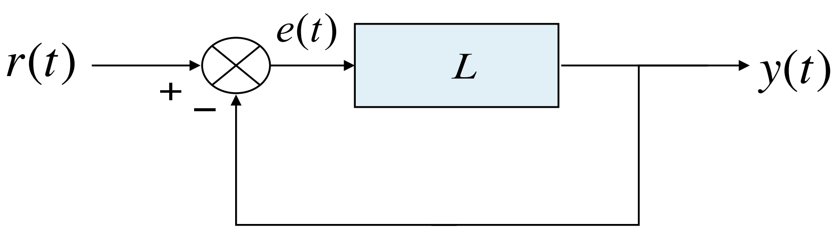


Figura 1: Unity feedback system.

The closed loop transfer function is then (see Lecture 9 Slides)

$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)}.$$

We can establish three points.

- If as a ratio of two polynomials $L(s) = \frac{b(s)}{a(s)}$, we have $G_{\mathrm{cl}}(s) = \frac{b(s)}{a(s)+b(s)}$. The closed-loop poles are thus given by $a(s) + b(s) = 0$, i.e.,
$$1 + L(s) = 0.$$
- For reference signal $r(t) = e^{j\omega t}$, the signal
$$y(t) = G_{\mathrm{cl}}(j\omega)r(t) = \frac{L(j\omega)}{1 + L(j\omega)}r(t)$$
is a particular solution to the closed-loop dynamics. Intuitively, when $r(t) = 0$, the only possibility where $y(t)$ can still be a sustained oscillation for some ω is the case with $1 + L(j\omega) = 0$.
- If indeed $L(j\omega) = -1$ for some ω, it means we have a CL pole right on the imaginary axis. That is the boundary between stability and instability.

Nyquist successfully converted these points into a fundamental stability criterion so that we can tell the CL stability directly from $L(s)$.

1.2 Nyquist Plot

The definition of Nyquist plot is very simple.

Nyquist Plot
The Nyquist plot of a transfer function $H(s)$ is the plot of $H(j\omega)$ in the complex plane for $-\infty < \omega < \infty$.

The intuition is, for any fixed ω, $H(j\omega)$ is a complex number corresponding to a unique point in the s-domain. Surely we can then let ω range over all frequencies and get a trajectory of $H(j\omega)$.

Example. For the transfer function $G(s) = \frac{1}{s+2}$, we can use the following Matlab code to carry out Nyquist plot.

```
g=tf(1, [1 2])
nyquist(g)
```

The result is shown in Figure 2.

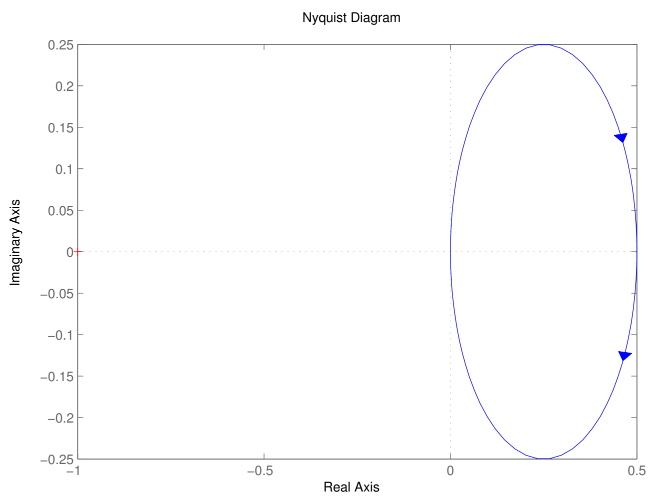


Figura 2: The Nyquist Plot of $G(s) = 1/(s + 2)$.

1.3 Nyquist Stability Criterion

Nyquist Stability Criterion:
For the unity feedback system shown in (1), the number of unstable CL poles Z can be represented as $Z = N + P$, where

- N = number of *clockwise* encirclements of -1 by $L(s)$
- P = number of open loop poles in the right half plane, i.e., poles with a positive real part.

For stability we want $Z = 0$. Note:

- $N > 0$ for clockwise encirclements
- $N < 0$ for anticlockwise encirclements
- $N = 0$ for no encirclements

Procedure of applying Nyquist Stability Criterion:
1. Plot L for $-j\infty \leq s \leq j\infty$. Note that $L(s) \approx 0$ for s with a large magnitude. This plot will always be symmetric with respect to the real axis and is called the Nyquist plot.
2. Evaluate the number N of clockwise encirclements of -1.
3. Determine the number of unstable poles of $L(s)$, denoted by P.
4. The number of unstable closed loop poles $Z = N + P$.

In many cases, we have $L(s) = KG(s)$ where K is a constant. Then

Every encirclement of -1 for $L(s)$ is in fact an encirclement of $-1/K$ for $G(s)$.

Therefore, we can just carry out the Nyquist plot of $G(s)$, and then we can apply the Nyquist Stability Criterion for any $L(s) = KG(s)$ by counting the encirclement of $-1/K$ for $G(s)$.

1.4 Examples

Example 1. Let $G(s) = \frac{1}{s+2}$. This transfer function has a stable open loop pole at $s = -2$. There are no unstable open loop poles, so $P = 0$. The Nyquist plot is shown in Figure 2. No clockwise encirclement of -1, so $N = 0$. Therefore there are $Z = 0 + 0 = 0$ unstable closed loop poles.

Example 2. Let $G(s) = \frac{1}{s-2}$. There is an unstable open loop pole at $s = +2$, so $P = 1$.

From Figure 3, there is no clockwise encirclement of -1, so $N = 0$. Therefore there are $Z = 0 + 1 = 1$ unstable closed loop poles.

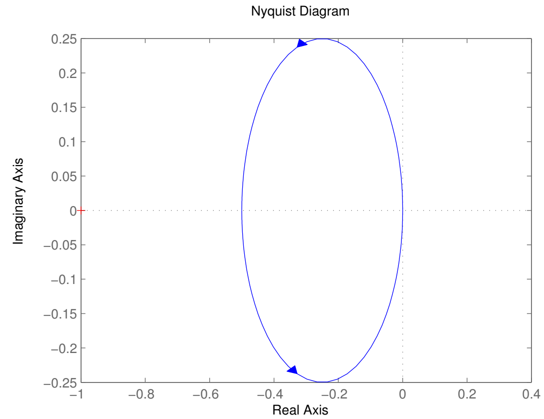


Figura 3: The Nuqist Plot of $G(s) = 1/(s-2)$.

2 Stability Margins

Many practical systems have the property that the system is stable for all small gain values and becomes unstable if the gain increases past a certain critical point. For the unity feedback system shown in Figure (1), a common case is $L(s) = KG(s)$ and the following trial stability test holds:

Trial Stability Test: $|KG(j\omega)| < 1$ at $\angle G(j\omega) = -180°$

Gain margin (GM) is the factor by which the gain can be raised before instability results.

Phase margin (PM) is the amount by which the phase of $G(j\omega)$ exceeds $-180°$ when $|KG(j\omega)| = 1$.

Let $\omega_{-180°}$ be the frequency at which $\angle G(j\omega_{-180°}) = -180°$. Then

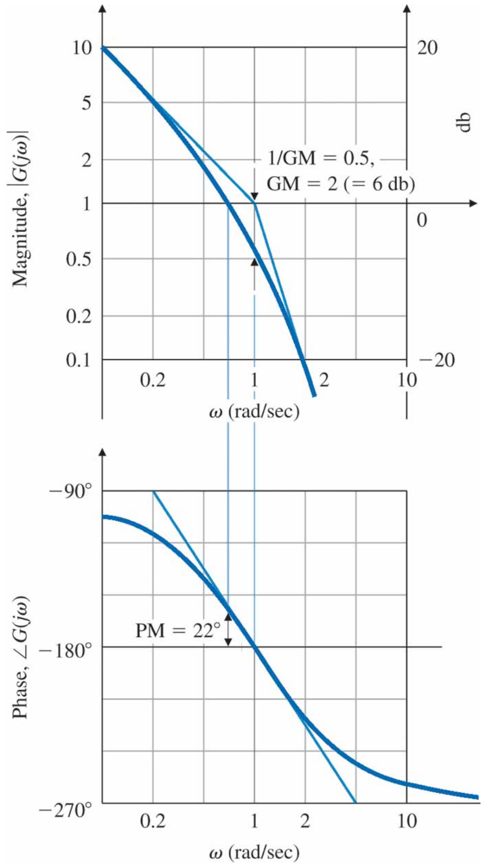
$$GM = \frac{1}{|G(j\omega_{-180°})|}$$

Let ω_c (the crossover frequency) be the frequency where $|H(j\omega_c)| = 1 (= 0\,db)$. Then

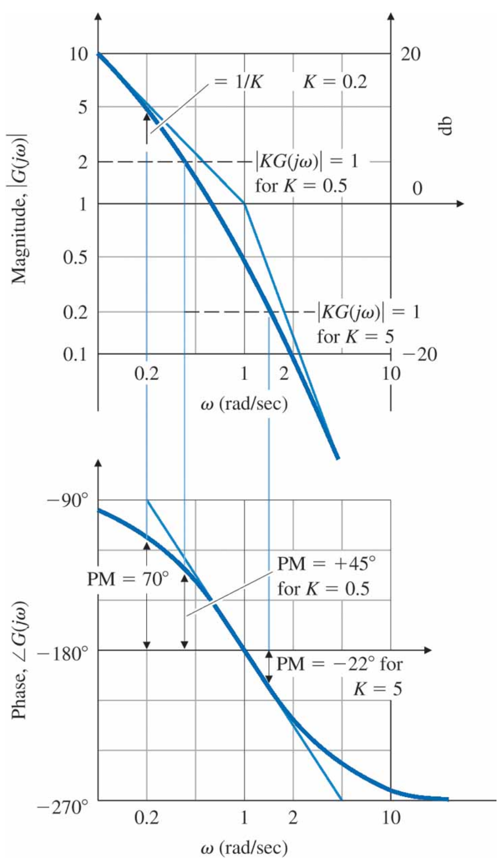
$$PM = \angle G(j\omega_c) + 180°$$

2.1 GM/PM from Bode Diagram

GM and PM can easily be found from the Bode diagram ($K = 1$):



Other gain values ($K = 0.2$):

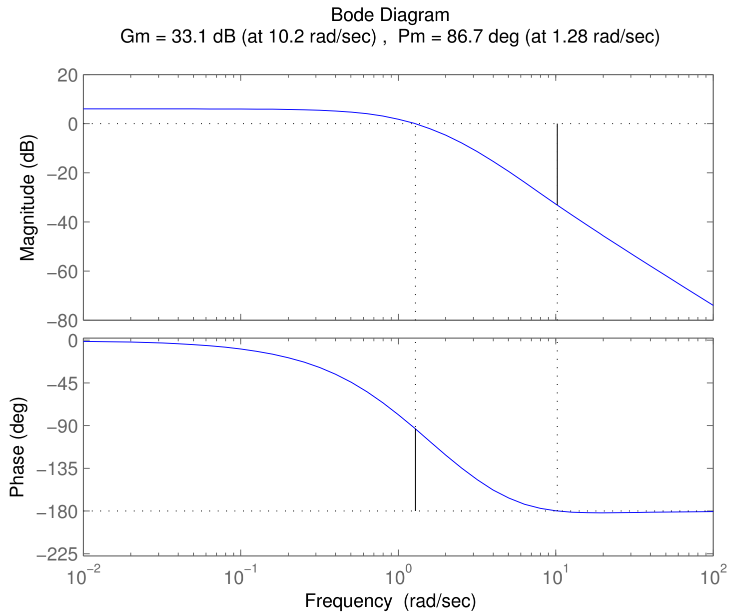


The stability margins can be easily obtained from Matlab for any given transfer function.

Example. (PM and GM from Matlab)

$$G(s) = \frac{2s + 16}{(s+1)(s+2)(s+4)}$$

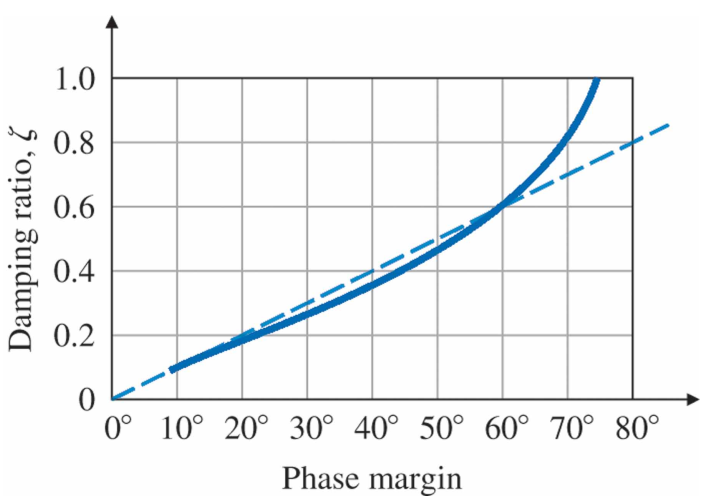
```
g=tf(2*poly(-8), poly([-1 -2 -4]))
margin(g)
```



2.2 PM and Step Specs

For a second order system PM is related to damping ratio by

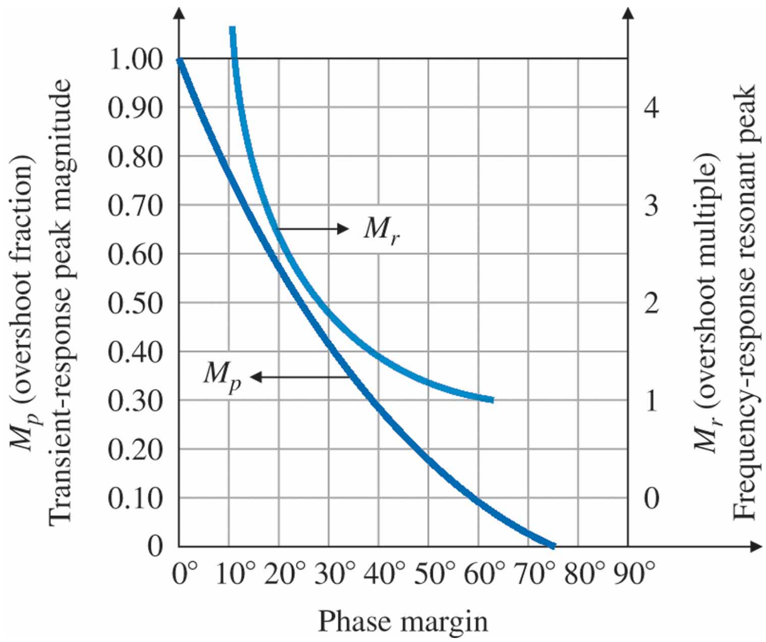
$$PM = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right]$$



For PM below 60° we have

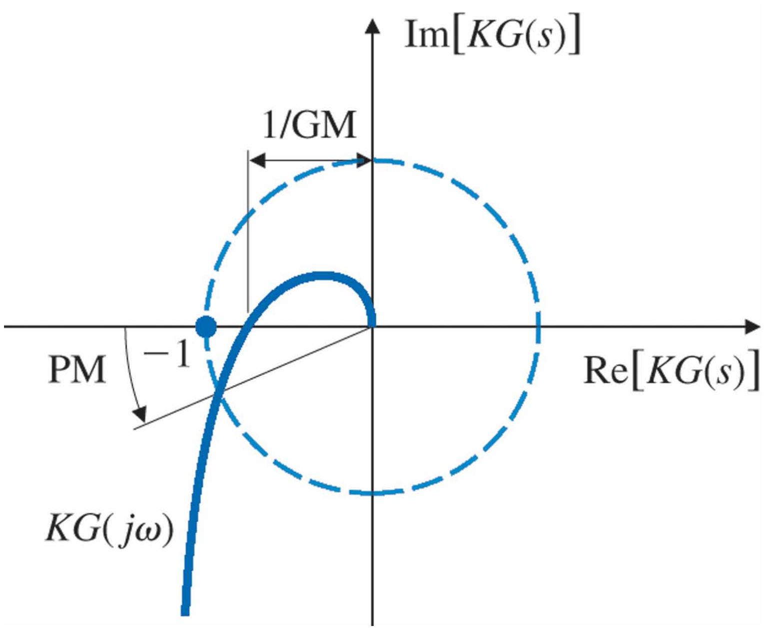
$$\zeta \approx \frac{PM}{100}.$$

The transient response overshoot M_p and resonance peak M_r are related to PM as follows:



Thus PM will often be used to assess and specify performance.

2.3 PM/GM from Nyquist plot



The stability margins may also be defined in terms of the Nyquist plot. The figure shows that GM and PM are measures of how close the Nyquist plot comes to encircling the -1 point. As from the figure: the GM indicates how much the gain can be raised before instability results in a system; the PM is the difference between the phase of $G(j\omega)$ and 180° when $KG(j\omega)$ crosses the circle $|KG(s)| = 1$.