Lecture Notes 05

Feedback Control of Second-order Systems

1 The Control Design Process

The process of designing a feedback control system has a number of well-defined steps, some of which will be iterated:

1. Study the system to be controlled: what can be measured? what can be manipulated directly? what are the control objectives?.
2. Model the system as a set of differential equations, e.g. via principles of physics and/or parameter estimation from experiment.
3. Simplify the model (e.g. linearize nonlinear dynamics) if necessary.
4. Choose a controller structure (e.g. PID) that can meet the objectives.
5. Choose controller gains/parameters based on the objectives (e.g. step response specifications) and the linear model.
6. Simulate the resulting closed-loop system on the more complex model, if needed. Are the requirements met? If not repeat steps 3, 4, or 5 as necessary.
7. Test and validate on the real physical system, tune controllers on-line if necessary, and if the result is not acceptable repeat steps 2-6 as required.

When we speak of "control design", sometimes we will mean this whole process, but often we will just mean steps 4 and 5: choosing a controller structure, and choosing the parameters of that structure to meet certain objectives.

2 Step Response – Complex Poles

Consider the second-order system

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = cu(t) \tag{1}$$

with $a > 0$, $b > 0$, and a unit step input $u(t) = \mathbf{1}(t)$.

This week we will focus on the case that the characteristic equation admits two stable complex poles $\lambda_1 = -\sigma + j\omega_d$ and $\lambda_2 = -\sigma - j\omega_d$ with $j = \sqrt{-1}$, $\sigma > 0$, and $\omega_d > 0$. The relationship between model parameters and pole locations is as follows:

$$\sigma = \frac{a}{2},\ \omega_d = \frac{\sqrt{4b - a^2}}{2}.$$

It is common to also work with the following parameters: $\omega_n = \sqrt{\sigma^2 + \omega_d^2} = \sqrt{b},\ \zeta = \sigma/\omega_n = a/2\omega_n$.

The system equation becomes

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = c.$$

The parameters ζ, ω_n can be related to a polar representation of the pole locations. Clearly ω_n is the magnitude. The parameter ζ, termed *damping ratio*, is the sine of the angle of the pole locations to the imaginary axis. Therefore σ, ω_d define the cartesian coordinates of the poles, and ω_n, ζ define the polar coordinates. This is illustrated in Figure 1.

Note that when ζ is small, poles are close to the imaginary axis, and when ζ is large, the poles are close to the (negative) real line. This is illustrated in Figure 2.

Figure 1: Two complex poles.

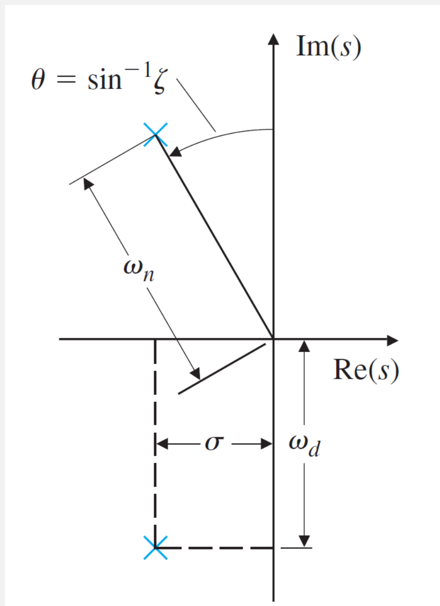
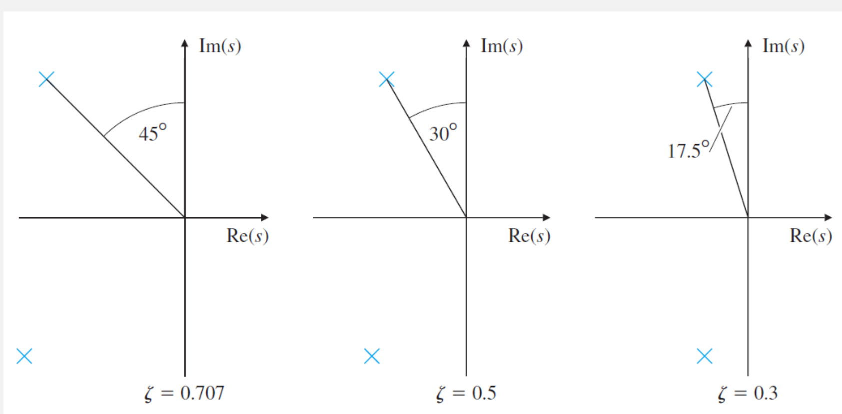


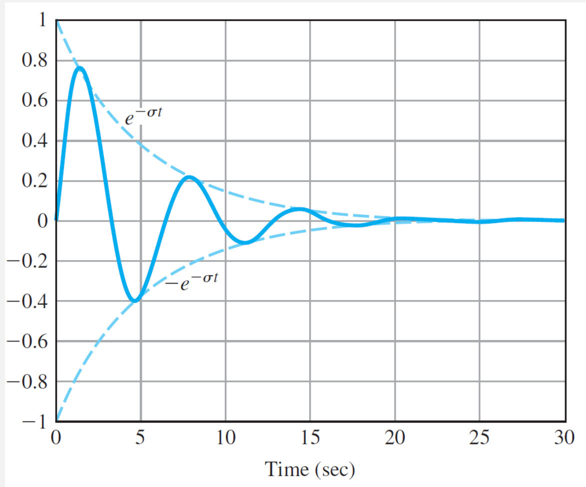
Figure 2: Poles and damping ratio.



2.1 Step Response vs. Damping

To understand the effect of ζ it is helpful to understand the shape of an exponentially-decaying oscillation, e.g. $x(t) = e^{-\sigma t} \sin(\omega_d t)$. This is illustrated in Figure 3.

Figure 3: Exponential envelope of $e^{-\sigma t} \sin(\omega_d t)$



We know that under zero initial values $x(0) = \dot{x}(0) = 0$, the response of $x(t)$ is

$$x(t) = \frac{c}{\omega_n^2}\Big(1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \beta)\Big) \tag{2}$$

$$= \frac{c}{\omega_n^2}\Big(1 - e^{-\sigma t}\big(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\big)\Big) \tag{3}$$

where $\beta = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$.

Since we have a linear system, the response of the system will be linear with respect to the input signal under zero initial values. We can therefore use the convenience to assume $c = \omega_n^2$, and then the step response (2) becomes

$$\begin{aligned} x(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \beta) \\ &= 1 - e^{-\sigma t}\Big(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\Big) \\ &= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right). \end{aligned} \tag{4}$$

When ζ is small, $\frac{\zeta}{\sqrt{1-\zeta^2}}$ is also small, so the step response is approximated by

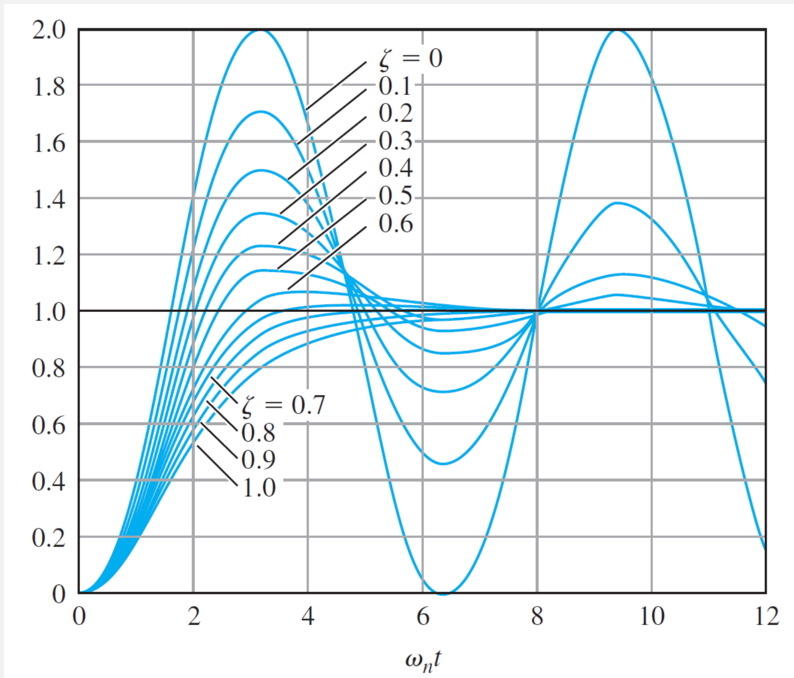
$$x(t) \approx 1 - e^{-\sigma t}\cos(\omega_d t)$$

which should be possible to visualize as an oscillation between 0 and 2, that eventually settles down to 1. See Figure 4 for an illustration of step responses for several values of ζ.

The two boundary cases, $\zeta = 0$ and $\zeta = 1$, have clear intuitions. Here the case $\zeta = 0$ corresponds to the scenario with a pair of purely imaginary poles, i.e., $\sigma = 0$; in that case the response becomes a sine wave.

In the case with $\zeta = 1$, oscillation disappears completely. When ζ is in the middle of 0 and 1, the larger ζ is, the less overshoot and oscillation occurs.

Figure 4: Step responses with different damping ratio ζ



3 Response Specifications

As we have seen, the step response (2) and its simplification (4) can demonstrate quite different behaviors. Such behaviors can be quantified by a number of specifications. We focus on Eq. (4) but such specifications apply to general second-order systems.

3.1 Overshoot and Peak Time

The *overshoot* M_p is the maximum amount that the system overshoots its final value in terms of ratio, usually represented by percentages. The *peak time* t_p is the time when the response reaches maximum value.

The time t_p that reaches the maximum value of $x(t)$ in (4) must satisfy

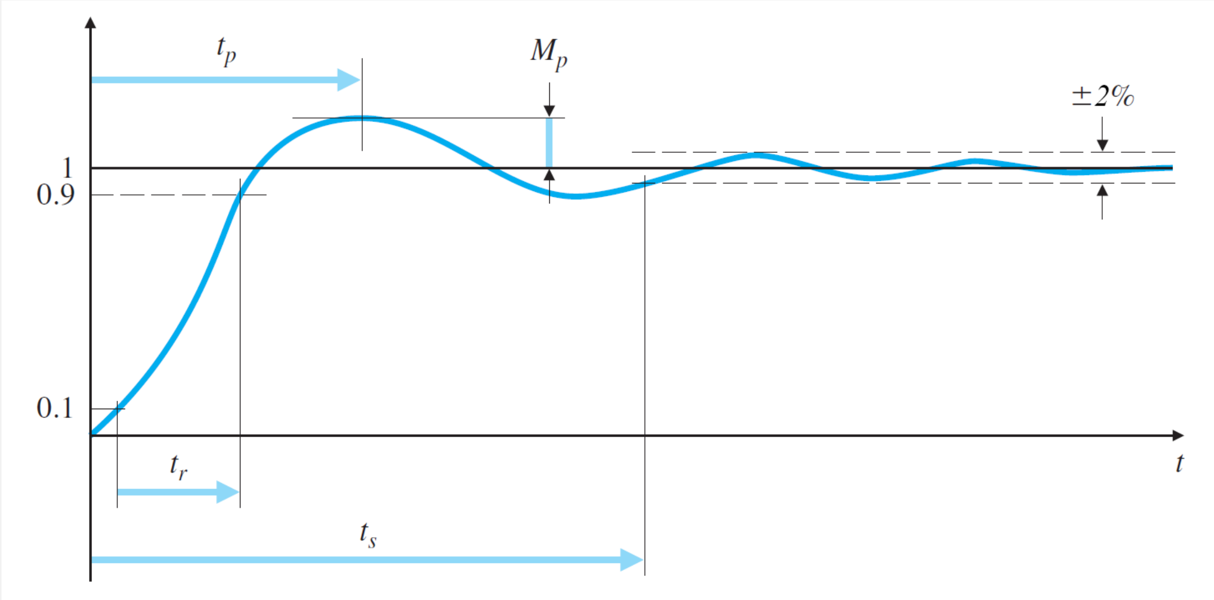
$$\dot{x}(t_p) = 0.$$

The equation

$$\dot{x}(t) = e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d} + \omega_d\right)\sin(\omega_d t) = 0 \tag{5}$$

leads to $\sin(\omega_d t) = 0$. Of course, this has multiple solutions for t, but the natural one to choose for peak time is the first one strictly greater than zero, since it is obviously not zero and later ones will generally have more decayed oscillations (for a stable system.

Figure 5: Specifications of step response in Eq. (4).



We can therefore get $\omega_d t_p = \pi$ and thus

$$t_p = \frac{\pi}{\omega_d}. \tag{6}$$

The maximum value of $x(t)$ is in turn

$$\begin{aligned} x(t_p) :&= 1 + M_p \\ &= 1 - e^{-\sigma\pi/\omega_d}\big(\cos\pi + \frac{\sigma}{\omega_d}\sin\pi\big) \\ &= 1 + e^{-\sigma\pi/\omega_d}. \end{aligned} \tag{7}$$

Therefore, we have

$$M_p = e^{-\sigma\pi/\omega_d} = e^{-\pi\zeta/\sqrt{1-\zeta^2}}. \tag{8}$$

Notice that overshoot is *only* a function of ζ. This relationship is illustrated in the figure below, and we see that $\zeta = 0.5, 0.6, 0.7$ correspond to about $16\%, 10\%$ and 5% overshoot, respectively. These are good to remember.

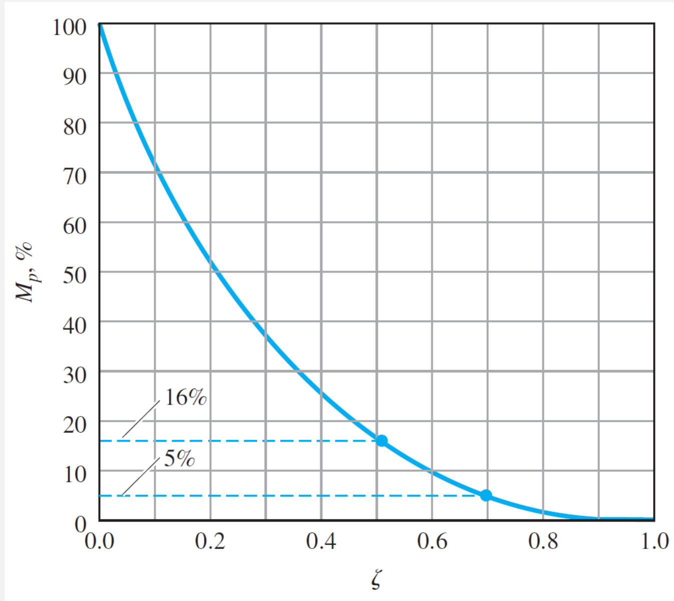
Also, notice that an *undamped* system ($\zeta = 0$) has 100% overshoot.

3.2 Settling time

The *settling time* t_s is the time it takes for the transients to decay to a small value, i.e. for the step response to reach a certain narrow range around its final value and stay there (i.e., not the first time it reaches that value). We can use any particular value for the range, but we will use 2% to define such a small value. The true value of settling time would take into account the oscillations, but a *conservative* estimate can be made based on the exponential envelope illustrated in Figure 3. Using the formula (4), this means

$$e^{-\sigma t_s} = 0.02.$$

Figure 6: Overshoot vs Damping



This gives us

$$t_s = \frac{3.93}{\sigma} = \frac{3.93}{\omega_n \zeta}. \tag{9}$$

I.e., the settling time (under this approximation) depends only on the real part of the poles. Note that this is *only* valid for underdamped systems ($\zeta < 1$).

This is very similar to the first order case, and again we can sometimes use the even looser (and more conservative) approximation $t_s \approx \frac{4}{\sigma}$.

3.3 Rise Time

The *rise time* t_r is the time it takes for the system to do the bulk of its motion from zero to one.. Particularly, we can use the rise time as the time it takes for the trajectory (4) to reach 0.9 from 0.1. The value of t_r is related to both ζ and ω_n in quite a complex way, however the following approximation is reasonably accurate:

$$t_r \approx \frac{1.8}{\omega_n}. \tag{10}$$

I.e., to a good approximation, rise time for underdamped systems depends only on natural frequency.

3.4 Discussions

These time-domain specifications are commonly used by engineers to specify requirements of the dynamical responses. Apparently we want the overshoot, rise time, and settling time to be as small as possible if the system allows. So we can see the use of the polar and cartesian representation of the pole locations

- To reduce settling time, we want to increase σ, i.e. move the poles further to the left in the complex plane.
- To reduce rise time, we want to increase ω_n, i.e. increase the *magnitude* of the pole locations, considered in polar representation.
- To reduce overshoot, we want to increase ζ, i.e. increase the *angle from the vertical* of the pole locations, considered in polar representation.
- It is less common in engineering practice to specify *peak time*, but nevertheless it is interesting to note that increasing ω_d, the imaginary component of the pole location, will decrease the peak time.

Recall that for first order system, the response will simply be an exponential function so there will be no overshoot. In that case we use *time constant* for specifications instead. For higher-order systems, to relate the values of the specifications to positions of poles is difficult. However, many higher-order systems can be approximated by second-order systems through "dominant" poles, i.e., poles that play a dominant role in the responses, and then the above calculations are still valid in the approximate sense.

3.5 Pole Placement

From (10), (6), (8), and (9), all the response specifications can be determined by the position of the poles in terms of σ, ζ, ω_n. Particularly, small t_r, M_p, and t_s can be achieved by large σ, ζ, and ω_n.

Exercise. Find the positions of the poles so that $t_r \leq 0.6$ sec, $t_s \leq 4$ sec, and $M_p \leq 10\%$.

4 Feedback Design

Suppose we have a second-order dynamical system

$$\ddot{x}(t) + \dot{x}(t) = u(t) \tag{11}$$

and the goal is to design a feedback controller so that the response $x(t)$ can track a constant reference $r\mathbf{1}(t)$.

4.1 Proportional Control

Let us try proportional control with

$$u(t) = Ke(t) = K(r - x(t)).$$

The closed-loop system becomes

$$\ddot{x}(t) + \dot{x}(t) + Kx(t) = Kr. \tag{12}$$

Then it is obvious that for any $K > 0$, the system is stable and yields a zero steady state error in the sense that[1] $x(\infty) = r$.

On the other hand, we have $\omega_n = \sqrt{K}$, $\zeta = \frac{1}{2\sqrt{K}}$, and $\sigma = \frac{1}{2}$. Therefore, as K increases, the rise time will decrease because ω_n increases, but the overshoot will increase because ζ decreases. While the settling time will not be affected by the value of K. These observations point out that different gain K will lead to different transient behaviors of the system response, and there exist limitations that the response specifications can possibly achieve by this proportional control.

4.2 PD Control

We can also design a controller as certain combination of a proportion of the error itself (P-control) and the derivative of the error (D-control). This is called PD control described by

$$u(t) = K_P e(t) + K_D \dot{e}(t) = K_P(r - x(t)) + K_D(\dot{r}(t) - \dot{x}(t)).$$

The closed-loop system becomes

$$\ddot{x}(t) + (1 + K_D)\dot{x}(t) + K_P x(t) = K_P r(t) + K_D \dot{r}(t). \tag{13}$$

It is evident that we can now assign the closed-loop system any characteristic equation by choosing different values of K_D and K_P, and therefore the closed-loop poles can be placed anywhere in the left half plain of the complex domain. This in turn means, at least on paper, the response specifications can be met at any required values! It is good to think about the practical realities that may limit this, however.

On the other hand, we also see that the transfer function is

$$G(s) = \frac{K_P + K_D s}{s^2 + (1 + K_D)s + K_P}$$

there is now a *zero* in the system, i.e. a value for s at which $G(s) = 0$. In particular, the zero is at $s = -K_P/K_D$. This will affect the system response.

Remark 1 *Sometimes PD control refers to the controller of the following form*

$$u(t) = K_P(r - x(t)) - K_D \dot{x}(t),$$

i.e. the reference signal is not differentiated. This is also sometimes called rate feedback. Verify that this form does not result in a closed-loop zero.

4.3 PI Control of a First-Order System

Last week, we saw that proportional plus integral (PI) control:

$$u(t) = K_P(r(t) - y(t)) + K_I \int_0^t (r(\tau) - y(\tau))d\tau.$$

[1] Again, at steady state $x(\infty)$ we have $\dot{x}(\infty) = \ddot{x}(\infty) = 0$. Therefore $Kx(\infty) = Kr$ from (12).

of a first order system:

$$\dot{y} + ay = u$$

resulted in

$$\ddot{y} + (a + K_P)\dot{y} + K_I y = K_P \dot{r} + K_I r$$

We see again that the poles of the system can be placed arbitrarily by choice of K_P, K_I, but a zero will be introduced.

5 Effect of Zeros

Consider a second-order system

$$\ddot{x} + a_1\dot{x} + a_0 x = b_1\dot{u} + b_0 u$$

with transfer function

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}.$$

For example, this form showed up in the closed-loop system for the PD and PI control examples above. By now we have quite a clear understanding of the effect of pole locations, now we will discuss the zero at $s = -b_0/b_1$.

We can do this by imagining another system without the zero,

$$\ddot{x} + a_1\dot{x} + a_0 x = v$$

but subjected to a new input

$$v = b_1\dot{u} + b_0 u.$$

Intuitively, if $b_1 > 0$ is large relative to $b_0 > 0$, then any sudden changes in u will be amplified by the $\dot{u}$ term. In particular, the step response will receive an additional "kick" when it changes from zero to one[2]. Geometrically, this corresponds to the zero being in the left half of the complex plane, but close to the unit circle.

Now consider the case that $b_1 < 0$ and $b_0 > 0$. In this case, the "initial kick" is in the opposite direction to the steady-state response. This means that the step response will initially go in one direction, and then go in the other. This is illustrated in the figure below, which shows the step response for the system

$$G(s) = \frac{-s + 1}{s^2 + s + 1}.$$

Geometrically, this corresponds to a zero in the right-half of the complex plane. Systems with right-half-plane zeros are called *non-minimum phase* and are quite difficult to control. An example you may be familiar with is reversing a car with a trailer attached: in order to turn one way, you first need to turn the other. A motorcycle at high speed is also non-minimum-phase: to turn left a rider will first turn the handle bars right to induce a roll to the left, after which they can turn left.

[2] Formally the derivative of a unit step function is a *unit impulse* or *dirac delta*, which is not a true function but a generalized function or distribution. Although it is not a true function, it does make sense as the input to a linear differential equation.

Figure 7: Step Response with Right-half-plane Zero