Lecture 12 – Frequency analysis of water extremes

ENCI 608: Sustainable Water Systems

Updates and Information

D2L — Lecture notes 12 are posted

Assignment 3 — Assignment 3 is available and due March 11th, 2025

Office hours — Mondays, from 10:00 to 12:00 hrs

Today's lecture

Learning objectives:

- Grasp the principles, assumptions, and methods underlying frequency analysis (FA).
- Develop the ability to apply FA in real-world scenarios.

Expected outcomes:

- Successfully estimate design events using FA for practical engineering applications.

Why does analyzing water extremes matter?

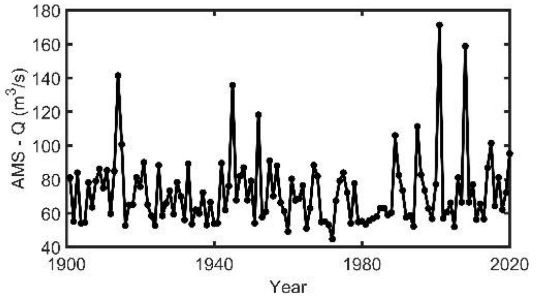
Floods are one of the most common and dangerous natural hazards:

- Floods affected 2.8 billion people globally and caused ~540,000 deaths from 1980 – 2009 ([Doocy et al., 2013](#)).
- Human settlements in floodplain zones have nearly doubled since 1985, increasing flood exposure ([Andreadis et al., 2022](#)).
- Flooding is Canada's most common and costly disaster ([Public Safety Canada](#)).
- In Canada, floods average $800 million annually in insured losses ([Insurance Bureau of Canada, 2024](#)), which are doubled by uninsured ones ([Honegger and Oehy, 2016](#)).

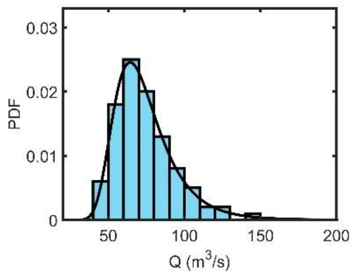
Frequency analysis (FA) of water extremes

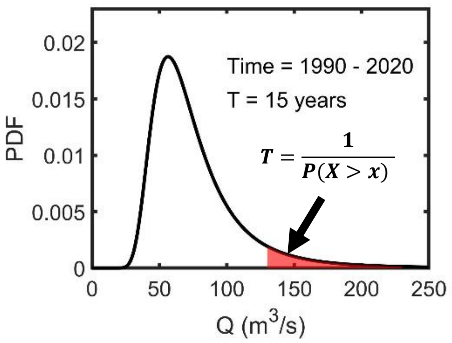
Extreme-event data

Streamflow extreme events



Select and fit a probability distribution

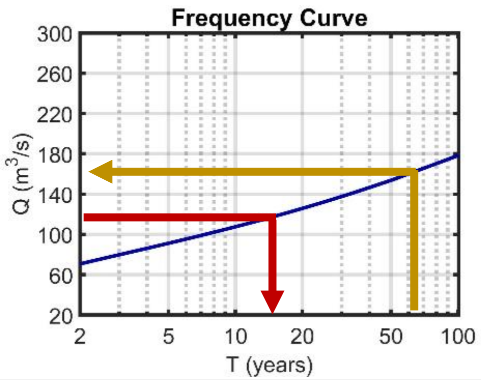




Communicate FA

How large...?

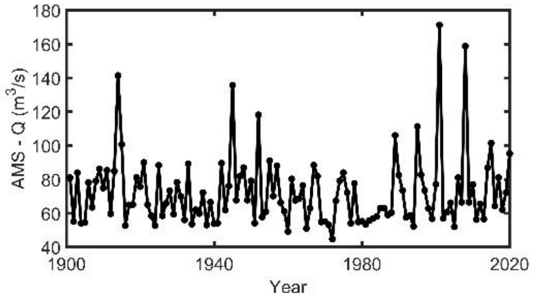
How often...?



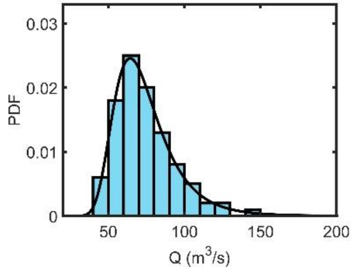
Frequency analysis (FA) of water extremes

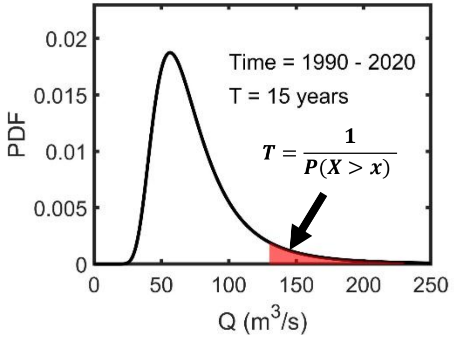
Extreme-event data

Streamflow extreme events



Select and fit a probability distribution

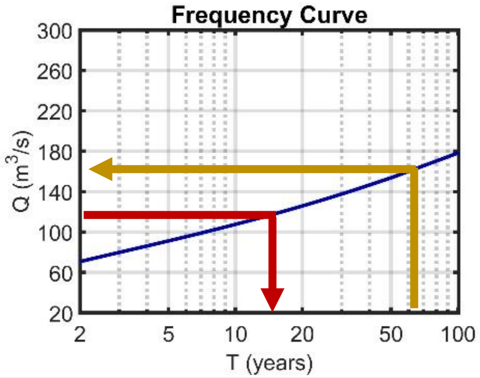




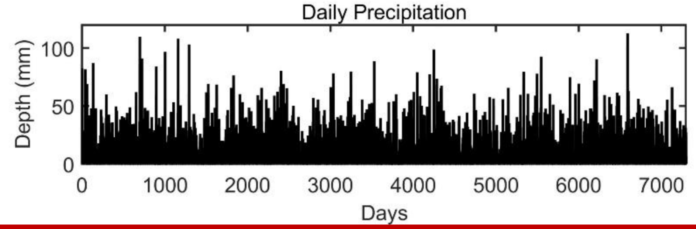
Communicate FA

How large...?

How often...?



Sampling data types



Annual Maxima

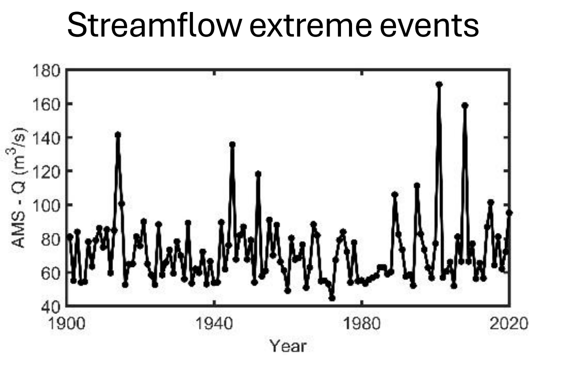
- ➢It employs the largest event per year.
- ➢Pioneered by Fréchet (1927) and Fisher and Tippett (1928) and applied in engineering by Gumbel (1958).

Peaks-over-threshold

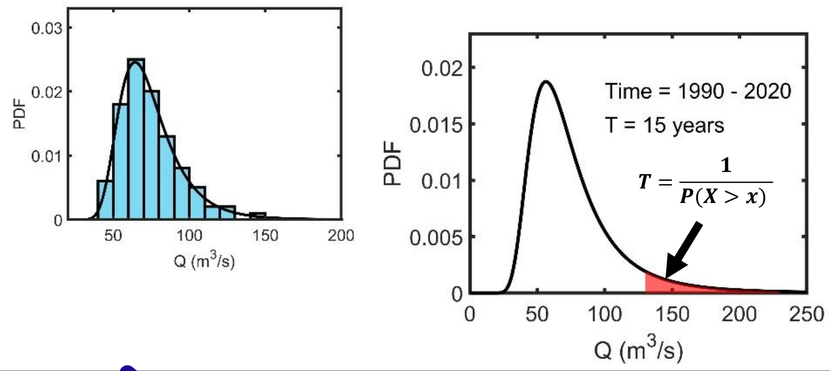
- ➢It uses independent events exceeding a high threshold.
- ➢The basis is the Pickands-Balkema-de Haan theorem (Balkema and de Haan 1974; Pickands III 1975).

Frequency analysis (FA) of water extremes

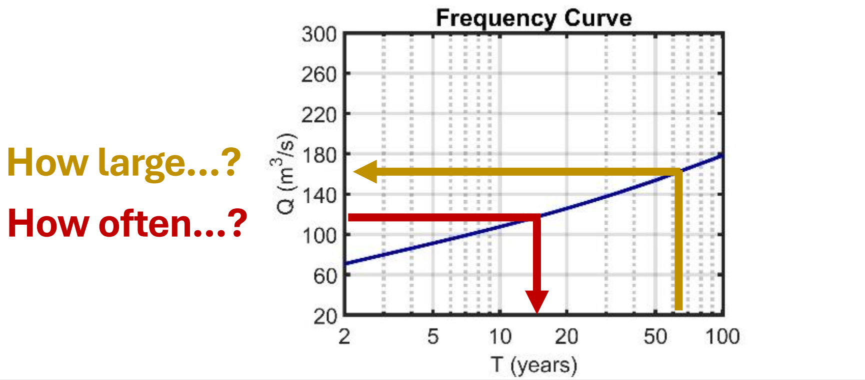
Extreme-event data



Select and fit a probability distribution



Communicate FA



FA conventional assumptions

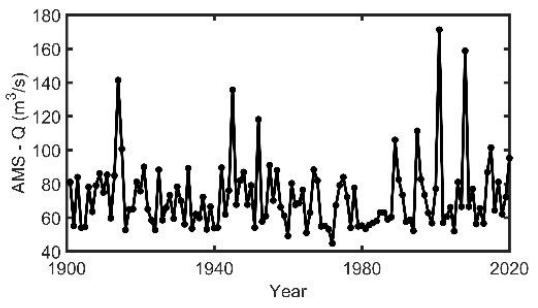
The conventional assumption in FA is that observations are independent and identically distributed (i.i.d.).

The i.i.d. assumption implies that observations are realizations from a **stationary random process** representing a **single physical generating mechanism**.

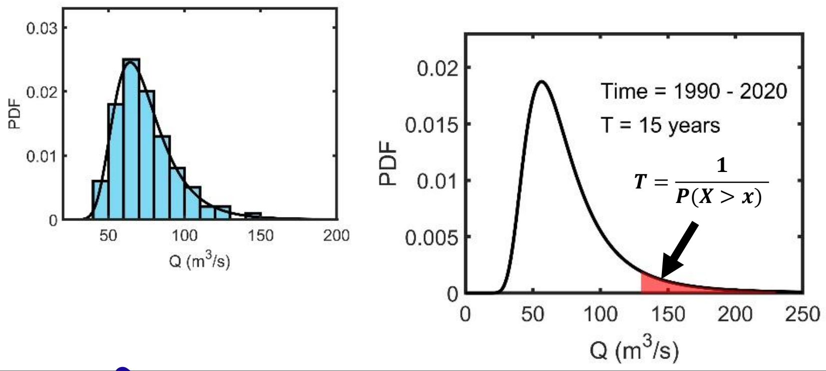
Frequency analysis (FA) of water extremes

Extreme-event data

Streamflow extreme events



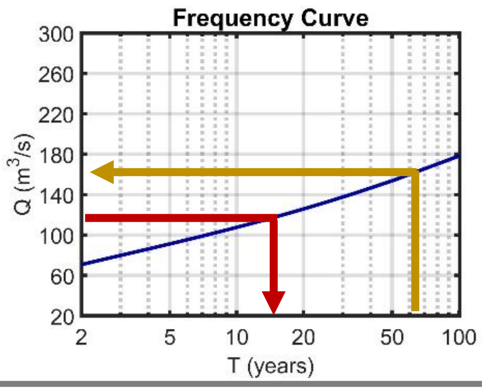
Select and fit a probability distribution



Communicate FA

How large...?

How often...?



FA distributional approaches

Non-parametric: often relies on plotting position equations and is less popular due to its lower out-of-sample prediction capability.

Parametric: assumes a probability distribution family whose mathematical structure (expression) is known. This is the most widely used approach.

Non-parametric FA: plotting positions

A **plotting position** is a distribution-free estimator commonly used to derive empirical exceedance probabilities.

Let $x_{n:n} \geq x_{n-1:n} \geq \cdots \geq x_{1:n}$ be the sample observations arranged in descending order of magnitude. The plotting position is given by:

$$p_{i:n} = \frac{i-a}{n+1-2a}, \qquad i = 1,2,\ldots,n$$

where $p_{i:n}$ is the exceedance probability, i is the rank of each sample point, n is the sample size, and the coefficient a is given by the selected plotting position formula (see Table 1).

Table 1. Popular plotting position formulas used in FA.

Plotting position formula	$\boldsymbol{a}$	**Simplified equation**
Weibull	0	$p_{i:n} = \frac{i}{n+1}$
Blom	0.375	$p_{i:n} = \frac{i-0.375}{n+0.25}$
Cunnane	0.4	$p_{i:n} = \frac{i-0.4}{n+0.2}$
Gringorten	0.44	$p_{i:n} = \frac{i-0.44}{n+0.12}$
Hazen	0.5	$p_{i:n} = \frac{i-0.5}{n}$

Non-parametric FA: plotting positions

Steps:

1. Sort the annual maxima values in decreasing order (i.e., from largest to smallest).
2. Assign ranks (i) to the sorted data.
3. Select a plotting position formula.
4. Calculate the exceedance probability of each data point using the plotting position formula.
5. Estimate the return periods T associated with the exceedance probability.
6. Plot observations against empirical return periods (optional).

Let's see an example…

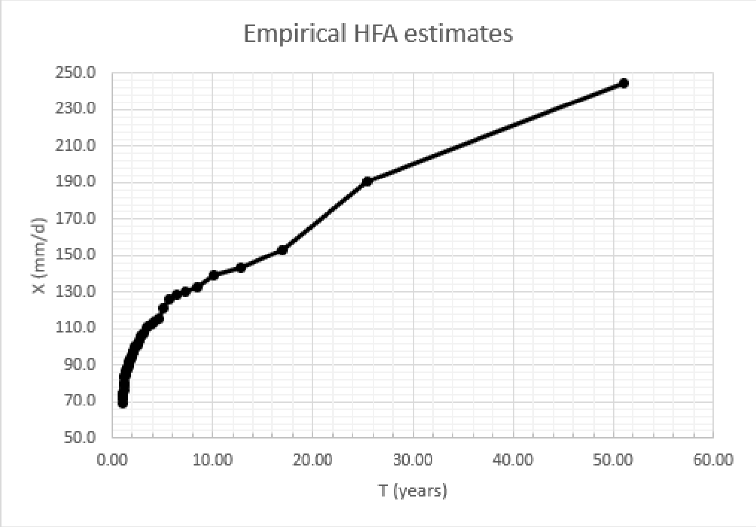
Non-parametric FA: plotting positions

Example: Use Excel to estimate the empirical return periods of a given sample (use the data from *ENCI608_L12_Examples.xlsx*)

	A	B
1	Year	X (mm/d)
2	1971	105.8
3	1972	130.5
4	1973	73.1
5	1974	76.7
6	1975	190.4
7	1976	92.1
8	1977	153.3
9	1978	69.4
10	1979	96.8
11	1980	94.8
12	1981	101.2
13	1982	71.7
14	1983	74.2
15	1984	106.5
16	1985	98.9
17	1986	113.6
18	1987	85.6
19	1988	93.4
20	1989	79.7
21	1990	107.3
22	1991	86.9
23	1992	139.1
24	1993	143.0
25	1994	126.2
26	1995	79.1
27	1996	115.8
28	1997	80.7
29	1998	110.7
30	1999	111.6
31	2000	244.9
32	2001	92.9

	A	B	C	D	E
1	Year	X (mm/d)	i (rank)	Pe	T (return period)
2	2000	244.9	1	0.020	51.00
3	1975	190.4	2	0.039	25.50
4	1977	153.3	3	0.059	17.00
5	1993	143.0	4	0.078	12.75
6	1992	139.1	5	0.098	10.20
7	2004	133.0	6	0.118	8.50
8	1972	130.5	7	0.137	7.29
9	2019	128.3	8	0.157	6.38
10	1994	126.2	9	0.176	5.67
11	2009	121.1	10	0.196	5.10
12	1996	115.8	11	0.216	4.64
13	1986	113.6	12	0.235	4.25
14	2020	112.6	13	0.255	3.92
15	1999	111.6	14	0.275	3.64
16	1998	110.7	15	0.294	3.40
17	1990	107.3	16	0.314	3.19
18	1984	106.5	17	0.333	3.00
19	1971	105.8	18	0.353	2.83
20	2010	103.7	19	0.373	2.68
21	1981	101.2	20	0.392	2.55
22	2007	100.6	21	0.412	2.43
23	2017	100.0	22	0.431	2.32
24	1985	98.9	23	0.451	2.22
25	2006	97.3	24	0.471	2.13
26	1979	96.8	25	0.490	2.04
27	1980	94.8	26	0.510	1.96
28	2005	94.3	27	0.529	1.89
29	1988	93.4	28	0.549	1.82
30	2001	92.9	29	0.569	1.76
31	1976	92.1	30	0.588	1.70
32	2002	91.2	31	0.608	1.65

Plotting position formula	
a =	0
n =	50



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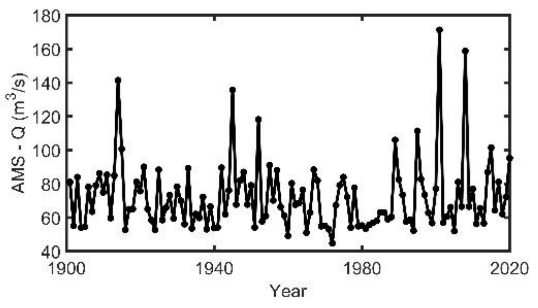
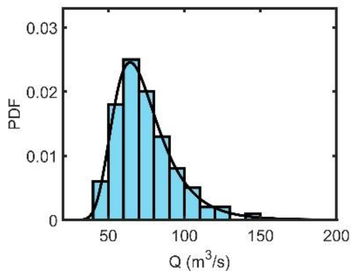
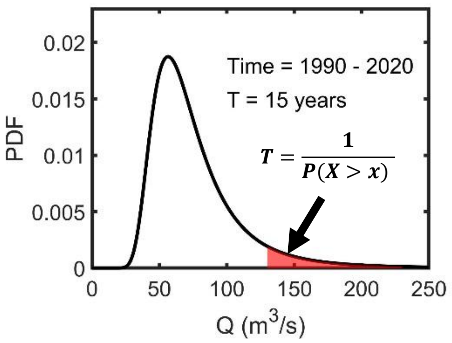
Extreme-event data



Select and fit a probability distribution

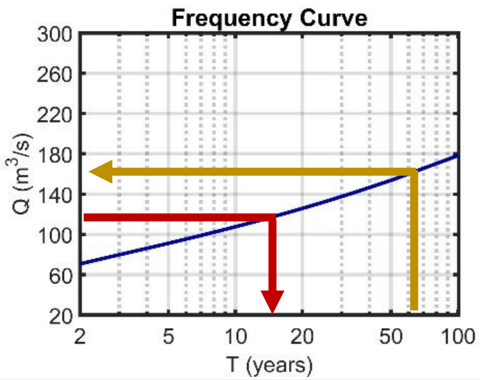




Communicate FA

How large...?

How often...?



FA distributional approaches

Non-parametric: often relies on plotting position equations and is less popular due to its lower out-of-sample prediction capability.

Parametric: assumes a probability distribution family whose mathematical structure (expression) is known. This is the most widely used approach.

What probability distribution $[F(x)]$ can we use in FA?

Annual maxima theoretically follow the **Generalized Extreme Value (GEV) distribution** as the number of events per year →∞.

Since convergence to GEV may not hold in hydrology, we also use several other popular empirical distributions:

- **Generalized Logistic (GLO)**
- **Generalized Normal (GNO/3LN)**
- **Log-Pearson type III (LP3)**
- **Pearson type III (PE3)**

These include simpler forms as special cases: GEV, LP3, and PE3 include **Gumbel**, **Log-normal**, and **Gamma**, respectively.

Methods for distribution selection:

- Arbitrary/subjective selection.
- Goodness-of-fit metrics (e.g., RMSE).
- Statistical tests (e.g., Kolmogorov-Smirnov and Anderson-Darling tests).
- Statistics of the sample and distributions.
  - The **L-moments** are widely used due to their robustness.

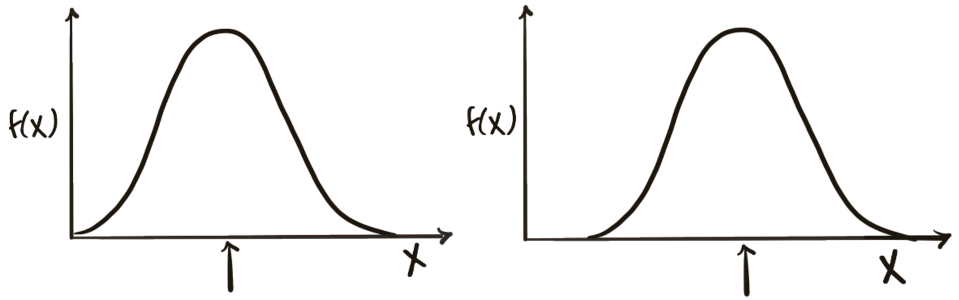
*“**L-moments** rely on expectations of linear combinations of order statistics.”*

Intuitive overview of L-moments

Conventional moments use powers of data values to describe distribution's features. **L-moments** also capture the distribution characteristics but rely on linear combinations of the data, which makes them less sensitive to outliers.

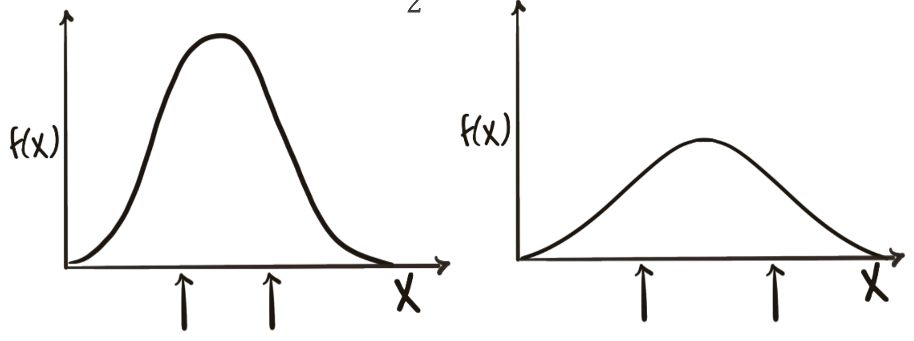
First L-moment: central tendency (location)

$$\lambda_1 = \mathrm{E}(X_{1:1})$$



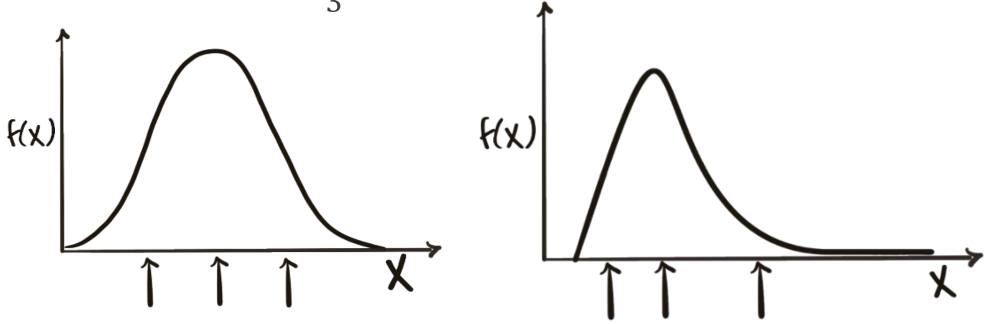
Second L-moment: dispersion (scale)

$$\lambda_2 = \frac{1}{2}\mathrm{E}(X_{2:2} - X_{1:2})$$



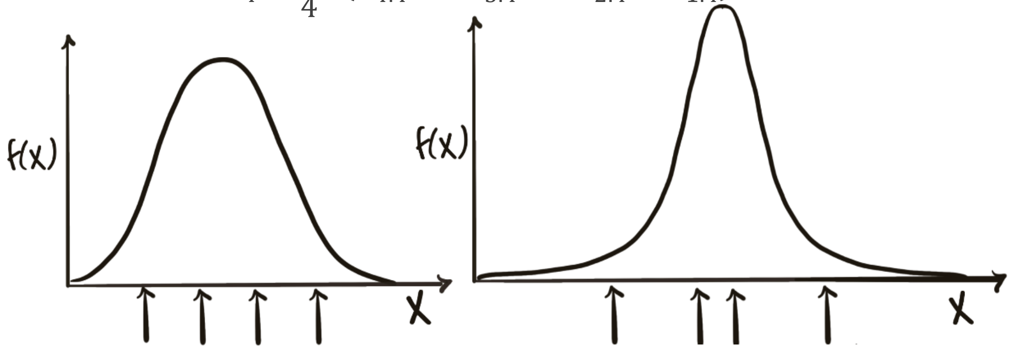
Third L-moment: skewness/asymmetry (shape)

$$\lambda_3 = \frac{1}{3}\mathrm{E}(X_{3:3} - 2X_{2:3} + X_{3:3})$$



Fourth L-moment: peakedness/flatness

$$\lambda_4 = \frac{1}{4}\mathrm{E}(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$$



Sample L-moments, step-by-step

In practice, we calculate the sample L-moments as follows.

Let $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$ be the ordered sample in ascending order of magnitude.

1) Calculate the unbiased estimators of the probability weighted moments (b_r) for $r = 0,1,2 \text{ and } 3$:

$$b_0 = \sum_{i=1}^{n} \frac{1}{n} x_{i:n}$$

$$b_1 = \sum_{i=2}^{n} \frac{(i-1)}{n(n-1)} x_{i:n}$$

$$b_2 = \sum_{i=3}^{n} \frac{(i-1)(i-2)}{n(n-1)(n-2)} x_{i:n}$$

$$b_3 = \sum_{i=4}^{n} \frac{(i-1)(i-2)(i-3)}{n(n-1)(n-2)(n-3)} x_{i:n}$$

2) Compute the first four sample L-moments by

$$l_1 = b_0$$

$$l_2 = 2b_1 - b_0$$

$$l_3 = 6b_2 - 6b_1 + b_0$$

$$l_4 = 20b_3 - 30b_2 + 12b_1 - b_0$$

3) Obtain the sample L-skewness (t_3) and L-kurtosis (t_4) are calculated by

$$t_3 = \frac{l_3}{l_2} \qquad t_4 = \frac{l_4}{l_2}$$

Sample L-moments

Example: Use Excel to estimate the sample L-moments and ratios (use the data from *ENCI608_L12_Examples.xlsx*)

Year	X (m3/s)
1971	105.8
1972	130.5
1973	73.1
1974	76.7
1975	190.4
1976	92.1
1977	153.3
1978	69.4
1979	96.8
1980	94.8
1981	101.2
1982	71.7
1983	74.2
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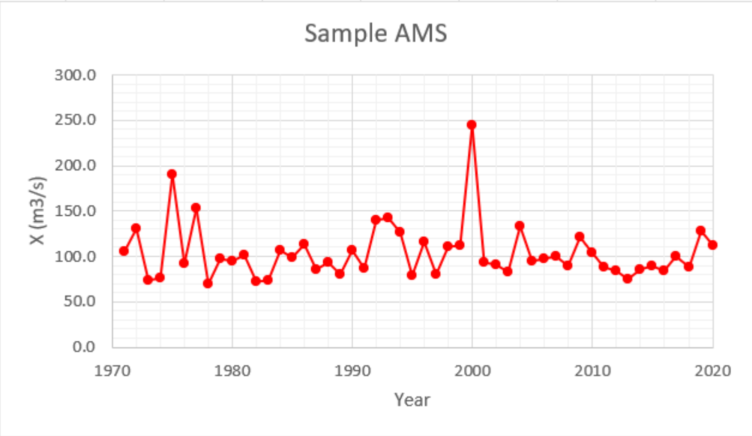


i	X (m3/s)	b0_calc	b1_calc	b2_calc	b3_calc
1	69.4	1.3878	0.0000	0.0000	0.0000
2	71.7	1.4338	0.0293	0.0000	0.0000
3	73.1	1.4623	0.0597	0.0012	0.0000
4	74.2	1.4843	0.0909	0.0038	0.0001
5	75.1	1.5028	0.1227	0.0077	0.0003
6	76.7	1.5335	0.1565	0.0130	0.0008
7	79.1	1.5825	0.1938	0.0202	0.0017
8	79.7	1.5933	0.2276	0.0285	0.0030
9	80.7	1.6138	0.2635	0.0384	0.0049
10	83.5	1.6696	0.3067	0.0511	0.0076
11	83.9	1.6784	0.3425	0.0642	0.0109
12	84.4	1.6871	0.3787	0.0789	0.0151
13	85.2	1.7043	0.4174	0.0956	0.0204
14	85.6	1.7128	0.4544	0.1136	0.0266
15	86.9	1.7382	0.4966	0.1345	0.0343
16	87.8	1.7550	0.5373	0.1567	0.0433
17	88.2	1.7635	0.5758	0.1799	0.0536
18	89.0	1.7804	0.6177	0.2059	0.0657
19	89.4	1.7889	0.6571	0.2327	0.0792
20	91.2	1.8234	0.7070	0.2651	0.0959
21	92.1	1.8410	0.7514	0.2974	0.1139
22	92.9	1.8590	0.7967	0.3320	0.1342
23	93.4	1.8681	0.8388	0.3670	0.1562
24	94.3	1.8867	0.8856	0.4059	0.1814
25	94.8	1.8962	0.9287	0.4450	0.2083
26	96.8	1.9355	0.9875	0.4938	0.2416
27	97.3	1.9458	1.0325	0.5377	0.2746
28	98.9	1.9777	1.0897	0.5903	0.3140
29	100.0	2.0000	1.1429	0.6429	0.3556
30	100.6	2.0116	1.1905	0.6945	0.3990
31	101.2	2.0234	1.2388	0.7485	0.4459
32	103.7	2.0738	1.3120	0.8200	0.5060

n =	50
b0 =	103.8903
b1 =	59.4408
b2 =	42.9657
b3 =	34.1721

sample L-moments

l_1 =	103.8903
l_2 =	14.9914
l_3 =	5.0396
l_4 =	3.8708

L-ratios

t_3 =	0.3362	L-skewness
t_4 =	0.2582	L-kurtosis

Sample L-moments for distribution selection

We select the distribution by comparing the sample L-moment ratios (t_3 and t_4) to theoretical L-moment ratios (τ_4 and τ_3) of distribution families.

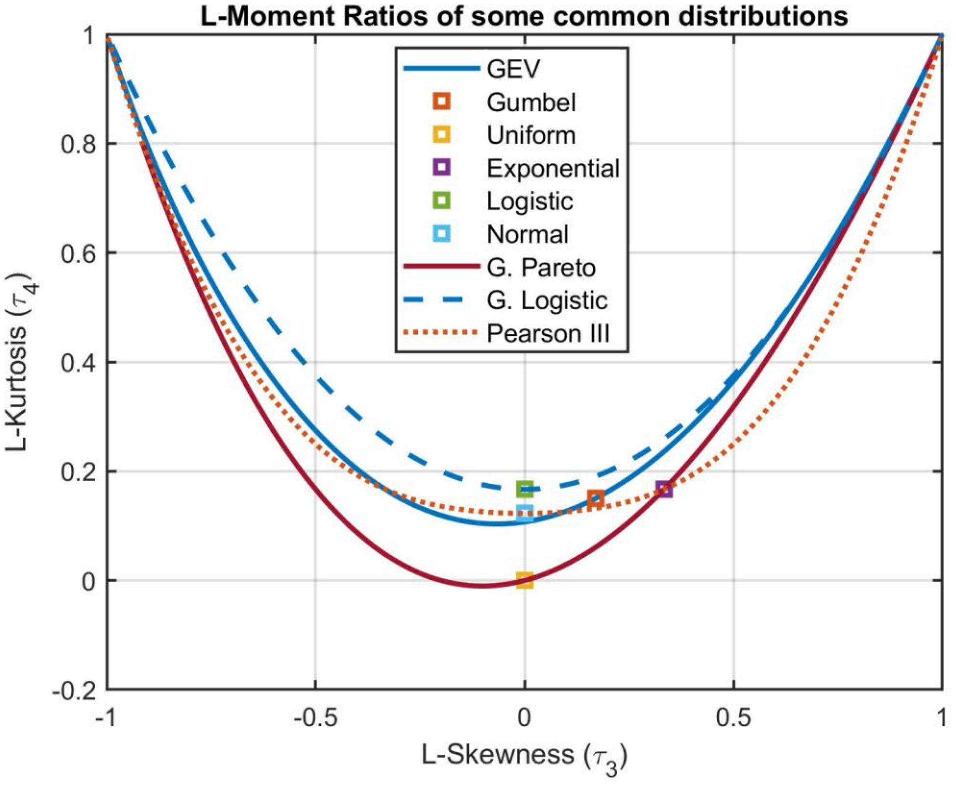
Table 2. L-moments of some common distributions (Hosking [1990]).

Distribution	*F(x) or x(F)*	*L-moments*
Uniform	$x=\alpha+(\beta-\alpha)F$	$\lambda_1=\frac{1}{2}(\alpha+\beta),\ \lambda_2=\frac{1}{6}(\beta-\alpha),\ \tau_3=0,\ \tau_4=0$
Exponential	$x=\xi-\alpha\log(1-F)$	$\lambda_1=\xi+\alpha,\ \lambda_2=\frac{1}{2}\alpha,\ \tau_3=\frac{1}{3},\ \tau_4=\frac{1}{6}$
Gumbel	$x=\xi-\alpha\log(-\log F)$	$\lambda_1=\xi+\gamma\alpha,\ \lambda_2=\alpha\log 2,\ \tau_3=0.1699,\ \tau_4=0.1504$
Logistic	$x=\xi+\alpha\log\{F/(1-F)\}$	$\lambda_1=\xi,\ \lambda_2=\alpha,\ \tau_3=0,\ \tau_4=\frac{1}{6}$
Normal	$F=\Phi\left(\frac{x-\mu}{\sigma}\right)$	$\lambda_1=\mu,\ \lambda_2=\pi^{-1}\sigma,\ \tau_3=0,\ \tau_4=30\pi^{-1}\tan^{-1}\sqrt{2}-9=0.1226$
Generalized Pareto	$x=\xi+\alpha\{1-(1-F)^k\}/k$	$\lambda_1=\xi+\alpha/(1+k),\ \lambda_2=\alpha/(1+k)(2+k),$ $\tau_3=(1-k)/(3+k),\ \tau_4=(1-k)(2-k)/(3+k)(4+k)$
Generalized extreme value	$x=\xi+\alpha\{1-(-\log F)^k\}/k$	$\lambda_1=\xi+\alpha\{1-\Gamma(1+k)\}/k,\ \lambda_2=\alpha(1-2^{-k})\Gamma(1+k)/k,$ $\tau_3=2(1-3^{-k})/(1-2^{-k})-3,$ $\tau_4=(1-6.2^{-k}+10.3^{-k}-5.4^{-k})/(1-2^{-k})$
Generalized logistic	$x=\xi+\alpha[1-\{(1-F)/F\}^k]/k$	$\lambda_1=\xi+\alpha\{1-\Gamma(1+k)\,\Gamma(1-k)\}/k,\ \lambda_2=\alpha\Gamma(1+k)\,\Gamma(1-k),$ $\tau_3=-k,\ \tau_4=(1+5k^2)/6$
Log-normal	$F=\Phi\left(\frac{\log(x-\xi)-\mu}{\sigma}\right)$	$\lambda_1=\xi+\exp(\mu+\sigma^2/2),\ \lambda_2=\exp(\mu+\sigma^2/2)\ \mathrm{erf}(\sigma/2),$ $\tau_3=6\pi^{-1/2}\int_0^{\sigma/2}\mathrm{erf}(x/\sqrt{3})\exp(-x^2)dx/\mathrm{erf}(\sigma/2)$
Gamma	$F=\beta^{-\alpha}\int_0^x t^{\alpha-1}\exp(-t/\beta)dt/\Gamma(\alpha)$	$\lambda_1=\alpha\beta,\ \lambda_2=\pi^{-1/2}\beta\Gamma(\alpha+\frac{1}{2})/\Gamma(\alpha),\ \tau_3=6I_{1/3}(\alpha,2\alpha)-3$

†γ is Euler's constant; Φ is the standard normal distribution function; $I_x(p,q)$ is the incomplete beta function. Expressions for τ_4 for the gamma and log-normal distributions are given by Hosking (1986).

Note: correction for τ4 (in Table 2) for Generalized Extreme Value distribution:

$$\tau_4=\frac{1-6\times 2^{-k}+10\times 3^{-k}-5\times 4^{-k}}{1-2^{-k}}$$



Sample L-moments for distribution selection

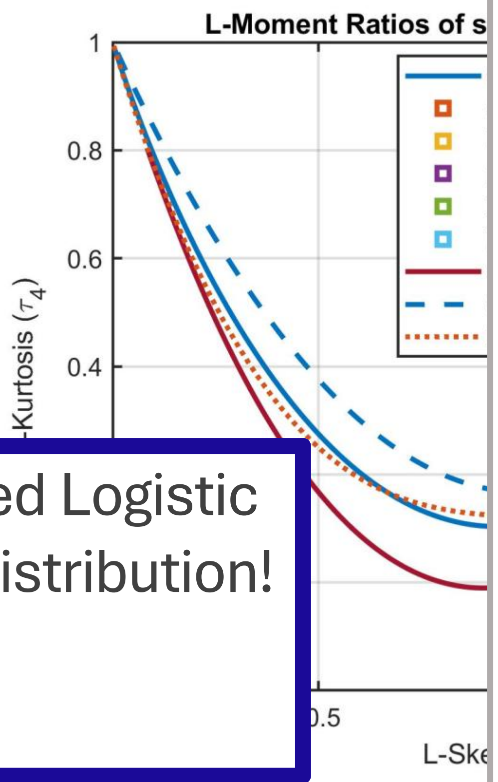
Example: Using the previously estimated sample L-moment ratios and the L-moment ratios diagram, select the best-fit distribution (use the data from *ENCI608_L12_Examples.xlsx*)

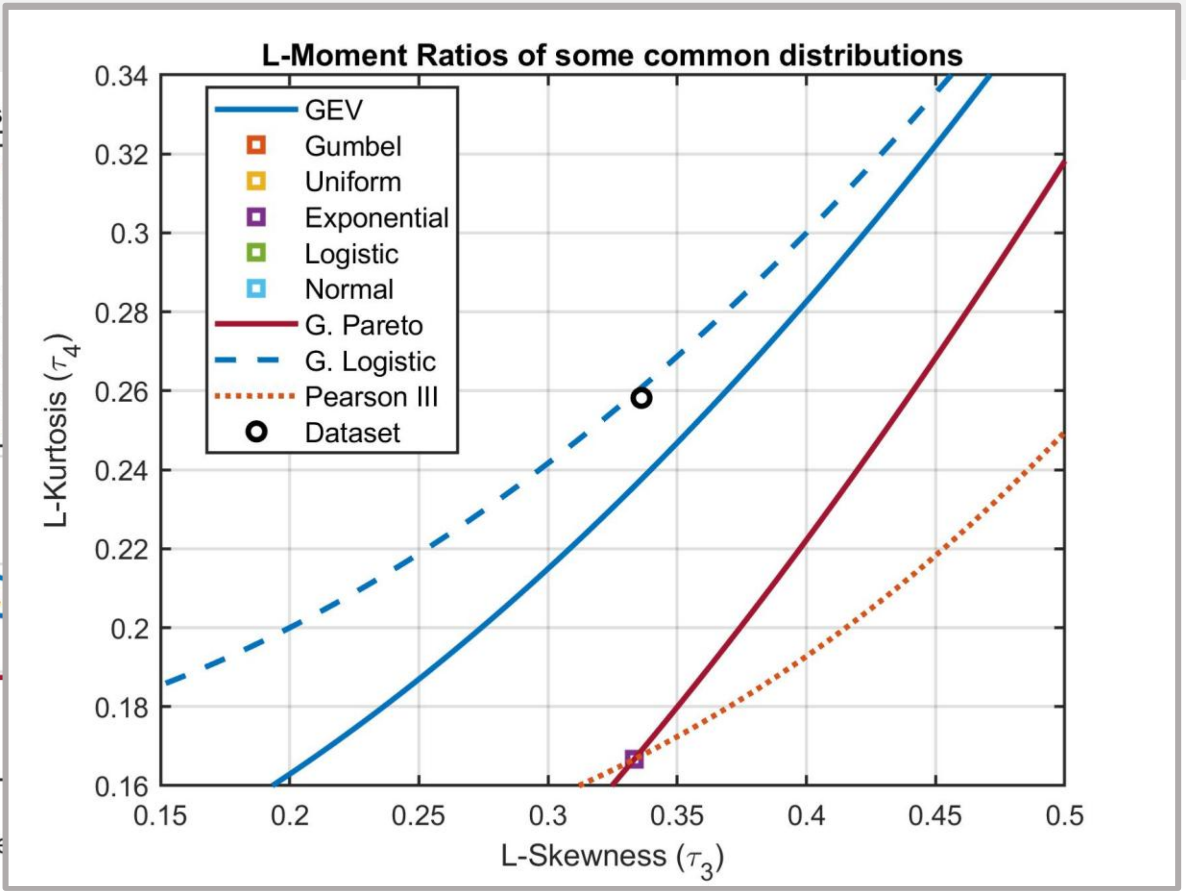
sample L-moments

l_1 =	103.8903	
l_2 =	14.9914	
l_3 =	5.0396	
l_4 =	3.8708	

L-ratios

t_3 =	0.3362	L-skewness
t_4 =	0.2582	L-kurtosis





The Generalized Logistic is the best-fit distribution!

What probability distribution [$F(x)$] can we use in FFA?

Annual maxima theoretically follow the **Generalized Extreme Value (GEV) distribution** as the number of events per year →∞.

Since convergence to GEV may not hold in hydrology, we also use several other popular empirical distributions:

- **Generalized Logistic (GLO)**
- **Generalized Normal (GNO/3LN)**
- **Log-Pearson type III (LP3)**
- **Pearson type III (PE3)**

These include simpler forms as special cases: GEV, LP3, and PE3 include **Gumbel**, **Log-normal**, and **Gamma**, respectively.

Methods for distribution selection:

- Arbitrary/subjective selection.
- Goodness-of-fit metrics (e.g., RMSE).
- Statistical tests (e.g., Kolmogorov-Smirnov and Anderson-Darling tests).
- Statistics of the sample and distributions.
  - The **L-moments** are widely used due to their robustness.

*"**L-moments** rely on expectations of linear combinations of order statistics."*

What probability distribution [$F(x)$] can we use in FFA?

Methods for parameter estimation:

- Maximum likelihood method.
- Method of moments.
- **L-moments**: less sensitive to sampling variability and measurement errors.

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- Arbitrary/subjective selection.
- Goodness-of-fit metrics (e.g., RMSE).
- Statistical tests (e.g., Kolmogorov-Smirnov and Anderson-Darling tests).
- Statistics of the sample and distributions.
  - The **L-moment ratio diagram** is widely used when the sample size < 100.

“L-moments rely on expectations of linear combinations of order statistics.”

The idea behind parameter estimation:

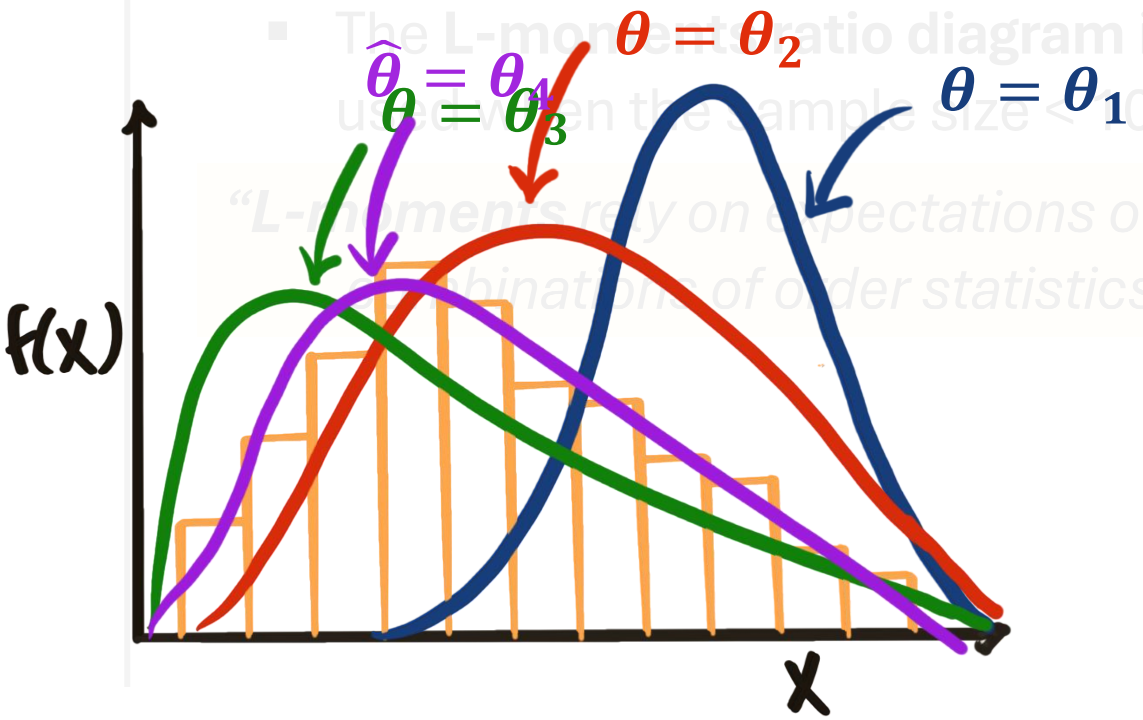
Is θ_1 a good estimate?

Is θ_2 a good estimate?

Is θ_3 a good estimate?

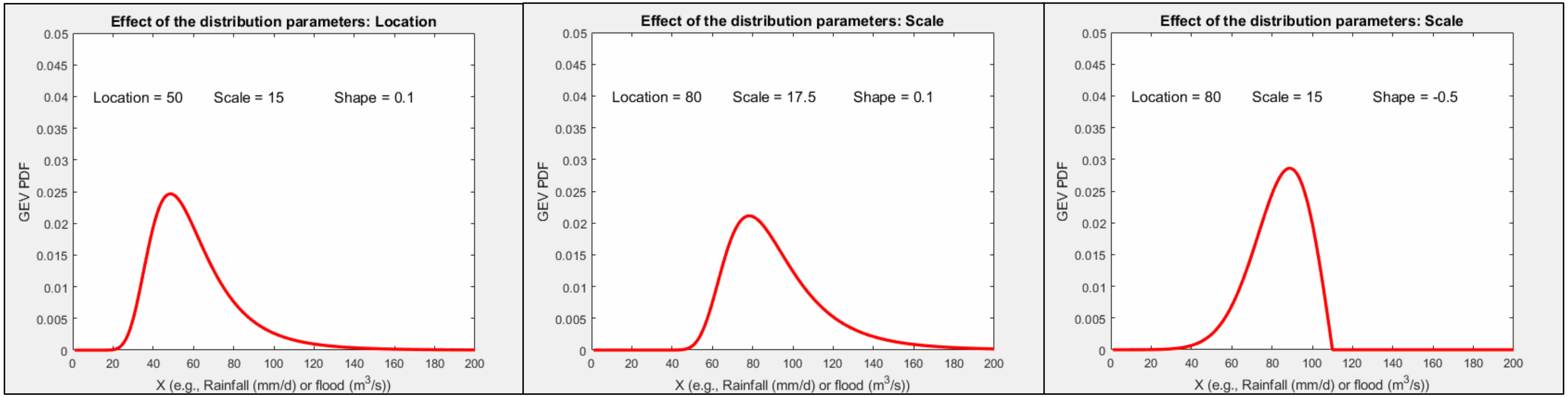
Is θ_4 a good estimate?

It is a good estimate!!! 🥳



CE 319.3 Hydrology

What is the role of the distribution parameters?



$GLO(x; \xi, \alpha, \kappa)$

ξ – location parameter
α – scale parameter
κ – shape parameter

$PE3(x; \mu, \sigma, \xi)$
$LPE3(\log x\,; \mu, \sigma, \xi)$

μ – location parameter
σ – scale parameter
ξ – shape parameter

$GEV(x; \xi, \alpha, \kappa)$

ξ – location parameter
α – scale parameter
κ – shape parameter

Parameter estimation using L-moments

We estimate parameters by equating the sample L-moments to their distribution counterparts ($\lambda_1, \lambda_2, .., \tau_3$, and τ_4).

Table 3. Parameter estimation via L-moments for some common distributions (from Hosking (1990); see also Appendix of Hosking and Wallis (1997)).

Parameter estimation via L-moments for some common distributions†

Distribution	*Estimators*
Exponential	(ξ known) $\hat{\alpha} = l_1$
Gumbel	$\hat{\alpha} = l_2/\log 2,\ \hat{\xi} = l_1 - \gamma\hat{\alpha}$
Logistic	$\hat{\alpha} = l_2,\ \hat{\xi} = l_1$
Normal	$\hat{\sigma} = \pi^{1/2} l_2,\ \hat{\mu} = l_1$
Generalized Pareto	(ξ known) $\hat{k} = l_1/l_2 - 2,\ \hat{\alpha} = (1+\hat{k}) l_1$
Generalized extreme value	$z = 2/(3+t_3) - \log 2/\log 3,\ \hat{k} \approx 7.8590z + 2.9554z^2,$ $\hat{\alpha} = l_2 \hat{k}/(1-2^{-\hat{k}})\Gamma(1+\hat{k}),\ \hat{\xi} = l_1 + \hat{\alpha}\{\Gamma(1+\hat{k}) - 1\}/\hat{k}$
Generalized logistic	$\hat{k} = -t_3,\ \hat{\alpha} = l_2/\Gamma(1+\hat{k})\Gamma(1-\hat{k}),\ \hat{\xi} = l_1 + (l_2 - \hat{\alpha})/\hat{k}$
Log-normal	$z = \sqrt{(8/3)}\Phi^{-1}\left(\frac{1+t_3}{2}\right),\ \hat{\sigma} \approx 0.999\,281z - 0.006\,118z^3 + 0.000\,127z^5,$ $\hat{\mu} = \log\{l_2/\mathrm{erf}(\sigma/2)\} - \hat{\sigma}^2/2,\ \hat{\xi} = l_1 - \exp(\hat{\mu} + \hat{\sigma}^2/2)$
Gamma	(ξ known) $t = l_2/l_1$; if $0 < t < \frac{1}{2}$ then $z = \pi t^2$ and $\hat{\alpha} \approx (1 - 0.3080z)/(z - 0.058\,12z^2 + 0.017\,65z^3)$; if $\frac{1}{2} \leqslant t < 1$ then $z = 1 - t$ and $\hat{\alpha} \approx (0.7213z - 0.5947z^2)/(1 - 2.1817z + 1.2113z^2)$; $\hat{\beta} = l_1/\hat{\alpha}$

†γ is Euler's constant; Φ^{-1} is the inverse standard normal distribution function.

Exercise: Estimate the distribution parameters for the previous sample and best-fit distribution (*ENCI608_L12_Examples.xlsx*)

Parameters of the Generalized Logistic (GLO) distribution:

$$\kappa = -\tau_3 \qquad \alpha = \frac{\lambda_2 \sin(\kappa\pi)}{\kappa\pi}$$

$$\xi = \lambda_1 - \alpha\left(\frac{1}{\kappa} - \frac{\pi}{\sin(\kappa\pi)}\right)$$

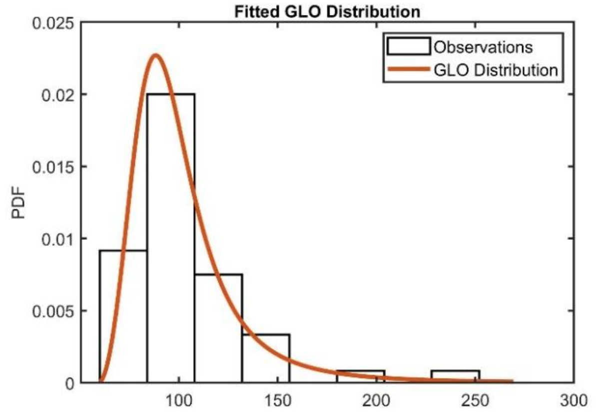
sample L-moments		
l_1 =	103.8903	
l_2 =	14.9914	
l_3 =	5.0396	
l_4 =	3.8708	
L-ratios		
t_3 =	0.3362	L-skewness
t_4 =	0.2582	L-kurtosis

Solution:

$$\boldsymbol{\kappa = -0.3362}$$

$$\boldsymbol{\alpha = 12.3560}$$

$$\boldsymbol{\xi = 96.0507}$$



Quantile estimation

The **quantile function (x_T)** yields the flood magnitude for a specific T, and is generally expressed as:

$$x_T = F^{-1}\left(1 - \frac{1}{T}\right)$$

For example, for a GEV distribution:

$$F_{GEV}(x) = \exp\left\{-\left[1 + \kappa\left(\frac{x-\xi}{\alpha}\right)\right]^{\frac{-1}{\kappa}}\right\}$$

$$\therefore \quad x_T = \xi - \frac{\alpha}{\kappa}\left\{1 - \left[-\ln\left(1 - \frac{1}{T}\right)\right]^{-\kappa}\right\}$$

Exercise: Now, calculate the design flood associated with a return period of T = 50 years (use *ENCI608_L12_Examples.xlsx*)

$$x_T = \xi + \frac{\alpha}{\kappa}\left\{1 - \left[\frac{1 - F(x)}{F(x)}\right]^{\kappa}\right\}$$

$$F(x) = 1 - \frac{1}{T}$$

$\kappa = -0.3362$

$\alpha = 12.3560$

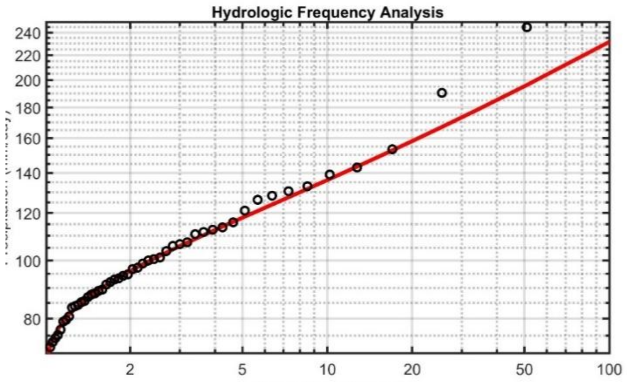
$\xi = 96.0507$

Solution:

$$F(x) = 1 - \frac{1}{50}$$

$$\therefore \ F(x) = 0.98$$

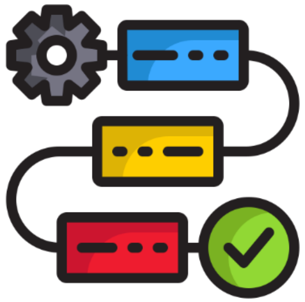
$$\boldsymbol{x_T = 195.29\ m^3/s}$$



Wrapping up activity

Final exercise – concept mapping:

Write keywords and drawings onto sticky notes and arrange them into a workflow depicting FA based on non-parametric (plotting positions) and parametric (L-moments-based) approaches.



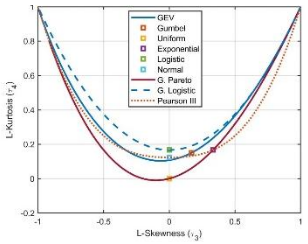
Wrapping up activity

Estimate sample L-moments and ratios

1. Estimate unbiased estimators of probability weighted moments (b_r)
2. Estimate the first four sample L-moments (l_r)
3. Estimate the sample L-moment ratios (t_r)

Select the distribution

4. Plot the sample L-skewness (t_3) and L-kurtosis (t_4) in the L-moment ratio diagram.
5. Identify the best-fit distribution



Parameterize the distribution

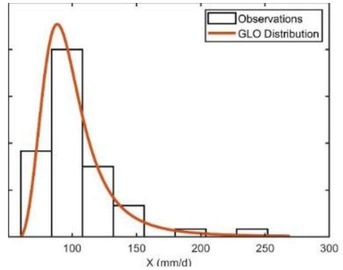
6. Estimate distribution parameters using the distribution-specific equations

$\boldsymbol{\theta} = \xi, \alpha, \kappa$

$\kappa = f(\tau_3)$

$\alpha = f(\lambda_2, \kappa)$

$\xi = f(\lambda_2, \kappa, \alpha)$

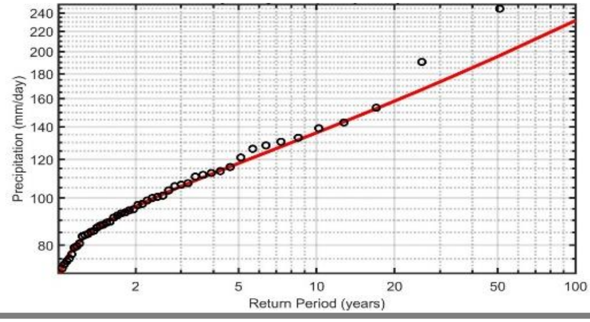


Estimate quantiles

7. Use the distribution-specific quantile function (or CDF function)

$F^{-1}(T; \boldsymbol{\theta})$ (or $F(x; \boldsymbol{\theta})$)

Communicate FA



Estimate the empirical exceedance probabilities

1(b). Sort the data in decreasing order

2(b). Rank the sorted data

3(b). Select a plotting position formula

4(b). Calculate the exceedance probabilities

Take-home messages

- We use FA to relate magnitudes of extreme water events to their return periods (annual exceedance probabilities).
  - We usually conduct FA using a parametric probability distribution.
  - We can also use non-parametric plotting position formulas for comparison.

- The **L-moments method** is advantageous for selecting the best-fit distribution and estimating its parameters in FA (***and you learned it today!***).

FA is fundamental in hydrology to design infrastructure and assess hazards and risks.

More references beyond the textbook

- Coles, S. (2001). An introduction to statistical modeling of extreme values. Springer-Verlag, London (2001).
- Hosking, J. R. (1990). L-moments: Analysis and estimation of distributions using linear combinations of order statistics. Journal of the Royal Statistical Society: Series B (Methodological), 52(1), 105-124.
- Hosking, J. R. M., and Wallis, J. R. (1997). Regional frequency analysis: an approach based on L-Moments. New York, USA.: Cambridge University Press.
- Kidson, R. and Richards, K.S. 2005. Flood frequency analysis: assumptions and alternatives. Progress in Physical Geography, 29, 392-410
- Stedinger, J. R. (1993). Frequency analysis of extreme events. in Handbook of Hydrology.
- Stedinger, J. R. (2017). Ch. 76—flood frequency analysis. Handbook of applied hydrology, 2nd edn. McGraw-Hill, New York.

That’s all for today’s lecture

If you have questions, please contact me:

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See you at the next class.

Take care!