1 Formal Logic

1. (a) *It will either be sunny and warm, or it will be cloudy. If it is not sunny then it will not be cloudy. Therefore it will be cloudy.*

   i. *Translate the argument into the formal language for propositional logic.*
   Note: Here we were unsure weather to represent *"it is warm"* as a seperate propositional variable from *"it is sunny"*, but we chose to err on the side of doing more work rather than less.

$$\begin{aligned} \text{Let } S &= \text{It will be sunny} \\ \text{Let } W &= \text{It will be warm} \\ \text{Let } C &= \text{It will be cloudy} \end{aligned}$$

$$(S \wedge W) \vee C, \neg C \to \neg S \vdash C \tag{1}$$

   ii. *The argument is propositionally valid; provide a natural deduction proof.*

$$\dfrac{\dfrac{\dfrac{(S \wedge W) \vee C \quad \dfrac{\dfrac{\dfrac{\overline{S \wedge W}^{\,1}}{S}\wedge E}{S \vee C}\vee I}{(S \wedge W) \to (S \vee C)}\to I,1 \quad \dfrac{\dfrac{\overline{C}^{\,2}}{S \vee C}\vee I}{C \to (S \vee C)}\to I,2}{S \vee C}\vee E \quad \dfrac{(C \to \bot) \to (S \to \bot) \quad \overline{C \to \bot}^{\,3}}{S \to \bot}\to E \quad \overline{C \to \bot}^{\,3}}{\bot}\vee E}{\mathrm{C}}\bot, 3$$

   iii. *Alter, add, or remove premises (or change the conclusion) in the argument to make it formally invalid, for the same narrative.*
   Change the conclusion to be $\neg C$.

   iv. *Give an explicit counterexample to your argument in (iii), and explain why it is a counterexample.*
   If we alter the conclusion to be $\neg C$, we have a counterexample as marked on the truth table below. This case is a counterexample because all of the premises are true, but the conclusion ($\neg C$) is not.

	S	W	C	$\neg C$	$\neg S$	$(S \wedge W) \vee C$	$\neg C \to \neg S$
	0	0	0	1	1	0	1
$\to$	0	0	1	0	1	1	1
	0	1	0	1	1	0	1
	0	1	1	0	1	1	1
	1	0	0	1	0	0	0
	1	0	1	0	0	1	1
	1	1	0	1	0	1	0
	1	1	1	0	0	1	1

   (b) *Consider the sequent $\neg\alpha \vee \beta \vdash \alpha \to \beta$.*

   i. *Provide a natural deduction proof.*

$$\dfrac{\dfrac{(\alpha \to \bot) \vee \beta \quad \dfrac{\dfrac{\dfrac{\overline{\alpha}^{\,1} \quad \overline{\alpha \to \bot}^{\,2}}{\bot}\to E}{\beta}\bot}{(\alpha \to \bot) \to \beta}\to I,2 \quad \dfrac{\overline{\beta}^{\,3}}{\beta \to \beta}\to I,3}{\beta}\vee E}{\alpha \to \beta}\to I,1$$

2. (a) *Explain in your own words what the Soundness and Completeness theorems for natural deduction mean.*
   Soundness: Soundness is the property of a logical system which means that anything provable with the system will be correct. That is to say, if we start with well formed formulae and follow the rules of that system (eg $\to I, \to E, \vee I, \vee E, \wedge I, \wedge E$ for minimal logic), any formula we derive will hold true in any case where all the premises hold.

   Completeness: Completeness is the property of a logical system which means that any statement which is true given a set of premises can be proven using the system. In other words, if $\Gamma \vdash \beta$ there is a deduction $\overset{\alpha}{\underset{\Gamma}{}}\mathcal{D}$ possible in the system.

   (b) *Is, in your opinion, Soundess or Completeness more important?*
   In our opinion soundness is more useful than completeness. A system which is complete but not sound could include

the rule: $\forall \alpha, \beta : \alpha \leftrightarrow \beta$. A proof given in this system has no value as it doesn't guarantee correctness. A proof in this system could be:

$$\alpha \vdash \neg\alpha$$

$$\frac{\alpha}{\neg\alpha}$$

On the other hand a system which is sound but not complete is not very useful for finding proofs, as for any particular truth one might not be possible, but any provided in such a system can at least be relied on to be correct.

(c) *Is it possible to make an invalid argument valid by adding extra premises? How about by removing premises?*
Adding premises: Take the example invalid argument $\alpha \vdash \beta$, then add the premise $\alpha \to \beta$, giving the valid argument $\alpha, \alpha \to \beta \vdash \beta$. Proof:

$$\begin{prooftree}
\AxiomC{$\alpha \to \beta$}
\AxiomC{α}
\RightLabel{$\to E$}
\BinaryInfC{β}
\end{prooftree}$$

Removing premises: Any invalid argument has a counterexample C by definition. This is a set of truth values for all the logical variables such that all the premises are true, but the conclusion is false. Removing a premise will still leave the counterexample: it will still be true for all the remaining premises, therefore the argument will remain invalid.

3. (a) *Use first order natural deduction to prove:*

i. $\forall x(Fx \lor Gx) \vdash \forall xFx \lor \exists xGx$

$$\begin{prooftree}
\AxiomC{$\forall xFx \lor \neg\forall xFx$}
\AxiomC{$\forall x(Fx \lor Gx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa \lor Ga$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\neg\forall xFx$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\neg\exists x\neg Fx$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg Fa$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x\neg Fx$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot, 1$}
\UnaryInfC{Fa}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall xFx$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot, 2$}
\UnaryInfC{$\exists x(Fx \to \bot)$}
\RightLabel{$\exists E$}
\UnaryInfC{$Fa \to \bot$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{$Ga \to \bot$}
\RightLabel{$\lor E$}
\TrinaryInfC{$\bot$}
\RightLabel{$\bot, 4$}
\UnaryInfC{Ga}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists xGx$}
\RightLabel{$\lor I$}
\UnaryInfC{$\forall xFx \lor \exists xGx$}
\RightLabel{$\to I, 3$}
\UnaryInfC{$\neg\forall xFx \to (\forall xFx \lor \exists xGx)$}
\AxiomC{}
\RightLabel{5}
\UnaryInfC{$\forall xFx$}
\RightLabel{$\lor I$}
\UnaryInfC{$\forall xFx \lor \exists xGx$}
\RightLabel{$\to I, 5$}
\UnaryInfC{$\forall xFx \to (\forall xFx \lor \exists xGx)$}
\RightLabel{$\lor E$}
\TrinaryInfC{$\forall xFx \lor \exists xGx$}
\end{prooftree}$$

ii. $\vdash \forall xQx \lor \exists y\neg Qy$

$$\begin{prooftree}
\AxiomC{$\forall xQx \lor \neg\forall xQx$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\forall xQx$}
\RightLabel{$\lor I$}
\UnaryInfC{$\forall xQx \lor \exists y\neg Qy$}
\RightLabel{$\to I, 1$}
\UnaryInfC{$\forall xQx \to \forall xQx \lor \exists y\neg Qy$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\neg\forall xQx$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\neg\exists x\neg Qx$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{$\neg Qa$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x\neg Qx$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot, 4$}
\UnaryInfC{Qa}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall xQx$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot, 3$}
\UnaryInfC{$\exists y\neg Qy$}
\RightLabel{$\lor I$}
\UnaryInfC{$\forall xQx \lor \exists y\neg Qy$}
\RightLabel{$\to I, 2$}
\UnaryInfC{$\neg\forall xQx \to \forall xQx \lor \exists y\neg Qy$}
\RightLabel{$\lor E$}
\TrinaryInfC{$\forall xQx \lor \exists y\neg Qy$}
\end{prooftree}$$

(b) *Translate the following argument into the formal language for first-order logic.*

Some things are neither fluffy nor cute, but anything that is fluffy is cute. So my pet cat is both fluffy and cute, because anything that isn't fluffy isn't my pet cat.

Let $Fx = x$ is fluffy, $Cx = x$ is cute, $p =$ my pet cat

$$\exists x(\neg Fx \land \neg Cx), \forall x(Fx \to Cx), \forall x(\neg Fx \to \neg(x = p)) \vdash Fp \land Cp$$

(c) *Give an explicit counterexample which shows the following:*

$\exists xPx \land \exists yQy \quad \nvdash \quad \exists x(Px \land Qx)$

$$U_{\mathcal{A}} = \{0, 1\}$$
$$I_{\mathcal{A}}(P) = \{0\}$$
$$I_{\mathcal{A}}(Q) = \{1\}$$

In this counterexample, the premise $\exists xPx \land \exists yQy$ holds when $x = 0$ and $y = 1$. The conclusion $\exists x(Px \land Qx)$ does not hold because $I_{\mathcal{A}}(P) \cup I_{\mathcal{A}}(Q) = \emptyset$.

2 Docendo discimus

Design a question suitable for a final exam on formal logic.

1. Explain in your own words the difference between classical and intuitionistic logic.
   In classical logic we can use the classical absurdity rule, meaning we can make an assumption, come to an absurdity, and therefore conclude that the original assumption was false. In intuitionistic logic, we cannot assume that every variable is either true or false (that is to say the law of excluded middle does not hold), which also means we cannot derive a number of common classical rules such as double negation elimenation. The relation between classical and intuitionistic logic is that intuitionistic is a subset of classical logic, meaning any derivation in intuitionistic logic is also valid in classical logic.

2. Provide a deduction in classical logic which cannot be made in intuitionistic Logic.
   $\neg\neg\alpha \vdash \alpha$

$$\dfrac{\dfrac{\dfrac{}{\neg\alpha}\,1 \qquad \neg\neg\alpha}{\bot}\rightarrow E}{\alpha}\,\bot, 1$$

1. *Explain carefully what part of the course your question examines, and state why that topic is important.*
   This question is examining the difference between intuitionistic and classical logic. This is important because it helps to demonstrate a fundamental understanding of the differences between the fields of formal logic. It also tests understanding of both the intuitionistic and classical absurdity rules.

2. *Write a model solution for your question.*
   This is included with the question. Part 2 could be answered with any valid classical deduction which is invalid in intuitionistic logic.

3. *In light of your answers above, comment on the suitability of your question for a time limited final exam.*
   We believe this question is very suitable for an early question in a time limited exam, as it is made up of two small parts which can easily be answered in a short time, but it tests deeper understanding of the underlying principals if the student chooses to provide more detailed answers.