Particle in the tunnel: analytical results

A.Werpachowska

September 17, 2013

Task parameters:

- tunnel length: $L = 100$ m
- tunnel height: $h = 5$ m
- particle velocity: $|\vec{v}| = 1$ m/s
- shooting angle: $\alpha \in U[0, \pi/4]$
- velocity reduction upon reflection: $v_{\mathrm{tr}} = v \sin\alpha \to \left(\frac{1}{1+\eta}\right) v_{\mathrm{tr}}$, where $\eta \in S(\lambda)$ (exponential distribution with rate $\lambda = 1/0.15$) and $v_{\mathrm{lng}} = v\cos\alpha = \mathrm{const}$

1 Java program simulation results

Averaging over 100 millions Monte-Carlo paths I obtained:

- average time of flight and the Monte-Carlo estimation error: 112.22131 ± 0.00118 s
- average number of reflections and the Monte-Carlo estimation error: 5.18047 ± 0.00029

2 Time of flight

The time of flight depends strictly on the variable $\alpha \in U[0, \pi/4]$ and is independent of the η parameter according to the problem description. Therefore, its average is given by

$$\frac{\int_0^{\pi/4} \frac{L}{v\cos\alpha'} \mathrm{d}\alpha}{\int_0^{\pi/4} \mathrm{d}\alpha} = \frac{L}{v\pi}(-\log(\cos(\alpha/2) - \sin(\alpha/2)) + \log(\cos(\alpha/2) + \sin(\alpha/2)))|_0^{\pi/4} \approx 112.21997 \text{ s}$$

The numerical simulation result agrees with the above exact result within 1.14 standard deviations.

3 Number of reflections

The number of particle reflections on the walls of the tunnel during the flight depends on both α and η, and it is determined by changes of the transversal velocity $v_{\mathrm{tr}}(\alpha, \eta)$. A particle shot at an angle α_n travels in the transversal direction with velocity $v_{\mathrm{tr}}(\alpha_n)$. If it experiences a reflection on the wall, it slows down in transversal direction to $v_{\mathrm{tr},ni} = v \sin\alpha_n \frac{1}{1+\eta_i}$. The velocities $v_{\mathrm{tr},ni}$ represent microstates of the particle in the phase space of all possible values of v_{tr}. In thermodynamics, the probability of finding the particle in each microstate is given by the partition function, which allows to perform a weighted averaging over all microstates in order to calculate their macroscopic, i.e. observable, “outcome”. However, we know from the cosmic background radiation measurements that the temperature of the world without gravity would be around 2.7 K and hence the partition function can be replaced with the probability density function, while averaging over microstates accounts for calculating expectation values :)

Since variables α and η are independent, it is easy and convenient to perform averaging over them separately. First I calculate the expectation value of $v_{\rm tr}$ for a given α after the first reflection:

$$\langle v_{\rm tr}(\alpha)\rangle = \int_0^\infty \frac{v\sin\alpha}{1+\eta}s(\lambda;\eta)\mathrm{d}\eta = v\sin\alpha\int_0^\infty \frac{\lambda e^{-\lambda\eta}}{1+\eta}\mathrm{d}\eta = \xi v\sin\alpha\ , \tag{1}$$

where $\xi = 0.881933$, $s(\lambda;\eta)$ is the exponential probability density function and η in the integral limits varies from 0 (associated with no change in $v_{\rm tr}$ upon the reflection) to ∞ (reduction of $v_{\rm tr}$ to zero). I estimate that every consecutive reflection on the wall will cause the same average change to the transversal velocity:

$$\langle v_{\rm tr}(\alpha)\rangle_k = \xi^k v\sin\alpha\ .$$

I can use the above results to estimate the expected number of reflections. The expected time the particle spends in the tunnel can be calculated in two ways: from longitudinal and transversal velocity, which leads to the following equality:

$$\frac{L}{v_{\rm lng}(\alpha)} = \sum_{k=0}^{N-1}\frac{h}{\langle v_{\rm tr}(\alpha)\rangle_k} = \frac{h}{\langle v_{\rm tr}(\alpha)\rangle_0}\sum_{k=0}^{N-1}\frac{1}{\xi^k} = \frac{h}{\langle v_{\rm tr}(\alpha)\rangle_0}\frac{1-(1/\xi)^N}{1-1/\xi}\ ,$$

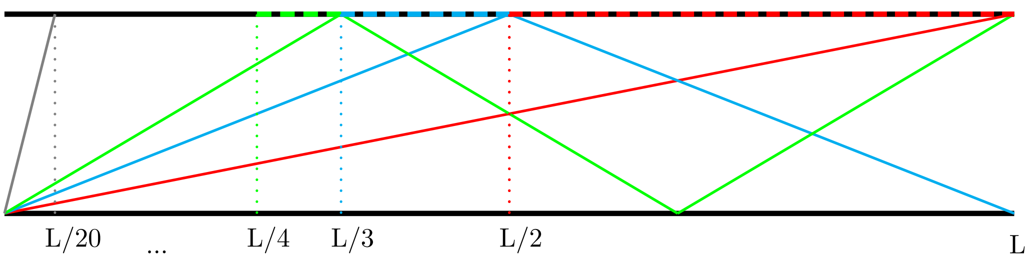
where $N-1$ is the expected number of reflections before exiting the tunnel and the expression for the sum of geometric sequence was used in the last step. I gather all α-dependent properties on one side and calculate their expectation value with respect to α:

$$\frac{L}{v_{\rm lng}(\alpha)}\frac{\langle v_{\rm tr}(\alpha)\rangle_0}{h} = \frac{L}{h}\frac{v\sin\alpha}{v\cos\alpha} = \frac{1-(1/\xi)^N}{1-1/\xi}$$
$$\frac{L}{h}\langle\tan\alpha\rangle = \frac{L}{h}\int_0^{\pi/4}\frac{\tan\alpha'}{\pi/4}\mathrm{d}\alpha = 8.82542 = \frac{1-(1/\xi)^N}{1-1/\xi}$$

Solving the above equation I estimate the expected number of reflections as equal to $N-1 = 5.2083$. This result remains "in agreement" with the numerical simulation outcome within 96 standard deviations (5.18047 for 100 million Monte-Carlo paths), however the relative error is just 0.5%.

4 Solutions for $\eta = 0$

- Time of flight does not depend on η so the derivation and results in Sec.2 remain valid.
- In this case the law of reflection (the angle of incidence equals the angle of reflection) applies. Consequently, the number of reflections can be calculated as $n = \left\lfloor \frac{L}{h/\tan\alpha}\right\rfloor$. Its maximal value is $n_{20} = \frac{L}{h\tan(\pi/4)} = 20$ and the minimal value is $n_0 = 0$ for $\alpha \le \arctan(h/L)$. The following graph demonstrates how to calculate the expected number of reflections:



The red, green and cyan lines are paths of the particle in the tunnel experiencing 1, 2 and 3 reflections, respectively. The red dashed segment $[L/2, L]$ along the tunnel wall indicates the range in which the first reflection has to take place for the particle to experience only

1 reflection during the whole flight. Similarly, if the particle hits the wall for the first time in the cyan segment $[L/3, L/2]$, it experiences 2 reflections during the flight and if it hits the green segment $[L/4, L/3]$, it experiences 3 reflections. We can continue selecting the segments on the wall until the maximum number of 20 reflections (grey lines). Each coloured segment is associated with a range of α angles, which can be calculated as $\arctan(\frac{h}{L/(k+1)}) - \arctan(\frac{h}{L/(k)})$ for $k+1^{\text{th}}$ reflection during the flight. Thus, the expected number of reflections can be obtained as

$$N = \sum_{k=1}^{19} \frac{\arctan(\frac{h}{L/(k+1)}) - \arctan(\frac{h}{L/k})}{\pi/4} k$$

($k = 20$ reflections is a limiting case for a single value of $\alpha = \pi/4$). I obtain $N = 8.3281$, which remains in agreement with the Monte-Carlo simulation result of 8.32879 ± 0.00056 reflections for 100 million paths within 1.2321 standard deviations.