Numeric certified algorithm for the topology of resultant and discriminant curves

Rémi Imbach, Guillaume Moroz and Marc Pouget

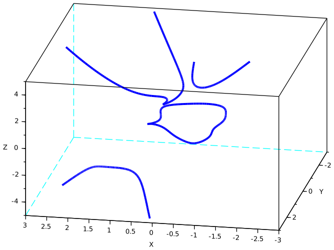
Vegas

Visibility Surfaces

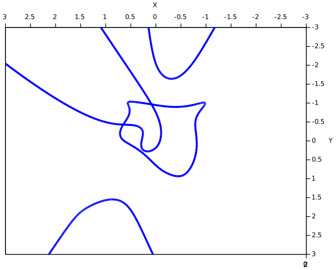
Resultant curves

$$\begin{cases} p(x,y,z) = 0 \\ q(x,y,z) = 0 \end{cases}$$

$$Res_z(p,q) = 0$$



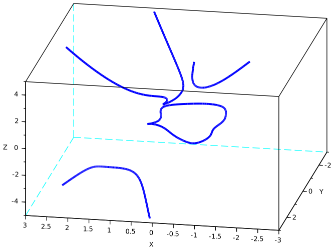
Smooth curves of $\mathbb{R}^3$



Singular curves of $\mathbb{R}^2$

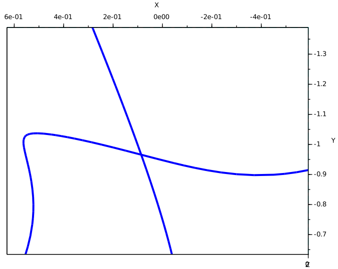
Resultant curves

$$\begin{cases} p(x,y,z) = 0 \\ q(x,y,z) = 0 \end{cases}$$



Smooth curves of $\mathbb{R}^3$

$$Res_z(p,q) = 0$$



Singular curves of $\mathbb{R}^2$

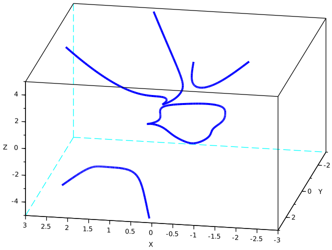
Singularities are:

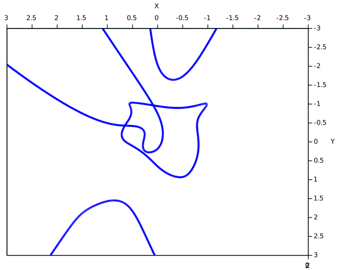
- nodes,

Resultant curves

$$\begin{cases} p(x,y,z) = 0 \\ q(x,y,z) = 0 \end{cases}$$

$$Res_z(p,q) = 0$$





Smooth curves of $\mathbb{R}^3$

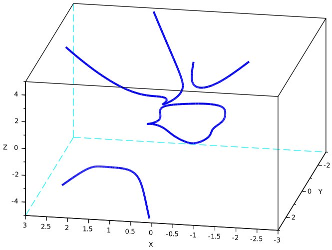
Singular curves of $\mathbb{R}^2$
Singularities are:

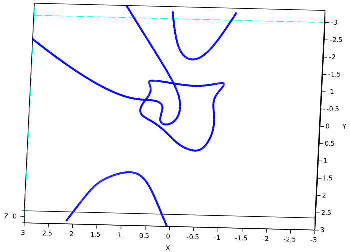
- nodes,
- stable.

Resultant curves

$$\begin{cases} p(x,y,z) = 0 \\ q(x,y,z) = 0 \end{cases}$$

$$Res_z(p,q) = 0$$





Smooth curves of $\mathbb{R}^3$

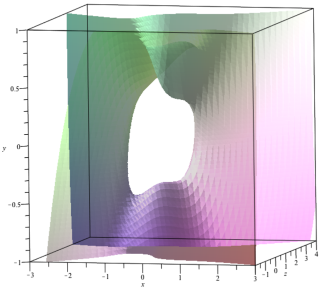
Singular curves of $\mathbb{R}^2$
Singularities are:

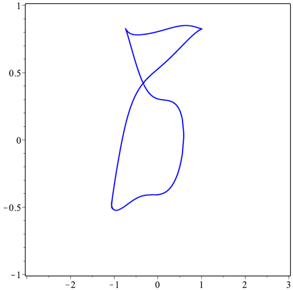
- nodes,
- stable.

Discriminant curves

$$\left\{ \; p(x,y,z) = 0 \right.$$

$$Res_z(p, \tfrac{\partial p}{\partial z}) = 0$$





Smooth surfaces of $\mathbb{R}^3$

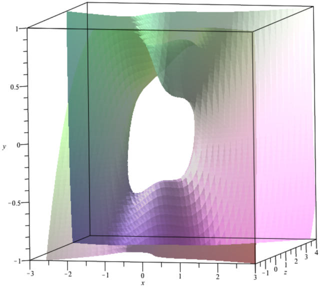
Singular curves of $\mathbb{R}^2$
Singularities are:

- nodes, cusps,
- stable.

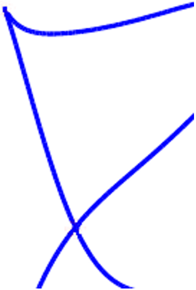
Discriminant curves

$$\{ \; p(x, y, z) = 0$$

$$Res_z(p, \frac{\partial p}{\partial z}) = 0$$



Smooth surfaces of $\mathbb{R}^3$

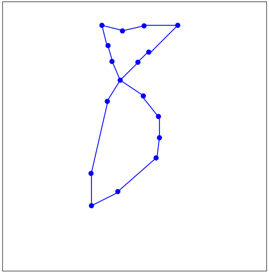


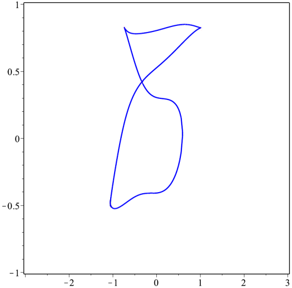
Singular curves of $\mathbb{R}^2$
Singularities are:

- nodes, cusps,
- stable.

Computing topology of $\mathcal{C}$

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$$

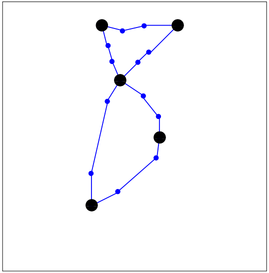




- Purely numerical methods fails near singularities
- Purely symbolic methods
  - Cylindrical Algebraic Decomposition requires : computing with algebraic numbers

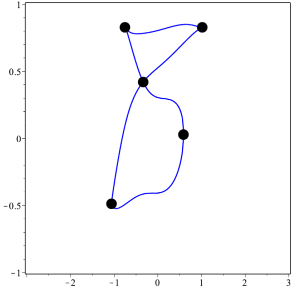
Computing topology of $\mathcal{C}$

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$$



$$f(x, y) = 0$$

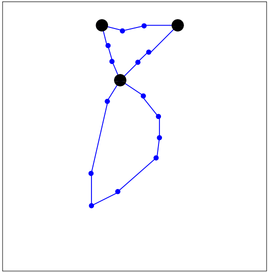
$$\frac{\partial f}{\partial y}(x, y) = 0$$



- Purely numerical methods fails near singularities
- Purely symbolic methods
  - Cylindrical Algebraic Decomposition requires : computing with algebraic numbers

Computing topology of $\mathcal{C}$

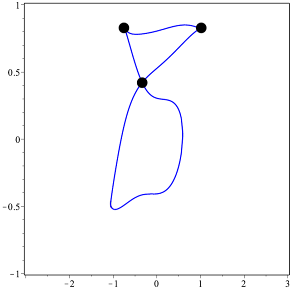
$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$$



$$f(x, y) = 0$$

$$\frac{\partial f}{\partial y}(x, y) = 0$$

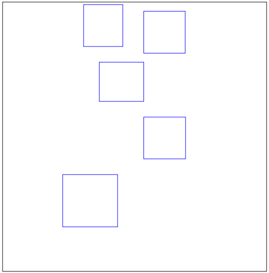
$$\frac{\partial f}{\partial x}(x, y) = 0$$

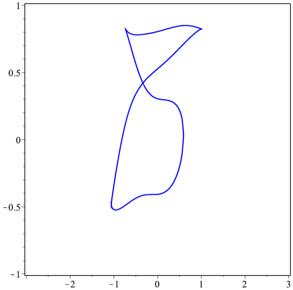


- Purely numerical methods fails near singularities
- Purely symbolic methods
  - Cylindrical Algebraic Decomposition requires : computing with algebraic numbers

A general framework

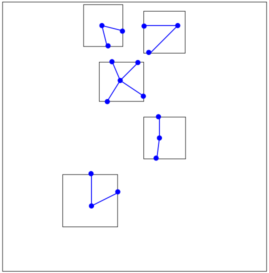
1. Isolating singularities and critical points in boxes
   $\rightarrow$ x-critical points system: $f = \frac{\partial f}{\partial y} = 0$
   $\rightarrow$ Singularities are x-critical points s.t $\frac{\partial f}{\partial x} = 0$

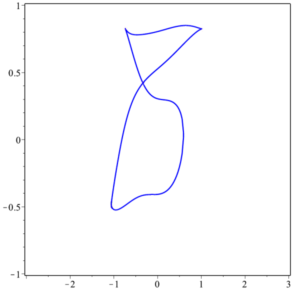




A general framework

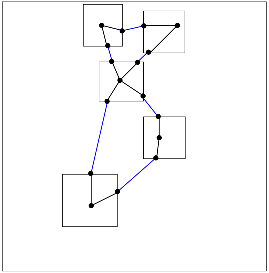
1. Isolating singularities and critical points in boxes
2. Computing topology around singularities
   → number of branches

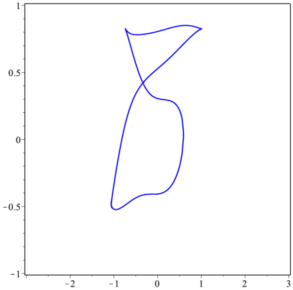




A general framework

1. Isolating singularities and critical points in boxes
2. Computing topology around singularities
3. Connecting the boxes by segments





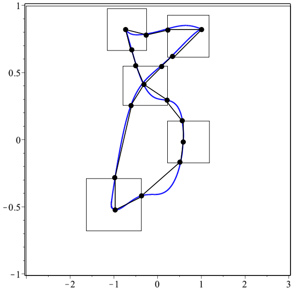
A general framework

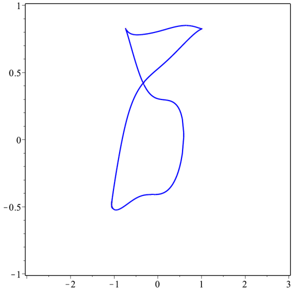
ISOTOP

1. Isolating singularities and critical points in boxes
2. Computing topology around singularities
3. Connecting the boxes by segments

} Gröbner basis and RUR

Sweeping algorithm





When $\mathcal{C}$ is a resultant or a discriminant curve

1. Isolating singularities and critical points in boxes
2. Computing topology around singularities
3. Connecting the boxes by segments

When $\mathcal{C}$ is a resultant or a discriminant curve

1. Isolating singularities and critical points in boxes
   $\rightarrow$squared deflation system
2. Computing topology around singularities
3. Connecting the boxes by segments

When $\mathcal{C}$ is a resultant or a discriminant curve

1. Isolating singularities and critical points in boxes
   $\rightarrow$squared deflation system
2. Computing topology around singularities
   $\rightarrow$interval based numerical criterion
3. Connecting the boxes by segments

Approach implemented with a certified numerical algorithm
(branch and bound algorithm)

Interval arithmetic

	floating point arithmetic	interval arithmetic
$f(x) = x^2$	$f(2) = 4$	$\mathcal{I} = [1.9, 2.1]$ $\Box f(\mathcal{I}) = [3.61, 4.41]$

- interval extension of usual operators $+, -, *, /, pow, sqrt, \ldots$
- interval evaluation $\Box f$ of f: $\forall \mathcal{I}, \forall x \in \mathcal{I}, f(x) \in \Box f(\mathcal{I})$

Interval arithmetic

	floating point arithmetic	interval arithmetic
$f(x) = x^2$	$f(2) = 4$	$\mathcal{I} = [1.9, 2.1]$ $\Box f(\mathcal{I}) = [3.60, 4.42]$

- interval extension of usual operators $+, -, *, /, pow, sqrt, \ldots$
- interval evaluation $\Box f$ of f: $\forall \mathcal{I}, \forall x \in \mathcal{I}, f(x) \in \Box f(\mathcal{I})$

Interval arithmetic

	floating point arithmetic	interval arithmetic
$f(x) = x^2$	$f(2) = 4$	$\mathcal{I} = [1.9, 2.1]$ $\square f(\mathcal{I}) = [3.60, 4.42]$
$f(x) = (x - 2^{24}) - 1/2$	$f(2^{24} + 1/2) = -0.5$	$\mathcal{I} = [2^{24}, 2^{24} + 2]$ $\square f(\mathcal{I}) = [-0.5, 3.5]$

- interval extension of usual operators $+, -, *, /, pow, sqrt, \ldots$
- interval evaluation $\square f$ of f: $\forall \mathcal{I}, \forall x \in \mathcal{I}, f(x) \in \square f(\mathcal{I})$
- certification of non-vanishing:
  - $0 \notin \square f(\mathcal{I}) \Rightarrow \forall x \in \mathcal{I}, f(x) \neq 0$

Interval arithmetic

	floating point arithmetic	interval arithmetic
$f(x) = x^2$	$f(2) = 4$	$\mathcal{I} = [1.9, 2.1]$ $\Box f(\mathcal{I}) = [3.60, 4.42]$
$f(x) = (x - 2^{24}) - 1/2$	$f(2^{24} + 1/2) = -0.5$	$\mathcal{I} = [2^{24}, 2^{24} + 2]$ $\Box f(\mathcal{I}) = [-0.5, 3.5]$

- interval extension of usual operators $+, -, *, /, pow, sqrt, \ldots$
- interval evaluation $\Box f$ of f: $\forall \mathcal{I}, \forall x \in \mathcal{I}, f(x) \in \Box f(\mathcal{I})$
- certification of non-vanishing:
  - $0 \notin \Box f(\mathcal{I}) \Rightarrow \forall x \in \mathcal{I}, f(x) \neq 0$
- existence of a unique zero in $\mathcal{I}$:
  - newton operator N
  - $N(\mathcal{I}) \subset \mathcal{I} \Rightarrow \mathcal{I}$ contains a unique zero of f

Interval arithmetic: bivariate extension

- box extension of usual operators $+, -, ., ^{-1}, \ldots$
- interval evaluation $\Box f$ of f: $\forall \mathcal{I}, \forall x \in \mathcal{I}, f(x) \in \Box f(\mathcal{I})$
- certification of non-vanishing:
  - $0 \notin \Box f(\mathcal{I}) \Rightarrow \forall x \in \mathcal{I}, f(x) \neq 0$
- existence of a unique zero in $\mathcal{I}$:
  - newton operator N
  - $N(\mathcal{I}) \subset \mathcal{I} \Rightarrow \mathcal{I}$ contains a unique zero of f

Interval arithmetic: bivariate extension

Krawczik operator:

$$N(\mathcal{I}) = y - (Yf(y) + (Id - Y\square f(\mathcal{I}))(\mathcal{I} - y)$$
$$\text{where } y = center(\mathcal{I})$$
$$\text{and } Y \simeq (J_f(y))^{-1}$$

- box extension of usual operators $+, -, ., ^{-1}, \ldots$
- interval evaluation $\square f$ of f: $\forall \mathcal{I}, \forall x \in \mathcal{I}, f(x) \in \square f(\mathcal{I})$
- certification of non-vanishing:
  - $0 \notin \square f(\mathcal{I}) \Rightarrow \forall x \in \mathcal{I}, f(x) \neq 0$
- existence of a unique zero in $\mathcal{I}$:
  - newton operator N
  - $N(\mathcal{I}) \subset \mathcal{I} \Rightarrow \mathcal{I}$ contains a unique zero of f
  - requires the inversion of the interval jacobian matrix
    $\rightarrow$ classical operators as Krawczik, Hansen-Sengputa, ...

Interval arithmetic: bivariate extension

Krawczik operator: $f : \mathbb{R}^2 \to \mathbb{R}^2$

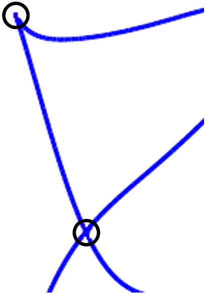
$$N(\mathcal{I}) = y - (Yf(y) + (Id - Y\Box f(\mathcal{I}))(\mathcal{I} - y)$$
where $y = center(\mathcal{I})$
and $Y \simeq (J_f(y))^{-1}$

- box extension of usual operators $+, -, ., ^{-1}, \ldots$
- interval evaluation $\Box f$ of f: $\forall \mathcal{I}, \forall x \in \mathcal{I}, f(x) \in \Box f(\mathcal{I})$
- certification of non-vanishing:
  - $0 \notin \Box f(\mathcal{I}) \Rightarrow \forall x \in \mathcal{I}, f(x) \neq 0$
- existence of a unique zero of multiplicity 1 in $\mathcal{I}$:
  - newton operator N
  - $N(\mathcal{I}) \subset \mathcal{I} \Rightarrow \mathcal{I}$ contains a unique zero of f
  - requires the inversion of the interval jacobian matrix
    $\rightarrow$ classical operators as Krawczik, Hansen-Sengputa, ...

Isolating singularities

$f \in \mathbb{Q}[x, y]$

Singularities of f are the solutions of $\begin{cases} f(x,y) &= 0 \\ f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{cases}$



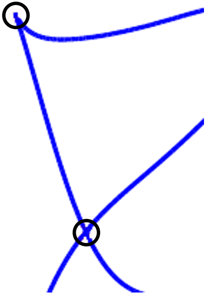
Isolating singularities

$f \in \mathbb{Q}[x, y]$

Singularities of f are the solutions of $\left\{ \begin{array}{ll} f(x,y) & = 0 \\ f_x(x,y) & = 0 \\ f_y(x,y) & = 0 \end{array} \right.$

...that is over-determined.

Singularities of f are the solutions of $\left\{ \begin{array}{ll} f_x(x,y) & = 0 \\ f_y(x,y) & = 0 \end{array} \right.$



Isolating singularities

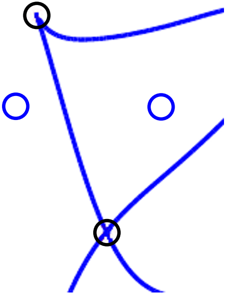
$f \in \mathbb{Q}[x, y]$

Singularities of f are the solutions of $\left\{ \begin{array}{ll} f(x,y) & = 0 \\ f_x(x,y) & = 0 \\ f_y(x,y) & = 0 \end{array} \right.$

…that is over-determined.

Singularities of f are the solutions of $\left\{ \begin{array}{ll} f_x(x,y) & = 0 \\ f_y(x,y) & = 0 \end{array} \right.$ s.t $f(x,y) = 0$

…that has spurious solutions,



Isolating singularities

$f \in \mathbb{Q}[x, y]$

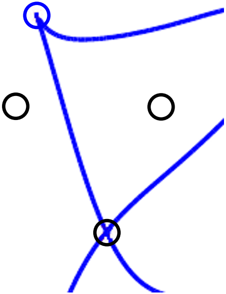
Singularities of f are the solutions of $\begin{cases} f(x,y) &= 0 \\ f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{cases}$

...that is over-determined.

Singularities of f are the solutions of $\begin{cases} f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{cases}$ s.t $f(x,y) = 0$

...that has spurious solutions,

...that has solutions of multiplicity 2.



Deflation system using sub-resultant chain

$p, q \in \mathbb{Q}[x, y, z]$

Sub-resultant chain of p, q:

$$\begin{array}{rcccccc}
S_z^0 = & & & & & & (Res_z(p,q))(x,y) \\
S_z^1 = & & & s_{11}(x,y)z & + & & s_{10}(x,y) \\
S_z^2 = & s_{22}(x,y)z^2 & + & s_{21}(x,y)z & + & & s_{20}(x,y) \\
\ldots = & & & \ldots & & &
\end{array}$$

S_z^i are

- reminders of pseudo-divisions of p, q

Deflation system using sub-resultant chain

$p, q \in \mathbb{Q}[x, y, z]$

Sub-resultant chain of p, q:

$$
\begin{array}{rcccccc}
S_z^0 = & & & & & & (Res_z(p,q))(x,y) \\
S_z^1 = & & & s_{11}(x,y)z & + & & s_{10}(x,y) \\
S_z^2 = & s_{22}(x,y)z^2 & + & s_{21}(x,y)z & + & & s_{20}(x,y) \\
\ldots = & & & \ldots & & &
\end{array}
$$

S_z^i are

- reminders of pseudo-divisions of p, q
- strongly related to $gcd(p, q)$, in particular:

Proposition: p and q have a gcd of degree 2 iff $s_{22} \neq 0$ and $s_{11} = s_{10} = 0$.

Deflation system using sub-resultant chain

$p, q \in \mathbb{Q}[x, y, z]$

Sub-resultant chain of p, q:

$$
\begin{array}{lccccc}
S_z^0 = & & & & & (Res_z(p,q))(x,y) \\
S_z^1 = & & & s_{11}(x,y)z & + & s_{10}(x,y) \\
S_z^2 = & s_{22}(x,y)z^2 & + & s_{21}(x,y)z & + & s_{20}(x,y) \\
\dots = & & & \dots & &
\end{array}
$$

Theorem: Assuming regularity conditions on p, q, the singularities of $Res_z(p, q) = 0$ are exactly the solutions (x, y) of

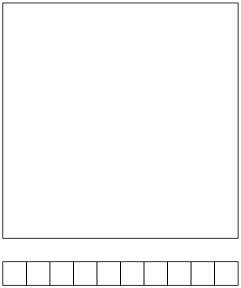
$$
(S) \left\{ \begin{array}{l} s_{10}(x,y) = 0 \\ s_{11}(x,y) = 0 \\ s_{22}(x,y) \neq 0 \end{array} \right.
$$

Moreover, solutions of (S) have multiplicity 1.

Subdivision based isolation of singularities using (S)

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x, y, z]$
Output: A list R of boxes containing singularities
$\rightarrow$ each box of R contains a singularity of $Res_z(p, q)$,
$\rightarrow$ each singularity of $Res_z(p, q)$ in $\mathcal{I}_0$ is in a box of R.

$$(S)\begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \\ s_{22}(x, y) \neq 0 \end{cases}$$



$\mathcal{I}_0$

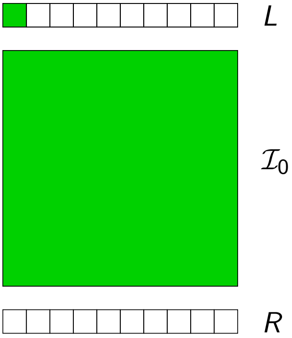
R

Subdivision based isolation of singularities using (S)

$$(S)\begin{cases} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \\ s_{22}(x,y)\neq 0 \end{cases}$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x,y,z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

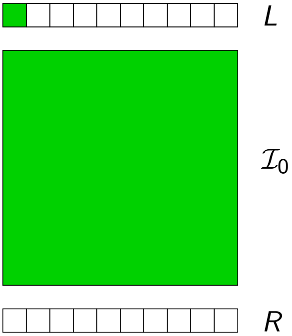


Subdivision based isolation of singularities using (S)

$$(S)\begin{cases} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \\ s_{22}(x,y)\neq 0 \end{cases}$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x,y,z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

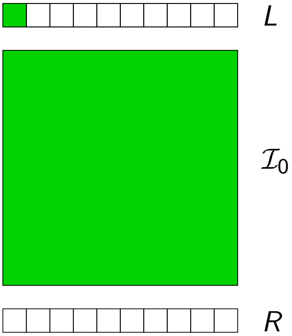


Subdivision based isolation of singularities using (S)

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x, y, z]$
Output: A list R of boxes containing singularities

$$(S) \begin{cases} s_{10}(x,y) = 0 \\ s_{11}(x,y) = 0 \\ s_{22}(x,y) \neq 0 \end{cases}$$

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

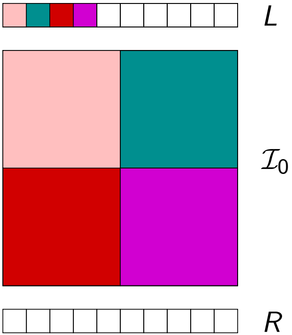


Subdivision based isolation of singularities using (S)

$$(S)\left\{\begin{array}{l} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \\ s_{22}(x,y)\neq 0 \end{array}\right.$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x, y, z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

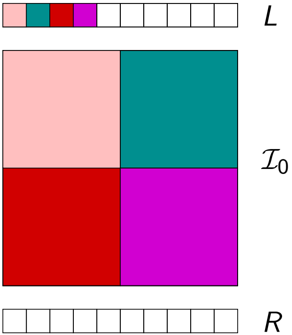


Subdivision based isolation of singularities using (S)

$$(S)\left\{\begin{array}{l} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \\ s_{22}(x,y)\neq 0 \end{array}\right.$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x,y,z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

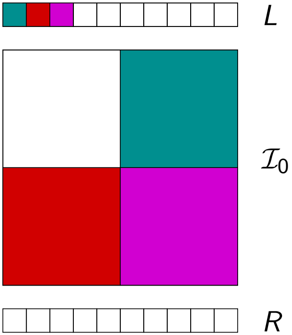


Subdivision based isolation of singularities using (S)

$$(S)\begin{cases} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \\ s_{22}(x,y)\neq 0 \end{cases}$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x,y,z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

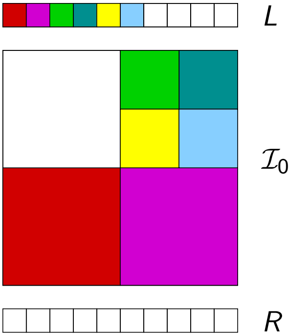


Subdivision based isolation of singularities using (S)

$$(S)\begin{cases} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \\ s_{22}(x,y)\neq 0 \end{cases}$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x,y,z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

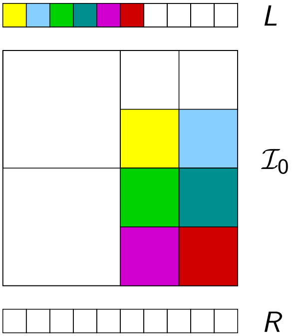


Subdivision based isolation of singularities using (S)

$$(S)\begin{cases} s_{10}(x,y) = 0 \\ s_{11}(x,y) = 0 \\ s_{22}(x,y) \neq 0 \end{cases}$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x,y,z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

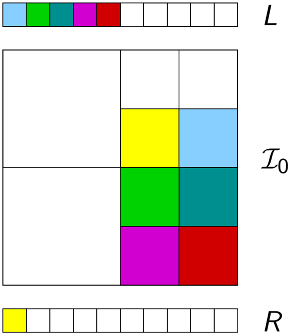


Subdivision based isolation of singularities using (S)

$$(S)\begin{cases} s_{10}(x,y) = 0 \\ s_{11}(x,y) = 0 \\ s_{22}(x,y) \neq 0 \end{cases}$$

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x,y,z]$
Output: A list R of boxes containing singularities

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p,q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R

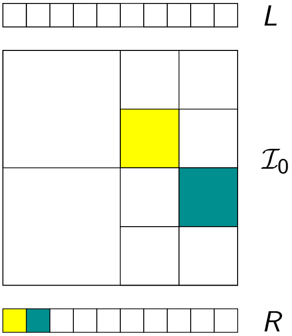


Subdivision based isolation of singularities using (S)

Input: A box $\mathcal{I}_0 \subset \mathbb{R}^2$, $p, q \in \mathbb{Q}[x, y, z]$
Output: A list R of boxes containing singularities

$$(S) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \\ s_{22}(x, y) \neq 0 \end{cases}$$

$L := \{\mathcal{I}_0\}$
Repeat:
1. $\mathcal{I} := L.pop$
2. If $0 \notin s_{10}(\mathcal{I})$ **or** $0 \notin s_{11}(\mathcal{I})$ **or** $0 \notin Res_z(p, q)(\mathcal{I})$ **then**
2.1. discard $\mathcal{I}$
3. Else
3.1. If $N(\mathcal{I}) \subset int(\mathcal{I})$ **and** $0 \notin s_{22}(\mathcal{I})$ **then**
3.1.1. insert $\mathcal{I}$ in R
3.2. Else
3.2.1. subdivide $\mathcal{I}$ and insert its children in L
3.3. End if
4. End if
Until $L = \emptyset$
Return R



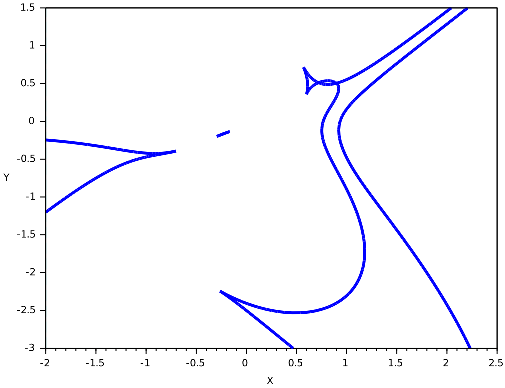
Implementation: a critical issue

$p \in \mathbb{Q}[x, y, z]$, degree 6, bit-size 8, nb of monomials 84

$$p = 158x^6 - 186x^5y + 205x^5z - 160x^4y^2 + 105x^4yz + 116x^4z^2 - 69x^3y^3 - 161x^3y^2z - 8x^3yz^2 + 107x^3z^3 + 144x^2y^4 - 193x^2y^3z + 130x^2y^2z^2 + x^2yz^3 + 47x^2z^4 + 165xy^5 - 220xy^4z - 21xy^3z^2 + 50xy^2z^3 - 130xyz^4 - 77xz^5 + 66y^6 - 55y^5z + 219y^4z^2 - 30y^3z^3 - 162y^2z^4 - 182yz^5 - 145z^6 + 105x^5 + 241x^4y - 177x^4z - 127x^3y^2 - 97x^3yz + 223x^3z^2 - 46x^2y^3 - 213x^2y^2z + 39x^2yz^2 + 191x^2z^3 + 65xy^4 - 105xy^3z - 248xy^2z^2 + 158xyz^3 - 183xz^4 + 48y^5 - 240y^4z + 235y^3z^2 + 194y^2z^3 - 45yz^4 + 159z^5 - 81x^4 - 230x^3y - 247x^3z - 38x^2y^2 + 106x^2yz + 184x^2z^2 + 49xy^3 - 197xy^2z - 182xyz^2 - 223xz^3 - 205y^4 - 225y^3z - 14y^2z^2 - 17yz^3 + 73z^4 - 234x^3 - 82x^2y + 179x^2z + 46xy^2 - 222xyz - 95xz^2 + 139y^3 + 168y^2z + 8yz^2 + 156z^3 + 159x^2 - 147xy - 22xz - 104y^2 + 181yz + 26z^2 - 90x + 250y + 19z + 19$$

Implementation: a critical issue

$p \in \mathbb{Q}[x, y, z]$, degree 6, bit-size 8, nb of monomials 84
$Res_z(p, \frac{\partial p}{\partial z}) \in \mathbb{Q}[x, y]$, degree 30, bit-size 111, nb of monomials 496



Implementation: a critical issue

$p \in \mathbb{Q}[x, y, z]$, degree 6, bit-size 8, nb of monomials 84
$Res_z(p, \frac{\partial p}{\partial z}) \in \mathbb{Q}[x, y]$, degree 30, bit-size 111, nb of monomials 496

$$Res_z(p, \frac{\partial p}{\partial z}) = 2537851751382193098537472618 5\, x^{30} - 1950289566984849821762662644 60\, x^{29} y +$$
$$6694606608938608139216045541 00\, x^{28} y^2 - 6313231163041522510562021480 00\, x^{27} y^3 -$$
$$1028704563680432990245022354280\, x^{26} y^4 + 4597797015605117908624008082 0\, x^{25} y^5 +$$
$$3554469553406371293751987742270\, x^{24} y^6 + 3711031010928440039666656612920\, x^{23} y^7 -$$
$$5634442800184514383998916600260\, x^{22} y^8 - 11658591855069381144706595841060\, x^{21} y^9 -$$
$$4387874939266072948066332459470\, x^{20} y^{10} + 16408843461038228420223023180230\, x^{19} y^{11} +$$
$$23700165794251777062304009772915\, x^{18} y^{12} + 43163241809977488659018002016 20\, x^{17} y^{13} -$$
$$24929137305247653219088728498740\, x^{16} y^{14} - 33372908351021778030492119654810\, x^{15} y^{15} -$$
$$9633448028150975870147511674570\, x^{14} y^{16} + 20500155431790235158403374001190\, x^{13} y^{17} +$$
$$31668089060759309350684716458350\, x^{12} y^{18} + 16544278550218652616250018398520\, x^{11} y^{19} -$$
$$5014730522275651771719575652535\, x^{10} y^{20} - 16590111614945163714073974823320\, x^{9} y^{21} -$$
$$13546083341149182083464535866425\, x^{8} y^{22} - 4754759946941791724566012110130\, x^{7} y^{23} +$$
$$1997218841447789708607357568825\, x^{6} y^{24} + 3898998021968250822246999603270\, x^{5} y^{25} +$$

Implementation: a critical issue

$p \in \mathbb{Q}[x, y, z]$, degree 6, bit-size 8, nb of monomials 84
$Res_z(p, \frac{\partial p}{\partial z}) \in \mathbb{Q}[x, y]$, degree 30, bit-size 111, nb of monomials 496
$s11 \in \mathbb{Q}[x, y]$, degree 20, bit-size 89, nb of monomials 231

$s11 = -140117848627008812531220\,x^{20} - 610153133593349354171040\,x^{19}y + 39516518923021733844070\,x^{18}y^2 +$

$3342883727033466620154170\,x^{17}y^3 + 2891274355142589403901890\,x^{16}y^4 + 112794729750527524649840\,x^{15}y^5 -$

$11340692490521298700125220\,x^{14}y^6 - 11062911106388945165447000\,x^{13}y^7 -$

$2946445042372334921153850\,x^{12}y^8 + 12890641493062475757808370\,x^{11}y^9 +$

$20482823881470123106468370\,x^{10}y^{10} + 11024860229216130931420010\,x^9y^{11} -$

$1126962434297495978162860\,x^8y^{12} - 12884485324685747664432680\,x^7y^{13} -$

$9059725287074848327234580\,x^6y^{14} - 4941320817429025658253850\,x^5y^{15} + 2122391146412348698406760\,x^4y^{16} +$

$2384112136850068775369540\,x^3y^{17} + 2363347796938811648578260\,x^2y^{18} + 735933941537801203166720\,xy^{19} +$

$293011939904302120871210\,y^{20} + 693411445688541987909840\,x^{19} + 4819667434476299196422270\,x^{18}y -$

$854531603999857310010090\,x^{17}y^2 - 4588903065796097271527060\,x^{16}y^3 - 12454540077632985887041990\,x^{15}y^4 -$

$19038809918580772113933260\,x^{14}y^5 - 5255594134400598288192960\,x^{13}y^6 + 1174005266404773044076220\,x^{12}y^7 +$

$39658021585466235582243720\,x^{11}y^8 + 49141822061980186469013340\,x^{10}y^9 +$

$5125145051120039185666690\,x^9y^{10} + 11669318785950916496923050\,x^8y^{11} -$

Implementation: a critical issue

$p \in \mathbb{Q}[x, y, z]$, degree 6, bit-size 8, nb of monomials 84
$Res_z(p, \frac{\partial p}{\partial z}) \in \mathbb{Q}[x, y]$, degree 30, bit-size 111, nb of monomials 496
$s11 \in \mathbb{Q}[x, y]$, degree 20, bit-size 89, nb of monomials 231

Polynomials are evaluated :

- at least once for each box
- in order to discard or validate boxes

Implementation: a critical issue

$p \in \mathbb{Q}[x, y, z]$, degree 6, bit-size 8, nb of monomials 84
$Res_z(p, \frac{\partial p}{\partial z}) \in \mathbb{Q}[x, y]$, degree 30, bit-size 111, nb of monomials 496
$s11 \in \mathbb{Q}[x, y]$, degree 20, bit-size 89, nb of monomials 231

Polynomials are evaluated :	Evaluation has to be:
• at least once for each box	quick
• in order to discard or validate boxes	sharp

Implementation: a critical issue

$p \in \mathbb{Q}[x, y, z]$, degree 6, bit-size 8, nb of monomials 84
$Res_z(p, \frac{\partial p}{\partial z}) \in \mathbb{Q}[x, y]$, degree 30, bit-size 111, nb of monomials 496
$s11 \in \mathbb{Q}[x, y]$, degree 20, bit-size 89, nb of monomials 231

Polynomials are evaluated : Evaluation has to be:

- at least once for each box — quick

  horner form

- in order to discard or validate boxes — sharp

  second order evaluation

$$\Box f(\mathcal{I}) = f(y) + J_f(y)(\mathcal{I} - y) + \frac{1}{2}\Box H_f(\mathcal{I})(\mathcal{I} - y)^2$$

where $y = center(\mathcal{I})$, $J_f(y) =$ jacobian matrix of f at y
and $\Box H_f(\mathcal{I}) =$ interval evaluation of hessian of f at $\mathcal{I}$.

Results: isolating singularities of a resultant curve

	RS4	HOM4PS		Bertini	Subdivision	
	$\mathbb{R}^2$	$\mathbb{C}^2$		$\mathbb{C}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d,σ	t	t	nsol/deg	t	t	t
$4,8$	0.214	0.078	98.6%	3.2	0.4	1.0
$5,8$	2.845	1.543	96.3%	124	0.6	2.6
$6,8$	23.90	15.18	90.3%	1604	3.0	9.6
$7,8$	137.9	97.95	75.5%	83120	8.4	27
$8,8$	725.7	-	-	382200*	43	82
$9,8$	2720	-	-	2766400*	47	273

Benches on 5 exemples for each (d,σ). Means of sequential times in seconds, on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

* interpolated times

Results: isolating singularities of a resultant curve

	RS4	HOM4PS		Bertini	Subdivision	
	$\mathbb{R}^2$	$\mathbb{C}^2$		$\mathbb{C}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d, σ	t	t	nsol/deg	t	t	t
4, 8	0.214	0.078	98.6%	3.2	0.4	1.0
5, 8	2.845	1.543	96.3%	124	0.6	2.6
6, 8	23.90	15.18	90.3%	1604	3.0	9.6
7, 8	137.9	97.95	75.5%	83120	8.4	27
8, 8	725.7	-	-	382200*	43	82
9, 8	2720	-	-	2766400*	47	273

Benches on 5 exemples for each (d, σ). Means of sequential times in seconds, on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

* interpolated times

Results: isolating singularities of a resultant curve

	RS4	HOM4PS		Bertini	Subdivision	
	$\mathbb{R}^2$	$\mathbb{C}^2$		$\mathbb{C}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d,σ	t	t	nsol/deg	t	t	t
$4,8$	0.214	0.078	98.6%	3.2	0.4	1.0
$5,8$	2.845	1.543	96.3%	124	0.6	2.6
$6,8$	23.90	15.18	90.3%	1604	3.0	9.6
$7,8$	137.9	97.95	75.5%	83120	8.4	27
$8,8$	725.7	-	-	382200*	43	82
$9,8$	2720	-	-	2766400*	47	273

Benches on 5 exemples for each (d,σ). Means of sequential times in seconds, on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

* interpolated times

Results: isolating singularities of a resultant curve

	RS4	HOM4PS		Bertini	Subdivision	
	$\mathbb{R}^2$	$\mathbb{C}^2$		$\mathbb{C}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d,σ	t	t	nsol/deg	t	t	t
$4,8$	0.214	0.078	98.6%	3.2	0.4	1.0
$5,8$	2.845	1.543	96.3%	124	0.6	2.6
$6,8$	23.90	15.18	90.3%	1604	3.0	9.6
$7,8$	137.9	97.95	75.5%	83120	8.4	27
$8,8$	725.7	-	-	382200*	43	82
$9,8$	2720	-	-	2766400*	47	273

Benches on 5 exemples for each (d,σ). Means of sequential times in seconds, on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

* interpolated times

Results: isolating singularities of a resultant curve

	RS4	HOM4PS		Bertini	Subdivision	
	$\mathbb{R}^2$	$\mathbb{C}^2$		$\mathbb{C}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d,σ	t	t	nsol/deg	t	t	t
$4,8$	0.214	0.078	98.6%	3.2	0.4	1.0
$5,8$	2.845	1.543	96.3%	124	0.6	2.6
$6,8$	23.90	15.18	90.3%	1604	3.0	9.6
$7,8$	137.9	97.95	75.5%	83120	8.4	27
$8,8$	725.7	-	-	382200*	43	82
$9,8$	2720	-	-	2766400*	47	273

Benches on 5 exemples for each (d,σ). Means of sequential times in seconds, on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

* interpolated times

Results: isolating singularities of a resultant curve

	RS4	HOM4PS		Bertini	Subdivision	
	$\mathbb{R}^2$	$\mathbb{C}^2$		$\mathbb{C}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d,σ	t	t	nsol/deg	t	t	t
$4,8$	0.214	0.078	98.6%	3.2	0.4	1.0
$5,8$	2.845	1.543	96.3%	124	0.6	2.6
$6,8$	23.90	15.18	90.3%	1604	3.0	9.6
$7,8$	137.9	97.95	75.5%	83120	8.4	27
$8,8$	725.7	-	-	382200*	43	82
$9,8$	2720	-	-	2766400*	47	273

$\rightarrow$ faster than state of the art methods when $d \geq 6$

$\rightarrow$ results come with a certification

Benches on 5 exemples for each (d,σ). Means of sequential times in seconds, on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

* interpolated times

Perspectives

- What about projections of curves of $\mathbb{R}^n$, with $n > 3$?
- What about projections of curves defined by non-polynomials functions?

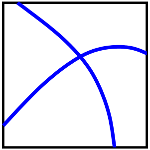
Questions?

Classification of singularities

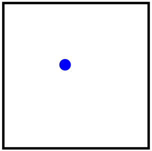
Resultant curve: $f = Res_z(p, q)$

Nodes are singularities such that $det(H_f) \neq 0$

- Nodes such that $det(H_f) < 0$: 4 real branches
- Nodes such that $det(H_f) > 0$: 0 real branches



$x^2 - y^2 = 0$



$x^2 + y^2 = 0$

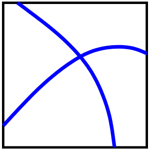
Classification of singularities

Discriminant curve: $f = Res_z(p, \frac{\partial p}{\partial z})$

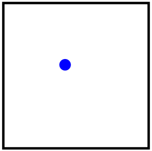
Nodes are singularities such that $det(H_f) \neq 0$

- Nodes such that $det(H_f) < 0$: 4 real branches
- Nodes such that $det(H_f) > 0$: 0 real branches

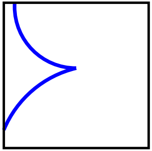
Cusps are singularities such that $\left\{ \begin{array}{c} det(H_f) = 0 \\ p = \frac{\partial p}{\partial z} = \frac{\partial^2 p}{\partial z^2} = 0 (S') \end{array} \right.$



$x^2 - y^2 = 0$



$x^2 + y^2 = 0$



$x^3 + y^2 = 0$

Classification of singularities

Discriminant curve: $f = Res_z(p, \frac{\partial p}{\partial z})$

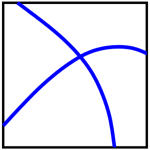
Nodes are singularities such that $det(H_f) \neq 0$

- Nodes such that $det(H_f) < 0$: 4 real branches
- Nodes such that $det(H_f) > 0$: 0 real branches

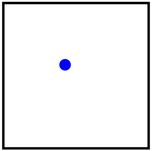
Cusps are singularities such that $\begin{cases} det(H_f) = 0 \\ p = \frac{\partial p}{\partial z} = \frac{\partial^2 p}{\partial z^2} = 0 (S') \end{cases}$

Algorithm: Computing topology at a singularity

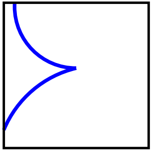
Input: A box $\mathcal{I}$ containing a singularity
Output: its type and number of branches
While True do:
1. If $0 \notin det(H_f(\mathcal{I}))$ **then**
1.1. If $det(H_f(\mathcal{I})) < 0$ **Return** node, 4
1.2. Else Return node, 0
2. Else estimate $\mathcal{I}_z$
2.1. If $\mathcal{I} \times I_z$ contains a unique roots of (S') **then**
2.1.1. Return cusp, 2
3. $\mathcal{I} \leftarrow$ reduce $\mathcal{I}$



$x^2 - y^2 = 0$



$x^2 + y^2 = 0$



$x^3 + y^2 = 0$

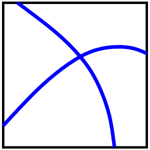
Classification of singularities

Discriminant curve: $f = Res_z(p, \frac{\partial p}{\partial z})$

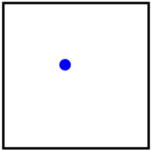
Nodes are singularities such that $det(H_f) \neq 0$

- Nodes such that $det(H_f) < 0$: 4 real branches
- Nodes such that $det(H_f) > 0$: 0 real branches

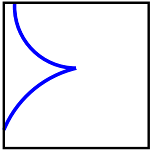
Cusps are singularities such that $\begin{cases} det(H_f) = 0 \\ p = \frac{\partial p}{\partial z} = \frac{\partial^2 p}{\partial z^2} = 0 (S') \end{cases}$



$x^2 - y^2 = 0$



$x^2 + y^2 = 0$



$x^3 + y^2 = 0$