New Progress in Univariate Polynomial Root Finding

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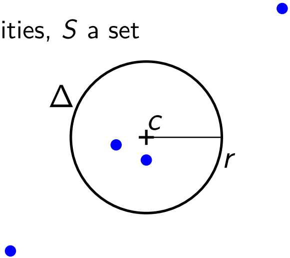
[3] NSF Grants # CCF-1563942 and # CCF-1564132
[4] NSF Grants # CCF-1116736 and # CCF-1563942 and PSC CUNY Award 698130048.

Root Counters

Complex disk: $\Delta = D(c, r) := \{z \text{ s.t. } |z - c| \leq r\}$

Polynomial: $p \in \mathbb{C}[z]$ of degree d
$\#(S, p) :=$ nb of roots of p in S counted with multiplicities, S a set

Counting test:



Exclusion test:

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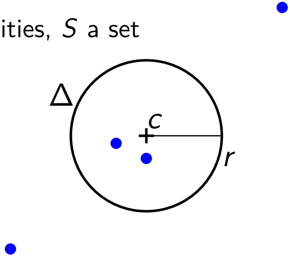

Counting test:

$C^*(p,\Delta,\ldots)$

Input: $p \in \mathbb{C}[z]$ of degree d, $\Delta = D(c,r)$ a disk
Output: $\#(\Delta,p)$ or -1 (**can not decide**)

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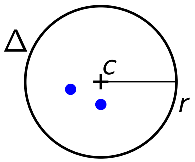

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Exclusion test:

$C^0(p, \Delta, \ldots)$

Input: $p \in \mathbb{C}[z]$ of degree d, $\Delta = D(c, r)$ a disk
Output: **true** $(\#(\Delta, p) = 0)$ or **can not decide**

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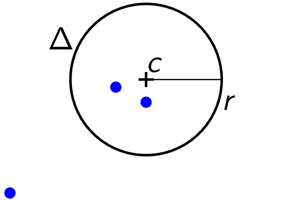

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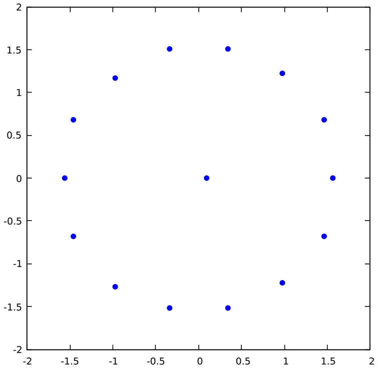
Core tools for root-finding algorithms based on subdivision

Root Clustering Problem

Input: a polynomial $p \in \mathbb{C}[z]$ of degree $d > 1$

Output:



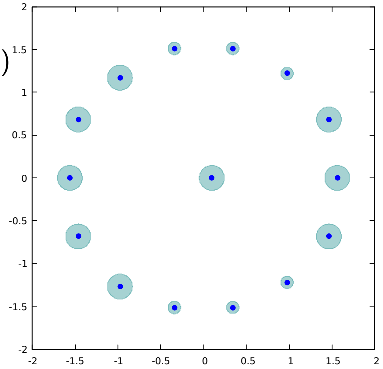
Example: Mignotte-like polynomial: $z^d - 2(2^\sigma z - 1)^2$, where $d = 16, \sigma = 4$

Root Clustering Problem

Input: a polynomial $p \in \mathbb{C}[z]$ of degree $d > 1$

Output: a set of pairs $\{(\Delta^1, m^1), \ldots, (\Delta^\ell, m^\ell)\}$ where

- the Δ_j's are pairwise-disjoint disks
- $\forall j$, $\#(\Delta^j, p) = m^j$, and $\#(3\Delta^j, p) = m^j$ (natural clusters)
- $Z(\mathbb{C}, p) = \bigcup_{j=1}^{\ell} Z(\Delta^j, p)$ and $\ell > 1$



Notations: $Z(S, p)$: roots of p in S

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Root Clustering Problem Local Version

Input: a polynomial $p \in \mathbb{C}[z]$, a complex box B

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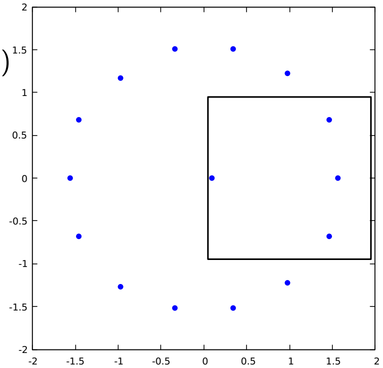

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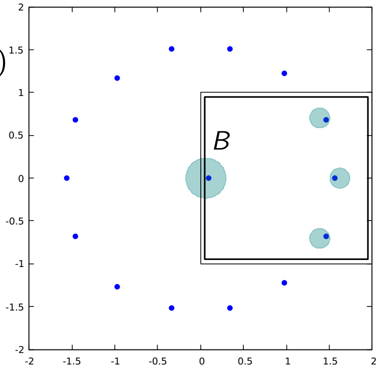
- $Z(B, p) \subseteq \bigcup_{j=1}^{\ell} Z(\Delta^j, p) \subseteq Z(\delta B, p)$,
  for $\delta > 1$



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Oracle Polynomials

Let $\alpha \in \mathbb{C}$.

Oracle for α: function $\mathcal{O}_\alpha : \mathbb{N} \rightarrow \mathbb{C}$ s.t. $|\alpha - \mathcal{O}_\alpha(L)| \leq 2^{-L}$

Let $p \in \mathbb{C}[z]$.

Oracle for p: function $\mathcal{O}_p : \mathbb{N} \rightarrow \mathbb{C}[z]$ s.t. $\|p - \mathcal{O}_p(L)\|_\infty \leq 2^{-L}$

Local Root Clustering Algorithm

[BSS+16] Ruben Becker, Michael Sagraloff, Vikram Sharma, Juan Xu, and Chee Yap.
Complexity analysis of root clustering for a complex polynomial.
In *Proceedings of the ACM on International Symposium on Symbolic and Algebraic Computation*, pages 71–78. ACM, 2016.

Input polynomial: p given as an oracle

Near optimal: bit complexity $\widetilde{O}(d^2(\sigma + d))$
for the benchmark problem ($p \in \mathbb{Z}[z]$, degree d, bit-size σ)

Implementation: `C` package `Ccluster`[1]
interface for julia: `Ccluster.jl`[2]
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In practice: Still not the users choice for global problems (`MPsolve`)

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Counting test:

Exclusion test:

Subdivision approach:

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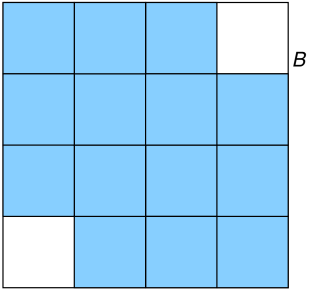

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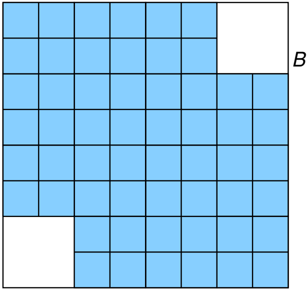

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Counting test:

Exclusion test: $C^0(\Delta, p)$

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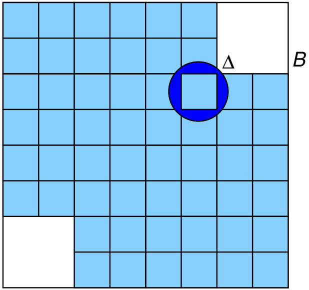

Outline of [BSS+16]

Counting test: $C^*(\Delta, p)$

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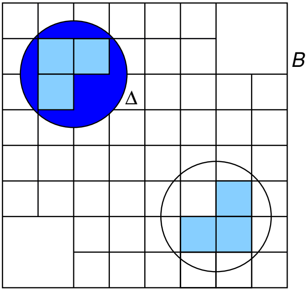


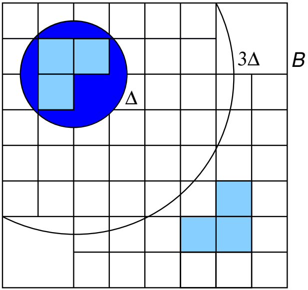
Outline of [BSS$^+$16]

Counting test: $C^*(\Delta, p)$

Exclusion test: $C^0(\Delta, p)$

Based on Pellet's Theorem: for $\Delta = D(c, r)$, requires to compute the coefficients of $p(c + rz)$

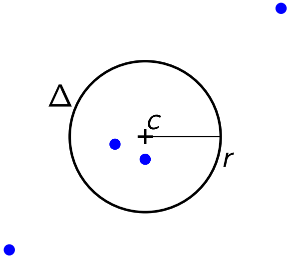
Subdivision approach:



Our Contributions

New Counting and Exclusion tests: based on power sums approximations

- require Δ to be "well isolated"
- do not require to shift p in $c + rz$
- evaluate p on a small nb. of points



Unsure Exclusion test:

- assume that Δ is well isolated
- check necessary conditions to verify the result
- experimentally reliable

Algorithm for the Global Root Clustering Problem:

- uses our unsure exclusion test
- the output is checked as a post-procedure

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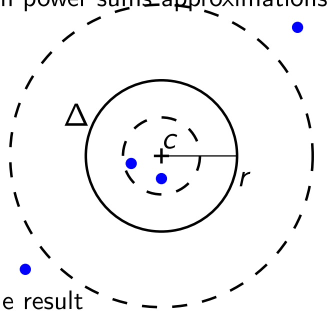

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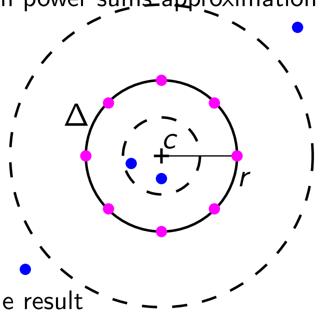

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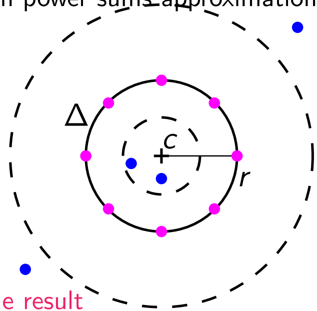

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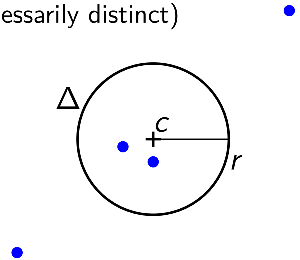
Let $\Delta = D(c, r)$, $p \in \mathbb{C}[z]$ of degree d
Let $\alpha_1, \ldots, \alpha_{d_\Delta}$ be the roots of p in Δ (non necessarily distinct)
Let $h \in \mathbb{Z}$

h-th power sum of p in Δ:

$$s_h(\Delta, p) = \alpha_1^h + \ldots + \alpha_{d_\Delta}^h$$



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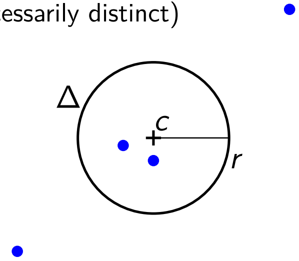

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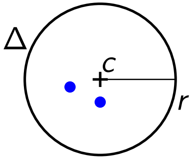

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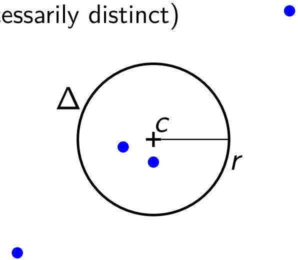

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Let $p_\Delta = p(c + rz)$:

(iii) $\#(\Delta, p) = s_0(D(0, 1), p_\Delta)$

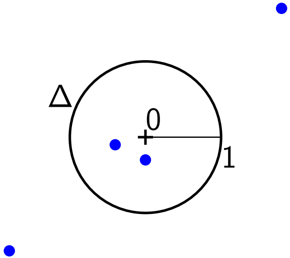
(iv) $\#(\Delta, p) = 0 \Rightarrow s_h(D(0, 1), p_\Delta) = 0$ for any h

Approximation of the Power Sums in $D(0,1)$

Let $h \in \mathbb{Z}$, $q \in \mathbb{N}_*$ s.t. $q > h$ and define

$$s_h^* = \frac{1}{q} \sum_{g=0}^{q-1} \zeta^{g(h+1)} \frac{p'(\zeta^g)}{p(\zeta^g)}$$

where ζ is a primitive q-th root of unity.



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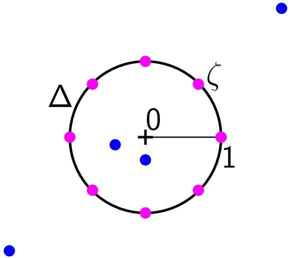

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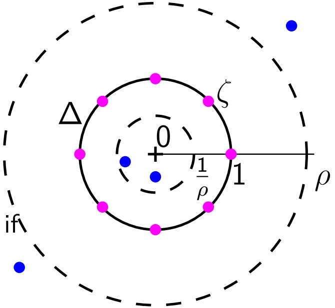

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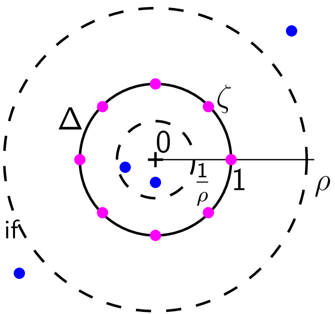

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Theorem [Sch82]: Let $\rho > 1$;
suppose $D(0,1)$ is ρ-isolated and contains d_Δ roots. Then

$$|s_h^* - s_h(D(0,1), p)| \leq \frac{d_\Delta \theta^{q+h} + (d - d_\Delta)\theta^{q-h}}{1 - \theta^q} \text{ where } \theta = \frac{1}{\rho}$$

[Sch82] Arnold Schönhage.
The fundamental theorem of algebra in terms of computational complexity.
Manuscript. Univ. of Tübingen, Germany, 1982.

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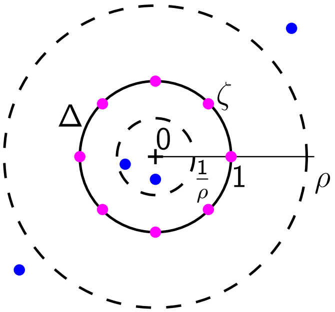

Corollary: Let $\rho > 1$;
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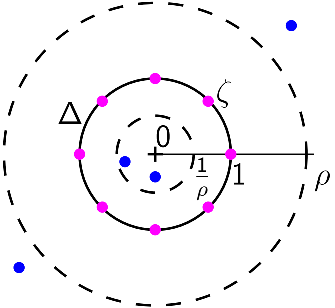

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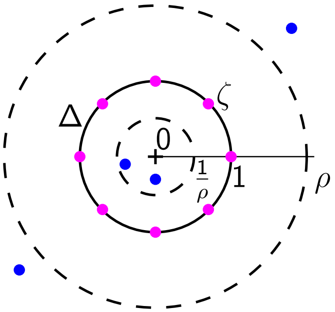

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Remark: $s_0(D(0,1), p)$ is an integer, thus error $e < \frac{1}{4}$ is enough to recover it from s_0^* !

Example: when $\rho = 2$ and $d = 500$, $q = 11$ in enough!

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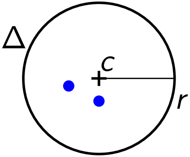

h-th power sum of p in Δ:

$$s_h(\Delta, p) = \alpha_1^h + \ldots + \alpha_{d_\Delta}^h$$

Remarks:

(i) $\#(\Delta, p) = s_0(\Delta, p)$

(ii) $\#(\Delta, p) = 0 \Rightarrow s_h(\Delta, p) = 0$ for any h

Let $p_\Delta = p(c + rz)$:

(iii) $\#(\Delta, p) = s_0(D(0, 1), p_\Delta)$

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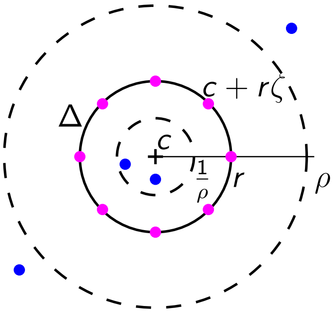
Approximation of the Power Sums in $\Delta = D(c, r)$

Let $h \in \mathbb{Z}$, $q \in \mathbb{N}_*$ s.t. $q > h$ and define

$$s_h^* = \frac{r}{q} \sum_{g=0}^{q-1} \zeta^{g(h+1)} \frac{p'(c + r\zeta^g)}{p(c + r\zeta^g)}$$

where ζ is a primitive q-th root of unity.



Corollary: Let $\rho > 1$;
suppose $D(c, r)$ is ρ-isolated. Then

(i) $|s_0^* - s_0(D(c, r), p)| \leq \dfrac{d\theta^q}{1 - \theta^q}$ where $\theta = \dfrac{1}{\rho}$

(ii) Fix $e > 0$. If $q = \lceil \log_\theta(\frac{e}{d+e}) \rceil$ then $|s_0^* - s_0(D(c, r), p)| \leq e$.

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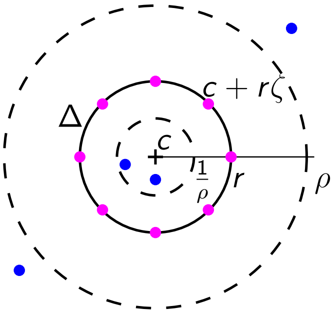

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Remark: Computing $s_0(D(c, r), p)$ does **not** require to compute the coefficients of $p_\Delta = p(c + rz)$!

Counting and Exclusion Tests

$P^*(p, \Delta, \rho)$ //*Output in* $\{0, 1, \ldots, d\}$

Input: $p \in \mathbb{C}[z]$ of degree d, $\rho > 1$, Δ a ρ-isolated disk
Output: $\#(\Delta, p)$

1. $e \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{e}{d+e}) \rceil$
3. compute $s_0^* = \frac{r}{q} \sum_{g=0}^{q-1} \zeta^g \frac{p'(c+r\zeta^g)}{p(c+r\zeta^g)}$
4. **return** the unique integer in $D(s_0^*, e)$

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1. an implementable version for oracle polynomials
2. an exclusion test:

$P^0(p, \Delta, \rho)$ *//Output in* { **true**, **false** }

Input: $p \in \mathbb{C}[z]$ of degree d, $\rho > 1$, Δ a ρ-isolated disk
Output: **true** iff p has no root in Δ

1. return $P^*(p, \Delta, \rho) == 0$

Counting and Exclusion Tests

$P^*(p, \Delta, \rho)$ *//Output in* $\{0, 1, \dots, d\}$

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Output: $\#(\Delta, p)$

Remarks: One can derive

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2. an exclusion test:

Question: What if ρ is not known?

Unsure Exclusion Test

$\widetilde{P^0}(p, \Delta)$

Input: $p \in \mathbb{C}[z]$ of degree d, Δ a disk
Output: in { **true**, **can not decide** }

0. Let $\rho = \frac{4}{3}$, and assume Δ is ρ-isolated
1. $e \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{e}{d+e}) \rceil$
3. compute $s_0^* = \frac{r}{q} \sum_{g=0}^{q-1} \zeta^g \frac{p'(c+r\zeta^g)}{p(c+r\zeta^g)}$
4. **if** $D(s_0^*, e)$ does not contain zero
5. **return can not decide**
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Remark: Even if the output of $\widetilde{P^0}(p, \Delta)$ is **true**, it may be wrong

Unsure Exclusion Test: Experiments

		C^0-tests	$\widetilde{P}^0$-tests			
d	n	t_0/t (%)	t_1/t_0	#F		
100 random dense polynomials per degree						
64	116302	87.2	1.0	4		
128	227842	90.5	.54	21		
191	340348	92.0	.42	26		
100 random sparse (10 monomials) polynomials per degree						
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Legend: d: degree
n: total number of exclusion tests
t: sequential time of `Ccluster`
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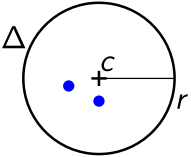
Power Sums

Let $\Delta = D(c, r)$, $p \in \mathbb{C}[z]$ of degree d
Let $\alpha_1, \ldots, \alpha_{d_\Delta}$ be the roots of p in Δ (non necessarily distinct)
Let $h \in \mathbb{Z}$



h-th power sum of p in Δ:

$$s_h(\Delta, p) = \alpha_1^h + \ldots + \alpha_{d_\Delta}^h$$

Remarks:

(i) $\#(\Delta, p) = s_0(\Delta, p)$

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Let $p_\Delta = p(c + rz)$:

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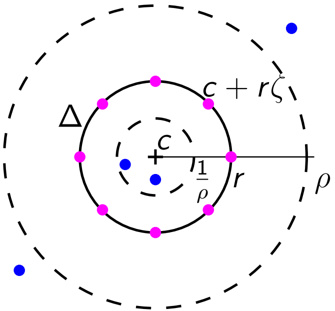
Approximation of the Power Sums in $\Delta = D(c, r)$ (II)

Let $h \in \mathbb{Z}$, $q \in \mathbb{N}_*$ s.t. $q > h$ and define

$$s_h^* = \frac{r}{q} \sum_{g=0}^{q-1} \zeta^{g(h+1)} \frac{p'(c + r\zeta^g)}{p(c + r\zeta^g)}$$

where ζ is a primitive q-th root of unity.



Corollary: Let $\rho > 1$;
suppose $D(c, r)$ is ρ-isolated. Then

(i) $|s_h^* - s_h(D(c, r), p)| \leq \frac{d\theta^{q-h}}{1 - \theta^q}$ where $\theta = \frac{1}{\rho}$

(ii) Fix $e > 0$. If $q = \lceil \log_\theta(\frac{e}{d+e}) \rceil + h$ then $|s_h^* - s_h(D(c, r), p)| \leq e$.

Unsure Exclusion Test

$\widetilde{P^0}(p, \Delta, k)$

Input: $p \in \mathbb{C}[z]$ of degree d, Δ a disk, k an integer ≥ 0
Output: in { **true**, **can not decide** }

0. Let $\rho = \frac{4}{3}$, and assume Δ is ρ-isolated
1. $e \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{e}{d+e}) \rceil + k$
3. evaluate p and p' at $c + r\zeta^g$ for $g = 0, \ldots, q-1$
4. **for** $h = 0, \ldots, k$ **do**
5. compute $s_h^* = \frac{r}{q} \sum_{g=0}^{q-1} \zeta^{g(h+1)} \frac{p'(c+r\zeta^g)}{p(c+r\zeta^g)}$
6. **if** $0 \notin D(s_h^*, e)$ **then**
7. **return can not decide**
8. **return true**

Unsure Exclusion Test: Experiments

		C^0-tests	$\widetilde{P}^0$-tests, $k=0$		$\widetilde{P}^0$-tests, $k=1$		$\widetilde{P}^0$-tests, $k=2$	
d	n	t_0/t (%)	t_1/t_0	#F	t'_1/t_0	#F'	t''_1/t_0	#F"
100 random dense polynomials per degree								
64	116302	87.2	1.0	4	1.0	0	1.1	0
128	227842	90.5	.54	21	.57	0	.59	0
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191	331966	92.1	.25	11	.26	2	.28	0

Legend: d: degree
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#F: nb of wrong res. in $\widetilde{P}^0$-tests with $k=0$

t'_1: time spent in $\widetilde{P}^0$-tests with $k=1$
#F': nb of wrong res. in $\widetilde{P}^0$-tests with $k=1$
t''_1: time spent in $\widetilde{P}^0$-tests with $k=2$
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Unsure Exclusion Test: Experiments

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Subdivision Algorithm with Unsure Exclusion Test

- for the (global) Root Clustering Problem
- uses $\widetilde{P^0}$-test with $k = 2$
- always terminates, but may fail: in this case, reports failure
- implemented in `C` within `Ccluster`: `CclusterF`

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- for the (global) Root Clustering Problem
- uses $\widetilde{P^0}$-test with $k = 2$
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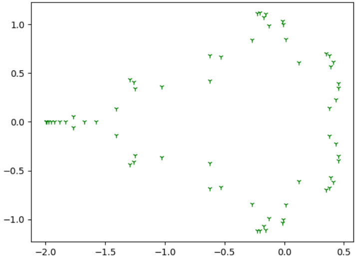
- implemented in `C` within `Ccluster`: `CclusterF`
- faster for sparse and procedural polynomial

Procedure: $\texttt{Mandelbrot}_k(z)$
Input: $k \in \mathbb{N}^*, z \in \mathbb{C}$
Output: $\alpha \in \mathbb{C}$

1. **if** $k = 1$ **then**
2. **return** z
3. **else**
4. **return** $z\texttt{Mandelbrot}_{k-1}(z)^2 + 1$



$k = 6$ (deg $= 63$)

Subdivision Algorithm with Unsure Exclusion Test

Results:

	Ccluster	CclusterF		
d	t	#Fails	t'	t'/t (%)
100 random dense polynomials per degree				
64	31.5	0	41.2	130
128	222	0	149	67.3
191	665	0	340	51.1
100 random sparse (10 monomials) polynomials per degree				
64	27.9	0	31.7	113
128	216	0	100	46.3
191	638	0	209	32.7
Mandelbrot polynomials				
127	3.46	0	0.56	16.1
255	18.4	0	1.79	9.70
511	118	0	7.61	6.42

Legend: t, t': seq. times in s. on an
Intel(R) Core(TM) i7-8700 CPU @ 3.20GHz machine with Linux

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Remarks and Future Works

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- Probabilistic and deterministic support of our $\widetilde{P^0}$-test

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Thank you!