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Clustering Complex Zeros of Triangular Systems of Polynomials

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[3] European Union's H2020 No. 676541 (OpenDreamKit)
[4] NSF Grants # CCF-1563942, # CCF-1564132 and # CCF-1708884

Triangular systems of polynomials:

$$\begin{cases} f_1(z_1) & = & 0 \\ f_2(z_1, z_2) & = & 0 \\ \dots & & \\ f_n(z_1, z_2, \dots, z_n) & = & 0 \end{cases}, deg_{z_i}(f_i) \geq 1$$

with: finite number of sols

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	Isolate RC, `Maple`		solve.lib, `Singular`		
type	symbolic	numeric $\mathbb{R}$	symbolic	numeric $\mathbb{C}$	
$\mathcal{S}_4$	3.8	3.7	0.6	0.18	
$\mathcal{S}_5$	24.2	>1000	42.9	0.57	

seq. times in s on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine
asked precision: 53 bits

$$\mathcal{S}_4 \begin{cases} z1^4 - 57 * z1^2 * z2 - 86 * z1 * z2^2 - 160 * z2^3 + 95 * z2^2 * z3 + 35 * z1^2 - 106 * z3 & = 0 \\ z2^4 - 64 * z2^3 - 190 * z1 * z2 + 186 * z1 * z3 - 119 * z2 * z3 + 188 * z3 + 93 & = 0 \\ z3^4 + 116 * z1 * z2^2 - 168 * z1 * z2 * z3 + 135 * z1 * z3^2 + 29 * z3^3 - 8 * z1 * z3 + 119 * z2 * z3 & = 0 \end{cases}$$

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with: finite number of sols
possibly sols with multiplicity

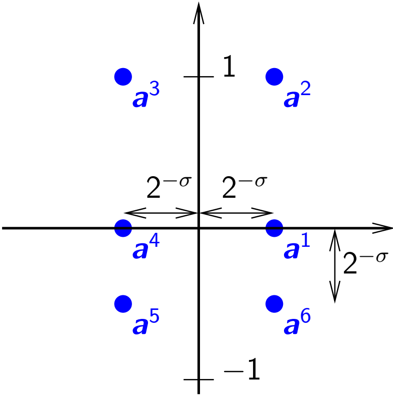
Example

System: Let $\sigma \geq 3$ and $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ be:

$$\begin{cases} (z_1 - 2^{-\sigma})\ (z_1 + 2^{-\sigma}) & = & 0 \\ (z_2 + 2^{\sigma} z_1^2)\ (z_2 - 1) z_2 & = & 0 \end{cases}$$

Solutions: $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has 6 solutions, all real:

$$\begin{aligned} \boldsymbol{a}^1 &= (2^{-\sigma}, 0) \\ \boldsymbol{a}^2 &= (2^{-\sigma}, 1) \\ \boldsymbol{a}^3 &= (-2^{-\sigma}, 1) \\ \boldsymbol{a}^4 &= (-2^{-\sigma}, 0) \\ \boldsymbol{a}^5 &= (-2^{-\sigma}, -2^{-\sigma}) \\ \boldsymbol{a}^6 &= (2^{-\sigma}, -2^{-\sigma}) \end{aligned}$$



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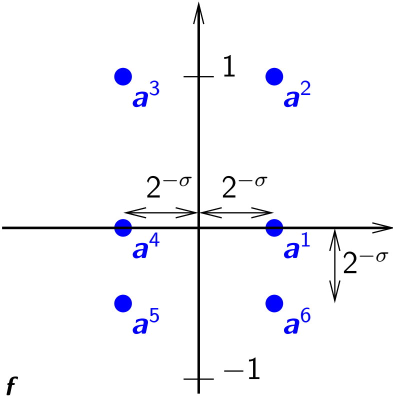

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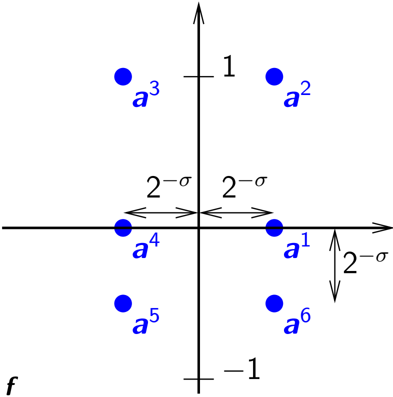

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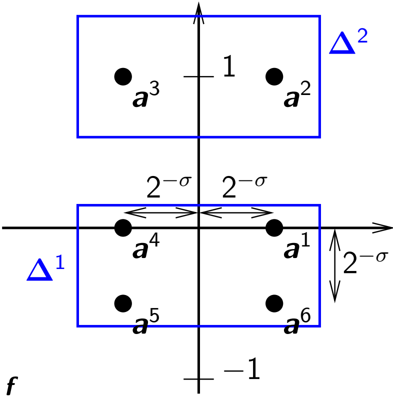

Natural clusters:

$(\boldsymbol{\Delta}^1, 9)$
$(\boldsymbol{\Delta}^2, 3)$

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Local solution Clustering Problem (LCP)

Input: a polynomial map $\boldsymbol{f} : \mathbb{C}^n \to \mathbb{C}^n$ (assume $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ is 0-dim),
a polybox $\boldsymbol{B} \subset \mathbb{C}^n$, the Region of Interest (RoI),
$\epsilon > 0$

Output:

Notations: $\boldsymbol{f} = (f_1, \ldots, f_n)$,
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Output: a set of pairs $\{(\boldsymbol{\Delta}^1, m^1), \ldots, (\boldsymbol{\Delta}^\ell, m^\ell)\}$ where:

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Definition: a pair $(\boldsymbol{\Delta}, m)$ is called natural cluster (relative to $\boldsymbol{f}$)
when it satisfies:

$$m = \#(\boldsymbol{\Delta}, \boldsymbol{f}) = \#(3\boldsymbol{\Delta}, \boldsymbol{f}) \geq 1$$

if $r(\boldsymbol{\Delta}) \leq \epsilon$, it is a natural ϵ-cluster

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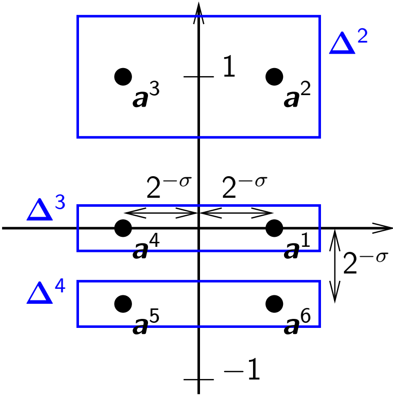

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$\boldsymbol{\Delta}^3$, $\boldsymbol{\Delta}^4$ are not natural clusters

Motivations for solutions clustering

Clustering can be done:

- $\rightarrow$ numerically with guarantee of results
- $\rightarrow$ locally
- $\rightarrow$ for systems of oracle polynomials
- $\rightarrow$ with sols. with multiplicity

Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

Oracle for α: function $\mathcal{O}_\alpha : \mathbb{Z} \to \square\mathbb{C}$
s.t. $\alpha \in \mathcal{O}_\alpha(L)$ and $w(\mathcal{O}_\alpha(L)) \leq 2^{-L}$

Notations: $\square\mathbb{C}$: set of complex interval

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Let $f \in \mathbb{C}[z_1, \dots, z_n]$

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Let $\square f \in \square\mathbb{C}[z_1, \ldots, z_n]$ and $(\square\alpha_1, \ldots, \square\alpha_{n-1}) \in \square\mathbb{C}^n$

Partial specialization of $\square f$: $\square f(\square\alpha_1, \ldots, \square\alpha_{n-1}) \in \square\mathbb{C}[z_n]$

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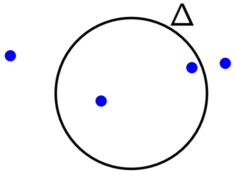
The soft Pellet's test

Pellet's Theorem: Let Δ be a complex disc centered in c and radius r.
Let $f \in \mathbb{C}[z]$, $d = deg(f)$ and $f_\Delta = f(c + rz)$.

If $\exists\ 0 \leq m \leq d$ s.t.

$$|(f_\Delta)_m| > \sum_{i \neq k} |(f_\Delta)_i| \tag{1}$$

then f has exactly m roots in Δ.



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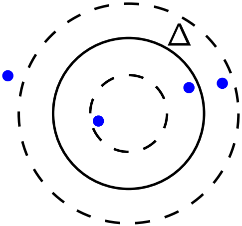
If f has no root in this annulus $\rightarrow$
$\exists m$ s.t. eq. (1) holds.



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PelletTest(Δ, f) $\qquad$ *//Output in* $\{-1, 0, 1, \ldots, d\}$

1. compute f_Δ
2. **for** m **from** 0 **to** d **do**
3. **if** $|(f_\Delta)_m| > \sum_{i \neq k} |(f_\Delta)_i|$
4. **return** m $\qquad$ *//m roots (with mult.) in* Δ
5. **return** -1 $\qquad$ *//Roots near the boundary of* Δ

The soft Pellet's test: for interval polynomials

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SoftCompare($\square a$, $\square b$) *// $\square a$, $\square b$ are real intervals*

Input: $\square a$, $\square b$ real intervals

Output: a number in $\{-2, -1, 1\}$ s.t.:

$1 \Rightarrow \square a > \square b$

$-1 \Rightarrow \square a < \square b$ or $\square a, \square b$ are too close

$-2 \Rightarrow \square a \cap \square b \neq \emptyset$

(meant to be embedded in a loop to compare oracle numbers)

The soft Pellet's test: for interval polynomials

SoftPelletTest($\Delta, \square f$) *//Output in $\{-2, -1, 0, 1, \ldots, d\}$*

1. compute $\square f_\Delta$
2. **for** m **from** 0 **to** d **do**
3. $\quad R \leftarrow$ SoftCompare($|(\square f_\Delta)_m|, \sum_{i \neq k} |(\square f_\Delta)_i|$)
4. $\quad$ **if** $R \geq 0$ **then return** m *//any $f \in \square f$ has m roots* // (with mult.) in Δ
5. $\quad$ **if** $R = -2$ **then return** -2 *//$\square f$ is too wide*
6. **return** -1 *//Roots near the boundary of Δ*

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(meant to be embedded in a loop to compare oracle numbers)

The soft Pellet's test: for interval polynomials

SoftPelletTest(Δ, $\square f$) *//Output in* $\{-2, -1, 0, 1, \ldots, d\}$

1. compute $\square f_\Delta$
2. **for** m **from** 0 **to** d **do**
3. $R \leftarrow$ SoftCompare$(|(\square f_\Delta)_m|, \sum_{i \neq k} |(\square f_\Delta)_i|)$
4. **if** $R \geq 0$ **then return** m *//any $f \in \square f$ has m roots*
   // (with mult.) in Δ
5. **if** $R = -2$ **then return** -2 *//$\square f$ is too wide*
6. **return** -1 *//Roots near the boundary of Δ*

[BSSY18] Ruben Becker, Michael Sagraloff, Vikram Sharma, and Chee Yap.
A near-optimal subdivision algorithm for complex root isolation based on Pellet test and Newton iteration.
JSC 86:51–96, May-June 2018.

Univariate root clustering algorithm

[BSS+16]: solves the LCP in 1D
SoftPelletTest embedded in a subdivision framework
accepts oracle polynomials in input
near optimal complexity (benchmark problem)

Implemented in [IPY18]

[BSS+16] Ruben Becker, Michael Sagraloff, Vikram Sharma, Juan Xu, and Chee Yap.
Complexity analysis of root clustering for a complex polynomial.
In *ISSAC 16*, pages 71–78. ACM, 2016.

[IPY18] Rémi Imbach, Victor Y. Pan, and Chee Yap.
Implementation of a near-optimal complex root clustering algorithm.
In *Mathematical Software – ICMS 2018*, pages 235–244, Cham, 2018.

LCP for triangular systems

Main tool: the T^*-test:

$T_*(\Delta, \square f)$

Input: Δ complex disc, $\square f \in \square\mathbb{C}[z]$ with $degree(f) = d$
Output: integer in $\{-2, -1, 0, 1, \ldots, d\}$ s.t.:

- $m \geq 1$: $\forall f \in \square f$, $\#(\Delta, f) = \#(3\Delta, f) = m$ (natural cluster)
- 0 : $\forall f \in \square f$, f has no root in Δ
- -1 : roots near the boundary of Δ (can not decide)
- -2 : not enough precision on $\square f$ (can not decide)

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Specialization to the bivariate case: $\boldsymbol{f} = (f_1, f_2)$
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ be is triangular system.

Number of solutions in a polydisc

Let $\boldsymbol{\Delta} = (\Delta_1, \Delta_2)$ and $\boldsymbol{m} = (m_1, m_2)$.

Proposition 1: Suppose

(i) f_1 has m_1 roots in Δ_1 with multiplicity

(ii) $\forall \alpha \in Z(\Delta_1, f_1)$, $f_2(\alpha, z_2)$ has m_2 roots in Δ_2 with multiplicity

Then $\boldsymbol{f}(\boldsymbol{z}) = \mathbf{0}$ has $m_2 \times m_1$ solutions in $\boldsymbol{\Delta}$ with multiplicity.

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Proof: direct consequence of

Theorem [ZFX11]: Let $\boldsymbol{\alpha} \in Z(\mathbb{C}^2, \boldsymbol{f})$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. Then

$$m(\boldsymbol{\alpha}, \boldsymbol{f}) = m(\alpha_2, f_2(\alpha_1, z_2)) \times m(\alpha_1, f_1)$$

[ZFX11] Zhihai Zhang, Tian Fang, and Bican Xia.
Real solution isolation with multiplicity of zero-dimensional triangular systems.
Science China Information Sciences, 54(1):60–69, 2011.

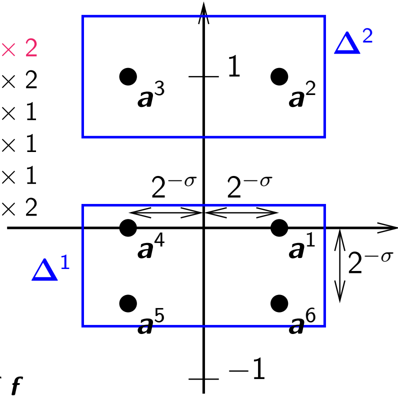
Example

System: Let $\sigma \geq 3$ and $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ be:

$$\left\{ \begin{array}{lcl} (z_1 - 2^{-\sigma})^2 (z_1 + 2^{-\sigma}) & = & 0 \\ (z_2 + 2^{\sigma} z_1^2)^2 (z_2 - 1) z_2 & = & 0 \end{array} \right.$$

Solutions: $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has 6 solutions, all real:

$$\begin{array}{llll} \boldsymbol{a}^1 = (2^{-\sigma} & , & 0) & \leftarrow m(\boldsymbol{a}^1, \boldsymbol{f}) = 2 = 1 \times 2 \\ \boldsymbol{a}^2 = (2^{-\sigma} & , & 1) & \leftarrow m(\boldsymbol{a}^2, \boldsymbol{f}) = 2 = 1 \times 2 \\ \boldsymbol{a}^3 = (-2^{-\sigma} & , & 1) & \leftarrow m(\boldsymbol{a}^3, \boldsymbol{f}) = 1 = 1 \times 1 \\ \boldsymbol{a}^4 = (-2^{-\sigma} & , & 0) & \leftarrow m(\boldsymbol{a}^4, \boldsymbol{f}) = 1 = 1 \times 1 \\ \boldsymbol{a}^5 = (-2^{-\sigma} & , & -2^{-\sigma}) & \leftarrow m(\boldsymbol{a}^5, \boldsymbol{f}) = 2 = 2 \times 1 \\ \boldsymbol{a}^6 = (2^{-\sigma} & , & -2^{-\sigma}) & \leftarrow m(\boldsymbol{a}^6, \boldsymbol{f}) = 4 = 2 \times 2 \end{array}$$



Natural clusters:
$(\boldsymbol{\Delta}^1, 9)$
$(\boldsymbol{\Delta}^2, 3)$

Notations: $m(\boldsymbol{a}, \boldsymbol{f})$: multiplicity of $\boldsymbol{a}$ as a sol. of $\boldsymbol{f}$

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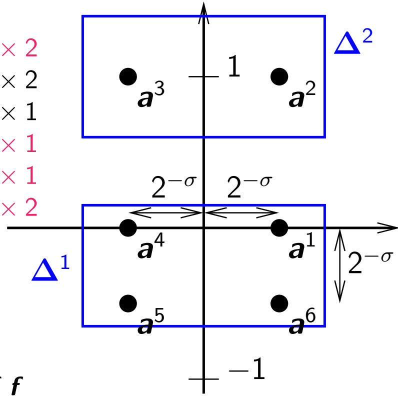

Natural clusters:

$(\boldsymbol{\Delta}^1, 9) \leftarrow 9 = 3 \times 3$

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Notations: $m(\boldsymbol{a}, \boldsymbol{f})$: multiplicity of $\boldsymbol{a}$ as a sol. of $\boldsymbol{f}$

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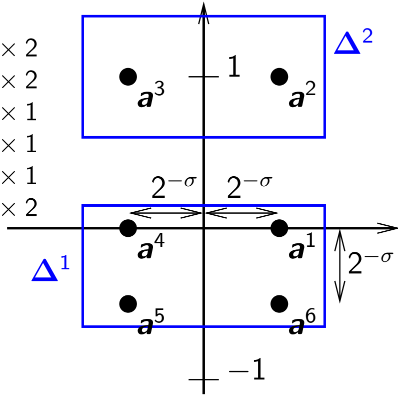

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Natural towers:

$(\boldsymbol{\Delta}^1, (3, 3))$

$(\boldsymbol{\Delta}^2, (1, 3))$

T^*-test and natural towers

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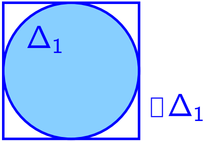
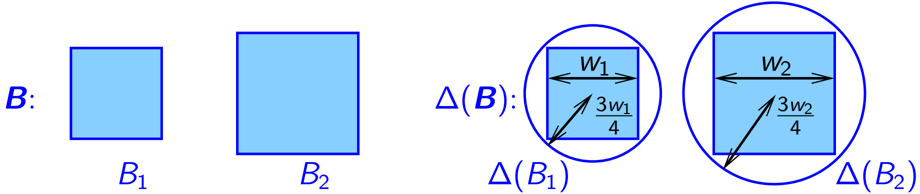


Geometry and subdivision

Containing polydisk of a polybox:



Geometry and subdivision

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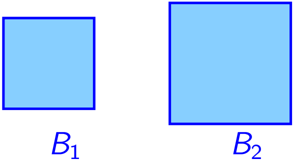
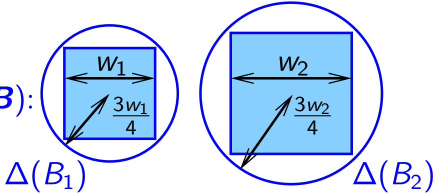
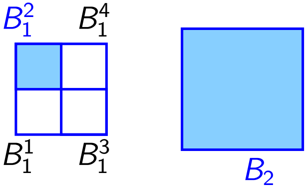


$\Delta(\boldsymbol{B})$:



Splits of a polybox:

$$split_1(\boldsymbol{B}) = \{(B_1^1, B_2), \ldots, (B_1^4, B_2)\}$$



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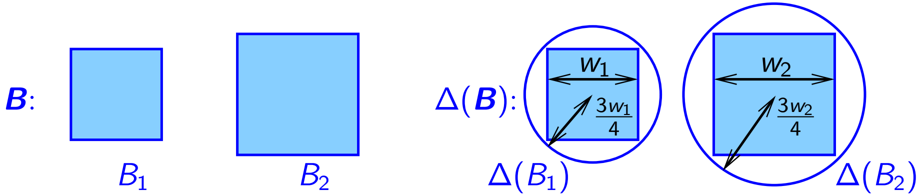

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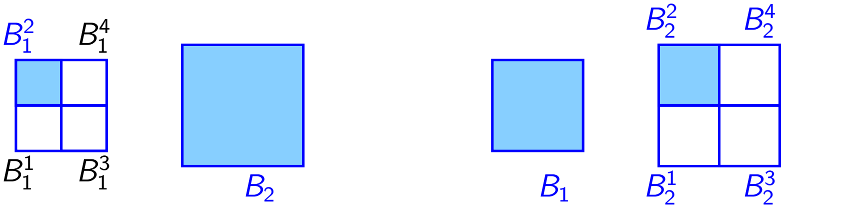

Lifting a polybox to a natural ϵ-tower

(case where $\boldsymbol{f}$ is known exactly)

Lift($\boldsymbol{f}$, $\boldsymbol{B}$, ϵ)

Input: a triangular system $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$, a polybox $\boldsymbol{B}$, $\epsilon > 0$
Output: two sets R and Q

1. $(\Delta_1, \Delta_2) \leftarrow \Delta(\boldsymbol{B})$
2. $m_1 \leftarrow T^*(\Delta_1, f_1)$ $\quad //m_1 \geq -1$ since f_1 is known exactly

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Lifting a polybox to a natural ϵ-tower

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5. **else** //*$m_1 \geq 1$ and $r(\Delta_1) \leq \epsilon$: natural ϵ-cluster*
7. $m_2 \leftarrow T^*(\Delta_2, f_2(\square\Delta_1))$
8. **if** $m_2 = -2$ **then return** $\emptyset$, $split_1(\boldsymbol{B})$
9. **if** $m_2 = 0$ **then return** $\emptyset, \emptyset$
10. **if** $m_2 = -1$ **or** $r(\Delta_2) > \epsilon$ **then return** $\emptyset$, $split_2(\boldsymbol{B})$
11. **else** //*$m_2 \geq 1$ and $r(\Delta_2) \leq \epsilon$: natural ϵ-cluster*
12. **return** $\{(\Delta(\boldsymbol{B}), (m_1, m_2))\}, \emptyset$

Lifting a polybox to a natural ϵ-tower

(case where $\boldsymbol{f}$ is known exactly)

Lift($\boldsymbol{f}$, $\boldsymbol{B}$, ϵ)

Input: a triangular system $\boldsymbol{f}(\boldsymbol{z}) = \mathbf{0}$, a polybox $\boldsymbol{B}$, $\epsilon > 0$
Output: two sets R and Q

1. $(\Delta_1, \Delta_2) \leftarrow \Delta(\boldsymbol{B})$
2. $m_1 \leftarrow T^*(\Delta_1, f_1)$ $\quad$ $//m_1 \geq -1$ since f_1 is known exactly
3. **if** $m_1 = 0$ **then return** $\emptyset, \emptyset$
4. **if** $m_1 = -1$ **or** $r(\Delta_1) > \epsilon$ **then return** $\emptyset$, $split_1(\boldsymbol{B})$
5. **else** $\quad$ $//m_1 \geq 1$ *and* $r(\Delta_1) \leq \epsilon$*: natural* ϵ*-cluster*
7. $\quad$ $m_2 \leftarrow T^*(\Delta_2, f_2(\Box\Delta_1))$
8. $\quad$ **if** $m_2 = -2$ **then return** $\emptyset$, $split_1(\boldsymbol{B})$
9. $\quad$ **if** $m_2 = 0$ **then return** $\emptyset, \emptyset$
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Solving the LCP problem for triangular systems

ClusterTri($\boldsymbol{f}$, $\boldsymbol{B}_0$, ϵ)

Input: a triangular system $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$, a RoI $\boldsymbol{B}_0$, $\epsilon > 0$
Output: a set of natural ϵ-towers solving the LCP

1. $R \leftarrow \emptyset$
2. $Q \leftarrow \{\boldsymbol{B}_0\}$
3. **while** Q is not empty **do**
4. $\quad\quad \boldsymbol{B} \leftarrow Q.pop()$
5. $\quad\quad R', Q' \leftarrow Lift(\boldsymbol{f}, \boldsymbol{B}, \epsilon)$
6. $\quad\quad R \leftarrow R \cup R'$
7. $\quad\quad Q \leftarrow Q \cup Q'$
8. remove duplicates from R
9. **return** R

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8. remove duplicates from R
9. **return** R

Our implementation

`Ccluster`: library in `C` based on

- FLINT[1]: arithmetic for the geometric algorithm
- Arb[2]: arbitrary precision floating arithmetic with error bounds

Available at `https://github.com/rimbach/Ccluster`

`Ccluster.jl`: package for julia[3] based on $\mathbb{N}e^{m}\mathcal{O}$[4]

- interface for `Ccluster`
- `tcluster`: implemetation of ClusterTri

Available at `https://github.com/rimbach/Ccluster.jl`

[1] `https://github.com/wbhart/flint2`
[2] `http://arblib.org/`
[3] `https://julialang.org/`
[4] `http://nemocas.org/`

Benchmark: systems

Type of a triangular system:
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense sys.
seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine.

type								
Systems with only simple solutions								
(9,9,9)								
(6,6,6,6)								
(9,9,9,9)								
(6,6,6,6,6)								
(9,9,9,9,9)								
(2,2,2,2,2,2,2,2,2,2)								
Systems with multiple solutions								
(9,9)								
(6,6,6)								
(9,9,9)								
(6,6,6,6)								

Benchmark: local vs global comparison

Type of a triangular system:
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense sys.
seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine.

	tcluster **local**		tcluster **global**					
type	(#Sols:#Clus)	t (s)	(#Sols:#Clus)	t (s)				
Systems with only simple solutions								
(9,9,9)	(149 : 149)	0.24	(729 : 729)	1.43				
(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	2.21				
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	14.6				
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	13.8				
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	130				
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.92				
Systems with multiple solutions								
(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.17				
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.26				
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.10				
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.29				

`tcluster` **local** : $\boldsymbol{B} = (\ [-1,1] + \boldsymbol{i}[-1,1] \qquad\qquad , \ldots), \ \epsilon = 2^{-53}$
`tcluster` **global**: $\boldsymbol{B} = (([-5,5] + \boldsymbol{i}[-5,5]) \times 10^5, \ldots), \ \epsilon = 2^{-53}$

Benchmark: extern comparison

Type of a triangular system:

$\boldsymbol{f}(\boldsymbol{z}) = \mathbf{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense sys.

seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine.

	`tcluster` **local**		`tcluster` **global**		`HomCont.jl`			
type	(#Sols:#Clus)	t (s)	(#Sols:#Clus)	t (s)	#Sols	t (s)		
Systems with only simple solutions								
(9,9,9)	(149 : 149)	0.24	(729 : 729)	1.43	729	4.21		
(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	2.21	1296	4.70		
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	14.6	6561	14.0		
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	13.8	7776	11.5		
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	130	59049	116		
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.92	1024	4.84		
Systems with multiple solutions								
(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.17	33.6	3.27		
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.26	53.2	2.75		
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.10	159	28.4		
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.29	134	8.06		

`tcluster` **local** : $\boldsymbol{B} = (\ [-1,1] + \boldsymbol{i}[-1,1] \qquad , \ldots), \ \epsilon = 2^{-53}$

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`HomCont.jl`: `HomotopyContinuation.jl`

Benchmark:

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(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	2.21	1296	4.70		
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	14.6	6561	14.0		
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	13.8	7776	11.5		
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	130	59049	116		
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.92	1024	4.84		
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(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.17	33.6	3.27		
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.26	53.2	2.75		
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.10	159	28.4		
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.29	134	8.06		

`tcluster` **local** : $\boldsymbol{B} = (\; [-1,1] + \boldsymbol{i}[-1,1] \qquad , \ldots), \; \epsilon = 2^{-53}$

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`HomCont.jl`: `HomotopyContinuation.jl`

Benchmark:

Type of a triangular system:
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense sys.
seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine.

	tcluster **local**		tcluster **global**		HomCont.jl		triang_solve	
type	(#Sols:#Clus)	t (s)	(#Sols:#Clus)	t (s)	#Sols	t (s)	#Sols	t (s)
Systems with only simple solutions								
(9,9,9)	(149 : 149)	0.24	(729 : 729)	1.43	729	4.21	729	0.37
(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	2.21	1296	4.70	1296	0.93
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	14.6	6561	14.0	6561	8.57
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	13.8	7776	11.5	7776	19.1
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	130	59049	116	59049	702
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.92	1024	4.84	1024	3.9
Systems with multiple solutions								
(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.17	33.6	3.27	45	0.03
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.26	53.2	2.75	54	0.05
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.10	159	28.4	225	0.23
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.29	134	8.06	162	0.15

`tcluster` **local** : $\boldsymbol{B} = (\ [-1,1] + \boldsymbol{i}[-1,1] \qquad , \ldots), \ \epsilon = 2^{-53}$
`tcluster` **global**: $\boldsymbol{B} = (([-5,5] + \boldsymbol{i}[-5,5]) \times 10^5, \ldots), \ \epsilon = 2^{-53}$
`HomCont.jl`: `HomotopyContinuation.jl`
`triang_solve`: `Singular solver for triangular systems`

Conclusion and future works

Contributions in MCS paper:

- definition of the Local complex solutions Clustering Problem
- algorithm for solving the LCP for triangular systems
- termination based on error analysis of partial substitution
- numerical experiments

Implementation:

- available for julia
- efficient

Future work:

- complexity analysis

Thank you for your attention!