A Certified Numerical Approach to Describe the Topology of Projected Curves

Rémi Imbach, Guillaume Moroz and Marc Pouget

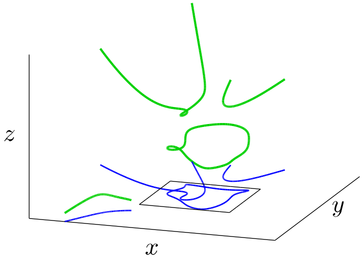
Projection and Apparent Contour

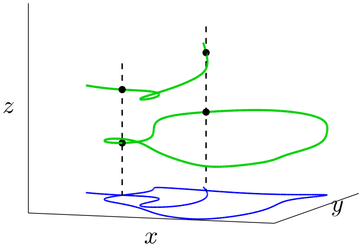
P, Q two polynomial maps $\mathbb{R}^3 \to \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) &= 0 \\ Q(x,y,z) &= 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\pi_{(x,y)}(\mathcal{C})$





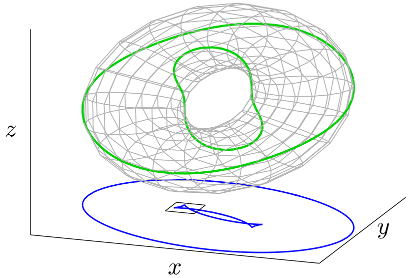
Projection and Apparent Contour

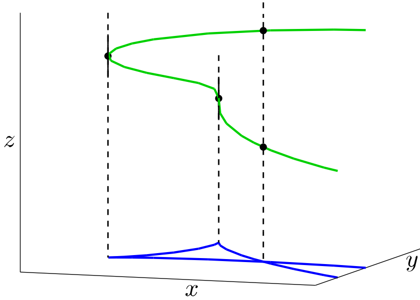
P, Q two polynomial maps $\mathbb{R}^3 \to \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) & = 0 \\ P_z(x,y,z) & = 0 \end{cases}, (x,y,z) \in \mathbb{R}^3, \qquad P_z = \frac{\partial P}{\partial z}$$

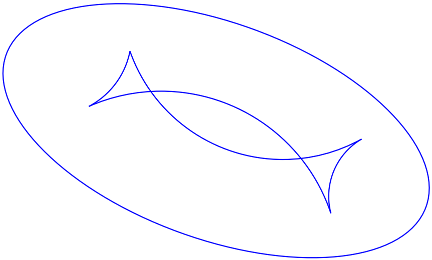
Apparent contour: $\pi_{(x,y)}(\mathcal{C})$

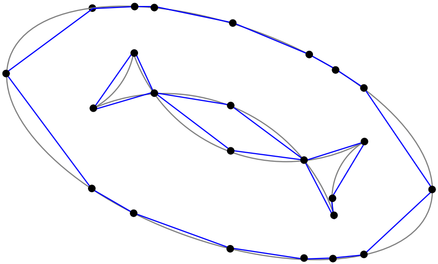




Computing topology of a real plane curve $\mathcal{B}$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$





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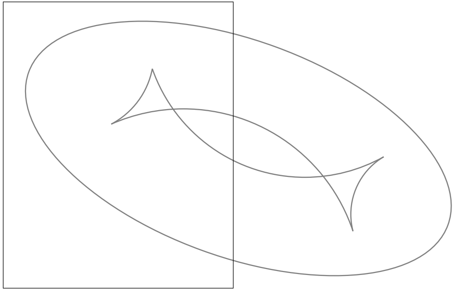

A general framework

0. Restrict to a compact $\mathbf{B}_0$
1. Isolate in boxes:
   - boundary points
   - x-critical points
   - singularities
2. Compute topology around singularities
3. Connect boxes

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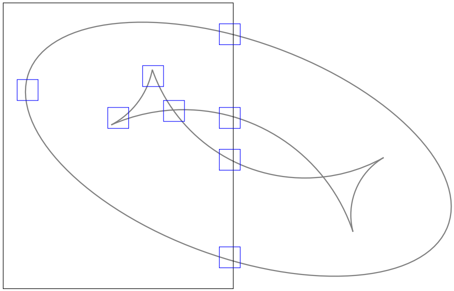

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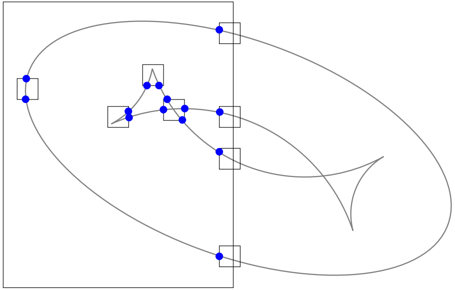

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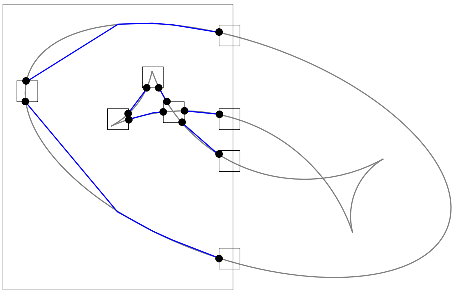

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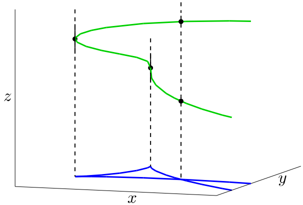
When $\mathcal{B}$ is a projection or an apparent contour



Geometric characterization of nodes and cusps:

- 4D square system
- 0-dim solver

❶ Isolate in boxes:

- boundary points
- x-critical points
- singularities

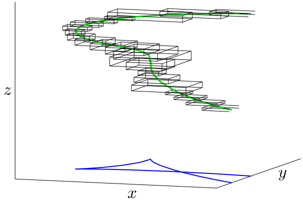
Certified numerical tools:

- 0-dim solver: subdivision

When $\mathcal{B}$ is a projection or an apparent contour

Enclosing $\mathcal{C}$ in a sequence of boxes:

- 1-dim solver
- 1 point on each C.C.: 0-dim solver



Geometric characterization of nodes and cusps:

- 4D square system
- 0-dim solver
- Restriction of the solving domain

❶ Isolate in boxes:

- boundary points
- x-critical points
- singularities

Certified numerical tools:

- 0-dim solver: subdivision
- 1-dim solver: path tracker

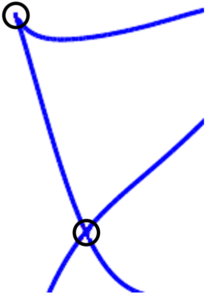
Isolating singularities

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\},$$

Singularities of $\mathcal{B}$ are the solutions of:

$$\left\{ \begin{array}{l} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \\ \frac{\partial r}{\partial y}(x,y) = 0 \end{array} \right.$$

...that is over-determined
...that has solutions of multiplicity 2



symbolic approaches: Gröbner Basis, RUR

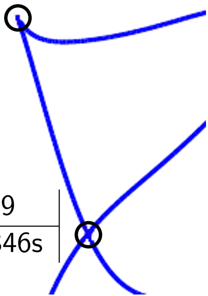
Isolating singularities of apparent contours

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the solutions of:

$$\begin{cases} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \\ \frac{\partial r}{\partial y}(x,y) = 0 \end{cases}$$

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$,	degree 29,	bit-size 115,	465 monomials



symbolic approaches: Gröbner Basis, RUR

degree of P	5	6	7	8	9
time with RSCube*	3.1s	32s	254s	1898s	9346s

* F. Rouillier

Isolating singularities of apparent contours

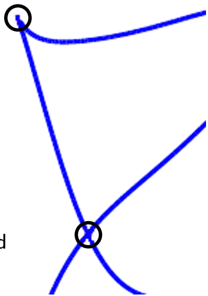
$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_2) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \end{cases} \quad \text{s.t. } s_{22}(x, y) \neq 0$$

... where s_{10}, s_{11}, s_{22} are coefficients in the subresultant chain.

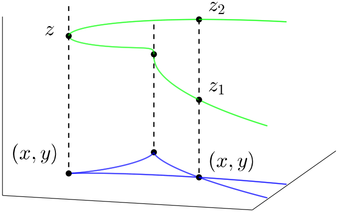
P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$,	degree 29,	bit-size 115,	465 monomials
s_{11}, s_{10},	degree 20,	bit-size 89,	231 monomials
s_{22},	degree 12,	bit-size 65,	91 monomials



[IMP15a] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Numeric certified algorithm for the topology of resultant and discriminant curves.
Research Report RR-8653, Inria, April 2015.

Isolating singularities:

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



Lemma 1: (x, y) is a node of $\mathcal{B} \Leftrightarrow (x, y, z_1, z_2)$ satisfies:

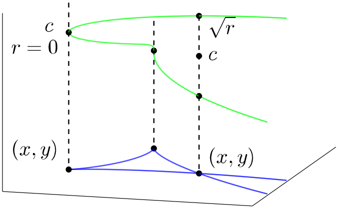
$$P(x, y, z_1) = Q(x, y, z_1) = P(x, y, z_2) = Q(x, y, z_2) = 0$$

Lemma 2: (x, y) is a cusp of $\mathcal{B} \Leftrightarrow (x, y, z)$ satisfies:

$$P(x, y, z) = Q(x, y, z) = P_z(x, y, z) = Q_z(x, y, z) = 0$$

Isolating singularities: the Ball system

$$\mathcal{B} = \{(x,y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x,y,z) \in \mathcal{C}\}$$



c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4)\left\{\begin{array}{rl} \frac{1}{2}(P(x,y,c+\sqrt{r}) + P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r}) - P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r}) + Q(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r}) - Q(x,y,c-\sqrt{r})) & = 0 \end{array}\right.$$

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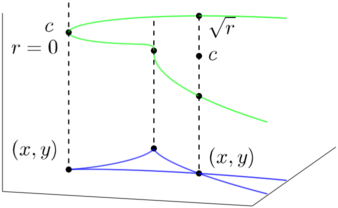


c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

when $r \to 0$

$$(\mathcal{S}_4)\left\{\begin{array}{rl} P(x,y,c) & = 0 \\ P_z(x,y,c) & = 0 \\ Q(x,y,c) & = 0 \\ Q_z(x,y,c) & = 0 \end{array}\right.$$

Isolating singularities: the Ball system

[IMP15b] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Numeric and certified isolation of the singularities of the projection of a smooth space curve.
In *Proceedings of the 6th International Conferences on Mathematical Aspects of Computer and Information Sciences*, MACIS'15, 2015.

Lemma 4. Under some genericity assumptions, all the solutions of $\mathcal{S}_4$ in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+$ are regular.

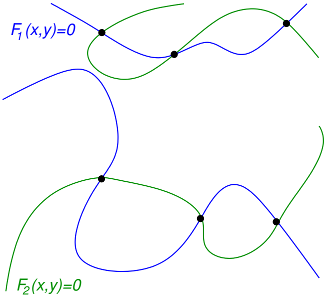
Lemma 3. Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4) \begin{cases} \frac{1}{2}(P(x,y,c+\sqrt{r}) + P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r}) - P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r}) + Q(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r}) - Q(x,y,c-\sqrt{r})) & = 0 \end{cases}$$

Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \to \mathbb{R}^n$, F polynomial,

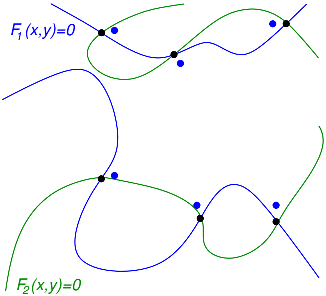
- find zeros of F: find $\{X \in \mathbb{R}^n | F(X) = 0\}$



Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, F polynomial,

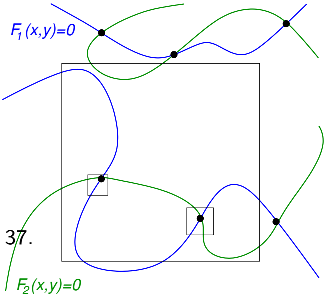
- find zeros of F: find $\{X \in \mathbb{R}^n | F(X) = 0\} \rightsquigarrow \{X \in \mathbb{R}^n | \|F(X)\| \leq \epsilon\}$



Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \to \mathbb{R}^n$, F polynomial, $\mathbf{X}_0$ a compact of $\mathbb{R}^n$

- find zeros of F: find $\{X \in \mathbb{R}^n | F(X) = 0\}$
- Isolate zeros of F in boxes $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ such that
  - each $\mathbf{X}_k$ contains a unique zero of F
  - each zero of F in $\mathbf{X}_0$ is in a unique box $\mathbf{X}_k$



[Neu90] Arnold Neumaier.
Interval methods for systems of equations, volume 37.
Cambridge university press, 1990.

Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \to \mathbb{R}^n$, F polynomial, $\mathbf{X}_0$ a compact of $\mathbb{R}^n$

Interval Arithmetic: $\mathbf{x} \subset \mathbb{R}$,$\mathbf{X} \subset \mathbb{R}^n$

- multi-dimensional extension of interval : box $\mathbf{X} \subset \mathbb{R}^n$

$$\mathbf{X} = \mathbf{x_1} \times \ldots \times \mathbf{x_n} = [l(x_1), r(x_1)] \times \ldots \times [l(x_n), r(x_n)]$$

[Neu90] Arnold Neumaier.
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Interval Arithmetic: $\mathbf{x} \subset \mathbb{R}$,$\mathbf{X} \subset \mathbb{R}^n$

- multi-dimensional extension of interval : box $\mathbf{X} \subset \mathbb{R}^n$
- interval arithmetic operators

$$\mathbf{x} = [l(x), r(x)], \mathbf{y} = [l(y), r(y)], \mathbf{x}+\mathbf{y} = [l(x)+l(y), r(x)+r(y)]$$

[Neu90] Arnold Neumaier.
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Certified numerical tools: 0-dim solver

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Interval Arithmetic: $\mathbf{x} \subset \mathbb{R}$,$\mathbf{X} \subset \mathbb{R}^n$,$F(\mathbf{X}) \supseteq \{F(X) | X \in \mathbf{X}\}$

- multi-dimensional extension of interval : box $\mathbf{X} \subset \mathbb{R}^n$
- interval arithmetic operators
- interval evaluation of $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$: $F(\mathbf{X}) \supseteq \{F(X) | X \in \mathbf{X}\}$

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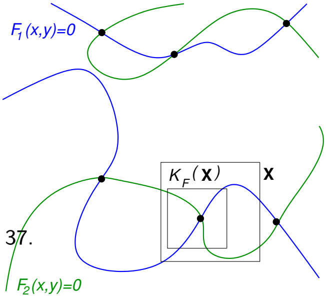
Krawczik criterion: $K_F : \mathbf{X} \subset \mathbb{R}^n \mapsto K_F(\mathbf{X}) \subset \mathbb{R}^n$

$K_F(\mathbf{X}) \subset Int(\mathbf{X}) \Rightarrow K_F(\mathbf{X})$ contains a unique zero of F

$K_F(\mathbf{X}) \cap \mathbf{X} = \emptyset \;\Rightarrow \mathbf{X}$ contains no zero of F

consequence of the Brouwer fixed point theorem.



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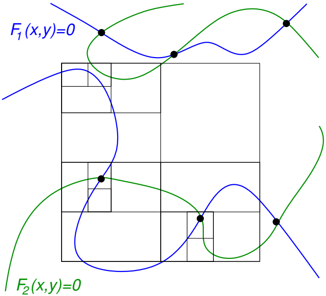
Subdivision method:

Input: $F : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{X}_0$ box of $\mathbb{R}^n$
Output: A list R of boxes containing solutions in $\mathbf{X}_0$ of $F = 0$
$L := \{\mathbf{X}_0\}$
Repeat:
  $\mathbf{X} := L.pop$
  If $0 \in F(\mathbf{X})$ **then**
    If $K_F(\mathbf{X}) \subset Int(\mathbf{X})$ **then**
      insert $\mathbf{X}$ in R
    Else If $K_F(\mathbf{X}) \cap \mathbf{X} \neq \emptyset$ **then**
      bisect $\mathbf{X}$ and insert its sub-boxes in L
    End if
  End if
Until $L = \emptyset$
Return R



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Krawczik criterion: $K_F : \mathbf{X} \subset \mathbb{R}^n \mapsto K_F(\mathbf{X}) \subset \mathbb{R}^n$

Subdivision method:

- terminates with a correct result when
  - $F = 0$ has only regular solutions,
  - working at arbitrary precision.
- can be extended to unbounded initial box $\mathbf{X}_0$
- its cost grows exponentially with n

[Neu90] Arnold Neumaier.
Interval methods for systems of equations, volume 37.
Cambridge university press, 1990.

Certified numerical isolation of singularities

Datas: Random dense polynomials of degree d, bit-size 8

Subdivision solver: home made in `C++`, with `boost` interval library

- evaluation of polynomials with horner scheme $\rightarrow$ quick
- evaluation of polynomials at order 2 $\rightarrow$ sharp

Numerical results: Subdivision solving within $[-1,1] \times [-1,1]$

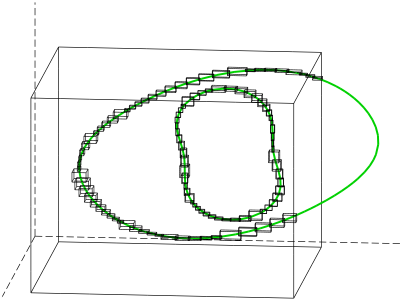
	Sub-resultant system $\mathcal{S}_2$	Ball system $\mathcal{S}_4$
d	t	t
5	0.05	24.8
6	0.50	8.40
7	4.44	43.8
8	37.9	70.2
9	23.1	45.6

means on 5 examples of sequential times in seconds on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine.

Restriction of the solving domain

Enclose $\mathcal{C}$: find a sequence $\{\mathbf{C}_k\}_{1\leq k\leq l}$ such that

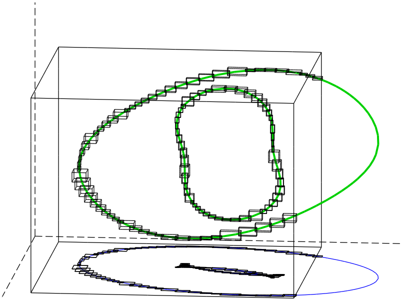
- $\mathcal{C} \subset \bigcup_k \mathbf{C}_k$,
- in each $\mathbf{C}_k$, $\mathcal{C} \cap \mathbf{C}_k$ is diffeomorphic to a close segment,
- each $\mathbf{C}_k$ has width less than η.



Restriction of the solving domain

Enclose $\mathcal{C}$: find a sequence $\{\mathbf{C}_k\}_{1\leq k\leq l}$ $\mathbf{C}_k = (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)$

$\rightarrow$ Enclose $\mathcal{B}$: each $B \in \mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$



Restriction of the solving domain

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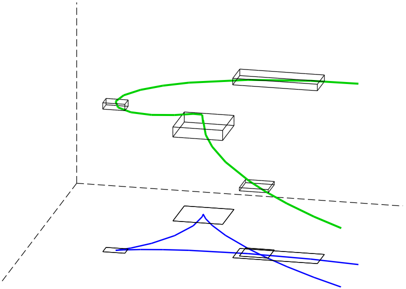
$\rightarrow$ Enclose $\mathcal{B}$: each $B \in \mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$ $\mathbf{B}_k = (\mathbf{x}_k, \mathbf{y}_k)$

$\rightarrow$ Enclose singularities:

- each cusp is in a $\mathbf{B}_k$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$



Restriction of the solving domain

Enclose $\mathcal{C}$: find a sequence $\{\mathbf{C}_k\}_{1\leq k\leq l}$ $\qquad \mathbf{C}_k = (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)$

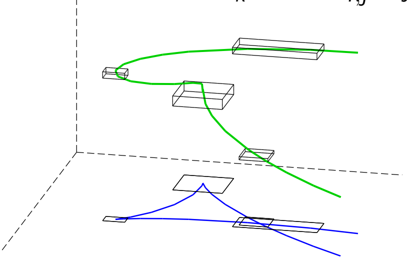
$\rightarrow$ Enclose $\mathcal{B}$: each $B \in \mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$ $\qquad \mathbf{B}_k = (\mathbf{x}_k, \mathbf{y}_k)$

$\rightarrow$ Enclose singularities:

- each cusp is in a $\mathbf{B}_k$ $\qquad \mathbf{D}_k = (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k, [0, (\frac{w(\mathbf{z}_k)}{2})^2])$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$ $\qquad \mathbf{D}_{ij} = (\mathbf{x}_{ij}, \mathbf{y}_{ij}, \frac{(\mathbf{z}_i + \mathbf{z}_j)}{2}, [0, (\frac{(\mathbf{z}_i - \mathbf{z}_j)}{2})^2])$

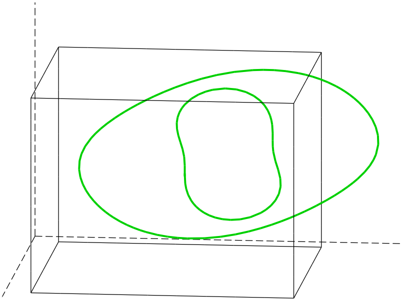
$\rightarrow$ Enclose solutions of the ball system:

Solutions of the ball system are in $\bigcup_k \mathbf{D}_k \cup \bigcup_{i,j} \mathbf{D}_{ij}$



Certified numerical tools: path tracker

$F : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\mathbf{X}_0$ a box of $\mathbb{R}^n$
$\mathcal{X} = \{X \in \mathbf{X}_0 | F(X) = 0\}$ is a smooth curve of $\mathbb{R}^n$
$\mathcal{X}^1, \ldots, \mathcal{X}^m$: connected components of $\mathcal{X}$



Certified numerical tools: path tracker

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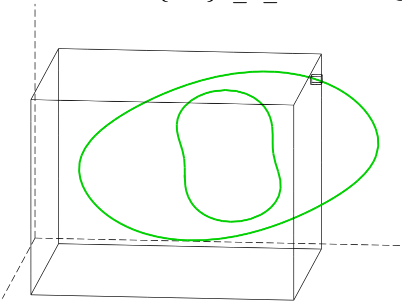
$\mathcal{X}^1, \ldots, \mathcal{X}^m$: connected components of $\mathcal{X}$

Certified path-tracker:

Input: $F : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\mathbf{X}_0$ box of $\mathbb{R}^n$, $\eta \in \mathbb{R}^+_*$
An initial box $\mathbf{X} \in \mathcal{X}^i$

Output: a sequence of boxes $\{\mathbf{X}_k\}_{1 \leq k \leq l}$ enclosing $\mathcal{X}^i$.



Certified numerical tools: path tracker

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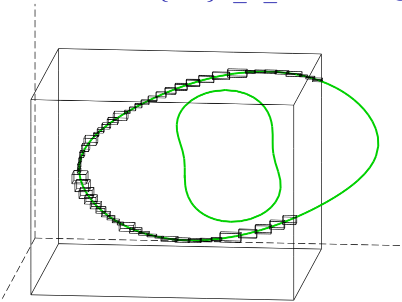
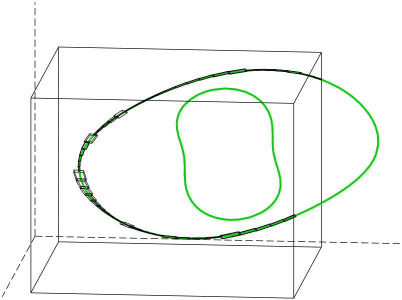


Certified numerical tools: path tracker

[MGGJ13] Benjamin Martin, Alexandre Goldsztejn, Laurent Granvilliers, and Christophe Jermann.
Certified parallelotope continuation for one-manifolds.
SIAM Journal on Numerical Analysis, 51(6):3373–3401, 2013.

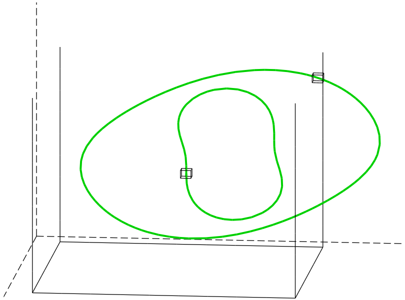


Enclosing $\mathcal{C}$

$F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{B}_0$ a box of $\mathbb{R}^2$
$\mathcal{C} = \{C \in \mathbf{B}_0 \times \mathbb{R} | F(X) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \dots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$
(A1) holds for generic polynomials P, Q

Finding one point on each connected component

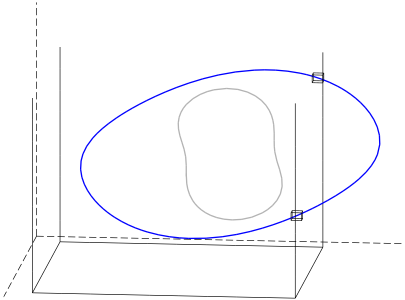


Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0, 1]$
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points



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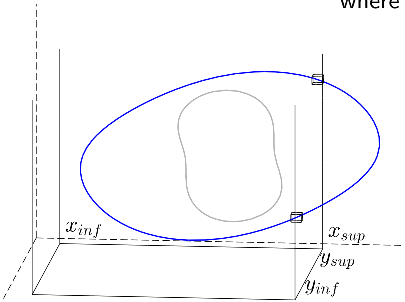
$\mathcal{C} \cap (\partial\mathbf{B}_0 \times \mathbb{R})$ are the solutions of the 4 systems:

$$\begin{cases} P(x = a, y, z) = 0 \\ Q(x = a, y, z) = 0 \end{cases}$$

$$\begin{cases} P(x, y = b, z) = 0 \\ Q(x, y = b, z) = 0 \end{cases}$$

where $a \in \{x_{inf}, x_{sup}\}$,
$b \in \{y_{inf}, y_{sup}\}$



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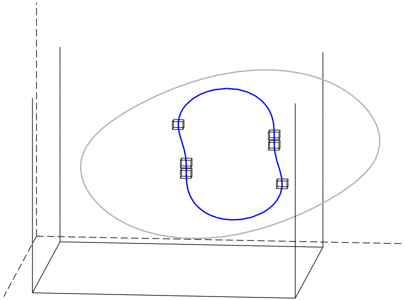

Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

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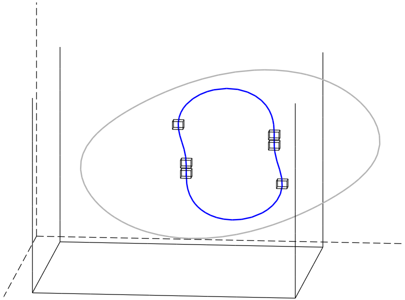
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points

x-critical points of $\mathcal{C}$ are the solutions of the system:

$$\begin{cases} Px,y,z) & = 0 \\ Q(x,y,z) & = 0 \\ \begin{vmatrix} P_y & P_z \\ Q_y & Q_z \end{vmatrix}(x,y,z) & = 0 \end{cases}$$



Certified numerical isolation of singularities

Path tracker: prototype in `python/cython`

Numerical results: solving within $[-1, 1] \times [-1, 1]$

	Sub-resultant system $\mathcal{S}_2$	Ball system $\mathcal{S}_4$	$\mathcal{S}_4$ with curve tracking
d	t	t	t
5	0.05	24.8	1.25
6	0.50	8.40	2.36
7	4.44	43.8	4.13
8	37.9	70.2	5.91
9	23.1	45.6	5.30

means on 5 examples of sequential times in seconds on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine.

Perspectives

- Projections of curves of $\mathbb{R}^n$, with $n > 3$
- Projections of surfaces

Questions?