Numerical and certified computation of the topology of projected curves

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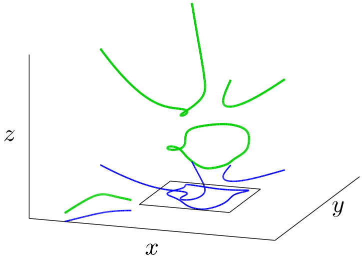
Projection and Apparent Contour

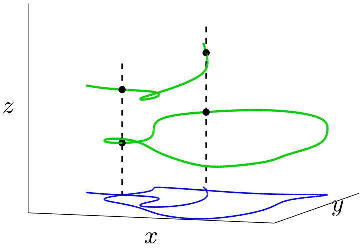
P, Q two (polynomial or analytic) maps $\mathbb{R}^3 \rightarrow \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) & = 0 \\ Q(x,y,z) & = 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\mathcal{B} = \pi_{(x,y)}(\mathcal{C})$





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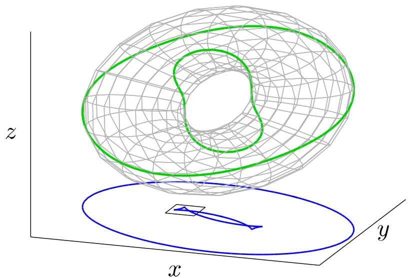
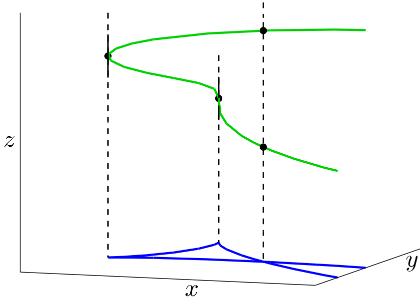
P, Q two (polynomial or analytic) maps $\mathbb{R}^3 \to \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) & = 0 \\ P_z(x,y,z) & = 0 \end{cases}, (x,y,z) \in \mathbb{R}^3, \qquad P_z = \frac{\partial P}{\partial z}$$

Apparent contour: $\mathcal{B} = \pi_{(x,y)}(\mathcal{C})$



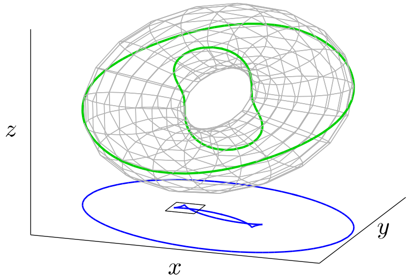


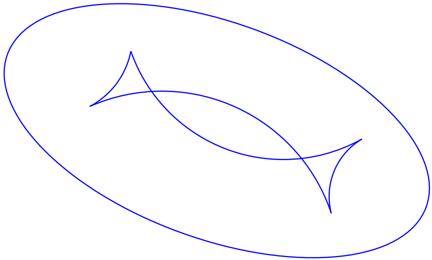
Computing the topology of the projected curve $\mathcal{B}$

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 | P(x, y, z) = Q(x, y, z) = 0\}$$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Our problem: given P, Q (with possibly $Q = P_z$),
compute a geometric approximation of $\mathcal{B}$
compute the topology of $\mathcal{B}$





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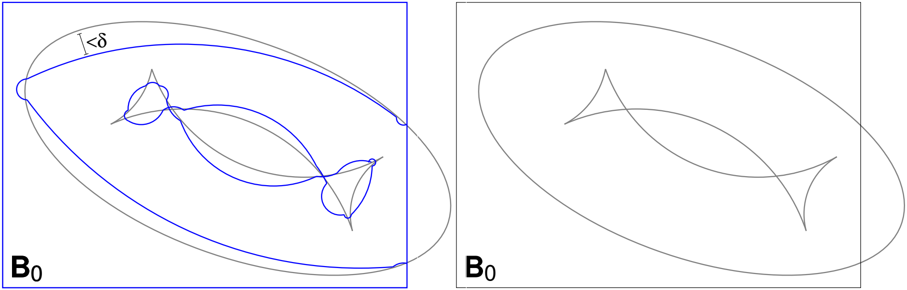

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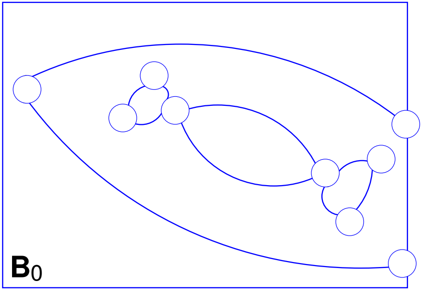
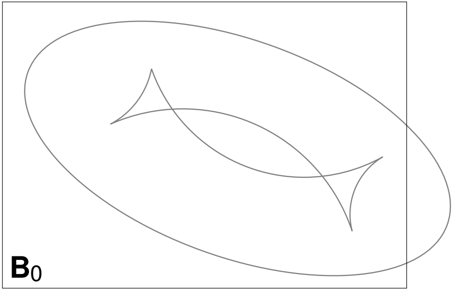


faces of $\mathcal{B}$: C.C. of $\mathbf{B}_0 \setminus \mathcal{B}$



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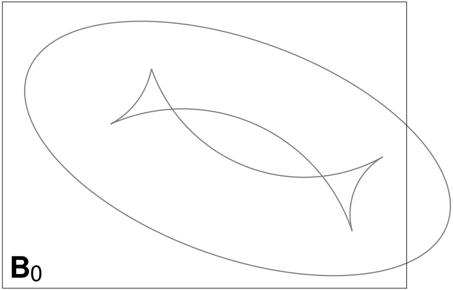
A general framework

0. Restrict to a box $\mathbf{B}_0$
1. Isolate special points:
    - boundary points
    - x-critical points
    - singularities
2. Loc. topology around special points
3. Connect special points
4. Embed the graph



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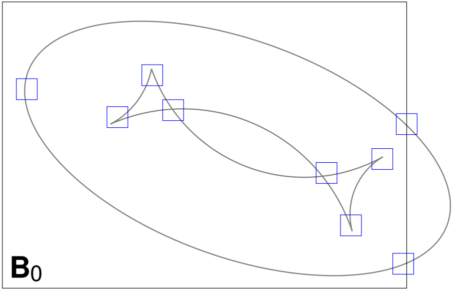

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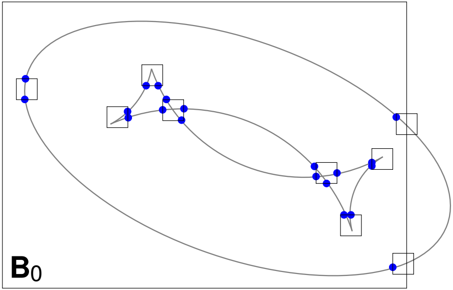

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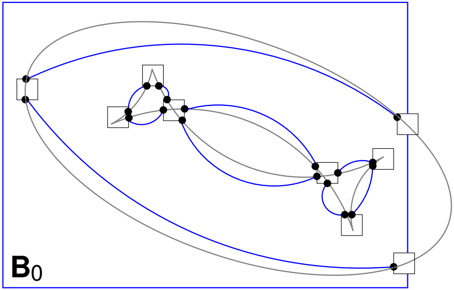

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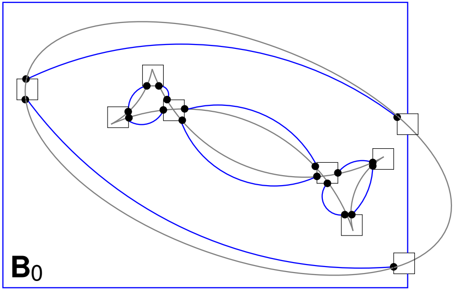

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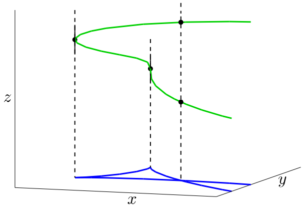

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Characterization of nodes and cusps:

- Resultant approaches
- Geometric approach

Certified numerical tools:

- 0-dim solver: subdivision solver

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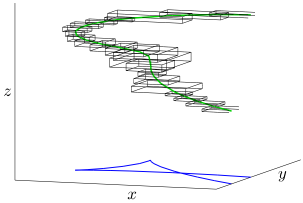

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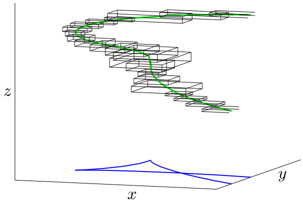

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Isolating singularities

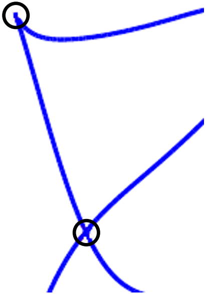
$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\},$$

Singularities of $\mathcal{B}$ are the solutions of:

$$(\mathcal{S}) \begin{cases} r(x, y) = 0 \\ r_x(x, y) = 0 \\ r_y(x, y) = 0 \end{cases}$$

...that is over-determined

...that has solutions of multiplicity 2



symbolic approaches: Gröbner Basis, RUR

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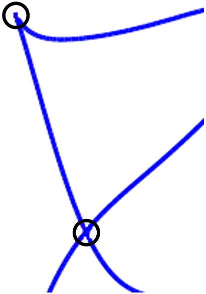

symbolic approaches: Gröbner Basis, RUR

Example

P, degree 6, bit-size 8, 84 monomials

$$P = 158x^6 - 186x^5y + 205x^5z - 160x^4y^2 + 105x^4yz + 116x^4z^2 - 69x^3y^3 - 161x^3y^2z - 8x^3yz^2 + 107x^3z^3 + 144x^2y^4 - 193x^2y^3z + 130x^2y^2z^2 + x^2yz^3 + 47x^2z^4 + 165xy^5 - 220xy^4z - 21xy^3z^2 + 50xy^2z^3 - 130xyz^4 - 77xz^5 + 66y^6 - 55y^5z + 219y^4z^2 - 30y^3z^3 - 162y^2z^4 - 182yz^5 - 145z^6 + 105x^5 + 241x^4y - 177x^4z - 127x^3y^2 - 97x^3yz + 223x^3z^2 - 46x^2y^3 - 213x^2y^2z + 39x^2yz^2 + 191x^2z^3 + 65xy^4 - 105xy^3z - 248xy^2z^2 + 158xyz^3 - 183xz^4 + 48y^5 - 240y^4z + 235y^3z^2 + 194y^2z^3 - 45yz^4 + 159z^5 - 81x^4 - 230x^3y - 247x^3z - 38x^2y^2 + 106x^2yz + 184x^2z^2 + 49xy^3 - 197xy^2z - 182xyz^2 - 223xz^3 - 205y^4 - 225y^3z - 14y^2z^2 - 17yz^3 + 73z^4 - 234x^3 - 82x^2y + 179x^2z + 46xy^2 - 222xyz - 95xz^2 + 139y^3 + 168y^2z + 8yz^2 + 156z^3 + 159x^2 - 147xy - 22xz - 104y^2 + 181yz + 26z^2 - 90x + 250y + 19z + 19$$

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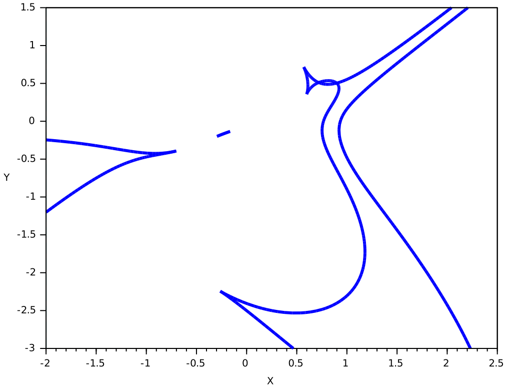

Example

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials

$$Res(P, P_z, z) = 2537851751382193098537472618 5\,x^{30} - 1950289566984849821762662644 60\,x^{29}y + 669460660893860813921604554100\,x^{28}y^2 - 631323116304152251056202148000\,x^{27}y^3 - 1028704563680432990245022354280\,x^{26}y^4 + 4597797015605117908624008082 0\,x^{25}y^5 + 3554469553406371293751987742270\,x^{24}y^6 + 3711031010928440039666656612920\,x^{23}y^7 - 5634442800184514383998916600260\,x^{22}y^8 - 11658591855069381144706595841060\,x^{21}y^9 - 4387874939266072948066332459470\,x^{20}y^{10} + 16408843461038228420223023180230\,x^{19}y^{11} + 23700165794251777062304009772915\,x^{18}y^{12} + 43163241809977488659018002016 20\,x^{17}y^{13} - 24929137305247653219088728498740\,x^{16}y^{14} - 33372908351021778030492119654810\,x^{15}y^{15} - 9633448028150975870147511674570\,x^{14}y^{16} + 20500155431790235158403374001190\,x^{13}y^{17} + 31668089060759309350684716458350\,x^{12}y^{18} + 16544278550218652616250018398520\,x^{11}y^{19} - 5014730522275651771719575652535\,x^{10}y^{20} - 16590111614945163714073974823320\,x^{9}y^{21} - 13546083341149182083464535866425\,x^{8}y^{22} - 4754759946941791724566012110130\,x^{7}y^{23} + 1997218841447789708607357568825\,x^{6}y^{24} + 3898998021968250822246999603270\,x^{5}y^{25} +$$

Isolating singularities of apparent contours

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the solutions of:

$$(\mathcal{S}) \begin{cases} r(x, y) = 0 \\ r_x(x, y) = 0 \\ r_y(x, y) = 0 \end{cases}$$

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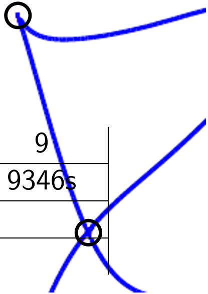


symbolic approaches: Gröbner Basis, RUR

degree of P	6	7	8	9
$(\mathcal{S})$ with `RSCube`*	32s	254s	1898s	9346s

* F. Rouillier

Sub-resultant based deflation system

(α, β) node of $\mathcal{B}$:

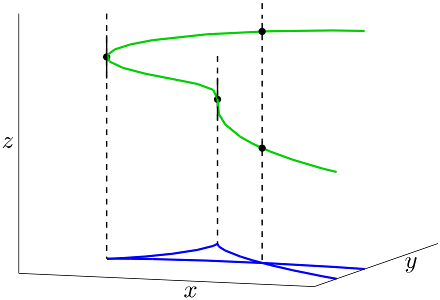
$\leftrightsquigarrow P(\alpha, \beta, z_1) = P_z(\alpha, \beta, z_1) = P(\alpha, \beta, z_2) = P_z(\alpha, \beta, z_2) = 0$

$\leftrightsquigarrow P(\alpha, \beta, z)$ and $P_z(\alpha, \beta, z)$ have two common roots z_1, z_2

(α, β) cusp of $\mathcal{B}$:

$\leftrightsquigarrow P(\alpha, \beta, z_1) = P_z(\alpha, \beta, z_1) = P_{zz}(\alpha, \beta, z_1) = 0$

$\leftrightsquigarrow P(\alpha, \beta, z)$ and $P_z(\alpha, \beta, z)$ have a common double root z_1



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$P(\alpha,\beta,z), P_z(\alpha,\beta,z)$ have a *gcd* of degree 2

Sub-resultant based deflation system

Sub-resultant chain of P, P_z, z:

$$
\begin{array}{lcccccc}
S^0 = & & & & & Res(P, P_z, z)(x,y) = r(x,y) \\
S^1 = & & & s_{11}(x,y)z & + & s_{10}(x,y) \\
S^2 = & s_{22}(x,y)z^2 & + & s_{21}(x,y)z & + & s_{20}(x,y) \\
\ldots = & & & \ldots & &
\end{array}
$$

where $s_{l,k} = det(A)$, $A \in \mathcal{M}_{(m+n-l)\times(m+n-l)}(\mathbb{Q}[x,y])$

Proposition $P(\alpha,\beta,z), P_z(\alpha,\beta,z)$ have a *gcd* of degree 2 iff $r(\alpha,\beta) = s_{11}(\alpha,\beta) = s_{10}(\alpha,\beta) = 0$ and $s_{22}(\alpha,\beta) \neq 0$.

Genericity assum.: $P(\alpha,\beta,z), P_z(\alpha,\beta,z)$ have a *gcd* of degree ≤ 2 then $s_{22}(x,y) \neq 0 \Leftrightarrow r(x,y) = 0$.

Sub-resultant based deflation system

$$(\mathcal{S}_2)\begin{cases} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \end{cases} \text{ s.t. } \begin{array}{l} s_{22}(x,y)\neq 0 \\ (\Leftrightarrow r(x,y)=0) \end{array}$$

Proposition $P(\alpha,\beta,z), P_z(\alpha,\beta,z)$ have a *gcd* of degree 2 iff $r(\alpha,\beta)=s_{11}(\alpha,\beta)=s_{10}(\alpha,\beta)=0$ and $s_{22}(\alpha,\beta)\neq 0$.

Genericity assum.: $P(\alpha,\beta,z), P_z(\alpha,\beta,z)$ have a *gcd* of degree ≤ 2 then $s_{22}(x,y)\neq 0 \Leftrightarrow r(x,y)=0$.

Isolating singularities of apparent contours

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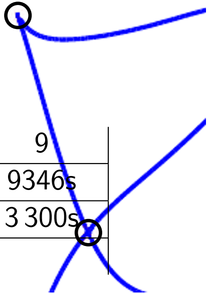
Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_2) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \end{cases} \text{s.t.} \quad \begin{array}{l} s_{22}(x, y) \neq 0 \\ (\Leftrightarrow r(x, y) = 0) \end{array}$$

... where s_{10}, s_{11}, s_{22} are coefficients in the subresultant chain.

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
s_{11}, s_{10},	degree 20,	bit-size 90,	231 monomials



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$(\mathcal{S}_2)$ with RSCube	15s	105s	620s	3 300s

* F. Rouillier

Isolating singularities of apparent contours

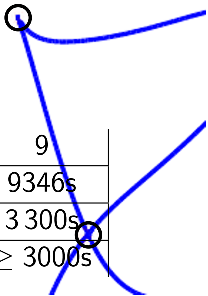
$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_2) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \end{cases} \text{s.t.} \quad \begin{array}{l} s_{22}(x, y) \neq 0 \\ (\Leftrightarrow r(x, y) = 0) \end{array}$$

. . . where s_{10}, s_{11}, s_{22} are coefficients in the subresultant chain.

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
s_{11}, s_{10},	degree 20,	bit-size 90,	231 monomials

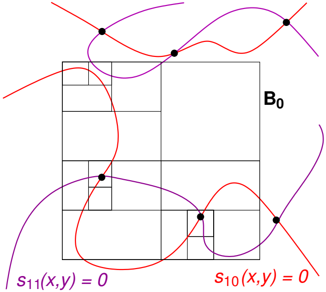


degree of P	6	7	8	9
$(\mathcal{S})$ with RSCube*	32s	254s	1898s	9346s
$(\mathcal{S}_2)$ with RSCube	15s	105s	620s	3 300s
$(\mathcal{S}_2)$ with Bertini	1005s	$\geq$ 3000s	$\geq$ 3000s	$\geq$ 3000s

* F. Rouillier

A subdivision solver for systems of large polynomials

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (s_{10}(x, y), s_{11}(x, y))$$
find (x, y) s.t. $F(x, y) = 0$



[Neu90] A. Neumaier.
Interval methods for systems of equations.
Cambridge University Press, 1990.

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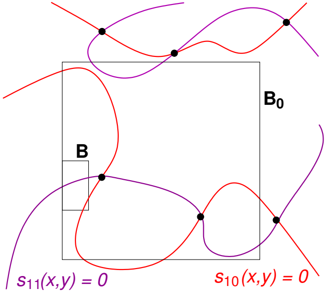
Interval extension $\square F$ of F:

$0 \notin \square F(\mathbf{B}) \Rightarrow$ no solution in $\mathbf{B}$



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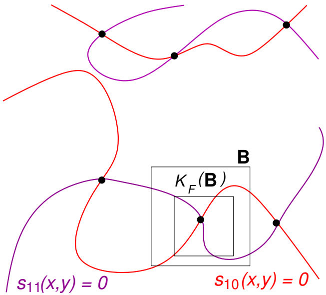

Interval extension $\Box F$ of F:

$0 \notin \Box F(\mathbf{B}) \Rightarrow$ no solution in $\mathbf{B}$

Interval newton operators $K_F(\mathbf{B})$:

$K_F(\mathbf{B}) \subset int(\mathbf{B}) \Rightarrow \exists!$ solution in $\mathbf{B}$

$\mathbf{B}_i = K_F(\mathbf{B}_{i-1})$ converges quadratically

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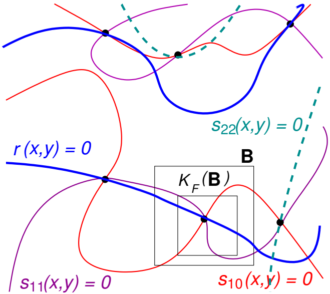
A subdivision solver for systems of large polynomials

$F : \mathbb{R}^2 \to \mathbb{R}^2$
$(x, y) \mapsto (s_{10}(x, y), s_{11}(x, y))$
find (x, y) s.t. $F(x, y) = 0$
and $s_{22}(x, y) \neq 0$



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$\mathbf{B}_i = K_F(\mathbf{B}_{i-1})$ converges quadratically

Constraints: at $F(x, y) = 0$,
either $s_{22}(x, y) \neq 0$ or $r(x, y) \neq 0$

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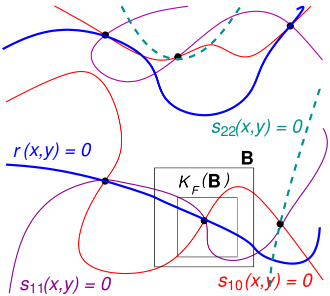


Interval extension $\square F$ of F:
$0 \notin \square F(\mathbf{B}) \Rightarrow$ no solution in $\mathbf{B}$

Interval newton operators $K_F(\mathbf{B})$:
$K_F(\mathbf{B}) \subset int(\mathbf{B}) \Rightarrow \exists!$ solution in $\mathbf{B}$
$\mathbf{B}_i = K_F(\mathbf{B}_{i-1})$ converges quadratically

Constraints: at $F(x, y) = 0$,
either $s_{22}(x, y) \neq 0$ or $r(x, y) \neq 0$

if $\exists!$ sol. in $\mathbf{B}$ and $0 \notin \square s_{22}(\mathbf{B})$
$\rightarrow$ singularity
if $\exists!$ sol. in $\mathbf{B}$ and $0 \in \square s_{22}(\mathbf{B})$
$\rightarrow$ refine $\mathbf{B}$ (with K_F) until
$0 \notin \square s_{22}(\mathbf{B})$ or $0 \notin \square r(\mathbf{B})$

A subdivision solver for systems of large polynomials

$s11 = -14011784862700881253122\,0 x^{20} - 6101531335933493541710\,40 x^{19}y +$

$3951651892302173384407\,0 x^{18}y^2 + 3342883727033466620154170 x^{17}y^3 +$

$2891274355142589403901890 x^{16}y^4 + 1127947297505275246498\,40 x^{15}y^5 -$

$1134069249052129870012522\,0 x^{14}y^6 - 1106291110638894516544700\,0 x^{13}y^7 -$

$2946445042372334921153850 x^{12}y^8 + 1289064149306247575780837\,0 x^{11}y^9 +$

$2048282388147012310646837\,0 x^{10}y^{10} + 1102486022921613093142001\,0 x^9y^{11} -$

$1126962434297495978162860 x^8y^{12} - 1288448532468574766443268\,0 x^7y^{13} -$

$905972528707484832723458\,0 x^6y^{14} - 494132081742902565825385\,0 x^5y^{15} +$

$2122391146412348698406760 x^4y^{16} + 238411213685006877536954\,0 x^3y^{17} +$

$2363347796938811648578260 x^2y^{18} + 735933941537801203166720 xy^{19} +$

$293011939904302120871210 y^{20} + 693411445688541987909840 x^{19} +$

$481966743447629919642227\,0 x^{18}y - 854531603999857310010090 x^{17}y^2 -$

$4588903065796097271527060 x^{16}y^3 - 1245454007763298588704199\,0 x^{15}y^4 -$

$1903880991858077211393326\,0 x^{14}y^5 - 525559413440059828819296\,0 x^{13}y^6 +$

$1174005266404773044076220 x^{12}y^7 + 3965802158546623558224372\,0 x^{11}y^8 +$

$4914182206198018646901334\,0 x^{10}y^9 + 5125145051120039185666669\,0 x^9y^{10} +$

$1166931878595091649692305\,0 x^8y^{11} - 2830149553650679222698808\,0 x^7y^{12} -$

Interval extension $\Box F$ of F:

$0 \notin \Box F(\mathbf{B}) \Rightarrow$ no solution in $\mathbf{B}$

Interval newton operators $K_F(\mathbf{B})$:

$K_F(\mathbf{B}) \subset int(\mathbf{B}) \Rightarrow \exists!$ solution in $\mathbf{B}$

$\mathbf{B}_i = K_F(\mathbf{B}_{i-1})$ converges quadratically

Constraints: at $F(x,y) = 0$,

either $s_{22}(x,y) \neq 0$ or $r(x,y) \neq 0$

Main Issues:

Evaluating $\Box F$:

- quickly
- sharply (with low over-estimation)

Adapting arithmetic precision

Results:

Datas: Random dense polynomials of degree d, bit-size 8

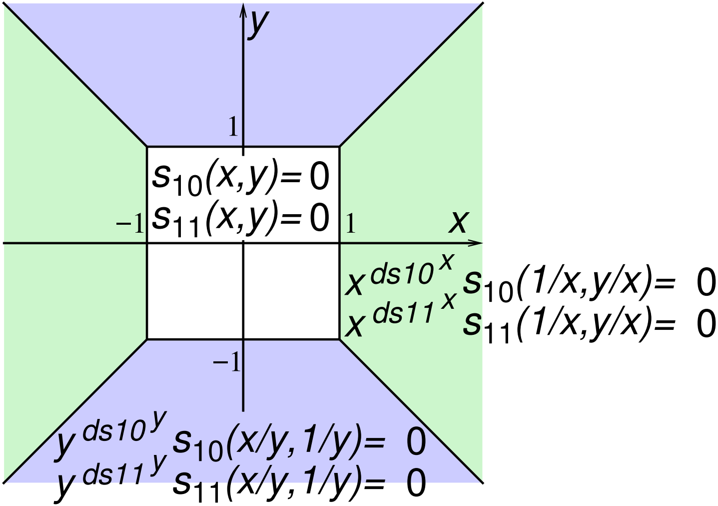
0-dim solver: multi-precision subdivision solver, `c++/cython/sage`
IA libraries: `BOOST` for double precision, `MPFI` otherwise

[Imb16] Rémi Imbach.
A Subdivision Solver for Systems of Large Dense Polynomials.
Technical Report 476, INRIA Nancy, March 2016.

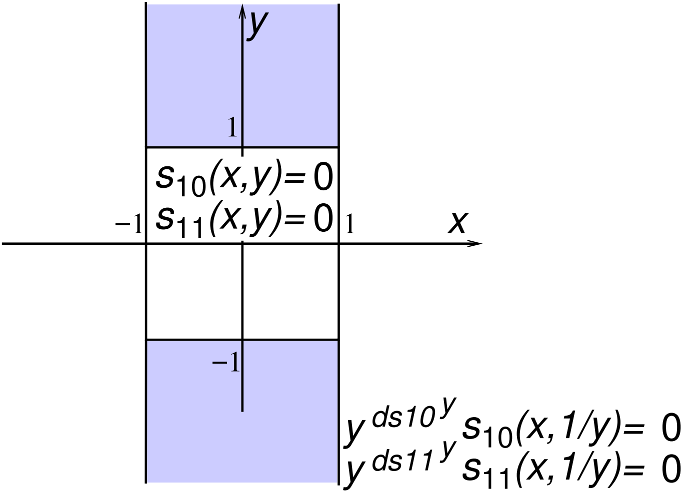
Numerical results: Isolating singularities of an apparent contour

system domain d	$\mathcal{S}_2$, RSCube $\mathbb{R}^2$	$\mathcal{S}_2$, subd. $[-1,1]\times[-1,1]$	
6	15	0.5	
7	105	4.44	
8	620	37.9	
9	3300	23.2	

means on 5 examples of sequential times.



[Sta95] Volker Stahl.
Interval Methods for Bounding the Range of Polynomials and Solving Systems of Nonlinear Equations.
PhD thesis, Johannes Kepler University, Linz, Austria, 1995.



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Results:

[IMP16] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
A certified numerical algorithm for the topology of resultant and discriminant curves.
Journal of Symbolic Computation, 2016.

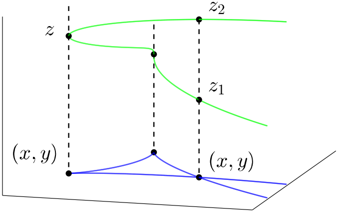
Numerical results: Isolating singularities of an apparent contour

system	$\mathcal{S}_2$, RSCube	$\mathcal{S}_2$, subd.		
domain	$\mathbb{R}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$	
d				
6	15	0.5	1.35	
7	105	4.44	21.9	
8	620	37.9	57.7	
9	3300	23.2	54.7	

means on 5 examples of sequential times.

Characterizing singularities:

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



Lemma 1: (x, y) is a node of $\mathcal{B} \Leftrightarrow (x, y, z_1, z_2)$ satisfies:

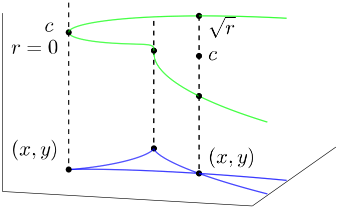
$$P(x, y, z_1) = P_z(x, y, z_1) = P(x, y, z_2) = P_z(x, y, z_2) = 0$$

Lemma 2: (x, y) is a cusp of $\mathcal{B} \Leftrightarrow (x, y, z)$ satisfies:

$$P(x, y, z) = P_z(x, y, z) = P_{zz}(x, y, z) = 0$$

Characterizing singularities:

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



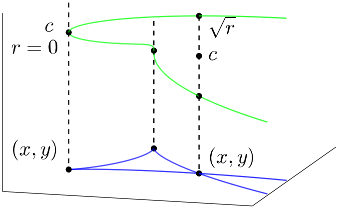
c: center of z_1, z_2
$r = \|c - z_1\|_2^2$
$z_1, z_2 = c \pm \sqrt{r}$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4) \begin{cases} \frac{1}{2}(P(x, y, c + \sqrt{r}) + P(x, y, c - \sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P(x, y, c + \sqrt{r}) - P(x, y, c - \sqrt{r})) & = 0 \\ \frac{1}{2}(P_z(x, y, c + \sqrt{r}) + P_z(x, y, c - \sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P_z(x, y, c + \sqrt{r}) - P_z(x, y, c - \sqrt{r})) & = 0 \end{cases}$$

Characterizing singularities: the Ball system

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



c: center of z_1, z_2
$r = \|c - z_1\|_2^2$
$z_1, z_2 = c \pm \sqrt{r}$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

when $r \to 0$

$$(\mathcal{S}_4) \left\{ \begin{array}{rl} P(x, y, c) & = 0 \\ P_z(x, y, c) & = 0 \\ P_z(x, y, c) & = 0 \\ P_{zz}(x, y, c) & = 0 \end{array} \right.$$

Characterizing singularities: the Ball system

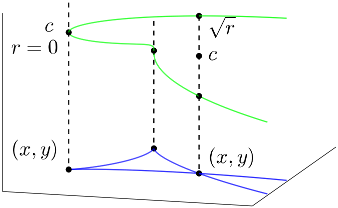
[IMP15] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Numeric and certified isolation of the singularities of the projection of a smooth space curve.
In *Proceedings of the 6th International Conferences on Mathematical Aspects of Computer and Information Sciences*, MACIS'15, 2015.

Lemma 4. Under some genericity assumptions, all the solutions of $\mathcal{S}_4$ in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+$ are regular.

Lemma 3. Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4)\left\{\begin{array}{rl} \frac{1}{2}(P(x,y,c+\sqrt{r})+P(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r})-P(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2}(P_z(x,y,c+\sqrt{r})+P_z(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2\sqrt{r}}(P_z(x,y,c+\sqrt{r})-P_z(x,y,c-\sqrt{r})) & =0 \end{array}\right.$$

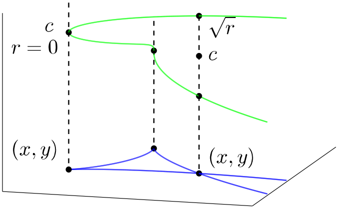
Isolating singularities: solving the Ball system



Finding the singularities of $\mathcal{B}$ in $\mathbf{B}_0 = (\mathbf{x}_0, \mathbf{y}_0)$:

$\leftrightsquigarrow$ solving the ball system on $\mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$

Isolating singularities: solving the Ball system



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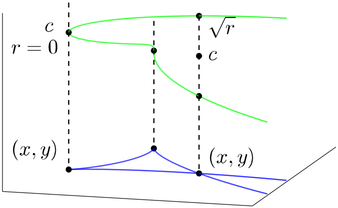
$\leftrightsquigarrow$ 3 systems solving on $\mathbf{B}_0 \times [-1, 1] \times [0, 1]$

Isolating singularities: solving the Ball system



Finding the singularities of $\mathcal{B}$ in $\mathbf{B}_0 = (\mathbf{x}_0, \mathbf{y}_0)$:

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Finding the singularities of $\mathcal{B}$ in $\mathbf{B} = \mathbb{R}^2$:

$\leftrightsquigarrow$ 5 systems solving on $[-1, 1]^2 \times [-1, 1] \times [0, 1]$

Results:

Datas: Random dense polynomials of degree d, bit-size 8

0-dim solver: multi-precision subdivision solver, `c++/cython/sage`
IA libraries: `BOOST` for double precision, `MPFI` otherwise

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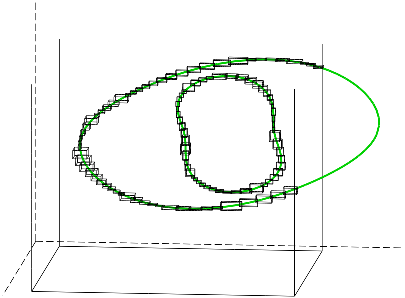
system	$\mathcal{S}_2$, RSCube	$\mathcal{S}_2$, subd.		$\mathcal{S}_4$, subd.	
domain	$\mathbb{R}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d					
6	15	0.5	1.35	8.4	11.3
7	105	4.44	21.9	43.8	54.2
8	620	37.9	57.7	70.2	99.2
9	3300	23.2	54.7	45.6	95.1

means on 5 examples of sequential times.

Filtering the domain where singularities of $\mathcal{B}$ are sougth

Enclosure of $\mathcal{C}$: a sequence $\{\mathbf{C}_k\}_{1\leq k\leq l}$ such that

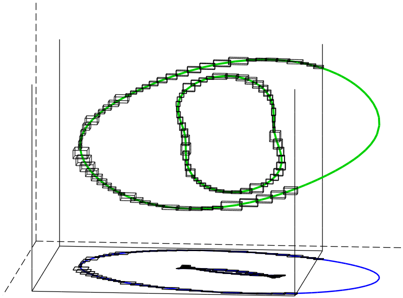
- $\mathcal{C} \subset \bigcup_k \mathbf{C}_k$,
- in each $\mathbf{C}_k$, $\mathcal{C} \cap \mathbf{C}_k$ is diffeomorphic to a close segment,
- each $\mathbf{C}_k$ has width less than δ.



Filtering the domain where singularities of $\mathcal{B}$ are sougth

Enclosure of $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

$\rightarrow \delta$ approximation of $\mathcal{B}$: each point of $\mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$



Filtering the domain where singularities of $\mathcal{B}$ are sougth

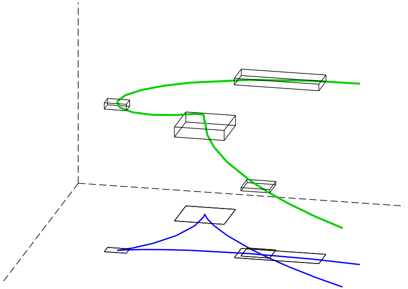
Enclosure of $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

δ approximation of $\mathcal{B}$: $\{\mathbf{B}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{1\leq k\leq l}$

$\rightarrow$ Location of singularities:

- each cusp is in a $\mathbf{B}_k$
- each node is in a $\mathbf{B}_k$ or in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$

$\rightarrow$ Singularities are in $\bigcup_k \mathbf{B}_k \cup \bigcup_{i,j} \mathbf{B}_{ij}$



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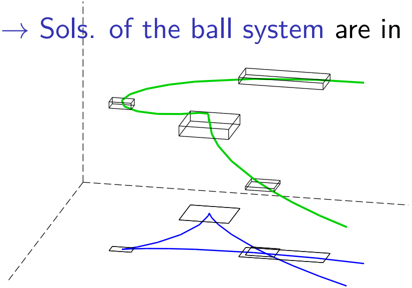
$\rightarrow$ Singularities are in $\bigcup_k \mathbf{B}_k \cup \bigcup_{i,j} \mathbf{B}_{ij}$

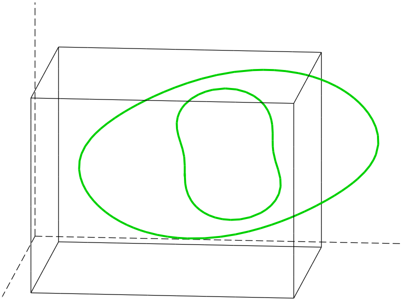
$\rightarrow$ Sols. of the ball system are in $\bigcup_k \mathbf{D}_k \cup \bigcup_{i,j} \mathbf{D}_{ij}$



$$\mathbf{D}_k = (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k, [0, (\tfrac{w(\mathbf{z}_k)}{2})^2])$$
$$\mathbf{D}_{ij} = (\mathbf{x}_{ij}, \mathbf{y}_{ij}, \tfrac{(\mathbf{z}_i+\mathbf{z}_j)}{2}, [0, (\tfrac{(\mathbf{z}_i-\mathbf{z}_j)}{2})^2])$$

Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{C}_0 = \mathbf{B}_0 \times \mathbb{R}$ a box of $\mathbb{R}^3$
$\mathcal{C} = \{C \in \mathbf{C}_0 | F(C) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$



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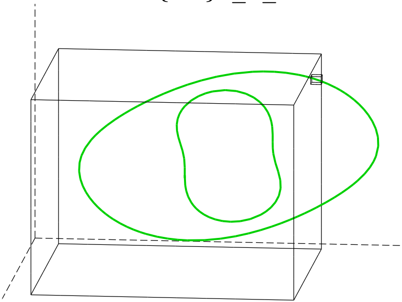
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Certified path-tracker:

Input: $F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{C}_0 = \mathbf{B}_0 \times \mathbb{R}$, $\delta \in \mathbb{R}^+_*$
An initial box $\mathbf{C} \in \mathcal{C}^i$

Output: a sequence of boxes $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ of width less than δ enclosing $\mathcal{C}^i$.



Certified numerical tools: 1-dim solver

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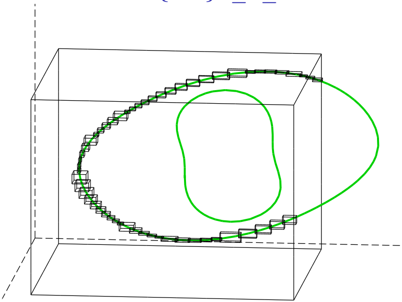
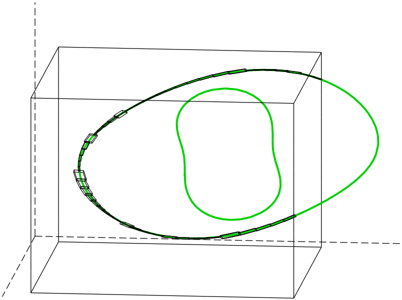
Input: $F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{C}_0 = \mathbf{B}_0 \times \mathbb{R}$, $\delta \in \mathbb{R}^+_*$
An initial box $\mathbf{C} \in \mathcal{C}^i$

Output: a sequence of boxes $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ of width less than δ enclosing $\mathcal{C}^i$.



Certified numerical tools: 1-dim solver

[MGGJ13] B. Martin, A. Goldsztejn, L. Granvilliers, and C. Jermann.
Certified parallelotope continuation for one-manifolds.
SIAM Journal on Numerical Analysis, 51(6):3373–3401, 2013.

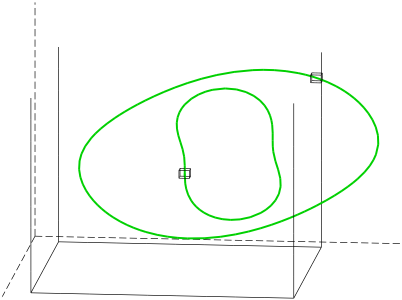


Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\mathbf{B}_0$ a box of $\mathbb{R}^2$
$\mathcal{C} = \{C \in \mathbf{B}_0 \times \mathbb{R} | F(X) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$
(A1) holds for generic polynomials P

Finding one point on each connected component

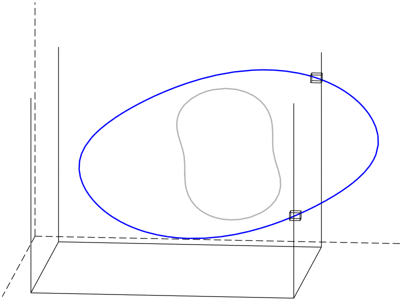


Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0, 1]$
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points



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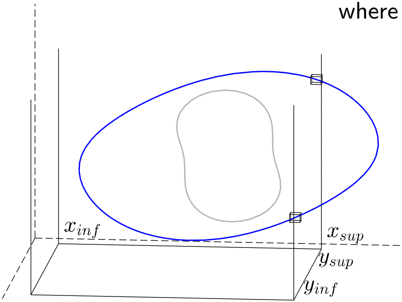

$\mathcal{C} \cap (\partial\mathbf{B}_0 \times \mathbb{R})$ are the solutions of the 4 systems:

$$\begin{cases} P(x=a,y,z)=0 \\ P_z(x=a,y,z)=0 \end{cases}$$

$$\begin{cases} P(x,y=b,z)=0 \\ P_z(x,y=b,z)=0 \end{cases}$$

where $a \in \{x_{inf}, x_{sup}\}$,
$b \in \{y_{inf}, y_{sup}\}$

Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

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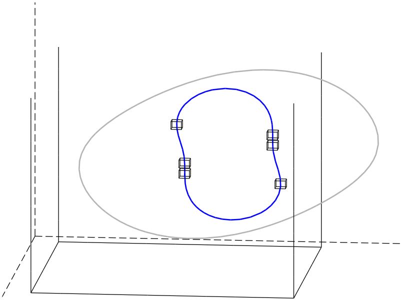

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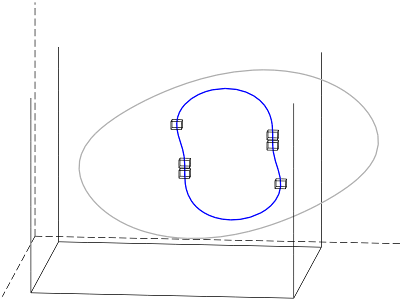
x-critical points of $\mathcal{C}$ are the solutions of the system:

$$\begin{cases} P(x,y,z) & = 0 \\ P_z(x,y,z) & = 0 \\ \begin{vmatrix} P_y & P_z \\ P_{zy} & P_{zz} \end{vmatrix}(x,y,z) & = 0 \end{cases}$$



Results:

Datas: Random dense polynomials of degree *d*, bit-size 8

0-dim solver: multi-precision subdivision solver, `c++/cython/sage`

Path tracker: prototype in `python/cython`

[MGGJ13] B. Martin, A. Goldsztejn, L. Granvilliers, and C. Jermann.
Certified parallelotope continuation for one-manifolds.
SIAM Journal on Numerical Analysis, 51(6):3373–3401, 2013.

Numerical results: Isolating singularities of an apparent contour

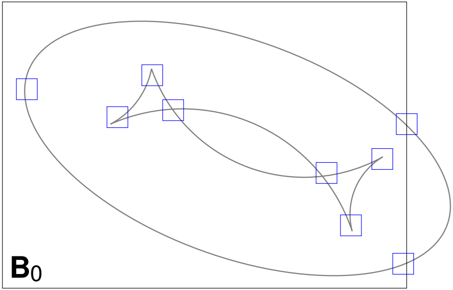
system domain *d*	$\mathcal{S}_2$, RSCube $\mathbb{R}^2$	$\mathcal{S}_2$, subd. $[-1,1]\times[-1,1]$	$\mathcal{S}_4$, subd. $[-1,1]\times[-1,1]$	with $\mathcal{C}$ $[-1,1]\times[-1,1]$
6	15	0.5	8.4	2.36
7	105	4.44	43.8	4.13
8	620	37.9	70.2	5.91
9	3300	23.2	45.6	5.30

means on 5 examples of sequential times.

Computing the topology of the projected curve $\mathcal{B}$

A general framework

0. Restrict to a box $\mathbf{B}_0$
1. Isolate special points:
   - boundary points
   - x-critical points
   - singularities
2. Loc. topology around special points
3. Connect special points
4. Embed the graph



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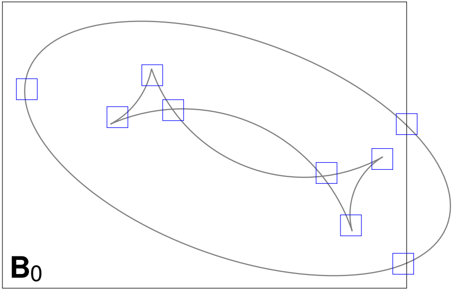
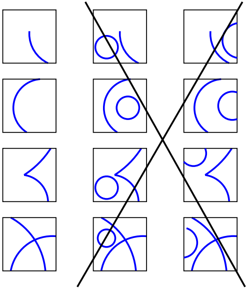
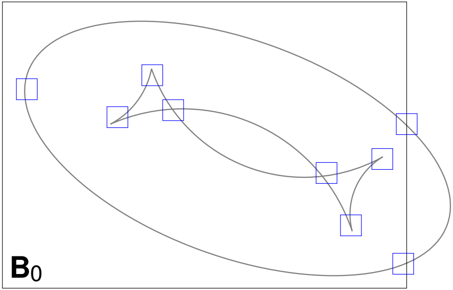


Local topology at special points

B^s = 2D boxes isolating the special points
Assumption: the topology in each $\mathbf{B}_i^s \in B^s$ is *simple*
i.e. $\mathcal{B} \cap \mathbf{B}_0 \cap \mathbf{B}_i^s$ is diffeomorphic to:





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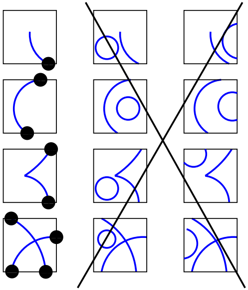
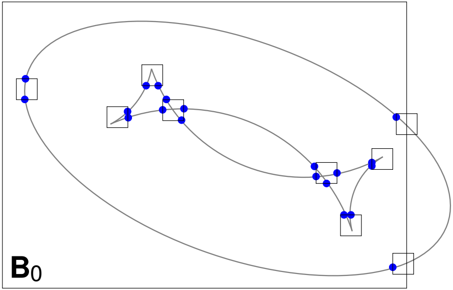
Assumption: the topology in each $\mathbf{B}_i^s \in B^s$ is *simple*

Definition: we call *connection* of $\mathcal{B} \cap \mathbf{B}_0$ in $\mathbf{B}_i^s$ the points $\mathcal{B} \cap \mathbf{B}_0 \cap \partial \mathbf{B}_i^s$





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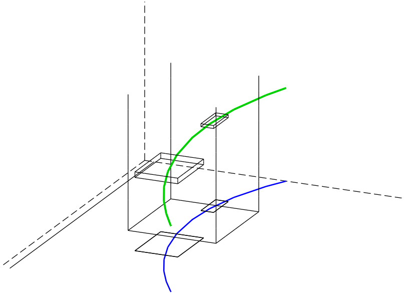
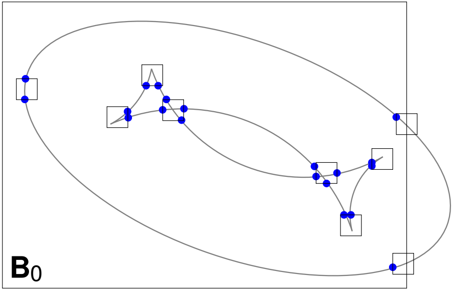
Cylinder $\mathbf{C}_i^s$ above $\mathbf{B}_i^s$: $\mathbf{B}_i^s \times \mathbb{R}$
Definition: we call *connection* of $\mathcal{C} \cap \mathbf{C}_0$ in $\mathbf{C}_i^s$ the points $\mathcal{C} \cap \mathbf{C}_0 \cap \partial \mathbf{C}_i^s$





Computing the topology of the projected curve $\mathcal{B}$

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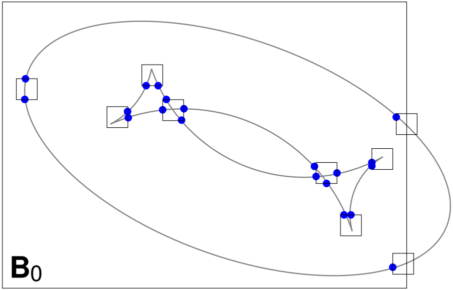
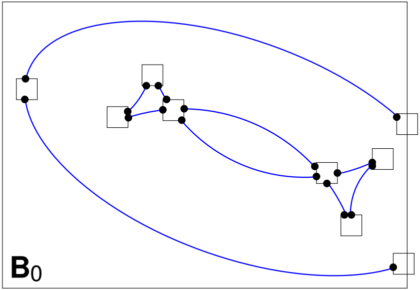


Connecting special points:

Computing a graph G s.t. any embedding $\mathcal{G}$ of G is homeo. to $\mathcal{B} \cap \mathbf{B}_0$

- Vertices of G: boxes of B^s
- Edges of G: connected components of $(\mathcal{B} \cap \mathbf{B}_0) \setminus \bigcup_{\mathbf{B}_i^s \in B^s} \mathbf{B}_i^s$



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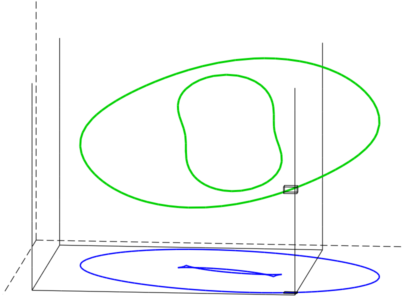
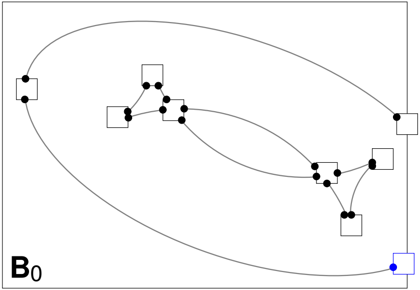
- Edges of G: connected components of $(\mathcal{B} \cap \mathbf{B}_0) \setminus \bigcup_{\mathbf{B}_i^s \in B^s} \mathbf{B}_i^s$

For each $\mathbf{C}_i^s \in C^s$

For each connection c of $\mathcal{C} \cap \mathbf{C}_0$ in $\mathbf{C}_i^s$

Track $\mathcal{C}$ from c until it reaches a connection c' of $\mathcal{C} \cap \mathbf{C}_0$ in $\mathbf{C}_j^s$





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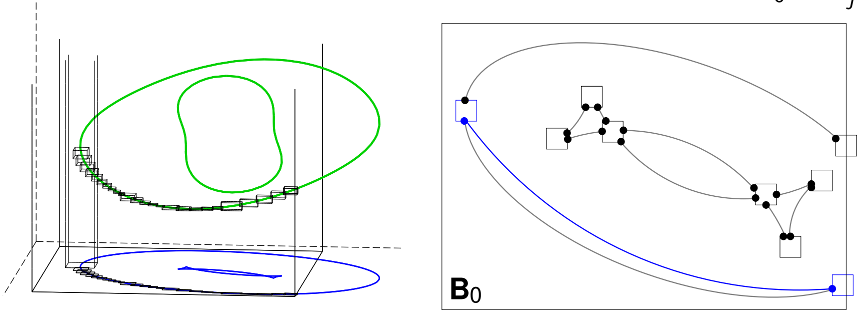

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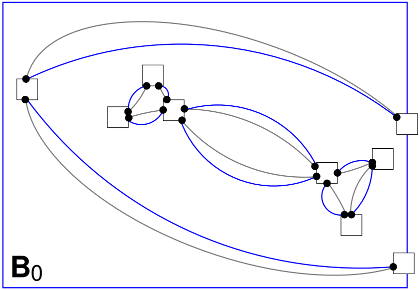

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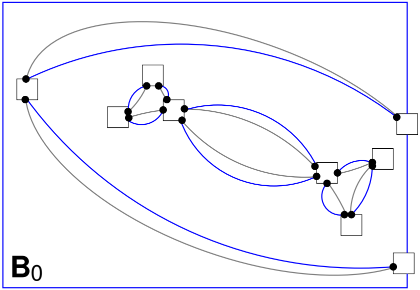

Computing the topology of the projected curve $\mathcal{B}$

A general framework

⓿ Restrict to a box $\mathbf{B}_0$

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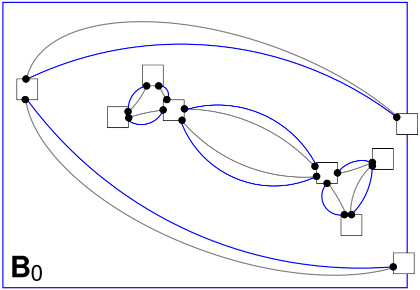
- singularities

❷ Loc. topology around special points

❸ Connect special points

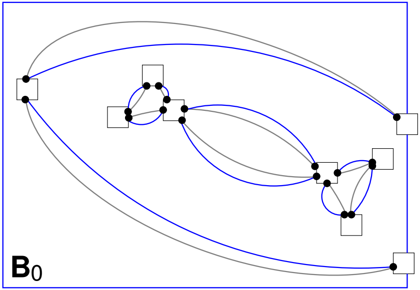
❹ Embed the graph



[Köt02] Ullrich Köthe.
Xpmaps and topological segmentation-a unified approach to finite topologies in the plane.
In *International Conference on Discrete Geometry for Computer Imagery*, pages 22–33. Springer, 2002.

Computing the topology of the projected curve $\mathcal{B}$

[IMP17] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Reliable location with respect to the projection of a smooth space curve.
Research report, to appear in reliable computing, INRIA, November 2017.



Results:

Datas: Random dense polynomials P of degree d, bit-size 8

Extern comparison: `Isotop` with input $Res(P, P_z, z)$, uses RSCube.

[CLP+10] J. Cheng, S. Lazard, L. Peñaranda, M. Pouget, F. Rouillier, and E. Tsigaridas.
On the topology of real algebraic plane curves.
Mathematics in Computer Science, 4:113–137, 2010.

Numerical results: Topology of an apparent contour

method domain	`Isotop` $\mathbb{R}^2$	Our approach $\mathbf{B}_0 = [-1,1] \times [-1,1]$
d	t	t
5	4.78	3.17
7	251	8.13
9	–	24.1
11	–	75.5
13	–	90.6
15	–	169

means on 5 examples of seq. times in seconds.