4
The scaling hypothesis

4.1 The homogeneity assumption

In the previous chapters the singular behavior in the vicinity of a continuous transition was characterized by a set of critical exponents $\{\alpha, \beta, \gamma, \delta, \nu, \eta, \cdots\}$. The saddle-point estimates of these exponents were found to be unreliable due to the importance of fluctuations. Since the various thermodynamic quantities are related, these exponents can not be independent of each other. The goal of this chapter is to discover the relationships between them, and to find the minimum number of independent exponents needed to describe the critical point.

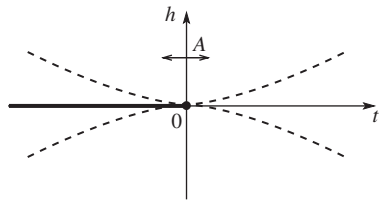

Fig. 4.1 The vicinity of the critical point in the (t, h) plane, with crossover boundaries indicated by dashed lines.

The non-analytical structure is a coexistence line for $t < 0$ and $h = 0$ that terminates at the critical point $t = h = 0$. The various exponents describe the leading singular behavior of a thermodynamic quantity $Q(t, h)$, in the vicinity of this point. A basic quantity in the canonical ensemble is the free energy, which in the saddle point approximation is given by

$$f(t,h) = \min\left[\frac{t}{2}m^2 + um^4 - h.m\right]_m = \begin{cases} -\dfrac{1}{16}\dfrac{t^2}{u} & \text{for } h = 0,\ t < 0 \\ -\dfrac{3}{4^{4/3}}\dfrac{h^{4/3}}{u^{1/3}} & \text{for } h \neq 0,\ t = 0\ . \end{cases} \tag{4.1}$$

The singularities in the free energy can in fact be described by a single *homogeneous function* in t and h, as[1]

$$f(t,h) = |t|^2 g_f\left(h/|t|^{\Delta}\right). \tag{4.2}$$

The function g_f only depends on the combination $x \equiv h/|t|^{\Delta}$, where Δ is known as the *gap exponent*. The asymptotic behavior of g_f is easily obtained by comparing Eqs. (4.1) and (4.2). The $h = 0$ limit is recovered if $\lim_{x\to 0} g_f(x) \sim 1/u$, while to get the proper power of h, we must set $\lim_{x\to\infty} g_f(x) \sim x^{4/3}/u^{1/3}$. The latter implies $f \sim |t|^2 h^{4/3}/(u^{1/3}|t|^{4\Delta/3})$. Since f can have no t dependence along $t = 0$, the gap exponent (corresponding to Eq. 4.1) has the value

$$\Delta = \frac{3}{2}. \tag{4.3}$$

The assumption of homogeneity is that, on going beyond the saddle point approximation, the singular form of the free energy (and any other thermodynamic quantity) retains the homogeneous form

$$f_{\text{sing}}(t,h) = |t|^{2-\alpha} g_f\left(h/|t|^{\Delta}\right). \tag{4.4}$$

The actual exponents α and Δ depend on the critical point being considered. The dependence on t is chosen to reproduce the heat capacity singularity at $h = 0$. The singular part of the energy is obtained from (say for $t > 0$)

$$\begin{aligned} E_{\text{sing}} &\sim \frac{\partial f}{\partial t} \sim (2-\alpha)|t|^{1-\alpha} g_f\left(h/|t|^{\Delta}\right) - \Delta h |t|^{1-\alpha-\Delta} g_f'\left(h/|t|^{\Delta}\right) \\ &\sim |t|^{1-\alpha}\left[(2-\alpha) g_f\left(h/|t|^{\Delta}\right) - \frac{\Delta h}{|t|^{\Delta}} g_f'\left(h/|t|^{\Delta}\right)\right] \equiv |t|^{1-\alpha} g_E\left(h/|t|^{\Delta}\right). \end{aligned} \tag{4.5}$$

Thus the derivative of one homogeneous function is another. Similarly, the second derivative takes the form (again for $t > 0$)

$$C_{\text{sing}} \sim -\frac{\partial^2 f}{\partial t^2} \sim |t|^{-\alpha} g_C\left(h/|t|^{\Delta}\right), \tag{4.6}$$

reproducing the scaling $C_{\text{sing}} \sim |t|^{-\alpha}$, as $h \to 0$.

It may appear that we have the freedom to postulate a more general form

$$C_{\pm}(t,h) = |t|^{-\alpha_{\pm}} g_{\pm}\left(h/|t|^{\Delta_{\pm}}\right), \tag{4.7}$$

with different functions and exponents for $t > 0$ and $t < 0$ that match at $t = 0$. However, this is ruled out by the condition that the free energy is analytic everywhere except on the coexistence line for $h = 0$ and $t < 0$, as shown as

[1] In general, a function $f(x_1, x_2, \cdots)$ is homogeneous if

$$f\left(b^{p_1} x_1, b^{p_2} x_2, \cdots\right) = b^{p_f} f(x_1, x_2, \cdots),$$

for any rescaling factor b. With the proper choice of b one of the arguments can be removed, leading to the scaling forms used in this section.

follows: Consider a point at $t = 0$ and finite h. By assumption, the function C is perfectly analytic in the vicinity of this point, expandable in a Taylor series,

$$C\left(t \ll h^{\Delta}\right) = \mathcal{A}(h) + t\mathcal{B}(h) + \mathcal{O}(t^2). \tag{4.8}$$

Furthermore, the same expansion should be obtained from both C_+ and C_-. But Eq. (4.7) leads to the expansions,

$$C_{\pm} = |t|^{-\alpha_{\pm}} \left[A_{\pm} \left(\frac{h}{|t|^{\Delta_{\pm}}} \right)^{p_{\pm}} + B_{\pm} \left(\frac{h}{|t|^{\Delta_{\pm}}} \right)^{q_{\pm}} + \cdots \right], \tag{4.9}$$

where $\{p_{\pm}, q_{\pm}\}$ are the leading powers in asymptotic expansions of $g_{\pm}$ for large arguments, and $\{A_{\pm}, B_{\pm}\}$ are the corresponding prefactors. Matching to the Taylor series in Eq. (4.8) requires $p_{\pm}\Delta_{\pm} = -\alpha_{\pm}$ and $q_{\pm}\Delta_{\pm} = -(1 + \alpha_{\pm})$, and leads to

$$C_{\pm}\left(t \ll h^{\Delta}\right) = A_{\pm} h^{-\alpha_{\pm}/\Delta_{\pm}} + B_{\pm} h^{-(1+\alpha_{\pm})/\Delta_{\pm}} |t| + \cdots \tag{4.10}$$

Continuity at $t = 0$ now forces $\alpha_+/\Delta_+ = \alpha_-/\Delta_-$, and $(1 + \alpha_+)/\Delta_+ = (1 + \alpha_-)/\Delta_-$, which in turn implies

$$\begin{cases} \alpha_+ = \alpha_- \equiv \alpha \\ \Delta_+ = \Delta_- \equiv \Delta. \end{cases} \tag{4.11}$$

Despite using $|t|$ in the postulated scaling form, we can still ensure the analyticity of the function at $t = 0$ for finite h by appropriate choice of parameters, e.g. by setting $B_- = -B_+$ to match Eq. (4.10) to the analytic form in Eq. (4.8). Having established this result, we can be somewhat careless henceforth in replacing $|t|$ in the scaling equations with t. Naturally these arguments apply to any quantity $Q(t, h)$.

Starting from the free energy in Eq. (4.4), we can compute the singular parts of other quantities of interest:

- The *magnetization* is obtained from

$$m(t, h) \sim \frac{\partial f}{\partial h} \sim |t|^{2-\alpha-\Delta} g_m\left(h/|t|^{\Delta}\right). \tag{4.12}$$

  In the limit $x \to 0$, $g_m(x)$ is a constant, and

$$m(t, h = 0) \sim |t|^{2-\alpha-\Delta}, \quad \Longrightarrow \quad \beta = 2 - \alpha - \Delta. \tag{4.13}$$

  On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and

$$m(t = 0, h) \sim |t|^{2-\alpha-\Delta} \left(\frac{h}{|t|^{\Delta}} \right)^{p}. \tag{4.14}$$

  Since this limit is independent of t, we must have $p\Delta = 2 - \alpha - \Delta$. Hence

$$m(t, h = 0) \sim h^{(2-\alpha-\Delta)/\Delta} \quad \Longrightarrow \quad \delta = \Delta/(2 - \alpha - \Delta) = \Delta/\beta. \tag{4.15}$$

- Similarly, the *susceptibility* is computed as

$$\chi(t,h) \sim \frac{\partial m}{\partial h} \sim |t|^{2-\alpha-2\Delta} g_\chi(h/|t|^\Delta) \Rightarrow \chi(t,h=0) \sim |t|^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha. \tag{4.16}$$

Thus, the consequences of the homogeneity assumption are:

(1) The singular parts of all critical quantities $Q(t,h)$ are homogeneous, with the same exponents above and below the transition.

(2) Because of the interconnections via thermodynamic derivatives, the same gap exponent, Δ, occurs for all such quantities.

(3) All (bulk) critical exponents can be obtained from only *two* independent ones, e.g. α and Δ.

(4) As a result of the above, there are a number of *exponent identities*. For example, Eqs. (4.13), (4.15), and (4.16) imply

$$\begin{aligned} \alpha + 2\beta + \gamma &= \alpha + 2(2-\alpha-\Delta) + (2\Delta - 2 + \alpha) = 2 \quad \text{(Rushbrooke's identity)}, \\ \delta - 1 &= \frac{\Delta}{2-\alpha-\Delta} - 1 = \frac{2\Delta - 2 + \alpha}{2-\alpha-\Delta} = \frac{\gamma}{\beta} \quad \text{(Widom's identity)}. \end{aligned} \tag{4.17}$$

These identities can be checked against the following table of critical exponents. The first three rows are based on a number of theoretical estimates in $d = 3$; the last row comes from an exact solution in $d = 2$. The exponent identities are completely consistent with these values, as well as with all reliable experimental data.

	α	β	γ	δ	ν	η
$n=1$	0.11	0.32	1.24	4.9	0.63	0.04
$n=2$	−0.01	0.35	1.32	4.7	0.67	0.04
$n=3$	−0.11	0.36	1.39	4.9	0.70	0.04
$n=1$	0	1/8	7/4	15	1	1/4

4.2 Divergence of the correlation length

The homogeneity assumption relates to the free energy and quantities derived from it. It says nothing about the behavior of correlation functions. An important property of a critical point is the divergence of the correlation length which is responsible for, and can be deduced from, diverging response functions. In

order to obtain an identity involving the exponent ν for the divergence of the correlation length, we replace the homogeneity assumption for the free energy with the following *two* conditions:

(1) The correlation length ξ is a homogeneous function,

$$\xi(t, h) \sim |t|^{-\nu} g\left(h/|t|^{\Delta}\right). \tag{4.18}$$

(For $t = 0$, ξ diverges as $h^{-\nu_h}$ with $\nu_h = \nu/\Delta$.)

(2) Close to criticality, the correlation length ξ is the most important length in the system, and is *solely* responsible for singular contributions to thermodynamic quantities.

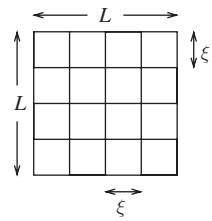

Fig. 4.2 A system of linear size L, presented as (approximately) independent components of size ξ, the correlation length.

The second condition determines the singular part of the free energy. Since $\ln Z(t, h)$ is *extensive* and *dimensionless*, it must take the form

$$\ln Z = \left(\frac{L}{\xi}\right)^{d} \times g_s + \cdots + \left(\frac{L}{a}\right)^{d} \times g_a, \tag{4.19}$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). The leading singular part of the free energy comes from the first term, and behaves as

$$f_{\text{sing}}(t, h) \sim \frac{\ln Z}{L^d} \sim \xi^{-d} \sim |t|^{d\nu} g_f\left(h/|t|^{\Delta}\right). \tag{4.20}$$

A simple interpretation of the above result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$, leading to Eq. (4.19).

The consequences of the above assumptions are:

(1) The homogeneity of $f_{\text{sing}}(t, h)$ emerges naturally.

(2) We obtain the additional exponent relation

$$2 - \alpha = d\nu \quad \text{(Joshephson's identity)}. \tag{4.21}$$

Identities obtained from the generalized homogeneity assumption involve the space dimension d, and are known as *hyperscaling relations*. The relation between α and ν is consistent with the exponents in the above table. However, it does not agree with the saddle point values, $\alpha = 0$ and $\nu = 1/2$, which are

valid for $d > 4$. Any theory of critical behavior must thus account for the validity of this relation in low dimensions, and its breakdown for $d > 4$.

4.3 Critical correlation functions and self-similarity

One exponent that has not so far been accounted for is η, describing the decay of correlation functions at criticality. Exactly at the critical point, the correlation length is infinite, and there is no other length scale (except sample size) to cut off the decay of correlation functions. Thus all correlations decay as a power of the separation. As discussed in the previous chapter, the magnetization correlations fall off as

$$G^c_{m,m}(\mathbf{x}) \equiv \langle m(\mathbf{x})m(\mathbf{0})\rangle - \langle m\rangle^2 \sim 1/|\mathbf{x}|^{d-2+\eta}. \tag{4.22}$$

Similarly, we can define an exponent η' for the decay of energy–energy correlations as

$$G^c_{E,E}(\mathbf{x}) = \langle \mathcal{H}(\mathbf{x})\mathcal{H}(\mathbf{0})\rangle - \langle \mathcal{H}\rangle^2 \sim 1/|\mathbf{x}|^{d-2+\eta'}. \tag{4.23}$$

Away from criticality, the power laws are cut off for distances $|\mathbf{x}| \gg \xi$. As the response functions can be obtained from integrating the connected correlation functions, there are additional exponent identities, such as (Fisher's identity)

$$\chi \sim \int \mathrm{d}^d\mathbf{x} G^c_{mm}(\mathbf{x}) \sim \int^{\xi} \frac{\mathrm{d}^d x}{|x|^{d-2+\eta}} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)} \quad \Longrightarrow \quad \gamma = (2-\eta)\nu. \tag{4.24}$$

Similarly, for the heat capacity,

$$C \sim \int \mathrm{d}^d\mathbf{x} G^c_{EE}(\mathbf{x}) \sim \int^{\xi} \frac{\mathrm{d}^d x}{|x|^{d-2+\eta'}} \sim \xi^{2-\eta'} \sim |t|^{-\nu(2-\eta')}, \quad \Longrightarrow \quad \alpha = (2-\eta')\nu. \tag{4.25}$$

As before, two *independent* exponents are sufficient to describe all singular critical behavior.

An important consequence of these scaling ideas is that the critical system has an additional *dilation symmetry*. Under a change of scale, the critical correlation functions behave as

$$G_{\text{critical}}(\lambda\mathbf{x}) = \lambda^p G_{\text{critical}}(\mathbf{x}). \tag{4.26}$$

This implies a *scale invariance* or *self-similarity*: if a snapshot of the critical system is blown up by a factor of λ, apart from a change of contrast (multiplication by λ^p), the resulting snapshot is statistically similar to the original one. Such statistical self-similarity is the hallmark of *fractal* geometry. As discussed by Mandelbrot, many naturally occurring forms (clouds, shore-lines, river basins, etc.) exhibit such behavior. The Landau–Ginzburg probability was constructed on the basis of *local* symmetries such as rotation invariance. If we could add to the list of constraints the requirement of *dilation symmetry*, the resulting probability would indeed describe the critical point. Unfortunately,

it is not possible to directly see how such a requirement constrains the effective Hamiltonian. One notable exception is in $d = 2$, where dilation symmetry implies conformal invariance, and a lot of information can be obtained by constructing conformally invariant theories. We shall instead prescribe a less direct route of following the effects of the dilation operation on the effective energy: the *renormalization group* procedure.

4.4 The renormalization group (conceptual)

Success of the scaling theory in correctly predicting various exponent identities strongly supports the assumption that close to the critical point the correlation length, ξ, is the only important length scale, and that microscopic length scales are irrelevant. The critical behavior is dominated by fluctuations that are self-similar up to the scale ξ. The self-similarity is of course only statistical, in that a magnetization configuration is generated with a weight $W[\vec{m}(\mathbf{x})] \propto \exp\{-\beta\mathcal{H}[\vec{m}(\mathbf{x})]\}$. Kadanoff suggested taking advantage of the self-similarity of the fluctuations to gradually eliminate the correlated degrees of freedom at length scales $x \ll \xi$, until one is left with the relatively simple, uncorrelated degrees of freedom at scale ξ. This is achieved through a procedure called the *renormalization group* (RG), whose conceptual foundation is the three steps outlined in this section.

(1) ***Coarse grain:*** There is an implicit short distance length cutoff scale a for allowed variations of $\vec{m}(\mathbf{x})$ in the system. This is the lattice spacing for a model of spins, or the coarse graining scale that underlies the Landau–Ginzburg Hamiltonian. In a digital picture of the system, a corresponds to the pixel size. The first step of the RG is to decrease the resolution by changing this minimum scale to ba ($b > 1$). The *coarse-grained* magnetization is then given by

$$m_i(\mathbf{x}) = \frac{1}{b^d} \int_{\text{Cell centered at } \mathbf{x}} \mathrm{d}^d\mathbf{x}' m_i(\mathbf{x}'). \tag{4.27}$$

(2) ***Rescale:*** Due to the change in resolution, the coarse grained "picture" is grainier than the original. The original resolution of a can be restored by decreasing all length scales by a factor of b, i.e. by setting

$$\mathbf{x}_{\text{new}} = \frac{\mathbf{x}_{\text{old}}}{b}. \tag{4.28}$$

(3) ***Renormalize:*** The variations of fluctuations in the rescaled magnetization profile is in general different from the original, i.e. there is a difference in contrast between the pictures. This can be remedied by introducing a change of contrast by a factor ζ, through defining a *renormalized* magnetization

$$\vec{m}_{\text{new}}(\mathbf{x}_{\text{new}}) = \frac{1}{\zeta b^d} \int_{\text{Cell centered at } b\mathbf{x}_{\text{new}}} \mathrm{d}^d\mathbf{x}' \vec{m}(\mathbf{x}'). \tag{4.29}$$

By following these steps, for each configuration $\vec{m}_{\text{old}}(\mathbf{x})$, we generate a renormalized configuration $\vec{m}_{\text{new}}(\mathbf{x})$. Equation (4.29) can be regarded as a mapping from one set of random variables to another, and can be used to construct the probability distribution, or weight $W_b[\vec{m}_{\text{new}}(\mathbf{x})] \equiv \exp\{-\beta\mathcal{H}_b[\vec{m}_{\text{new}}(\mathbf{x})]\}$. Kadanoff's insight was that since on length scales less than ξ, the renormalized configurations are statistically similar to the original ones, they may be distributed by a Hamiltonian $\beta\mathcal{H}_b$ that is also "close" to the original. In particular, the original Hamiltonian becomes critical by tuning the two parameters t and h to zero: at this point the original configurations are statistically similar to those of the rescaled system. The critical Hamiltonian should thus be invariant under rescaling and renormalization. In the original problem, one moves away from criticality for finite t and h. Kadanoff postulated that the corresponding renormalized Hamiltonian is similarly described by non-zero t_{new} and/or h_{new}.

The assumption that the closeness of the original and renormalized Hamiltonians to criticality is described by the two parameters t and h greatly simplifies the analysis. The effect of the RG transformation on the probability of configurations is now described by the two parameter mappings $t_{\text{new}} \equiv t_b(t_{\text{old}}, h_{\text{old}})$ and $h_{\text{new}} \equiv h_b(t_{\text{old}}, h_{\text{old}})$. The next step is to note that since the transformation only involves changes at the shortest length scales, it cannot cause any singularities. The renormalized parameters must be *analytic* functions of the original ones, and hence expandable as

$$\begin{cases} t_b(t,h) = A(b)t + B(b)h + \cdots \\ h_b(t,h) = C(b)t + D(b)h + \cdots \end{cases} \tag{4.30}$$

Note that there are no constant terms in the above Taylor expansions. This expresses the condition that if $\beta\mathcal{H}$ is at its critical point ($t = h = 0$), then $\beta\mathcal{H}_b$ is also at criticality, and $t_{\text{new}} = h_{\text{new}} = 0$. Furthermore, due to rotational symmetry, under the combined transformation ($m(x) \mapsto -m(x)$, $h \mapsto -h$, $t \mapsto t$) the weight of a configuration is unchanged. As this symmetry is preserved by the RG, the coefficients B and C in the above expression must be zero, leading to the further simplifications

$$\begin{cases} t_b(t,h) = A(b)t + \cdots \\ h_b(t,h) = D(b)h + \cdots \end{cases} \tag{4.31}$$

The remaining coefficients $A(b)$ and $D(b)$ depend on the (arbitrary) rescaling factor b, and trivially $A(1) = D(1) = 1$ for $b = 1$. Since the above transformations can be carried out in sequence, and the net effect of rescalings of b_1 and b_2 is a change of scale by $b_1 b_2$, the RG procedure is sometimes referred to as a *semi-group*. The term applies to the action of RG on the space of configurations: each magnetization profile is mapped uniquely to one at larger scale, but the inverse process is non-unique as some short scale information is lost in the coarse graining. (There is in fact no problem with inverting the transformation in the space of the parameters of the Hamiltonian.) The dependence of A and

D in Eqs. (4.31) on b can be deduced from this group property. Since at $b = 1$, $A = D = 1$, and $t(b_1 b_2) \approx A(b_1)A(b_2)t \approx A(b_1 b_2)t$, we must have $A(b) = b^{y_t}$, and similarly $D(b) = b^{y_h}$, yielding

$$\begin{cases} t' \equiv t_b = b^{y_t} t + \cdots \\ h' \equiv h_b = b^{y_h} h + \cdots \end{cases} \tag{4.32}$$

If $\beta\mathcal{H}_{\text{old}}$ is slightly away from criticality, it is described by a large but finite correlation length ξ_{old}. After the RG transformation, due to the rescaling in Eq. (4.28), the new correlation length is smaller by a factor of b. Hence the renormalized Hamiltonian is less critical, and the RG procedure moves the parameters further away from the origin, i.e. the exponents y_t and y_h must be positive.

We can now explore some consequences of the assumptions leading to Eqs. (4.32).

(1) ***The free energy:*** The RG transformation is a many to one map of the original configurations to new ones. Since the weight of a new configuration, $W'([m'])$, is the sum of the weights $W([m])$, of old configurations, the partition function is preserved, i.e.

$$Z = \int Dm W([m]) = \int Dm' W'([m']) = Z'. \tag{4.33}$$

Hence $\ln Z = \ln Z'$, and the corresponding free energies are related by

$$V f(t, h) = V' f(t', h'). \tag{4.34}$$

In d dimensions, the rescaled volume is smaller by a factor of b^d, and

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_t} h), \tag{4.35}$$

where we have made use of the assumption that the two free energies are obtained from the *same Hamiltonian* in which only the parameters t and h have changed according to Eqs. (4.32). Equation (4.35) describes a *homogeneous function* of t and h. This is made apparent by choosing a rescaling factor b such that b^{y_t} is a constant, say unity, i.e. $b = t^{-1/y_t}$, leading to

$$f(t, h) = t^{d/y_t} f(1, h/t^{y_h/y_t}) \equiv t^{d/y_t} g_f(h/t^{y_h/y_t}). \tag{4.36}$$

We have thus recovered the scaling form in Eq. (4.4), and can identify the exponents as

$$2 - \alpha = d/y_t, \quad \Delta = y_h/y_t. \tag{4.37}$$

(2) ***Correlation length:*** All length scales are reduced by a factor of b during the RG transformation. This is also true of the correlation length, $\xi' = \xi/b$, implying

$$\xi(t, h) = b\xi(b^{y_t} t, b^{y_h} h) = t^{-1/y_t} \xi(1, h/t^{y_h/y_t}) \sim t^{-\nu}. \tag{4.38}$$

This identifies $\nu = 1/y_t$, and using Eq. (4.37) the hyperscaling relation, $2 - \alpha = d\nu$, is recovered.

(3) ***Magnetization:*** From the homogenous form of the free energy (Eq. 4.36), we can obtain other bulk quantities such as magnetization. Alternatively, from the RG results for Z, V, and h, we may directly conclude

$$m(t,h) = -\frac{1}{V}\frac{\partial \ln Z(t,h)}{\partial h} = -\frac{1}{b^d V'}\frac{\partial \ln Z'(t',h')}{b^{-y_h}\partial h'} = b^{y_h-d} m(b^{y_t}t, b^{y_h}h). \quad (4.39)$$

Choosing $b = t^{-1/y_t}$, we obtain $\beta = (y_h - d)/y_t$, and $\Delta = y_h/y_t$ as before.

It is thus apparent that quite generally, the singular part of any quantity X has a homogeneous form

$$X(t,h) = b^{y_X} X(b^{y_t}t, b^{y_h}h). \quad (4.40)$$

For any conjugate pair of variables, contributing a term $\int \mathrm{d}^d\mathbf{x} F\cdot X$ to the Hamiltonian, the *scaling dimensions* are related by $y_X = y_F - d$, where $F' = b^{y_F}F$ under RG.

4.5 The renormalization group (formal)

In the previous section we noted that all critical properties can be obtained from the recursion relations in Eqs. (4.32). Though conceptually appealing, it is not clear how such a procedure can be formally carried out. In particular, why should the forms of the two Hamiltonians be identical, and why are two parameters t and h sufficient to describe the transformation? In this section we outline a more formal procedure for identifying the effects of the dilation operation on the Hamiltonian. The various steps of the program are as follows:

(1) Start with the most general Hamiltonian allowed by symmetries. For example, in the presence of rotational symmetry,

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x}\left[\frac{t}{2}m^2 + um^4 + vm^6 + \cdots + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right]. \quad (4.41)$$

A particular system with such symmetry is therefore completely specified by a point in the (infinite-dimensional) parameter space $S \equiv (t, u, v, \cdots, K, L, \cdots)$.

(2) Apply the three steps of renormalization in configuration space: (i) coarse grain by b; (ii) rescale, $\mathbf{x}' = \mathbf{x}/b$; and (iii) renormalize, $m' = m/\zeta$. This accomplishes the change of variables,

$$m'(\mathbf{x}') = \frac{1}{\zeta b^d}\int_{\text{Cell of size } b \text{ centered at } b\mathbf{x}'} \mathrm{d}^d\mathbf{x} m(\mathbf{x}). \quad (4.42)$$

Given the probabilities $\mathcal{P}[m(\mathbf{x})] \propto \exp(-\beta\mathcal{H}[m(\mathbf{x})])$, for the original configurations, we can use the above change of variables to construct the corresponding probabilities $\mathcal{P}'[m'(\mathbf{x}')]$, for the new configurations. Naturally this is the most difficult step in the program.

(3) Since rotational symmetry is preserved by the RG procedure, the rescaled Hamiltonian must also be described by a point in the parameter space of Eq. (4.41), i.e.

$$\beta\mathcal{H}'[m'(\mathbf{x}')] \equiv \ln \mathcal{P}'[m'(\mathbf{x}')] = f_b + \int \mathrm{d}^d\mathbf{x}' \left[\frac{t'}{2} m'^2 + u' m'^4 + v' m'^6 + \cdots + \frac{K'}{2}(\nabla m')^2 + \frac{L'}{2}(\nabla^2 m')^2 + \cdots \right]. \tag{4.43}$$

The renormalized parameters are functions of the original ones, i.e. $t' = t_b(t, u, ...)$; $u' = u_b(t, u, ...)$, etc., defining a mapping $S' = \Re_b S$ in parameter space.

(4) The operation $\Re_b$ describes the effects of dilation on the Hamiltonian of the system. Hamiltonians that describe statistically self-similar configurations must thus correspond to *fixed points* S^*, such that $\Re_b S^* = S^*$. Since the correlation length, a function of Hamiltonian parameters, is reduced by b under the RG operation (i.e. $\xi(S) = b\xi(\Re_b S)$), the correlation length at a fixed point must be zero or infinity. Fixed points with $\xi^* = 0$ describe independent fluctuations at each point and correspond to complete disorder (infinite temperature), or complete order (zero temperature). A fixed point with $\xi^* = \infty$ describes a critical point ($T = T_c$).

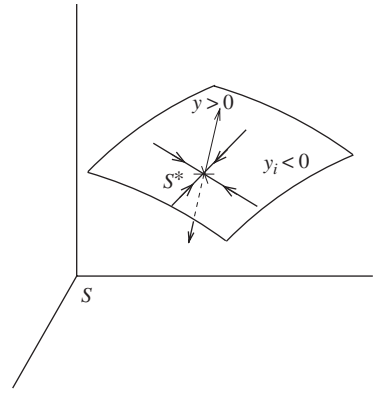

Fig. 4.3 The fixed point S^* has a basin of attraction spanned by the irrelevant directions of negative eigenvalues y_i, and is unstable in the relevant direction with $y > 0$.

(5) Equations (4.32) represent a simplified case in which the parameter space is two dimensional. The point $t = h = 0$ is a fixed point, and the lowest order terms in these equations describe the behavior in its neighborhood. In general, we can study the stability of a fixed point by *linearizing* the recursion relations in its vicinity: Under RG, a point $S^* + \delta S$ is transformed to

$$S^*_\alpha + \delta S'_\alpha = S^*_\alpha + (\Re^L_b)_{\alpha\beta}\delta S_\beta + \cdots, \quad \text{where} \quad (\Re^L_b)_{\alpha\beta} \equiv \left.\frac{\partial S'_\alpha}{\partial S_\beta}\right|_{S^*}. \tag{4.44}$$

We now diagonalize the matrix $(\Re_b^L)_{\alpha\beta}$ to get the eigenvectors $\mathcal{O}_i$, and corresponding eigenvalues $\lambda(b)_i$. Because of the group property[2],

$$\Re_b^L \Re_{b'}^L \mathcal{O}_i = \lambda(b)_i \lambda(b')_i \mathcal{O}_i = \Re_{bb'}^L \mathcal{O}_i = \lambda(bb')_i \mathcal{O}_i. \tag{4.45}$$

Together with the condition $\lambda(1)_i = 1$, the above equation implies

$$\lambda(b)_i = b^{y_i}. \tag{4.46}$$

The vectors $\mathcal{O}_i$ are called scaling directions associated with the fixed point S^*, and y_i are the corresponding *anomalous dimensions*. Any Hamiltonian in the vicinity of the fixed point is described by a point $S = S^* + \Sigma_i g_i \mathcal{O}_i$. The renormalized Hamiltonian has interaction parameters $S' = S^* + \Sigma_i g_i b^{y_i} \mathcal{O}_i$. The following terminology is used to classify the eigenoperators:

- If $y_i > 0$, g_i *increases* under scaling, and $\mathcal{O}_i$ is a *relevant operator*.
- If $y_i < 0$, g_i *decreases* under scaling, and $\mathcal{O}_i$ is an *irrelevant operator*.
- If $y_i = 0$, g_i is called a *marginal operator*, and higher order terms are necessary to track its behavior.

The subspace spanned by the irrelevant operators is the *basin of attraction* of the fixed point S^*. Since ξ always decreases under RG, and $\xi(S^*) = \infty$, then ξ is also infinite for any point on the basin of attraction of S^*. For a general point in the vicinity of S^*, the correlation length satisfies

$$\xi(g_1, g_2, \cdots) = b\xi(b^{y_1} g_1, b^{y_2} g_2, \cdots). \tag{4.47}$$

For a sufficiently large b, all the irrelevant operators scale to zero. The leading singularities of ξ are then determined by the remaining set of *relevant* operators. In particular, if the operators are indexed in order of decreasing dimensions, we can choose b such that $b^{y_1} g_1 = 1$. In this case, Eq. (4.47) implies

$$\xi(g_1, g_2, \cdots) = g_1^{-1/y_1} f(g_2/g_1^{y_2/y_1}, \cdots). \tag{4.48}$$

We have thus obtained an exponent $\nu_1 = 1/y_1$, for the divergence of ξ, and a generalized set of gap exponents $\Delta_\alpha = y_\alpha/y_1$, associated with g_α.

Let us imagine that the fixed point S^* describes the critical point of the magnet in Eq. (4.41) at zero magnetic field. As the temperature, or some other control parameter, is changed, the coefficients of the effective Hamiltonian are altered, and the point S moves along a trajectory in parameter space. Except for a single point (at the critical temperature) the magnet has a finite correlation length. This can be achieved if the trajectory taken by the point S intersects the basis of attraction of S^* only at one point. To achieve this, the basin of attraction must have co-dimension one, i.e. the fixed point S^* must have one and only one relevant operator. This

[2] The group property $\Re_b^L \Re_{b'}^L = \Re_{bb'}^L = \Re_{b'}^L \Re_b^L$ also implies that the linearized matrices for different b commute. It is thus possible to diagonalize them simultaneously, implying that the eigenvectors $\{\mathcal{O}_i\}$ are independent of b.

provides an explanation of *universality*, in that the very many microscopic details of the system make up the huge space of irrelevant operators comprising the basin of attraction. In the presence of a magnetic field, two system parameters must be adjusted to reach the critical point ($T = T_c$ and $h = 0$). Thus the magnetic field corresponds to an additional relevant operator at S^*. Again, other "odd" interactions, such as $\{m^3, m^5, \cdots\}$, should not lead to any other relevant operators.

Although the formal procedure outlined in this section is quite rigorous, it suffers from some quite obvious shortcomings: How do we actually implement the RG transformations of step (2) analytically? There are an infinite number of interactions allowed by symmetry, and hence the space of parameters S is inconveniently large. How do we know a priori that there are fixed points for the RG transformation; that $\Re_b$ can be linearized; that relevant operators are few, etc.? Following the initial formulation of RG by Kadanoff, there was a period of uncertainty until Wilson showed how these steps can be implemented (at least perturbatively) in the Landau–Ginzburg model.

4.6 The Gaussian model (direct solution)

The RG approach will be applied to the *Gaussian model* in the next section. For the sake of later comparison, here we provide the direct solution of this problem. The Gaussian model is obtained by keeping only the quadratic terms in the Landau–Ginzburg expansion. The resulting partition function is

$$Z = \int \mathcal{D}\vec{m}(\mathbf{x}) \exp\left\{ - \int \mathrm{d}^d\mathbf{x} \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 + \frac{L}{2} (\nabla^2 m)^2 + \cdots - \vec{h} \cdot \vec{m} \right] \right\}. \quad (4.49)$$

Clearly the model is well defined only for $t \geq 0$, since there is no m^4 term to insure its stability for $t < 0$. The partition function still has a singularity at $t = 0$, and we can regard this as representing approaching a phase transition from the disordered side.

The quadratic form is easily evaluated by the usual rules of Gaussian integration. The kernel is first diagonalized by the Fourier modes: The allowed $\mathbf{q}$ values are discretized in a finite system of size L, with spacing of $2\pi/L$. The largest $\mathbf{q}$ are limited by the lattice spacing, and confined to a *Brillouin zone* whose shape is determined by the underlying lattice. We shall in fact use a slightly different normalization for the Fourier modes, and keep careful track of the volume factors, by setting

$$\begin{cases} \vec{m}(\mathbf{q}) = \int \mathrm{d}^d\mathbf{x} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \vec{m}(\mathbf{x}) \\ \vec{m}(\mathbf{x}) = \sum_{\mathbf{q}} \frac{\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}}{V} \vec{m}(\mathbf{q}) = \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \vec{m}(\mathbf{q}). \end{cases} \quad (4.50)$$

(We should really use a different symbol, such as $\tilde{m}_i(\mathbf{q})$ to indicate the Fourier modes. For the sake of brevity we use the same symbol, but explicitly include the argument $\mathbf{q}$ as the indicator of the Fourier transformed function.) The last transformation applies to the infinite size limit ($L \to \infty$), and V is the system volume.

In re-expressing the Hamiltonian in terms of Fourier modes, we encounter expressions such as

$$\int \mathrm{d}^d\mathbf{x} m(\mathbf{x})^2 = \int \mathrm{d}^d\mathbf{x} \sum_{\mathbf{q},\mathbf{q}'} \frac{\mathrm{e}^{-\mathrm{i}(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}}{V^2} \vec{m}(\mathbf{q})\cdot\vec{m}(\mathbf{q}') = \sum_{\mathbf{q}} \frac{\vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q})}{V}. \quad (4.51)$$

The last expression follows from the vanishing of the integral over $\mathbf{x}$ unless $\mathbf{q}+\mathbf{q}' = \mathbf{0}$, in which case it equals the system volume. Similar manipulations lead to the Hamiltonian

$$\beta\mathcal{H} = \sum_{\mathbf{q}} \left(\frac{t + Kq^2 + Lq^4 + \cdots}{2V} \right) |m(\mathbf{q})|^2 - \vec{h}\cdot\vec{m}(\mathbf{q}=\mathbf{0}). \quad (4.52)$$

With the choice of the normalization in Eq. (4.50), the Jacobian of the transformation to Fourier modes is $1/\sqrt{V}$ per mode, and the partition function equals

$$Z = \prod_{\mathbf{q}} V^{-n/2} \int \mathrm{d}\vec{m}(\mathbf{q}) \exp\left[-\frac{t + Kq^2 + Lq^4 + \cdots}{2V} |m(\mathbf{q})|^2 + \vec{h}\cdot\vec{m}(\mathbf{q}=0) \right]. \quad (4.53)$$

The integral for $\mathbf{q} = \mathbf{0}$ is

$$Z_0 = V^{-n/2} \int_{-\infty}^{\infty} \mathrm{d}\vec{m}(\mathbf{0}) \exp\left[-\frac{t}{2V} |m(\mathbf{0})|^2 + \vec{h}\cdot\vec{m}(\mathbf{0}) \right] = \left(\frac{2\pi}{t} \right)^{n/2} \exp\left[\frac{Vh^2}{2t} \right]. \quad (4.54)$$

After performing the integrations for $\mathbf{q} \neq \mathbf{0}$, we obtain

$$Z = \exp\left[\frac{Vh^2}{2t} \right] \prod_{\mathbf{q}} \left(\frac{2\pi}{t + Kq^2 + Lq^4 + \cdots} \right)^{n/2}. \quad (4.55)$$

The total number of modes, N, equals the number of original lattice points. Apart from a constant factor resulting from $(2\pi)^{nN/2}$, the free energy is

$$f(t,h) = -\frac{\ln Z}{V} = \frac{n}{2} \int_{BZ} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \ln\left(t + Kq^2 + Lq^4 + \cdots \right) - \frac{h^2}{2t}. \quad (4.56)$$

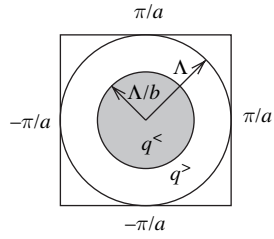

Fig. 4.4 The Brillouin zone is approximated by a hypersphere of radius Λ, which is then reduced by a factor of b in this RG scheme.

The integral in Eq. (4.56) is over the Brillouin zone, which for a hypercubic lattice of spacing a, is a cube of side $2\pi/a$ centered on the origin. However, we expect the singularities to originate from the long wavelength modes close to $\mathbf{q} = \mathbf{0}$. The contributions from the vicinity of the Brillouin zone edge are clearly analytic, since the logarithm can be simply expanded in powers of t for

a finite q^2. Thus to simplify the extraction of the singular behavior as $t \to 0$, we approximate the shape of the Brillouin zone by a hypersphere of radius $\Lambda \approx \pi/a$. The spherical symmetry of the integrand then allows us to write

$$f_{\rm sing}(t,h) = \frac{n}{2} K_d \int_0^\Lambda {\rm d}q q^{d-1} \ln\left(t + Kq^2 + Lq^4 + \cdots\right) - \frac{h^2}{2t}, \tag{4.57}$$

where $K_d \equiv S_d/(2\pi)^d$, and S_d is the d-dimensional solid angle. The leading dependence of the integral on t can be extracted after rescaling q by a factor of $\sqrt{t/K}$, as

$$\begin{aligned} f_{\rm sing}(t,h) = \frac{n}{2} K_d \left(\frac{t}{K}\right)^{d/2} \int_0^{\Lambda\sqrt{K}/\sqrt{t}} {\rm d}x x^{d-1} \\ \times \left[\ln t + \ln\left(1 + x^2 + Ltx^4/K^2 + \cdots\right)\right] - \frac{h^2}{2t}. \end{aligned} \tag{4.58}$$

Ignoring analytic contributions in t, the leading singular dependence of the free energy can be written as

$$f_{\rm sing}(t,h) = -t^{d/2}\left[A + \frac{h^2}{2t^{1+d/2}}\right] \equiv t^{2-\alpha} g_f(h/t^\Delta). \tag{4.59}$$

Note that the higher order gradients, i.e. terms proportional to $L, \cdots$, do not effect the singular behavior in Eq. (4.59). On approaching the point $h = 0$ for $t = 0^+$, the singular part of the free energy is described by a homogeneous scaling form, with exponents

$$\alpha_+ = 2 - d/2, \quad \Delta = 1/2 + d/4. \tag{4.60}$$

Since the ordered phase for $t < 0$ is not stable, the exponent β is undefined. The susceptibility $\chi \propto \partial^2 f/\partial h^2 \propto 1/t$, however, diverges with the exponent $\gamma_+ = 1$.

4.7 The Gaussian model (renormalization group)

The renormalization of the Gaussian model is most conveniently performed in terms of the Fourier modes. The goal is to evaluate the partition function

$$Z \sim \int \mathcal{D}\vec{m}(\mathbf{q}) \exp\left[-\int_0^\Lambda \frac{{\rm d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t + Kq^2 + Lq^4 + \cdots}{2}\right)|m(\mathbf{q})|^2 + \vec{h}\cdot\vec{m}(\mathbf{0})\right] \tag{4.61}$$

indirectly via the three steps of RG. Note that the Brillouin zone is approximated by a hypersphere of radius Λ.

(1) ***Coarse grain:*** Eliminating fluctuations at scales $a < x < ba$ is similar to removing Fourier modes with wavenumbers $\Lambda/b < q < \Lambda$. We thus break up the momenta into two subsets,

$$\{\vec{m}(\mathbf{q})\} = \{\vec{\sigma}(\mathbf{q}^>)\} \oplus \{\tilde{\vec{m}}(\mathbf{q}^<)\}, \tag{4.62}$$

and write

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}^<) \int \mathcal{D}\vec{\sigma}(\mathbf{q}^>) {\rm e}^{-\beta\mathcal{H}[\tilde{\vec{m}},\vec{\sigma}]}. \tag{4.63}$$

Since the two sets of modes are decoupled in the Gaussian model, the integration is trivial, and

$$\begin{aligned} Z \sim \exp\left[-\frac{n}{2}V\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\ln(t+Kq^2+Lq^4+\cdots)\right]\times\int\mathcal{D}\tilde{\vec{m}}(\mathbf{q}^<) \\ \times\exp\left[-\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2+Lq^4+\cdots}{2}\right)|\tilde{m}(\mathbf{q})|^2+\vec{h}\cdot\tilde{\vec{m}}(\mathbf{0})\right]. \end{aligned} \tag{4.64}$$

(2) ***Rescale:*** The partition function for the modes $\tilde{\vec{m}}(\mathbf{q}^<)$ is similar to the original, except that the upper cutoff has decreased to Λ/b, reflecting the coarsening in resolution. The rescaling $\mathbf{x}'=\mathbf{x}/b$ in real space is equivalent to $\mathbf{q}'=b\mathbf{q}$ in *momentum space*, and restores the cutoff to its original value. The rescaled partition function is

$$\begin{aligned} Z=\mathrm{e}^{-V\delta f_b(t)}\times\int\mathcal{D}\tilde{\vec{m}}(\mathbf{q}')\times\exp\left[-\int_0^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}'}{(2\pi)^d}b^{-d}\right. \\ \left.\times\left(\frac{t+Kb^{-2}q'^2+Lb^{-4}q'^4+\cdots}{2}\right)|\tilde{m}(\mathbf{q}')|^2+\vec{h}\cdot\tilde{\vec{m}}(\mathbf{0})\right]. \end{aligned} \tag{4.65}$$

(3) ***Renormalize:*** The final step of RG in real space is the renormalization of magnetization, $\vec{m}'(\mathbf{x}')=\tilde{\vec{m}}(\mathbf{x}')/\zeta$. Alternatively, we can renormalize the Fourier modes according to $\vec{m}'(\mathbf{q}')=\tilde{\vec{m}}(\mathbf{q}')/z$, resulting in

$$\begin{aligned} Z=\mathrm{e}^{-V\delta f_b(t)}\times\int\mathcal{D}\vec{m}'(\mathbf{q}')\times\exp\left[-\int_0^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}'}{(2\pi)^d}b^{-d}z^2\right. \\ \left.\times\left(\frac{t+Kb^{-2}q'^2+Lb^{-4}q'^4+\cdots}{2}\right)|m'(\mathbf{q}')|^2+z\vec{h}\cdot\vec{m}'(\mathbf{0})\right]. \end{aligned} \tag{4.66}$$

(Note that the factors ζ and z, for rescalings of magnetization in real and Fourier space, are different.)

Equation (4.66) indicates the renormalized modes are distributed according to a Gaussian Hamiltonian, with renormalized parameters,

$$\begin{cases} t'=z^2b^{-d}t \\ h'=zh \\ K'=z^2b^{-d-2}K \\ L'=z^2b^{-d-4}L \\ \cdots \end{cases} \tag{4.67}$$

The singular point $t=h=0$ is mapped onto itself as expected. To make the fluctuations scale invariant at this point, we must insure that the remaining Hamiltonian stays fixed. This is achieved by the choice of $z=b^{1+d/2}$, which sets $K'=K$, and makes the remaining parameters $L,\cdots$ scale to zero. Away from criticality, the two relevant directions now scale as

$$\begin{cases} t'=b^2t \\ h'=b^{1+d/2}h \end{cases} \implies \begin{cases} y_t=2 \\ y_h=1+d/2. \end{cases} \tag{4.68}$$

Using the results of Section 3.4, we can identify the critical exponents,

$$\begin{aligned}\nu &= 1/y_t = 1/2,\\ \Delta &= \frac{y_h}{y_t} = \frac{1+d/2}{2} = \frac{1}{2}+\frac{d}{4},\\ \alpha &= 2-d\nu = 2-d/2,\end{aligned}$$

in agreement with the direct solution in the previous section.

The fixed point Hamiltonian ($t^* = h^* = L^* = \cdots = 0$) in *real space* is

$$\beta\mathcal{H}^* = \frac{K}{2}\int_a \mathrm{d}^d\mathbf{x}(\nabla m)^2. \tag{4.69}$$

(The subscript a is placed on the integral as a reminder of the implicit short distance cutoff.) Under a simple rescaling $\mathbf{x} \mapsto \mathbf{x}'$, and $\vec{m}(\mathbf{x}) \mapsto \zeta\vec{m}'(\mathbf{x}')$; $K \mapsto K' = b^{d-2}\zeta^2 K$. Scale invariance is achieved with the choice $\zeta = b^{1-d/2}$. Forgetting the coarse-graining step, a general power of $\vec{m}(\mathbf{x})$, added as a small perturbation to $(\beta\mathcal{H})^*$, behaves as

$$\beta\mathcal{H}^* + u_n\int \mathrm{d}^d\mathbf{x}m^n \quad \mapsto \quad \beta\mathcal{H}^* + u_n b^d\zeta^n\int \mathrm{d}^d\mathbf{x}'m'^n, \tag{4.70}$$

suggesting that such perturbations scale as

$$u_n' = b^d b^{n\left(\frac{2-d}{2}\right)}u_n, \quad \Longrightarrow \quad y_n = n - d\left(\frac{n}{2}-1\right). \tag{4.71}$$

The values $y_1 = 1+d/2$, and $y_2 = 2$, reproduce the exponents for y_h and y_t in Eq. (4.68). The operators with higher powers are less relevant. The next most important operator in a system with spherical symmetry is $y_4 = 4-d$, which is irrelevant for $d > 4$ but relevant for $d < 4$; $y_6 = 6-2d$ is relevant only for $d < 3$. Indeed the majority of operators are irrelevant at the Gaussian fixed point for $d > 2$.

Problems for chapter 4

1. *Scaling in fluids:* Near the liquid–gas critical point, the free energy is assumed to take the scaling form $F/N = t^{2-\alpha}g(\delta\rho/t^\beta)$, where $t = |T-T_c|/T_c$ is the reduced temperature, and $\delta\rho = \rho - \rho_c$ measures deviations from the critical point density. The leading singular behavior of any thermodynamic parameter $Q(t,\delta\rho)$ is of the form t^x on approaching the critical point along the isochore $\rho = \rho_c$; or $\delta\rho^y$ for a path along the isotherm $T = T_c$. Find the exponents x and y for the following quantities:

(a) The internal energy per particle $\langle H\rangle/N$, and the entropy per particle $s = S/N$.

(b) The heat capacities $C_V = T\partial s/\partial T\,|_V$, and $C_P = T\partial s/\partial T\,|_P$.

(c) The isothermal compressibility $\kappa_T = \partial\rho/\partial P\,|_T/\rho$, and the thermal expansion coefficient $\alpha = \partial V/\partial T\,|_P/V$.

Check that your results for parts (b) and (c) are consistent with the thermodynamic identity $C_P - C_V = TV\alpha^2/\kappa_T$.

(d) Sketch the behavior of the latent heat per particle L on the coexistence curve for $T < T_c$, and find its singularity as a function of t.

2. *The Ising model:* The differential recursion relations for temperature T, and magnetic field h, of the Ising model in $d = 1 + \epsilon$ dimensions are (for $\mathrm{b} = \mathrm{e}^{\ell}$)

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{d}\ell} = -\epsilon\, T + \frac{1}{2} T^2 \\ \dfrac{\mathrm{d}h}{\mathrm{d}\ell} = dh. \end{cases}$$

(a) Sketch the renormalization group flows in the (T, h) plane (for $\epsilon > 0$), marking the fixed points along the $h = 0$ axis.

(b) Calculate the eigenvalues y_t and y_h, at the critical fixed point, to order of ϵ.

(c) Starting from the relation governing the change of the correlation length ξ under renormalization, show that $\xi(t, h) = t^{-\nu} g_{\xi}\left(h/|t|^{\Delta}\right)$ (where $t = T/T_c - 1$), and find the exponents ν and Δ.

(d) Use a hyperscaling relation to find the singular part of the free energy $f_{\mathrm{sing}}(t, h)$, and hence the heat capacity exponent α.

(e) Find the exponents β and γ for the singular behaviors of the magnetization and susceptibility, respectively.

(f) Starting with the relation between susceptibility and correlations of local magnetizations, calculate the exponent η for the critical correlations ($\langle m(\mathbf{0}) m(\mathbf{x}) \rangle \sim |\mathbf{x}|^{-(d-2+\eta)}$).

(g) How does the correlation length diverge as $T \to 0$ (along $h = 0$) for $d = 1$?

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3. *The nonlinear σ model* describes n component unit spins. As we shall demonstrate later, in $d = 2$ dimensions, the recursion relations for temperature T, and magnetic field h, are (for $\mathrm{b} = \mathrm{e}^{\ell}$)

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{d}\ell} = \frac{(n-2)}{2\pi} T^2 \\ \dfrac{\mathrm{d}h}{\mathrm{d}\ell} = 2h. \end{cases}$$

(a) How does the correlation length diverge as $T \to 0$?

(b) Write down the singular form of the free energy as $T, h \to 0$.

(c) How does the susceptibility χ diverge as $T \to 0$ for $h = 0$?

4. *Coupled scalars:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x} \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 - hm + \frac{L}{2} (\nabla^2 \phi)^2 + v \nabla m . \nabla \phi \right],$$

coupling two one-component fields m and ϕ.

(a) Write $\beta\mathcal{H}$ in terms of the Fourier transforms $m(\mathbf{q})$ and $\phi(\mathbf{q})$.

(b) Construct a renormalization group transformation as in the text, by rescaling distances such that $\mathbf{q}' = b\mathbf{q}$; and the fields such that $m'(\mathbf{q}') = \tilde{m}(\mathbf{q})/z$ and $\phi'(\mathbf{q}') = \tilde{\phi}(\mathbf{q})/y$. Do not evaluate the integrals that just contribute a constant additive term.

(c) There is a fixed point such that $K' = K$ and $L' = L$. Find y_t, y_h and y_v at this fixed point.

(d) The singular part of the free energy has a scaling from $f(t, h, v) = t^{2-\alpha} g(h/t^{\Delta}, v/t^{\omega})$ for t, h, v close to zero. Find α, Δ, and ω.

(e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point, and how do they scale?

5
Perturbative renormalization group

5.1 Expectation values in the Gaussian model

Can we treat the Landau–Ginzburg Hamiltonian as a perturbation to the Gaussian model? In particular, for zero magnetic field, we shall examine

$$\begin{aligned}\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} \equiv \int \mathrm{d}^d\mathbf{x}\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right] \\ + u\int \mathrm{d}^d\mathbf{x} m^4 + \cdots .\end{aligned} \tag{5.1}$$

The *unperturbed* Gaussian Hamiltonian can be decomposed into independent Fourier modes, as

$$\begin{aligned}\beta\mathcal{H}_0 &= \frac{1}{V}\sum_{\mathbf{q}} \frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2 \\ &\equiv \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2.\end{aligned} \tag{5.2}$$

The *perturbative interaction* which mixes up the normal modes has the form

$$\begin{aligned}\mathcal{U} &= u\int \mathrm{d}^d\mathbf{x} m(\mathbf{x})^4 + \cdots \\ &= u\int \mathrm{d}^d\mathbf{x}\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)}m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(\mathbf{q}_4) \\ &\quad + \cdots ,\end{aligned} \tag{5.3}$$

where summation over α and β is implicit. The integral over $\mathbf{x}$ sets $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 = \mathbf{0}$, and

$$\mathcal{U} = u\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) + \cdots . \tag{5.4}$$

From the variance of the Gaussian weights, the two-point expectation values in a finite sized system with discretized modes are easily obtained as

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\mathbf{q},-\mathbf{q}'}\,\delta_{\alpha,\beta}\,V}{t + Kq^2 + Lq^4 + \cdots}. \tag{5.5}$$

In the limit of infinite size, the spectrum becomes continuous, and Eq. (5.5) goes over to

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\alpha,\beta}(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t + Kq^2 + Lq^4 + \cdots}. \tag{5.6}$$

The subscript 0 is used to indicate that the expectation values are taken with respect to the unperturbed (Gaussian) Hamiltonian. Expectation values involving any product of m's can be obtained starting from the identity

$$\left\langle \exp\left[\sum_i a_i m_i\right]\right\rangle_0 = \exp\left[\sum_{i,j}\frac{a_i a_j}{2}\langle m_i m_j\rangle_0\right], \tag{5.7}$$

which is valid for any set of Gaussian distributed variables $\{m_i\}$. (This is easily seen by "completing the square.") Expanding both sides of the equation in powers of $\{a_i\}$ leads to

$$\begin{aligned} 1 \;+\; & a_i\langle m_i\rangle_0 + \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k}{6}\langle m_i m_j m_k\rangle_0 + \frac{a_i a_j a_k a_l}{24}\langle m_i m_j m_k m_k\rangle_0 + \cdots = \\ 1 \;+\; & \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k a_l}{24}\left(\langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0\right. \\ & \left. +\; \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0\right) + \cdots \end{aligned} \tag{5.8}$$

Matching powers of $\{a_i\}$ on the two sides of the above equation gives

$$\left\langle \prod_{i=1}^{\ell} m_i\right\rangle_0 = \begin{cases} 0 & \text{for } \ell \text{ odd} \\ \text{sum over all pairwise contractions} & \text{for } \ell \text{ even.} \end{cases} \tag{5.9}$$

This result is known as *Wick's theorem*; and for example,

$$\langle m_i m_j m_k m_l\rangle_0 = \langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0.$$

5.2 Expectation values in perturbation theory

In the presence of an interaction $\mathcal{U}$, the expectation value of any operator $\mathcal{O}$ is computed perturbatively as

$$\begin{aligned} \langle\mathcal{O}\rangle &= \frac{\int \mathcal{D}\vec{m}\,\mathcal{O}\mathrm{e}^{-\beta\mathcal{H}_0-\mathcal{U}}}{\int \mathcal{D}\vec{m}\,\mathrm{e}^{-\beta\mathcal{H}_0-\mathcal{U}}} = \frac{\int \mathcal{D}\vec{m}\,\mathrm{e}^{-\beta\mathcal{H}_0}\mathcal{O}[1-\mathcal{U}+\mathcal{U}^2/2-\cdots]}{\int \mathcal{D}\vec{m}\,\mathrm{e}^{-\beta\mathcal{H}_0}[1-\mathcal{U}+\mathcal{U}^2/2-\cdots]} \\ &= \frac{Z_0[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \langle\mathcal{O}\mathcal{U}^2\rangle_0/2-\cdots]}{Z_0[1-\langle\mathcal{U}\rangle_0+\langle\mathcal{U}^2\rangle_0/2-\cdots]}. \end{aligned} \tag{5.10}$$

Inverting the denominator by an expansion in powers of $\mathcal{U}$ gives

$$\begin{aligned} \langle\mathcal{O}\rangle &= \left[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \frac{1}{2}\langle\mathcal{O}\mathcal{U}^2\rangle_0 - \cdots\right]\left[1+\langle\mathcal{U}\rangle_0+\langle\mathcal{U}\rangle_0^2 - \frac{1}{2}\langle\mathcal{U}^2\rangle_0 - \cdots\right] \\ &= \langle\mathcal{O}\rangle_0 - (\langle\mathcal{O}\mathcal{U}\rangle_0 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0) + \frac{1}{2}\left(\langle\mathcal{O}\mathcal{U}^2\rangle_0 - 2\langle\mathcal{O}\mathcal{U}\rangle_0\langle\mathcal{U}\rangle_0\right. \\ &\quad \left. +2\langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0^2 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}^2\rangle_0\right) + \cdots \\ &\equiv \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle\mathcal{O}\mathcal{U}^n\rangle_0^c. \end{aligned} \tag{5.11}$$

The *connected averages* (cumulants) are defined as the combination of unperturbed expectation values appearing at various orders in the expansion. Their significance will become apparent in diagrammatic representations, and from the following example.

Let us calculate the two-point correlation function of the Landau–Ginzburg model to first order in the parameter u. (In view of their expected irrelevance, we shall ignore higher order interactions, and also only keep the lowest order Gaussian terms.) Substituting Eq. (5.4) into Eq. (5.11) yields

$$\begin{aligned}\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 - u \int \frac{\mathrm{d}^d\mathbf{q}_1 \mathrm{d}^d\mathbf{q}_2 \mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} \\ \times[\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}') m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ -\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 \langle m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0] \\ + \mathcal{O}(u^2). \qquad (5.12)\end{aligned}$$

To calculate $\langle \mathcal{O}\mathcal{U}\rangle_0$ we need the unperturbed expectation value of the product of six m's. This can be evaluated using Eq. (5.9) as the sum of all pair-wise contractions, 15 in all. Three contractions are obtained by first pairing m_α to m_β, and then the remaining four m's in $\mathcal{U}$. Clearly these contractions cancel exactly with corresponding ones in $\langle \mathcal{O}\rangle_0 \langle \mathcal{U}\rangle_0$. The only surviving terms involve contractions that connect $\mathcal{O}$ to $\mathcal{U}$. This cancellation persists at all orders, and $\langle \mathcal{O}\mathcal{U}^n\rangle_0^c$ contains only terms in which all $n+1$ operators are connected by contractions. The remaining 12 pairings in $\langle \mathcal{O}\mathcal{U}\rangle_0$ fall into two classes:

(1) Four pairings involve contracting m_α and m_β to m's with the same index, e.g.

$$\begin{aligned}&\langle m_\alpha(\mathbf{q})\, m_i(\mathbf{q}_1)\rangle_0 \langle m_\beta(\mathbf{q}')\, m_i(\mathbf{q}_2)\rangle_0 \langle m_j(\mathbf{q}_3)\, m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ &= \frac{\delta_{\alpha i}\,\delta_{\beta i}\,\delta_{jj}\,(2\pi)^{3d}\,\delta^d(\mathbf{q}+\mathbf{q}_1)\,\delta^d(\mathbf{q}'+\mathbf{q}_2)\,\delta^d(\mathbf{q}_1+\mathbf{q}_2)}{(t+Kq^2)\,(t+Kq'^2)\,(t+Kq_3^2)}, \qquad (5.13)\end{aligned}$$

where we have used Eq. (5.6). After summing over i and j, and integrating over $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, these terms make a contribution

$$-4u \frac{n\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2} \int \frac{\mathrm{d}^d\mathbf{q}_3}{(2\pi)^d} \frac{1}{t+Kq_3^2}. \qquad (5.14)$$

(2) Eight pairings involve contracting m_α and m_β to m's with different indices, e.g.

$$\begin{aligned}&\langle m_\alpha(\mathbf{q}) m_i(\mathbf{q}_1)\rangle_0 \langle m_\beta(\mathbf{q}') m_j(\mathbf{q}_3)\rangle_0 \langle m_i(\mathbf{q}_2) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ &= \frac{\delta_{\alpha i}\delta_{\beta j}\delta_{ij}(2\pi)^{3d}\delta^d(\mathbf{q}+\mathbf{q}_1)\delta^d(\mathbf{q}'+\mathbf{q}_3)\delta^d(\mathbf{q}_1+\mathbf{q}_3)}{(t+Kq^2)(t+Kq'^2)(t+Kq_2^2)}. \qquad (5.15)\end{aligned}$$

Summing over all indices, and integrating over the momenta leads to an overall contribution of

$$-8u \frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2} \int \frac{\mathrm{d}^d\mathbf{q}_2}{(2\pi)^d} \frac{1}{t+Kq_2^2}. \qquad (5.16)$$

Adding up both contributions, we obtain

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{t+Kq^2}\left[1-\frac{4u(n+2)}{t+Kq^2}\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\,\frac{1}{t+Kk^2}+\mathcal{O}(u^2)\right]. \tag{5.17}$$

5.3 Diagrammatic representation of perturbation theory

The calculations become more involved at higher orders in perturbation theory. A diagrammatic representation can be introduced to help keep track of all possible contractions. To calculate the ℓ-point expectation value $\langle\prod_{i=1}^{\ell} m_{\alpha_i}(\mathbf{q}_i)\rangle$, at pth order in u, proceed according to the following rules:

(1) Draw ℓ *external points* labeled by $(\mathbf{q}_i, \alpha_i)$ corresponding to the coordinates of the required correlation function. Draw p *vertices* with four legs each, labeled by *internal* momenta and indices, e.g. $\{(\mathbf{k}_1, i), (\mathbf{k}_2, i), (\mathbf{k}_3, j), (\mathbf{k}_4, j)\}$. Since the four legs are not equivalent, the four point vertex is indicated by two solid branches joined by a dotted line. (The extension to higher order interactions is straightforward.)

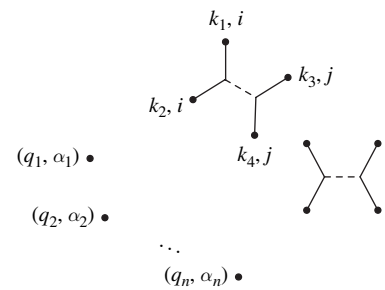
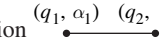
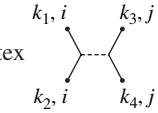


Fig. 5.1 Elements of the diagrammatic representation of perturbation theory.

(2) Each point of the graph now corresponds to one factor of $m_{\alpha_i}(\mathbf{q}_i)$, and the unperturbed average of the product is computed by Wick's theorem. This is implemented by joining all external and internal points *pair-wise*, by lines connecting one point to another, in all topologically distinct ways; see (5) below.

(3) The algebraic value of each such graph is obtained as follows: (i) A line joining a pair of points represents the two point average;[1] e.g. a connection 

, corresponds to $\delta_{\alpha_1\alpha_2}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2)/(t+Kq_1^2)$; (ii) A vertex 

stands for a term $u(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$ (the delta-function insures that momentum is conserved).

[1] Because of its original formulation in quantum field theory, the line joining two points is usually called a *propagator*. In this context, the line represents the world-line of a particle in time, while the perturbation $\mathcal{U}$ is an "interaction" between particles. For the same reason, the Fourier index is called a "momentum".

(4) Integrate over the $4p$ internal momenta $\{\mathbf{k}_i\}$, and sum over the $2p$ internal indices. Note that each closed loop produces a factor of $\delta_{ii} = n$ at this stage.

(5) There is a numerical factor of

$$\frac{(-1)^p}{p!} \times \text{number of different pairings leading to the same topology.}$$

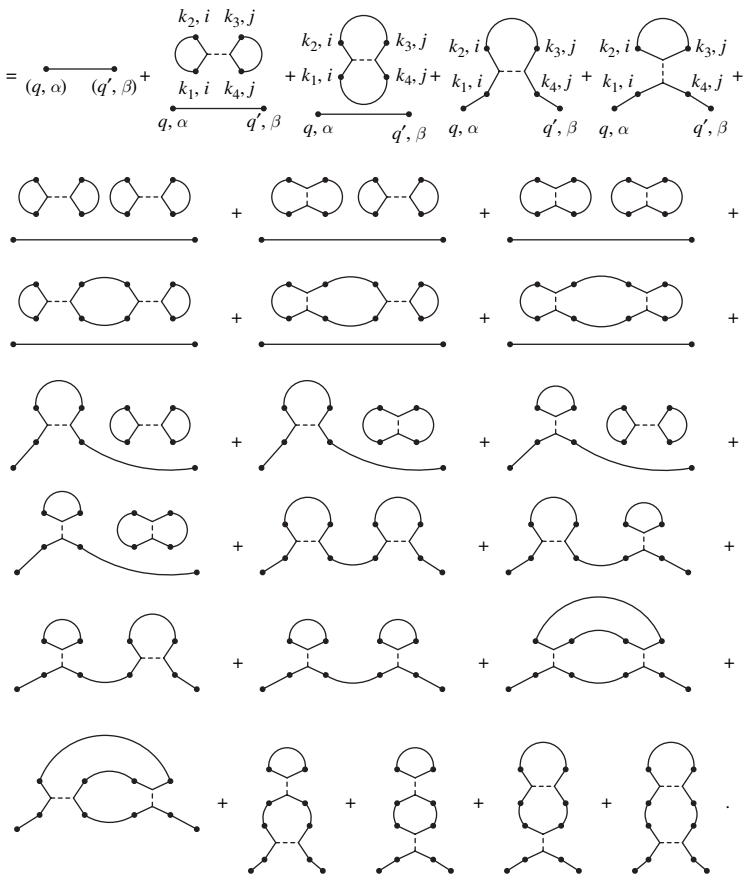
The first contribution comes from the expansion of the exponential; the second merely states that graphs related by symmetry give the same result, and can be calculated once.

(6) When calculating cumulants, only fully connected diagrams (without disjoint pieces) need to be included. This is a tremendous simplification.

For example, the diagrams appearing in the expansion for the propagator

$$\langle m_\alpha(q) m_\beta(q') \rangle \equiv \underset{(q,\alpha) \quad (q^l,\beta)}{\text{⟨double line⟩}},$$

to second order are



5.4 Susceptibility

It is no accident that the correction term in Eq. (5.17) is similar in form to the unperturbed value. This is because the form of the two point correlation function is constrained by symmetries, as can be seen from the identity

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \int \mathrm{d}^d\mathbf{x} \int \mathrm{d}^d\mathbf{x}' \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'} \langle m_\alpha(\mathbf{x})\, m_\beta(\mathbf{x}')\rangle. \tag{5.18}$$

The two-point correlation function in real space must satisfy translation and rotation symmetry, and (in the high temperature phase)$\langle m_\alpha(\mathbf{x})\, m_\beta(\mathbf{x}')\rangle = \delta_{\alpha\beta}\langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle$. Transforming to center of mass and relative coordinates, the above integral becomes,

$$\begin{aligned}
&\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle \\
&= \int \mathrm{d}^d \left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) \mathrm{d}^d\,(\mathbf{x}-\mathbf{x}')\, \mathrm{e}^{i(\mathbf{q}+\mathbf{q}')\cdot(\mathbf{x}+\mathbf{x}')/2} \mathrm{e}^{i(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{q}-\mathbf{q}')/2} \delta_{\alpha\beta}\langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle \\
&\equiv (2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')\delta_{\alpha\beta} S(q),
\end{aligned} \tag{5.19}$$

where

$$S(q) = \left\langle |m_1(\mathbf{q})|^2 \right\rangle = \int \mathrm{d}^d\mathbf{x} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}} \langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle \tag{5.20}$$

is the quantity observed in scattering experiments (Section 2.4).

From Eq. (5.17) we obtain

$$S(q) = \frac{1}{t+Kq^2}\left[1 - \frac{4u(n+2)}{t+Kq^2}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2)\right]. \tag{5.21}$$

It is useful to examine the expansion of the inverse quantity

$$S(q)^{-1} = t + Kq^2 + 4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2). \tag{5.22}$$

In the high-temperature phase, Eq. (5.20) indicates that the $q \to 0$ limit of $S(q)$ is just the magnetic susceptibility χ. For this reason, $S(q)$ is sometimes denoted by $\chi(q)$. From Eq. (5.22), the inverse susceptibility is given by

$$\chi^{-1}(t) = t + 4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2). \tag{5.23}$$

The susceptibility no longer diverges at $t = 0$, since

$$\begin{aligned}
\chi^{-1}(0) &= 4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{Kk^2} = \frac{4(n+2)u}{K}\frac{S_d}{(2\pi)^d}\int_0^\Lambda dk k^{d-3} \\
&= \frac{4(n+2)u}{K} K_d \left(\frac{\Lambda^{d-2}}{d-2}\right)
\end{aligned} \tag{5.24}$$

is a finite number ($K_d \equiv S_d/(2\pi)^d$). This is because in the presence of u the critical temperature is reduced to a negative value. The modified critical point is obtained by requiring $\chi^{-1}(t_c) = 0$, and hence from Eq. (5.23), to order of u,

$$t_c = -4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t_c+Kk^2} \approx -\frac{4u(n+2)K_d\Lambda^{d-2}}{(d-2)K} < 0. \tag{5.25}$$

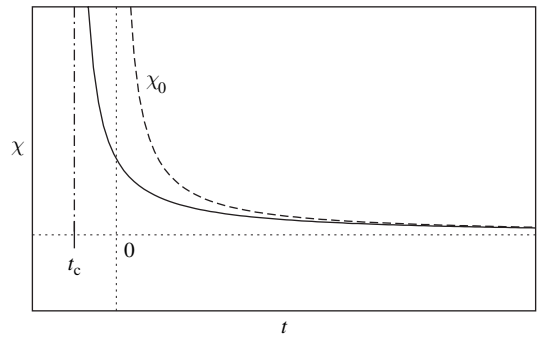


Fig. 5.2 The divergence of susceptibility occurs at a lower temperature due to the interaction u.

How does the perturbed susceptibility diverge at the shifted critical point? From Eq. (5.23),

$$\begin{aligned}\chi^{-1}(t)-\chi^{-1}(t_c) &= t-t_c+4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\left(\frac{1}{t+Kk^2}-\frac{1}{t_c+Kk^2}\right)\\ &= (t-t_c)\left[1-\frac{4u(n+2)}{K^2}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{k^2(k^2+(t-t_c)/K)}+\mathcal{O}(u^2)\right].\end{aligned} \tag{5.26}$$

In going from the first equation to the second, we have changed the position of t_c from one denominator to another. Since $t_c = \mathcal{O}(u)$, the corrections due to this change only appear at $\mathcal{O}(u^2)$. The final integral has dimensions of $\left[k^{d-4}\right]$. For $d > 4$ it is dominated by the largest momenta and scales as Λ^{d-4}. For $2 < d < 4$, the integral is convergent at both limits. Its magnitude is therefore set by the momentum scale $\xi^{-1} = \sqrt{(t-t_c)/K}$, which can be used to make the integrand dimensionless. Hence, in these dimensions,

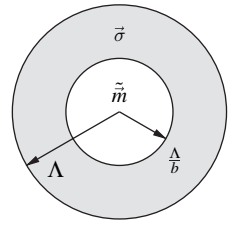
$$\chi^{-1}(t) = (t-t_c)\left[1-\frac{4u(n+2)}{K^2}c\left(\frac{K}{t-t_c}\right)^{2-d/2}+\mathcal{O}(u^2)\right], \tag{5.27}$$

where c is a constant. For $d < 4$, the correction term at the order of u diverges at the phase transition, masking the unperturbed singularity of χ with $\gamma = 1$. Thus the perturbation series is inherently inapplicable for describing the divergence of susceptibility in $d < 4$. The same conclusion arises in calculating any other quantity perturbatively. Although we start by treating u as the perturbation parameter, it is important to realize that it is not dimensionless; u/K^2 has dimensions of $(\text{length})^{d-4}$. The perturbation series for any quantity then takes the form $X(t,u) = X_0(t)[1+f(ua^{4-d}/K^2, u\xi^{4-d}/K^2)]$, where f is a power series. The two length scales a and ξ are available to construct dimensionless variables. Since ξ diverges close to the critical point, there is an inherent failure of the perturbation series. The effective (dimensionless) perturbation parameter diverges at t_c and is not small, making it an inherently ineffective expansion parameter.

5.5 Perturbative RG (first order)

The last section demonstrates how various expectation values associated with the Landau–Ginzburg Hamiltonian can be calculated perturbatively in powers of u. However, the perturbative series is inherently divergent close to the critical point and cannot be used to characterize critical behavior in dimensions $d \leq 4$. K.G. Wilson showed that it is possible to combine perturbative and renormalization group approaches into a systematic method for calculating critical exponents. Accordingly, we shall extend the RG calculation of Gaussian model in Section 3.7 to the Landau–Ginzburg Hamiltonian, by treating $\mathcal{U} = u \int \mathrm{d}^d\mathbf{x} m^4$ as a perturbation.

(1) ***Coarse grain:*** This is the most difficult step of the RG procedure. As before, subdivide the fluctuations into two components as,



$$\vec{m}(\mathbf{q}) = \begin{cases} \tilde{\vec{m}}(\mathbf{q}) & \text{for } 0 < q < \Lambda/b \\ \vec{\sigma}(\mathbf{q}) & \text{for } \Lambda/b < q < \Lambda. \end{cases} \tag{5.28}$$

In the partition function,

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}) \mathcal{D}\vec{\sigma}(\mathbf{q}) \exp\left\{ -\int_0^\Lambda \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) \left(|\tilde{m}(\mathbf{q})|^2 + |\sigma(\mathbf{q})|^2 \right) - \mathcal{U}[\tilde{\vec{m}}(\mathbf{q}), \vec{\sigma}(\mathbf{q})] \right\}, \tag{5.29}$$

the two sets of modes are mixed by the operator $\mathcal{U}$. Formally, the result of integrating out $\{\vec{\sigma}(\mathbf{q})\}$ can be written as

$$\begin{aligned} Z &= \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}) \exp\left\{ -\int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |\tilde{m}(\mathbf{q})|^2 \right\} \\ &\quad \times \exp\left\{ -\frac{nV}{2} \int_{\Lambda/b}^\Lambda \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \ln\left(t + Kq^2\right) \right\} \left\langle \mathrm{e}^{-\mathcal{U}[\tilde{\vec{m}}, \vec{\sigma}]} \right\rangle_\sigma \\ &\equiv \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}) \mathrm{e}^{-\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}]}. \end{aligned} \tag{5.30}$$

Here we have defined the partial averages

$$\langle \mathcal{O} \rangle_\sigma \equiv \int \frac{\mathcal{D}\vec{\sigma}(\mathbf{q})}{Z_\sigma} \mathcal{O} \exp\left[-\int_{\Lambda/b}^\Lambda \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |\sigma(\mathbf{q})|^2 \right], \tag{5.31}$$

with $Z_\sigma = \int \mathcal{D}\vec{\sigma}(\mathbf{q}) \exp\{-\beta\mathcal{H}_0[\vec{\sigma}]\}$, being the *Gaussian* partition function associated with the short wavelength fluctuations. From Eq. (5.30), we obtain

$$\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |\tilde{m}(\mathbf{q})|^2 - \ln\left\langle \mathrm{e}^{-\mathcal{U}[\tilde{\vec{m}}, \vec{\sigma}]} \right\rangle_\sigma. \tag{5.32}$$

The final expression can be calculated perturbatively as,

$$\ln\left\langle \mathrm{e}^{-\mathcal{U}}\right\rangle_\sigma = -\langle\mathcal{U}\rangle_\sigma + \frac{1}{2}\left(\langle\mathcal{U}^2\rangle_\sigma - \langle\mathcal{U}\rangle_\sigma^2\right) + \cdots + \frac{(-1)^\ell}{\ell!} \times \ell\text{th cumulant of } \mathcal{U} + \cdots \tag{5.33}$$

The cumulants can be computed using the rules set in the previous sections. For example, at the first order we need to compute

$$\begin{aligned}\left\langle \mathcal{U}\left[\tilde{\vec{m}},\vec{\sigma}\right]\right\rangle_\sigma = u\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\\ \left\langle\left[\tilde{\vec{m}}(\mathbf{q}_1)+\vec{\sigma}(\mathbf{q}_1)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_2)+\vec{\sigma}(\mathbf{q}_2)\right]\right.\\ \left.\times\left[\tilde{\vec{m}}(\mathbf{q}_3)+\vec{\sigma}(\mathbf{q}_3)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_4)+\vec{\sigma}(\mathbf{q}_4)\right]\right\rangle_\sigma .\end{aligned} \tag{5.34}$$

The following types of terms result from expanding the product:

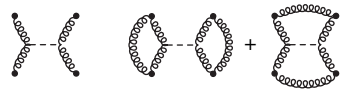
[1] 1 $\left\langle \tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\ \tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$  $\mathcal{U}[\tilde{m}]$

[2] 4 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\ \tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$  0

[3] 2 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\,\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$ 

[4] 4 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\ \vec{\sigma}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$ 

[5] 4 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\ \vec{\sigma}(\mathbf{q}_3)\cdot\vec{m}(\mathbf{q}_4)\right\rangle_\sigma$  0

[6] 1 $\langle\vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\ \vec{\sigma}(\mathbf{q}_3)\cdot\vec{\sigma}(\mathbf{q}_4)\rangle_\sigma$  (5.35)

The second element in each line is the number of terms with a given "symmetry". The total of these coefficients is $2^4 = 16$. Since the averages $\langle\mathcal{O}\rangle_\sigma$ involve only the short wavelength fluctuations, only contractions with $\vec{\sigma}$ appear. The resulting internal momenta are integrated from Λ/b to Λ.

Term [1] has no $\vec{\sigma}$ factors and evaluates to $\mathcal{U}[\tilde{\vec{m}}]$. The second and fifth terms involve an odd number of $\vec{\sigma}$s and their average is zero. Term [3] has one contraction and evaluates to

$$- u \times 2 \int \frac{\mathrm{d}^d\mathbf{q}_1 \cdots \mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}} (2\pi)^d \delta^d(\mathbf{q}_1 + \cdots + \mathbf{q}_4) \frac{\delta_{jj}(2\pi)^d \delta^d(\mathbf{q}_1 + \mathbf{q}_2)}{t + Kq_1^2} \tilde{\vec{m}}(\mathbf{q}_3) \cdot \tilde{\vec{m}}(\mathbf{q}_4)$$
$$= -2nu \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} |\tilde{m}(\mathbf{q})|^2 \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{t + Kk^2}. \tag{5.36}$$

Term [4] also has one contraction but there is no closed loop (the factor δ_{jj}) and hence no factor of n. The various contractions of 4 $\vec{\sigma}$ in term [6] lead to a number of terms with no dependence on $\tilde{\vec{m}}$. We shall denote the sum of these terms by $uV\delta f_b^1$. Collecting all terms, the coarse-grained Hamiltonian at order of u is given by

$$\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}] = V\left(\delta f_b^0 + u\delta f_b^1\right) + \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \left(\frac{\tilde{t} + Kq^2}{2}\right) |\tilde{m}(\mathbf{q})|^2$$
$$+ u \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}_1 \mathrm{d}^d\mathbf{q}_2 \mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} \tilde{\vec{m}}(\mathbf{q}_1) \cdot \tilde{\vec{m}}(\mathbf{q}_2) \tilde{\vec{m}}(\mathbf{q}_3) \cdot \tilde{\vec{m}}(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3), \tag{5.37}$$

where

$$\tilde{t} = t + 4u(n+2) \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{t + Kk^2}. \tag{5.38}$$

The coarse-grained Hamiltonian is thus again described by three parameters $\tilde{t}$, $\tilde{K}$, and $\tilde{u}$. The last two parameters are unchanged, and

$$\tilde{K} = K, \quad \text{and} \quad \tilde{u} = u. \tag{5.39}$$

(2) ***Rescale*** by setting $\mathbf{q} = b^{-1}\mathbf{q}'$, and
(3) ***Renormalize***, $\tilde{\vec{m}} = z\vec{m}'$, to get

$$(\beta\mathcal{H})'[m'] = V\left(\delta f_b^0 + u\delta f_b^1\right) + \int_0^{\Lambda} \frac{\mathrm{d}^d\mathbf{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{\tilde{t} + Kb^{-2}q'^2}{2}\right) |m'(\mathbf{q}')|^2$$
$$+ uz^4 b^{-3d} \int_0^{\Lambda} \frac{\mathrm{d}^d\mathbf{q}'_1 \mathrm{d}^d\mathbf{q}'_2 \mathrm{d}^d\mathbf{q}'_3}{(2\pi)^{3d}} \vec{m}'(\mathbf{q}'_1) \cdot \vec{m}'(\mathbf{q}'_2) \vec{m}'(\mathbf{q}'_3) \cdot \vec{m}'(-\mathbf{q}'_1 - \mathbf{q}'_2 - \mathbf{q}'_3). \tag{5.40}$$

The renormalized Hamiltonian is characterized by the triplet of interactions (t', K', u'), such that

$$t' = b^{-d} z^2 \tilde{t}, \quad K' = b^{-d-2} z^2 K, \quad u' = b^{-3d} z^4 u. \tag{5.41}$$

As in the Gaussian model there is a fixed point at $t^* = u^* = 0$, provided that we set $z = b^{1+\frac{d}{2}}$, such that $K' = K$. The recursion relations for t and u in the vicinity of this point are given by

$$\begin{cases} t'_b = b^2 \left[t + 4u(n+2) \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{t + Kk^2} \right] \\ u'_b = b^{4-d} u. \end{cases} \tag{5.42}$$

While the recursion relation for u at this order is identical to that obtained by dimensional analysis, the one for t is different. It is common to convert the discrete

recursion relations to continuous differential flow equations by setting $b = \mathrm{e}^{\ell}$, such that for an infinitesimal $\delta\ell$,

$$t'_b \equiv t(b) = t(1+\delta\ell) = t + \delta\ell \frac{\mathrm{d}t}{\mathrm{d}\ell} + \mathcal{O}(\delta\ell^2), \quad u'_b \equiv u(b) = u + \delta\ell \frac{\mathrm{d}u}{\mathrm{d}\ell} + \mathcal{O}(\delta\ell^2).$$

Expanding Eqs. (5.42) to order of $\delta\ell$, gives

$$\begin{cases} t + \delta\ell \dfrac{\mathrm{d}t}{\mathrm{d}\ell} = (1+2\delta\ell)\left(t + 4u(n+2)\dfrac{S_d}{(2\pi)^d}\dfrac{1}{t+K\Lambda^2}\Lambda^d\delta\ell\right) \\ u + \delta\ell \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = (1+(4-d)\delta\ell)\,u. \end{cases} \tag{5.43}$$

The differential equations governing the evolution of t and u under rescaling are then

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\ell} = 2t + \dfrac{4u(n+2)K_d\Lambda^d}{t+K\Lambda^2} \\ \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = (4-d)u. \end{cases} \tag{5.44}$$

The recursion relation for u is easily integrated to give $u(\ell) = u_0\mathrm{e}^{(4-d)\ell} = u_0 b^{(4-d)}$.

The recursion relations can be linearized in the vicinity of the fixed point $t^* = u^* = 0$, by setting $t = t^* + \delta t$ and $u = u^* + \delta u$, as

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix} = \begin{pmatrix} 2 & \dfrac{4(n+2)K_d\Lambda^{d-2}}{K} \\ 0 & 4-d \end{pmatrix}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix}. \tag{5.45}$$

In the differential form of the recursion relations, the eigenvalues of the matrix determine the relevance of operators. Since the above matrix has zero elements on one side, its eigenvalues are the diagonal elements, and as in the Gaussian model we can identify $y_t = 2$, and $y_u = 4-d$. The results at this order are identical to those obtained from dimensional analysis on the Gaussian model. The only difference is in the eigendirections. The exponent $y_t = 2$ is still associated with $u = 0$, while $y_u = 4-d$ is actually associated with the direction $t = -4u(n+2)K_d\Lambda^{d-2}/K$. This agrees with the shift in the transition temperature calculated to order of u from the susceptibility.

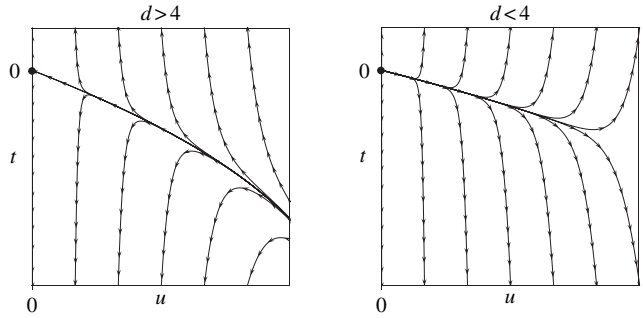


Fig. 5.3 RG flows obtained perturbatively to first order.

For $d > 4$ the Gaussian fixed point has only one unstable direction associated with y_t. It thus correctly describes the phase transition. For $d < 4$ it has two relevant directions and is unstable. Unfortunately, the recursion relations have no other fixed point at this order and it appears that we have learned little from the perturbative RG. However, since we are dealing with an alternating series we can *anticipate* that the recursion relations at the next order are modified to

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\ell} = 2t + \dfrac{4u(n+2)K_d\Lambda^d}{t+K\Lambda^2} - Au^2 \\ \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = (4-d)u - Bu^2, \end{cases} \tag{5.46}$$

with A and B positive. There is now an additional fixed point at $u^* = (4-d)/B$ for $d < 4$. For a systematic perturbation theory we need to keep the parameter u small. Thus the new fixed point can be explored systematically only for small $\epsilon = 4-d$; we are led to consider an expansion in the dimension of space in the vicinity of $d = 4$! For a calculation valid at $\mathcal{O}(\epsilon)$ we have to keep track of terms of second order in the recursion relation for u, but only to first order in that of t. It is thus unnecessary to calculate the term A in the above recursion relation.

5.6 Perturbative RG (second order)

The coarse-grained Hamiltonian at second order in $\mathcal{U}$ is

$$\beta\tilde{\mathcal{H}}[\tilde{\tilde{m}}] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \left(\frac{t+Kq^2}{2}\right) |\tilde{m}(\mathbf{q})|^2 + \langle \mathcal{U} \rangle_\sigma - \frac{1}{2}\left(\langle \mathcal{U}^2 \rangle_\sigma - \langle \mathcal{U} \rangle_\sigma^2\right) + O(\mathcal{U}^3). \tag{5.47}$$

To calculate $\left(\langle \mathcal{U}^2 \rangle_\sigma - \langle \mathcal{U} \rangle_\sigma^2\right)$ we need to consider all possible decompositions of two $\mathcal{U}$s into $\tilde{\tilde{m}}$ and $\vec{\sigma}$ as in Eq. (5.34). Since each $\mathcal{U}$ can be broken up into six types of terms as in Eq. (5.35), there are 36 such possibilities for two $\mathcal{U}$s which can be arranged in a 6×6 matrix, as below. Many of the elements of this matrix are either zero, or can be neglected at this stage, due to a number of considerations:

(1) All the 11 terms involving at least one factor of type [1] are zero because they cannot be contracted into a *connected* piece, and the disconnected elements cancel in calculating the cumulant.

(2) An additional 12 terms (such as $[2] \times [3]$) involve an *odd* number of $\vec{\sigma}$s and are zero due to their *parity*.

(3) Two terms, $[2] \times [5]$ and $[5] \times [2]$, involve a vertex where two $\vec{\sigma}$s are contracted together, leaving a $\tilde{\tilde{m}}(\mathbf{q}^<)$ and a $\vec{\sigma}(\mathbf{q}^>)$. This configuration is not allowed by the δ-function which ensures momentum conservation for the vertex, as by construction $\mathbf{q}^> + \mathbf{q}^< \neq \mathbf{0}$.

(4) Terms $[3]\times[6]$, $[4]\times[6]$, and their partners by exchange have two factors of $\tilde{\tilde{m}}$. They involve *two-loop* integrations, and appear as corrections to the coefficient $\tilde{t}$. We shall denote their net effect by A, which as noted earlier does not need to be known precisely at this order.

(5) The term $[5]\times[5]$ also involves two factors of $\tilde{\tilde{m}}$, while $[2]\times[2]$ includes six such factors. The latter is important as it indicates that the space of parameters *is not closed* at this order. Even if initially zero, a term proportional to m^6 is generated under RG. In fact, considerations of momentum conservation indicate that both these terms are zero for $\mathbf{q}=0$, and are thus contributions to q^2m^2 and q^2m^6, respectively. We shall comment on their effect later on.

(6) The contributions resulting from $[6]\times[6]$ are constants, and will be collectively denoted by $u^2V\delta f_b^2$.

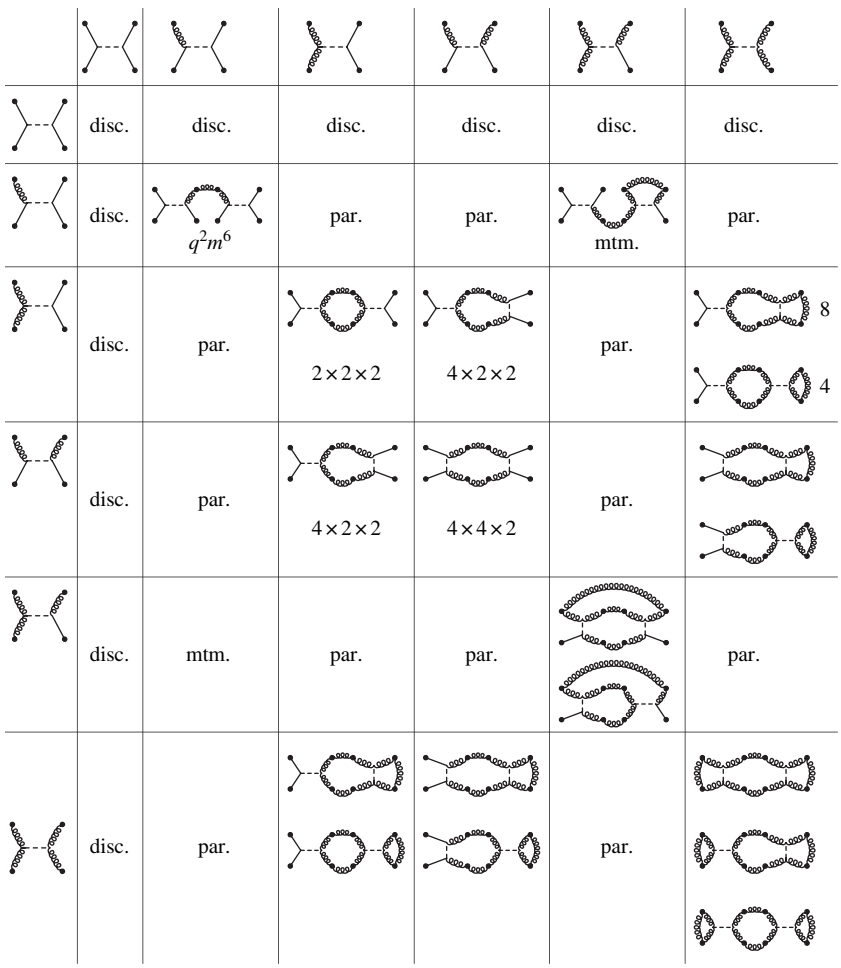
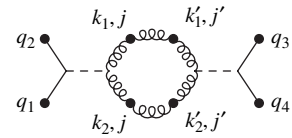


Fig. 5.4 Diagrams appearing in the second-order RG calculation (par. and disc. indicate contributions that are zero due to parity considerations, or being disconnected and mtm. is used to label diagrams that appear at higher order in q^2 due to momentum conservation).

(7) The terms $[3]\times[3]$, $[3]\times[4]$, $[4]\times[3]$, and $[4]\times[4]$ contribute to $\tilde{\tilde{m}}^4$. For example, $[3]\times[3]$ results in



$$\begin{aligned}
&\frac{u^2}{2}\times 2\times 2\times 2\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\cdots\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}_1\mathrm{d}^d\mathbf{k}_2\mathrm{d}^d\mathbf{k}_1'\mathrm{d}^d\mathbf{k}_2'}{(2\pi)^{4d}}\\
&\quad\times(2\pi)^{2d}\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k}_1+\mathbf{k}_2)\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{q}_3+\mathbf{q}_4)\\
&\quad\times\frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_1')}{t+Kk_1^{2'}}\,\frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_2')}{t+Kk_2^{2'}}\tilde{\tilde{m}}(\mathbf{q}_1)\cdot\tilde{\tilde{m}}(\mathbf{q}_2)\tilde{\tilde{m}}(\mathbf{q}_3)\cdot\tilde{\tilde{m}}(\mathbf{q}_4)\\
&=4nu^2\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\cdots\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\tilde{\tilde{m}}(\mathbf{q}_1)\cdot\tilde{\tilde{m}}(\mathbf{q}_2)\tilde{\tilde{m}}(\mathbf{q}_3)\cdot\tilde{\tilde{m}}(\mathbf{q}_4)\\
&\qquad\times\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t+Kk^2)\,(t+K(\mathbf{q}_1+\mathbf{q}_2-\mathbf{k})^2)}.
\end{aligned}\tag{5.48}$$

The contractions from terms $[3]\times[4]$, $[4]\times[3]$, and $[4]\times[4]$ lead to similar expressions with prefactors of 8, 8, and 16 respectively. Apart from the dependence on $\mathbf{q}_1$ and $\mathbf{q}_2$, the final result has the form of $\mathcal{U}[\tilde{\tilde{m}}]$. In fact the last integral can be expanded as

$$f(\mathbf{q}_1+\mathbf{q}_2)=\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t+Kk^2)^2}\left[1-\frac{2K\mathbf{k}\cdot(\mathbf{q}_1+\mathbf{q}_2)-K(\mathbf{q}_1+\mathbf{q}_2)^2}{(t+Kk^2)}+\cdots\right].\tag{5.49}$$

After fourier transforming back to real space we find in addition to m^4, such terms as $m^2(\nabla m)^2$, $m^2\nabla^2 m^2,\cdots$.

Putting all contributions together, the coarse grained Hamiltonian at order of u^2 takes the form

$$\begin{aligned}
\beta\tilde{\mathcal{H}}=&V\left(\delta f_b^0+u\delta f_b^1+u^2\delta f_b^2\right)+\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}|\tilde{m}(\mathbf{q})|^2\\
&\left[\frac{t+Kq^2}{2}+2u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}-\frac{u^2}{2}A(t,K,q^2)\right]\\
&+\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}}\tilde{\tilde{m}}(\mathbf{q}_1)\cdot\tilde{\tilde{m}}(\mathbf{q}_2)\tilde{\tilde{m}}(\mathbf{q}_3)\cdot\tilde{\tilde{m}}(\mathbf{q}_4)\times\left[u-\frac{u^2}{2}(8n+64)\right.\\
&\left.\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t+Kk^2)^2}+\mathcal{O}(u^2q^2)\right]+\mathcal{O}(u^2\tilde{m}^6q^2,\cdots)+\mathcal{O}(u^3).
\end{aligned}\tag{5.50}$$

5.7 The ϵ-expansion

The parameter space (K, t, u) is no longer closed at this order; several new interactions proportional to m^2, m^4, and m^6, all consistent with symmetries of the problem, appear in the coarse-grained Hamiltonian at second order in u. Ignoring these interactions for the time being, the coarse grained parameters are given by

$$\begin{cases} \tilde{K} = K - u^2 A''(0) \\ \tilde{t} = t + 4(n+2)\, u \int_{\Lambda/b}^{\Lambda} \dfrac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \dfrac{1}{t + Kk^2} - u^2 A(0) \\ \tilde{u} = u - 4(n+8) u^2 \int_{\Lambda/b}^{\Lambda} \dfrac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \dfrac{1}{(t + Kk^2)^2}, \end{cases} \tag{5.51}$$

where $A(0)$ and $A''(0)$ correspond to the first two terms in the expansion of $A(t, K, q^2)$ in Eq. (5.50) in powers of q.

After the *rescaling* $\mathbf{q} = b^{-1}\mathbf{q}'$, and *renormalization* $\tilde{\vec{m}} = z\vec{m}'$, steps of the RG procedure, we obtain

$$K' = b^{-d-2} z^2 \tilde{K}, \quad t' = b^{-d} z^2 \tilde{t}, \quad u' = b^{-3d} z^4 \tilde{u}. \tag{5.52}$$

As before, the renormalization parameter z is chosen such that $K' = K$, leading to

$$z^2 = \frac{b^{d+2}}{(1 - u^2 A''(0)/K)} = b^{d+2} \left(1 + O(u^2)\right). \tag{5.53}$$

The value of z does depend on the fixed point position u^*. But as u^* is of the order of ϵ, $z = b^{1+\frac{d}{2}+\mathcal{O}(\epsilon^2)}$, it is not changed at the lowest order. Using this value of z, and following the previous steps for constructing differential recursion relations, we obtain

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\ell} = 2t + \dfrac{4u(n+2)K_d \Lambda^d}{t + K\Lambda^2} - A(t, K, \Lambda) u^2 \\ \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = (4-d)u - \dfrac{4(n+8)K_d \Lambda^d}{(t + K\Lambda^2)^2} u^2. \end{cases} \tag{5.54}$$

The fixed points are obtained from $\mathrm{d}t/\mathrm{d}\ell = \mathrm{d}u/\mathrm{d}\ell = 0$. In addition to the Gaussian fixed point at $u^* = t^* = 0$, discussed in the previous section, there is now a non-trivial fixed point located at

$$\begin{cases} u^* = \dfrac{(t^* + K\Lambda^2)^2}{4(n+8)K_d \Lambda^d} \epsilon = \dfrac{K^2}{4(n+8)K_4} \epsilon + \mathcal{O}(\epsilon^2) \\ t^* = -\dfrac{2u^*(n+2)K_d \Lambda^d}{t^* + K\Lambda^2} = -\dfrac{(n+2)}{2(n+8)} K\Lambda^2 \epsilon + \mathcal{O}(\epsilon^2). \end{cases} \tag{5.55}$$

The above expressions have been further simplified by systematically keeping terms to first order in $\epsilon = 4 - d$.

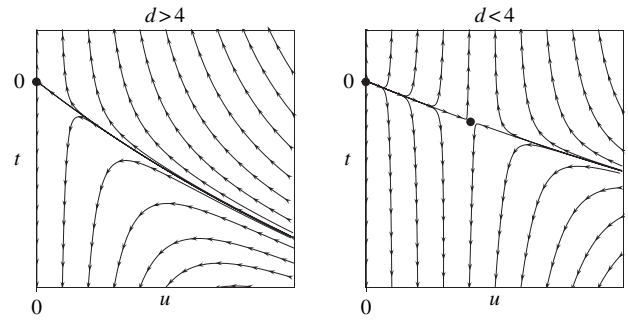


Fig. 5.5 RG flows obtained perturbatively to second order.

Linearizing the recursion relations in the vicinity of the fixed point results in

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}=\begin{pmatrix}2-\dfrac{4(n+2)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^*-A'u^{*2} & \dfrac{4(n+2)K_d\Lambda^d}{t^*+K\Lambda^2}-2Au^*\\ \dfrac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^3}u^{*2} & \epsilon-\dfrac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^*\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}. \quad (5.56)$$

At the Gaussian fixed point, $t^*=u^*=0$, and Eq. (5.45) is reproduced. At the new fixed point of Eqs. (5.55),

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}=\begin{pmatrix}2-\dfrac{4(n+2)K_4\Lambda^4}{K^2\Lambda^4}\dfrac{K^2\epsilon}{4(n+8)K_4} & \cdots\cdots\\ \mathcal{O}(\epsilon^2) & \epsilon-\dfrac{8(n+8)K_4\Lambda^4}{K^2\Lambda^4}\dfrac{K^2\epsilon}{4(n+8)K_4}\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}. \quad (5.57)$$

We have not explicitly calculated the top element of the second column as it is not necessary for calculating the eigenvalues. This is because the lower element of the first column is zero to order of ϵ. Hence the eigenvalues are determined by the diagonal elements alone. The first eigenvalue is positive, controlling the instability of the fixed point,

$$y_t=2-\frac{(n+2)}{(n+8)}\epsilon+\mathcal{O}(\epsilon^2). \quad (5.58)$$

The second eigenvalue,

$$y_u=-\epsilon+\mathcal{O}(\epsilon^2), \quad (5.59)$$

is negative for $d<4$. The new fixed point thus has co-dimension of one and can describe the phase transition in these dimensions. It is quite satisfying that while various intermediate results, such as the position of the fixed point, depend on such microscopic parameters as K and Λ, the final eigenvalues are pure numbers, only depending on n and $d=4-\epsilon$. These eigenvalues characterize the *universality classes* of rotational symmetry breaking in $d<4$, with short-range interactions. (As discussed in the problem section, long-range interaction may lead to new universality classes.)

The divergence of the correlation length, $\xi \sim (\delta t)^{-\nu}$, is controlled by the exponent

$$\nu = \frac{1}{y_t} = \left\{ 2\left[1 - \frac{(n+2)}{2(n+8)}\epsilon \right] \right\}^{-1} = \frac{1}{2} + \frac{1}{4}\frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2). \tag{5.60}$$

The singular part of the free energy scales as $f \sim (\delta t)^{2-\alpha}$, and the heat capacity diverges with the exponent

$$\alpha = 2 - d\nu = 2 - \frac{(4-\epsilon)}{2}\left[1 + \frac{1}{2}\frac{n+2}{n+8}\epsilon \right] = \frac{4-n}{2(n+8)}\,\epsilon + \mathcal{O}(\epsilon^2). \tag{5.61}$$

To complete the calculation of critical exponents, we need the eigenvalue associated with the (relevant) symmetry breaking field h. This is easily found by adding a term $-\vec{h}\cdot\int \mathrm{d}^d\mathbf{x}\vec{m}(\mathbf{x}) = -\vec{h}\cdot\vec{m}(\mathbf{q}=\mathbf{0})$ to the Hamiltonian. This term is not affected by coarse graining or rescaling, and after the renormalization step changes to $-z\vec{h}\cdot\vec{m}'(\mathbf{q}'=\mathbf{0})$, implying

$$h' = zh = b^{1+\frac{d}{2}}h, \quad \Longrightarrow \quad y_h = 1 + \frac{d}{2} + \mathcal{O}(\epsilon^2) = 3 - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2). \tag{5.62}$$

The vanishing of magnetization as $T \to T_c^-$ is controlled by the exponent

$$\begin{aligned} \beta &= \frac{d-y_h}{y_t} = \left(\frac{4-\epsilon}{2} - 1 \right) \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2) \right) \\ &= \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2), \end{aligned} \tag{5.63}$$

while the susceptibility diverges as $\chi \sim (\delta t)^{-\gamma}$, with

$$\gamma = \frac{2y_h - d}{y_t} = 2 \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon \right) = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2). \tag{5.64}$$

Using the above results, we can estimate various exponents as a function of d and n. For example, for $n = 1$, by setting $\epsilon = 1$ or 2 in Eqs. (5.60) and Eqs. (5.63) we obtain the values $\nu(1) \approx 0.58$, $\nu(2) \approx 0.67$, and $\beta(1) \approx 0.33$, $\beta(2) \approx 0.17$. The best estimates of these exponents in $d = 3$ are $\nu \approx 0.63$, and $\beta \approx 0.32$. In $d = 2$ the exact values are known to be $\nu = 1$ and $\beta = 0.125$. The estimates for β are quite good, while those for ν are less reliable. It is important to note that in all cases these estimates are an improvement over the mean field (saddle point) values. Since the expansion is around four dimensions, the results are more reliable in $d = 3$ than in $d = 2$. In any case, they correctly describe the decrease of β with lowering dimension, and the increase of ν. They also correctly describe the trends with varying n at a fixed d as indicated by the following table of exponents $\alpha(n)$.

Although the sign of α is incorrectly predicted at this order for $n = 2$ and 3, the decrease of α with increasing n is correctly described.

	$n=1$	$n=2$	$n=3$	$n=4$
$\mathcal{O}(\epsilon)$ at $\epsilon=1$	0.17	0.11	0.06	0
Experiments in $d=3$	0.11	−0.01	−0.12	–

5.8 Irrelevance of other interactions

The fixed point Hamiltonian at $\mathcal{O}(\epsilon)$ (from Eqs. 5.55) has only three terms

$$\beta\mathcal{H}^* = \frac{K}{2}\int_\Lambda \mathrm{d}^d\mathbf{x}\left[(\nabla m)^2 - \frac{(n+2)}{(n+8)}\epsilon\Lambda^2 m^2 + \frac{\epsilon\Lambda^{-\epsilon}}{2(n+8)}\frac{K}{K_4}m^4\right], \tag{5.65}$$

and explicitly depends on the imposed cutoff $\Lambda \sim 1/a$ (unlike the exponents). However, as described in Section 3.4, the starting point for RG must be the most general Hamiltonian consistent with symmetries. We also discovered that even if some of these terms are left out of the original Hamiltonian, they are generated under coarse graining. At second order in u, terms proportional to m^6 were generated; higher powers of m will appear at higher orders in u.

Let us focus on a rotationally symmetric Hamiltonian for $\vec{h}=0$. We can incorporate all terms consistent with this symmetry in a perturbative RG by setting $\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$, where

$$\beta\mathcal{H}_0 = \int \mathrm{d}^d\mathbf{x}\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right] \tag{5.66}$$

includes all quadratic (Gaussian terms), while the remaining higher order terms are placed in the perturbation

$$\mathcal{U} = \int \mathrm{d}^d\mathbf{x}\left[um^4 + vm^2(\nabla m)^2 + \cdots + u_6 m^6 + \cdots + u_8 m^8 + \cdots\right]. \tag{5.67}$$

After coarse graining, and steps (ii) and (iii) of RG in real space, $\mathbf{x} = b\mathbf{x}'$ and $\tilde{\vec{m}} = \zeta\vec{m}'$, the renormalized weight depends on the parameters

$$\begin{cases} t \mapsto & b^d\zeta^2\tilde{t} = b^2\tilde{t} \\ K \mapsto & b^{d-2}\zeta^2\tilde{K} = K \\ L \mapsto & b^{d-4}\zeta^2\tilde{L} = b^{-2}\tilde{L} \\ & \vdots \\ u \mapsto & b^d\zeta^4\tilde{u} = b^{4-d}\tilde{u} \\ v \mapsto & b^{d-2}\zeta^4\tilde{v} = b^{2-d}\tilde{v} \\ & \vdots \\ u_6 \mapsto & b^d\zeta^6\tilde{u_6} = b^{6-2d}\tilde{u_6} \\ u_8 \mapsto & b^d\zeta^8\tilde{u_8} = b^{8-3d}\tilde{u_8} \\ & \vdots \end{cases} \tag{5.68}$$

The second set of equalities are obtained by choosing $\zeta^2 = b^{2-d}K/\tilde{K} = b^{2-d}[1 + \mathcal{O}(u^2, uv, v^2, \cdots)]$, such that $K' = K$. By choosing an infinitesimal rescaling, the recursion relations take the differential forms

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\ell} = & 2t + \mathcal{O}(u, v, u_6, u_8, \cdots) \\ \dfrac{\mathrm{d}K}{\mathrm{d}\ell} = & 0 \\ \dfrac{\mathrm{d}L}{\mathrm{d}\ell} = & -2L + \mathcal{O}(u^2, uv, v^2, \cdots) \\ & \vdots \\ \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = & \epsilon u - Bu^2 + \mathcal{O}(uv, v^2, \cdots) \\ \dfrac{\mathrm{d}v}{\mathrm{d}\ell} = & (-2+\epsilon)v + \mathcal{O}(u^2, uv, v^2, \cdots) \\ & \vdots \\ \dfrac{\mathrm{d}u_6}{\mathrm{d}\ell} = & (-2+2\epsilon)u_6 + \mathcal{O}(u^3, u_6^2, \cdots) \\ \dfrac{\mathrm{d}u_8}{\mathrm{d}\ell} = & (-4+3\epsilon)u_8 + \mathcal{O}(u^3, u^2u_6, \cdots) \\ & \vdots \end{cases} \tag{5.69}$$

These recursion relations describe two fixed points:

(1) The Gaussian fixed point, $t^* = L^* = u^* = v^* = \cdots = 0$, and $K \neq 0$, has eigenvalues

$$\begin{aligned} &y_t^0 = 2,\ y_L^0 = -2,\ \cdots,\ y_u^0 = +\epsilon,\ y_v^0 = -2+\epsilon,\ \cdots, \\ &y_6^0 = -2+2\epsilon,\ y_8^0 = -4+3\epsilon,\ \cdots. \end{aligned} \tag{5.70}$$

(2) Setting Eqs. (5.69) to zero, a non-trivial fixed point is located at

$$t^* \sim u^* \sim \mathcal{O}(\epsilon), \quad L^* \sim v^* \sim \cdots \sim \mathcal{O}(\epsilon^2), \quad u_6^* \sim \cdots \sim \mathcal{O}(\epsilon^3), \quad \cdots. \tag{5.71}$$

The stability of this fixed point is determined by the matrix,

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix} \delta t \\ \delta L \\ \vdots \\ \delta u \\ \delta v \\ \vdots \end{pmatrix} = \begin{pmatrix} 2-\mathcal{O}(u^*) & \mathcal{O}(\epsilon) & \cdots & \mathcal{O}(1) & \mathcal{O}(1) & \cdots \\ \mathcal{O}(\epsilon^2) & -2+\mathcal{O}(\epsilon) & & & & \\ \vdots & & \ddots & & & \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & & & & \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & & & & \\ \vdots & & & & & \end{pmatrix}\begin{pmatrix} \delta t \\ \delta L \\ \vdots \\ \delta u \\ \delta v \\ \vdots \end{pmatrix}. \tag{5.72}$$

Note that as $\epsilon \to 0$, the non-trivial fixed part, its eigenvalues and eigendirections continuously go over to the Gaussian fixed points. Hence the eigenvalues can only be corrected by order of ϵ, and Eq. (5.70) is modified to

$$y_t = 2 - \frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2), \qquad y_L = -2 + \mathcal{O}(\epsilon), \cdots,$$
$$y_u = -\epsilon + \mathcal{O}(\epsilon^2), y_v = -2 + \mathcal{O}(\epsilon), \cdots, y_6 = -2 + \mathcal{O}(\epsilon), y_8 = -4 + \mathcal{O}(\epsilon), \cdots \tag{5.73}$$

While the eigenvalues are still labeled with the coefficients of the various terms in the Landau–Ginzburg expansion, we must remember that the actual eigendirections are now rotated away from the axes of this parameter space, although their largest projection is still parallel to the corresponding axis.

Whereas the Gaussian fixed point has two relevant directions in $d < 4$, the generalized $O(n)$ fixed point has only one relevant direction corresponding to y_t. At least perturbatively, this fixed point has a basin of attraction of co-dimension one, and thus describes the phase transition. The original concept of Kadanoff scaling is thus explicitly realized and the universality of exponents is traced to the irrelevance (at least perturbatively) of the multitude of other possible interactions. The perturbative approach does not exclude the existence of other fixed points at finite values of these parameters. The uniqueness of the critical exponents observed so far for each universality class, and their proximity to the values calculated from the ϵ-expansion, suggests that postulating such *non-perturbative* fixed points is unnecessary.

5.9 Comments on the ϵ-expansion

The perturbative implementation of RG for the Landau–Ginzburg Hamiltonian was achieved by K.G. Wilson in the early 1970s; the ϵ-expansion was developed jointly with M.E. Fisher. This led to a flurry of activity in the topic which still continues. Wilson was awarded the Nobel Prize in 1982. Historical details can be found in his Nobel lecture reprinted in Rev. Mod. Phys. **55**, 583 (1983). A few comments on the ϵ-expansion are in order at this stage.

(1) ***Higher orders, and convergence of the series:*** Calculating the exponents to $\mathcal{O}(\epsilon)^2$ and beyond, by going to order of $\mathcal{U}^3$ and higher, is quite complicated as we have to keep track of many more interactions. It is in fact quite unappealing that the intermediate steps of the RG explicitly keep track of the cutoff scale Λ, while the final exponents must be independent of it. In fact there are a number of field theoretical RG schemes (dimensional regularization, summing leading divergences, etc.) that avoid many of these difficulties. These methods are harder to visualize and will not be described here. All higher order calculations are currently performed using one of these schemes. It is sometimes (but not always) possible to prove that these approaches are consistent with each other, and can be carried out to all orders. In principle, the problem of evaluating critical exponents in $d = 3$ is now solved: simple computations lead to approximate results, while more refined calculations should provide better answers. The situation is somewhat like finding the energy levels of a He atom, which cannot be done exactly, but which may be obtained with sufficient accuracy using various approximation methods.

To estimate how reliable the exponents are, we need some information on the convergence of the series. The ϵ expansion has been carried out to the fifth order, and the results for the exponent γ, for $n = 1$ at $d = 3$, are

$$\gamma = 1 + 0.167\epsilon + 0.077\epsilon^2 - 0.049\epsilon^3 + 0.180\epsilon^4 - 0.415\epsilon^5$$
$$1.2385 \pm 0.0025 = 1.000, 1.167, 1.244, 1.195, 1.375, 0.96. \tag{5.74}$$

The second line compares the values obtained at different orders by substituting $\epsilon = 1$, with the best estimate of $\gamma \approx 1.2385$ in $d = 3$. Note that the elements of the series have alternating signs. The truncated series evaluated at $\epsilon = 1$ improves up to third order, beyond which it starts to oscillate, and deviates from the left hand side. These are characteristics of an *asymptotic series*. It can be proved that for large p, the coefficients in the expansion of most quantities scale as $|f_p| \sim cp!a^{-p}$. As a result, the ϵ-expansion series is *non-convergent*, but can be evaluated by the *Borel summation* method, using the identity $\int_0^\infty \mathrm{d}x\, x^p \mathrm{e}^{-x} = p!$, as

$$f(\epsilon) = \sum_p f_p \epsilon^p = \sum_p f_p \epsilon^p \frac{1}{p!} \int_0^\infty \mathrm{d}x\, x^p \mathrm{e}^{-x} = \int_0^\infty \mathrm{d}x \mathrm{e}^{-x} \sum_p \frac{f_p (\epsilon x)^p}{p!}. \tag{5.75}$$

The final summation (which is convergent) results in a function of x which can be integrated to give $f(\epsilon)$. Very good estimates of exponents in $d = 3$, such as the one for γ quoted above, are obtained by this summation method. There is no indication of any singularity in the exponents up to $\epsilon = 2$, corresponding to the lower critical dimension $d = 2$.

(2) ***The 1/n expansion:*** The fixed point position,

$$u^* = \frac{(t^* + K\Lambda^2)^2 (4-d)}{4(n+8) K_d \Lambda^d},$$

vanishes as $n \to \infty$. This suggests that a controlled $1/n$ expansion of the critical exponents is also possible. Indeed such an expansion can be developed by a number of methods, such as a saddle point expansion that takes advantage of the exponential dependence of the Hamiltonian on n, or by an exact resummation of the perturbation series. Equation (5.58) in this limit gives,

$$y_t = \lim_{n\to\infty} \left[2 - \frac{n+2}{n-8}(4-d) \right] = d - 2 \quad \Longrightarrow \quad \nu = \frac{1}{d-2}. \tag{5.76}$$

This result is exact in dimensions $4 < d < 2$. Above four dimensions the mean field value of 1/2 is recovered, while for $d < 2$ there is no order.

Problems for chapter 5

1. *Longitudinal susceptibility:* While there is no reason for the longitudinal susceptibility to diverge at the mean-field level, it in fact does so due to fluctuations in dimensions $d < 4$. This problem is intended to show you the origin of this divergence in perturbation theory. There are actually a number of subtleties in this calculation

which you are instructed to ignore at various steps. You may want to think about why they are justified.

Consider the Landau–Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2\right],$$

describing an n-component magnetization vector $\vec{m}(\mathbf{x})$, in the ordered phase for $t < 0$.

(a) Let $\vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expand $\beta\mathcal{H}$ *keeping all terms in the expansion.*

(b) Regard the quadratic terms in ϕ_ℓ and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling ϕ_ℓ and $\vec{\phi}_t$ as a perturbation U; i.e.

$$U = 4u\overline{m}\int \mathrm{d}^d\mathbf{x}\phi_\ell(\mathbf{x})\vec{\phi}_t(\mathbf{x})^2.$$

Write U in Fourier space in terms of $\phi_\ell(\mathbf{q})$ and $\vec{\phi}_t(\mathbf{q})$.

(c) Calculate the Gaussian (bare) expectation values $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle_0$ and $\langle\phi_{t,\alpha}(\mathbf{q})\phi_{t,\beta}(\mathbf{q}')\rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_\ell(\mathbf{q})_0$ and $\chi_t(\mathbf{q})_0$.

(d) Calculate $\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\ \ \vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\rangle_0$ using Wick's theorem. (Don't forget that $\vec{\phi}_t$ is an $(n-1)$ component vector.)

(e) Write down the expression for $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle$ to second order in the perturbation U. Note that since U is odd in ϕ_ℓ, only two terms at the second order are non-zero.

(f) Using the form of U in Fourier space, write the correction term as a product of two four-point expectation values similar to those of part (d). Note that only connected terms for the longitudinal four-point function should be included.

(g) Ignore the disconnected term obtained in (d) (i.e. the part proportional to $(n-1)^2$), and write down the expression for $\chi_\ell(\mathbf{q})$ in second order perturbation theory.

(h) Show that for $d < 4$, the correction term diverges as q^{d-4} for $q \to 0$, implying an infinite longitudinal susceptibility.

2. *Crystal anisotropy:* Consider a ferromagnet with a tetragonal crystal structure. Coupling of the spins to the underlying lattice may destroy their full rotational symmetry. The resulting anisotropies can be described by modifying the Landau–Ginzburg Hamiltonian to

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u\left(\vec{m}^2\right)^2 + \frac{r}{2}m_1^2 + v\,m_1^2\vec{m}^2\right],$$

where $\vec{m} \equiv (m_1, \cdots, m_n)$, and $\vec{m}^2 = \sum_{i=1}^n m_i^2$ ($d = n = 3$ for magnets in three dimensions). Here $u > 0$, and *to simplify calculations we shall set* $v = 0$ throughout.

(a) For a fixed magnitude $|\vec{m}|$, what directions in the n component magnetization space are selected for $r > 0$, and for $r < 0$?

(b) Using the saddle point approximation, calculate the free energies ($\ln Z$) for phases uniformly magnetized *parallel* and *perpendicular* to direction 1.

(c) Sketch the phase diagram in the (t, r) plane, and indicate the phases (type of order), and the nature of the phase transitions (continuous or discontinuous).

(d) Are there Goldstone modes in the ordered phases?

(e) For $u = 0$, and positive t and r, calculate the unperturbed averages $\langle m_1(\mathbf{q}) m_1(\mathbf{q}')\rangle_0$ and $\langle m_2(\mathbf{q}) m_2(\mathbf{q}')\rangle_0$, where $m_i(\mathbf{q})$ indicates the Fourier transform of $m_i(\mathbf{x})$.

(f) Write the fourth order term $\mathcal{U} \equiv u \int \mathrm{d}^d\mathbf{x} (\vec{m}^2)^2$, in terms of the Fourier modes $m_i(\mathbf{q})$.

(g) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_1(\mathbf{q}) m_1(\mathbf{q}')\rangle$. (You can leave your answers in the form of some integrals.)

(h) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_2(\mathbf{q}) m_2(\mathbf{q}')\rangle$.

(i) Using the above answer, identify the inverse susceptibility χ_{22}^{-1}, and then find the transition point, t_c, from its vanishing to first order in u.

(j) Is the critical behavior different from the isotropic $O(n)$ model in $d < 4$? In RG language, is the parameter r *relevant* at the $\mathcal{O}(n)$ fixed point? In either case indicate the universality classes expected for the transitions.

3. *Cubic anisotropy – mean-field treatment:* Consider the modified Landau–Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x} \left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + v\sum_{i=1}^{n} m_i^4 \right],$$

for an n-component vector $\vec{m}(\mathbf{x}) = (m_1, m_2, \cdots, m_n)$. The "cubic anisotropy" term $\sum_{i=1}^{n} m_i^4$ breaks the full rotational symmetry and selects specific directions.

(a) For a fixed magnitude $|\vec{m}|$, what directions in the n component magnetization space are selected for $v > 0$ and for $v < 0$? What is the degeneracy of easy magnetization axes in each case?

(b) What are the restrictions on u and v for $\beta\mathcal{H}$ to have finite minima? Sketch these regions of stability in the (u, v) plane.

(c) In general, higher order terms (e.g. $u_6(\vec{m}^2)^3$ with $u_6 > 0$) are present and insure stability in the regions not allowed in part (b) (as in the case of the tricritical point discussed in earlier problems). With such terms in mind, sketch the saddle point phase diagram in the (t, v) plane for $u > 0$; clearly identifying the phases, and order of the transition lines.

(d) Are there any Goldstone modes in the ordered phases?

4. *Cubic anisotropy ε-expansion:*

(a) By looking at diagrams in a second order perturbation expansion in both u and v show that the recursion relations for these couplings are

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = \varepsilon u - 4C\left[\ (n+8)u^2 + 6uv\ \right] \\ \dfrac{\mathrm{d}v}{\mathrm{d}\ell} = \varepsilon v - 4C\left[\ 12uv + 9v^2\ \right], \end{cases}$$

where $C = K_d \Lambda^d/(t + K\Lambda^2)^2 \approx K_4/K^2$ is approximately a constant.

(b) Find all fixed points in the (u, v) plane, and draw the flow patterns for $n < 4$ and $n > 4$. Discuss the relevance of the cubic anisotropy term near the stable fixed point in each case.

(c) Find the recursion relation for the reduced temperature, t, and calculate the exponent ν at the stable fixed points for $n < 4$ and $n > 4$.

(d) Is the region of stability in the (u, v) plane calculated in part (b) of the previous problem enhanced or diminished by inclusion of fluctuations? Since in reality higher order terms will be present, what does this imply about the nature of the phase transition for a small negative v and $n > 4$?

(e) Draw schematic phase diagrams in the (t, v) plane $(u > 0)$ for $n > 4$ and $n < 4$, identifying the ordered phases. Are there Goldstone modes in any of these phases close to the phase transition?

5. *Exponents:* Two critical exponents at second order are,

$$\begin{cases} \nu = \dfrac{1}{2} + \dfrac{(n+2)}{4(n+8)}\epsilon + \dfrac{(n+2)(n^2+23n+60)}{8(n+8)^3}\epsilon^2 \\ \eta = \dfrac{(n+2)}{2(n+8)^2}\epsilon^2. \end{cases} ,$$

Use scaling relations to obtain ϵ-expansions for two or more of the remaining exponents $\alpha,\ \beta,\ \gamma,\ \delta$ and Δ. Make a table of the results obtained by setting $\epsilon = 1,\ 2$ for $n = 1,\ 2$ and 3; and compare to the best estimates of these exponents that you can find by other sources (series, experiments, etc.).

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6. *Anisotropic criticality:* A number of materials, such as liquid crystals, are anisotropic and behave differently along distinct directions, which shall be denoted parallel and perpendicular, respectively. Let us assume that the d spatial dimensions are grouped into n parallel directions $\mathbf{x}_\parallel$, and $d - n$ perpendicular directions $\mathbf{x}_\perp$. Consider a one-component field $m(\mathbf{x}_\parallel, \mathbf{x}_\perp)$ subject to a Landau–Ginzburg Hamiltonian, $\beta\mathcal{H} = \beta\mathcal{H}_0 + U$, with

$$\beta\mathcal{H}_0 = \int \mathrm{d}^n\mathbf{x}_\parallel \mathrm{d}^{d-n}\mathbf{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm\right],$$

$$\text{and} \quad U = u\int \mathrm{d}^n\mathbf{x}_\parallel \mathrm{d}^{d-n}\mathbf{x}_\perp\ m^4\ .$$

(Note that $\beta\mathcal{H}$ depends on the **first** *gradient in the $\mathbf{x}_\parallel$ directions, and on the* **second** *gradient in the $\mathbf{x}_\perp$ directions.)*

(a) Write $\beta\mathcal{H}_0$ in terms of the Fourier transforms $m(\mathbf{q}_\parallel, \mathbf{q}_\perp)$.

(b) Construct a renormalization group transformation for $\beta\mathcal{H}_0$, by rescaling coordinates such that $\mathbf{q}'_\parallel = b\ \mathbf{q}_\parallel$ and $\mathbf{q}'_\perp = c\ \mathbf{q}_\perp$ and the field as $m'(\mathbf{q}') = m(\mathbf{q})/z$. *Note that parallel and perpendicular directions are scaled differently.* Write down the recursion relations for $K,\ L,\ t$, and h in terms of $b,\ c$, and z. (The exact shape of the Brillouin zone is immaterial at this stage, and you do not need to evaluate the integral that contributes an additive constant.)

(c) Choose $c(b)$ and $z(b)$ such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h for the rescalings of t and h.

(d) Write the relationship between the (singular parts of) free energies $f(t, h)$ and $f'(t', h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form $f(t, h) = t^{2-\alpha} g_f(h/t^{\Delta})$, and identify the exponents α and Δ.

(e) How does the unperturbed zero-field susceptibility $\chi(t, h = 0)$ diverge as $t \to 0$?

In the remainder of this problem set $h = 0$, and treat U as a perturbation.

(f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(q)m(q')\rangle_0$, and the corresponding susceptibility $\chi_0(q) = \langle |m_q|^2\rangle_0$, where q stands for $(\mathbf{q}_\parallel, \mathbf{q}_\perp)$.

(g) Write the perturbation U, in terms of the normal modes $m(q)$.

(h) Using RG, or any other method, find the upper critical dimension d_u, for validity of the Gaussian exponents.

(i) Write down the expansion for $\langle m(q)m(q')\rangle$, to first order in U, and reduce the correction term to a product of two point expectation values.

(j) Write down the expression for $\chi(q)$, in first order perturbation theory, and identify the transition point t_c at order of u. (Do not evaluate any integrals explicitly.)

7. *Long-range interactions* between spins can be described by adding a term

$$\int \mathrm{d}^d\mathbf{x} \int \mathrm{d}^d\mathbf{y} J(|\mathbf{x}-\mathbf{y}|)\vec{m}(\mathbf{x})\cdot\vec{m}(\mathbf{y}),$$

to the usual Landau–Ginzburg Hamiltonian.

(a) Show that for $J(r) \propto 1/r^{d+\sigma}$, the Hamiltonian can be written as

$$\beta\mathcal{H} = \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \frac{t + K_2 q^2 + K_\sigma q^\sigma + \cdots}{2} \vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q}) + u\int \frac{\mathrm{d}^d\mathbf{q}_1 \mathrm{d}^d\mathbf{q}_2 \mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} \vec{m}(\mathbf{q}_1)\cdot\vec{m}(\mathbf{q}_2)\,\vec{m}(\mathbf{q}_3)\cdot\vec{m}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \quad .$$

(b) For $u = 0$, construct the recursion relations for (t, K_2, K_σ) and show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K'_\sigma = K_\sigma$, in which case K_2 is irrelevant. What is the fixed Hamiltonian now?

(d) For $\sigma < 2$, calculate the generalized Gaussian exponents ν, η, and γ from the recursion relations. Show that u is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.

(e) For $\sigma < 2$, use a perturbation expansion in u to construct the recursion relations for (t, K_σ, u) as in the text.

(f) For $d < 2\sigma$, calculate the critical exponents ν and η to first order in $\epsilon = d - 2\sigma$.

[See M.E. Fisher, S.-K. Ma and B.G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).]

(g) What is the critical behavior if $J(r) \propto \exp(-r/a)$? Explain!

6
Lattice systems

6.1 Models and methods

While Wilson's perturbative RG provides a systematic approach to probing critical properties, carrying out the ϵ-expansion to high orders is quite cumbersome. Models defined on a discrete lattice provide a number of alternative computational routes that can complement the perturbative RG approach. Because of universality, we expect that all models with appropriate microscopic symmetries and range of interactions, no matter how simplified, lead to the same critical exponents. Lattice models are convenient for visualization, computer simulation, and series expansion purposes. We shall thus describe models in which an appropriate "spin" degree of freedom is placed on each site of a lattice, and the spins are subject to simple interaction energies. While such models are formulated in terms of explicit "microscopic" degrees of freedom, depending on their degree of complexity, they may or may not provide a more accurate description of a specific material than the Landau–Ginzburg model. The point is that universality dictates that both descriptions describe the same *macroscopic* physics, and the choice of continuum or discrete models is then a matter of computational convenience.

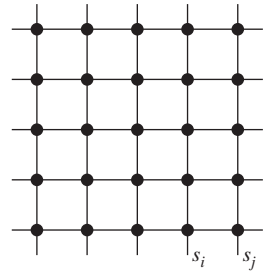

Fig. 6.1 Interacting "spins" $\{s_i\}$ defined on a square lattice.

Some commonly used lattice models are described here:

(1) ***The Ising model*** is the simplest and most widely applied paradigm in statistical mechanics. At each site i of a lattice, there is a spin σ_i which takes the two values

of $+1$ or -1. Each state may correspond to one of two species in a binary mixture, or to empty and occupied cells in a lattice approximation to an interacting gas. The simplest possible *short-range* interaction involves only neighboring spins, and is described by a Hamiltonian

$$\mathcal{H} = \sum_{\langle i,j \rangle} \hat{B}(\sigma_i, \sigma_j), \tag{6.1}$$

where the notation $\langle i, j \rangle$ is commonly used to indicate the sum over all *nearest neighbor* pairs on the lattice. Since $\sigma_i^2 = 1$, the most general interaction between two spins is

$$\hat{B}(\sigma, \sigma') = -\hat{g} - \frac{\hat{h}}{z}(\sigma + \sigma') - J\sigma\sigma'. \tag{6.2}$$

For N spins, there are 2^N possible *microstates*, and the (Gibbs) partition function is

$$Z = \sum_{\{\sigma_i\}} \mathrm{e}^{-\beta\mathcal{H}} = \sum_{\{\sigma_i\}} \exp\left[K \sum_{\langle i,j \rangle} \sigma_i\sigma_j + h\sum_i \sigma_i + g \right], \tag{6.3}$$

where we have set $K = \beta J$, $h = \beta\hat{h}$, and $g = z\beta\hat{g}/2$ ($\beta = 1/k_{\mathrm{B}}T$, and z is the number of bonds per site, i.e. the coordination number of the lattice). For $h = 0$ at $T = 0$, the ground state has a two fold degeneracy with all spins pointing up or down ($K > 0$). This order is destroyed at a critical $K_c = J/k_{\mathrm{B}}T_c$ with a phase transition to a disordered state. The field h breaks the *up–down symmetry* and removes the phase transition. The parameter g merely shifts the origin of energy, and has no effect on the relative weights of microstates, or the macroscopic properties.

All the following models can be regarded as generalizations of the Ising model.

(2) ***The O(n) model:*** Each lattice site is now occupied by an n-component *unit* vector, i.e

$$S_i \equiv (S_i^1, S_i^2, \cdots, S_i^n), \quad \text{with} \quad \sum_{\alpha=1}^{n} (S_i^\alpha)^2 = 1. \tag{6.4}$$

A nearest-neighbor interaction can be written as

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - \hat{\vec{h}} \cdot \sum_i \vec{S}_i. \tag{6.5}$$

In fact, the most general interaction consistent with spherical symmetry is $f(\vec{S}_i \cdot \vec{S}_j)$ for an arbitrary function f. Similarly, the *rotational symmetry* can be broken by a number of "fields" such as $\sum_i (\vec{h}_p \cdot \vec{S}_i)^p$. Specific cases are the Ising model ($n = 1$), the *XY model* ($n = 2$), and the *Heisenberg model* ($n = 3$).

(3) ***The Potts model:*** Each site of the lattice is occupied by a q-valued spin $S_i \equiv 1, 2, \cdots, q$. The interactions between the spins are described by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{S_i,S_j} - \hat{h} \sum_i \delta_{S_i,1}. \tag{6.6}$$

The field h now breaks the *permutation symmetry* amongst the q-states. The Ising model is recovered for $q = 2$, since $\delta_{\sigma,\sigma'} = (1 + \sigma\sigma')/2$. The three state Potts

model can for example describe the distortion of a cube along one of its faces. Potts models with $q > 2$ represent new universality classes not covered by the $O(n)$ model. Actually, the transitions for $q \geq 4$ in $d = 2$, and $q > 3$ in $d = 3$ are discontinuous.

(4) ***Spin s-models:*** The spin at each site takes the $2s + 1$ values, $s_i = -s, -s + 1, \cdots, +s$. A general nearest-neighbor Hamiltonian is

$$\mathcal{H} = \sum_{\langle i,j \rangle} \left(J_1 s_i s_j + J_2 (s_i s_j)^2 + \cdots + J_{2s} (s_i s_j)^{2s} \right) - \hat{h} \sum_i s_i. \tag{6.7}$$

The Ising model corresponds to $s = 1/2$, while $s = 1$ is known as the Blume–Emery–Griffith (BEG) model. It describes a mixture of non-magnetic ($s = 0$) and magnetic ($s = \pm 1$) elements. This model exhibits a tricritical point separating continuous and discontinuous transitions. However, since the ordered phase breaks an up–down symmetry, the phase transition belongs to the Ising universality class for all values of s.

Some of the computational tools employed in the study of discrete models are:

(1) ***Exact solutions*** can be obtained for a very limited subset of lattice models. These include many one dimensional systems that can be solved by the transfer matrix method described next, and the two-dimensional Ising model discussed in the next chapter.

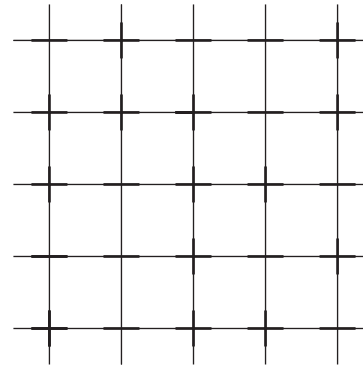

Fig. 6.2 A configuration of Ising spins on a square lattice.

(2) ***Position space renormalizations:*** These are implementations of Kadanoff's RG scheme on lattice models. Some approximation is usually necessary to keep the space of interactions tractable. Most such approximations are uncontrolled; a number of them will be discussed in this chapter.

(3) ***Monte Carlo simulations:*** The aim of such methods is to generate configurations of spins that are distributed with the correct Boltzmann weight $\exp(-\beta\mathcal{H})$. There are a number of methods, most notably the Metropolis algorithm, for achieving this aim. Various expectation values and correlation functions are then directly computed from these configurations. With the continuing increase of computer

power, numerical simulations have become increasingly popular. Limitations of the method are due to the size of systems that can be studied, and the amount of time needed to ensure that the correctly weighted configurations are generated. There is an extensive literature on numerical simulations which will only be touched upon briefly.

(4) ***Series expansions:*** Low-temperature expansions start with the ordered ground state and examine the lowest energy excitations around it (see the next chapter). High temperature expansions begin with the collection of non-interacting spins at infinite temperature and include the interactions between spins perturbatively. Critical behavior is then extracted from the singularities of such series.

6.2 Transfer matrices

Consider a linear chain of N Ising spins ($\sigma_i = \pm 1$), with a nearest-neighbor coupling K, and a magnetic field h. To simplify calculations, we assume that the chain is closed upon itself such that the first and last spins are also coupled (periodic boundary conditions), resulting in the Hamiltonian

$$-\beta\mathcal{H} = K\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \cdots + \sigma_{N-1}\sigma_N + \sigma_N\sigma_1\right) + h\sum_{i=1}^{N}\sigma_i\ . \tag{6.8}$$

The corresponding partition function, obtained by summing over all states, can be expressed as the product of matrices, since

$$\begin{aligned} Z &= \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}\prod_{i=1}^{N}\exp\left[K\sigma_i\sigma_{i+1} + \frac{h}{2}\left(\sigma_i + \sigma_{i+1}\right)\right] \\ &\equiv \mathrm{tr}\left[\langle\sigma_1|T|\sigma_2\rangle\langle\sigma_2|T|\sigma_3\rangle\cdots\langle\sigma_N|T|\sigma_1\rangle\right] = \mathrm{tr}\left[T^N\right], \end{aligned} \tag{6.9}$$

where we have introduced the 2×2 *transfer matrix* T, with elements

$$\langle\sigma_i|T|\sigma_j\rangle = \exp\left[K\sigma_i\sigma_j + \frac{h}{2}(\sigma_i + \sigma_j)\right], \quad \text{i.e.} \quad T = \begin{pmatrix} \mathrm{e}^{K+h} & \mathrm{e}^{-K} \\ \mathrm{e}^{-K} & \mathrm{e}^{K-h} \end{pmatrix}. \tag{6.10}$$

The expression for trace of the matrix can be evaluated in the basis that diagonalizes T, in which case it can be written in terms of the two eigenvalues $\lambda_\pm$ as

$$Z = \lambda_+^N + \lambda_-^N = \lambda_+^N\left[1 + \left(\lambda_-/\lambda_+\right)^N\right] \approx \lambda_+^N. \tag{6.11}$$

We have assumed that $\lambda_+ > \lambda_-$, and since in the limit of $N \to \infty$ the larger eigenvalue dominates the sum, the free energy is

$$\beta f = -\ln Z/N = -\ln\lambda_+. \tag{6.12}$$

Solving the characteristic equation, we find the eigenvalues

$$\lambda_\pm = \mathrm{e}^K\cosh h \pm \sqrt{\mathrm{e}^{2K}\sinh^2 h + \mathrm{e}^{-2K}}. \tag{6.13}$$

We shall leave a discussion of the singularities of the resulting free energy (at zero temperature) to the next section, and instead look at the averages and correlations in the limit of $h = 0$.

To calculate the average of the spin at site i, we need to evaluate

$$\begin{aligned}\langle\sigma_i\rangle &= \frac{1}{Z}\sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}\sigma_i\prod_{j=1}^{N}\exp\left(K\sigma_j\sigma_{j+1}\right)\\ &\equiv \frac{1}{Z}\mathrm{tr}\left[\langle\sigma_1|T|\sigma_2\rangle\cdots\langle\sigma_{i-1}|T|\sigma_i\rangle\sigma_i\langle\sigma_i|T|\sigma_{i+1}\rangle\cdots\langle\sigma_N|T|\sigma_1\rangle\right]\\ &= \frac{1}{Z}\mathrm{tr}\left[T^{i-1}\hat{\sigma}_z T^{N-i+1}\right]\\ &= \frac{1}{Z}\mathrm{tr}\left[T^N\hat{\sigma}_z\right],\end{aligned} \tag{6.14}$$

where have permuted the matrices inside the trace, and $\hat{\sigma}_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the usual 2×2 Pauli matrix. One way to evaluate the final expression in Eq. (6.14) is to rotate to a basis where the matrix T is diagonal. For $h = 0$, this is accomplished by the unitary matrix $U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, resulting in

$$\langle\sigma_i\rangle = \frac{1}{Z}\mathrm{tr}\left[\begin{pmatrix}\lambda_+^N & 0\\ 0 & \lambda_-^N\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \frac{1}{Z}\begin{pmatrix}0 & \lambda_+^N\\ \lambda_-^N & 0\end{pmatrix} = 0. \tag{6.15}$$

Note that under this transformation the Pauli matrix $\hat{\sigma}_z$ is rotated into $\hat{\sigma}_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

The vanishing of the magnetization at zero field is of course expected by symmetry. A more interesting quantity is the two-spin correlation function

$$\begin{aligned}\langle\sigma_i\sigma_{i+r}\rangle &= \frac{1}{Z}\sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}\sigma_i\sigma_{i+r}\prod_{j=1}^{N}\exp\left(K\sigma_j\sigma_{j+1}\right)\\ &= \frac{1}{Z}\mathrm{tr}\left[T^{i-1}\hat{\sigma}_z T^r\hat{\sigma}_z T^{N-i-r+1}\right] = \frac{1}{Z}\mathrm{tr}\left[\hat{\sigma}_z T^r\hat{\sigma}_z T^{N-r}\right].\end{aligned} \tag{6.16}$$

Once again rotating to the basis where T is diagonal simplifies the trace to

$$\begin{aligned}\langle\sigma_i\sigma_{i+r}\rangle &= \frac{1}{Z}\mathrm{tr}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda_+^r & 0\\ 0 & \lambda_-^r\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda_+^{N-r} & 0\\ 0 & \lambda_-^{N-r}\end{pmatrix}\right]\\ &= \frac{1}{Z}\mathrm{tr}\begin{pmatrix}\lambda_+^{N-r}\lambda_-^r & 0\\ 0 & \lambda_-^{N-r}\lambda_+^r\end{pmatrix} = \frac{\lambda_+^{N-r}\lambda_-^r + \lambda_-^{N-r}\lambda_+^r}{\lambda_+^N + \lambda_-^N}.\end{aligned} \tag{6.17}$$

Note that because of the periodic boundary conditions, the above answer is invariant under $r \to (N-r)$. We are interested in the limit of $N \gg r$, for which

$$\langle\sigma_i\sigma_{i+r}\rangle \approx \left(\frac{\lambda_-}{\lambda_+}\right)^r \equiv \mathrm{e}^{-r/\xi}, \tag{6.18}$$

with the correlation length

$$\xi = \left[\ln\left(\frac{\lambda_+}{\lambda_-}\right)\right]^{-1} = -\frac{1}{\ln\tanh K}. \tag{6.19}$$

The above transfer matrix approach can be generalized to any one dimensional chain with variables $\{s_i\}$ and nearest-neighbor interactions. The partition function can be written as

$$Z = \sum_{\{s_i\}} \exp\left[\sum_{i=1}^{N} B(s_i, s_{i+1})\right] = \sum_{\{s_i\}} \prod_{i=1}^{N} \mathrm{e}^{B(s_i, s_{i+1})}, \tag{6.20}$$

where we have defined a *transfer matrix* T with elements,

$$\langle s_i|T|s_j\rangle = \mathrm{e}^{B(s_i, s_j)}. \tag{6.21}$$

In the case of *periodic boundary conditions*, we then obtain

$$Z = \mathrm{tr}\left[T^N\right] \approx \lambda_{\max}^N. \tag{6.22}$$

Note that for $N \to \infty$, the trace is dominated by the largest eigenvalue $\lambda_{\max}$. Quite generally the largest eigenvalue of the transfer matrix is related to the free energy, while the correlation lengths are obtained from ratios of eigenvalues. *Frobenius' theorem* states that for any finite matrix with finite positive elements, the largest eigenvalue is always non-degenerate. This implies that $\lambda_{\max}$ and Z are analytic functions of the parameters appearing in B, and that one dimensional models can exhibit singularities (and hence a phase transition) only at zero temperature (when some matrix elements become infinite).

While the above formulation is framed in the language of discrete variables $\{s_i\}$, the method can also be applied to continuous variables as illustrated by problems at the end of this chapter. As an example of the latter, let us consider three component *unit* spins $\vec{s}_i = \left(s_i^x, s_i^y, s_i^z\right)$, with the *Heisenberg model* Hamiltonian

$$-\beta\mathcal{H} = K\sum_{i=1}^{N} \vec{s}_i \cdot \vec{s}_{i+1}. \tag{6.23}$$

Summing over all spin configurations, the partition function can be written as

$$Z = \underset{\vec{s}_i}{\mathrm{tr}}\, \mathrm{e}^{K\sum_{i=1}^{N} \vec{s}_i\cdot\vec{s}_{i+1}} = \underset{\vec{s}_i}{\mathrm{tr}}\, \mathrm{e}^{K\vec{s}_1\cdot\vec{s}_2}\mathrm{e}^{K\vec{s}_2\cdot\vec{s}_3}\cdots\mathrm{e}^{K\vec{s}_N\cdot\vec{s}_1} = \mathrm{tr}\, T^N, \tag{6.24}$$

where $\langle\vec{s}_1|T|\vec{s}_2\rangle = \mathrm{e}^{K\vec{s}_1\cdot\vec{s}_2}$ is a transfer function. Quite generally we would like to bring T into the diagonal form $\sum_\alpha \lambda_\alpha|\alpha\rangle\langle\alpha|$ (in Dirac notation), such that

$$\langle\vec{s}_1|T|\vec{s}_2\rangle = \sum_\alpha \lambda_\alpha \langle\vec{s}_1|\alpha\rangle\langle\alpha|\vec{s}_2\rangle = \sum_\alpha \lambda_\alpha f_\alpha(\vec{s}_1) f_\alpha^*(\vec{s}_2). \tag{6.25}$$

From studies of plane waves in quantum mechanics you may recall that the exponential of a dot product can be decomposed in terms of the spherical harmonics $Y_{\ell m}$. In particular,

$$\mathrm{e}^{K\vec{s}_1\cdot\vec{s}_2} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} 4\pi i^\ell j_\ell(-iK) Y_{\ell m}^*(\vec{s}_1) Y_{\ell m}(\vec{s}_2) \tag{6.26}$$

is precisely in the form of Eq. (6.25), from which we can read off the eigenvalues $\lambda_{\ell m}(K) = 4\pi i^\ell j_\ell(-iK)$, which do not depend on m. The partition function is now given by

$$Z = \operatorname{tr} T^N = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \lambda_{\ell m}^N = \sum_{\ell=0}^{\infty} (2\ell+1)\lambda_\ell^N \approx \lambda_0^N, \tag{6.27}$$

with $\lambda_0 = 4\pi j_0(-iK) = 4\pi \sinh K/K$ as the largest eigenvalue. The second largest eigenvalue is three fold degenerate, and given by $\lambda_1 = 4\pi j_1(-iK) = 4\pi\left[\cosh K/K - \sinh K/K^2\right]$.

6.3 Position space RG in one dimension

An exact RG treatment can be carried out for the Ising model with nearest-neighbor interactions (Eq. 6.1) in one dimension. The basic idea is to find a transformation that reduces the number of degrees of freedom by a factor b, while preserving the partition function, i.e.

$$Z = \sum_{\{\sigma_i | i=1,\cdots,N\}} \mathrm{e}^{-\beta \mathcal{H}[\sigma_i]} = \sum_{\left\{\sigma'_{i'} | i'=1,\cdots,N/b\right\}} \mathrm{e}^{-\beta \mathcal{H}'[\sigma'_{i'}]}. \tag{6.28}$$

There are many mappings $\{\sigma_i\} \mapsto \{\sigma'_{i'}\}$ that satisfy this condition. The choice of the transformation is therefore guided by the simplicity of the resulting RG. With $b = 2$, for example, one possible choice is to group pairs of neighboring spins and define the renormalized spin as their average. This *majority rule*, $\sigma'_i = (\sigma_{2i-1} + \sigma_{2i})/2$, is in fact not very convenient as the new spin has three possible values $(0, \pm 1)$ while the original spins are two valued. We can remove the ambiguity by assigning one of the two spins, e.g. σ_{2i-1}, the role of tie-breaker whenever the sum is zero. In this case the transformation is simply $\sigma'_i = \sigma_{2i-1}$. Such an RG procedure effectively removes the even numbered spins, $s_i = \sigma_{2i}$ and is usually called a *decimation*.

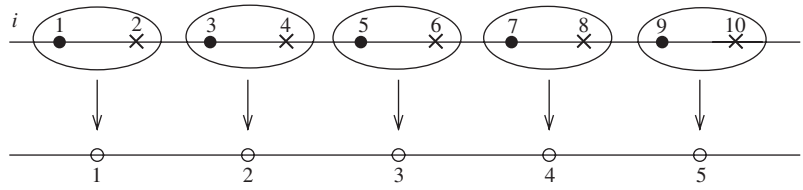

Fig. 6.3 Renormalization treatment of a one-dimensional chain via decimation by a factor of $b = 2$.

Note that since $\sigma' = \pm 1$ as in the original model, no renormalization factor ζ is necessary in this case. Since the interaction is over adjacent neighbors, the partition function can be written as

$$Z = \sum_{\{\sigma_i\}} \exp\left[\sum_{i=1}^{N} B(\sigma_i, \sigma_{i+1})\right] = \sum_{\left\{\sigma'_i\right\}}^{N/2} \sum_{\{s_i\}}^{N/2} \exp\left[\sum_{i=1}^{N/2} \left[B(\sigma'_i, s_i) + B(s_i, \sigma'_{i+1})\right]\right]. \tag{6.29}$$

Summing over the decimated spins, $\{s_i\}$, leads to

$$e^{-\beta\mathcal{H}'[\sigma'_i]} \equiv \prod_{i=1}^{N/2}\left[\sum_{s_i=\pm1} e^{\left[B(\sigma'_i,s_i)+B(s_i,\sigma'_{i+1})\right]}\right] \equiv e^{\sum_{i=1}^{N/2} B'(\sigma'_i,\sigma'_{i+1})}, \tag{6.30}$$

where following Eq. (6.2)

$$B(\sigma_1,\sigma_2) = g + \frac{h}{2}(\sigma_1+\sigma_2) + K\sigma_1\sigma_2, \tag{6.31}$$

and

$$B'(\sigma'_1,\sigma'_2) = g' + \frac{h'}{2}(\sigma'_1+\sigma'_2) + K'\sigma'_1\sigma'_2 \tag{6.32}$$

are the most general interaction forms for Ising spins.

Following Eq. (6.30), the renormalized interactions are obtained from

$$\begin{aligned} R(\sigma'_1,\sigma'_2) &\equiv \exp\left[K'\sigma'_1\sigma'_2 + \frac{h'}{2}(\sigma'_1+\sigma'_2) + g'\right] \\ &= \sum_{s_1=\pm1} \exp\left[Ks_1(\sigma'_1+\sigma'_2) + \frac{h}{2}(\sigma'_1+\sigma'_2) + hs_1 + 2g\right]. \end{aligned} \tag{6.33}$$

To solve for the renormalized interactions it is convenient to set

$$\begin{cases} x = e^K, \quad y = e^h, \quad z = e^g \\ x' = e^{K'}, \quad y' = e^{h'}, \quad z' = e^{g'}. \end{cases} \tag{6.34}$$

The four possible configurations of the bond are

$$\begin{cases} R(+,+) = x'y'z' = z^2y(x^2y + x^{-2}y^{-1}) \\ R(-,-) = x'y'^{-1}z' = z^2y^{-1}(x^{-2}y + x^2y^{-1}) \\ R(+,-) = x'^{-1}z' = z^2(y + y^{-1}) \\ R(-,+) = x'^{-1}z' = z^2(y + y^{-1}). \end{cases} \tag{6.35}$$

The last two equations are identical, resulting in three equations in three unknowns, with the solutions,

$$\begin{cases} z'^4 = z^8(x^2y + x^{-2}y^{-1})(x^{-2}y + x^2y^{-1})(y + y^{-1})^2 \\ y'^2 = y^2\dfrac{x^2y + x^{-2}y^{-1}}{x^{-2}y + x^2y^{-1}} \\ x'^4 = \dfrac{(x^2y + x^{-2}y^{-1})(x^{-2}y + x^2y^{-1})}{(y + y^{-1})^2}. \end{cases} \tag{6.36}$$

Taking the logarithms, we find recursion relations of the form

$$\begin{cases} g' = 2g + \delta g(K,h) \\ h' = h + \delta h(K,h) \\ K' = K'(K,h). \end{cases} \tag{6.37}$$

The parameter g is just an additive constant to the Hamiltonian. It does not affect the probabilities and hence does not appear in the recursion relations for K and h; $\delta g(K,h)$ is the contribution of the decimated spins to the overall free energy.

(1) ***Fixed points:*** The $h = 0$ subspace is closed by symmetry, and it can be checked that for $y = 1$ Eqs. (6.36) imply $y' = 1$, and

$$e^{4K'} = \left(\frac{e^{2K} + e^{-2K}}{2}\right)^2, \quad \Longrightarrow \quad K' = \frac{1}{2}\ln\cosh 2K. \tag{6.38}$$

The recursion relation for K has the following fixed points:

(a) An infinite temperature fixed point at $K^* = 0$, which is the *sink* for the disordered phase. If K is small, $K' \approx \ln(1 + 4K^2/2)/2 \approx K^2$, is even smaller, indicating that this is a *stable* fixed point with zero correlation length.

(b) A zero temperature fixed point at $K^* \to \infty$, describing the ordered phase. For a large but finite K, the renormalized interaction $K' \approx \ln\left(e^{2K}/2\right)/2 \approx K - \ln 2/2$ is somewhat smaller. This fixed point is thus unstable with an infinite correlation length.

Fig. 6.4 Fixed points and RG flows for the coupling K in one dimension.

Clearly any finite interaction renormalizes to zero, indicating that the one-dimensional chain is always disordered at sufficiently long length scales. The absence of any other fixed point is apparent by noting that the recursion relation of Eq. (6.38) can alternatively be written as $\tanh K' = (\tanh K)^2$.

(2) ***Flow diagrams*** indicate that in the presence of a field h, all flows terminate on a line of fixed points with $K^* = 0$ and arbitrary h^*. These fixed points describe *independent* spins and all have zero correlation length. The flows originate from the fixed point at $h^* = 0$ and $K^* \to \infty$ which has two unstable directions in the (K, h) parameter space.

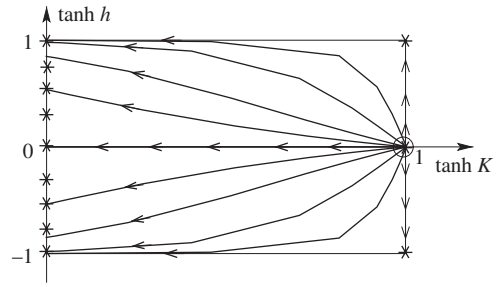

Fig. 6.5 Fixed points and RG flows in the space of coupling $(\tanh h, \tanh K)$ in one dimension.

(3) ***Linearizing*** the recursion relations around this fixed point ($x \to \infty$) yields

$$\begin{cases} x'^4 \approx x^4/4 \\ y'^2 \approx y^4 \end{cases} \quad \Longrightarrow \quad \begin{cases} e^{-K'} = \sqrt{2}e^{-K} \\ h' = 2h. \end{cases} \tag{6.39}$$

We can thus regard e^{-K} and h as scaling fields. Since $\xi' = \xi/2$, the correlation length in the vicinity of the fixed point satisfies the homogeneous form ($b = 2$)

$$\begin{aligned}\xi\left(e^{-K}, h\right) &= 2\xi\left(\sqrt{2}e^{-K}, 2h\right) \\ &= 2^{\ell}\xi\left(2^{\ell/2}e^{-K}, 2^{\ell}h\right).\end{aligned} \tag{6.40}$$

The second equation is obtained by repeating the RG procedure ℓ times. Choosing ℓ such that $2^{\ell/2}e^{-K} \approx 1$, we obtain the scaling form

$$\xi\left(e^{-K}, h\right) = e^{2K} g_{\xi}\left(he^{2K}\right). \tag{6.41}$$

The correlation length diverges on approaching $T = 0$ for $h = 0$. However, its divergence is not a power law of temperature. There is thus an ambiguity in identifying the exponent ν related to the choice of the measure of vicinity to $T = 0$ ($1/K$ or e^{-K}).

The hyperscaling assumption states that the singular part of the free energy in d dimensions is proportional to ξ^{-d}. Hence we expect

$$f_{\text{sing}}(K, h) \propto \xi^{-1} = e^{-2K} g_f\left(he^{2K}\right). \tag{6.42}$$

At zero field, the magnetization is always zero, while the susceptibility behaves as

$$\chi(K) \sim \left.\frac{\partial^2 f}{\partial^2 h}\right|_{h=0} \sim e^{2K}. \tag{6.43}$$

On approaching zero temperature, the divergence of the susceptibility is proportional to that of the correlation length. Using the general forms, $\langle s_i, s_{i+x}\rangle \sim e^{-x/\xi}/x^{d-2+\eta}$, and $\chi \sim \int d^d\mathbf{x}\,\langle s_0 s_{\mathbf{x}}\rangle_c \sim \xi^{2-\eta}$, we conclude that $\eta = 1$.

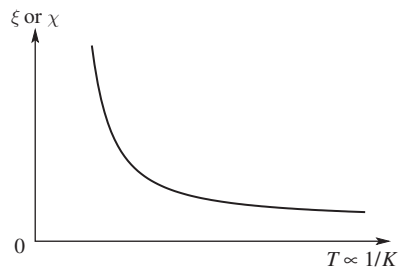

Fig. 6.6 The susceptibility (and correlation length) diverge similarly at zero temperature in the one-dimensional Ising model.

The transfer matrix method also provides an alternative RG scheme for all general one-dimensional chains with nearest-neighbor interactions. For decimation by a factor b, we can use $Z = \text{tr}\left[\left(T^b\right)^{N/b}\right]$ to construct the rescaled bond energy from

$$e^{B(s_i', s_j')} \equiv \left\langle s_i'|T'|s_j'\right\rangle = \left\langle s_i'|T^b|s_j'\right\rangle. \tag{6.44}$$

6.4 The Niemeijer–van Leeuwen cumulant approximation

Unfortunately, the decimation procedure cannot be performed exactly in higher dimensions. For example, the square lattice can be divided into two sublattices. For an RG with $b = \sqrt{2}$, we can start by decimating the spins on one sublattice. The interactions between the four spins surrounding each decimated spin are obtained by generalizing Eq. (6.33). If initially $h = g = 0$, we obtain

$$R(\sigma_1', \sigma_2', \sigma_3', \sigma_4') = \sum_{s=\pm 1} \mathrm{e}^{Ks(\sigma_1'+\sigma_2'+\sigma_3'+\sigma_4')} = 2\cosh\left[K(\sigma_1'+\sigma_2'+\sigma_3'+\sigma_4')\right]. \quad (6.45)$$

Clearly the four spins appear symmetrically in the above expression, and hence are subject to the same two-body interaction. This implies that new interactions along the diagonals of the renormalized lattice are also generated, and the nearest-neighbor form of the original Hamiltonian is not preserved. There is also a four-point interaction, and

$$R = \exp\left[g' + K'(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_4' + \sigma_4'\sigma_1' + \sigma_1'\sigma_3' + \sigma_2'\sigma_4') + K_4'\sigma_1'\sigma_2'\sigma_3'\sigma_4'\right]. \quad (6.46)$$

The number (and range) of new interactions increases with each RG step, and some truncating approximation is necessary. Two such schemes are described in the following sections.

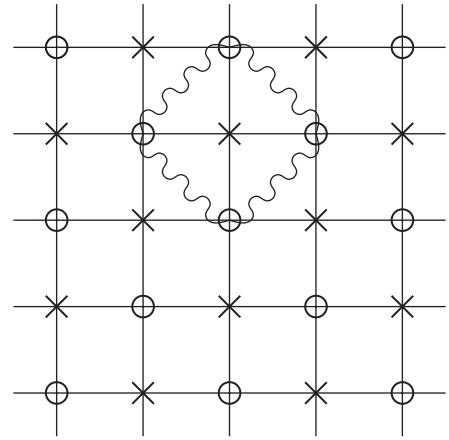

Fig. 6.7 Removal of a spin in two (and higher) dimensions results in more than nearest-neighbor interactions.

One of the earliest approaches was developed by Niemeijer and van Leeuwen (NvL) for treating the Ising model on a *triangular lattice*, subject to the usual nearest-neighbor Hamiltonian $-\beta\mathcal{H} = K\sum_{\langle ij\rangle}\sigma_i\sigma_j$. The original lattice sites are grouped into *cells* of three spins (e.g. in alternating up pointing triangles). Labeling the three spins in cell α as $\{\sigma_\alpha^1, \sigma_\alpha^2, \sigma_\alpha^3\}$, we can use a *majority rule* to define the renormalized cell spin as

$$\sigma_\alpha' = \mathrm{sign}\left[\sigma_\alpha^1 + \sigma_\alpha^2 + \sigma_\alpha^3\right]. \quad (6.47)$$

(There is no ambiguity in the rule for any odd number of sites, and the renormalized spin is two-valued.) The renormalized interactions corresponding to the above map are obtained from the constrained sum

$$\mathrm{e}^{-\beta\mathcal{H}'[\sigma'_\alpha]} = \sum_{\{\sigma^i_\alpha \mapsto \sigma'_\alpha\}}{}' \mathrm{e}^{-\beta\mathcal{H}[\sigma^i_\alpha]}. \tag{6.48}$$

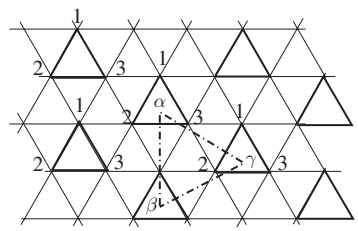

Fig. 6.8 Each cell spin is assigned from the majority of its three site spins.

To truncate the number of interactions in the renormalized Hamiltonian, NvL introduced a perturbative scheme by setting $\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$. The unperturbed Hamiltonian

$$-\beta\mathcal{H}_0 = K\sum_\alpha \left(\sigma^1_\alpha\sigma^2_\alpha + \sigma^2_\alpha\sigma^3_\alpha + \sigma^3_\alpha\sigma^1_\alpha\right) \tag{6.49}$$

involves only *intracell interactions*. Since the cells are decoupled, this part of the Hamiltonian can be treated exactly. The remaining *intercell interactions* are treated as a perturbation

$$-\mathcal{U} = K\sum_{\langle\alpha,\beta\rangle}\left(\sigma^{(1)}_\beta\sigma^{(2)}_\alpha + \sigma^{(1)}_\beta\sigma^{(3)}_\alpha\right). \tag{6.50}$$

The sum is over all neighboring cells, each connected by two bonds. (The actual spins involved depend on the relative orientations of the cells.) Equation (6.48) is now evaluated perturbatively as

$$\mathrm{e}^{-\beta\mathcal{H}'[\sigma'_\alpha]} = \sum_{\{\sigma^i_\alpha \mapsto \sigma'_\alpha\}}{}' \mathrm{e}^{-\beta\mathcal{H}_0[\sigma^i_\alpha]}\left[1 - \mathcal{U} + \frac{\mathcal{U}^2}{2} - \cdots\right]. \tag{6.51}$$

The renormalized Hamiltonian is given by the cumulant series

$$\beta\mathcal{H}'[\sigma'_\alpha] = -\ln Z_0[\sigma'_\alpha] + \langle\mathcal{U}\rangle_0 - \frac{1}{2}\left(\langle\mathcal{U}^2\rangle_0 - \langle\mathcal{U}\rangle_0^2\right) + \mathcal{O}(\mathcal{U}^3), \tag{6.52}$$

where $\langle\ \rangle_0$ refers to the expectation values with respect to $\beta\mathcal{H}_0$, with the restriction of fixed $[\sigma'_\alpha]$, and Z_0 is the corresponding partition function.

To proceed, we construct a table of all possible configurations of spins within a cell, their renormalized value, and contribution to the cell energy:

	σ'_α	σ^1_α	σ^2_α	σ^3_α	$\exp[-\beta\mathcal{H}_0]$	
	$+$	$+$	$+$	$+$	e^{3K}	
	$+$	$-$	$+$	$+$	e^{-K}	
	$+$	$+$	$-$	$+$	e^{-K}	
	$+$	$+$	$+$	$-$	e^{-K}	
	$-$	$-$	$-$	$-$	e^{3K}	
	$-$	$+$	$-$	$-$	e^{-K}	
	$-$	$-$	$+$	$-$	e^{-K}	
	$-$	$-$	$-$	$+$	e^{-K}	

The restricted partition function is the product of contributions from the independent cells,

$$Z_0[\sigma'_\alpha] = \prod_\alpha \left[{\sum_{\{\sigma^i_\alpha \mapsto \sigma'_\alpha\}}}' \mathrm{e}^{K(\sigma^1_\alpha\sigma^2_\alpha + \sigma^2_\alpha\sigma^3_\alpha + \sigma^3_\alpha\sigma^1_\alpha)} \right] = \left(\mathrm{e}^{3K} + 3\mathrm{e}^{-K}\right)^{N/3}. \tag{6.53}$$

It is *independent* of $[\sigma'_\alpha]$, thus contributing an additive constant to the Hamiltonian. The first cumulant of the interaction is

$$-\langle \mathcal{U} \rangle_0 = K \sum_{\langle \alpha,\beta \rangle} \left[\langle \sigma^1_\beta \rangle_0 \langle \sigma^2_\alpha \rangle_0 + \langle \sigma^1_\beta \rangle_0 \langle \sigma^3_\alpha \rangle_0 \right] = 2K \sum_{\langle \alpha,\beta \rangle} \langle \sigma^i_\alpha \rangle_0 \left\langle \sigma^j_\beta \right\rangle_0, \tag{6.54}$$

where we have taken advantage of the equivalence of the three spins in each cell. Using the table, we can evaluate the restricted average of site spins as

$$\langle \sigma^i_\alpha \rangle_0 = \left\{ \begin{array}{ll} \dfrac{+\mathrm{e}^{3K} - \mathrm{e}^{-K} + 2\mathrm{e}^{-K}}{\mathrm{e}^{3K} + 3\mathrm{e}^{-K}} & \text{for } \sigma'_\alpha = +1 \\ \dfrac{-\mathrm{e}^{3K} + \mathrm{e}^{-K} - 2\mathrm{e}^{-K}}{\mathrm{e}^{3K} + 3\mathrm{e}^{-K}} & \text{for } \sigma'_\alpha = -1 \end{array} \right\} \equiv \frac{\mathrm{e}^{3K} + \mathrm{e}^{-K}}{\mathrm{e}^{3K} + 3\mathrm{e}^{-K}} \sigma'_\alpha. \tag{6.55}$$

Substituting in Eq. (6.54) leads to

$$-\beta\mathcal{H}'[\sigma'_\alpha] = \frac{N}{3} \ln\left(\mathrm{e}^{3K} + 3\mathrm{e}^{-K}\right) + 2K \left(\frac{\mathrm{e}^{3K} + \mathrm{e}^{-K}}{\mathrm{e}^{3K} + 3\mathrm{e}^{-K}} \right)^2 \sum_{\langle \alpha\beta \rangle} \sigma'_\alpha \sigma'_\beta + \mathcal{O}(\mathcal{U}^2). \tag{6.56}$$

At this order, the renormalized Hamiltonian involves only nearest-neighbor interactions, with the recursion relation

$$K' = 2K \left(\frac{\mathrm{e}^{3K} + \mathrm{e}^{-K}}{\mathrm{e}^{3K} + 3\mathrm{e}^{-K}} \right)^2. \tag{6.57}$$

(1) Equation (6.57) has the following *fixed points:*

(a) The high-temperature sink at $K^* = 0$. If $K \ll 1$, $K' \approx 2K(2/4)^2 = K/2 < K$, i.e. this fixed point is *stable*, and has zero correlation length.

(b) The low-temperature sink at $K^* = \infty$. If $K \gg 1$, then $K' \approx 2K > K$, i.e. unlike the one-dimensional case, this fixed point is also stable with zero correlation length.

(c) Since both of the above fixed points are stable, there must be at least one unstable fixed point at finite $K' = K = K^*$. From Eq. (6.57), the fixed point position satisfies

$$\frac{1}{\sqrt{2}} = \frac{e^{3K^*} + e^{-K^*}}{e^{3K^*} + 3e^{-K^*}}, \quad \Longrightarrow \quad \sqrt{2}e^{4K^*} + \sqrt{2} = e^{4K^*} + 3. \tag{6.58}$$

The fixed point value

$$K^* = \frac{1}{4}\ln\left(\frac{3-\sqrt{2}}{\sqrt{2}-1}\right) \approx 0.3356 \tag{6.59}$$

can be compared to the exactly known value of 0.2747 for the triangular lattice.

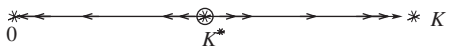

Fig. 6.9 Fixed points and RG flows for the coupling K in two dimensions.

(2) Linearizing the recursion relation around the non-trivial fixed point yields,

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2\left(\frac{e^{4K^*}+1}{e^{4K^*}+3}\right)^2 + 32K^*e^{4K^*}\frac{(e^{4K^*}+1)}{(e^{4K^*}+3)^3} \approx 1.624. \tag{6.60}$$

The fixed point is indeed unstable as required by the continuity of flows. This RG scheme removes 1/3 of the degrees of freedom, and corresponds to $b = \sqrt{3}$. The thermal eigenvalue is thus obtained as

$$b^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K^*} \quad \Longrightarrow \quad y_t \approx \frac{\ln(1.624)}{\ln(\sqrt{3})} \approx 0.883. \tag{6.61}$$

This can be compared to the exactly known value of $y_t = 1$, for the two-dimensional Ising model. It is certainly better than the mean-field (Gaussian) estimate of $y_t = 2$. From this eigenvalue we can estimate the exponents

$$\nu = 1/y_t \approx 1.13 \quad (1), \quad \text{and} \quad \alpha = 2 - 2/y_t = -0.26 \quad (0),$$

where the exact values are given in the brackets.

(3) To complete the calculation of exponents, we need the *magnetic eigenvalue* y_h, obtained after adding a magnetic field to the Hamiltonian, i.e. from

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} - h\sum_i \sigma_\alpha^i. \tag{6.62}$$