Assignment 2

Tuesday, January 23, 2024 11:22 AM

Part A) Problems and Proofs (6 points)

Rewrite the following claim as a universal proposition in symbolic form:

"The algorithm bruteForceSearch() is a linear-time algorithm".

and use the strategy of generalizing from generic particular to prove that the claim is correct. For a detailed explanation of the claim and related concepts in algorithm analysis, see the figure on the next page.

```
public int bruteForceSearch(int[] values, int value){
   int numOfSteps = 0;
   boolean found = false;

   while( !found && numOfSteps < values.size){
     numOfSteps++;
     if(values[numOfSteps] == value) found = true;
   }

   return numOfSteps;
}
```

Figure 1

Generalizing:

Let $A(x)$: x is an algorithm
$B(x)$: x is bruteForceSearch()
$L(x)$: x is a linear algorithm.

Then the claim can be rewritten as

$$(\forall x \in \mathbb{Z}),\ (A(X) \land B(x)) \Rightarrow L(x)$$

Generic particular: bruteForceSearch() (☆)

From Figure 1, it shows that (☆) iterates over array values[] until it finds the value. So, time complexity is $O(n)$ where n is size of array, given worst case, it will still be $O(n)$ where value is at the last element of values[]. Therefore, it is a Linear-time algorithm.

$$\therefore (A(x) \land B(x)) \Rightarrow L(x). \quad \square$$

Part B) Proof by Cases (6 points)

Use the method of proof by cases to prove the following statement

$$\forall n \geq 3,\ n^2 - 3n + 3 \text{ is odd.}$$

Proof by cases.

Case 1: $n \in \mathbb{Z}_{odd} : n \geq 3$,

Then, $n = 2k+1$.

We have, $(2k+1)^2 - 3(2k+1) + 3$ for $k \geq 3$

$\Leftrightarrow \quad 4k^2 + 4k + 1 - 6k - 3 + 3$

$\Leftrightarrow \quad 4k^2 - 2k + 1$

$\Leftrightarrow \quad 2\underbrace{(2k^2 - k)}_{\in \mathbb{Z}.} + 1 \quad \text{II}$

Case 2: let $n \in \mathbb{Z}_{even} : n \geq 3$,

Then, by def, $n = 2k$.

$\Rightarrow (2k)^2 - 3(2k) + 3$ for $k \geq 3$

$\Leftrightarrow \quad 4k^2 - 6k + 3$

$\Leftrightarrow \quad 2\underbrace{(2k^2 - 3k)}_{\in \mathbb{Z}.} + 3$

$\therefore$ The statement in Part B is true by case 1 & 2,

for $\forall n \geq 3$. $\square$

4.52 Let n and m be integers. If nm is not evenly divisible by 3, then neither n nor m is evenly divisible by 3. (In fact, the converse is true too, but you don't have to prove it.)

Proof by Contrapositive

"n and m is evenly divisible by 3 $\Rightarrow$ nm is divisible by 3."

Let $(n, m) \in \mathbb{R}$,

then by def, $n = 3k$ and $m = 3k$. $(k \in \mathbb{Z})$

$nm = 3k \cdot 3k = (3k)^2 = 9k^2 = 3(\underbrace{3k^2}_{\in \mathbb{Z}})$

$\therefore (n \in (3k) \wedge m \in (3k)) \Rightarrow (nm) \in (3k)$ where $k \in \mathbb{Z}$. $\square$

4.56 Let x, y be positive real numbers. If $x^2 - y^2 = 1$, then x or y (or both) is not an integer.

Proof by contradiction

Let's assume $(x, y) \in \mathbb{Z}^+ : x^2 - y^2 = 1$ (for the sake of contradiction)

$\Rightarrow (x+y)(x-y) = 1$

$\Leftrightarrow \quad (x+y) = 1 \quad \wedge \quad (x-y) = 1$

$\Leftrightarrow (x = 1 - y) \quad \wedge \quad (x = 1 + y)$

$\Leftrightarrow \quad 1 - y = 1 + y \Leftrightarrow 2y = 0 \Leftrightarrow y = 0$!!!

Which is absurd because we assumed $(x, y) \in \mathbb{Z}^+$

$\therefore$ The original claim is true b/c assuming $(x, y) \in \mathbb{Z}^+$ creates a contradiction $\square$

5.2 $\sum_{i=0}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4}$

Proof by induction

Base Case: Let $S(n) = (5.2)$ where $n=0$.

$$S(n) = \sum_{i=0}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4}$$

$$S(0) = \sum_{i=0}^{0} i^3 = \frac{0^4 + 2\,0^3 + 0^2}{4}$$

$$0^3 = \frac{0}{4}$$

$$0 = 0 \qquad \Rightarrow LHS = RHS$$

Inductive Hypothesis: If $S(n)$ holds, then $S(n+1)$ will hold for some $n \geq 0$.

Inductive Step: $n = n+1$

$$S(n+1) = \sum_{i=0}^{n+1} i^3 = \frac{(n+1)^4 + 2(n+1)^3 + (n+1)^2}{4}$$

By Ind hypo, LHS =

$$\sum_{i=0}^{n} i^3 + (n+1)^3 = \frac{n^4 + 2n^3 + n^2}{4} + (n+1)^3$$

Now, we simplify RHS

$$= \frac{(n+1)^4 + 2(n+1)^3 + (n+1)^2}{4}$$

$$= \frac{n^4 + 4n^3 + 6n^2 + 4n + 1 + 2n^3 + 6n^2 + 6n + 2 + n^2 + 2n + 1}{4}$$

$$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

$$= \frac{n^4 + 2n^3 + n^2 + 4n^3 + 12n^2 + 4}{4}$$

$$= \frac{n^4 + 2n^3 + n^2}{4} + n^3 + 6n^2 + 1$$

$$= \frac{n^4 + 2n^3 + n^2}{4} + (n+1)^3$$

RHS = LHS

$\therefore$ By Principle of Math Induction, we have proven that $S(n)$ is true for some $n \geq 0$. $\square$