COSC221 Assignment 3

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March 26, 2024

Part A) *Proof.* By induction on $n \in \mathbb{Z}^+$. Here, we choose $\mathbb{Z}^+$ because it is intuitive to have a positive integer of $n \times n$ chessboard.

Remark: $\mathbb{Z}^+$ is the set of all positive integers; $\mathbb{Z}^+ = \{1, 2, 3, \ldots, n\}_{n \to \infty}$.

Base Case: $n = 1$

$S(1) = 1 \times 1$ chessboard, so there exists one cell only on that chessboard, which makes it the top-left and the bottom-left cell, where the rook starts and ends at the same cell. Hence, there exists the rook's tour, even if $n = 1$ is a little bit trivial.

Inductive Hypothesis: If our constructed $S(n)$ holds, then there exists some $n \in \mathbb{Z}^+$ such that it satisfies $S(n+1)$ as well.

Inductive Step: $n \to n + 1$

$S(n+1) = (n+1) \times (n+1)$ chessboard. By the given information on page 234 for rook's tour and rook's move, we let $(r, c, x) \in \mathbb{Z}^+$, then it is true that rook can start at $\langle r, c \rangle$. Then for $n \times n$ chessboard to $(n+1) \times (n+1)$ chessboard, there are two possible movement,

Vertical movement $\Rightarrow \langle r, c \pm x \rangle \to \langle r, n+1 \rangle$, so this means that $c \pm x = n + 1 \Leftrightarrow c = n + 1 \pm x$

Horizontal movement $\Rightarrow \langle r \pm x, c \rangle \to \langle n+1, c \rangle$, so this also means that $r \pm x = n + 1 \Leftrightarrow r = n + 1 \pm x$

Now, we know that, if we let

$$x = min(\mathbb{Z}^+) = 1$$

and,

$$min(r) = min(c) = n + 1 - x$$

therefore,

$$min(r) = min(c) = n + 1 - 1 = n$$

Then, our movement for both vertical and horizontal are both within the bounds of $\mathbb{Z}^+$, as we said above, $n \in \mathbb{Z}^+$. Now, let these coordinates be true that $\langle 1, 1\rangle \Rightarrow$ top-left, $\langle n+1, 1\rangle \Rightarrow$ top-right, $\langle 1, n+1\rangle \Rightarrow$ bottom-left and, $\langle n+1, n+1\rangle \Rightarrow$ bottom-right.

Now to traverse through the board, there are two possible cases:

Case $\boldsymbol{\alpha}$: $(n+1) \times (n+1)$ chessboard where $(n+1) \in \mathbb{Z}_{\text{even}}$

Then if we let rook start at $\langle 1, 1\rangle$, then we follow a horizontal "zig-zag" pattern below:

$$\begin{array}{l}
(1,1) \rightarrow (2,1) \rightarrow \ldots \rightarrow (n,1) \rightarrow \underbrace{(n+1,1)}_{\downarrow} \\
\underbrace{(1,2)}_{\downarrow} \leftarrow (2,2) \leftarrow \ldots \leftarrow (n,2) \leftarrow (n+1,2) \\
(1,3) \rightarrow (2,3) \rightarrow \ldots \rightarrow (n,3) \rightarrow \underbrace{(n+1,3)}_{\downarrow} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
(1,n+1) \leftarrow (2,n+1) \leftarrow \ldots \leftarrow (n+1,n+1)
\end{array}$$

By this visual, we can clearly see that the rook arrives at the left side every time $c \in \mathbb{Z}_{\text{even}}$, so it is entirely possible that the rook can end up on the bottom-left side, because $c = (n+1) \in \mathbb{Z}^+_{\text{even}}$.

Case $\boldsymbol{\beta}$: $(n+1) \times (n+1)$ chessboard where $(n+1) \in \mathbb{Z}^+_{\text{odd}}$

Then if we let rook start at $\langle 1, 1\rangle$, then we follow a vertical "zig-zag" pattern below:

$$\begin{array}{lllll}
\underbrace{(1,1)}_{\downarrow} & (2,1) \rightarrow & \underbrace{(\ldots)}_{\downarrow} & (n,1) \rightarrow & \underbrace{(n+1,1)}_{\downarrow} \\
 & \overbrace{\qquad}^{\uparrow} & & \overbrace{\qquad}^{\uparrow} & \\
\underbrace{\vdots}_{\downarrow} & \vdots & \underbrace{\vdots}_{\downarrow} & (n, \vdots) & \underbrace{(n+1,2)}_{\downarrow}
\end{array}$$

$$(1,n) \rightarrow \overbrace{(2,n)}^{\uparrow} \quad (\ldots) \rightarrow \overbrace{(n,n)}^{\uparrow} \qquad \underbrace{(n+1,n)}_{\downarrow}$$

$$\vdots$$

$$(1,n+1) \leftarrow (2,n+1) \leftarrow \ldots \leftarrow (n+1,n+1)$$

By this visual, we can clearly see that whenever $r \in \mathbb{Z}_{\text{odd}}$, it will be making a downwards movement, so since $(n+1) \in \mathbb{Z}^{+}_{\text{odd}}$, we know that $n \in \mathbb{Z}^{+}_{\text{even}}$, that means that it will be making an upwards movement so that it can turn right towards $(n+1,1)$ where it can start moving down again. Note that we have never really visited $c = n+1$ for these traversal, so let it be known that, after reaching $(n+1,n+1)$, it can go directly to $(1,n+1)$, where the rook will be at the bottom-left of the chessboard.

$\therefore$ By mathematical induction, and by **Case $\boldsymbol{\alpha}$** and $\boldsymbol{\beta}$, it is true that the rook's tour exists for the $n \times n$ chessboard, for some $n \in \mathbb{Z}^{+}$.

$\square$

Part B) **11.36** We know that in a simple graph,

$$\textit{directed} \Rightarrow \textit{density} = \frac{|E|}{\frac{n(n-1)}{2}} = \frac{2|E|}{n(n-1)}$$

$$\textit{undirected} \Rightarrow \textit{density} = \frac{|E|}{n(n-1)}$$

Where n is the vertex or node, and $|E|$ is the edges.

As a function of n the density of an n-node path means, $|E| \Rightarrow (n-1)$, where n is vertices, so we have,

$$f(n) = \frac{2(n-1)}{n(n-1)} = \frac{2}{n}$$

11.37 As a function of n the density of an n-node cycle means, $|E| \Rightarrow n$, where n is vertices, so we have,

$$f(n) = \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

11.38 As a function of n the density of a graph that consists of $\frac{n}{3}$ disconnected triangles, $|E| \Rightarrow n$, where n is vertices, so we have,

$$f(n) = \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

which is essentially the same as a cycle.

Part C) **Remark:** Let it be known that clique 1, clique 2 and clique 3 represents the "clusters" of nodes from left to right respectively in **Figure 11.33d**

11.54 We can see that each "clusters" or cliques represents its own movie, then there must be at least, 3 movies, but it is also shown that there is two curves on the top of the graph, where the actor from the clique 3, knows actor from clique 1 and clique 2. which means that there must exists at least another 2 movies. Therefore, there must exists, at the minimum, five movies that could be generated by this collaboration network.

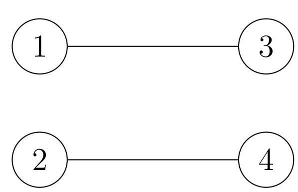
11.55 No, it is uncertain. It is entirely possible that there are more than 5 movies generated in this graph because each cliques can have its own subset, where the actors are in another movie different than the minimum 5 movies.

Part D) **11.95**

Proof. By counterexample, assume G is a bipartite graph where,

$$G = \langle L \cup R, E \rangle$$

and where, $|L| = |R|$, so we can take, $L = \{1, 2\}$ and $R = \{3, 4\}$ and since all nodes from L and R has at least one neighbours, we can construct G such that $E = \{\{1, 3\}, \{2, 4\}\}$, visually:



and we can see that the graph we made, G, consists of two disjoint sets which is absurd, because the original statement say that it is connected.

$\therefore$ It is disproved that if $G = \langle L \cup R, E \rangle$ where G is a bipartite graph with $|L| = |R|$, then it is impossible for G to be a connected graph, if there is at minimum, one neighbour for every node in L and R.x‘ $\square$

End of Assignment 3.