COSC221 Assignment 3

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Part A) **Remark:**

- $\mathbb{Z}^+ = \{\forall n \in \mathbb{Z} : n \geq 1\} = \{1, 2, 3, \ldots\}$
- $\mathbb{Z}^+_{even} = \{n \in \mathbb{Z}^+ : n \bmod 2 = 0\} = \{2, 4, 6, \ldots\}$
- $\mathbb{Z}^+_{odd} = \mathbb{Z}^+ \setminus \mathbb{Z}^+_{even} = \{1, 3, 5, \ldots\}$

Proof. By induction on $n \in \mathbb{Z}^+$. Here, we choose $\mathbb{Z}^+$ because it is more intuitive to have a positive integer of $n \times n$ chessboard.

Let $S(n) = n \times n$ chessboard, where there exists a rook's tour from top-left to bottom-left.

Base Case: $n = 1$. It is known that a 1×1 chessboard is trivial, but ambiguous, so we can still proceed:

$S(1) = 1 \times 1$ chessboard, so there exists one cell only on that chessboard, which makes it the top-left and the bottom-left cell at the same time, where the rook starts and ends at the same cell. Hence, there exists the rook's tour.

Inductive Hypothesis: If our constructed $S(n)$ holds, then there exists some $n \in \mathbb{Z}^+$ such that it satisfies $S(n + 1)$ as well.

Inductive Step: $n \to n + 1$.

$S(n + 1) = (n + 1) \times (n + 1)$ chessboard. By the given information on page 234 for rook's tour and rook's move, we let $(x, y, k) \in \mathbb{Z}^+$, then it is true that rook can start at $\langle x, y \rangle$. Then from $n \times n$ chessboard to $(n + 1) \times (n + 1)$ chessboard, there are two possible movements:

- **Vertical movement:**

$$\langle x, y \pm k \rangle \to \langle x, n+1 \rangle \Rightarrow y \pm k = n+1 \Leftrightarrow y = n+1 \pm k$$

- **Horizontal movement:**

$$\langle x \pm k, y \rangle \to \langle n+1, y \rangle \Rightarrow x \pm k = n+1 \Leftrightarrow x = n+1 \pm k$$

Now, we know that, if we let

$$k = min(\mathbb{Z}^+) = 1$$

and,

$$min(x) = min(y) = n + 1 - k$$

therefore,

$$min(x) = min(y) = n + 1 - 1 = n$$

Then, our movement for both vertical and horizontal are both within the bounds of $\mathbb{Z}^+$, as we said above, $k \in \mathbb{Z}^+$. Now, let these coordinates be true that $\langle 1, 1 \rangle \Rightarrow$ top-left, $\langle n+1, 1 \rangle \Rightarrow$ top-right, $\langle 1, n+1 \rangle \Rightarrow$ bottom-left and, $\langle n+1, n+1 \rangle \Rightarrow$ bottom-right.

Now to traverse through the board, there are two possible cases:

Case $\boldsymbol{\alpha}$: $(n+1) \times (n+1)$ chessboard where $(n+1) \in \mathbb{Z}^+_{\text{even}}$

Then we let the rook start at $(1, 1)$, we follow a horizontal "zig-zag" pattern below:

$$\begin{array}{c}
(1,1) \to (2,1) \to \ldots \to (n,1) \to \underbrace{(n+1,1)}_{\downarrow} \\
\underbrace{(1,2)}_{\downarrow} \leftarrow (2,2) \leftarrow \ldots \leftarrow (n,2) \leftarrow (n+1,2) \\
(1,3) \to (2,3) \to \ldots \to (n,3) \to \underbrace{(n+1,3)}_{\downarrow} \\
\vdots \\
(1,n+1) \leftarrow (2,n+1) \leftarrow \ldots \leftarrow (n+1,n+1)
\end{array}$$

With this visual, we can clearly see that the rook arrives at the left side every time $y \in \mathbb{Z}^+_{\text{even}}$, so it is entirely possible that the rook can end up on the bottom-left side, because $max(y) = (n+1) \in \mathbb{Z}^+_{\text{even}}$.

Case $\boldsymbol{\beta}$: $(n+1) \times (n+1)$ chessboard where $(n+1) \in \mathbb{Z}^+_{\text{odd}}$

Then we let the rook start at $(1, 1)$, we follow a vertical "zig-zag" pattern below:

$$\begin{array}{ccccc}
\underbrace{(1,1)}_{\downarrow} & (2,1) \rightarrow & \underbrace{(\ldots)}_{\downarrow} & (n,1) \rightarrow & \underbrace{(n+1,1)}_{\downarrow} \\
\underbrace{\vdots}_{\downarrow} & \overbrace{\vdots}^{\uparrow} & \underbrace{\vdots}_{\downarrow} & \overbrace{(n,\vdots)}^{\uparrow} & \underbrace{(n+1,2)}_{\downarrow} \\
(1,n) \rightarrow & \overbrace{(2,n)}^{\uparrow} & (\ldots) \rightarrow & \overbrace{(n,n)}^{\uparrow} & \underbrace{(n+1,3)}_{\downarrow} \\
 & & & & \vdots \\
\end{array}$$

$$(1, n+1) \leftarrow (2, n+1) \leftarrow \ldots \leftarrow (n+1, n+1)$$

With this visual, we can clearly see that whenever $x \in \mathbb{Z}^+_{\text{odd}}$, it will be making a downwards movement, so since $max(x) = (n + 1) \in \mathbb{Z}^+_{\text{odd}}$, then we know that $x = n \in \mathbb{Z}^+_{\text{even}}$, that means that it will be making an upwards movement so that it can turn right towards $(n + 1, 1)$ where it can move down towards $(n + 1, n)$. Note that we have never really visited $y = n + 1$ for these traversal, so let it be known that, after reaching $(n + 1, n + 1)$, it can go directly to $(1, n + 1)$, where the rook will be at the bottom-left of the chessboard.

$\therefore$ By mathematical induction, and by **Case $\boldsymbol{\alpha}$** and $\boldsymbol{\beta}$, it is true that the rook's tour exists for the $n \times n$ chessboard, for some $n \in \mathbb{Z}^+$. $\square$

Part B) **Remark:** We know that in a simple graph,

$$directed \Rightarrow density = \frac{|E|}{\frac{n(n-1)}{2}} = \frac{2|E|}{n(n-1)}$$

$$undirected \Rightarrow density = \frac{|E|}{n(n-1)}$$

Where n is the vertex or node, and $|E|$ is the edges.

11.36 As a function of n the density of an n-node path means, $|E| \Rightarrow (n - 1)$, where n is vertices, so we have,

$$f(n) = \frac{2(n-1)}{n(n-1)} = \frac{2}{n}$$

11.37 As a function of n the density of an n-node cycle means, $|E| \Rightarrow n$, where n is vertices, so we have,

$$f(n) = \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

11.38 As a function of n the density of a graph that consists of $\frac{n}{3}$ disconnected triangles, $|E| \Rightarrow n$, where n is vertices, so we have,

$$f(n) = \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

which is essentially the same as a cycle.

Part C) **Remark:** Let it be known that clique 1, clique 2 and clique 3 represents the "clusters" of nodes from left to right respectively in **Figure 11.33d**

11.54 We can see that each "clusters" or cliques represents its own movie, then there must be at least, 3 movies, but it is also shown that there is 2 curves on the top of the graph, where the actor from the clique 3, knows actor from clique 1 and clique 2. which means that there must exists at least another 2 movies. Therefore, there must exists, at the minimum, five movies that could be generated by this collaboration network.

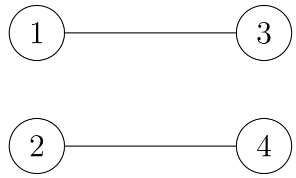
11.55 No, it is uncertain. It is entirely possible that there are more than 5 movies generated in this graph because each cliques can have its own subset, where the actors are in another movie different than the minimum 5 movies.

Part D) **11.95**

Proof. By counterexample, assume G is a bipartite graph where,

$$G = \langle L \cup R, E \rangle$$

and where, $|L| = |R|$, so we can take, $L = \{1, 2\}$ and $R = \{3, 4\}$ and since all nodes from L and R has at least one neighbours, we can construct G such that $E = \{\{1, 3\}, \{2, 4\}\}$, visually:



We can see that the graph we made, G, consists of two disjoint sets which is absurd, because the original statement say that it is connected.

$\therefore$ It is disproved that if $G = \langle L \cup R, E \rangle$ where G is a bipartite graph with $|L| = |R|$, then it is impossible for G to be a connected graph, if there is at minimum, one neighbour for every node in L and R. $\square$

End of Assignment 3.