STAT230 Assignment 3

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1. (a) *Proof.* Given: X is a continuous variable that follows a normal distribution with mean μ and standard deviation σ. Which also means

$$X \sim N(\mu, \sigma^2)$$

We want to show 95% of the area under the normal density curve lies between 2 standard deviation of the mean. By definition of the given statement, we can safely assume that

$$Z = \frac{X - \mu}{\sigma}$$

and standardized normal distribution, we will have

$$Z \sim N(0, 1)$$

and it is also true that to compute $P(X \leq x)$, we can use the fact that

$$P(X \leq x) = P(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma})$$

.

$$= P(z \leq \frac{x - \mu}{\sigma})$$

$$\Rightarrow P(X \leq \pm 2\sigma) = P(z \leq \pm \frac{2\sigma - \mu}{\sigma})$$

the $\pm$ is because we want to show "between" 2σ. Now, if we substitute $\mu = 0$ and $\sigma = 1$ we will have the following:

$$P(X \leq \pm 2(1)) = P(z \leq \pm \frac{2(1) - 0}{1}) = P(z \leq \pm 2)$$

Because we want a between value, we would need to subtract $P(z \leq 2)$ from $P(z \leq -2)$ as per z table. So now we have,

$$P(X \leq 2\sigma) = P(z \leq 2) - P(z \leq -2)$$

$$= 0.9772 - 0.0228 = 0.9544 \simeq 0.95$$

With this value, 0.95, it implies that the 95% of the area lies between $P(z \leq 2)$ and $P(z \leq -2)$.

$\therefore$ It is true that, 95% of the area under the normal density curve lies between 2σ of the mean. $\square$

(b) *Proof.* Given: Y_1 and Y_2 are independent random variables, $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$.

We want to show $Cov(Y_1, Y_2)$ in terms of Y_1 and Y_2. The covariance formula is given by:

$$Cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)] = E[XY] - E[X]E[Y]$$

Since Y_1 and Y_2 are independent random variables, it is true that,

$$Cov(Y_1, Y_2) = 0 = E[Y_1 Y_2] - E[Y_1]E[Y_2]$$

Rearranging our U_1, and U_2, we find out that

$$Y_1 = U_1 - Y_2$$

$$\begin{aligned}
Y_1 &= U_2 + Y_2 \\
U_1 - Y_2 &= U_2 + Y_2 \\
U_1 - U_2 &= 2Y_2 \\
Y_2 &= \frac{U_1 - U_2}{2} \\
\Rightarrow Y_1 = U_2 + \frac{U_1 - U_2}{2} &= \frac{2U_2}{2} + \frac{U_1 - U_2}{2} \\
Y_1 &= \frac{U_1 + U_2}{2}
\end{aligned}$$

Following our formula for covariance above, we have to find $E[Y_1]$ and $E[Y_2]$ to solve it. Let us remind ourselves that $E[aX] = aE[X]$

and that $E[aX + bY + c] = aE[X] + bE[Y] + c$ where a, b, and c are some arbitrary constant. Then we will have:

$$E[Y_1] = E[\frac{U_1 + U_2}{2}] = E[\frac{1}{2}(U_1 + U_2)]$$

$$= \frac{1}{2}E[(U_1 + U_2)]$$

and $E[Y_2]$ will follow the same concept as follows:

$$E[Y_2] = E[\frac{U_1 - U_2}{2}] = E[\frac{1}{2}(U_1 - U_2)]$$

$$= \frac{1}{2}E[(U_1 - U_2)]$$

Now we substitute it into the formula

$$Cov(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]$$

$$= E[\frac{U_1 + U_2}{2} \times \frac{U_1 - U_2}{2}] - \frac{1}{2}E[(U_1 + U_2)]\frac{1}{2}E[(U_1 - U_2)]$$

$$= E[\frac{U_1^2 - U_2^2}{4}] - \frac{1}{4}E[(U_1 + U_2)]E[(U_1 - U_2)]$$

$$= \frac{1}{4}E[U_1^2 - U_2^2] - \frac{1}{4}((E[U_1] + E[U_2]) \times (E[U_1] - E[U_2]))$$

$$= \frac{1}{4}E[U_1^2 - U_2^2] - \frac{1}{4}(E[U_1]^2 - E[U_2]^2)$$

$$= \frac{1}{4}(E[U_1^2 - U_2^2] - (E[U_1]^2 - E[U_2]^2))$$

$$= \frac{1}{4}(E[U_1^2] - E[U_2^2] - E[U_1]^2 + E[U_2]^2)$$

$$= \frac{1}{4}((E[U_1^2] - E[U_1]^2) - (E[U_2^2] - E[U_2]^2))$$

We recall that $Var[X] = E[X^2] - E[X]^2$ so then we have,

$$Cov(Y_1, Y_2) = \frac{1}{4}(Var[U_1] - Var[U_2])$$

$\therefore$ We have expressed $Cov(Y_1, Y_2)$ in terms of the variance of Y_1 and Y_2. $\square$

2. The $Cov(X,Y) = E[XY] - E[X]E[Y]$ and $Cov(X,Y) = 0$ when X and Y are independent but it is also given that the pdf is,

$$f(x,y) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Which ultimately implies that the variable x and y is independent because $f(x,y)$ is not affected for any value of y in $0 \leq y \leq 1$. Which leads to our conclusion that $Cov(X,Y) = 0$.

3. *Proof.* Let us denote the given joint probability distribution table with ϕ

We want to show that X and Y are dependent, then we need to proof the following

$$\exists (X,Y) \in \phi : P(X,Y) \neq P(X)P(Y)$$

so then we can take the pair $(0,0) \in \phi$ where

$$P(0,0) = 0$$

Now, we find the marginal probability of the following,

$$P(X=0) = \frac{3}{16} + 0 + \frac{3}{16} = \frac{6}{16}$$

$$P(Y=0) = \frac{3}{16} + 0 + \frac{3}{16} = \frac{6}{16}$$

Then,

$$P(X=0)P(Y=0) = \frac{6}{16} \times \frac{6}{16} = \frac{36}{256}$$

So,

$$P(0,0) \neq P(X=0)P(Y=0) \Rightarrow 0 \neq \frac{36}{256}$$

Hence, we have proven that X and Y are not independent.

Now we want to show that X and Y have zero covariance, then we need to proof the following

$$\forall (X,Y) \in \phi, C(X,Y) = E[XY] - E[X]E[Y] = 0$$

First we find the marginal sum of

$$E[X] = \sum xP(X = x)$$

$$= -1(\frac{1}{16} + \frac{3}{16} + \frac{1}{16}) + 0(\frac{3}{16} + 0 + \frac{3}{16}) + 1(\frac{1}{16} + \frac{3}{16} + \frac{1}{16})$$

$$= -\frac{5}{16} + \frac{5}{16} = 0$$

$$E[Y] = \sum yP(Y = y)$$

$$= -1(\frac{1}{16} + \frac{3}{16} + \frac{1}{16}) + 0(\frac{3}{16} + 0 + \frac{3}{16}) + 1(\frac{1}{16} + \frac{3}{16} + \frac{1}{16})$$

$$= -\frac{5}{16} + \frac{5}{16} = 0$$

and now we find,

$$E[XY] = \sum xyP(X = x, Y = y)$$

$$= (-1)(-1)(\frac{1}{16}) + (0)(-1)(\frac{3}{16}) + (1)(-1)(\frac{1}{16})$$

$$+(-1)(0)(\frac{3}{16}) + (0)(0)(0) + (1)(0)(\frac{3}{16})$$

$$+(-1)(1)(\frac{1}{16}) + (0)(1)(\frac{3}{16}) + (1)(1)(\frac{1}{16})$$

$$= \frac{1}{16} + 0 - \frac{1}{16} + 0 + 0 + 0 - \frac{1}{16} + 0 + \frac{1}{16} = 0$$

Now, we can finally substitute it in the equation,

$$4C(X, Y) = E[XY] - E[X]E[Y] = 0 - 0 * 0 = 0$$

Hence, we have proven that the $Cov(X, Y) = 0$.

$\therefore$ We have shown that X and Y are dependent, but they have zero covariance. $\square$