Stat 230 Introductory Statistics
Continuous Distribution

University of British Columbia Okanagan

Sampling Distribution of Sampling Mean (review)

Recall:

- if $X_1, X_2, \ldots, X_n$ are random variables which follow the same distribution with known mean μ and standard deviation σ
- The Central Limit Theorem tells us that if $X_1, \ldots, X_n$ come from ANY distribution with mean μ and standard deviation σ, as long as n is 'large enough', then $\bar{X}$ is approximately normally distributed.
- Specifically, for large n, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

Sampling Distribution of Other Statistics

- Are there any other statistics for which we can find the sampling distribution?
- Consider, for example, the sample variance for random samples from a normal population.
- Since we know this random variable cannot be negative, you may have guess that it does follow a normal or t-distribution.
- We need another distribution to describe this distribution ... the Chi-square distribution.

Chi-squared Random Variables

Definition 7.1 (χ^2 Random Variables On 1 Degree of Freedom)

Suppose Z is a standard normal random variable, and set

$$X = Z^2.$$

Then X is a χ^2 random variable on 1 degree of freedom ($\chi^2_{(1)}$).

Definition 7.2 (PDF of $\chi^2_{(1)}$ Distribution)

The probability density function of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}, \; x \geq 0$$

$f(x) = 0$, for $x < 0$.

Chi-squared Random Variables with more Degrees of Freedom

Definition 7.3

Suppose $Z_1, Z_2, \ldots, Z_k$ are independent standard normal random variables, and

$$X = Z_1^2 + Z_2^2 + \cdots + Z_k^2.$$

Then X is a χ^2 random variable on k degrees of freedom.

Definition 7.4 (PDF of $\chi^2_{(k)}$ Distribution)

The probability density function of X is

$$f(x) = \frac{1}{\Gamma(k/2)} x^{k/2-1} e^{-x/2}, \ x \geq 0$$

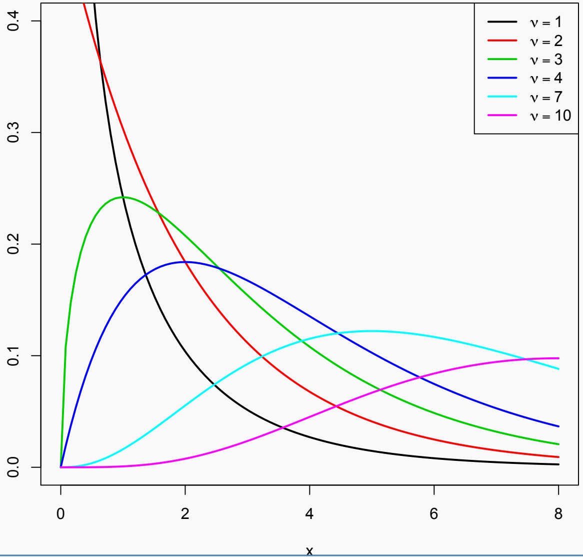
$f(x) = 0$, for $x < 0$. $\Gamma(k/2) = \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{k}{2}-1\right)\sqrt{\pi}$, if k is odd and $(k/2-1)!$, if k is even.

χ^2-Distribution

- The χ^2-distribution (read "chi-squared" distribution) is a continuous distribution parameterized by its degrees of freedom parameter.
- Like the standard normal, we can use R that help us find probabilities.

- The chi-squared distribution is used primarily in hypothesis testing and is one of the most widely used probability distributions in inferential statistics.

Chi-squared Random Variables PDFs



Chi-squared Random Variables

- The chi-square distribution curve is skewed to the right, and its shape depends on the degrees of freedom df.
- See the plot of the density functions for 1 (black), 2 (red), 3 (green), 4 (blue), 7 (turquoise) and 10 (pink) degrees of freedom. All are skewed to the right, but as the degrees of freedom increase, the bulk of the probability mass shifts to the right.
- If $k = 2$, then

$$f(x) = \frac{1}{2} e^{-x/2}$$

  This is the exponential density function with $\lambda = 1/2$.

Example 7.1

Using R to find the value of $P(\chi^2_{\nu=17} > 27.587)$

Chi-squared Random Variables

- Expected value of a Chi-squared Random Variable on k degrees of freedom:

$$E[X] = k$$

The expected value of a χ^2 random variable is its degrees of freedom.

- Variance:

$$\mathrm{Var}(X) = 2k$$

The variance of a χ^2 random variable is twice its degrees of freedom.

Chi-square Random Variables and Z Scores

Suppose $X_1, X_2, \ldots, X_n$ are independent normal random variables with mean μ and variance σ^2. Then

$$Z_i = \frac{X_i - \mu}{\sigma}, i = 1, 2, \ldots, n$$

are independent standard normal random variables.

$$Y = \sum_{i=1}^{n} Z_i^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2$$

is a χ^2 random variable on n degrees of freedom.

Contrast Y with the sample variance of the $X's$

The sample variance uses the estimate $\bar{X}$ instead of μ:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

In fact,

$$\frac{(n-1)S^2}{\sigma^2}$$

has a χ^2 distribution on $n-1$ degrees of freedom. We lose 1 degree of freedom because we estimated μ.

Sampling Distribution of S^2

Theorem 7.5

If S^2 is the variance of a random sample of size n taken from a normal population having variance σ^2, then

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2}$$

is a random variable having the chi square distribution with the parameter $n-1$.

The parameter is the degrees of freedom.

Summary of Theoretical Background

- $X_1, X_2, \ldots, X_n$ is a random sample from a normal population with mean or expected value μ and variance σ^2.
- S^2 is the sample variance and $\bar{X}$ is the sample mean.
- $(n-1)S^2/\sigma^2$ is a $\chi^2_{(n-1)}$ random variable.

t random variable

Definition 7.6 (t random variable)

If Z is standard normal and X is $\chi^2_{(\nu)}$, and Z and X are independent, then $T = \frac{Z}{\sqrt{X/\nu}}$ is a t random variable on ν degrees of freedom.

A t Random Variable

- If $X_1, X_2, \ldots, X_n$ are independent normal measurements with mean μ and variance σ^2, then

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

  is standard normal.
- We estimate σ with S:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

  is <u>not</u> standard normal.
- T has a t distribution on $n - 1$ degrees of freedom.

A t Random Variable

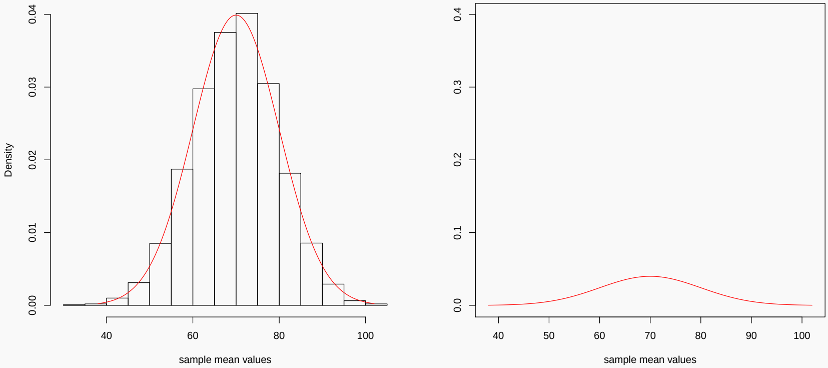
- The number of degrees of freedom = denominator of S^2.
- The density of the t distribution is bell-shaped like the normal, but it is flatter and more spread out.
- As the number of degrees of freedom increases, a t random variable becomes more and more like a standard normal random variable:

$$t_{n-1} \to z \text{ as } n \text{ increases}$$

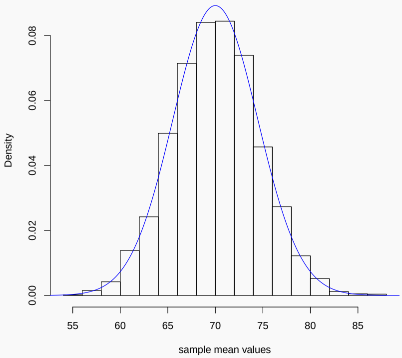
Some More Vocab (review)

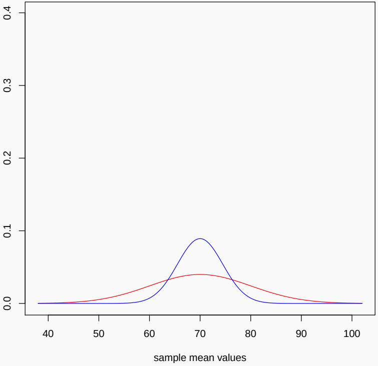
- Concentrating on the variance term, σ^2/n, the implication is that as our sample size n increases, the variability in our statistic $\bar{X}$ decreases.
- Consider a continuous random variable Y that measures Vancouver commute times.
- These times are believed to follow a normal distribution with a mean of 70 and standard deviation of 10.
- Let's see what happens to the variance of $\overline{X}$ when we increase our sample size n, from 1, to 5, 10, 30, 50, and 100.

Single sample n=1 - i.e. averages of samples of size 1

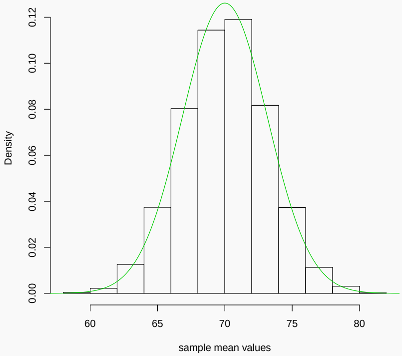


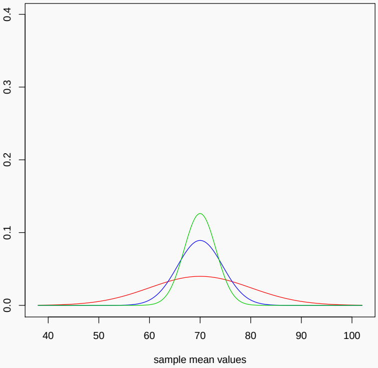
Sample n=5 - i.e. averages of samples of size 5



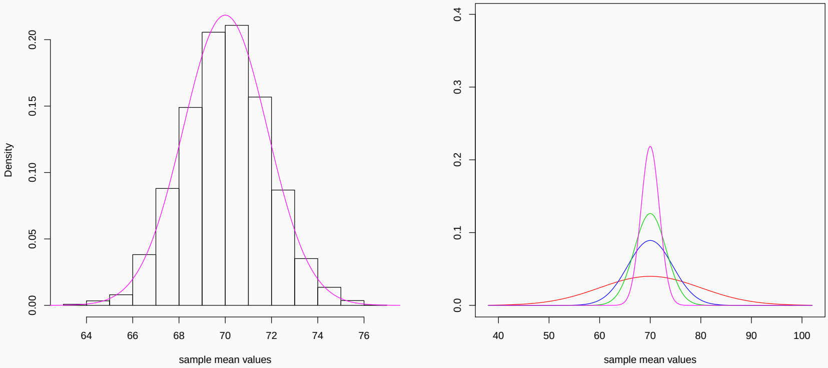


Sample n=10 - i.e. averages of samples of size 10

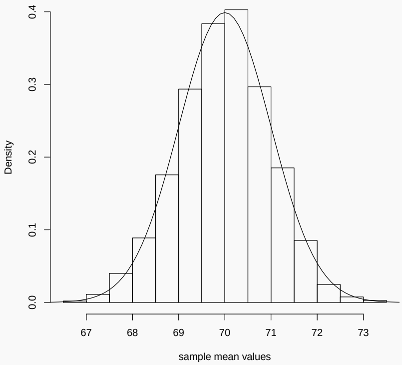


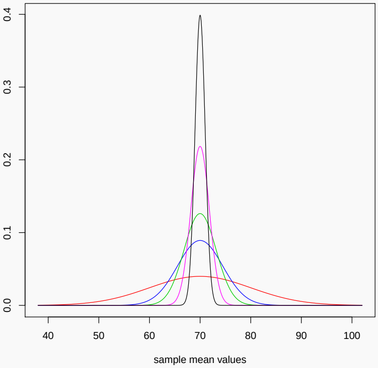


Sample n=30 - i.e. averages of samples of size 30



Sample n=100 - i.e. averages of samples of size 100





Recap

- Recall that the distribution of $\overline{X}$ is normal with a mean of μ and a standard deviation of $\sigma_{\overline{X}} = \sigma/\sqrt{n}$.
- Since we don't know σ, we could *estimate* $\sigma_{\overline{X}}$ by $\frac{s}{\sqrt{n}}$, where s is the *sample* standard deviation.
- This estimate of our standard deviation is called **standard error**. (Note: it is still a standard deviation, but of $\bar{X}$, not X.)

Definition 7.7 (standard error of the sample mean)

The standard error of the sample mean is the estimate of the standard deviation in our sampling distribution of $\overline{X}$ and is given by

$$SE_{\overline{X}} = \frac{s}{\sqrt{n}}.$$

Sampling Distribution of the mean (σ unknown)

- So, instead of using the "true" standard deviation $\sigma_{\overline{X}}$, we estimate it with the standard error $SE_{\overline{X}}$.

- So, instead of using the statistic $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ we are using

$$\frac{\overline{X} - \mu}{s/\sqrt{n}} \tag{1}$$

- (1) does NOT follow a standard normal distribution

- There is added uncertainty associated with estimating σ by s. We therefore need to turn to another distribution . . . which we saw before is the t distribution

Sampling Distribution of the mean (σ unknown)

Theorem 7.8

If $\overline{X}$ is the mean of a random sample of size n take from a normal population having mean μ and variance σ^2, and $S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$, then

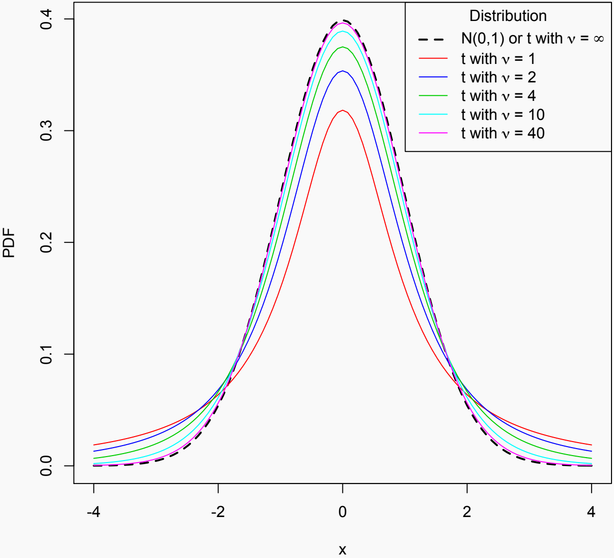
$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

is a random variable having the t distribution with the parameter $n-1$

The parameter is the number of degrees of freedom.

Student's t-Distribution

- First published by WS Gosset in 1908.
- Gosset was an Oxford-trained chemist who worked in the Guinness brewery in Dublin, Ireland.
- He was not allowed to publish his work under his own name, so he used the pseudonym 'Student'.
- The t-distribution looks more and more like the normal distribution as the sample size gets larger. For smaller sample sizes, it has relatively thick tails.


Distribution
N(0,1) or t with ν = ∞
t with ν = 1
t with ν = 2
t with ν = 4
t with ν = 10
t with ν = 40
PDF
0.0
0.1
0.2
0.3
0.4
-4
-2
0
2
4
x

Student's t-Distribution

- The shape of the distribution is determined by its degrees of freedom parameter
- Like the standard normal, we have tables that help us find probabilities
- Be careful because the t-table works a little different from the Z-table!

Example 7.2

Using R find $P(t_8 > 1.397)$

Example 7.3

Using R find t^* such that $P(t_{12} < t*) = 0.05$

Confidence Intervals (σ unknown)

Definition 7.9

Based on a sample of size n, a $100(C)\%$ confidence interval for μ when the population is normal with standard deviation is unknown is given by

$$\overline{x} \pm m \qquad \text{or}$$

$$(\overline{x} - m, \overline{x} + m)$$

- where m is called the **margin of error** given by $t^* s/\sqrt{n}$ and
- t^* is the the value such that $P(t^* < T < t^*) = C$ and
- T follows a t distribution with $n - 1$ degrees of freedom.
- Many textbooks use the notation $t_{\nu}^{\alpha/2}$ to denote the t^* defined above, where $\alpha = 1 - C$.

Example

Example 7.4

Experimental determinations of spring constants were made for three spring types; Type 1 (4 inch, tsc=1.86), Type 2 (6 inch, tsc=2.63) and Type 3 (6inch, tsc=2.63). Note that tsc means "theoretical spring constant". The following summary statistics were obtained. Six springs were used in each case and you may assume normality.

Statistic	Type 1	Type 2	Type 3
$\overline{x}$	2.03	2.55	2.34
s	0.068	0.084	0.064

Example 7.4

Construct a 95% confidence interval for the first string (Type 1)

Example contd.

A 95% confidence interval for μ_1 as follows.

Exercise 7.1 Construct 95% CIs for Types 2 and 3

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Another Look at Hypothesis Testing

We introduced the four steps for performing a hypothesis test for the population mean:

1. State the null and alternative hypotheses
2. Calculate your test statistic
3. Make a decision based on *critical values*[1] or p-values
4. State the decision in the context of the original problem

Notice how we have another way to base our decision, and that is to use critical values.

[1] See the next slide.

Critical Values

- We can do hypothesis testing with p-values or critical values. To test with critical values, we specify a significance level α.
- A critical value is a number which represents the probability of rejecting the null hypothesis (incorrectly) when it is really true.
- For example, suppose we would like to test the null hypothesis that $\mu = \mu_0$ versus the alternative that $\mu > \mu_0$.
- To do this test, we compute the test statistic $t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$ and we choose to reject the null hypothesis at significance level α by rejecting H_0 when $t > t^*_\alpha$ where

$$P(T > t^*_\alpha) = \alpha.$$

The critical value t^*_α can usually be read directly from a t-table.

Reading Critical Values from Tables

- For example, suppose we have a sample of 22 measurements and we would like to test the one-sided hypothesis $\mu > 14$ (the null hypothesis is $\mu = 14$). If we want to use a 5% significance level for our test, we read off the tabled value that tells us $P(T > t^*) = .05$ with 21 degrees of freedom: 1.721. In other words, if $t > 1.721$, we reject the null hypothesis.
- Suppose, now that we want to test the hypothesis that $\mu < 4$ using a sample of 7 observations and a significance level of 1%. Now we want the value of t^* where $P(T < t^*) = .01$, but by symmetry, this would be $P(T > -t^*) = .01$: 3.143, using 6 degrees of freedom. In other words, we reject the null hypothesis if $t < -3.143$.

Example 7.5

Suppose that Snickers claims that their chocolate bars contain on average 6 peanuts, with a standard deviation of 1.2 peanuts. We would like to investigate whether their claim is true at the $\alpha = 0.05$ significance level.

1. State the null and alternative hypothesis for this test.
2. We sampled 40 chocolate bars and found an average of 5.67 peanuts per bar. Calculate the test statistic.
3. Compare to a critical value for a 95% test, and make a decision regarding your hypotheses.
4. State your decision regarding the peanut content of Snickers bars.

Example 7.5

5. Find the 95% confidence interval for the mean number of peanuts in snicker bars.

Two-Tailed Test Summary: Critical Value Approach

- For a two-tailed test with known σ^2:
- $H_0 : \mu = \mu_0$
- $H_a : \mu \neq \mu_0$
- Test statistic:

$$Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

- Comparison and decision:
  - If $Z \leq -z_{\alpha/2}$ or $Z \geq z_{\alpha/2}$, then reject H_0 at the α significance level.
  - Otherwise, fail to reject at the α significance level.

One-Tailed Tests

- Sometimes, we're only interested in a one-sided alternative hypothesis...
- For instance, do we care if Snickers is giving us too many peanuts? Maybe we only want to know if they're short-changing us...
- In this case, perhaps we'd rather investigate the following hypotheses:
  - $H_0 : \mu = \mu_0$
  - $H_a : \mu < \mu_0$

Left-Tailed Test Summary: Critical Value Approach

- For a left-tailed test with known σ^2:
- $H_0 : \mu = \mu_0$
- $H_a : \mu < \mu_0$
- Test statistic:

$$Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

- Comparison and decision:
  - If $Z \leq -z_\alpha$, then reject H_0 at the α significance level.
  - Otherwise, fail to reject at the α significance level.

Right-Tailed Test Summary: Critical Value Approach

- For a right-tailed test with known σ^2:
- $H_0 : \mu = \mu_0$
- $H_a : \mu > \mu_0$
- Test statistic:

$$Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

- Comparison and decision:
  - If $Z \geq z_\alpha$, then reject H_0 at the α significance level.
  - Otherwise, fail to reject at the α significance level.

Example 7.5

6. Conduct a left-tailed test for the mean of peanuts in a Snickers bar.

Example 7.5

7. Conduct a right-tailed test for the mean of peanuts in a Snickers bar.

Unknown σ

- But what if we don't know σ?
- Analogous to our confidence intervals, we need to adjust our methods when the population standard deviation (σ) is unknown.
- Namely, if we don't know σ, we can't use the tests we just learned (and we often won't know σ).
- So we will once again use Student's t-distribution.

Student's t-Test

The set up for the hypothesis test is going to be similar to the so called Z-test we learned from our previous lecture, with the exception of the second step:

1. State the null and alternative hypotheses
2. Calculate your test statistic
3. Make a decision based on critical values or p-values
4. State the decision in the context of the original problem

The t-Test (σ unknown)

- Given a sample mean $\overline{x}$ and a sample standard deviation s, one can test hypotheses about the value of μ, using the following test statistic;

$$T = \frac{\overline{x} - \mu_0}{s/\sqrt{n}},$$

  where t follows a Student-t distribution with degrees of freedom $\nu = n - 1$.

  Recall that we use t-tables to find probabilities for this distribution.

Example

Example 7.6

A drug company introduces a new drug that lowers blood pressure. They must prove at the 99% confidence level that their drug is more effective than an established drug which has been shown to reduce systolic blood pressure by an average of 25.5 mmHg. A sample of 39 patients show an average blood pressure decrease of 28.4 mmHg with a standard deviation of 10.2 mmHg. Perform an appropriate hypothesis test about the population mean.

Hypothesis Testing for mean (σ unknown) - p-value method

1. *State the null and alternative hypothesis*

$$H_0 : \mu = \mu_0 \qquad \begin{cases} H_1 : \mu \neq \mu_0 & \text{two-sided test} \\ H_1 : \mu < \mu_0 & \text{one-sided (lower-tail) test} \\ H_1 : \mu > \mu_0 & \text{one-sided (upper-tail) test} \end{cases}$$

2. *Compute the test statistic* $t_{obs} = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}$.
3. *Calculate the p-value*

$$\begin{cases} 2P(Z \geq |t_{obs}|) & \text{if } H_1 : \mu \neq \mu_0 \\ P(Z \leq t_{obs}) & \text{if } H_1 : \mu < \mu_0 \\ P(Z \geq t_{obs}) & \text{if } H_1 : \mu > \mu_0 \end{cases}$$

4. *State the conclusion*

$$\begin{cases} \text{If the } p\text{-value} \leq \alpha & \text{we } \textbf{reject} \text{ the null hypothesis} \\ \text{Otherwise} & \text{we } \textbf{fail to reject} \text{ the null hypothesis} \end{cases}$$

Example

Example 7.7

Experimental determinations of spring constants were made for three spring types; Type 1 (4 inch, tsc=1.86), Type 2 (6 inch, tsc=2.63) and Type 3 (6inch, tsc=2.63). Note that tsc means "theoretical spring constant". The following summary statistics were obtained. Six springs were used in each case and you may assume normality.

Statistic	Type 1	Type 2	Type 3
$\overline{x}$	2.03	2.55	2.34
s	0.068	0.084	0.064

Example 7.8

Test whether or not the long-run mean for Type 1 springs equals the tsc of 1.86.

Exercise 7.2 Carry out the t-tests for Types 2 and 3.

Example contd.

The test looks like this:

Hypotheses:

Test Statistic:

p-value:

Conclusion:

Again, since this test is **two-sided** ($H_1 : \mu_1 \neq 1.86$) this should agree with the conclusion we make based on a 95% CI.

To test whether $\mu_1 = 1.86$ at a significance level of 5%, we simply need to check if 1.86 falls with the corresponding 95% CI

We already computed this CI to be (1.96, 2.10)

Since 1.86 is not contained in the above interval, we reject the null hypothesis that $\mu_1 = 1.86$ (as expected).

Note, this method will NOT correspond to a conclusion made in a **one-sided** hypothesis test.

Possible Errors

- When we decide to reject the null hypothesis, we are always running the risk that the null hypothesis was true all along.
- This is called Type I Error.
- When we decide not to reject the null hypothesis, we are always running the risk that null hypothesis was wrong all along.
- This is called Type II Error

Types of Error

These errors have a clear analogy between the errors we make in a courtroom trial.

Mistakes can happen if either

- an innocent person is convicted, or
- a guilty person is set free.

In hypothesis testing, the two types of errors we can make are:

- we reject a claim when it is actually true (Type I)
- we fail to reject a claim when it is actually false (Type II)

Power and the Probability of Error

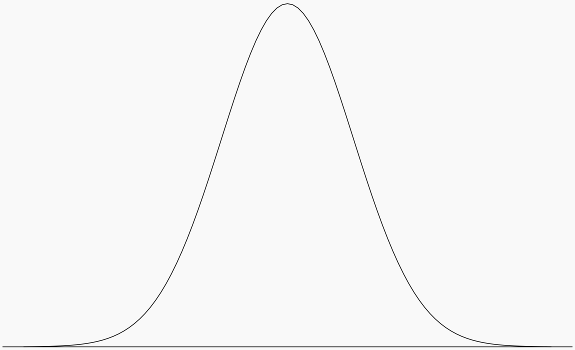
Definition 7.10

We make a **Type I** error when we reject H_0 when it is true. The P(Type I error) $= \alpha$, where α is our significance level for a hypothesis test.

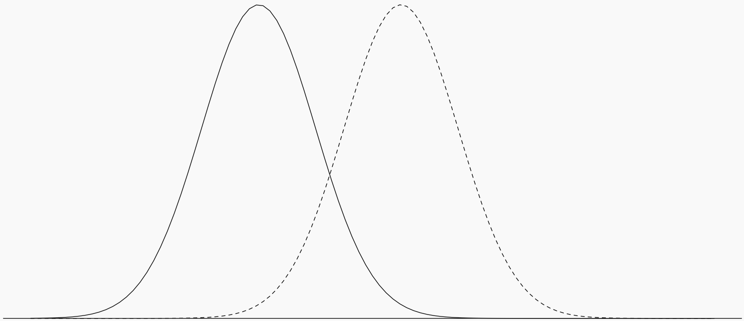
Definition 7.11

We make a **Type II** error when we fail to reject H_0 when it is false. The P(Type II error) $= \beta$.

Type I error



Type II error



These terms can be summarized in the following chart.

	H_0 **True**	H_0 **False**
Reject H_0	Type I Error	Correct Decision
Fail to Reject H_0	Correct Decision	Type II Error

- We consider Type I Error the gravest of mistakes...akin to convicting an innocent person.
- As in civilized societies, we assume innocence until there is sufficient evidence to convict.
- For hypothesis tests, we assume H_0 until there is enough evidence to reject H_0.

Let's Add Some Rigor

- We will control the Type I Error rate by fixing an acceptable probability of making this mistake.
- That is controlled by our significance level, α.
- Which means we have a confidence level for our tests once again, found by $1 - \alpha$.
- In theory, if the hypothesis is true and we conducted 100 different hypothesis tests using 100 different samples, 95 of these test would (correctly) fail to reject the null, while 5 of them would make a Type I error.

F random variable

Definition 7.12 (F random variable)

If $X_1 \sim \chi^2_{\nu_1}$ and $X_2 \sim \chi^2_{\nu_2}$, and X_1 and X_2 are independent, then

$$F = \frac{X_1/\nu_1}{X_2/\nu_2}$$

has an F distribution on ν_1 and ν_2 degrees of freedom.

The F distribution

- Recall that Z^2 is $\chi^2_{(1)}$ and $(n-1)S^2/\sigma^2$ is $\chi^2_{(n-1)}$, and S^2 and $\bar{X}$ are independent, if the underlying sample is random and normal with mean μ and variance σ^2.
- Therefore,
$$\frac{(\bar{X}-\mu)^2}{S^2/n}$$
has an F distribution on 1 and $n-1$ degrees of freedom.
- Percentiles of the F distribution can be read from the F tables at the back of the book or use `qf(1-a, n1, n2)`
- e.g. $f_{.95,1,7} = 5.59$ or `qf(.95, 1, 7); [1] 5.591448`

Example: Testing μ (using the F distribution)

- Is the expected value of the concentration measurements different from 207?
- We test this by contradiction:
  Suppose $\mu = 207$. Then

$$f = \frac{(\bar{x} - \mu)^2}{s^2/n} = \frac{(201 - 207)^2}{5.5^2/8} = 9.52$$

  Conclusion: This value exceeds 5.59, so the probability of observing an f value this large, under our assumption (called the *Null Hypothesis*) is less than .05. We have evidence that the true expected concentration differs from 207.