(1) $v' = (-1, 0, 3, -2) \quad w = (2, 3, -1, -3)'$

a) $v' = (-1, 0, 3, -2) \quad w = (2, 3, -1, -3)' = \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix}$

$$v' \cdot w = (-1, 0, 3, -2) \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix} = -1 \cdot 2 + 0 \cdot 3 + 3 \cdot (-1) + (-2) \cdot (-3) =$$
$$= 1$$

b) $$v' \cdot v = (-1, 0, 3, -2) \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \\ -2 \end{pmatrix} = (-1) \cdot (-1) + 0 \cdot 0 + 3 \cdot 3 + (-2) \cdot (-2) =$$
$$= 14$$

c) $$w' \cdot w \cdot v = \cancel{(2, 3, -1, -3)} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \\ -2 \end{pmatrix} = 23 \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \\ -2 \end{pmatrix} =$$

$$(2, 3, -1, -3) \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix} \qquad \cancel{\begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix} \cdot (2, 3, -1, -3)} \qquad = \begin{pmatrix} -23 \\ 0 \\ 69 \\ -46 \end{pmatrix}$$

d) $$w \cdot v' \cdot w = \cancel{(2, 3, -1, -3) \cdot (-1, 0, 3, -2)}$$

$$\begin{array}{cc} 4 \times 4 & 4 \times 1 \end{array}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix} \cdot (-1, 0, 3, -2) \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 6 & -4 \\ -3 & 0 & 9 & -6 \\ 1 & 0 & -3 & 2 \\ 3 & 0 & -9 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix}$$

(2)

$$v_1 \cdot v_2 = \underset{3\times 1}{\begin{pmatrix}1\\2\\-1\end{pmatrix}} \cdot \underset{3\times 1}{\begin{pmatrix}4\\2\\x\end{pmatrix}}$$

$$v_1 \cdot v_2 = 1\cdot 4 + 2\cdot 2 + (-1)\cdot x = 0 \Rightarrow x = 8$$

$$v_1 \cdot v_3 = 1\cdot y + 2z - 1 = 0$$

$$v_2 \cdot v_3 = 4y + 2z + 8 = 0$$

$$\begin{cases} y + 2z - 1 = 0 \\ 4y + 2z + 8 = 0 \end{cases} \Rightarrow$$

$$3y + 9 = 0 \qquad y = -3$$

$$-3 + 2z - 1 = 0$$

$$2z = 4 \Rightarrow z = 2$$

$$v_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix} \quad v_2 = \begin{pmatrix}4\\2\\8\end{pmatrix} \quad v_3 = \begin{pmatrix}-3\\2\\1\end{pmatrix}$$

(3)

$$w = (1,-2,5)' = a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\2\\3\end{pmatrix} + c\begin{pmatrix}2\\-1\\1\end{pmatrix} =$$

$$= \begin{pmatrix}a+b+2c\\a+2b-c\\a+3b+c\end{pmatrix} = \begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} a & b & 2c & 1 \\ a & 2b & -c & -2 \\ a & 3b & +c & 5 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \left(\begin{array}{ccc|c} a & b & 2c & 1 \\ 0 & b & -3c & -3 \\ 0 & 2b & -c & 4 \end{array}\right) R_3 \to R_3 - 2R_2$$

$$\begin{matrix} a & b & 2c \\ a & 2b & -c \\ a & 3b & \end{matrix}$$

$$\begin{pmatrix} a & b & 2c \\ 0 & b & -3c \\ 0 & & 5c \end{pmatrix}$$

③ $w = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \left(\begin{matrix} x + y + 2z \\ x + 2y - z \\ x + 3y + z \end{matrix} \middle| \begin{matrix} 2 \\ -1 \\ 1 \end{matrix}\right) \Rightarrow$

$$\left(\begin{matrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{matrix} \middle| \begin{matrix} 2 \\ -1 \\ 1 \end{matrix}\right) \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \left(\begin{matrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & -1 \end{matrix} \middle| \begin{matrix} 1 \\ -3 \\ 4 \end{matrix}\right) R_3 \to R_3 - 2R_2$$

$$\left(\begin{matrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{matrix} \middle| \begin{matrix} 1 \\ -3 \\ 10 \end{matrix}\right)$$

$x + y + 2z = 1 \Rightarrow x + 3 + 2 \cdot 2 = 1 \Rightarrow x = -6$

$y - 3z = -3 \Rightarrow y - 3 \cdot 2 = -3 \Rightarrow y = 3$

$5z = 10 \Rightarrow z = 2$

$$X = \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}.$$

4) a) $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \end{pmatrix}$ $\quad B = \begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 4 \end{pmatrix}$

$$A \cdot B = \underset{2\times3}{\begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \end{pmatrix}} \cdot \underset{3\times4}{\begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 4 \end{pmatrix}} = \begin{pmatrix} -1 & 1 & 4 & 3 \\ 4 & 8 & -2 & 12 \end{pmatrix}$$

$$B \cdot A = \underset{3\times4}{\begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 4 \end{pmatrix}} \underset{2\times3}{\begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \end{pmatrix}}$$

$3 \times \underline{4}$ and $\underline{2} \times 3$

we can't multiply.

b) $A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 2 & -3 \end{pmatrix}$ $\quad B = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

$$A \cdot B = \underset{2\times3}{\begin{pmatrix} -2 & 1 & 2 \\ 1 & 2 & -3 \end{pmatrix}} \cdot \underset{2\times2}{\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}}$$ we can't.

$$B \times A = \underset{2\times\underline{2}}{\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}} \cdot \underset{\underline{2}\times3}{\begin{pmatrix} -2 & 1 & 2 \\ 1 & 2 & -3 \end{pmatrix}} = \begin{pmatrix} -4 & 7 & 0 \\ 0 & 5 & -4 \end{pmatrix}$$

c) $A = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad B = (-2, 0, 1)$

$$\begin{array}{cc} \cancel{2\times3}\ 3\times1 & 1\times3 \end{array}$$

$$A \cdot B = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot (-2, 0, 1) = \begin{pmatrix} -2 & 0 & 1 \\ -4 & 0 & 2 \\ -4 & 0 & 2 \end{pmatrix}$$

$$B \cdot A = \quad 1\times3 \quad 3\times1 \quad (-2, 0, 1) \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -2\cdot1 + 0\cdot2 + 1\cdot2 = 0$$

d) $A = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 & 2 \\ -4 & -9 & 2 \\ -2 & -8 & 1 \end{pmatrix}$

$$\begin{array}{cc} 3\times3 & 3\times3 \end{array}$$

$$A \cdot B = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 & 2 \\ -4 & -9 & 2 \\ -2 & -8 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 3 & 5 & 2 \\ -4 & -9 & 2 \\ -2 & -8 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

e) $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$$\begin{array}{cc} 2\times2 & 2\times2 \end{array}$$

$$A \cdot B = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5

1)

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 6 & 6 \\ 3 & 7 & 4 \\ 0 & 8 & 2 \end{pmatrix} \qquad B = A' = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 5 & 6 & 7 & 8 \\ 0 & 6 & 4 & 2 \end{pmatrix}$$

$$A \cdot B = \underset{4 \times 3}{\begin{pmatrix} 1 & 5 & 0 \\ 2 & 6 & 6 \\ 3 & 7 & 4 \\ 0 & 8 & 2 \end{pmatrix}} \underset{3 \times 4}{\begin{pmatrix} 1 & 2 & 3 & 0 \\ 5 & 6 & 7 & 8 \\ 0 & 6 & 4 & 2 \end{pmatrix}} = \underset{4 \times 4}{\begin{pmatrix} 26 & 32 & 38 & 40 \\ 32 & 76 & 72 & 60 \\ 38 & 72 & 74 & 64 \\ 40 & 60 & 64 & 68 \end{pmatrix}}$$

$$B \cdot A = \underset{3 \times 4}{\begin{pmatrix} 1 & 2 & 3 & 0 \\ 5 & 6 & 7 & 8 \\ 0 & 6 & 4 & 2 \end{pmatrix}} \underset{4 \times 3}{\begin{pmatrix} 1 & 5 & 0 \\ 2 & 6 & 6 \\ 3 & 7 & 4 \\ 0 & 8 & 2 \end{pmatrix}} = \underset{3 \times 3}{\begin{pmatrix} 14 & 38 & 24 \\ 38 & 174 & 80 \\ 24 & 80 & 56 \end{pmatrix}}$$

Ex 5

a) $A(BA)^{-1}B = \underbrace{A A^{-1}}_{I} \cdot \underbrace{B^{-1} B}_{I} = I$

b) $\cancel{(AB)'} \cdot (AB')' (BA')^{-1} C = (B')' \overbrace{A' (A')^{-1}}^{I} B^{-1} C =$

$= B \cdot B^{-1} \cdot C = C$

c) $AB'(B^{-1})' A^{-1} = \cancel{(AB A')'} \cdot \cancel{AB'}$

We can change -1 and $'$ and use $(A \cdot B)^{-1} = B^{-1} A^{-1}$

$= (AB') \cdot (AB')^{-1} = I$

Ex 6

a) $A'X' = [A(I+B)]'$ $\qquad (AB)' = B'A'$

$A'X' = (I+B)'A'$

$(A')^{-1} \qquad (A')^{-1}$

$X' = (A')^{-1}(I+B)'A'$

$X' = (A^{-1})'[A(I+B)]'$

$X' = ([A(I+B)]A^{-1})'$

$X' = [(A+AB)A^{-1}]' = [(AA^{-1}+ABA^{-1})]' =$

$= (I+ABA^{-1})' \Rightarrow X = (I+ABA^{-1})$

b) $(XA+IX)' = A'+I$

$(XA+X)' = A'+I$

$A'X'+X' = A'+I$

$(A'+I)X' = A'+I$

$(A'+I)^{-1} \qquad (A'+I)^{-1}$

$X' = (A'+I)^{-1}(A'+I) = I$

$X = I$

ⓒ $X(A+I) = I + A^{-1}$

$X = (I + A^{-1})(A+I)^{-1}$

Ex7)

a) $\det A = \begin{vmatrix} 4 & 1 \\ -4 & 2 \end{vmatrix} = 4 \cdot 2 - (-4 \cdot 1) = 12$

b) $\det B = \begin{vmatrix} 1 & -4 & -10 \\ 10 & -8 & 2 \\ 0 & -1 & 6 \end{vmatrix} = (1 \cdot (-8) \cdot 6 + 10 \cdot (-1) \cdot (-10) +$

$+ (-4) \cdot 2 \cdot 0) - (0 \cdot (-8) \cdot (-10) +$

$+ 10 \cdot (-4) \cdot 6 + (-1) \cdot 2 \cdot 1) =$

$= 294$

c) $\det C = \begin{pmatrix} -7 & 1 & -10 \\ 1 & 10 & 2 \\ 1 & 0 & 6 \end{pmatrix} = \overset{-420}{-7 \cdot 10 \cdot 6} + \overset{0}{1 \cdot 0 \cdot (-10)} +$

$+ \overset{2}{1 \cdot 1 \cdot 2} + \overset{-100}{1 \cdot (-10) \cdot 10} + \overset{6}{1 \cdot 1 \cdot 6} +$

$- \overset{0}{(-7) \cdot 2 \cdot 0} = -324$

d)

$$\det D = \begin{vmatrix} -3 & 0 & -8 & 7 \\ -7 & 1 & -4 & -10 \\ 1 & 10 & -8 & 2 \\ 1 & 0 & -1 & 6 \end{vmatrix} = (-1)^{1+1}(-3)\underbrace{\begin{vmatrix} 1 & -4 & -10 \\ 10 & -8 & 2 \\ 0 & -1 & 6 \end{vmatrix}}_{\det_1} +$$

$$+ (-1)^{1+2}\cdot 0 \begin{vmatrix} -7 & -4 & -10 \\ 1 & -8 & 2 \\ 1 & -1 & 6 \end{vmatrix} + \cancel{(-1)^{1+3}}\cdot(-8)\underbrace{\begin{vmatrix} -7 & 1 & -10 \\ 1 & 10 & 2 \\ 1 & 0 & 6 \end{vmatrix}}_{\det_2} +$$

$$+ \cancel{(-1)^{1+4}}\cdot 7 \underbrace{\begin{vmatrix} -7 & 1 & -4 \\ 1 & 10 & -8 \\ 1 & 0 & -1 \end{vmatrix}}_{\det_3} =$$

det 1

$$\cancel{(-1)^{2}}\cdot(-3)\left[\begin{vmatrix} -8 & 2 \\ -1 & 6 \end{vmatrix}\cdot(-1)^{1+1}\cdot 1 + \cancel{(-1)^{1+2}}\cdot(-4)\begin{vmatrix} 10 & 2 \\ 0 & 6 \end{vmatrix} +\right.$$

$$\left. + \cancel{(-1)^{1+3}}(-10)\begin{vmatrix} 10 & -8 \\ 0 & -1 \end{vmatrix}\right] = -3(-46+240+100) =$$

$$= 294\cdot(-3) = -882$$

det 2

$$-8\left(\overset{1}{\cancel{(-1)^{1+1}}}\cdot(-7)\begin{vmatrix}10 & 2\\ 0 & 6\end{vmatrix} + \overset{-1}{\cancel{(-1)^{1+2}}}\cdot 1\begin{vmatrix}\cancel{-1} & -2\\ 1 & 6\end{vmatrix} +\right.$$

$$\left.+\overset{1}{\cancel{(-1)^{1+3}}}\cdot(-10)\cdot\begin{vmatrix}\cancel{-1} & 10\\ 1 & 0\end{vmatrix}\right) =$$

$$= -8\left[(-7)\cdot 60 + 1\cdot 24 + (-10)(-10)\right] =$$

$$= \cancel{4028}\ 2592$$

det 3

$$-7\left(\overset{1}{\cancel{(-1)^{1+1}}}\cdot(-7)\begin{vmatrix}10 & -8\\ 0 & -1\end{vmatrix} + \overset{-1}{\cancel{(-1)^{1+2}}}\cdot 1\begin{vmatrix}1 & -8\\ 1 & -1\end{vmatrix} +\right.$$

$$\left.+\overset{1}{\cancel{(-1)^{1+3}}}\cdot(-4)\begin{vmatrix}1 & 10\\ 1 & 0\end{vmatrix}\right) =$$

$$= -7\left[(-7)\cdot(-10) + (-1)\cdot(7) + (-4)(-10)\right] =$$

$$= -721$$

$\det D = -882 + 2592 + (-721) = 989$

e) ~~B~~

1) $\det F = \begin{vmatrix} 3 & 0 & 8 & -7 \\ 7 & -1 & 4 & 10 \\ -1 & -10 & 8 & -2 \\ -1 & 0 & 1 & -6 \end{vmatrix} = \underbrace{(-1)^{1+1}}_{1} \cdot 3 \cdot \begin{vmatrix} -1 & 4 & 10 \\ -10 & 8 & -2 \\ 0 & 1 & -6 \end{vmatrix} + \underbrace{(-1)^{1+3}}_{1} \cdot 8 \begin{vmatrix} 7 & -1 & 10 \\ -1 & -10 & -2 \\ -1 & 0 & -6 \end{vmatrix}$

$+ \underbrace{(-1)^{1+4}}_{-1} \cdot (-7) \begin{vmatrix} 7 & -1 & 4 \\ -1 & -10 & 8 \\ -1 & 0 & 1 \end{vmatrix}$

det 1

$3\begin{vmatrix} -1 & 4 & 10 \\ -10 & 8 & -2 \\ 0 & 1 & -6 \end{vmatrix} = 3\Big(\underbrace{(-1)^{1+1}}_{1} \cdot (-1) \begin{vmatrix} 8 & -2 \\ 1 & -6 \end{vmatrix} + \underbrace{(-1)^{1+2}}_{-1} \cdot 4 \begin{vmatrix} -10 & -2 \\ 0 & -6 \end{vmatrix} +$

$+ \underbrace{(-1)^{1+3}}_{1} \cdot 10 \begin{vmatrix} -10 & 8 \\ 0 & 1 \end{vmatrix} \Big) = 3\left[(-1)(-46) + (-4) \cdot 60 + 10(-10)\right] =$

$= 3 \cdot (-294) = -882$

det 2

$8 \begin{vmatrix} 7 & -1 & 10 \\ -1 & -10 & -2 \\ -1 & 0 & -6 \end{vmatrix} = 8 \Big((-1)^{1+1} \cdot 7 \begin{vmatrix} -10 & -2 \\ 0 & -6 \end{vmatrix} + (-1)^{1+2} \cdot (-1) \cdot$

$\cdot \begin{vmatrix} -1 & -2 \\ -1 & -6 \end{vmatrix} + (-1)^{1+3} \cdot 10 \begin{vmatrix} -1 & -10 \\ -1 & 0 \end{vmatrix} \Big) = 8 \cdot 324 = 2592$

det 3

$$\cancel{(-1)^{1+4}}\,-7\begin{vmatrix}-7 & 1 & -4\\ 1 & 10 & -8\\ 1 & 0 & -1\end{vmatrix} = -7\left((-1)^{1+1}\cdot(-7)\begin{vmatrix}10 & -8\\ 0 & -1\end{vmatrix} +\right.$$

$$\left.+(-1)^{1+2}\cdot 1\begin{vmatrix}1 & -8\\ 1 & -1\end{vmatrix} + (-1)^{1+3}\cdot(-4)\begin{vmatrix}1 & 10\\ 1 & 0\end{vmatrix}\right) =$$

$$= -7\cdot 103 = -721$$

$$-721 + 2592 + (-882) = \cancel{}\ 989$$

F is ~~actually~~ F is just $-D$ (every entry is neg

for any $n \times n$ ~~a~~ matrix $\det(-A) = (-1)^n \det(A)$

$\hookrightarrow F = \det(-D) = (-1)^4 \det(D) = 989$!

Ex 8

a) $\det A = \begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{vmatrix} = \cancel{(-1)^{1+2}} \overset{-1}{} \cdot 1 \begin{vmatrix} 5 & 7 \\ 3 & 1 \end{vmatrix} + \cancel{(-1)^{1+3}} \overset{1}{} \cdot 3 \begin{vmatrix} 2 & 5 \\ 3 & 0 \end{vmatrix} =$

$= \cancel{-1 \cdot 5 + 3 \cdot (-15) = \ldots} \quad -1 \cdot (-19) + 3 \cdot (-15) = -26$

b) $\det(A_{swapped}) = -\det(A) = 26 = \begin{vmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix}$

c) $|A^T| = \begin{vmatrix} 0 & 2 & 3 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{vmatrix} = 2 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} =$

$= -2 \cdot \cancel{5} + 3 \cdot (-8) = -26 \rightarrow \det(A^T) = \det(A)$

d) $|2A| = \begin{vmatrix} 0 & 2 & 6 \\ 4 & 10 & 14 \\ 6 & 0 & 2 \end{vmatrix} = 2 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 14 \\ 6 & 2 \end{vmatrix} + 6(-1)^{1+3} \begin{vmatrix} 4 & 10 \\ 6 & 0 \end{vmatrix} =$

$= -2 \cdot (-76) + 6(-60) = -208 = 2^3 \cdot (-26) = 2^n \det(A)$

f) $\det(AB) = \det(A)\cdot\det(B)$

$$\det(B) = \begin{vmatrix} 2 & 3 & 1 \\ 0 & 2 & 5 \\ 1 & 0 & 1 \end{vmatrix} = (-1)^{2+2}\cdot 2 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + (-1)^{2+3}\cdot 5 \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 2\cdot 1 - 5\cdot(-3) = 17$$

$$\det(A\cdot B) = 17\cdot(26) - 442$$

g) $\det(A+B)$

$$A+B = \begin{pmatrix} 0+2 & 1+3 & 3+1 \\ 2+0 & 5+2 & 7+5 \\ 3+1 & 0+0 & 1+1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 4 \\ 2 & 7 & 12 \\ 4 & 0 & 2 \end{pmatrix}$$

$$\det(A+B) = \begin{vmatrix} 2 & 4 & 4 \\ 2 & 7 & 12 \\ 4 & 0 & 2 \end{vmatrix} = 4\cdot(-1)^{3+1} \begin{vmatrix} 4 & 4 \\ 7 & 12 \end{vmatrix} + 2\cdot(-1)^{3+3} \begin{vmatrix} 2 & 4 \\ 2 & 7 \end{vmatrix}$$

$$= 4\cdot 20 + 2\cdot 6 = 92$$

$\det(A) + \det(B) = -9 \quad \det(A+B) \neq \det(A) + \det(B)$

Ex 9

a) $A = \begin{pmatrix} 4 & 1 \\ -4 & 2 \end{pmatrix} \circ A^{-1} \; \dfrac{1}{\underset{\downarrow\, 12}{8+4}} \begin{pmatrix} 2 & -1 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{12} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

b) $B = \begin{pmatrix} 4 & 1 \\ -2 & -0{,}5 \end{pmatrix} \circ B^{-1} \; \dfrac{1}{-2+2}$

$\Rightarrow = 0 \quad det = 0 \quad$ not invertible!

c) $C^{-1} = \dfrac{1}{-4-6} \begin{pmatrix} -4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{10} & -\frac{1}{10} \end{pmatrix}$

d) $D^{-1} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 \end{array}\right) R_3 \to R_3 + \frac{1}{2} R_2 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1{,}5 & 0 & 0{,}5 & 1 \end{array}\right)$

$R_2 \to R_2 + 2R_3$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1{,}5 & 0 & 0{,}5 & 1 \end{array}\right) \begin{array}{l} R_2 \to \frac{R_2}{2} \\ R_3 \to \frac{2}{3} R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}\right)$

$D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

Ex 10

a) $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 4 & -6 \end{pmatrix} R_3 \to R_3 + 4R_2$

$\begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -10 \end{pmatrix}$ rank $= 3$

b)

$\det A = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{vmatrix} = \cancel{1 \cdot 0 \cdot 2} + \cancel{1 \cdot 2 \cdot 0} + (-1) \cdot 3 \cdot 4 -$
$- \cancel{2 \cdot 0 \cdot 4} - (-1) \cdot 1 \cdot 2 + \cancel{0 \cdot 3 \cdot 1} = -10$

so A is not singular it is regular

c) Invers

$$A^{-1} = \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -10 & 0 & -4 & 1 \end{array}\right) \begin{array}{l} R_2 \to -R_2 \\ R_3 \to -\frac{1}{10} R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & +1 & +1 & -1 & +1 & 0 \\ 0 & 0 & 1 & 0 & +\frac{2}{5} & -\frac{1}{10} \end{array}\right) \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$0,4 \quad -0,1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -0,2 & -0,3 \\ 0 & 1 & 0 & -1 & 0,6 & 0,1 \\ 0 & 0 & 1 & 0 & 0,4 & -0,1 \end{array}\right) \qquad A^{-1} = \begin{pmatrix} 0 & -0,2 & -0,3 \\ -1 & 0,6 & 0,1 \\ 0 & 0,4 & -0,1 \end{pmatrix}$$

Ex11

a)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 3 & 4 & 5 & -2 \\ 4 & 6 & 8 & -4 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 4 & 5 & -2 \\ 0 & 1 & 1 & -1 \\ 4 & 6 & 8 & -4 \end{array}\right) \xrightarrow{R_1 \to \frac{R_1}{3}} \left(\begin{array}{ccc|c} 1 & \frac{4}{3} & \frac{5}{3} & -\frac{2}{3} \\ 0 & 1 & 1 & -1 \\ 4 & 6 & 8 & -4 \end{array}\right)$$

$R_3 \to R_3 - 4R_1$

$$\left(\begin{array}{ccc|c} 1 & \frac{4}{3} & \frac{5}{3} & -\frac{2}{3} \\ 0 & 1 & 1 & -1 \\ 0 & \frac{2}{3} & \frac{4}{3} & -\frac{4}{3} \end{array}\right) R_3 \to R_3 - \left(\frac{2}{3}\right) R_2 \left(\begin{array}{ccc|c} 3 & 4 & 5 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & \frac{2}{3} & -\frac{2}{3} \end{array}\right)$$

$\frac{2}{3} x_3 = \frac{-2}{3} \rightarrow x_3 = -1$

$x_2 + (-1) = -1 \rightarrow x_2 = 0$

$3x_1 + 4 \cdot 0 + 5 \cdot (-1) = -2$

$x_1 = 1 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

c) $\Delta = \begin{vmatrix} 0 & 1 & 1 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{vmatrix} = (1)^{1+2} \cdot 1 \begin{vmatrix} 3 & 5 \\ 4 & 8 \end{vmatrix} + (1)^{1+3} \cdot 1 \begin{vmatrix} 3 & 4 \\ 4 & 6 \end{vmatrix} = -2$

$\Delta_1 = \begin{vmatrix} -1 & 1 & 1 \\ -2 & 4 & 5 \\ -4 & 6 & 8 \end{vmatrix} = (-1) \cdot (-1)^{1+1} \begin{vmatrix} 4 & 5 \\ 6 & 8 \end{vmatrix} + (-1)^{1+2} \cdot 1 \begin{vmatrix} -2 & 5 \\ -4 & 8 \end{vmatrix} +$

$+ (-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} -2 & 5 \\ -4 & 8 \end{vmatrix} = -2$

$\Delta_2 = \begin{vmatrix} 0 & -1 & 1 \\ 3 & -2 & 5 \\ 4 & -4 & 8 \end{vmatrix} = (-1)^{1+2} \cdot (-1) \begin{vmatrix} -2 & 5 \\ -4 & 8 \end{vmatrix} + (-1)^{1+3} \cdot 1 \begin{vmatrix} 3 & 5 \\ 4 & 8 \end{vmatrix} =$

$= 0$

$\Delta_3 = \begin{vmatrix} 0 & 1 & -1 \\ 3 & 4 & -2 \\ 4 & 6 & -4 \end{vmatrix} = (-1)^{1+2} \cdot 1 \begin{vmatrix} 4 & -2 \\ 6 & -4 \end{vmatrix} + (-1)^{1+3} \cdot (-1) \begin{vmatrix} 3 & 4 \\ 4 & 6 \end{vmatrix}$

$= 2$

$x_1 = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1 \qquad x_2 = \frac{\Delta_2}{\Delta} = \frac{0}{-2} = 0 \qquad x_3 = \frac{\Delta_3}{\Delta} = \frac{2}{-2}$

$X = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Ex 12

a) $\begin{pmatrix} 2 & 1 & 0 & 1 & 2 \\ -1 & 2 & 1 & 0 & -1 \\ 0 & 4 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} +1 & -2 & -1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -2 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 & 0 \\ 0 & 4 & 1 & 2 & 1 \end{pmatrix}$

$R_3 - \frac{4}{5} R_2$

$\begin{pmatrix} 1 & -2 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 & 0 \\ 0 & 0 & -\frac{3}{5} & \frac{6}{5} & 1 \end{pmatrix} \to$ Rank A = 3.

b) $B = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & -1 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 4 & 2 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ $\begin{matrix} R_2 + 2R_1 \\ R_3 - 4R_1 \\ R_4 + R_1 \end{matrix}$ $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & -2 & -11 \\ 0 & -1 & 2 \end{pmatrix}$ $\begin{matrix} R_3 + 2R_2 \\ R_4 + R_2 \end{matrix}$

$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & -27 \\ 0 & 0 & 10 \end{pmatrix} R_4 + \frac{10}{27} R_3 \quad \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & -27 \\ 0 & 0 & 0 \end{pmatrix}$ Rank B = 3.

ex13

$$A=\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 5 & 1 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 2 & 1/2 & 3/2 \\ 1 & 4 & 2 \\ 3 & 5 & 1 \end{pmatrix} \begin{matrix} \\ R_2-R_1 \\ R_3-3R_1 \end{matrix} \begin{pmatrix} 2 & 1/2 & 3/2 \\ 0 & 7/2 & 1/2 \\ 0 & 7/2 & -7/2 \end{pmatrix} R_3$$

$$\begin{pmatrix} 2 & 1/2 & 3/2 \\ 0 & 7/2 & 1/2 \\ 0 & 0 & -4 \end{pmatrix} \quad \det A = 2\cdot\frac{7}{2}\cdot(-4) = -28 \neq 0 \text{ invert}$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ R_2 \to R_2-2R_1 \\ R_3-3R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & -7 & -1 & 1 & -2 & 0 \\ 0 & -7 & -5 & 0 & -3 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3-R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & -7 & -1 & 1 & -2 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{array}\right) R_3\left(-\frac{1}{4}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & -7 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right) \begin{matrix} R_1-2R_3 \\ R_2+R_3 \\ \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & -7 & 0 & 5/4 & -9/4 & -1/4 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right) \begin{matrix} R_1+\frac{4}{7}R \\ R_2/-7 \\ \end{matrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/14 \\ & & & \\ & & & \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1/2} \left(\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 7/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 7/2 & -7/2 & -3/2 & 0 & 1 \end{array}\right) R_2 \cdot \frac{2}{7} \left(\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/7 & -1/7 & 2/7 & 0 \\ 0 & 7/2 & -7/2 & -3/2 & 0 & 1 \end{array}\right) R_3 - \frac{7}{2} R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/7 & -1/7 & 2/7 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{array}\right) R_3/-4 \left(\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/7 & -1/7 & 2/7 & 0 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right) R_2 - \frac{1}{7} R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -5/28 & 1/4 & 1/28 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right) R_1 - \frac{3}{2} R_3 \left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/8 & -3/8 & 3/8 \\ 0 & 1 & 0 & -5/28 & 1/4 & 1/28 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right) R_1 - \frac{1}{2} R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/14 & -1/2 & 5/14 \\ 0 & 1 & 0 & -5/28 & 1/4 & -1/28 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right)$$

Inverse

$A \cdot A^{-1} = ?$

$$(2,1,3)\begin{pmatrix} \frac{3}{14} \\ -\frac{5}{28} \\ 1/4 \end{pmatrix} = 1 \qquad (2\ 1\ 3)\begin{pmatrix} -1/2 \\ 1/4 \\ 1/4 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 5 & 1 \end{pmatrix}\begin{pmatrix} 3/14 & -1/2 & 5/14 \\ -5/28 & 1/4 & 1/28 \\ 1/4 & 1/4 & -1/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex 14

$w = (X^T X)^{-1} X^T y$

① $X^T X = \underset{2\times3}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}} \underset{3\times2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}} = \begin{pmatrix} 35 & 44 \\ 44 & 56 \end{pmatrix}$

② $X^T \cdot y = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 76 \\ 100 \end{pmatrix}$

③ $(X^T X)^{-1} = \frac{1}{24}\begin{pmatrix} 56 & -44 \\ -44 & 35 \end{pmatrix}$

④ $\frac{1}{24}\begin{pmatrix} 56 & -44 \\ -44 & 35 \end{pmatrix}\begin{pmatrix} 76 \\ 100 \end{pmatrix} = \frac{1}{24}\begin{pmatrix} -144 \\ 156 \end{pmatrix} = \begin{pmatrix} -6 \\ 6,5 \end{pmatrix}$