Journal of Knot Theory and Its Ramifications, Vol. 3 No. 2 (1994) 121–140

LISSAJOUS KNOTS

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Received 2 December 1993

Abstract

A *Lissajous knot* is defined to be one isotopic to a knot which admits a parametrization (for $0 \le t \le 2\pi$)

$$x(t) = \cos(n_x t + \phi_x)$$
$$y(t) = \cos(n_y t + \phi_y)$$
$$z(t) = \cos(n_z t + \phi_z)\ .$$

[1] J.E.H. was supported in part by National Institutes of Health grant GM47945 and by the Office of Basic Energy Science in the Biological Energy Division of the Department of Energy under contract DE-ACO30-76F000978. J.E.H. thanks David Hearst and Georges Papadopoulos for their assistance in the graphics programming.

[2] Partially supported by U.S. Air Force contract FDF 49620-92-J-0067 and National Science Foundation grant DMS 87-20208.

[3] L.S. was supported in part by National Institutes of Health grant GM 47945.

Motivation for considering Lissajous knots came originally from the study of DNA molecular configurations. We will show that a Lissajous knot necessarily has Kervaire invariant zero so that the trefoil, figure-8 and the (2,5) torus knot are not Lissajous. The knot 5_2 can be realized with $n_x = 2,\ n_y = 3,\ n_z = 7$.

0. Definition and Motivation

0.1 Introduction. Two harmonic oscillators moving in perpendicular directions generate what is known in elementary physics as a "Lissajous figure". The frequencies and phases of the two oscillators may vary independently though by a choice of origin in time, only the relative phase is relevant. Note in two dimensions that all maxima and minima of x and y occur on the same straight lines.

In two dimensions the Lissajous figure necessarily intersects itself. For three oscillators in three dimensions this is not so and for generic values of the relative phases one obtains a (parametrized) knot. The observation about maxima and minima of Lissajous figures shows that such knots are plats ([Bi]) in three directions simultaneously.

Definition. A knot in $\mathbb{R}^3$ is a *Lissajous knot* (of type $(n_x, n_y, n_z) \in \mathbb{N}^3$) if it is isotopic to one parametrized in a one-to-one fashion by:

$$\begin{aligned} x &= \cos(n_x t + \phi_x) \\ y &= \cos(n_y t + \phi_y) \\ z &= \cos(n_z t) \end{aligned}$$

for $0 \le t < 2\pi$.

Note that we have deliberately set the z-phase equal to zero so there is no freedom of time origin left. The projection of the knot onto the $x - y$ plane depends only on the relative phase $\phi_y - \phi_x$ but the knot type will depend on ϕ_y and ϕ_x (mod 2π).

Lemma 1. *If K is a Lissjou knot of type (n_x, n_y, n_z) then n_x, n_y and n_z are relatively prime, i.e. no integer > 1 divides any pair.*

PROOF. Since the parametrization is one-to-one there can be no integer > 1 dividing all n_x, n_y and n_z. So suppose without loss of generality that $n_x = qm_x$, $n_y = qm_y$ for $q > 1$ and q does not divide n_z. By inspection of the graphs of $z(t)$ and $z(t + \frac{2\pi}{q})$ there is a pair $a(b, t_1) \subset [0, \frac{\pi}{n_z})$ for which $z(t') = z(t' + \frac{2\pi}{q})$. Thus the points $(x(t'), y(t'), z(t'))$ and $(x(t' + \frac{2\pi}{q}),\ y(t' + \frac{2\pi}{q}), z(t' + \frac{2\pi}{q}))$ are equal.

□

WARNING. Lissajous knots are not necessarily torus knots. In fact it seems likely that the unknot is the only knot which is both Lissajous and a torus knot.

0.2 Motivation from DNA.

DNA Dynamics. A detailed understanding of the organization of DNA in the eukaryotic cell remains unknown. It is, nevertheless, commonly accepted that DNA *in vivo* is always at a high negative linking number density (-0.05). Since it is highly likely that various conformational changes at approximate constant linking number density occur during the regulation of genes and during the condensation of chromosomes at metaphase, a mathematics that enables one to model such changes would be of significance. Based on an earlier theory by Harris and Hearst for the worm-like coil, Lissajous functions are the first such three-dimensional functions which we have investigated with these objectives in mind [HH].

Lissajous functions. In an effort to further the development of the statistical mechanics of supercoiled DNA in solution, Hearst and Hunt investigated the properties of the following three-dimensional Lissajous curves as models for plectonemic closed supercoils [HeH].

$$\begin{aligned} x &= \varepsilon_x \sin(n_x s) \\ y &= \varepsilon_y \sin(n_y s) \\ z &= \varepsilon_z \cos(n_z s)\ . \end{aligned}$$

The parameter s has a range from 0 to 2π, and the amplitudes ε_x, ε_y and ε_z are proportional to $(1/n_x)$, $(1/n_y)$ and $(1/n_z)$, respectively. In particular, curves which have a periodicity of one (arbitrarily taken as the z axis, $n_z = 1$) and arbitrary periodicity along the x and y axes were considered because they intuitively eliminate the possibility of knot formation. For this case, when z is either +1 or -1, x and y are both 0. The writhe of these three-dimensional Lissajous functions was then determined by numerical integration, with the conclusion that maximum absolute values of writhe for such Lissajous curves occur when $n_x = n_y + 1$ or when $n_x = n_y - 1$. Furthermore, because of coincidence in space, many combinations of the integers n_x and n_y have no writhe and must be rejected from the set.

This initial interest has led us to consider the broader class of Lissajous functions in the representation of unknots, and ultimately knots as well. The issue of "knottedness" is independent of amplitude (unlike writhe). Our investigations have continued using functions of the form:

$$\begin{aligned} x &= \cos[(n_x s) + \phi_x] \\ y &= \cos[(n_y s) + \phi_y] \\ z &= \cos(n_z s) \ . \end{aligned}$$

0.3 Motivation from Vassiliev theory. In [V], Vassiliev began the investigation of knots by a filtration on the space of all smooth functions from S^1 to $\mathbb{R}^3$. Subtracting all such functions with any kind of self intersection (the singular locus, or discriminant) one obtains the space of all parametrized knots, whose connected components form the space of all knot types. Thus its zero-th cohomology group is by definition the set of all knot invariants. The filtration chosen on the space of all functions could have been by the size of the highest Fourier mode in the x, y and z coordinates. Lissajous knots are those representable with precisely one Fourier frequency per coordinate.

For each (n_x, n_y, n_z) the parametrized Lissajous knot of this type gives a 2-torus inside the Vassiliev space of all parametrized knots. We will see exactly how these tori intersect the discriminant in §2.2. Thus we have a very low-dimensional picture of what the Vassiliev space looks like.

The Vassiliev theory suggests itself as a tool for analyzing Lissajous knots but so far we have been unable to use it in a significant way.

0.4 A false argument. There is a simple (and false) argument that suggests that any knot is Lissajous. First note that Lissajous figures can have an arbitrarily large number of self-intersections on a lattice-like grid. Thus if one is allowed to choose the over/under behavior at each crossing independently one should be able to obtain any knot.

If $t_1, t_2, \ldots t_k$ are the times at which the self-intersections occur in the parametrization, this boils down to being able to choose the signs of $\cos(t_i n_z)$ for $i = 1, 2, \ldots, k$ arbitrarily. But by the ergodicity of the irrational flows on high dimensional tori, there will be an irrational r with $\cos(t_i r)$ having the desired signs. Approximating r by rationals should give the desired n_z.

This argument was believed for an long time by the third author, but must be false, as we will see in §3.1.

1. Some Examples

Lemma 2. *The unknot is Lissajous of type* (n_x, n_y, n_z) *for any three relatively prime integers* n_x, n_y, n_z.

PROOF. When $\phi_x = \phi_y = 0$ we have the curve $x(t) = \cos n_x t$, $y(t) = \cos(n_y t)$, $z(t) = \cos(n_z t)$. Since $\cos\theta = \cos(-\theta)$, for these phase values the curve is doubly parametrized by t in $[0, \pi)$. Simple algebra (see §2.1 for the general situation) shows that there is no pair t_1, t_2 in $[0, 2\pi]$ for which $x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$, $z(t_1) = z(t_2)$ so we may choose a tubular neighborhood of the curve. A small perturbation by ϕ_x, ϕ_y then obviously stays in the tubular neighborhood and gives an unknot. See Figure 1.

1.1 The knot 5_2 (Conway-Rolfsen numbering) is Lissajous of type (2,3,7) with $\phi_x = 0.2$, $\phi_y = 0.7$. See Figure 2.

1.2 The knot 6_1 is Lissajous of type (2,3,5) with $\phi_x = 1$, $\phi_y = 0.5$. See Figure 3.

1.3 The knot 7_4 is Lissajous of type (2,3,7) with $\phi_x = 0.4$, $\phi_y = 1$. See Figure 5 (table 1). Amusingly enough, this is the usual picture of the 7_4 knot, found in the knot tables (e.g. [R]).

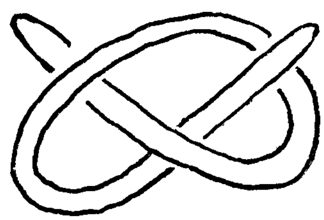

Fig. 1. Proof of Lemma 2.

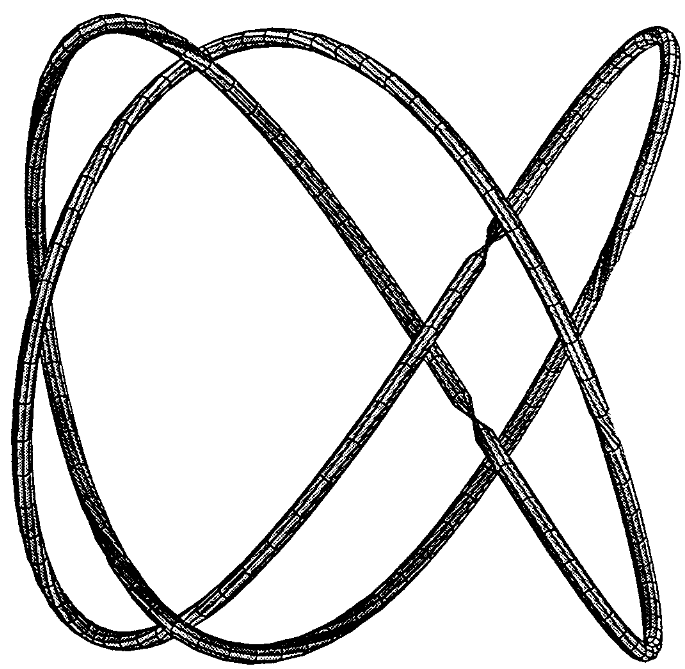

Fig. 2. The knot 5_2 (Conway–Rolfsen numbering) is a Lissajous of type (2,3,7) with $\phi_x = 0.2$, $\phi_y = 0.5$.

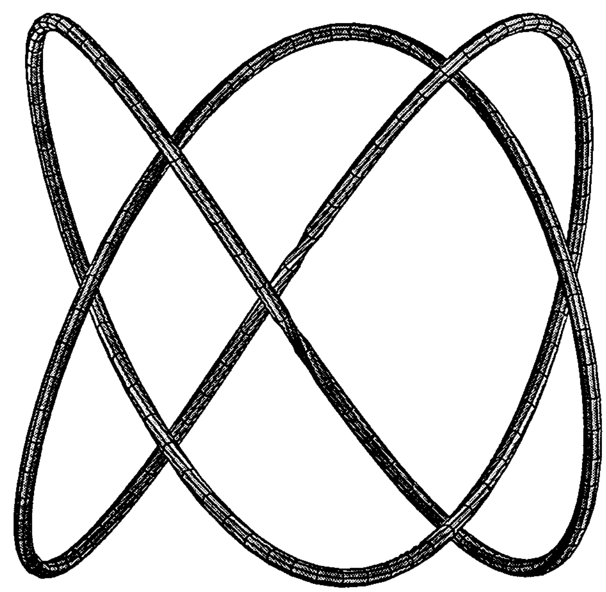

Fig. 3. The knot 6_1 is Lissajous of type (2,3,5) with $\phi_x = 1$, $\phi_y = 0.5$.

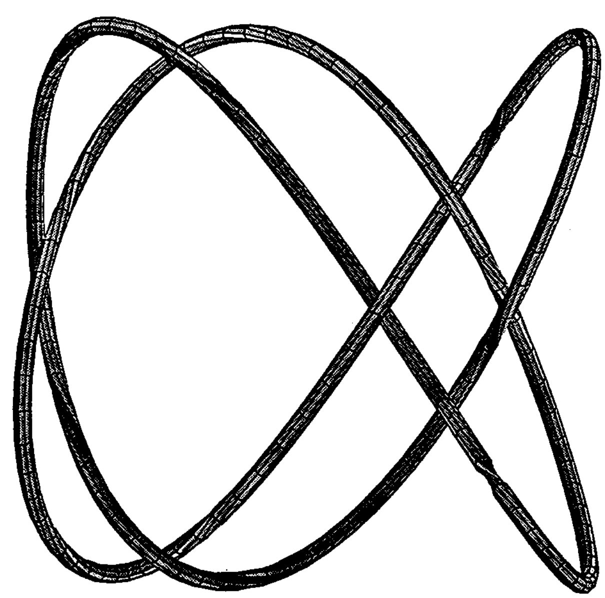

Fig. 4.

Figure 5 (Table 1)

Knot Thistlethwaite	Conway/ Rolfsen	n_x	n_y	n_z	ϕ_x	ϕ_y
		2	3	7	0.2	0.7
5_1	5_2	2	5	7	0.4	0.7
		2	5	9	0.8	0.6
6_3	6_1	2	3	5	0.2	1.5
		2	5	9	0.1	0.6
$3_1\#\overline{3_1}$	$3_1\#\overline{3_1}$	3	5	7	0.7	1
7_6	7_4	2	3	7	0.4	1
8_{20}	8_{21}	3	4	7	0.1	0.7
		5	8	9	0.1	0.6
8_2	8_{15}	3	7	10	3.08	0.72
$5_1\#\overline{5_1}$	$5_2\#\overline{5_2}$	3	5	7	0.1	0.7
10_{75}	10_1	2	5	7	0.1	0.5
10_{23}	10_{35}	2	5	7	0.1	0.9
10_{20}	10_{58}	3	4	7	0.1	0.8

There are possibly some omissions due to computer roundoff. The most interesting entry is the occurrence of $8_2/8_{15}$ as a 3, 7, 10 Lissajous knot. This demonstrates that it is probably very difficult to find a necessary and sufficient topological condition on a knot for it to be Lissajous. Note that there are many other Lissajous knots with $\max(n_x, n_y, n_z) \leq 10$ but they have more than ten crossings on their minimal crossing projection.

1.4 A census of knots on ≤ 10 crossings which are Lissajous knots with $\max(n_x, n_y, n_z) \leq 10$. Note that if a knot is Lissajous for some (n_x, n_y, n_z), then so are its mirror image and reverse, by simple changes of variable: for mirror image, interchange x and y, for reverse change ϕ_x and ϕ_y to $-\phi_x$ and $-\phi_y$. Thus in the table we have only recorded one of the four possible knot types. See Figure 5 (table 1).

1.5 Non-Lissajous projections. The Lissajous property is highly related to the rectilinear coordinate systems. Projections onto other than the coordinate planes look very different. Here is an example — it is an oblique projection of the Lissajous knot (3,4,7) with $\phi_x = 0.1$ and $\phi_y = 0.7$. Unlike the representations of Figures 3, 4 and 5, which are viewed along the z-axis (a Lissajous axis), this figure is viewed off axis from direction (1,1,5). See Figure 6.

2. Singular Points

2.1. Self-intersection points of Lissajous figures. Take the figure $x = \cos n_1 t$, $y = \cos(n_2 t + \alpha)$. If t', t'' are to give a crossing, we have

(1) $\cos n_1 t' = \cos n_1 t''$
(2) $\cos(n_2 t' + \alpha) = \cos(n_2 t'' + \alpha)$

or

(1′) $t' - t'' = \frac{2k\pi}{n_1}$
(2′) $t' - t'' = \frac{2\ell\pi}{n_2}$
(1″) $t' + t'' = \frac{2k\pi}{n_1}$
(2″) $t' + t'' = \frac{2\ell\pi}{n_2} - 2\alpha$

(1′) cannot be nontrivially combined with (2′) because n_1 and n_2 are relatively prime. For all but finitely many α's (1″) and (2″) have no solutions. Assuming α to be in this set, we obtain two families of double points.

(*)

(1′) and (2″) give $t' = \pi\left(\frac{k}{n_1} + \frac{\ell}{n_2}\right) - \alpha$, $\quad t'' = \pi\left(\frac{\ell}{n_2} - \frac{k}{n_1}\right) - \alpha$

(**)

(1″) and (2′) give $t' = \pi\left(\frac{k}{n_1} + \frac{\ell}{n_2}\right)$, $\quad t'' = \pi\left(\frac{k}{n_1} - \frac{\ell}{n_2}\right)$

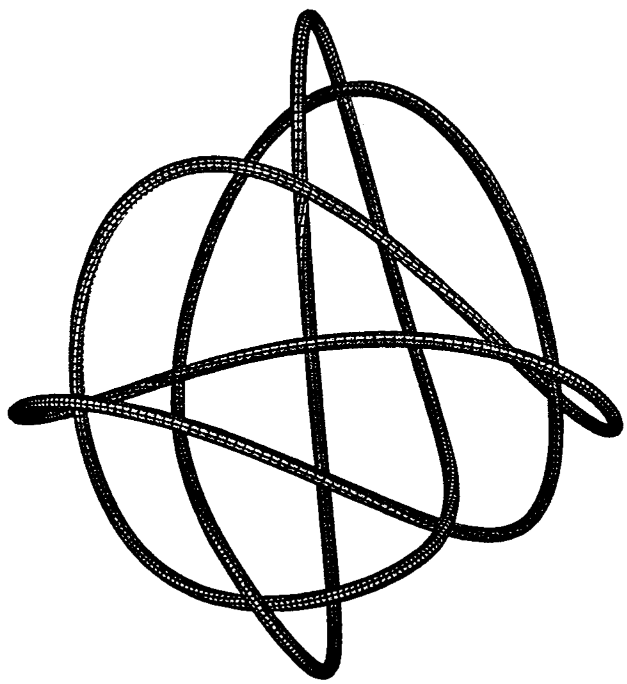

Fig. 6.

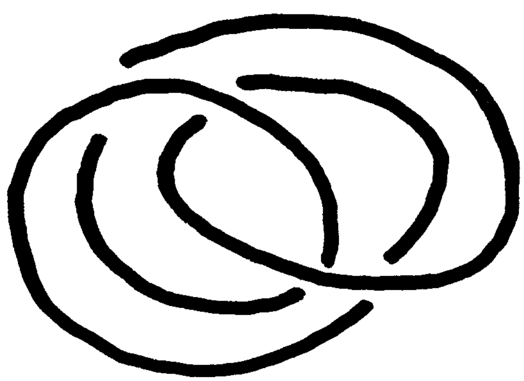

Fig. 7. The trefoil has period two.

In $(*)$ we should impose the requirement $n_1 \nmid k$ and in $(**)$, $n_2 \nmid \ell$ (note that these cases give the points of the figure on the boundary of the unit square).

To obtain each crossing only once, we can set

$$\begin{array}{lll} \text{in } (*) & k = \overline{1, n_1 - 1} & \ell = \overline{0, n_2 - 1} \\ \text{in } (**) & k = \overline{0, n_1 - 1} & \ell = \overline{1, n_2 - 1}\ . \end{array}$$

Altogether, we have $2n_1n_2 - n_1 - n_2$ crossings.

2.2. Singular locus on the torus

Consider the curve $x = \cos n_1 t$, $y = \cos(n_2 t + \alpha)$, $z = \cos(n_3 t + \beta)$. To find the values of α and β for which this curve in $\mathbb{R}^3$ has double points, i.e. is not a knot, we do the following analysis:

(a) Suppose α is such that equations (1") and (2") of the previous section are compatible, i.e.

$$\alpha = \left(-\frac{k}{n_1} + \frac{\ell}{n_2}\right)\pi\ , \qquad \text{for } k, \ell \in \mathbb{Z}\ , \text{ or}$$
$$\alpha = \frac{m}{n_1 n_2}\pi \qquad \text{for } m \in \mathbb{Z}\ .$$

Then the space curve has a double point.

By considering the xz-projection, we see that

$$\beta = \frac{m}{n_1 n_3}\,\pi \qquad \text{for } m \in \mathbb{Z}$$

also gives a curve with singularities.

(b) The only other singularities may come from a pair t', t'' satisfying (1') and (2") from the preceding section, and the corresponding equations for the xz-projection. This happens if and only if

$$\alpha - \beta = \frac{k}{n_2 n_3}\,\pi\ , \text{ where } k \in \mathbb{Z}\ .$$

The phase torus results from the (α, β)-plane after the following factorisations

$$(\alpha, \beta) \sim (\alpha + 2\pi, \beta)\ , \qquad (\alpha, \beta) \sim (\alpha, \beta + 2\pi)\ ,$$
$$(\alpha, \beta) \sim (\alpha + \frac{2\pi}{n_1}, \beta + \frac{2\pi}{n_2})\ .$$

The singular locus described above gives a triangulation of the torus. Note also that if we had phases ϕ_1, ϕ_2, ϕ_3, the equations for the singularities take the symmetric form

$$\phi_1 - \phi_2 = \frac{k}{n_1 n_2}\,\pi\,; \qquad \phi_1 - \phi_3 = \frac{k}{n_1 n_3}\,\pi\,; \qquad \phi_2 - \phi_3 = \frac{k}{n_2 n_3}\,\pi\,.$$

The three types of singular curves correspond to projections on the three coordinate planes that "bend on themselves", i.e. that are plane curves parametrised between 0 and π, and then retraced in the opposite direction.

We include below a picture of how the knot type may vary as the phases change on (part of) the phase torus. The different letters in the region designate the various knot types for the particular values of the phases given by the x- and y- coordinates of the point in the region. Here x varies from 0 to $\frac{\pi}{4}$ and y varies (downwards) from 0 to $\frac{\pi}{2}$. The x,y and z frequencies are 3,7 and 10 and the z-phase is zero.

3. Not All Knots are Lissajous

3.1. Strong plus amphicheirality. A knot K in S^3 is called *strongly plus amphicheiral* if there is a period two diffeomorphism ϕ of S^3 which reverses the orientation of S^3, preserves K, i.e. $\phi(K) = K$, and preserves the orientation of K.

Strongly plus amphicheiral knots seem to be quite rare (though see §5.2 for "all" examples). The only two with less than 11 crossings are 10_{99} and 10_{123} (Conway Rolfsen numbering).

Proposition. *If K is Lissajous of type n_x, n_y, n_z with all n's odd, then K is strongly plus amphicheiral.*

PROOF. Let ϕ be the diffeomorphism of $\mathbb{R}^3$ (and S^3) given by $\phi(x, y, z) = -(x, y, z)$. Then letting $t \to t + \pi$ in the parametrization of K we see that $\phi(K) = K$ and ϕ preserves the orientation of K.

$\square$

3.2. Period two. A knot K is said to have *period two* if there is a period two diffeomorphism ϕ of S^3 which preserves K and preserves the orientation of S^3.

Example. The trefoil has period two as can be seen from the picture illustrated by Figure 7.

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A  AABBBBBCCCCCDDD  EEEEEF
AAAA BBBBBCCCCCDDDDDGEEEEF
AAAAAH BBBCCCCCDDDDDGGI EF
AAAAAHHH BCCCCCDDDDDGGGGG
AAAAAHHHHH  CCCDDDDDGGGGGI
BBBBBCCCCCDDD  EEEEEFFFFFJ
BBBBBCCCCCDDDDD EEEEFFFFFJ
H BBBCCCCCDDDDDGG  EFFFFFJ
HHH BCCCCCDDDDDGGGGG FFFFJ
HHHHH CCCCDDDDDGGGGGII  FJ
CCCCCDDD  EEEEEFFFFFJJJJ
CCCCCDDDD  EEEEFFFFFJJJJJK
CCCCCDDDDDGG EEFFFFFJJJJJK
CCCCCDDDDDGGGGG FFFFJJJJJK
   CCCDDDDDGGGGGI   FJJJJJK
DDD  EEEEEFFFFFJJJJ  LLLLA
DDDD  EEEEFFFFFJJJJJK  LLA
DDDDDGG EEFFFFFJJJJJKKKK A
DDDDDGGGGI FFFFJJJJJKKKKKA
DDDDDGGGGGI  FFJJJJJKKKKKA
EEEEEFFFFFJJJJ  LLLLAAAAAA
 EEEEFFFFFJJJJJK  LLAAAAAAA
GG EEFFFFFJJJJJKKKK AAAAAA
GGGG  FFFFJJJJJKKKKKA AAAA
GGGGGI  FFJJJJJKKKKKAAAA A
FFFFFJJJJ LLLLLAAAAAAAAAAA
FFFFFJJJJJKKLLLAAAAAAAAAAA
FFFFFJJJJJKKKKLAAAAAAAAAAA
 FFFFJJJJJKKKKKA AAAAAAAAA
I  FFJJJJJKKKKKAAAAAAAAAAA
JJJJ  LLLLAAAAAAAAAAAAKAAAK
JJJJJK  LLAAAAAAAAAAAAAA  K
JJJJJKKK LAAAAAAAAAAAAAAA
JJJJJKKKKKA AAAAAAAAAAAAAL
JJJJJKKKKKAAAAAAAAAAAAAAAL
 LLLLAAAAAAAAAAAA AAAKKKKKJ
KK LLAAAAAAAAAAAA  AKKKKKJ
KKK LAAAAAAAAAAAAAA LKKKKJ
KKKKKA AAAAAAAAAAAAALLL KJ
KKKKKAAA AAAAAAAAAAALLLLL
AAAAAAAAAAAAAAAKKKKKJJJJJF
AAAAAAAAAAAA  AKKKKKJJJJJF
AAAAAAAAAAAAAA  KKKKJJJJJF
A AAAAAAAAAAAAALL  KJJJJJF
AAA AAAAAAAAAAALLLL  JJJJF
AAAAAAAAAAAKKKKKJJJJJFF IIG
AAAAAAAL AKKKKKJJJJJFFFF
AAAAAAAAAA KKKKJJJJJFFFFFE
AAAAAAAAAALL  KJJJJJFFFFFE
AAAAAAAAAALLLL   JJJFFFFFE
 AAAAKKKKKJJJJJFF IIGGGGGD
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Fig. 8. A picture of how the knot type may vary as the phases change on (part of) the phase torus. The different letters in the region designate the various knot types for the particular values of the phases given by the x- and y-coordinates of the point in the region. Here x varies from 0 to $\pi/4$ and y varies (downwards) from 0 to $\pi/2$. The x, y and z frequencies are 3, 7 and 10 and the z-phase is 0.

Proposition. *If K is a Lissajous knot of type (n_x, n_y, n_z) and one of the n's is even, then K has period two.*

PROOF. Assume n_x is even and let $\phi(x, y, z) = (x, -y, -z)$ (rotation about x-axis by π). Then letting $t \to t+\pi$ in the parametrization of K we see $\phi(K) = K$ (and ϕ preserves the orientation of K). $\square$

3.3. 8_{10} is not Lissajous. According to the table in [BZ], 8_{10} is the first knot which does not have period 2. We know that it is not strongly plus amphicheiral.

4. Lissajous Knots have Zero Arf Invariant

4.1 The Arf, Kervaire, or Robertello invariant. A Seifert surface Σ for a knot K is an embedded oriented surface having the knot as oriented boundary. The orientation determines a quadratic form q on $H_1(\Sigma, \mathbb{Z}/2)$ by $q(c) = \ell k(c, c^+)$ where ℓk denotes the linking number (mod 2) and c^+ is the result of pushing a curve c (representing a homology class) off the surface a small distance. The Arf invariant of this quadratic form is known as the Arf, Kervaire or Robertello invariant of the knot. It appears in many ways.

(1) It is 0 if $\Delta(-1) \equiv 1 \bmod 8$ and 1 if $\Delta(-1) \equiv 5 \bmod 8$ (Δ is the Alexander polynomial).
(2) It is $\frac{1-J(i)}{2}$ (J is the polynomial of [J]).
(3) It is $\frac{\Delta''(1)}{2} \bmod 2$ and $\frac{J''(1)}{6} \bmod 2$. We will use a formula for this Kervaire invariant due to Lannes in [L].

Consider a C^1 mapping: $\phi : S^1 \to \mathbb{R}^2$, let X be the set of its double points. Let $\tilde{X}$ be the inverse image of X under ϕ. The set of sections of $\phi \restriction \tilde{X}$ (i.e. of subsets of $\tilde{X}$ in bijection with X by ϕ) has a natural structure of a $|X|$-dimensional affine space over GF_2, which we shall denote with U. H becomes a vector space if we take a "descendent section" as the origin. Every section corresponds to a knot projection in an obvious way (for every double point let its preimage in the section lie above the other one).

A "descending section" is defined as follows. Pick an orientation on S^1. By taking away a simple point "a" of ϕ, we obtain an ordering "$<<$" of $S^1 \backslash \{a\}$ isomorphic to the ordered real line. Fixing the point a we define a "descending section" by picking for each element of X

the first of its preimages in the << ordering. A descending section is easily seen to give a projection of the unknot.

Introduce an order $\prec$ on X by

$$X_1 \prec X_2 \text{iff} \ \min_{<<} \phi^{-1}(X_1) << \min_{<<} \phi^{-1}(X_2)$$

Define

$$e(X_1, X_2) = \begin{cases} 1 & \text{if } \phi^{-1}(X_1) \text{ and } \phi^{-1}(X_2) \text{ are interlaced} \\ 0 & \text{otherwise} \end{cases}$$

(two pairs of points are interlaced iff the points of one of the pairs belong to different connected components of S^1 with the other pair taken away. Then, for a section v of $\phi \restriction \tilde{X}$, $v \in GF_2$, we define the Arf invariant Arv(v) as

$$\text{Arf}(v) = \sum_{x \prec y} e(x, y)(1 + v(x))v(y) \tag{1}$$

It can be seen that this expression does not depend on the point a used in defining the descending projection, essentially by the method in 4.2. Then it easy to check that (1) is not affected by Reidemaster moves.

Our next goal is

Theorem. *The Kervaire invariant of a Lissajous knot is 0.*

We shall consider the two cases separately: (1) all frequencies are odd; (2) one of the frequencies is even.

4.2. A Lissajous knot with odd frequencies. Take the knot $x(t) = \cos n_1 t$, $y(t) = \cos(n_2 t + \alpha)$, $z(t) = \cos(n_3 t + (\beta))$ where n_1, n_2, n_3 are odd. Then $x(t+\pi) = -x(t)$, $y(t+\pi) = -y(t)$, $z(t+\pi) = -z(t)$, i.e. the knot has a central symmetry, corresponding to the symmetry of the circle $e^{it} \longleftrightarrow e^{i(t+\pi)}$.

Choose for ϕ a projection onto a coordinate plane. Then $\phi(t+\pi) = -\phi(t)$. We prove

Lemma 3. *Let $\phi : S^1 \to \mathbb{R}^2$ satisfy $\phi(e^{i(t+\pi)}) = -\phi(e^{it})$. Then for any section producing a centrally symmetrical knot on ϕ, the Arf invariant of the resulting knot is 0.*

PROOF. Define a self-inverse bijection $\tilde{}$ on X by $\widetilde{\phi(t)} = \phi(t+\pi)$. It has at most one fixed point — the origin.

Set (in the notation of the previous paragraph) $a = 0$. Classify the double points as

type (11)	iff both their preimages lie in $(0, \pi)$
type (22)	iff both their preimages lie in $(\pi, 2\pi)$
type (12)	iff one of their preimages lies in $(0, \pi)$ and the other one in $(\pi, 2\pi)$.

Fact: If v is a section producing a centrally symmetric projection, and x a double point then

$$\text{if } x \text{ is of type (11) or (22), } v(x) = 1 + v(\overline{x})$$

$$\text{if } x \text{ is type (12), } v(x) = v(\overline{x})$$

PROOF OF FACT. Let x be of type 11, with preimages t_1 and t_2, $0 < t_1 < t_2 < \pi$. Then $t_1 << t_2$, $t_1 + \pi << t_2 + \pi$ and if t_1 passes above t_2 $t_2 + \pi$ will pass above $t_1 + \pi$ by central symmetry. Type (22) is completely analogous. In the case of type (12), the symmetry will reverse the order of the preimages (if $t_1 << t_2$ then $t_2 + \pi << t_1 + \pi$). □

Extend $\tilde{}$ to couples of double points by $\{\widetilde{x_1, x_2}\} = \{\tilde{x}_1, \tilde{x}_2\}$. Since $\tilde{}$ has at most one fixed double point, the only fixed couples are the ones of the form $\{x, \tilde{x}\}$. Noting that such couples are never interlaced, let's simplify the expression (1) for the Arf invariant.

If $x \prec y$ are interlaced, and both of type (12) then $\tilde{x} \prec \tilde{y}$, and

$$(1 + v(x))v(y) + (1 + v(\tilde{x}))v(\tilde{y}) = (1 + v(x))v(y) + (1 + v(x)v(y) = 0 \ .$$

If $x \prec y$ are interlaced, x is of type (11) and y is of type (12) then $\tilde{y} \prec \tilde{x}$, and

$$(1 + v(x))v(y) + (1 + v(\tilde{y}))v(\tilde{x}) = 1 + v(x) \ .$$

If $x \prec y$ are interlaced, and both of type (11), then $\tilde{x} \prec \tilde{y}$, and

$$(1 + v(x))v(y) + (1 + v(\tilde{x}))v(\tilde{y}) = v(x) + v(y) .$$

Thus, after grouping the terms corresponding to $\tilde{}$ related couples, and rearranging the sum, we have

$$\mathrm{Arf}(v) = \sum_{\substack{x \text{ of} \\ \text{type } (11)}} (\ell(x)(1 + v(x)) + m(x)v(x))$$

where $\ell(x)$ is the number of type (12) double points interlaced with x, and $m(x)$ is the number of type (11) knots interlaced with x (the terms $v(x)+v(y)$ resulting from interlacing (11) crossings we distribute between x and y). We show that both $\ell(x)$ and $m(x)$ are even for every x.

Fix x of type (11). Then $\ell(x)$ is the number of times the part of the knot parametrized in $[\pi, 2\pi]$ intersects the loop ℓ parametrized in $[t_{1x}, t_{2x}]$, where $t_{1x} << t_{2x}$, and $\phi(t_{1x}) = \phi(t_{2x}) = x$. Considering ℓ as a knot projection on its own (by "soldering" its ends), color its complement in black and white, so that neighboring domains have opposite colors, and let the outside be white. The arc from π to 2π starts and ends on white, changing color at intersections. Thus there must be an even number of intersections.

The argument for $m(x)$ is similar, by considering the arcs going from 0 to $t_{1x} - \varepsilon$ and from $t_{1x} + \varepsilon$ to π where ε is small. $\square$

4.3. The case of an even frequency. Take a knot parametrized as $x(t) = \cos n_1 t$, $y(t) = \cos(n_2 t + \alpha)$, $z(t) = \cos(n_3 t + \beta)$, n_1 even, n_2 and n_3 odd. Let ϕ be a projection onto the xy or xz plane. Then $\phi_1(t) = \phi_1(t + \pi)$, $\phi_2(t) = -\phi_2(t + \pi)$. Define $\tilde{}$ as before: $\widetilde{\phi(t)} = \phi(t + \pi)$. $\tilde{}$ now relates points symmetric with respect to the x-axis, and all the intersections of the knot with the x-axis are crossings of the knot fixed by $\tilde{}$ (conversely, all the fixed points of $\tilde{}$ are such crossings). The Fact from the preceding section remains unaltered. All the simplifications carried out there still go, with the caution that terms corresponding to couples $\{x, y\}$ where $x = \tilde{x}$ and $y = \tilde{y}$ are counted twice, thus we are left with the expression

$$Arf(v) = \sum_{\substack{x \prec y \\ x_2 = y_2 = 0}} (1 + v(x))v(y)$$

(all such couples are interlaced).

We have to show that for any Lissajous knot this expression is equal to 0. From

$$\cos(n_2 t + \alpha) = 0$$

we obtain

$$n_2 t + \alpha = \frac{\pi}{2} + k\pi \ , \quad \text{i.e} \ \ t = \frac{k}{n_2}\pi - \frac{\alpha}{n_2} + \frac{\pi}{2n_2} \ .$$

Thus the interval $[0, 2\pi]$ is cut by $2n_2$ points, the first n_2 of which give the smaller preimages of the crossings in question. On the circle these points cut $2n_2$ equal arcs. Then

$$v(x) = \begin{cases} 0 & \text{if } \cos(n_3 t_{1x} + \beta) > 0 \\ 1 & \text{if } \cos(n_3 t_{1x} + \beta) < 0 \ . \end{cases}$$

By now it is clear that the Arf invariant does not depend on the even frequency, so assume $n_1 = 2$.

Note that the third cosine cuts the circle into $2n_3$ pieces where it alternates between positive and negative.

The rest of the proof is by induction similar to that of the Euclid algorithm for finding the greatest common divisor of two numbers.

Let $n_3 > 2n_2, \ \ n_3 = 2n_2 q + r$.

Fact 1: Replacing the third cosine with $\cos(rt + \beta + q(\pi - 2\alpha))$ doesn't change the Arf invariant.

PROOF. Trivial substitution. □

Now let $n_2 < n_3 < 2n_2$.

Fact 2: Replacing the third cosine with $\cos((2n_2 - n_3)t + 2\alpha - \pi - \beta)$ doesn't change the Arf invariant. □

By Fact 1 and Fact 2 we are reduced to the case $n_3 < n_2$. Switching projections allows us to carry the induction to an end. □

5. Further Questions

5.1. Which knots are Lissajous? This seems to be quite a difficult question. Based on examination of many examples with $n_x, n_y, n_z \leq 20$ one might be tempted to conclude that Lissajous knots are quite rare. On the other hand, the Lissajous picture of $8_2/8_{15}$ (Thistlethwaite/Rolfsen) has 32 crossings so 24 crossings may be eliminated. The ideal answer to this question would be to find an algorithm for determining if a knot is Lissajous from a diagram of it, including the values $n_x, n_y, n_z, \phi_x, \phi_y$. In the meantime it would be nice to have more necessary conditions on a knot than the ones we have found in this paper.

5.2. More Fourier modes. By the completeness of Fourier series one may obviously obtain any knot by allowing linear combinations of sufficiently many terms $\cos(n_1 t + \phi)$, $\cos(n_2 t + \phi_2), \ldots$, etc. for the x, y and z coordinates. But is there a number k such that any knot may be obtained by linear combinations of at most k Fourier modes? At this stage it is even possible that $k = 2$ would do. Here for instance are frequencies, phases and amplitudes giving the trefoil and figure-8 knots.

$$\text{trefoil} \begin{cases} x = \cos(2t + 0.8) \\ y = \cos(2t + 0.15) \\ z = \cos(4t + 1) + \cos 5t \end{cases}$$

$$\text{figure-8} \begin{cases} x = \cos(2t + 6) \\ y = \cos(3t + 0.15) \\ z = \cos(4t + 1) + \cos 5t \end{cases}$$

It is even possible that the false argument of §0.4 could actually apply for some $k > 1$. Note though that strong plus amphicheirality is forced by n_x, n_y and n_z being odd, and any such choice of n_x, n_y and n_z gives a strongly plus amphicheiral knot.

5.3. Is there a knot basis of $L^2(\mathbb{R})$? Perhaps one should not be so tied down to sines and cosines. One might ask whether there is some smooth orthonormal basis of $L^2(S^1, \mathbb{R})$ such that any knot type can be represented by x, y, z as linear combinations of just two basis elements each — or perhaps even one basis element. It is clear that orthogonality is relevant here as one may a construct total countable family of linearly independent functions such that each knot type can be realized by x, y, z being functions in the family — simply choose any smooth representations of each knot type, enumerate the x, y, z functions and perturb them slightly if necessary to ensure linear independence without changing knot type. An affirmative answer to this question would only be of any real interest if the orthonormal basis were somehow natural, say, eigenfunctions for some natural differential operator.

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