Ph136b Homework 4 Solutions

Winter 2021

Kelvin-Helmholtz Instability with Gravity

a) We consider the scenario where the vorticity is confined to a thin plane between two immiscible fluids and look at small perturbations in the interface between the two flows. Note, we consider perturbations larger than the thickness of the boundary layer, and thus overlook any boundary layer effects. The two flows lie on top of each other in the z direction, with the interface in the x-y plane. Assuming invariance in the y direction, such that the $z > 0$ flow moves at velocity z in the $\hat{x}$ direction with $\rho = \rho_+$, and the $z < 0$ flows is at rest with $\rho = \rho_-$, consider the vertical perturbation $\delta z = \xi(x, t)$, with the associated pressure and velocity fluctuations

$$P(\vec{x}, t) = P_0 + \delta P(\vec{x}, t)\,, \qquad \vec{v} = v\Theta(z)\hat{x} + \delta\vec{v}(\vec{x}, t)\,,$$

where P_0 is the constant pressure and $\Theta(z)$ is the Heaviside step function. There are two ways to proceed in generalizing the derivation from the text to include gravity, which will alter the location of the interface. We can solve for perturbations ∇P and δv using Euler's equation with a $\nabla\Phi$ contribution and then match the displacement ξ at the interface $z = 0$. Alternatively, and more simply, we can proceed in solving Euler's equation for the perturbed variables just as in the text, and match the pressures at the interface, including a gravitational contribution $\rho g \xi$.

As the perturbation $\delta\vec{v}$ is incompressible and is related to δP by the Euler equation,

$$\frac{d\delta\vec{v}}{dt} = \frac{\partial\delta\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\delta\vec{v} = -\nabla\delta P/\rho \quad \rightarrow \quad \nabla^2\delta P = 0\,,$$

where gravity does not enter into the Euler equation for the perturbed variables. We thus posit

$$\delta P = \delta P_0 e^{-k|z|} e^{i(kx-\omega t)}\,,$$

and using Euler's equation, we find

$$\begin{aligned}
\delta v_z &= \frac{ik\delta P}{(\omega - kv)\rho_+} = \frac{ik\delta P_0}{(\omega - kv)\rho_+} e^{-k|z|} e^{i(kx-\omega t)} && \text{for} \quad z > 0 \\
\delta v_z &= \frac{-ik\delta P}{\omega\rho_-} = \frac{-ik\delta P_0}{\omega\rho_-} e^{-k|z|} e^{i(kx-\omega t)} && \text{for} \quad z < 0\,.
\end{aligned}$$

b) A similar calculation for δv_x yields $\delta v_x = k\delta P/[(\omega - kv)\rho_+]$ above the interface and $\delta v_x = k\delta P/(\omega\rho_-)$ below the interface. Incompressibility requires $\boldsymbol{\nabla}\cdot v = \partial v_x/\partial x + \partial v_z/\partial z$, and above the interface this is $ikv_x - kv_z = 0$ based on the expressions above. Below the interface this is $ikv_x + kv_z = 0$ using the equations above.

c) At the interface $z = 0$, the velocity perturbations are related to the displacement as $d\xi/dt = \delta v_z$. For a vertical perturbation of the form $\xi = Ae^{i(kx-\omega t)}$, we find

$$\xi = -\frac{k\delta P_0}{(\omega - kv)^2\rho_+}e^{i(kx-\omega t)} \quad \text{for} \quad z = 0_+\,, \qquad \xi = \frac{k\delta P_0}{\omega^2\rho_-}e^{i(kx-\omega t)} \quad \text{for} \quad z = 0_-\,.$$

or equivalently, the variations in pressure at the interface, in terms of the displacement

$$\delta P_+ = -\frac{(\omega - kv)^2\rho_+}{k}\xi \quad \text{for} \quad z = 0_+\,, \qquad \delta P_- = \frac{\omega^2\rho_-}{k}\xi \quad \text{for} \quad z = 0_-\,.$$

We now match the pressures along the interface, including a gravitational contribution to the pressure, evaluated at ξ,

$$P_+ = P_0 + \delta P_+ - \rho_+ g\xi\,, \qquad P_- = P_0 + \delta P_- - \rho_- g\xi\,,$$

gives us

$$\frac{(\omega - kv)^2\rho_+}{k}\xi + \rho_+ g\xi = -\frac{\omega^2\rho_-}{k}\xi + \rho_- g\xi$$

$$\rightarrow (\rho_+ + \rho_-)\omega^2 - 2kv\rho_+\omega + k^2v^2\rho_+ + (\rho_+ - \rho_-)gk = 0\,.$$

Solving the equation, we find the dispersion

$$\frac{\omega}{k} = \frac{v\rho_+}{(\rho_+ + \rho_-)} \pm \left(\frac{g}{k}\frac{(\rho_- - \rho_+)}{(\rho_+ + \rho_-)} - \frac{v^2\rho_+\rho_+}{(\rho_+ + \rho_-)^2}\right)^{1/2}\,,$$

In the limit where $g = 0$, this reduces to

$$\omega = kv\frac{\rho_+ \pm i(\rho_+\rho_-)^{1/2}}{\rho_+ + \rho_-}\,.$$

d) The more general expression with gravity is written above.

e) Unstable modes require

$$\frac{(\rho_- - \rho_+)}{(\rho_+ + \rho_-)} < \frac{kv^2}{g}\frac{\rho_+\rho_+}{(\rho_+ + \rho_-)^2}\,.$$

When the velocity is zero, this reduces to $\rho_+ > \rho_-$, i.e., the density of the overlying fluid exceeds that of the underlying fluid. This is the Rayleigh-Taylor instability.

When v is non-zero and $\rho_+ < \rho_-$, we can rewrite the condition above in terms of the wavelength $\lambda = 2\pi/k$, so that instability requires

$$\lambda < \frac{2\pi v^2}{g}\frac{\rho_+\rho_+}{\rho_-^2 - \rho_+^2}\,.$$

Thermal Stability

a) If the cooling time is longer than the sound-crossing time, the gas is able to come into force balance as it cools. Thus, the perturbation should evolve at roughly constant pressure. Quantitatively,

$$\frac{\delta P}{P} = \frac{\delta n}{n} + \frac{\delta T}{T} \ll \frac{\delta n}{n}, \frac{\delta T}{T}$$

so that $\delta n/n \sim -\delta T/T$.

To derive the dispersion relation, we'll use the energy equation: $\rho T ds/dt = \mathcal{H}$. After linearizing and Fourier-transforming this equation, it becomes

$$-i\omega\delta s = \frac{\delta\mathcal{H}}{\rho T} = \frac{1}{\rho T}\left(\frac{\partial\mathcal{H}}{\partial T}\right)_P \delta T,$$

where $\mathcal{H} = H - L$ and H and L represent heating and cooling rates, respectively. Note that there are several equivalent ways to express the change in the heating (e.g. $(\partial\mathcal{H}/\partial T)_n\delta T + (\partial\mathcal{H}/\partial n)_T\delta n$); for this problem, the derivative with respect to temperature at constant pressure is the most convenient.

The entropy per unit volume $s = c_V \ln(P/\rho^\gamma)$. So for an isobaric perturbation, we have

$$\delta s = -\gamma c_V \frac{\delta\rho}{\rho} = -c_P \frac{\delta\rho}{\rho}.$$

Putting this into the perturbed energy equation, and using $\delta\rho/\rho \sim -\delta T/T$, we find

$$p \equiv -i\omega = \frac{1}{\rho c_P}\left(\frac{\partial\mathcal{H}}{\partial T}\right)_P,$$

where perturbations evolve as e^{pt}. Note that the evolution timescale p^{-1} for the perturbations is proportional to $t_{\rm cool} \sim nkT/\mathcal{H}$.

The dispersion relation shows that the perturbations are unstable if $(\partial\mathcal{H}/\partial T)_P > 0$, so that $p > 0$. We might have guessed this more intuitively as follows: if this derivative is positive, $\delta\mathcal{H}$ and δT have the same sign, so that an increase in temperature leads to an increase in the net heating rate. Assuming the heat capacity of the gas is positive, this signals thermal instability.

b) If the cooling time is short compared to the sound-crossing time, the gas cannot dynamically adjust in response to cooling. Thus, the fluid elements are "frozen" and the perturbation evolves at roughly constant density.

In this case, $\delta s = c_V \delta T/T$, so the perturbed energy equation is

$$\begin{aligned} \frac{ds}{dt} &= \frac{\mathcal{H}}{\rho T} \\ -i\omega c_V \frac{\delta T}{T} &= \frac{\delta\mathcal{H}}{\rho T} = \frac{1}{\rho T}\left(\frac{\partial\mathcal{H}}{\partial T}\right)_\rho \delta T, \end{aligned}$$

so that the growth rate of the thermal instability is

$$p \equiv -i\omega = \frac{1}{\rho c_V}\left(\frac{\partial\mathcal{H}}{\partial T}\right)_\rho.$$

The interpretation is identical to the one in part a, expect the relevant derivative and the heat capacity are the ones with volume (or density) held fixed.

c) Including thermal conduction, the energy equation becomes:

$$\rho T\frac{ds}{dt} = \mathcal{H} - \nabla\cdot(\kappa\nabla T)$$

We can linearize and Fourier transform this as before to derive a dispersion relation. If the perturbation is isobaric, we obtain

$$p \equiv -i\omega = \frac{1}{\rho c_P}\left(\frac{\partial \mathcal{H}}{\partial T}\right)_P - \frac{\kappa}{\rho c_P}k^2.$$

Note that

$$\frac{\kappa}{\rho c_P} = \frac{\gamma - 1}{\gamma}\frac{\mu m \kappa}{k_\mathrm{B}\rho}.$$

The quantity $\mu m\kappa/k_\mathrm{B}\rho$ has the units of cm^2/s, i.e. of a diffusion coefficient. It is usually called χ. The growth rate for the thermal instability is therefore

$$p \equiv -i\omega = \frac{1}{\rho c_P}\left(\frac{\partial \mathcal{H}}{\partial T}\right)_P - \frac{\gamma-1}{\gamma}k^2\chi$$

The positive term is independent of scale, and the negative term $\propto k^2$ increases in magnitude as the perturbation becomes smaller. Since a positive value of p signals instability, we see that small scales are stabilized. Note that the positive term $= t_\mathrm{TI}^{-1}$ is the inverse of the thermal instbility time we derived in part a. The negative term is $= t_\mathrm{cond}^{-1}$ is the time it takes heat to diffuse across the wavelength of the perturbation. As expected, the stability of a particular scale is set by the competition between cooling and conduction.

By equating the two terms, we can determine the threshold for stability:

$$\lambda^2 = \frac{(2\pi)^2}{k^2} = (2\pi)^2\frac{nk_\mathrm{B}\chi}{\left(\frac{\partial\mathcal{H}}{\partial T}\right)_P} \equiv \lambda_\mathrm{F}^2.$$

λ_F is known as the "Field length" after James Field, who derived it in 1965. Intuitively, the Field length is the distance heat can diffuse in a cooling time. I don't know of a way to get the 2π's without solving the dispersion relation, though.

Turbulent Jets

a) Consider a two-dimensional, turbulent jet discharging into an ambient fluid. The scales in our problem are the jet velocity in the x-direction v_x, the distance from the jet nozzle x, and the spread of the jet in the y direction w. Both the jet velocity v_x and the lateral spread w, will depend on x.

From momentum conservation, where the drag force per unit length must equal the change in momentum per unit length,

$$C_D\frac{1}{2}\rho v^2 d = \int dy\,\rho v^2 \quad \rightarrow \quad \int dy\,\rho v_x^2 = \mathrm{const} \quad \rightarrow \quad v_x^2 \sim 1/w\,.$$

b) Ignoring pressure gradients and gravity, dimensional analysis of the Navier-Stokes equation $(\vec{v}\cdot\nabla)\vec{v} = \nu\nabla^2\vec{v}$, tells us

$$v_y\frac{\partial v_x}{\partial y} + v_x\frac{\partial v_x}{\partial x} = \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \quad \rightarrow \quad \frac{v_x v_y}{w} + \frac{v_x^2}{x} \sim \nu\frac{v_x}{x^2} + \nu\frac{v_x}{w^2}\,,$$

where $w < x$ means that the second term is dominant. From incompressibility

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad \rightarrow \quad v_y \sim \frac{w}{x}v_x\,,$$

we find $v_x \sim \nu x/w^2$. Equivalently,

$$w \sim \sqrt{\nu x/v_x}\,.$$

c) For turbulent viscosity $\nu_t = \frac{1}{3}\ell v_\ell$, where the relevant scales for the eddies are the jet velocity v_x and the transverse spread w, we have $\nu_t \sim v_x w$. Using these relations, we can say how the spread w and jet velocity depend on x

$$w \sim x \quad \text{and} \quad v_x \sim 1/\sqrt{x}\,.$$

d) Now consider a three-dimensional jet, with the jet velocity in the x-direction v_x, the distance from the jet nozzle x, and two transverse scales: the spread in the y-direction, and the spread in the z-direction. It is easiest to work in cylindrical coordinates, where we consider an axially symmetric spread in the r-direction with a scale w about the x-axis. Incompressibility gives the radial speed $v_r \sim w/x$. From Navier-Stokes in cylindrical coordinates,

$$\frac{v_r}{r}\frac{\partial(rv_x)}{\partial r} + v_x\frac{\partial v_x}{\partial x} = \nu\frac{\partial^2 v_x}{\partial x^2} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) \quad \rightarrow \quad \frac{v_x^2}{x} \sim \nu\frac{v_x}{x^2} + \nu\frac{v_x}{w^2}\,.$$

As $w < x$, the second term dominates and, just as in 2b, we find

$$w \sim \sqrt{\nu x/v_x}\,.$$

Momentum conservation in 3d now tells us that $F_{\text{drag}} \sim \rho v_x^2 A = \text{const}$, which gives

$$v_x^2 \sim 1/w^2 \quad \rightarrow \quad v_x \sim 1/x\,,$$

different from 2c, but with the same x dependence for transversal spread $w \sim x$.

e) We want to estimate the scaling of the rate of mass flow along the turbulent 3-dimensional jet. The change in mass $\Delta M = \rho\Delta\text{Vol} = \rho v_x A\Delta t$, means that

$$\frac{dM}{dt} \sim \frac{\Delta M}{\Delta t} \sim \rho v_x A \sim \rho v_x w^2 \sim \rho x\,.$$

Scaling linearly in x, means that the mass flow increases farther away from the origin of the jet. This is entrainment, where, farther away, more and more ambient fluid is being sucked into the flow and becomes turbulent.

Kolmogorov Spectrum

a) Consider a quantity, like the index of refraction n, which is transported with the fluid. We can find the k dependence of the spectral density of n, $S_n(k)$, by consider the quantities it depends on, including the rate of change of the variance

$$q_n = \frac{d\sigma_n^2}{dt} \quad \text{where} \quad \sigma_n^2 = \int \frac{d^3k}{(2\pi)^3}\,S_n(k)\,.$$

We see that q_n and q, the eddy velocity over the turnover time, have the following dependence

$$q_n \sim \frac{S_n k^3}{t} \qquad q \sim \frac{v^2}{\tau} \sim \frac{1}{t^3k^2}\,.$$

All together, we must have that

$$S_n \sim q_n k^{-3}(qk^2)^{-1/3} \sim q_n q^{-1/3}k^{-11/3}\,.$$

b) Consider the structure function, defined as twice the difference between the variance and the autocorrelation function (the Fourier transform of the spectral density)

$$D_n(\xi) = \left\langle \big(\delta n(\vec{x}+\vec{\xi}) - \delta n(\vec{x})\big)^2 \right\rangle = 2\big(\sigma_n^2 - C_n(\xi)\big) \,.$$

To find the ξ dependence of the structure function, we can Fourier transform $S_n(k)$ to find

$$S_n(\xi) \sim \frac{q_n}{q^{1/3}} \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{\xi}} k^{-11/3} \sim \frac{q_n}{q^{1/3}} \int dk\, k^2\, e^{ik\xi} k^{-11/3} \,,$$

where we use polar coordinates in k-space, define $k = |\vec{k}|$, and integrate over the sphere. We also ignore constant factors. Demanding that the spectral density $S_n(k)$ be zero below k_{min}, meaning that there are no eddies at wavelengths larger than $\lambda \sim 1/k_{\mathrm{min}}$, and be zero above k_{max}, as these eddies have Re ~ 1 and dissipate due to viscosity, effectively upper and lower bounds the above integral such that it is finite. But as we are not concerned with its exact value, just its ξ-dependence, we can perform a change of variables to the dimensionless $u = k\xi$

$$S_n(\xi) \sim \frac{q_n}{q^{1/3}} \xi^{2/3} \int du\, u^{-5/3}\, e^{iu} \sim \frac{q_n}{q^{1/3}} \xi^{2/3} \,,$$

where the integral gives some constant which we can ignore.

We can rapidly reach the same conclusion from dimensional analysis. To find the ξ dependence of $D_n(\xi)$, we first note that

$$D_n \sim \sigma_n^2 \sim k^3 S_n \sim \frac{q}{q^{1/3}} k^{-2/3} \,.$$

As ξ is the relevant length scale, and k has dimensions of inverse length, we find

$$D_n(\xi) \sim \frac{q}{q^{1/3}} \xi^{2/3} \,.$$