Answers to the final examination on Logic Circuit Design

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Questions.

1. Let be an n-bit binary number. How many numbers can it encode? What is the lowest? What is the highest? (Prove it.)

2. Same questions if the binary number is interpreted as a 2-complement.

3. When is the number $d_{n-1} \times 10^{n-1} + d_{n-2} \times 10^{n-2} + \cdots + d_1 \times 10 + d_0$ interpreted as the decimal number whose digits are $d_{n-1}d_{n-2} \ldots d_0$?

4. Prove that, in order to convert a binary number into an octal number, one groups the bits three by three from right to left and then convert each group into decimal, without worrying about carries.

5. Why is the multiplication of binary numbers easy?

6. Consider an n-bit 2-complement binary number. If the leftmost bit is 1, then the number is negative, whilst if it is 0, then the number is positive or nul. Why?

7. Can the negation of an n-bit 2-complement binary number fail?

8. For all bits x and y, the boolean function F is partially defined as

A	B	C	D	F
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	1	0	1
0	1	1	1	1
1	1	x	y	0
1	0	0	0	1
1	0	1	1	0

   Find both the simplified sum of products and product of sums of F.

Answers.

1. The general shape of an n-bit binary number, written $b_{n-1}b_{n-2}\dots b_0$, is, by definition

$$b_{n-1} \times 2^{n-1} + b_{n-2} \times 2^{n-2} + \cdots + b_0$$

where each b_i can be either 0 or 1.

Therefore, there is two choices for b_{n-1}, two for b_{n-2} etc. so the total number of choices for $\{b_{n-1}, b_{n-2}, \dots, b_0\}$ is $\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$, which is the number of numbers that can be represented by an n-bit binary number.

The highest number it can represent, called $\max_n$, is achieved when $b_i = 1$, for all the i, that is:

$$\max_n = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \tag{1}$$

If we multiply it by 2 each sides, we get

$$2 \times \max_n = 2^n + 2^{n-1} + \cdots + 2^2 + 2 \tag{2}$$

Forming (2) − (1) we get

$$\begin{aligned} 2 \times \max_n - \max_n &= 2^n - 1 \\ \max_n &= 2^n - 1 \end{aligned}$$

The smallest number that can be represented is achieved when the $b_i = 0$, for all i, i.e. $\min_n = 0$.

By the way, we can check that $\max_n - \min_n + 1 = 2^n$, as expected.

2. The general shape of an n-bit, 2-complement binary number written $b_{n-1}b_{n-2}\dots b_0$ is

$$-b_{n-1} \times 2^{n-1} + b_{n-2} \times 2^{n-2} + \cdots + b_0$$

Therefore, the lowest number that can be represented is achieved when $b_{n-1} = 1$ and $b_{n-2}, b_{n-3}, \dots, b_0$ are 0: $\min_n = -2^{n-1}$.

The highest number is achieved when $b_{n-1} = 0$ and $b_{n-2}, b_{n-3}, \dots, b_0$ are 1: $\max_n = 2^{n-2} + 2^{n-3} + \cdots + 1 = 2^{n-1} - 1$.

The total number of numbers that can be represented is thus

$$\max_n - \min_n + 1 = 2^{n-1} - 1 + 2^{n-1} + 1 = 2^n$$

This is, of course, the same number as with the (unsigned) binary numbers.

3. When $0 \leqslant d_i \leqslant 9$, for all i.

4. Consider the n-bit binary number general form:

$$b_{n-1}2^{n-1} + b_{n-2}2^{n-2} + \cdots + b_1 2 + b_0$$

We can group the bits by groups of three, from right to left:

$$\cdots + (b_8 2^8 + b_7 2^7 + b_6 2^6) + (b_5 2^5 + b_4 2^4 + b_3 2^3) + (b_2 2^2 + b_1 2 + b_0)$$

We can factorise 2^3, 2^6 etc. and get

$$\cdots + (b_8 2^2 + b_7 2 + b_6) \times 2^6 + (b_5 2^2 + b_4 2 + b_3) \times 2^3 + (b_2 2^2 + b_1 2 + b_0)$$

That is, since $8 = 2^3$ and $x^{pq} = (x^p)^q$, then $2^{3q} = (2^3)^q = 8^q$ and

$$\cdots + (b_8 2^2 + b_7 2 + b_6) \times 8^2 + (b_5 2^2 + b_4 2 + b_3) \times 8^1 + (b_2 2^2 + b_1 2 + b_0) \times 8^0$$

Therefore, in order to convert an octal number to its equivalent binary representation, we convert separately its digits into binary and simply catenate them.

5. Because there are no carries produced by the 1-bit multiplications. When multiplying a number N by M, from right to left, every time a bit of M is 1, then N is the partial result; else it is 0. The final addition can imply carries, of course, but the first stage is very easy (copy N or write 0).

6. The general shape of an n-bit, 2-complement binary number written $b_{n-1}b_{n-2}\ldots b_0$ is

$$-b_{n-1} \times 2^{n-1} + b_{n-2} \times 2^{n-2} + \cdots + b_0$$

If the leftmost bit is 1 then the number is negative. Why? Assume that in an n-bit 2-complement number, the leftmost bit is 1. Then the highest positive value that can be formed with the remaining $n-1$ bits is having only 1s. In other words, the n bits are all 1s. So this number is

$$N = -2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1$$

We proved earlier that

$$2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = 2^n - 1$$

So

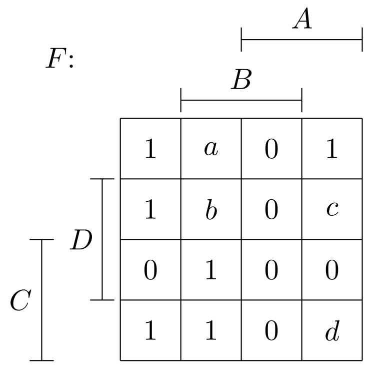
$$N = -2^{n-1} + (2^{n-1} - 1) = -1 < 0$$

If the leftmost bit is 0 then the number is positive. Why? Because it is simply of the form

$$0 \times 2^{n-1} + b_{n-2}2^{n-2} + b_{n-3}2^{n-3} + \cdots + b_1 2 + b_0 \geqslant 0$$

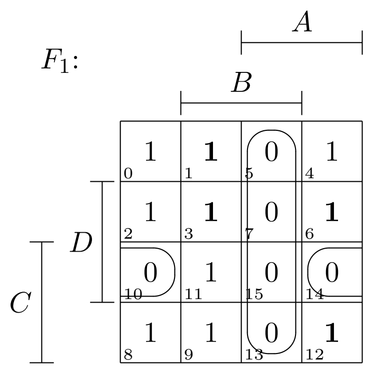
7. Yes, it can fail. Consider again question 2: the numbers that can be coded as an n-bit, 2-complement binary number range from -2^{n-1} to $2^{n-1}-1$. Therefore the negation of -2^{n-1} is out of range (overflow).

8. From the truth table, we build a Karnaugh map. The unspecified values of F are represented by a, b, c and d. The ranges where a given variable is 1 are given around the map.



In order to find the simplest sum of products, we can choose values for a, b, c and d that suit best the groupings of 1's in the map. There are several possibilities. An optimum covering is achieved if $a = b = c = d = 1$. The corresponding min-terms lead to F_1:

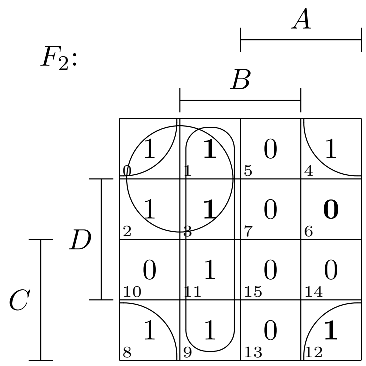
$$F_1 = \overline{A}\,B + \overline{B}\,\overline{C} + \overline{B}\,\overline{D}$$



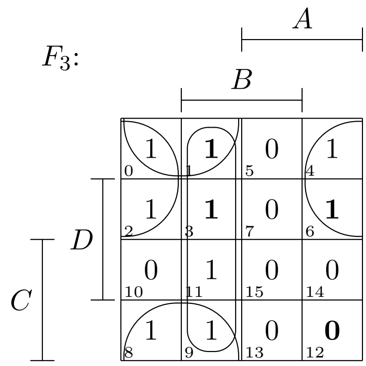
Other variable assignments with optimum covering are possible:

$$F_2 = \overline{A}\,B + \overline{A}\,\overline{C} + \overline{B}\,\overline{D}$$
$$F_3 = \overline{A}\,B + \overline{A}\,\overline{D} + \overline{B}\,\overline{C}$$

F_2:



F_3:



In order to find the simplest product of sums, we can choose *different* values for a, b, c and d, since the only constraint is given by the truth table and any choice would agree with it. A different value assignment would yield a different function F, but this does not matter, only the part of F which is constrained by the table.

An optimum covering of 0's is achieved if we take F_1 again:

$$F_1 = (\overline{A} + \overline{B})(B + \overline{C} + \overline{D})$$

F_1: