CS 663: Digital Image Processing
Assignment – 4

Question – 2,3

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Question – 2)

Qn: 2) To maximize : f^tCf

Constraints : $f^tf = 1$, $e^tf = 0$.

Soln: Lagrange formulation:

$$J(f) = f^tCf - \lambda_1 f^tf - \lambda_2 f^te$$

Taking derivative w.r.t. f & equate to 0

$$J'(f) = 2Cf - 2\lambda_1 f - \lambda_2 e = 0 \quad -(1)$$

$$\Rightarrow 2(e^tC)f - 2\lambda_1 e^tf - \lambda_2 e^te = 0$$

$e^te=1 \Rightarrow 2(C^te)^tf - 2\lambda_1 e^tf - \lambda_2 = 0$

C is a symmetric matrix $\Rightarrow C = C^t$.
Also, e is an eigen vector of C i.e. $Ce = \mu e$

$$\Rightarrow 2\mu e^tf - 2\lambda_1 e^tf - \lambda_2 = 0$$

$$\Rightarrow 2(\mu - \lambda_1)e^tf - \lambda_2 = 0$$

$\because e^tf = 0 \Rightarrow \lambda_2 = 0 \quad -(2)$

putting (2) in (1), we get

$Cf = \lambda_1 f \Rightarrow f$ is an eigenvector of C

Now, $f^tCf = \lambda_1 f^tf = \lambda_1$. ← to be maximized

So we observe that f^tCf is maximized when f is eigenvector and λ_1 is eigen value. Since all non-zero eigen values are distinct, λ_1 has to be the ~~last~~ second largest eigen value

Question – 3a)

Qn : 3) (a) $P = A^T A$, $Q = AA^T$. ; $A = m \times n$

To show : $y^T P y \geq 0$ and $y^T Q z \geq 0$

$y \rightarrow n$ dimensional vector
$z \rightarrow m$ dimensional vector.

Now, $y^T P y = y^T A^T A y$

$= (Ay)^T Ay$

$= ||Ay||^2 \geq 0$

$z^T Q z = z^T A A^T z$

$= (A^T z)^T A^T z$

$= ||A^T z||^2 \geq 0$

$\therefore$ P and Q are positive semidefinite (PSD)

Say P is a PSD matrix and V is its eigenvector with λ as eigen value, then,

$V^T P V = V^T \lambda V$

$= \lambda V^T V$

$= \lambda ||v||^2$

But, $V^T P V \geq 0 \Rightarrow \lambda ||v||^2 \geq 0$

$\Rightarrow \lambda \geq 0 \quad (\because ||v||^2 \geq 0)$

$\Rightarrow$ For a PSD, eigen values are always non-negative and P & Q are PSD.

Question – 3b)

$A = m \times n$

Qn: 3)(b) Given: $Pu = \lambda u$, $Qv = \mu v$.

To Prove: $Q(Au) = \lambda(Au)$, $P(A^T v) = \mu(A^T v)$

Proof: $Pu = \lambda u$

$\therefore A^T A u = \lambda u$

$\Rightarrow A A^T A u = A \lambda u$ (Multiplying both sides by A)

$\Rightarrow (A A^T) A u = \lambda (Au)$

$\Rightarrow Q(Au) = \lambda(Au)$

u = n-dimensional vector.

$Qv = \mu v$

$\Rightarrow A A^T v = \mu v$

$\Rightarrow (A^T A) A^T v = \mu (A^T v)$ (Multiplying both sides by A^T)

$\Rightarrow P(A^T v) = \mu (A^T v)$

v = m-dimensional vector.

Question – 3c)

Qn 3)(c) Say α is the eigen~~vector~~value corresponding to eigenvector v_i of matrix Q.

from section (a) Q is PSD $\Rightarrow \alpha \geq 0$.

$$\therefore \quad Qv_i = \alpha v_i \quad (\alpha \geq 0).$$

Now,

$$Au_i = A\left(\frac{A^T v_i}{\|A^T v_i\|_2}\right)$$

$$= \frac{(AA^T) v_i}{\|A^T v_i\|_2}$$

$$= \frac{Q v_i}{\|A^T v_i\|_2}$$

$$= \left(\frac{\alpha}{\|A^T v_i\|_2}\right) v_i$$

$$\Rightarrow \boxed{\sigma_i = \frac{\alpha}{\|A^T v_i\|_2}}.$$

Question – 3d)

Qn 3) d) Given: $Au_i = r_i v_i$ — ①

$U = [v_1 \ v_2 \ \dots \ v_m]$, $V = [u_1 \ u_2 \ \dots \ u_m]$

$\Gamma = \text{diag}(r_1, r_2, \dots r_m)$

Soln:

$$U\Gamma = U \times \text{diag}(r_1, \dots r_m)$$

$$= [v_1 \ v_2 \ \dots \ v_m] \begin{bmatrix} r_1 & \dots & 0 \\ \vdots & r_2 & \vdots \\ 0 & \dots & r_m \end{bmatrix}$$

$$= [r_1 v_1 \ \ r_2 v_2 \ \dots \ r_m v_m]$$

$$= [Au_1 \ \ Au_2 \ \dots \ Au_m] \quad (\text{from } ①)$$

$$= AV.$$

$$\therefore U\Gamma V^T = AVV^T.$$

$\because$ All the column vectors of V are orthonormal (given),

$$\therefore VV^T = I.$$

$$\Rightarrow \boxed{U\Gamma V^T = A}$$

Note: vi and ui are column vectors and not single elements. The notation might confuse the reader to consider them as single elements but they are vectors in the matrix. So, [v1 v2 v3 .. vn] = A matrix with vi as column vectors