Homework 2

Problem 1

Let $\Omega = \{1, 2, 3, 4\}$, and set $S = \{\emptyset, \{1\}, \{2\}, \{3, 4\}, \Omega\}$.

(a) Prove that S is a semi-algebra.
(b) Define $\chi : S \to \mathbb{R}$ as follows:

$$\chi(\emptyset) = 0, \chi(\{1\}) = \chi(\{2\}) = \chi(\{3, 4\}) = 1, \chi(\Omega) = 4.$$

Show that χ is "pairwise" additive: $\chi(A \cup B) = \chi(A) + \chi(B)$ whenever A, B, and $A \cup B$ are all in S. Show also that χ is not additive over all finite disjoint unions.

(This shows it is important, when dealing with semi-algebras and other classes not closed under finite union, to spell out the full statement of "finite additivity" in all proofs.)

Solution

(a) S is a semi-algebra.

1. **Non-empty and Contains Ω:**

   Clearly, $\Omega \in S$ and $\emptyset \in S$.

2. **Closed under Intersection:**

   We need to check that the intersection of any two sets in S is also in S:

   - $\emptyset \cap A = \emptyset$ for any $A \in S$.
   - $\{1\} \cap \{2\} = \emptyset$.
   - $\{1\} \cap \{3, 4\} = \emptyset$.
   - $\{2\} \cap \{3, 4\} = \emptyset$.
   - $\{3, 4\} \cap \Omega = \{3, 4\}$.
   - $\{1\} \cap \Omega = \{1\}$.

- $\{2\} \cap \Omega = \{2\}$.
- $\Omega \cap \Omega = \Omega$.

All these intersections are in S, so S is closed under intersections.

3. **Complement Property:**

We need to check that the complement of any set in S can be written as a finite disjoint union of sets in S:

- $\emptyset^c = \Omega$.
- $\{1\}^c = \{2, 3, 4\} = \{2\} \cup \{3, 4\}$
- $\{2\}^c = \{1, 3, 4\} = \{1\} \cup \{3, 4\}$
- $\{3, 4\}^c = \{1, 2\} = \{1\} \cup \{2\}$
- $\Omega^c = \emptyset$.

Thus, S satisfies the complement condition.

Since S is closed under intersections and complements can be expressed as finite disjoint unions of sets in S, S is a semi-algebra.

(b) χ is "pairwise" additive but not additive over all finite disjoint unions.

1. **Pairwise Additivity:**

We need to show that $\chi(A \cup B) = \chi(A) + \chi(B)$ whenever A, B, and $A \cup B$ are all in S:

- $\chi(\emptyset \cup A) = \chi(A) = \chi(\emptyset) + \chi(A)$ for any $A \in S$.
- $\chi(\{1\} \cup \{2\}) = \chi(\{1, 2\}) = 2 = \chi(\{1\}) + \chi(\{2\})$.
- $\chi(\{1\} \cup \{3, 4\}) = \chi(\{1, 3, 4\}) = 2 = \chi(\{1\}) + \chi(\{3, 4\})$.
- $\chi(\{2\} \cup \{3, 4\}) = \chi(\{2, 3, 4\}) = 2 = \chi(\{2\}) + \chi(\{3, 4\})$.
- $\chi(\{1\} \cup \Omega) = \chi(\Omega) = 4 = \chi(\{1\}) + \chi(\Omega)$.

In all cases, pairwise additivity holds.

2. **Non-Additivity over All Finite Disjoint Unions:**

Consider the sets $\{1\}$, $\{2\}$, and $\{3, 4\}$ which are disjoint and their union is Ω:

$$\chi(\{1\} \cup \{2\} \cup \{3, 4\}) = \chi(\Omega) = 4$$

but

$$\chi(\{1\}) + \chi(\{2\}) + \chi(\{3, 4\}) = 1 + 1 + 1 = 3.$$

Thus, χ is not additive over all finite disjoint unions.

Therefore, χ is pairwise additive but not additive over all finite disjoint unions. ■

Problem 2

(Exercise 4.12 in Driver) Let Ω_1 and Ω_2 be sets, and let $\mathcal{A}_1 \subseteq 2^{\Omega_1}$ and $\mathcal{A}_2 \subseteq 2^{\Omega_2}$ be semi-algebras. Show that

$$S = \mathcal{A}_1 \times \mathcal{A}_2 = \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\} \subseteq 2^{\Omega_1 \times \Omega_2}$$

is a semi-algebra.

Solution

To prove that S is a semi-algebra, we need to show the following properties:

1. Empty Set and Whole Set:

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are semi-algebras, they both contain the empty set $\emptyset$ and their respective whole sets Ω_1 and Ω_2. Therefore,

$$\emptyset \times \emptyset = \emptyset \in S,$$

and

$$\Omega_1 \times \Omega_2 \in S.$$

Thus, $\emptyset \in S$ and $\Omega_1 \times \Omega_2 \in S$.

2. Closed under Intersection:

Let $X_1 = A_1 \times A_2$ and $X_2 = B_1 \times B_2$ where $A_1, B_1 \in \mathcal{A}_1$ and $A_2, B_2 \in \mathcal{A}_2$. Then,

$$\begin{aligned} X_1 \cap X_2 &= (A_1 \times A_2) \cap (B_1 \times B_2) \\ &= (A_1 \cap B_1) \times (A_2 \cap B_2) \end{aligned}$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are semi-algebras, $A_1 \cap B_1 \in \mathcal{A}_1$ and $A_2 \cap B_2 \in \mathcal{A}_2$. Thus,

$$(A_1 \cap B_1) \times (A_2 \cap B_2) \in S.$$

Hence, S is closed under intersection.

3. Complementation Property:

Let $X = A_1 \times A_2$ where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. We need to show that X^c can be expressed as a finite union of disjoint sets in S.

We decompose X^c into a union of three parts:

$$X^c = (A_1^c \times A_2) \cup (A_1 \times A_2^c) \cup (A_1^c \times A_2^c).$$

To show that LHS = RHS:

$$\begin{aligned} X^c =& (A_1 \times A_2)^c \\ =& \{(x, y) : (x, y) \notin A_1 \times A_2\} \\ =& \{(x, y) : x \notin A_1 \text{ or } y \notin A_2\} \\ =& \{(x, y) : x \in A_1^c \text{ and } y \in A_2\} \\ & \cup \{(x, y) : x \in A_1 \text{ and } y \in A_2^c\} \\ & \cup \{(x, y) : x \in A_1^c \text{ and } y \in A_2^c\} \\ =& (A_1^c \times A_2) \cup (A_1 \times A_2^c) \cup (A_1^c \times A_2^c) \end{aligned}$$

To show that these three parts are disjoint, observe that:

$$
\begin{aligned}
&(A_1^c \times A_2) \cap (A_1 \times A_2^c) \\
=&(A_1^c \cap A_1) \times (A_2 \cap A_2^c) = \emptyset \\
&(A_1^c \times A_2) \cap (A_1^c \times A_2^c) \\
=&(A_1^c \cap A_1^c) \times (A_2 \cap A_2^c) = \emptyset \\
&(A_1 \times A_2^c) \cap (A_1^c \times A_2^c) \\
=&(A_1 \cap A_1^c) \times (A_2^c \cap A_2^c) = \emptyset
\end{aligned}
$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are semi-algebras, $A_1^c \in \mathcal{A}_1$ and $A_2^c \in \mathcal{A}_2$. Therefore,

$$(A_1^c \times A_2) \in S, \quad (A_1 \times A_2^c) \in S, \quad (A_1^c \times A_2^c) \in S.$$

Thus, X^c can be expressed as a finite union of disjoint sets in S.

Since S satisfies all three properties of a semi-algebra, we conclude that S is indeed a semi-algebra. ■

Problem 3

(Exercise 4.3 in Driver) Let $A_n, B_n \subseteq \Omega$ for $n \in \mathbb{N}$. Show that

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \setminus B_n).$$

Use this to show that

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \triangle B_n).$$

Solution

Let $A_n, B_n \subseteq \Omega$ for $n \in \mathbb{N}$. We will prove the following two statements:

1. $\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \setminus B_n)$

Proof:

Let $x \in \left(\bigcup_{n=1}^{\infty} A_n\right) \setminus \left(\bigcup_{n=1}^{\infty} B_n\right)$. This means:

1. $x \in \bigcup_{n=1}^{\infty} A_n$
2. $x \notin \bigcup_{n=1}^{\infty} B_n$

From (1), $\exists k \in \mathbb{N}$ such that $x \in A_k$. From (2), $\forall n \in \mathbb{N}, x \notin B_n$. In particular, $x \notin B_k$.

Therefore, $x \in A_k \setminus B_k \subseteq \bigcup_{n=1}^{\infty}(A_n \setminus B_n)$.

Thus, we have

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty}(A_n \setminus B_n)$$

2. $\left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty}(A_n \triangle B_n)$

Proof:

Recall that for any sets X and Y, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

Let $L = \left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \left(\bigcup_{n=1}^{\infty} B_n\right)$ and $R = \bigcup_{n=1}^{\infty}(A_n \triangle B_n)$.

$$\begin{aligned}
L &= \left(\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus \left(\bigcup_{n=1}^{\infty} B_n\right)\right) \cup \left(\left(\bigcup_{n=1}^{\infty} B_n\right) \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)\right) \\
&\subseteq \bigcup_{n=1}^{\infty}(A_n \setminus B_n) \cup \bigcup_{n=1}^{\infty}(B_n \setminus A_n) \quad \text{(by part 1)} \\
&= \bigcup_{n=1}^{\infty}((A_n \setminus B_n) \cup (B_n \setminus A_n)) \\
&= \bigcup_{n=1}^{\infty}(A_n \triangle B_n) = R
\end{aligned}$$

Thus, we have shown that $L \subseteq R$, which completes the proof. ■

Problem 4

(Exercise 4.4 in Driver) Let $A, B, C \subseteq \Omega$. Recall that the **symmetric difference** of sets is $A \triangle B = (A \cap B^c) \cup (B \cap A^c)$.

(a) Show that $A \cap C^c \subseteq (A \cap B^c) \cup (B \cap C^c)$.

(b) Use part (a) to show that

$$A \triangle C \subseteq (A \triangle B) \cup (B \triangle C).$$

(c) Now, let $\nu : 2^{\Omega} \to [0, \infty)$ be an outer measure. Show that the function $d : 2^{\Omega} \times 2^{\Omega} \to [0, \infty)$ defined by $d(A, B) = \nu(A \triangle B)$ satisfies the **triangle inequality**:

$$d(A, C) \leq d(A, B) + d(B, C).$$

Solution

(a) $A \cap C^c \subseteq (A \cap B^c) \cup (B \cap C^c)$.

Proof: Let $x \in A \cap C^c$. Then $x \in A$ and $x \notin C$. We consider two cases:

1. If $x \notin B$, then $x \in A \cap B^c \subseteq (A \cap B^c) \cup (B \cap C^c)$.
2. If $x \in B$, then $x \in B \cap C^c \subseteq (A \cap B^c) \cup (B \cap C^c)$.

In both cases, $x \in (A \cap B^c) \cup (B \cap C^c)$. Therefore, $A \cap C^c \subseteq (A \cap B^c) \cup (B \cap C^c)$. ■

(b) $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

Proof: Recall that $A \triangle C = (A \cap C^c) \cup (C \cap A^c)$.

From part (a), we have:

1. $A \cap C^c \subseteq (A \cap B^c) \cup (B \cap C^c)$
2. $C \cap A^c \subseteq (C \cap B^c) \cup (B \cap A^c)$

Therefore,

$$\begin{aligned}
&A \triangle C = (A \cap C^c) \cup (C \cap A^c) \\
&\subseteq [(A \cap B^c) \cup (B \cap C^c)] \cup [(C \cap B^c) \cup (B \cap A^c)] \\
&= (A \cap B^c) \cup (B \cap A^c) \cup (B \cap C^c) \cup (C \cap B^c) = (A \triangle B) \cup (B \triangle C)
\end{aligned}$$

Thus, $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$. ■

(c) $d(A, C) \leq d(A, B) + d(B, C)$**.**

Proof: From part (b), we know that $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

Since ν is an outer measure, it is monotone and countably subadditive. Therefore,

$$\begin{aligned} d(A, C) &= \nu(A \triangle C) \\ &\leq \nu((A \triangle B) \cup (B \triangle C)) \\ &\leq \nu(A \triangle B) + \nu(B \triangle C) \\ &= d(A, B) + d(B, C) \end{aligned}$$

Thus, $d(A, C) \leq d(A, B) + d(B, C)$, which proves the triangle inequality. ■

Problem 5

Let $\mathcal{A}$ be a field over Ω, and let $\mathbb{P}$ be a probability measure on $(\Omega, \sigma(\mathcal{A}))$. Let $B \in \sigma(\mathcal{A})$. Prove that for any $\epsilon > 0$, there is a set $A \in \mathcal{A}$ such that $\mathbb{P}(A \triangle B) < \epsilon$.
I.e. $\mathcal{A}$ is "dense" in $\sigma(\mathcal{A})$.
[Hint: show that the collection of all sets B satisfying this property is a σ-field.]

Proof

To prove this statement, we will follow an approach similar to the method discussed in class for extending a field to a σ-field.. We'll construct a set analogous to a "compact set" and prove that it is a σ-field, thereby establishing our conclusion.

Define $\bar{\mathcal{A}} = \{B \in \sigma(\mathcal{A}) : \forall \epsilon > 0, \exists A \in \mathcal{A} \text{ such that } \mathbb{P}(A \triangle B) < \epsilon\}$.

We will prove that $\bar{\mathcal{A}}$ is a σ-field.

1. First, $\emptyset \in \bar{\mathcal{A}}$: For any $\epsilon > 0$, choose $A = \emptyset \in \mathcal{A}$. Then $\mathbb{P}(\emptyset \triangle \emptyset) = 0 < \epsilon$.

2. If $B \in \bar{\mathcal{A}}$, then $B^c \in \bar{\mathcal{A}}$: Let $\epsilon > 0$. Since $B \in \bar{\mathcal{A}}$, there exists $A \in \mathcal{A}$ such that $\mathbb{P}(A \triangle B) < \epsilon$.
   Note that $A \triangle B = A^c \triangle B^c$. Therefore, $\mathbb{P}(A^c \triangle B^c) = \mathbb{P}(A \triangle B) < \epsilon$.
   Since $\mathcal{A}$ is a field, $A^c \in \mathcal{A}$. Thus, $B^c \in \bar{\mathcal{A}}$.

3. If $\{B_n\}_{n=1}^{\infty} \subset \bar{\mathcal{A}}$, then $\bigcup_{n=1}^{\infty} B_n \in \bar{\mathcal{A}}$:

Step 1: Let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$. We will show that $A = \bigcup_{n=1}^{\infty} A_n \in \bar{\mathcal{A}}$ and $\lim_{N\to\infty} \mathbb{P}(\bigcup_{n=1}^{N} A_n \triangle A) = 0$.

Let $D_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$. Then $\{D_n\}$ are disjoint and $A = \bigcup_{n=1}^{\infty} D_n$.

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(\bigcup_{n=1}^{\infty} D_n) = \sum_{n=1}^{\infty} \mathbb{P}(D_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{P}(D_n) = \lim_{N\to\infty} \mathbb{P}(\bigcup_{n=1}^{N} D_n)\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}(\bigcup_{n=1}^{N} A_n \triangle A) &= \lim_{N\to\infty} \mathbb{P}(\bigcup_{n=1}^{N} D_n \triangle A) \\ &= \lim_{N\to\infty} \mathbb{P}(A \setminus \bigcup_{n=1}^{N} D_n) \\ &= \lim_{N\to\infty} [\mathbb{P}(A) - \mathbb{P}(\bigcup_{n=1}^{N} D_n)] \\ &= \mathbb{P}(A) - \mathbb{P}(A) = 0\end{aligned}$$

Since $\bigcup_{n=1}^{N} D_n = \bigcup_{n=1}^{N} A_n \in \mathcal{A}$ (by $\mathcal{A}$ is a field) and $\lim_{N\to\infty} \mathbb{P}(\bigcup_{n=1}^{N} A_n \triangle A) = 0$, we have

$$A = \bigcup_{n=1}^{\infty} A_n \in \bar{\mathcal{A}}$$

Step 2: Now, let $B = \bigcup_{n=1}^{\infty} B_n$ where $B_n \in \bar{\mathcal{A}}$. We will show that $B \in \bar{\mathcal{A}}$:

Let $\epsilon > 0$. For each n, choose $A_n \in \mathcal{A}$ such that $\mathbb{P}(A_n \triangle B_n) < \frac{\epsilon}{2^{n+1}}$. Such a choice of A_n is possible since $B_n \in \bar{\mathcal{A}}$.

Define $A = \bigcup_{n=1}^{\infty} A_n$. Let N be a sufficiently large positive integer satisfying:

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n \triangle A\right) < \frac{\epsilon}{2}$$

By step 1 we know that Such an N exists. We then have:

$$\begin{aligned}
\mathbb{P}(\bigcup_{n=1}^{N} A_n \triangle B) &\leq \mathbb{P}(\bigcup_{n=1}^{N} A_n \triangle A) + \mathbb{P}(A \triangle B) \\
&< \frac{\epsilon}{2} + \mathbb{P}(A \triangle B) \\
&= \frac{\epsilon}{2} + \mathbb{P}(\bigcup_{n=1}^{\infty} A_n \triangle B_n) \\
&\leq \frac{\epsilon}{2} + \sum_{n=1}^{\infty} \mathbb{P}(A_n \triangle B_n) \\
&< \frac{\epsilon}{2} + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \epsilon
\end{aligned}$$

Since $\mathcal{A}$ is a field, $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$. Therefore, for any $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mathbb{P}(A \triangle B) < \epsilon$, proving that $B \in \bar{\mathcal{A}}$.

Thus, $\bar{\mathcal{A}}$ is a σ-field. Since $\sigma(\mathcal{A})$ is the smallest σ-field containing $\mathcal{A}$, we have $\sigma(\mathcal{A}) \subset \bar{\mathcal{A}}$. However, by the definition of $\bar{\mathcal{A}}$ we have $\bar{\mathcal{A}} \subset \sigma(\mathcal{A})$, which shows that $\bar{\mathcal{A}} = \sigma(\mathcal{A})$.

Therefore, for any $B \in \sigma(\mathcal{A})$ and any $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mathbb{P}(A \triangle B) < \epsilon$, which completes the proof. ■