Homework 1

Problem 1

(Exercise 1.1.4 in Durrett) Suppose $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are σ-fields over a sample space Ω, and suppose they are nested: $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ Prove that

$$\bigcup_{n=1}^{\infty} \mathcal{F}_n$$

is a field. Is it necessarily a σ-field?

Solution

Part 1: Proving $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a field

Proof:

1. $\Omega \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$:
   Since each $\mathcal{F}_n$ is a σ-field, $\Omega \in \mathcal{F}_n$ for all n. Therefore, $\Omega \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

2. Closure under complementation:
   Let $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then there exists an $m \in \mathbb{N}$ such that $A \in \mathcal{F}_m$. Since $\mathcal{F}_m$ is a σ-field, $A^c \in \mathcal{F}_m$. Therefore, $A^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

3. Closure under finite unions:
   Let $A, B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then there exist $m, n \in \mathbb{N}$ such that $A \in \mathcal{F}_m$ and $B \in \mathcal{F}_n$. Let $k = \max(m, n)$. Since the σ-fields are nested, $A, B \in \mathcal{F}_k$. As $\mathcal{F}_k$ is a σ-field, $A \cup B \in \mathcal{F}_k$. Therefore, $A \cup B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

Thus, $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a field.■

Part 2: $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is not necessarily a σ-field.

We will prove this by constructing a counterexample.

Counterexample

We construct a sequence of strictly nested σ-fields:

Let $\Omega = \mathbb{N}$, and define:

- $\mathcal{F}_1 = \sigma(\{\{1\}\})$
- $\mathcal{F}_2 = \sigma(\{\{1\}, \{2\}\})$
- $\mathcal{F}_3 = \sigma(\{\{1\}, \{2\}, \{3\}\})$
- ...

We assert that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is not a σ-field, since $\{2k\} \in \mathcal{F}_{2k}$ but

$$\bigcup_{k=1}^{\infty} \{2k\} \notin \bigcup_{n=1}^{\infty} \mathcal{F}_n$$

Therefore, we conclude that while $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is always a field, it is not necessarily a σ-field. ■

For part 2, we have a more general result:

Statement: Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a sequence of σ-algebras. If the inclusion $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$ is strict for all n, then $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ **is not** a σ-algebra.

See **The union of a strictly increasing sequence of σ-algebras is not a σ-algebra** for more details.

Problem 2

Let Ω be an infinite set. Let $\mathcal{F}$ denote the collection of all sets $E \subseteq \Omega$ such that either E or E^c is finite.

(a) Is $\mathcal{F}$ a field? Prove that your answer is correct.
(b) Is $\mathcal{F}$ a σ-field? Prove that your answer is correct.

Solution

(a) $\mathcal{F}$ is a field.

Proof:
We will show that $\mathcal{F}$ satisfies the three properties of a field:

1. $\Omega \in \mathcal{F}$:
   The complement of Ω is $\emptyset$, which is finite. Therefore, $\Omega \in \mathcal{F}$.

2. Closure under complementation:
   Let $E \in \mathcal{F}$. By definition, either E or E^c is finite, which means either E^c or $(E^c)^c$ is finite. Therefore, $E^c \in \mathcal{F}$.

3. Closure under finite unions:
   Let $A, B \in \mathcal{F}$. We consider two cases:

   Case 1: If both A and B are finite, then $A \cup B$ is finite, so $A \cup B \in \mathcal{F}$.

   Case 2: If at least one of A or B is infinite, without loss of generality, assume A is infinite.
   Then A^c is finite. We have:

   $$(A \cup B)^c = A^c \cap B^c$$

   Since A^c is finite, $A^c \cap B^c$ is also finite. Therefore, $(A \cup B)^c$ is finite, which implies $A \cup B \in \mathcal{F}$.

Thus, $\mathcal{F}$ satisfies all properties of a field. ■

(b) $\mathcal{F}$ is not necessarily a σ-field.

Proof:
We will provide a counterexample to show that $\mathcal{F}$ is not closed under countable unions.

Let $\Omega = \mathbb{N}$, the set of all natural numbers. Define $E_k = \{2k\}$ for $k \in \mathbb{N}$.

1. Each $E_k \in \mathcal{F}$ because E_k is finite (it contains only one element).

2. Consider $E = \bigcup_{k=1}^{\infty} E_k = \{2, 4, 6, 8, \dots\}$, the set of all even natural numbers.

3. E is infinite, and its complement $E^c = \{1, 3, 5, 7, \dots\}$ (the set of all odd natural numbers) is also infinite.

4. Therefore, $E \notin \mathcal{F}$, as neither E nor E^c is finite.

This demonstrates that $\mathcal{F}$ is not closed under countable unions, and thus is not a σ-field. ■

Problem 3

(Exercise 4.8 in Driver) Let Ω be a set, and let $\mathcal{E}_1, \mathcal{E}_2 \subseteq 2^\Omega$ be collections of subsets of Ω.
Show that $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$ if and only if $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$. Give an example where $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$, but $\mathcal{E}_1 \neq \mathcal{E}_2$.

Solution

Proof of "only if" part:

Suppose $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$. By definition of σ-algebra, $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$. Similarly, $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$. Therefore, $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

Proof of "if" part:

Suppose $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$. Since $\sigma(\mathcal{E}_1)$ is the smallest σ-algebra containing $\mathcal{E}_1$, and $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$, we have $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$. Similarly, we have $\sigma(\mathcal{E}_2) \subseteq \sigma(\mathcal{E}_1)$. Therefore, $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

Thus, $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$ if and only if $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$. ■

Example:

Let $\Omega = \{0, 1\}$. Consider the collections:

- $\mathcal{E}_1 = \{\emptyset, \{0\}, \Omega\}$

- $\mathcal{E}_2 = \{\emptyset, \{0\}, \{1\}, \Omega\}$

Here, $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) = \{\emptyset, \{0\}, \{1\}, \Omega\}$, but clearly $\mathcal{E}_1 \neq \mathcal{E}_2$.

Problem 4

(Exercise 4.9 in Driver) Verify that the Borel σ-field $\mathcal{B}(\mathbb{R})$ is generated by any of the following collections of intervals:

(a) $\mathcal{E}_1 = \{(a, \infty) : a \in \mathbb{R}\}$
(b) $\mathcal{E}_2 = \{(a, \infty) : a \in \mathbb{Q}\}$
(c) $\mathcal{E}_3 = \{[a, \infty) : a \in \mathbb{Q}\}$

Solution to Problem 4

We will verify that the Borel σ-field $\mathcal{B}(\mathbb{R})$ is generated by each of the given collections of intervals.

Preliminaries

Recall that:

1. $\mathcal{B}(\mathbb{R}) = \sigma(\text{all open sets of } \mathbb{R}) = \sigma(\{(a, b] : a < b, a, b \in \mathbb{R}\})$
2. For two collections $\mathcal{E}_1$ and $\mathcal{E}_2$, $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$ if and only if $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$

(a) $\mathcal{E}_1 = \{(a, \infty) : a \in \mathbb{R}\}$

Let $\mathcal{E}_4 = \{(a, b] : a < b, a, b \in \mathbb{R}\}$. We know that $\sigma(\mathcal{E}_4) = \mathcal{B}(\mathbb{R})$.

To prove $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_4)$, we need to show:

1. $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_4)$: $\mathcal{E}_1$ is a subset of all open sets of $\mathbb{R}$, and $\sigma(\mathcal{E}_4) = \sigma(\text{all open sets of } \mathbb{R})$. Thus, $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_4)$.

2. $\mathcal{E}_4 \subseteq \sigma(\mathcal{E}_1)$: For any $(a, b] \in \mathcal{E}_4$, we have $(a, b] = (a, \infty) \cap ((b, \infty))^c$. Since σ-fields are closed under complementation and finite intersection, $(a, b] \in \sigma(\mathcal{E}_1)$.

Therefore, $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_4) = \mathcal{B}(\mathbb{R})$.

(b) $\mathcal{E}_2 = \{(a, \infty) : a \in \mathbb{Q}\}$

We will prove $\sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_1)$ by showing:

1. $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$: This is obvious since $\mathcal{E}_2 \subseteq \mathcal{E}_1$.

2. $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$: For any $(a, \infty) \in \mathcal{E}_1$, by the density of rationals, we can construct a monotonically decreasing sequence of rationals $\{a_n\}_{n=1}^{\infty}$ converging to a. Then:

$$(a, \infty) = \bigcup_{n=1}^{\infty} (a_n, \infty)$$

   Since each $(a_n, \infty) \in \mathcal{E}_2$ and σ-fields are closed under countable unions, $(a, \infty) \in \sigma(\mathcal{E}_2)$.

Therefore, $\sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_1) = \mathcal{B}(\mathbb{R})$.

(c) $\mathcal{E}_3 = \{[a, \infty) : a \in \mathbb{Q}\}$

We will prove $\sigma(\mathcal{E}_3) = \sigma(\mathcal{E}_2)$ by showing:

1. $\mathcal{E}_3 \subseteq \sigma(\mathcal{E}_2)$: For any $[a, \infty) \in \mathcal{E}_3$, we can write:

$$[a, \infty) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty)$$

   Since $a \in \mathbb{Q}$, $a - \frac{1}{n} \in \mathbb{Q}$ for all $n \in \mathbb{N}$. Thus, $(a - \frac{1}{n}, \infty) \in \mathcal{E}_2$ for all n. As σ-fields are closed under countable intersections, $[a, \infty) \in \sigma(\mathcal{E}_2)$.

2. $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_3)$: For any $(a, \infty) \in \mathcal{E}_2$, we can write:

$$(a, \infty) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, \infty)$$

   Since $a \in \mathbb{Q}$, $a + \frac{1}{n} \in \mathbb{Q}$ for all $n \in \mathbb{N}$. Thus, $[a + \frac{1}{n}, \infty) \in \mathcal{E}_3$ for all n. As σ-

fields are closed under countable unions, $(a, \infty) \in \sigma(\mathcal{E}_3)$.

Therefore, $\sigma(\mathcal{E}_3) = \sigma(\mathcal{E}_2) = \mathcal{B}(\mathbb{R})$.

Thus, we have verified that the Borel σ-field $\mathcal{B}(\mathbb{R})$ is generated by each of the given collections of intervals. ■

Problem 5

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and suppose $A \in \mathcal{F}$ satisfies $P(A) > 0$. Define

$$\mathcal{F}_A := \{B \in \mathcal{F} : B \subseteq A\}, \text{ and for } B \in \mathcal{F}_A \text{ define } P_A(B) = \frac{P(B)}{P(A)}.$$

Prove that $(A, \mathcal{F}_A, P_A)$ is a probability space.

Solution

Proof:
We need to prove that $(A, \mathcal{F}_A, P_A)$ is a probability space. To do this, we will first show that $\mathcal{F}_A$ is a σ-algebra on A, and then demonstrate that P_A is a probability measure on $\mathcal{F}_A$.

Part 1: $\mathcal{F}_A$ is a σ-algebra on A

1. $A \in \mathcal{F}_A$: Since $A \subseteq A$ and $A \in \mathcal{F}$, we have $A \in \mathcal{F}_A$ by definition.

2. $\mathcal{F}_A$ is closed under complementation: Let $B \in \mathcal{F}_A$. We need to show that $B_A^c := A \setminus B \in \mathcal{F}_A$. We have

$$B_A^c = A \setminus B = A \cap B^c$$

   Since $B \in \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra, $B^c \in \mathcal{F}$. As $\mathcal{F}$ is closed under finite intersections, $A \cap B^c \in \mathcal{F}$. Besides, $A \cap B^c \subseteq A$. So by definition of $\mathcal{F}_A$, we have $B_A^c \in \mathcal{F}_A$.

3. $\mathcal{F}_A$ is closed under countable unions: Let $\{B_i\}_{i=1}^{\infty}$ be a countable collection of sets in $\mathcal{F}_A$. For each i, $B_i \in \mathcal{F}$ and $B_i \subseteq A$. Since $\mathcal{F}$ is a σ-algebra, $\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}$. Moreover, $\bigcup_{i=1}^{\infty} B_i \subseteq A$ since each $B_i \subseteq A$. Therefore, $\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}_A$.

Thus, $\mathcal{F}_A$ is a σ-algebra on A.

Part 2: P_A is a probability measure on $\mathcal{F}_A$

1. Non-negativity: For any $B \in \mathcal{F}_A$,

$$P_A(B) = \frac{P(B)}{P(A)} \geq 0$$

   since $P(B) \geq 0$ and $P(A) > 0$.

2. Countable additivity: Let $\{B_i\}_{i=1}^{\infty}$ be a countable collection of disjoint sets in $\mathcal{F}_A$. Then,

$$\begin{aligned} P_A(\bigcup_{i=1}^{\infty} B_i) &= \frac{P(\bigcup_{i=1}^{\infty} B_i)}{P(A)} \\ &= \frac{1}{P(A)} \sum_{i=1}^{\infty} P(B_i) \quad \text{(by countable additivity of } P) \\ &= \sum_{i=1}^{\infty} \frac{P(B_i)}{P(A)} = \sum_{i=1}^{\infty} P_A(B_i) \end{aligned}$$

3. Normalization:

$$P_A(A) = \frac{P(A)}{P(A)} = 1$$

Therefore, P_A is a probability measure on $\mathcal{F}_A$.

Since $\mathcal{F}_A$ is a σ-algebra on A and P_A is a probability measure on $\mathcal{F}_A$, we conclude that $(A, \mathcal{F}_A, P_A)$ is indeed a probability space. ■