Notes from Mr.Honner's Linear Algebra Class

Kellen Yu — Stuyvesant High School

2024 - 2024

Contents

0.1 Systems of Equations 3
  0.1.1 Ways to think about it 3
0.2 Vectors 4
  0.2.1 Vector operations 5
0.3 Plane Vectors 6
0.4 More Plane Vectors 7
0.5 Barycentric Coordinates 7

0.1 Systems of Equations

0.1.1 Ways to think about it

Given some system of n equations with m unknowns:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,m}x_m &= k_1 \qquad (1) \\ a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,m}x_m &= k_2 \\ &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + ... + a_{n,m}x_m &= k_n \end{aligned}$$

We can think of each variable x_i as a "degree of freedom", and each equation as a restriction on the system. Linearly combining equations not only preserves solutions, but has the ability to free up restrictions in the system.

Solving this system of n equations with m unknowns can end in 3 distinct ways:

1. One solution
2. No solutions
3. Infinitely many solutions

But we can also describe our system as either **dependent** or **independent** and **consistent** or **inconsistent**. What do these mean? In the context of systems of equations, dependence means that there are equations inside of our system can be described as linear combinations of other equations also with the system, and independence means the opposite of this. Consistence means that there exists a solution, and inconsistence means that no solution exists.

0.2 Vectors

Vectors are mathematical objects with a **direction** and **magnitude**. A more formal definition of a vector is that a vector is an element of a **vector space**.

Lets work with a real-n vector $\vec{u}$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The **magnitude** or **norm** or $\vec{u}$ is:

$$\|\vec{u}\| = \sqrt{{u_1}^2 + {u_2}^2 + \ldots + {u_n}^2}$$

Notice that:

$$\begin{aligned} \|c\vec{u}\| &= \sqrt{c{u_1}^2 + c{u_2}^2 + \ldots + c{u_n}^2} \\ &= \sqrt{c^2({u_1}^2 + {u_2}^2 + \ldots + {u_n}^2)} \\ &= c\sqrt{{u_1}^2 + {u_2}^2 + \ldots + {u_n}^2} \\ \|c\vec{u}\| &= c\,\|\vec{u}\| \end{aligned}$$

A vector with a magnitude of 1 is considered to be a **unit vector**. To turn any vector in a unit vector, also known as **normalizing** the vectorm, and essentially isolate its "direction":

$$\vec{u}_{norm} = \frac{\vec{u}}{\|\vec{u}\|}$$
$$\|\vec{u}_{norm}\| = 1$$

0.2.1 Vector operations

Adding vectors is component wise:

$$\vec{u} + \vec{v} = \sum_{i=1}^{n} u_i + v_i$$

From this definition of vector addition, for real-n vectors we can say that vector addition is **commutative** and **associative**. We can also say that scalar multiplication is **distributive** and **associative**.

The dot product of two vectors is defined as:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

Through this definition, we can realize some key properties of the dot product:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$
$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$
$$c\vec{u} \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$
$$\vec{u} \cdot \vec{u} = \|u\|^2$$
$$\vec{u} \cdot \vec{0} = 0$$

But lets think about this some more. We know that $\vec{v} = \begin{bmatrix} sin(\theta) \\ cos(\theta) \end{bmatrix}$ is a unit vector. (Think the pythagorean identity: $sin^2(\theta) + cos^2(\theta) = 1$). We can express every vector as its magnitude multiplied by its "direction" (its normalized version):

$$\vec{v} = \|\vec{v}\| \frac{\vec{v}}{\|\vec{v}\|}$$
$$\vec{v} = \|\vec{v}\| \begin{bmatrix} sin(\alpha) \\ cos(\alpha) \end{bmatrix}$$

Now lets introduce another vector $\vec{u} = \|\vec{u}\| \begin{bmatrix} sin(\beta) \\ cos(\beta) \end{bmatrix}$:

$$\begin{aligned} \vec{v} \cdot \vec{u} &= \|\vec{v}\| \, \|\vec{u}\| \, (sin(\alpha)sin(\beta) + cos(\alpha)cos(\beta)) \\ &= \|\vec{v}\| \, \|\vec{u}\| \, cos(\alpha - \beta) \\ \theta = \alpha - \beta & \\ \vec{v} \cdot \vec{u} &= \|\vec{v}\| \, \|\vec{u}\| \, cos(\theta) \end{aligned}$$

If $\vec{u} \cdot \vec{v} = 0$, and we know that $\|\vec{v}\|, \|\vec{u}\| \neq 0$, then $cos(\theta) = 0$ which means $\theta = \frac{\pi}{2}$. In other words, if the dot product of two vectors is 0, then they are **orthogonal**.

Taking this definition further:

$$cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

0.3 Plane Vectors

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the Cartesian plane, the definition of the plane vector $\vec{PQ}$ is:

$$\vec{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

Some properties of plane vectors:

$$\vec{PQ} + \vec{QW} = \vec{PW} \tag{2}$$
$$\vec{PQ} = -\vec{QP}$$
$$\vec{PP} = \vec{0}$$

0.4 More Plane Vectors

0.5 Barycentric Coordinates