Notes from Mr.Honner's Linear Algebra Class

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Contents

0.1 Systems of Equations

0.1.1 Ways to think about it

Given some system of n equations with m unknowns:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,m}x_m &= k_1 \qquad (1) \\ a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,m}x_m &= k_2 \\ &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + ... + a_{n,m}x_m &= k_n \end{aligned}$$

We can think of each variable x_i as a "degree of freedom", and each equation as a restriction on the system. Linearly combining equations not only preserves solutions, but has the ability to free up restrictions in the system.

Solving this system of n equations with m unknowns can end in 3 distinct ways:

1. One solution
2. No solutions
3. Infinitely many solutions

But we can also describe our system as either **dependent** or **independent** and **consistent** or **inconsistent**. What do these mean? In the context of systems of equations, dependence means that there are equations inside of our system can be described as linear combinations of other equations also with the system, and independence means the opposite of this. Consistence means that there exists a solution, and inconsistence means that no solution exists.

0.2 Vectors

Vectors are mathematical objects with a **direction** and **magnitude**. A more formal definition of a vector is that a vector is an element of a **vector space**.

Lets work with a real-n vector $\vec{u}$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The **magnitude** or **norm** or $\vec{u}$ is:

$$\|\vec{u}\| = \sqrt{{u_1}^2 + {u_2}^2 + \ldots + {u_n}^2}$$

Notice that:

$$\begin{aligned} \|c\vec{u}\| &= \sqrt{c{u_1}^2 + c{u_2}^2 + \ldots + c{u_n}^2} \\ &= \sqrt{c^2({u_1}^2 + {u_2}^2 + \ldots + {u_n}^2)} \\ &= c\sqrt{{u_1}^2 + {u_2}^2 + \ldots + {u_n}^2} \\ \|c\vec{u}\| &= c\,\|\vec{u}\| \end{aligned}$$

A vector with a magnitude of 1 is considered to be a **unit vector**. To turn any vector in a unit vector, also known as **normalizing** the vectorm, and essentially isolate its "direction":

$$\vec{u}_{norm} = \frac{\vec{u}}{\|\vec{u}\|}$$
$$\|\vec{u}_{norm}\| = 1$$

0.2.1 Vector operations

Adding vectors is component wise:

$$\vec{u} + \vec{v} = \sum_{i=1}^{n} u_i + v_i$$

From this definition of vector addition, for real-n vectors we can say that vector addition is **commutative** and **associative**. We can also say that scalar multiplication is **distributive** and **associative**.

The dot product of two vectors is defined as:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

Through this definition, we can realize some key properties of the dot product:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$
$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$
$$c\vec{u} \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$
$$\vec{u} \cdot \vec{u} = \|u\|^2$$
$$\vec{u} \cdot \vec{0} = 0$$

But lets think about this some more. We know that $\vec{v} = \begin{bmatrix} sin(\theta) \\ cos(\theta) \end{bmatrix}$ is a unit vector. (Think the pythagorean identity: $sin^2(\theta) + cos^2(\theta) = 1$). We can express every vector as its magnitude multiplied by its "direction" (its normalized version):

$$\vec{v} = \|\vec{v}\| \frac{\vec{v}}{\|\vec{v}\|}$$
$$\vec{v} = \|\vec{v}\| \begin{bmatrix} sin(\alpha) \\ cos(\alpha) \end{bmatrix}$$

Now lets introduce another vector $\vec{u} = \|\vec{u}\| \begin{bmatrix} sin(\beta) \\ cos(\beta) \end{bmatrix}$:

$$\begin{aligned} \vec{v} \cdot \vec{u} &= \|\vec{v}\| \, \|\vec{u}\| \, (sin(\alpha)sin(\beta) + cos(\alpha)cos(\beta)) \\ &= \|\vec{v}\| \, \|\vec{u}\| \, cos(\alpha - \beta) \\ \theta = \alpha - \beta & \\ \vec{v} \cdot \vec{u} &= \|\vec{v}\| \, \|\vec{u}\| \, cos(\theta) \end{aligned}$$

If $\vec{u} \cdot \vec{v} = 0$, and we know that $\|\vec{v}\|, \|\vec{u}\| \neq 0$, then $cos(\theta) = 0$ which means $\theta = \frac{\pi}{2}$. In other words, if the dot product of two vectors is 0, then they are **orthogonal**.

Taking this definition further:

$$cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

0.3 Plane Vectors

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the Cartesian plane, the definition of the plane vector $\vec{PQ}$ is:

$$\vec{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

Some properties of plane vectors:

$$\vec{PQ} + \vec{QW} = \vec{PW} \tag{2}$$
$$\vec{PQ} = -\vec{QP}$$
$$\vec{PP} = \vec{0}$$

0.4 More Plane Vectors

0.5 Barycentric Coordinates

0.6 Linear Functions

This is a linear algebra course. But what does linear even mean? It means alot of things, but regarding functions and transformations there is a very concrete definition for linearity.

0.6.1 Requirements of linear functions

Given a linear function $f : \mathbb{R} \to \mathbb{R}$, f is linear if it satisfies the following:

$$\begin{aligned} c \in \mathbb{R}, \forall x &: f(cx) = cf(x) \\ \forall x, y &: f(x+y) = f(x) + f(y) \end{aligned}$$

One thing we can notice is that under these conditions $f(0) = 0$ is always true. This means that the range of linear functions must contain zero. But why do we care about this? Why is knowing these conditions useful? What power do linear functions hold?

Lets look at a linear function $f : \mathbb{C} \to \mathbb{C}$. Lets say that z is some complex number $a + bi$.

$$\begin{aligned} f(z) &= f(a+bi) \\ &= f(a+0i) + f(0+bi) \\ &= af(1) + bf(i) \end{aligned}$$

We can see that if we know what f does to 1 and i, we know what it does to all complex numbers. Within linear functions is the structure of linear combination. The result of applying a linear function can be seen as a linear combination of other functions being applied.

0.7 Matrix Multiplication

Matrix multiplication is an interesting thing.

0.7.1 Proving matrix multiplication is associative

Given three matricies A, B, and C, where A is of size $n \times r$, B is of size $r \times q$ and C is of size $q \times m$, we will show that (AB)C = A(BC). Note that we can represent any element of A in the form $(a_{i,j})$ (this follows for B and C).

Let M be the product of A and B.

$$m_{i,j} = \sum_{k=1}^{r} a_{i,k} \cdot b_{k,j}$$

Let N be the product of M and C:

$$\begin{aligned} n_{i,j} &= \sum_{l=1}^{q} m_{i,l} \cdot c_{l,j} \\ &= \sum_{l=1}^{q} \left(\sum_{k=1}^{r} a_{i,k} \cdot b_{k,l} \right) \cdot c_{l,j} \\ &= \sum_{k=1}^{r} a_{i,k} \cdot \left(\sum_{l=1}^{q} b_{k,l} \cdot c_{l,j} \right) \end{aligned}$$

To explain why we can sort of "pull out" the $b_{k,l}$ from its summation is simply the distributive property. $c_{l,j}$ is being multiplied by the sum $a_{i,1} \cdot b_{1,l} + a_{i,2} \cdot b_{2,l} + ...$, which would be no different from multiplying $a_{i,k}$ by the sum $b_{k,1} \cdot c_{1,j} + b_{k,1} \cdot c_{1,j} + ...$ We can see that the summation within the parentheses looks very similar to out definition to our definition of the product AB. It turns out that this is the definition of the product BC. And therefore:

$$\mathrm{(AB)\,C = A\,(BC)}$$

Matrix multiplication is associative.