Vectors are mathematical objects with a **direction** and **magnitude**. A more formal definition of a vector is that a vector is an element of a **vector space**.

Lets work with a real-n vector $\vec{u}$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The **magnitude** or **norm** or $\vec{u}$ is:

$$\|\vec{u}\| = \sqrt{{u_1}^2 + {u_2}^2 + \ldots + {u_n}^2}$$

Notice that:

$$\begin{aligned} \|c\vec{u}\| &= \sqrt{c{u_1}^2 + c{u_2}^2 + \ldots + c{u_n}^2} \\ &= \sqrt{c^2({u_1}^2 + {u_2}^2 + \ldots + {u_n}^2)} \\ &= c\sqrt{{u_1}^2 + {u_2}^2 + \ldots + {u_n}^2} \\ \|c\vec{u}\| &= c\,\|\vec{u}\| \end{aligned}$$

A vector with a magnitude of 1 is considered to be a **unit vector**. To turn any vector in a unit vector, also known as **normalizing** the vectorm, and essentially isolate its "direction":

$$\vec{u}_{norm} = \frac{\vec{u}}{\|\vec{u}\|}$$
$$\|\vec{u}_{norm}\| = 1$$

0.0.1 Vector operations

Adding vectors is component wise:

$$\vec{u} + \vec{v} = \sum_{i=1}^{n} u_i + v_i$$

From this definition of vector addition, for real-n vectors we can say that vector addition is **commutative** and **associative**. We can also say that scalar multiplication is **distributive** and **associative**.

The dot product of two vectors is defined as:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

Through this definition, we can realize some key properties of the dot product:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u} \\ \vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\ c\vec{u} \cdot \vec{v} &= c(\vec{u} \cdot \vec{v}) \\ \vec{u} \cdot \vec{u} &= \|u\|^2 \\ \vec{u} \cdot \vec{0} &= 0 \end{aligned}$$

But lets think about this some more. We know that $\vec{v} = \begin{bmatrix} sin(\theta) \\ cos(\theta) \end{bmatrix}$ is a unit vector. (Think the pythagorean identity: $sin^2(\theta) + cos^2(\theta) = 1$). We can express every vector as its magnitude multiplied by its "direction" (its normalized version):

$$\begin{aligned} \vec{v} &= \|\vec{v}\| \frac{\vec{v}}{\|\vec{v}\|} \\ \vec{v} &= \|\vec{v}\| \begin{bmatrix} sin(\alpha) \\ cos(\alpha) \end{bmatrix} \end{aligned}$$

Now lets introduce another vector $\vec{u} = \|\vec{u}\| \begin{bmatrix} sin(\beta) \\ cos(\beta) \end{bmatrix}$:

$$\begin{aligned} \vec{v} \cdot \vec{u} &= \|\vec{v}\| \|\vec{u}\| (sin(\alpha)sin(\beta) + cos(\alpha)cos(\beta)) \\ &= \|\vec{v}\| \|\vec{u}\| cos(\alpha - \beta) \\ \theta = \alpha - \beta & \\ \vec{v} \cdot \vec{u} &= \|\vec{v}\| \|\vec{u}\| cos(\theta) \end{aligned}$$

If $\vec{u} \cdot \vec{v} = 0$, and we know that $\|\vec{v}\|, \|\vec{u}\| \neq 0$, then $cos(\theta) = 0$ which means $\theta = \frac{\pi}{2}$. In other words, if the dot product of two vectors is 0, then they are

orthogonal.

Taking this definition further:

$$cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$