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DSE 210

HW 3 - Worksheet 6 - Generative Models 2

① pair of r.v's are positive correlated, negatively correlated or uncorrelated?

ⓐ weight of new car and its price.

I would argue uncorrelated. Some very light cars are super expensive. On the other hand, heavy duty pickup trucks are very heavy but way less expensive than lets say a ferrari. But then some luxury SUV's are both heavy and expensive. Etc.

ⓑ weight of car and # of seats in it.

Overall would argue a positive correlation. SUV and Minivans and in general cars w/ more seats, require a larger size, and therefore heavier weight. But there could be outliers such as big trucks and Uhauls where they have only 2 seats, but very heavier to transport alot of stuff.

ⓒ age in years of a second-hand car and its current market value

would argue negative correlation. As age of car increases, value generally declines, as more years generally means more miles on car and higher mileage cars need more maintenance which can be costly. Also newer cars generally tend to have newer and more wanted features vs older cars.

② consider a population of married couples in which every wife is exactly 0.9 of her husband's age. What is correlation between husband and wife's age?

lets define. R.V's H = husbands age W = wifes age

$W = 0.9H$

since one R.V is a linear transformation of other, correlation = 1

$$\text{corr}(H, 0.9H) = \frac{\text{cov}(H, 0.9H)}{\text{std}(H)\,\text{std}(0.9H)} = \frac{\text{cov}(H, 0.9H)}{\sqrt{\text{var}(H)} \cdot \sqrt{\text{var}(0.9H)}}$$

since $\text{cov}(x, bx) = b\,\text{cov}(x, x) = b\,\text{Var}(x)$

then $\text{corr}(H, 0.9H) = \dfrac{0.9\,\text{Var}(x)}{\sqrt{\text{var}(x)} \cdot \sqrt{(0.9)^2\,\text{var}(x)}} = \dfrac{0.9\,\text{var}(x)}{0.9\,\text{var}(x)} = 1$ ✓

we know this by theorem $\text{cov}(ax, by) = ab\cdot\text{cov}(x, y)$

③ each following scenarios describe joint distribution (x,y). In each case give params of the (unique) bivariate gaussian that satisfies properties

ⓐ x has mean 2 and std 1, y has mean 2, and std = 0.5, correlation between x and y is -0.5

so $\mu_x = 2$ $\mu_y = 2$ $std(x) = 1$ $std(y) = 0.5$

$cov(x,y) = corr(x,y) \cdot std(x) \cdot std(y)$

$cov(x,y) = (-0.5)(1)(0.5) = -0.25$

$var(x) = (1)^2 = 1$ $var(y) = (0.5)^2 = 0.25$

$$\boxed{\mu = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{cov matrix } \Sigma = \begin{pmatrix} 1 & -0.25 \\ -0.25 & 0.25 \end{pmatrix}}$$

ⓑ x has mean 1 and std 1, y is equal to x so $(Y = x)$

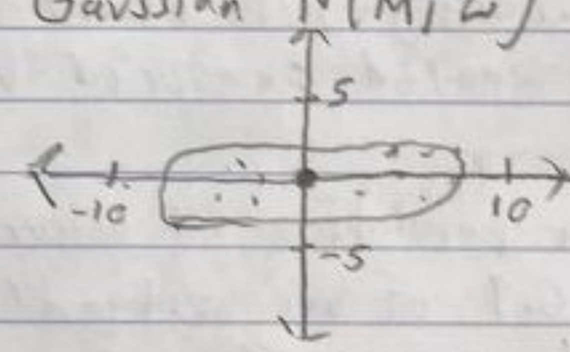
$\mu_x = 1$ $\mu_y = 1$ $std(x) = 1$ $std(y) = 1$ $cov(x,y) = cov(x,x) = var(x) = 1$

$corr(x,y) = 1$

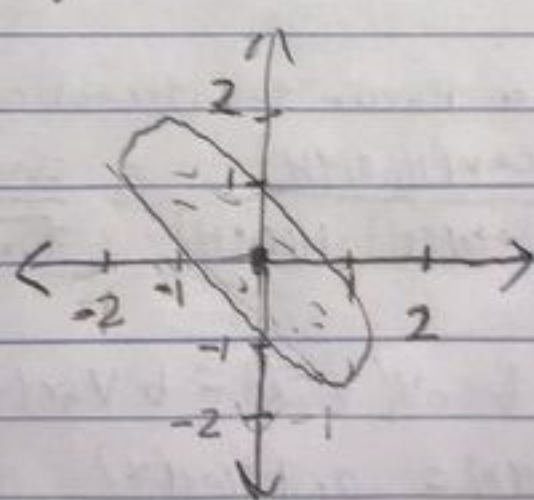
$$\boxed{\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{cov. matrix } \Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

④ sketch shape of following Gaussian $N(\mu, \Sigma)$

(a) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$



(b) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 1 & -0.75 \\ -0.75 & 1 \end{pmatrix}$



⑤ See github repo

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HW 3 - Worksheet 7 - Linear Algebra Primer

① Find unit vector in same direction as $x = (1, 2, 3)$

$\|x\| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}$

$Y = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

lets double check $\|Y\| = 1$

$\|Y\| = \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2$

$\|Y\| = \frac{1}{14} + \frac{4}{14} + \frac{9}{14} = \frac{14}{14} = 1$ ✓

② Find all unit vectors in $\mathbb{R}^2$ that are orthogonal to $(1,1)$

to find orthogonal need to find $(a,b) \cdot (1,1) = 0$

so $1a + 1b = 0 \rightarrow a = -b$

so orthogonal vectors, lets pick $(1,-1)$ and $(-1,1)$

now we need to convert vectors to unity vectors

let $x = (1,-1)$, then $\|x\| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

then $Y = (-1, 1)$ then $\|Y\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

so units vector in $\mathbb{R}^2$ that are orthogonal to $(1,1)$

are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

③ How would you describe set all points $x \in \mathbb{R}^d$ with $x \cdot x = 25$

$x \cdot x = \|x\|^2$ so $\|x\| = 5$

$25 = x_1^2 + x_2^2 + \cdots x_d^2$ any point whose length is 5, $(\|x\| = 5)$

④ function $f(x) = 2x_1 - x_2 + 6x_3$ can be written as $w \cdot x$ for $x \in \mathbb{R}^3$

What is w?

$w = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$

double check $\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} (x_1, x_2, x_3) = 2x_1 - x_2 + 6x_3$ ✓

⑧ For $x = (1, 3, 5)$ compute $x^T x$ and $x x^T$

$x^T x = x \cdot x = \|x\|^2 \quad \|x\| = \sqrt{(1)^2 + (3)^2 + (5)^2} = \sqrt{35}$ so $\|x\|^2 = 35 = x^T x$

then for $x \cdot x^T$

$\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} (1\ 3\ 5) = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 3 \cdot 3 & 3 \cdot 5 \\ 5 \cdot 1 & 5 \cdot 3 & 5 \cdot 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 9 & 15 \\ 5 & 15 & 25 \end{pmatrix} = x \cdot x^T$

⑨ vectors $x, y \in \mathbb{R}^d$ both have length 2. If $x^T y = 2$, what is the angle between x and y? $\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$ and $x^T y = x \cdot y$ $\|x\| = 2$ $\|y\| = 2$

so $\cos\theta = \frac{2}{(2)(2)}$ $\cos\theta = \frac{1}{2}$ $\boxed{\theta = 60^\circ}$

⑩ quadratic function $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$ can be written in form $x^T M x$ for some symmetric matrix M. What is M?

let $\boxed{M = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & 6 \end{pmatrix}}$ then $M^T = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & 6 \end{pmatrix}$ so symmetric ✓

$x^T M x = 3x^1 + 1x_1x_2 - 2x_1x_3 + 1x_2x_1 - 2x_3x_1 + 6x_3^2 = \begin{matrix} 3x_1^2 + 2x_1x_2 \\ -4x_1x_3 + 6x_3^2 \end{matrix}$

⑫ let $A = \text{diag}(1,2,3,4,5,6,7,8)$

(a) what is $|A|$? since diagonal matrix, the determinant is product of each diagonal element $|A| = 8! = 40320$

(b) What is A^{-1}? since diagonal matrix $A^{-1} = \text{diag}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8})$

⑭ Matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & z \end{pmatrix}$ is singular, what is z?

$$\begin{array}{cc} a & b \\ c & d \end{array}$$

singular matrix is a matrix that does not have an inverse

so we want to find z s.t $|A| = 0$

$|A| = ad - bc$ so $0 = (1)(z) - (2)(3)$

$0 = z - 6$

$\boxed{z = 6}$