Review Sheet 17

1) a. $\|-3\| = \sqrt{\langle -3,-3\rangle} = \sqrt{(-3)(-3)} = 3$

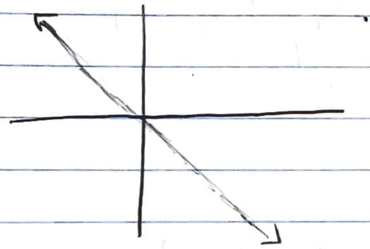
b. $\|3+4i\| = \sqrt{(3+4i)(3-4i)} = \sqrt{9+16} = 5$

c. $\|(3,4)\| = \sqrt{9+16} = 5$

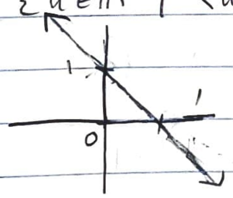
d. $\|(1+i, i\sqrt{7})\| = \sqrt{2+7} = 3$

2) $\mathbb{R}^2$, $v=(1,1)$.

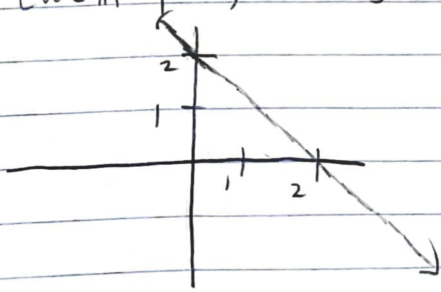
a. $\{u \in \mathbb{R}^2 \mid \langle u,v\rangle = 0\}$



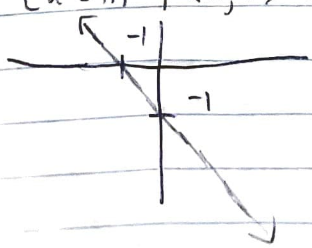
b. $\{u \in \mathbb{R}^2 \mid \langle u,v\rangle = 1\}$ $\quad u=(u_1,u_2) \rightarrow \langle u,v\rangle = u_1+u_2 = 1$



c. $\{u \in \mathbb{R}^2 \mid \langle u,v\rangle = 2\}$ $\quad u_1+u_2=2$



d. $\{u \in \mathbb{R}^2 \mid \langle u,v\rangle = -1\}$



3) Prove that for positive reals $c_1, \ldots, c_n$,

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = c_1 w_1 \bar{z}_1 + \ldots + c_n w_n \bar{z}_n$$

is an inner product.

- Conjugate symmetry: observe that $\bar{\bar{z}} = z$ (conjugate of conjugate is original). Then

$$\overline{\langle z, w \rangle} = \overline{c_1 z_1 \bar{w}_1 + \ldots + c_n z_n \bar{w}_n}$$
$$= \bar{c}_1 \bar{z}_1 w_1 + \ldots + \bar{c}_n \bar{z}_n w_n$$
$$= c_1 w_1 \bar{z}_1 + \ldots + c_n w_n \bar{z}_n,$$

since for any real number $c \in \mathbb{R}$, $c = \bar{c}$.

- Positive definiteness:

$$\langle z, z \rangle = c_1 z_1 \bar{z}_1 + \ldots + c_n z_n \bar{z}_n$$
$$= c_1 |z_1| + \ldots + c_n |z_n| \geq 0,$$

since $c_i > 0$, and $|z_i| \geq 0$ for any $z_i \in \mathbb{F}$. Recall that $|z_i| = 0$ iff $z_i = 0$; thus $\langle z, z \rangle = 0$ iff $z = 0$.

- Linearity in 1st slot: Let $\lambda_1, \lambda_2 \in \mathbb{F}$, $u, w, z \in \mathbb{F}^n$. Then

$$\langle \lambda_1 u + \lambda_2 w, z \rangle = \sum_{i=1}^{n} c_i (\lambda_1 u_i + \lambda_2 w_i) \bar{z}_i$$
$$= \sum_{i=1}^{n} (c_i \lambda_1 u_i + c_i \lambda_2 w_i) \bar{z}_i$$
$$= \sum_{i=1}^{n} c_i \lambda_1 u_i \bar{z}_i + c_i \lambda_2 w_i \bar{z}_i$$
$$= \lambda_1 \sum_{i=1}^{n} c_i u_i \bar{z}_i + \lambda_2 \sum_{i=1}^{n} c_i w_i \bar{z}_i$$
$$= \lambda_1 \langle u, z \rangle + \lambda_2 \langle w, z \rangle$$

Thus all three properties hold, and so

$$\langle w, z \rangle = \sum_{i=1}^{n} c_i w_i \bar{z}_i$$

is an inner product on $\mathbb{F}^n$.