Problem §2 Consider the following list of vectors in $\mathbb{R}^3$. From left to right, remove each vector if it is in the span of the vectors to the left of it. What list of vectors remains?

$$(1,1,0),\ (-2,-2,0),\ (0,0,0),\ (0,0,4),\ (1,1,1),\ (4,0,0),\ (0,1,1),\ (4,-5,3),\ (7,2,2).$$

Solution: $(1,1,0)$ remains. $(-2,-2,0)$, $(0,0,0)$ are removed. $(0,0,4)$ remains. $(1,1,1)$ is removed. $(4,0,0)$ remains. $(0,1,1)$, $(4,-5,3)$, $(7,2,2)$ are removed. Thus, the final list is

$$(1,1,0),\ (0,0,4),\ (4,0,0).$$

Problem §3 Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Prove the following statements briefly.

(a) If $\dim V > 0$, then V has a non-zero vector.

(b) If $\dim V = 0$, then $V = \{0\}$.

(c) If U is a subspace of V, then $\dim U \leq \dim V$.

(d) If $T : V \to W$ is injective, then $\dim V = \dim \operatorname{range} T$. Then, conclude that if W is finite-dimensional, $\dim V \leq \dim W$.

(e) If $T : V \to W$ is surjective, then $\dim V \geq \dim W$.

(f) If $T : V \to W$ is bijective, then $\dim V = \dim W$.

Solution:

(a) If $\dim V = n > 0$, then V has a length-n basis with non-zero vectors $v_1, \ldots, v_n$ (if $v_i = 0$, then the list would be linearly dependent and thus not a basis). Thus there exists a non-zero $v \in \operatorname{span}(v_1, \ldots, v_n) = V$.

(b) If $\dim V = 0$, then V only has basis () (the empty basis). Since the span of the empty basis is just $\{0\}$, we have $V = \{0\}$.

(c) If U is a subspace of V, then the basis of U is a linearly independent (not necessarily spanning) list of vectors in V. Thus the basis of V has at least as many elements as the basis of U, and so $\dim U \leq \dim V$.

(d) If $T : V \to W$ is injective, then T has a trivial kernel (that is, $\operatorname{null} T = \{0\}$, and so $\dim \operatorname{null} T = 0$); by the rank-nullity theorem, we have $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = 0 + \dim \operatorname{range} T = \dim \operatorname{range} T$, as required. Moreover, since $\operatorname{range} T$ is a subspace of W, by part (c), we have $\dim \operatorname{range} T = \dim V \leq \dim W$.

(e) If $T : V \to W$ is surjective, then $W = \operatorname{range} T$; by the rank-nullity theorem, we have $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \geq \dim \operatorname{range} T = \dim W$, as required.

(f) By part (d), $\dim V = \dim \operatorname{range} T$; and by surjectivity, $\operatorname{range} T = W$. Hence $\dim V = \dim \operatorname{range} T = \dim W$, as required.