

Problem §1 Let $S, T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear maps determined by

$$S(1, 1) = (2, -1, 3),$$

$$S(1, -1) = (2, -2, 4)$$

$$T(1, 0) = (4, 0, 0),$$

$$T(0, 1) = (-4, -1, 0).$$

Compute:

(a) $(-S)(4, 0)$

(b) $(-S + T)(4, 0)$

(c) $(-S + T)(4, 0) + (-S + T)(1, 3) + (-S + T)(-2, -2) + (-S + T)(-1, -1)$

(d) $\mathcal{M}(-S)$

(e) $\mathcal{M}(T)$

(f) $-\mathcal{M}(S) + \mathcal{M}(T)$

Solution: We first find the matrices of S and T with respect to the standard basis:

$$\mathcal{M}(S) = \begin{pmatrix} 2 & 0 \\ -\frac{3}{2} & \frac{1}{2} \\ \frac{7}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\mathcal{M}(T) = \begin{pmatrix} 4 & -4 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

(a) $-S(4, 0) = (-8, 6, -14).$

(b) $(-S + T)(4, 0) = (8, 6, -14).$

(c) $(-S + T)(4, 0) + (-S + T)(1, 3) + (-S + T)(-2, -2) + (-S + T)(-1, -1) = (-16, 12, -28).$

(d) $\mathcal{M}(-S) = \begin{pmatrix} -2 & 0 \\ \frac{3}{2} & -\frac{1}{2} \\ -\frac{7}{2} & \frac{1}{2} \end{pmatrix}$

(e) $\mathcal{M}(T) = \begin{pmatrix} 4 & -4 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$

(f) $-\mathcal{M}(S) + \mathcal{M}(T) = \begin{pmatrix} 2 & -4 \\ \frac{3}{2} & -\frac{3}{2} \\ -\frac{7}{2} & \frac{1}{2} \end{pmatrix}$

Problem §2 Write the linear map

$$T : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$$

given by

$$T(f) = f + f' + f''$$

as a matrix with respect to the basis

$$1, x - 2, (x - 2)^2, (x - 2)^3$$

for both the domain and codomain.

Solution: We have

$$T(1) = 1 + 0 + 0 = 1$$

$$T(x - 2) = x - 2 + 1 + 0 = x - 1$$

$$T((x - 2)^2) = (x - 2)^2 + 2(x - 2) + 2 = x^2 - 2x + 2$$

$$T((x - 2)^3) = (x - 2)^3 + 2(x - 2)^2 + 4(x - 2) = x^3 - 3x^2 + 6x - 8.$$

So,

$$T(1) = (1, 0, 0, 0)$$

$$T(x - 2) = x - 1 = (x - 2) + 1 = (1, 1, 0, 0)$$

$$T((x - 2)^2) = x^2 - 2x + 2 = (x - 2)^2 + 2x - 2 = (x - 2)^2 + 2(x - 2) + 2 = (2, 2, 1, 0)$$

$$T((x - 2)^3) = x^3 - 3x^2 + 6x - 8 = (x - 2)^3 + 3x^2 - 6x = (x - 2)^3 + 3(x - 2)^2 + 6(x - 2) = (0, 6, 3, 1)$$

and so

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to the basis $1, x - 2, (x - 2)^2, (x - 2)^3$.