Problem §1

(a) Prove that if α is a cut, then

$$-\alpha := \{c - b \mid c \in \mathbb{Q}, c < 0, b \in \mathbb{Q} \setminus \alpha\}$$

is a cut.

(b) Prove that for all cuts α, $\alpha \geq 0^*$ if and only if $-\alpha \leq 0^*$.

Solution:

(a) Clearly, $-\alpha \neq \varnothing$. Choose any $c \in \mathbb{Q}$, $c < 0$; since α is a cut, there exists a $b \in \mathbb{Q} \setminus \alpha$, and so $c - b \in \alpha \neq \varnothing$. Additionally, $-\alpha \neq \mathbb{Q}$, as one can choose any $a \in \alpha$; $c - \alpha \notin -\alpha$. Thus property (i) holds.

Observe that $c \in 0^*$. Suppose $r \in -\alpha$; then $r = c_1 - b_1$ for some $c_1 \in 0^*$, $b_1 \in \mathbb{Q} \setminus \alpha$. For any $s \in \mathbb{Q}$, if $s < r$, then either

- $s = c_2 - b_1,\ c_2 < c_1$. By property (ii) for 0^*, for any $c \in 0^*$, if $c' \in \mathbb{Q}$ and $c' < c$, then $c' \in 0^*$. Hence $s = c_2 - b_1 \in -\alpha$, as required.
- $s = c_1 - b_2,\ b_2 > b_1$. By property (ii), if $b_1 \in \mathbb{Q} \setminus \alpha$, and $b_2 > b_1$, then $b_2 \in \mathbb{Q} \setminus \alpha$ as well. Hence $s = c_1 - b_2 \in -\alpha$, as required.

Thus property (ii) holds.

For any $r \in -\alpha$, where $r = c_1 - b_1$, since $c_1 \in 0^*$, we can choose $c_2 \in 0^*$ such that $c_2 > c_1$ (by property (iii) of 0^*), and so there exists an $s \in -\alpha$, where $s = c_2 - b_1 > c_1 - b_1 = r$. Thus property (iii) holds.

Therefore all three properties hold, and so $-\alpha$ is a cut.

(b) Suppose $a \geq 0^*$. Then $0^* \subset \alpha$. Let $r \in -\alpha$. Then $r = c - b$ for some $c \in 0^*$, $b \in \mathbb{Q} \setminus \alpha$. Since $b \in \mathbb{Q} \setminus \alpha$, and $0 \in \alpha$, we have $0 < b$ by property (ii) (and so $-b < 0$); moreover, $c < 0$ for any $c \in 0^*$. Thus, we have

$$r = c - b < 0 - b = -b < 0,$$

and so $r < 0$; hence $r \in -\alpha$ implies that $r \in 0^*$, and so $-\alpha \subset 0^*$. Therefore $-\alpha \leq 0^*$.

Now, suppose that $-\alpha \leq 0^*$. Then $-\alpha \subset 0^*$; that is, for any $r \in -\alpha$, where $r = c - b$ for some $c \in 0^*$, $b \in \mathbb{Q} \setminus \alpha$, we have $r = c - b < 0$.

Since $b \in \mathbb{Q} \setminus \alpha$, any b satisfies $a < b$ for any $a \in \alpha$. Moreover, $r = c - b < 0$ implies that $c < b$ for all $c \in 0^*$. In other words, $b \geq 0$; and by the denseness of Q^*, there exists an $a \in \mathbb{Q}$ such that $a \leq b$ and $0 \leq a$. Since $a < b$, we have $a \in \alpha$; hence $0^* \subset \alpha$, and so $0^* \leq \alpha$.

Problem §2 Let α be a cut, $\alpha > 0^*$. Prove that

$$\alpha^{-1} := \{r \in \mathbb{Q} \mid r < 0\} \cup \{r \in \mathbb{Q} \mid 0 \leq r < t \text{ for some } t \in \mathbb{Q} \text{ such that } \frac{1}{t} \notin \alpha\ \}$$

is a cut and $\alpha^{-1} > 0^*$.

Solution: Clearly, $\alpha^{-1} \neq \varnothing$ (since $-1 \in \alpha^{-1}$) and $\alpha^{-1} \neq \mathbb{Q}$ (choose any $s > t$; $s \notin \alpha^{-1}$). Thus property (i) holds.

Suppose $r \in \alpha^{-1}$, and let $s \in \mathbb{Q}$ such that $s < r$. If $s < 0$, clearly $s \in \alpha^{-1}$, so choose $s \geq 0$. Since $0^* < \alpha$, there exists an $a \in \alpha$ that satisfies $a > 0$; and by properties of ordered fields, we know $a^{-1} = \frac{1}{a} > 0$. Hence if $\frac{1}{t} \notin \alpha$, then $\frac{1}{t} > a > 0$ and so $t > 0$. From this, we get $0 \leq s < r < t$, and so $s \in \alpha^{-1}$. Thus property (ii) holds.

Suppose $r \in alpha^{-1}$, and choose $s = \frac{r+t}{2}$. Since $s = \frac{r+t}{2} < \frac{t+t}{2} = t$, $s < t$; additionally, $s = \frac{r+t}{2} > \frac{r+r}{2} = r$, so $r < s$. From this, we get that $0 \leq r < s < t$, and so $s \in \alpha^{-1}$. Hence for any $r \in \alpha^{-1}$, there exists an $s \in \alpha^{-1}$ such that $r < s$.

Since $\alpha > 0$, any $a \in \alpha$ satisfies $\alpha > 0$; thus any $t \notin \alpha$ satisfies $t > \alpha > 0$. Choose any $0 \leq s < t$; clearly $s \notin 0^*$, so we have $0^* \neq alpha^{-1}$. Since we also have $0^* \leq \alpha^{-1}$ (one can easily see $0^* = \{r \in \mathbb{Q} \mid r < 0\} \subset \alpha^{-1}$), we necessarily have $0^* < \alpha^{-1}$.