

**Problem §1** (1.6) Prove that 7 divides  $11^n - 4^n$  for all  $n \in \mathbb{Z}^+$ .

*Proof.* Let  $P_n$ : “7 divides  $11^n - 4^n$  for some  $n \in \mathbb{Z}^+$ ”.

$P_1$  is true because  $11 - 4 = 7$ .

Now, assume  $P_n$  is true. Then

$$11^n - 4^n = 7k, \quad k \in \mathbb{Z}^+.$$

To prove  $P_{n+1}$  from  $P_n$ , we have

$$\begin{aligned} 11^{n+1} - 4^{n+1} &= 11^n \cdot 11 - 4^n \cdot 4 \\ &= 11^n \cdot 11 - 4^n \cdot 11 + 4^n \cdot 11 - 4^n \cdot 4 \\ &= (11^n \cdot 11 - 4^n \cdot 11) + (4^n \cdot 11 - 4^n \cdot 4) \\ &= 11 \cdot 7k + 7 \cdot 4^n \\ &= 7(11k + 4^n). \end{aligned}$$

Hence  $P_{n+1}$  is true whenever  $P_n$  is true, and thus by mathematical induction the statement is true.  $\square$

**Problem §2** (1.11) Let  $P_n$ : “ $n^2 + 5n + 1$  is even.”

(a) Show that  $P_{n+1}$  is true whenever  $P_n$  is true.

(b) For which  $P_n$  is the statement actually true? What is the moral of this exercise?

*Solution:*

(a) Assume  $P_n$  is true. Then

$$n^2 + 5n + 1 = 2k.$$

To prove  $P_{n+1}$  from  $P_n$ , we have

$$\begin{aligned} (n+1)^2 + 5(n+1) + 1 &= n^2 + 2n + 1 + 5n + 5 + 1 \\ &= (n^2 + 5n + 1) + 2n + 6 \\ &= 2k + 2(n+3) \\ &= 2(k+n+3). \end{aligned}$$

Hence  $P_{n+1}$  is true whenever  $P_n$  is true, and thus by mathematical induction the statement is true.

(b) None!  $P_1 = 7$ ,  $P_2 = 15, \dots$  In general,  $P_n$  is always odd:

- If  $n = 2k$  (even):  $4k^2 + 10k + 1 \implies \text{odd}$
- If  $n = 2k + 1$  (odd):  $4k^2 + 4k + 1 + 10k + 5 + 1 = 2(2k^2 + 7k + 3) + 1 \implies \text{odd}$

Thus, in order for mathematical induction to be valid, there must be a base case.

**Problem §3** Prove that

$$\left(1 - \frac{1}{\sqrt{2}}\right) \cdots \left(1 - \frac{1}{\sqrt{n}}\right) < \frac{2}{n^2}.$$

for all  $n \geq 2$ .

*Proof.* Let the above statement denote  $P_n$ .

$P_2$  is true:

$$\begin{aligned} 1 - \frac{1}{\sqrt{2}} &= 1 - \frac{\sqrt{2}}{2} < \frac{1}{2} \\ \frac{1}{2} &< \frac{\sqrt{2}}{2} \\ \frac{1}{4} &< \frac{2}{4}. \end{aligned}$$

Now, assume  $P_n$  is true. Then

$$\left(1 - \frac{1}{\sqrt{2}}\right) \cdots \left(1 - \frac{1}{\sqrt{n}}\right) < \frac{2}{n^2}.$$

Let  $a_n = 1 - \frac{1}{\sqrt{n}}$ . Then  $a_{n+1} = 1 - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1}-1}{\sqrt{n+1}} = \frac{n}{n+1+\sqrt{n+1}}$ . We can rewrite  $P_n$  using logarithms:

$$\log(a_1) + \cdots + \log(a_n) < \log(2) - 2\log(n).$$

To prove  $P_{n+1}$  from  $P_n$ , we have

$$\begin{aligned} \log(a_1) + \cdots + \log(a_n) + \log(a_{n+1}) &< \log(2) - \log(n) + \log(a_{n+1}) \\ &= \log(2) - \log(n) + \log(n) - \log(n+1+\sqrt{n+1}) \\ &= \log(2) - \log(n+1+\sqrt{n+1}). \end{aligned}$$

We now show that  $\log(n+1+\sqrt{n+1})$  is greater than  $2\log(n+1)$  (and so its reciprocal is less), which completes the proof.

$$\begin{aligned} \log(n+1+\sqrt{n+1}) &> 2\log(n+1) \\ n+1+\sqrt{n+1} &> (n+1)^2 = n^2+2n+1 \\ n\sqrt{n+1} &> n+1 \\ n^2(n+1) &> n^2+2n+1 \\ n^3+n^2 &> n^2+2n+1 \\ n^3 &> 2n+1 \\ n^3-2n &> 1, \end{aligned}$$

which is clearly true for all  $n \geq 2$ , and so  $n+1+\sqrt{n+1} > (n+1)^2 = n^2+2n+1$ . From this we get (after removing logs)

$$\frac{2}{n+1+\sqrt{n+1}} < \frac{2}{(n+1)^2},$$

completing the proof.

Hence  $P_{n+1}$  is true whenever  $P_n$  is true, and thus by mathematical induction the statement is true.  $\square$

**Problem §4** Prove that for all  $n \geq 3$ , there exist different natural numbers  $a_1, a_2, \dots, a_n$  such that

$$1 = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

*Proof.* We begin by observing  $n = 3, 4, 5$ .

- For  $n = 3$ :  $a_1 = 2, a_2 = 3, a_3 = 6$
- For  $n = 4$ :  $a_1 = 2, a_2 = 4, a_3 = 6, a_4 = 12$
- For  $n = 5$ :  $a_1 = 2, a_2 = 4, a_3 = 8, a_4 = 12, a_5 = 24$

From this, we get a pattern: for an  $a_{n-1}, a_n$ , we have  $a_n = 2a_{n-1}$ . Moreover, when we add a new number,  $a_{n-1}$  is updated, and  $a_{n+1} = 2a_n$ . Formally, we define

$P_n$  : There exists different natural numbers  $a_1, \dots, a_n$ , with  $a_n = 2a_{n-1}$ , such that  $1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{a_n}$ .

From above, we see that  $P_3$  is true.

Suppose  $P_n$  is true. Then

$$1 = \frac{1}{a_1} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{a_n}.$$

To prove  $P_{n+1}$  from  $P_n$ , we have

$$1 = \frac{1}{b_1} + \dots + \frac{1}{b_{n-1}} + \frac{1}{b_n} + \frac{1}{b_{n+1}}.$$

Choose  $a_i = b_i$  for all  $1 \leq i < n-1$  and  $i = n$ . Let  $b_{n-1} = \frac{2a_n}{3}$ ,  $b_{n+1} = 2a_n$ . Then

$$\begin{aligned} 1 &= \frac{1}{b_1} + \dots + \frac{1}{b_{n-1}} + \frac{1}{b_n} + \frac{1}{b_{n+1}} \\ 1 &= \frac{1}{a_1} + \dots + \frac{3}{2a_n} + \frac{1}{a_n} + \frac{1}{2a_n} \\ 1 &= \frac{1}{a_1} + \dots + \frac{2}{a_n} + \frac{1}{a_n} \\ 1 &= \frac{1}{a_1} + \dots + \frac{2}{2a_{n-1}} + \frac{1}{a_n} \\ 1 &= \frac{1}{a_1} + \dots + \frac{1}{a_{n-1}} + \frac{1}{a_n}. \end{aligned}$$

Hence  $P_{n+1}$  is true whenever  $P_n$  is true, and thus by mathematical induction the statement is true.  $\square$

**Problem §5** (2.4) Show that  $\sqrt[3]{5 - \sqrt{3}} \notin \mathbb{Q}$ .

*Solution:* Let  $a = \sqrt[3]{5 - \sqrt{3}}$ . Then

$$\begin{aligned} a^3 &= 5 - \sqrt{3} \\ \sqrt{3} &= 5 - a^3 \\ 3 &= 25 - 10a^3 + a^6 \\ 0 &= a^6 - 10a^3 + 22. \end{aligned}$$

By the Rational Roots Theorem, we see that the only possible rational solutions are  $\pm 1, \pm 2, \pm 11, \pm 22$ . Simple inspection by plugging in each possible rational solution indicates that none of them work, and so  $\sqrt[3]{5 - \sqrt{3}}$  is not rational.

**Problem §6** (2.7) Show that

(a)  $\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$

(b)  $\sqrt{6 + 4\sqrt{2}} - \sqrt{2}$

are actually rational.

*Solution:* We observe that the insides of the large square roots are actually perfect squares.

(a)

$$\begin{aligned} \sqrt{4 + 2\sqrt{3}} &= \sqrt{3 + 2\sqrt{3} \cdot 1 + 1} \\ &= \sqrt{(\sqrt{3} + 1)^2} \\ &= \sqrt{3} + 1. \end{aligned}$$

From this, we get  $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = \sqrt{3} + 1 - \sqrt{3} = 1 \in \mathbb{Q}$ .

(b)

$$\begin{aligned}\sqrt{6 + 4\sqrt{2}} &= \sqrt{4 + 2 \cdot 2 \cdot \sqrt{2} + 2} \\ &= \sqrt{(2 + \sqrt{2})^2} \\ &= 2 + \sqrt{2}.\end{aligned}$$

From this, we get  $\sqrt{6 + 4\sqrt{2}} - \sqrt{2} = 2 + \sqrt{2} - \sqrt{2} = 2 \in \mathbb{Q}$ .

Thus both are actually rational.

**Problem §7** Find all rational solutions of the equation  $3x^3 + x^2 - 8x + 4 = 0$ .

*Solution:* By the Rational Root Theorem, the only possible rational solutions are of the form  $\pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 4$ , and  $\pm \frac{4}{3}$ . By plugging in each possible value, we observe that  $1, -2, \frac{2}{3}$  satisfy the above equation, and thus are the three rational roots of  $3x^3 + x^2 - 8x + 4 = 0$ .