

**Problem §1** (Treil 5.4) Use cofactor formula to compute the inverses of the following matrices:

(a)  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(b)  $\begin{pmatrix} 19 & -17 \\ 3 & -2 \end{pmatrix}$

(c)  $\begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}$

(d)  $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

*Solution:*

(a)  $A^{-1} = \frac{1}{4-6} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}^T = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$

(b)  $A^{-1} = \frac{1}{-38-(-51)} \begin{pmatrix} -2 & -3 \\ 17 & -19 \end{pmatrix}^T = \begin{pmatrix} -\frac{2}{13} & \frac{17}{13} \\ -\frac{3}{13} & -\frac{19}{13} \end{pmatrix}$

(c)  $A^{-1} = \frac{1}{5} \begin{pmatrix} 5 & -3 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}$

(d)  $A^{-1} = \frac{1}{(1-2)-(2)} \begin{pmatrix} -1 & (-1)^3(2) & 2 \\ (-1)^3(1) & 1 & (-1)^5(1) \\ 2 & (-1)^5(2) & -1 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{3}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

**Problem §2** (Axler Problem 5.C.16) The **Fibonacci Sequence**  $F_1, F_2, \dots$  is defined by

$$F_1 = 1, F_2 = 1, F_n = F_{n-2} + F_{n-1}, n \geq 3.$$

Define  $T \in \mathcal{L}(\mathbb{R}^2)$  by  $T(x, y) = (y, x + y)$ .

- (a) Show that  $T^n(0, 1) = (F_n, F_{n+1})$  for all positive integers  $n$ .
- (b) Find the eigenvalues of  $T$ .
- (c) Find a basis of  $\mathbb{R}^2$  consisting of eigenvectors of  $T$ .
- (d) Use the solution to part (c) to compute  $T^n(0, 1)$ . Conclude that

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].$$

*Solution:*

- (a) We begin by induction. Clearly,  $T^1(0, 1) = (1, 1) = (F_1, F_2)$ . Now, assume  $T^n(0, 1) = (F_n, F_{n+1})$ . Then

$$T^{n+1}(0, 1) = T(T^n(0, 1)) = T(F_n, F_{n+1}) = (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2}),$$

as required. Thus

$$T^n(0, 1) = (F_n, F_{n+1})$$

for all positive integers  $n$ .

(b) Let  $A = \mathcal{M}(T)$  with respect to the standard basis. Then

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 1.$$

Thus

$$\lambda_1 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2},$$

since those are the roots of  $\lambda^2 - \lambda - 1$ .

(c) We now find the eigenspaces associated with  $\lambda_1, \lambda_2$ :

$$E(\lambda_1, T) = \{(x, y) \mid y = \left(\frac{1 + \sqrt{5}}{2}\right)x, x + y = \left(\frac{1 + \sqrt{5}}{2}\right)y\}.$$

From the first condition, we get that any eigenvector corresponding to  $\lambda_1$  is a scalar multiple of  $\left(1, \left(\frac{1 + \sqrt{5}}{2}\right)\right)$ . Similarly, we get that any eigenvector corresponding to  $\lambda_2$  is a scalar multiple of  $\left(1, \left(\frac{1 - \sqrt{5}}{2}\right)\right)$ . Thus

$$\left\{ \left(1, \frac{1 + \sqrt{5}}{2}\right), \left(1, \frac{1 - \sqrt{5}}{2}\right) \right\}$$

is a basis of eigenvectors in  $\mathbb{R}^2$  (since eigenvalues of distinct eigenvalues are linearly independent, and thus we have a list of 2 linearly independent vectors in  $\mathbb{R}^2$ , thus forming a basis).

(d) Let  $v_1 = \left(1, \frac{1 + \sqrt{5}}{2}\right)$ ,  $v_2 = \left(1, \frac{1 - \sqrt{5}}{2}\right)$ . Note that  $(0, 1) = \frac{v_1 - v_2}{\sqrt{5}}$ . Thus

$$T^n(0, 1) = \frac{1}{\sqrt{5}}T^n(v_1 - v_2) = \frac{1}{\sqrt{5}} \left[ \left(\frac{1 + \sqrt{5}}{2}\right)^n v_1 - \left(\frac{1 - \sqrt{5}}{2}\right)^n v_2 \right].$$

Since the first component is  $F_n$ , we get

$$F_n = \frac{1}{\sqrt{5}} \left[ \left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n \right],$$

as required.