REVIEW SHEET 13, Math 540, Summer 2021, Melody Chan

Due Weds June 30 at 11:59pm Eastern Time

Submit all of the following on Gradescope, and don't forget to tag each answer to its page. I put a copy of this review sheet in the Overleaf folder.

For each of the linear operators T in questions (1)–(7) of Review Sheet 12:

(a) I claim that $p(T) = 0$ for some nonzero polynomial $p(z) \in \mathcal{P}_2(\mathbb{R})$ of degree at most 2. Find such a polynomial $p(z)$.

(b) Factor $p(z)$ into linear factors over $\mathbb{R}$ *if possible*.[1]

(c) If $p(z)$ does factor into linear factors $p(z) = c(z - \lambda_1)(z - \lambda_2)$ for $c, \lambda_1, \lambda_2 \in \mathbb{R}$, then we argued in class that at least one of λ_1, λ_2 must be an eigenvalue of T. Identify which one(s) of λ_1 and λ_2 are in fact eigenvalues of T, in accordance with your answers in Review Sheet 12.

Example: $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is projection to the nearest point on the line $y = x$.

Answer:

Let $p(z) = z^2 - z$. Then $p(T) = T^2 - T = 0$.

We may factor $p(z) = z(z - 1)$. Therefore, at least one of the numbers $0, 1$ are eigenvalues of T. In fact, both 0 and 1 are eigenvalues of T, as shown on Review Sheet 12.

(1) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is reflection across the line $y = x$.

(2) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is 180° degree rotation about the origin.

[1] In other words, if possible, write $p(z) = c(z - \lambda_1)(z - \lambda_2)$ for some *real* numbers c, λ_1, λ_2. This may not always be possible; in class we stated that it can always be done provided you allow c, λ_1, and λ_2 to be *complex* numbers.

(3) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $60°$ degree counterclockwise rotation about the origin.

(4) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x,y) = (x+y,\, x+y)$.

(5) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by "walking vertically to the line $y = x$." That is, $T(x,y) = (x,x)$.

(6) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is projection to the xy-plane. That is, $T(x,y,z) = (x,y,0)$.

(7) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is $180°$ rotation about the x-axis.