Problem §3 Let V be a finite-dimensional $\mathbb{F}$-vector space and let $T \in \mathcal{L}(V)$ be an operator. Prove that the following are equivalent:

(a) $T^2 = T$.

(b) There exist subspaces U_1, U_2 of V such that $V = U_1 \oplus U_2$ and

$$T(u_1 + u_2) = u_1$$

for all $u_1 \in U_1, u_2 \in U_2$.

(c) T is diagonalizable and its set of eigenvalues is a subset of $\{0, 1\}$.

Solution: We start with a lemma.

Lemma 1. *Suppose V is a finite-dimensional $\mathbb{F}$-vector space, $T \in \mathcal{L}(V)$, and $T^2 = T$. Then $V = \operatorname{range} T \oplus \operatorname{null} T$.*

Proof. First, consider $v - T(v) \in V$. $T^2 = T$ implies $T(v - T(v)) = T(v) - T^2(v) = 0$, so $v - T(v) \in \operatorname{null} T$. Moreover, $T(v) \in \operatorname{range} T$ trivially. Thus for any vector $v \in V$,

$$v = T(v) + v - T(v) = u_1 + u_2, \text{ where } u_1 \in \operatorname{range} T,\ u_2 \in \operatorname{null} T,$$

and so $V = \operatorname{range} T + \operatorname{null} T$. Now, consider $v \in \operatorname{range} T \cap \operatorname{null} T$. Then for some $v' \in V$, $T(v') = v$; moreover, $T(v) = 0$. Thus

$$v = T(v') = T^2(v') = T(T(v')) = T(v) = 0$$

and so $v = 0$. Thus $\operatorname{range} T \cap \operatorname{null} T = \{0\}$, and so $V = \operatorname{range} T \oplus \operatorname{null} T$. $\square$

Assume $T^2 = T$. By the lemma, $V = \operatorname{range} T \cap \operatorname{null} T$. We know $\operatorname{range} T$ and $\operatorname{null} T$ are both subspaces of V. Let $u_1 \in \operatorname{range} T$ (and so $u_1 = T(v)$ for some $v \in V$), $u_2 \in \operatorname{null} T$. Then

$$T(u_1 + u_2) = T(u_1) + T(u_2) = T(T(v)) + 0 = T^2(v).$$

But $T^2 = T$, so $T^2(v) = T(v) = u_1$. Thus $T(u_1 + u_2) = u_1$, as required.

Now, assume there exist subspaces U_1, U_2 of V such that $V = U_1 \oplus U_2$ and

$$T(u_1 + u_2) = u_1$$

for all $u_1 \in U_1, u_2 \in U_2$. Recall that all subspaces U of V must have $\mathbf{0} \in U$. For any $u_2 \in U_2$,

$$0 = T(0 + u_2) = T(u_2) = 0u_2, \text{ so } \lambda_1 = 0.$$

Thus, if $U_2 \neq \{0\}$ and $v_1, \ldots, v_j$ form a basis of U_2, then T has an eigenvalue 0, and $v_1, \ldots, v_j \in E(0, T)$. Similarly, for any $u_1 \in U_1$,

$$T(u_1) = T(u_1 + 0) = u_1 = 1u_1, \text{ so } \lambda_2 = 1.$$

Thus, if $U_1 \neq \{0\}$, and $v_{j+1}, \ldots, v_n$ form a basis of U_1, then T has an eigenvalue 1, and $v_{j+1}, \ldots, v_n \in E(1, T)$.

Since $U_1 \oplus U_2 = V$, any vector $v \in V$ can be uniquely represented in terms of the two bases of U_1 and U_2: $a_1 v_1 + \ldots + a_j v_j + a_{j+1} v_{j+1} + \ldots + a_n v_n$. Moreover, since all v_i are eigenvectors, T is thus diagonalizable; additionally, any eigenvalue of T is either 0 or 1, as required.

Finally, assume T is diagonalizable and its set of eigenvalues are a subset of $\{0, 1\}$. Recall that T diagonalizable means that V has a basis of eigenvectors of T. Let $v_1, \ldots, v_n$ be a basis of V. T must clearly have eigenvalues (since otherwise it wouldn't be diagonalizable).

If $\lambda = 0$ is the only eigenvalue, then for any $v \in V$, $v = a_1 v_1 + \ldots + a_n v_n$ and

$$T(v) = T(a_1 v_1 + \ldots + a_n v_n) = 0 = T(0) = T(T(v)) = T^2(v).$$

Similarly, if $\lambda = 1$ is the only eigenvalue, then

$$T^2(v) = T(T(v)) = T(v).$$

Finally, suppose T has two eigenvalues $\lambda_1 = 0,\ \lambda_2 = 1$. Let $v_1, \ldots, v_j \in E(0,T),\ v_{j+1}, \ldots, v_n \in E(1,T)$. Then for any $v \in V$, we have

$$T(v) = T(a_1v_1 + \ldots + a_jv_j + a_{j+1}v_{j+1} + \ldots + a_nv_n) = a_1 \cdot 0v_1 + \ldots + a_j \cdot 0v_j + a_{j+1} \cdot 1v_{j+1} + \ldots + a_n \cdot 1v_n = a_{j+1}v_{j+1} + \ldots + a_nv_n$$

and

$$T(v^2) = T(T(a_1v_1 + \ldots + a_jv_j + a_{j+1}v_{j+1} + \ldots + a_nv_n)) = T(a_{j+1}v_{j+1} + \ldots + a_nv_n) = a_{j+1}v_{j+1} + \ldots + a_nv_n.$$

Thus for any $v \in V$, $T^2 = T$, as required.

Therefore (a) implies (b), (b) implies (c), and (c) implies (a), and so the three statements are equivalent.

Problem §4 Suppose $T \in \mathcal{L}(V)$, $\mathbb{F} = \mathbb{C}$, $p \in \mathcal{P}(\mathbb{C})$, and $\alpha \in \mathbb{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

Solution: Suppose α is an eigenvalue of $p(T)$, and consider $q(z) = p(z) - \alpha$. Since α is an eigenvalue, $p(T) - \alpha I$ is not injective; thus $(p(T) - \alpha I)(v) = 0$ for some non-zero $v \in V$, and so

$$q(T)(v) = \left(c(T - \lambda_1 I) \ldots (T - \lambda_m I)\right)(v) = 0.$$

Hence one of $T - \lambda_j I$ is not injective, and so λ_j is an eigenvalue of T. Moreover,

$$q(\lambda_j) = p(\lambda_j) - \alpha = 0,$$

so $\alpha = p(\lambda_j)$ for some eigenvalue λ_j of T.

Conversely, suppose $\alpha = p(\lambda)$ for some eigenvalue λ of T. Let $v \in E(\lambda, T)$. Then $T(v) = \lambda(v)$; moreover, $T^2(v) = T(T(v)) = \lambda T(v) = \lambda^2 v$. Trivial induction leads to $T^n(v) = \lambda^n v$. Thus for any polynomial

$$p(z) = a_0 + a_1z + \ldots + a_nz^n,$$

we have

$$p(T)(v) = (a_0I + a_1T + \ldots + a_nT^n)(v) = a_0v + a_1T(v) + \ldots + a_nT^n(v) = a_0v + a_1\lambda v + \ldots + a_n\lambda^n v = p(\lambda)(v).$$

Thus $p(T)(v) = p(\lambda)(v) = \alpha v$, and so $\alpha = p(\lambda)$ is an eigenvalue of $p(T)$.