Problem §1 Let $S, T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear maps determined by

$$S(1,1) = (2,-1,3), \qquad S(1,-1) = (2,-2,4)$$
$$T(1,0) = (4,0,0), \qquad T(0,1) = (-4,-1,0).$$

Compute:

(a) $(-S)(4,0)$

(b) $(-S+T)(4,0)$

(c) $(-S+T)(4,0) + (-S+T)(-1,3) + (-S+T)(-2,-2) + (-S+T)(-1,-1)$

(d) $\mathcal{M}(-S)$

(e) $\mathcal{M}(T)$

(f) $-\mathcal{M}(S) + \mathcal{M}(T)$

Solution: We first find the matrices of S and T with respect to the standard basis:

$$\mathcal{M}(S) = \begin{pmatrix} 2 & 0 \\ -\frac{3}{2} & \frac{1}{2} \\ \frac{7}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \mathcal{M}(T) = \begin{pmatrix} 4 & -4 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

(a) $-S(4,0) = (-8, 6, -14)$.

(b) $(-S+T)(4,0) = (8, 6, -14)$.

(c) $(-S+T)(4,0) + (-S+T)(1,3) + (-S+T)(-2,-2) + (-S+T)(-1,-1) = (0,0,0)$.

(d) $\mathcal{M}(-S) = \begin{pmatrix} -2 & 0 \\ \frac{3}{2} & -\frac{1}{2} \\ -\frac{7}{2} & \frac{1}{2} \end{pmatrix}$

(e) $\mathcal{M}(T) = \begin{pmatrix} 4 & -4 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$

(f) $-\mathcal{M}(S) + \mathcal{M}(T) = \begin{pmatrix} 2 & -4 \\ \frac{3}{2} & -\frac{3}{2} \\ -\frac{7}{2} & \frac{1}{2} \end{pmatrix}$

Problem §2 Write the linear map

$$T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$$

given by

$$T(f) = f + f' + f''$$

as a matrix with respect to the basis

$$1,\ x-2,\ (x-2)^2,\ (x-2)^3$$

for both the domain and codomain.

Solution: We have

$$\begin{aligned} T(1) &= 1 + 0 + 0 = 1 \\ T(x-2) &= x - 2 + 1 + 0 = x - 1 \\ T((x-2)^2) &= (x-2)^2 + 2(x-2) + 2 = x^2 - 2x + 2 \\ T((x-2)^3) &= (x-2)^3 + 2(x-2)^2 + 4(x-2) = x^3 - 3x^2 + 6x - 8. \end{aligned}$$

So,

$$\begin{aligned}
T(1) &= (1,0,0,0) \\
T(x-2) &= x-1 = (x-2)+1 = (1,1,0,0) \\
T((x-2)^2) &= x^2-2x+2 = (x-2)^2+2x-2 = (x-2)^2+2(x-2)+2 = (2,2,1,0) \\
T((x-2)^3) &= x^3-3x^2+6x-8 = (x-2)^3+3x^2-6x = (x-2)^3+3(x-2)^2+6(x-2) = (0,6,3,1)
\end{aligned}$$

and so

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to the basis $1, x-2, (x-2)^2, (x-2)^3$.