

Problem §1 (Treil 4.2) Let P be a **permutation matrix**, i.e. an $n \times n$ matrix consisting of zeroes and ones such that there is exactly one 1 in every row and every column.

- (a) Can you describe the corresponding linear transformation? That will explain the name.
- (b) Show that P is invertible. Can you describe P^{-1} ?
- (c) Show that for some $N > 0$

$$P^N := \underbrace{PP \dots P}_{N \text{ times}} = I.$$

Solution:

- (a) It swaps the basis vectors around (e.g. it *permutes* them); for instance, e_i becomes $Te_i = e_j$ for some j not necessarily equal to i .
- (b) One can clearly see that no column is a linear combination of other columns (since each row and column only has one 1 element); thus all columns are linearly independent, and so $\det A \neq 0$; i.e. P is invertible.

Alternatively, let $\sigma \in \mathcal{S}_n$ such that $P_{i,\sigma(i)} = 1$ for all $1 \leq i \leq n$. Since each row and column only has one j such that $\sigma(i) = j$ and $P_{i,j} = 1$, σ is unique (if another permutation φ does the same for all $1 \leq i \leq n$, then $\varphi = \sigma$). Additionally, for any other permutation $\tau \in \mathcal{S}_n$ with $\tau \neq \sigma$, we have $P_{i,\tau(i)} = 0$ (since otherwise, we must have $\sigma = \tau$). Thus,

$$\det P = \sum_{\phi \in \mathcal{S}_n} p_{1,\phi(1)} \dots p_{n,\phi(n)} \operatorname{sgn}(\phi)$$

becomes $p_{1,\sigma(1)} \dots p_{n,\sigma(n)} \operatorname{sgn}(\sigma)$, since all $\tau \in \mathcal{S}_n$, $\tau \neq \sigma$ has $p_{j,\tau(j)} = 0$ for some $1 \leq j \leq n$ (since that means $\tau(j) \neq \sigma(j)$ for some $1 \leq j \leq n$, and so $p_{j,\sigma(j)} \neq p_{j,\tau(j)} = 0$).

All $p_{i,\sigma(i)} = 1$, and $\operatorname{sgn}(\sigma) = \pm 1$, so

$$\det P = \pm 1 \neq 0,$$

so P is invertible.

P^{-1} essentially reverts any permutation that σ applies to i ; thus, we get that all 1s are in the indices

$$P_{\sigma(i),i}$$

which is in contrast to $P_{i,\sigma(i)}$; this aligns with our intuition of “reverting” a permutation.

- (c) Let $\sigma \in \mathcal{S}_n$ as defined above. In cycle notation,

$$\sigma = (p_1 p_2 \dots p_n), \quad \sigma(p_i) = p_{i+1} \text{ [with modular arithmetic at the edges]}$$

for $p_i \in \{1, \dots, n\}$. Then σ has order n (i.e. $\sigma^n = e$, the identity permutation), since applying σ^n on any p_i will cycle back to p_i .

Next, we observe that multiplying an $n \times n$ matrix A by P permutes the columns of A by σ :

$$(AP)_{i,\sigma(j)} = A_{i,j}$$

(one can check this: only $p_{j,\sigma(j)} = 1 \neq 0$, so we have $(AP)_{i,\sigma(j)} = \sum_{r=1}^n A_{i,r} P_{r,\sigma(j)} = A_{i,j} P_{j,\sigma(j)} = A_{i,j}$). Thus $P_{i,\sigma(j)}^2 = 1$ only when $j = \sigma(i)$, and so we get $P_{i,\sigma^2(i)}^2 = 1$. This repeats for powers, until we get

$$P_{i,\sigma(\sigma^{n-1}(1))}^n = P_{i,\sigma^n(i)}^n = P_{i,i}^n = 1.$$

In other words, for any row i , its (only) 1 entry is at $P_{i,i}^n$, and all $P_{i,j}^n$ with $i \neq j$ has value 0. Thus $P^n = I$, where n is the order of σ .

Problem §2 Let n be a positive integer, and let $A \in \mathbb{R}^{n,n}$ be a square matrix whose diagonal entries are odd integers, and whose off-diagonal entries are even integers. Prove that A is invertible.

Solution: We use induction to prove that any matrix $A_n \in \mathbb{R}^{n,n}$ with the above property (odd on diagonal, even elsewhere) has an odd (and thus non-zero) determinant. This is clearly true for $n = 1$, so assume the property holds for $1, \dots, n-1$.

Let $A_n \in \mathbb{R}^{n \times n}$ be a matrix with the above property. By co-factor expansion,

$$\det A_n = \sum_{j=1}^n a_{1,j} C_{1,j}.$$

But note that $a_{1,1}$ is odd, and all $a_{1,j}$ for $1 < j \leq n$ are even (since they're all off-diagonal), and that $C_{1,1} = A_{n-1}$ (since all of the diagonal entries of $C_{1,1}$ are also on the diagonal of A_n , which are all odd, and all of the off-diagonal entries of $C_{1,1}$ are also off the diagonal of A_n ; thus $C_{1,1}$ has the same property as above).

By induction, we know $\det A_{n-1}$ is odd. Thus $a_{1,1}C_{1,1}$ is odd (since for two odds $2i+1, 2j+1$, $(2i+1)(2j+1) = (4ij + 2i + 2j) + 1 = 2k + 1$), and all $a_{1,j}C_{1,j}$ for $1 < j \leq n$ are even (since for an even $2i$ and any integer, even $2j$ or odd $2k+1$, the resulting product, either $2i \cdot 2j = 4ij = 2(2ij)$ or $2i(2k+1) = 4ik + 2i = 2(2ik + i)$, is even). Therefore the determinant

$$\det A_n = \text{odd integer} + \sum_{j=2}^n \text{even integer} = \text{odd integer} \neq 0$$

is odd, and thus non-zero (clearly, odd + even is still odd).

Thus, by induction any $A_n \in \mathbb{R}^{n,n}$ with the above property has an odd (non-zero) determinant for all $n \in \mathbb{N}$, and so A_n is invertible.

Problem §3 Use Cramer's Rule to solve the following system of linear equations:

(a)

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 2x_1 + 3x_2 &= 4. \end{aligned}$$

(b)

$$\begin{aligned} -x_1 + 0x_2 + 2x_3 &= 0 \\ 0x_1 + 5x_2 + x_3 &= 4 \\ 3x_1 - x_2 - x_3 &= 7. \end{aligned}$$

Solution:

(a) $\det A = 3 \cdot 3 - 4 \cdot 2 = 1.$

(a) $B_1 = \begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$, so $\det B_1 = 5 \cdot 3 - 4 \cdot 4 = -1$, so $x_1 = -1$.

(b) $B_2 = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$, so $\det B_2 = 3 \cdot 4 - 5 \cdot 2 = 2$, so $x_2 = 2$.

(b) $\det A = -1(-5 - (-1)) + 2(-15) = -26.$

(a) $B_1 = \begin{pmatrix} 0 & 0 & 2 \\ 4 & 5 & 1 \\ 7 & -1 & -1 \end{pmatrix}$, so $\det B_1 = 2(-4 - 35) = -78$, so $x_1 = \frac{78}{26} = 3$.

$$(b) \ B_2 = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 4 & 1 \\ 3 & 7 & -1 \end{pmatrix}, \text{ so } \det B_2 = -(-4 - 7) + 2(-12) = -13, \text{ so } x_2 = \frac{1}{2}.$$

$$(c) \ B_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 4 \\ 3 & -1 & 7 \end{pmatrix}, \text{ so } \det B_3 = -(35 - (-4)) = -39, \text{ so } x_3 = \frac{39}{26} = \frac{3}{2}.$$