Problem §1 (4.4) Let $V,\ W$ be F-vector spaces, let

$$L_1 : V \longrightarrow W \text{ and } L_2 : V \longrightarrow W$$

be linear transformations from V to W, and let $c \in F$ be a scalar. We define new functions $L_1 + L_2$ and cL_1 that map V to W as follows:

$$(L_1 + L_2)(v) = L_1(v) + L_2(v) \text{ and } (cL_1)(v) = c(L_1(v)).$$

(a) Prove that $L_1 + L_2$ and cL_1 are linear transformations.

(b) We denote the set of F-linear transformations from V to W by

$$\mathrm{Hom}_F(V, W) = \{\text{linear transformations} L : V \to W\}.$$

Prove that the addition and scalar multiplication defined above make $\mathrm{Hom}_F(V, W)$ a vector space.

Solution:

(a) Pretty trivial by the definition, but here goes: for $a_1, a_2 \in F$ and $v_1, v_2 \in V$,

$$\begin{aligned}(L_1 + L_2)(a_1v_1 + a_2v_2) &= L_1(a_1v_1 + a_2v_2) + L_2(a_1v_1 + a_2v_2)\\ &= a_1L_1(v_1) + a_1L_2(v_1) + a_2L_1(v_2) + a_2L_2(v_2)\\ &= a_1(L_1 + L_2)(v_1) + a_2(L_1 + L_2)(v_2).\end{aligned}$$

and for $c \in F$,

$$(cL_1)(a_1v_1 + a_2v_2) = c(L_1(a_1v_1 + a_2v_2)) = ca_1L_1(v_1) + ca_2L_1(v_2) = a_1(cL_1)(v_1) + a_2(cL_1)(v_2)$$

by commutativity of multiplication in fields. Hence $L_1 + L_2$ and cL_1 are linear transformations... wow!

(b) $\mathrm{Hom}_F(V, W)$ clearly forms an Abelian group under addition:

- Associativity follows since W is a vector space (and so addition of vectors is associative)
- The zero map $0 : V \to W,\ v \mapsto 0$ is the identity, since $(L_1 + 0)(v) = L_1(v) + 0 = L_1(v)$ for any $L \in \mathrm{Hom}_F(V, W)$.
- For any $L \in \mathrm{Hom}_F(V, W)$, the map $(-L)(v) = -(L(v))$ is the inverse of L (existence and uniqueness follow trivially from W being a vector space).
- Commutativity also follows since W is a vector space.

Moreover, for any $L_1, L_2 \in \mathrm{Hom}_F(V, W)$, $c_1, c_2 \in F$,

- $(1L)(v) = 1(L(v)) = L(v)$, since $1 \in F$, $L(v) \in W$, and $1w = w$ for any $w \in W$; hence $1L = L$.
- $(c(L_1+L_2))(v) = c(L_1(v)+L_2(v)) = cL_1(v)+cL_2(v)$, by distributivity of W. Hence $c(L_1+L_2) = cL_1 + cL_2$.
- $((c_1 + c_2)L)(v) = (c_1 + c_2)(L(v)) = c_1L(v) + c_2L(v)$, again by distributivity of W. Hence $(c_1 + c_2)L = c_1L + c_2L$.
- $c_1(c_2L)(v) = c_1c_2(L(v)) = ((c_1c_2)L)(v)$, since $(cL)(v) = c(L(v))$ for all $c \in F$, $L \in \mathrm{Hom}_F(V, W)$, and F is closed under multiplication, so $c_1c_2 \in F$. Hence $c_1(c_2L) = (c_1c_2)L$.

Thus $\mathrm{Hom}_F(V, W)$ forms a vector space. . . . . .

Problem §2 (4.5) Let V be an F-vector space, and let

$$L_1 : V \longrightarrow V \text{ and } L_2 : V \longrightarrow V$$

be linear operators on V. Define $L_1 + L_2$ and L_1L_2 as follows:

$$(L_1 + L_2)(v) = L_1(v) + L_2(v) \text{ and } (L_1L_2)(v) = L_1(L_2(v)).$$

(a) Prove that $L_1 + L_2$ and L_1L_2 are linear transformations.

(b) Let $L : V \to V$ be another linear transformation. Prove the following formulas:

  (a) $(L_1 + L_2) + L_3 = L_1 + (L_2 + L_3)$

  (b) $(L_1L_2)L_3 = L_1(L_2L_3)$

  (c) $L_1(L_2 + L_3) = L_1L_2 + L_1L_3$ and $(L_1 + L_2)L_3 = L_1L_3 + L_2L_3$

(c) Prove that the set of linear transformations from V to V is a ring, where addition and multiplication are given above. What is the identity element of this ring? What is the additive inverse of a linear transformation L?

Solution:

(a) See above for $L_1 + L_2$; essentially the same steps. For $c_1, c_2 \in F$, $L_1, L_2 \in \mathcal{L}(V)$, $v_1, v_2 \in V$,

$$\begin{aligned}(L_1L_2)(c_1v_1 + c_2v_2) &= L_1(L_2(c_1v_1 + c_2v_2)) \\ &= L_1(c_1L_2(v_1) + c_2L_2(v_2)) \\ &= c_1L_1(L_2(v_1)) + c_2L_1(L_2(v_2)) \\ &= c_1(L_1L_2)(v_1) + c_2(L_1L_2)(v_2).\end{aligned}$$

Hence L_1L_2 is a linear transformation as well.

(b)
- Trivial: V is associative under addition, and any $L(v) \in V$. Thus $(L_1 + (L_2 + L_3))(v) = L_1(v) + L_2(v) + L_3(v) = (L_1 + L2)(v) + L_3(v)$.
- $((L_1L_2)L_3)(v) = (L_1L_2)(L_3(v)) = L_1(L_2(L_3(v))) = L_1((L_2L_3)(v)) = (L_1(L_2L_3))(v)$, as required.
- $L_1(L2 + L_3)(v) = L_1(L_2(v) + L_3(v)) = \underbrace{L_1(L_2(v)) + L_1(L_3(v))}_{\text{by linearity}} = (L_1L_2)(v) + (L_1L_3)(v)$, and

$$((L_1 + L_2)L_3)(v) = (L_1 + L_2)(L_3(v)) = L_1(L_3)(v)) + L_2(L_3(v)) = (L_1L_3)(v) + (L_2L_3)(v),$$

  as required.

(c) (a) proves closure/well-definedness (addition and multiplication make sense). (b)1,2 proves associativity of addition and multiplication. (b)3 proves distributivity. Commutativity of addition follows since V forms an Abelian group under vector addition (and since $L(v) \in V$, we have $L_1(v) + L_2(v) = L_2(v)+L_1(v)$). For any $L \in \mathcal{L}(V)$, since $L(v) \in V$ and V is a group, there exists some $w = -L(v) \in V$. So, if we define $-L$ as $(-L)(v) = -L(v)$,

$$(L + -L)(v) = L(v) + -L(v) = 0 = (-L + L)(v) = -L(v) + L(v).$$

Hence every element has an additive inverse. Clearly, $0 : V \to V$, $v \mapsto 0$ is in $\mathcal{L}(V)$, and $0 + L = L + 0 = L$ for any $L \in \mathcal{L}(V)$; hence additive identity exists. Moreover, for $I : V \to V$, $v \mapsto v$, $I \in \mathcal{L}(V)$, and clearly $IL = LI = L \in \mathcal{L}(V)$; hence multiplicative identity exists. Thus $\mathcal{L}(V)$ is an Abelian group under addition and a monoid under multiplication. Therefore $\mathcal{L}(V)$ is a ring.

Problem §3 (4.15) Suppose V is an F-vector space, $\mathcal{A}$ and $\mathcal{B}$ are subsets of V, and

- $\mathcal{B}$ is linearly independent.
- $|A| = |B|$.

- $\text{span}(B) \subseteq \text{span}(A)$.

Prove $\text{span}(A) = \text{span}(B)$.

Solution: We start with 2 lemmas.

Lemma 1. *Let V be an F-vector space. Given a subset $\mathcal{A} \subseteq V$,* $\text{span}(A)$ *is a subspace of V; and if $\mathcal{A}$ is linearly independent, then $\mathcal{A}$ forms a basis of* $\text{span}(A)$.

Proof. Clearly, given $v_1, v_2 \in \text{span}(A)$, we have $v_1 + v_2 \in \text{span}(A)$ (since for each component $a_i\alpha_i$ of v_1 and corresponding $a_i'\alpha_i$ of v_2, $(a_i + a_i') \in F$ by closure of addition in fields, so $v_1 + v_2$ is still in $\text{span}(A)$ by definition); and for any $c \in F$, $cv \in \text{span}(A)$ as well (like above, for each component $a_i\alpha_i$ of v, $(ca_i) \in F$, so $cv \in \text{span}(A)$ by definition). Finally, setting each a_i coefficient to 0 shows that $\mathbf{0} \in \text{span}(A)$. Hence $\text{span}(A)$ is a subspace of V.

If $\mathcal{A}$ is linearly independent, clearly $\mathcal{A}$ forms a basis for $\text{span}(A)$ (since $\mathcal{A}$ is both linearly independent and spans $\text{span}(A)$. Wow!) $\square$

Lemma 2. *Let a set $\mathcal{A}$ span a vector space V. If a linearly independent set $\mathcal{B}$ has the same number of elements as $\mathcal{A}$, $\mathcal{A}$ is also linearly independent, and is thus a basis for V.*

Proof. Suppose $\mathcal{A}$ is linearly dependent. Then for some $\alpha_i \in \mathcal{A}$, $\alpha_i \in \text{span}(\mathcal{A} \setminus \{\alpha_i\})$, so $\mathcal{A} \setminus \{\alpha_i\}$ spans V as well. But then $|\mathcal{A} \setminus \{\alpha_I\}| < |\mathcal{A}| = |\mathcal{B}|$ a linearly independent set in V, a contradiction of Lemma 4.24. Thus $\mathcal{A}$ is linearly independent in V as well, and thus is a basis for V. $\square$

Lemma 3. *Let $\mathcal{A}$ be a basis for a vector space V. If a linearly independent set $\mathcal{B}$ has the same number of elements as $\mathcal{A}$, then $\mathcal{B}$ is a basis for V as well.*

Proof. Suppose $\mathcal{B}$ is not a basis for V. Then for some $v \in V$ where $v \notin \text{span}(\mathcal{B})$, $\mathcal{B}' = \mathcal{B} \cup \{v\}$ is linearly independent. However, Lemma 4.24 tells us that the size of any linearly independent set in V is less than or equal to the length of any spanning set in V, and since $\mathcal{A}$ spans V and $|\mathcal{A}| = |\mathcal{B}| < |\mathcal{B}'|$, this is a contradiction. Hence $\mathcal{B}$ must span V as well, and so $\mathcal{B}$ is a basis for V. $\square$

Lemma 1 tells us that $\text{span}(\mathcal{A})$ is a subspace of V (and thus also a vector space, allowing us to apply theorems about bases and dimensions of vector spaces). Since $\text{span}(\mathcal{B}) \subseteq \text{span}(\mathcal{A})$, clearly $\mathcal{B} \subseteq \text{span}(\mathcal{A})$. Thus $\mathcal{B}$ is a linearly independent set in $\text{span}(\mathcal{A})$. Since $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements, Lemma 2 tells us that $\mathcal{A}$ is linearly independent in $\text{span}(\mathcal{A})$ as well, and thus is a basis. Lemma 3 then tells us that $\mathcal{B}$ is a basis for $\text{span}(\mathcal{A})$ too; thus $\text{span}(\mathcal{A}) = \text{span}(\mathcal{B})$, as required.

Problem §4 Suppose V, W are finite-dimensional F-vector spaces. Let $L : V \to W$ be a linear transformation.

- If L injective, prove $\dim V \leq \dim W$.
- If L surjective, prove $\dim V \geq \dim W$.

Solution: Let $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_m\}$ be basis for V and W respectively.

- Suppose $L : V \to W$ is injective. For any $v \in V$, we can write

$$v = \sum_{i=1}^{n} a_i v_i, \text{ where } a_i \in F.$$

Clearly, $\mathcal{B} = \{L(v_1), \ldots, L(v_n)\}$ spans $\text{range}\, L$, since for any $L(v) \in \text{range}\, L$, we have

$$L(v) = L(a_1 v_1 + \ldots + a_n v_n) = a_1 L(v_1) + \ldots + a_n L(v_n).$$

We claim that $\mathcal{B}$ is a basis for $\text{range}\, L$.

Suppose $L(v),\ L(v') \in \operatorname{range} L$ are different ways of representing a vector in $\operatorname{range} L$; in other words, $L(v) = L(v')$ and

$$L(v) = \sum_{i=1}^{n} a_i L(v_i),\ L(v') = \sum_{i=1}^{n} a_i' L(v_i),\ a_i \neq a_i'.$$

By linearity,

$$L(v) = \sum_{i=1}^{n} a_i L(v_i) = \sum_{i=1}^{n} L(a_i v_i)$$

and

$$L(v') = \sum_{i=1}^{n} a_i' L(v_i) = \sum_{i=1}^{n} L(a_i' v_i).$$

L injective then means

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i' v_i,$$

and since $\{v_1, \ldots, v_n\}$ is a basis for V (and thus is linearly independent), we necessarily have

$$\sum_{i=1}^{n} (a_i - a_i') v_i = 0,\ a_i - a_i' = 0.$$

and hence $a_i = a_i'$. Equivalently, $L(v)$ and $L(v')$ are the same, and thus every $L(v) \in \operatorname{range} L$ can be represented uniquely as

$$L(v) = \sum_{i=1}^{n} a_i L(v_i).$$

In other words, $\mathcal{B}$ is a basis for $\operatorname{range} L$.

Since $\operatorname{range} L \subseteq W$ and $\mathcal{B}$ is linearly independent in $\operatorname{range} L$ (and thus in W as well), by Lemma 4.24 any spanning set must have at least as many elements as $\mathcal{B}$. Hence any basis of W must have at least as many elements as $\mathcal{B}$; and since $|\mathcal{B}| = \dim V$, we get $\dim V \leq \dim W$, as required.

- Suppose $L : V \to W$ is surjective. From before, we know that $\mathcal{B} = \{L(v_1), \ldots, L(v_n)\}$ spans $\operatorname{range} L$; since L is surjective (and so $\operatorname{range} L = W$), $\mathcal{B}$ spans W as well. By Lemma 4.24, the size of any linearly independent set in W is less than or equal to the size of any spanning set of W. Since $\{w_1, \ldots, w_m\}$ is linearly independent, we thus have $\dim W \leq \operatorname{range} L = \dim V$. Therefore $\dim V \geq \dim W$, as required.

Problem §5 (4.18) Let V be a finite-dimensional F-vector space, and let $U \subseteq V$ be a vector subspace.

(a) Prove that U is finite-dimensional.

(b) Prove that $\dim_F U \leq \dim_F V$.

(c) Prove that

$$U = V \iff \dim_F U = \dim_F V.$$

Solution:

(a) If $U = \{0\}$, U is clearly finite-dimensional, so suppose $U \neq \{0\}$. Let $u_1 \in U$ be a non-zero vector. If $U = \operatorname{span}(\{u_1\})$, then we are done; otherwise, continue adding non-zero vectors $u_j \in U$ such that

$$u_j \notin \operatorname{span}(\{u_1, \ldots, u_{j-1}\}),$$

until $\{u_1, \ldots, u_j\}$ forms a spanning set of U. With each addition, $\{u_1, \ldots, u_j\}$ is a linearly independent set by construction (since each added vector was not in the span of the previous vectors). Moreover, every linearly independent set $\{u_1, \ldots, u_j\}$ is in V, since each $u_i \in U \subseteq V$.

Let $n = \dim V$. Since any basis of V is spanning, and by Lemma 4.24, the number of elements in any linearly independent set in V must be less than or equal to the length of any spanning set in V, the length of $\{u_1, \ldots, u_j\}$ must be less than or equal to n. Thus the above process will eventually terminate (it cannot repeat infinitely — or past $j = n$ — since the number of elements must be less than or equal to n), and so we are left with a finite linearly independent spanning set $\{u_1, \ldots, u_j\}$ of U. Thus U is finite-dimensional.

(b) From above, we see that a basis $\{u_1, \ldots, u_j\}$ of U cannot have more elements than $n = \dim V$. Hence $\dim_F U \leq \dim_F V$.

Alternatively, let $\{u_1, \ldots, u_m\}$ be a basis for U. Then $\{u_1, \ldots, u_m\} \subseteq U \subseteq V$ is a linearly independent set of vectors in V. By Lemma 4.24, any linearly independent set of vectors in V cannot have more elements than any spanning set of V. Since a basis $\{v_1, \ldots, v_n\}$ of V is the smallest spanning set of V and has $n = \dim V$ elements, any linearly independent set cannot have more than n elements. Thus $|\{u_1, \ldots, u_m\}| = \dim_F U \leq \dim_F V = n = |\{v_1, \ldots, v_n\}|$.

(c) Suppose $U = V$. Then a basis $\{v_1, \ldots, v_n\}$ of U is also a basis of V, and so $\dim_F U = \dim_F V$.

Conversely, suppose $\dim_F U = \dim_F V = n$, and let $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$ be bases for U and V respectively. Since U is a subspace of V, we know that $\{u_1, \ldots, u_n\}$ is a linearly independent set of vectors in V. From Lemma 3 (of Problem §3), since $\{u_1, \ldots, u_n\}$ is a linearly independent set of vectors with the same number of elements as a basis $\{v_1, \ldots, v_n\}$ of V, $\{u_1, \ldots, u_n\}$ is a basis of V as well. Thus

$$U = \operatorname{span}(\{u_1, \ldots, u_n\}) = V,$$

and so $U = V$, as required.