MATH 540 HONORS LINEAR ALGEBRA SUMMER 2021, Melody Chan
PROBLEM SET E

Due Monday June 21 at 11:59pm Eastern

Submit all of the following on Gradescope, and don't forget to tag each answer to its page. I put a copy of this problem set in the Overleaf folder too.

1. (3 points) Axler 3.B.30 on page 69. You are free to assume V is finite-dimensional, but you can also solve the problem without that assumption.

2. (4 points) A square matrix $A \in \mathbb{F}^{n,n}$ is called *upper triangular* if $A_{j,k} = 0$ whenever $j > k$. For example,
$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$
is an upper triangular matrix. Let A and C be $n \times n$ upper triangular matrices.

   (a) Prove that AC is upper triangular.

   (b) Also, prove that $(AC)_{j,j} = A_{j,j}C_{j,j}$ for each $j = 1, \ldots, n$.

3. (Hot potato, 5 points) Players $1, 2, 3, 4, 5, 6$ are seated around a circle, in that order. One of them holds a hot potato. At each time $t = 1, 2, 3, \ldots$ seconds from the beginning of the game, whoever is holding the potato passes it either to the person to their immediate right or the person to their immediate left, with equal probability.

   (a) Let $A \in \mathbb{R}^{6,6}$ be the matrix in which $A_{j,k}$ is the probability that, supposing player j has the potato at time 0, player k has the potato at time 1. Write down A.

   (b) Give an argument that $(A^2)_{j,k}$ is the probability that, supposing player j has the potato at time 0, player k has the potato at time 2. It's OK if your argument is a bit informal, since this is not a probability class, as long as you yourself are convinced.

   (c) Guess the matrices A^{10} and A^{11}. Once you have given it your best possible guess, compute A^{10} and A^{11} using a computer.[1] Can you give any interpretation?

   (d) Formulate a *conjecture* (guess) about locations of zero entries in A^n for all integers $n > 1$. Prove your conjecture by using the definition of matrix product, or by analyzing the rules of the hot potato game. Better yet, do both.

[1] You can compute many things online for free. For example, go to `https://www.wolframalpha.com` and type in

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{{1,1},{0,2}}^2
```

to compute the square of the matrix above.