Problem §1 Suppose N and d are integers, with $N > d \geq 0$. Let $a_1, \dots, a_N$ be distinct real numbers, and let $b_1, \dots, b_N$ be any real numbers. Prove that there exists a unique polynomial $f \in \mathcal{P}_d(\mathbb{R})$ that comes "closest" to satisfying

$$f(a_1) = b_1, \dots, f(a_N) = b_N.$$

More precisely, prove there exists a unique polynomial $f \in \mathcal{P}_d(\mathbb{R})$ minimizing

$$\sum_{i=1}^{N} (f(a_i) - b_i)^2.$$

Solution: Consider $\mathcal{P}_N(\mathbb{R})$ and its subspace $\mathcal{P}_d(\mathbb{R})$, and define an inner product on $\mathcal{P}_N(\mathbb{R})$ (the subspace $\mathcal{P}_d(\mathbb{R})$ will inherit the same inner product):

$$\langle p, q \rangle = \sum_{i=1}^{N} p(a_i) q(a_i).$$

We first verify that this is actually an inner product:

- Recall from Problem Set F that given $a_1, \dots, a_N$ distinct real values, a unique polynomial in $\mathcal{P}_N(\mathbb{R})$ threads real numbers $b_1, \dots, b_N$. Since $p(x) = \mathbf{0}$ achieves this, uniqueness of the polynomial means that no non-zero polynomial can satisfy $p(a_i) = 0$ for all $a_1, \dots, a_N$. Thus

  $$\langle p, p \rangle = \sum_{i=1}^{N} p(a_i)^2 \geq 0$$

  for all $p(x) \in \mathcal{P}_N(\mathbb{R})$, with equality holding if and only if $p(x) = \mathbf{0}$. Thus $\langle \cdot, \cdot \rangle$ is positive-definite.

- Commutativity of multiplication in $\mathbb{R}$ means

  $$\langle p, q \rangle = \sum_{i=1}^{N} p(a_i) q(a_i) = \sum_{i=1}^{N} q(a_i) p(a_i) = \langle q, p \rangle$$

  for every $p, q \in \mathcal{P}_N(\mathbb{R})$, so $\langle \cdot, \cdot \rangle$ is symmetric.

- For any $p, q, r \in \mathcal{P}_N(\mathbb{R})$, $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

  $$\langle \lambda_1 p + \lambda_2 q, r \rangle = \sum_{i=1}^{N} (\lambda_1 p(a_i) + \lambda_2 q(a_i)) r(a_i) = \lambda_1 \sum_{i=1}^{N} p(a_i) r(a_i) + \lambda_2 \sum_{i=1}^{N} q(a_i) r(a_i) = \lambda_1 \langle p, r \rangle + \lambda_2 \langle q, r \rangle.$$

  Hence $\langle \cdot, \cdot \rangle$ is linear in the first slot.

Thus, the above inner product is, in fact, an inner product.

Now, decompose the vector space $\mathcal{P}_N(\mathbb{R})$ into $\mathcal{P}_d(\mathbb{R})$ and its orthogonal complement:

$$\mathcal{P}_N(\mathbb{R}) = \mathcal{P}_d(\mathbb{R}) \oplus (\mathcal{P}_d(\mathbb{R}))^{\perp}.$$

Let $U = \mathcal{P}_d(\mathbb{R})$, let $e_0, \dots, e_d$ be an orthonormal basis of U, and let $g(x) \in \mathcal{P}_N(\mathbb{R})$ be the unique polynomial that satisfies

$$g(a_1) = b_1, \dots, g(a_N) = b_N$$

(existence and uniqueness come from Problem Set F, again). Project the polynomial onto U:

$$\mathcal{P}_U(g) = \langle g, e_0 \rangle e_0 + \langle g, e_1 \rangle e_1 + \ldots + \langle g, e_d \rangle e_d \in U.$$

From the minimization problem (Axler 6.56), $\mathcal{P}_U(g)$ satisfies

$$\|g - \mathcal{P}_U(g)\| \leq \|g - u\|$$

for any $u \in U$, with equality holding if and only if $u = \mathcal{P}_U(g)$ (in other words, $\mathcal{P}_U(g) \in \mathcal{P}_d(\mathbb{R})$ is the unique polynomial in $\mathcal{P}_d(\mathbb{R})$ that minimizes the norm of $g(x) - u(x)$ for any $u(x) \in \mathcal{P}_d(\mathbb{R})$). Squaring both sides, we get

$$\begin{aligned}
\|g - \mathcal{P}_U(g)\|^2 &\le \|g - u\|^2 \\
\langle g - \mathcal{P}_U(g), g - \mathcal{P}_U(g)\rangle &\le \langle g - u, g - u\rangle \\
\sum_{i=1}^{N} (g(a_i) - \mathcal{P}_U(g)(a_i))^2 &\le \sum_{i=1}^{N} (g(a_i) - u(a_i))^2 \\
\sum_{i=1}^{N} (\mathcal{P}_U(g)(a_i) - b_i)^2 &\le \sum_{i=1}^{N} (u(a_i) - b_i)^2
\end{aligned}$$

for any $u(x) \in \mathcal{P}_d(\mathbb{R})$, with equality holding if and only if $u(x) = \mathcal{P}_U(\mathbb{R})$ (we get the last equation since $g(a_i) = b_i$ for all $a_1, \dots, a_N$, and $(a-b)^2 = (b-a)^2$ for any real numbers $a, b \in \mathbb{R}$). In other words, $\mathcal{P}_U(g)$ is the unique polynomial f in $\mathcal{P}_d(\mathbb{R})$ that minimizes

$$\sum_{i=1}^{N} (f(a_i) - b_i)^2,$$

as desired.

Problem §2 Let $p(x) = x^{12} + x^2 - x + 7$. Let T be a self-adjoint operator on a finite-dimensional inner product space V over $\mathbb{R}$. Prove that $p(T)$ is invertible.

Solution: It suffices to show that $\left\langle (T^{12} + T^2 - T + 7I)v, v \right\rangle \neq 0$ for all non-zero $v \in V$ (recall that a trivial null space implies injectivity, and operators are invertible iff injective). We make two observations:

- For any integer $n \in \mathbb{Z}$, if T is a self-adjoint operator, then

  $$\left\langle T^{2n}v, w \right\rangle = \langle T^n v, T^n w\rangle.$$

  One can quickly verify this by repeating $\left\langle T^{2n}v, w \right\rangle = \left\langle T^{2n-1}v, Tw \right\rangle = \left\langle T^{2n-2}v, T^2 w \right\rangle = \ldots = \langle T^n v, T^n w\rangle$.

- For any two vectors $u, v \in V$, Cauchy-Schwarz gives us

  $$|\langle u, v\rangle| \le \|u\| \cdot \|v\|,$$

  or equivalently,

  $$-\|u\| \cdot \|v\| \le \langle u, v\rangle \le \|u\| \cdot \|v\|,$$

  using basic properties of absolute values.

Let $v \in V$ be a non-zero vector. Then

$$\begin{aligned}
\left\langle \left(T^{12} + T^2 - T + 7I\right), v \right\rangle &= \left\langle T^{12}v, v \right\rangle + \left\langle T^2 v, v \right\rangle - \langle Tv, v\rangle + 7\langle v, v\rangle \\
&= \left\langle T^6 v, T^6 v \right\rangle + \langle Tv, Tv\rangle - \langle Tv, v\rangle + 7\langle v, v\rangle \\
&\ge \|T^6 v\|^2 + \|Tv\|^2 + \|Tv\| \cdot \|v\| + 7\|v\|^2 \qquad \text{[by Cauchy-Schwarz; see observation]} \\
&> 0,
\end{aligned}$$

since v non-zero means $\|v\|^2 > 0$, and clearly $\|\cdot\| \ge 0$ for any vector in V. Thus $(T^{12} + T^2 - T + 7I)v \neq 0$ for any non-zero $v \in V$, so $p(T)$ is injective; in particular, it is invertible as well.

Problem §3 Find the singular values of the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (-4y, x)$.

Solution: We first find the value of $T^* \in \mathcal{L}(V)$. In particular, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we need

$$\begin{aligned} \langle (x_1, y_1), T^*(x_2, y_2) \rangle &= \langle T(x_1, y_1), (x_2, y_2) \rangle \\ &= \langle (-4y_1, x_1), (x_2, y_2) \rangle \\ &= -4x_2 y_1 + x_1 y_2 \\ &= \langle (x_1, y_1), (y_2, -4x_2) \rangle. \end{aligned}$$

Thus $T^*(x, y) = (y, -4x)$, so $T^*T(x) = (x, 16y)$, so $\sqrt{T^*T}(x, y) = (x, 4y)$. Thus the eigenvalues of $\sqrt{T^*T}$ are 1 and 4, each with a corresponding eigenspace $E(\lambda, \sqrt{T^*T})$ with dimension 1. Thus T has singular values 1 and 4.

Problem §4 Let V be an n-dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $T \in \mathcal{L}(V)$ be a linear opeartor, and let $s_1 \leq \cdots \leq s_n$ be its singular values. Prove that for all $v \in V \setminus \{0\}$,

$$s_1 \leq \frac{\|Tv\|}{\|v\|} \leq s_n.$$

Additionally, verify that both these upper and lower bounds for $\|Tv\|/\|v\|$ are achieved by some vectors $v_{\min}, v_{\max} \in V$ respectively.

Solution: Since $\sqrt{T^*T}$ is positive, in particular it is self-adjoint, so by the Spectral Theorem it has an orthonormal basis consisting of eigenvectors of T, say $e_1, \ldots, e_n$, with corresponding real, non-negative eigenvalues (since positive), say $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Then any vector $v \in V \setminus \{0\}$ can be written as a linear combination of these eigenvectors:

$$v = \langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n.$$

Applying $\sqrt{T^*T}v$, we get

$$\begin{aligned} \sqrt{T^*T}v &= \sqrt{T^*T}(\langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n) \\ &= \langle v, e_1 \rangle \sqrt{T^*T}e_1 + \ldots + \langle v, e_n \rangle \sqrt{T^*T}e_n \\ &= \lambda_1 \langle v, e_1 \rangle e_1 + \ldots + \lambda_n \langle v, e_n \rangle e_n. \end{aligned}$$

Since $\|\sqrt{T^*T}v\| = \|Tv\|$, we then have

$$\|Tv\| = \sqrt{\lambda_1^2 |\langle v, e_1 \rangle|^2 + \ldots + \lambda_n^2 |\langle v, e_n \rangle|^2}.$$

Let $\lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_n\}$. Then

$$\|Tv\| \leq \sqrt{\lambda_{\max}^2 |\langle v, e_1 \rangle|^2 + \ldots + \lambda_{\max}^2 \langle v, e_n \rangle} = |\lambda_{\max}| \sqrt{|\langle v, e_1 \rangle|^2 + \ldots + |\langle v, e_n \rangle|^2}$$

(by factoring out $\sqrt{\lambda_{\max}^2}$). However,

$$\|v\| = \sqrt{|\langle v, e_1 \rangle|^2 + \ldots + |\langle v, e_n \rangle|^2},$$

and $|\lambda_{\max}| = \lambda_{\max} = s_n$ (since singular values of T *are* eigenvalues of $\sqrt{T^*T}$, and all eigenvalues are non-negative real numbers, so $|\lambda| = \lambda$), so

$$\|Tv\| \leq \lambda_{\max} \|v\|,$$

or equivalently

$$\frac{\|Tv\|}{\|v\|} \leq s_n \quad [\text{since } v \neq \mathbf{0}, \|v\| > 0].$$

Similarly, if we define $\lambda_{\text{min}} = \min\{\lambda_1, \dots, \lambda_n\}$, then we get

$$\|Tv\| \geq |\lambda_{\text{min}}|\|v\| = s_1\|v\|,$$

or equivalently

$$s_1 \leq \frac{\|Tv\|}{\|v\|}.$$

Thus, for any $v \in V \setminus \{\mathbf{0}\}$, we get

$$s_1 \leq \frac{\|Tv\|}{\|v\|} \leq s_n.$$

Finally, let e_{max} be the eigenvector corresponding to $\lambda_{\text{max}} = s_n$. Then

$$\|Te_{\text{max}}\| = \|\sqrt{T^*T}e_{\text{max}}\| = \|\lambda_{\text{max}}e_{\text{max}}\| = \sqrt{|\lambda_{\text{max}}|^2 |\langle e_{\text{max}}, e_{\text{max}}\rangle|^2} = |\lambda_{\text{max}}| = \lambda_{\text{max}}$$

(since $\langle e_{\text{max}}, e_{\text{max}}\rangle = 1$). Clearly $\|e_{\text{max}}\| = 1$; thus

$$\frac{\|Te_{\text{max}}\|}{\|e_{\text{max}}\|} = \|Te_{\text{max}}\| = \lambda_{\text{max}} = s_n.$$

Similarly, if e_{min} is the eigenvector corresponding to $\lambda_{\text{min}} = s_1$, then

$$\|Te_{\text{min}}\| = |\lambda_{\text{min}}| = \lambda_{\text{min}}$$

(following the same steps as above), so

$$\frac{\|Te_{\text{min}}\|}{\|e_{\text{min}}\|} = \|Te_{\text{min}}\| = \lambda_{\text{min}} = s_1.$$

Thus the upper and lower bounds of $\|Tv\|/\|v\|$ are achieved by $e_{\text{max}}, e_{\text{min}} \in V$ respectively.