Problem §3 (4.15) Suppose V is an F-vector space, $\mathcal{A}$ and $\mathcal{B}$ are subsets of V, and

- $\mathcal{B}$ is linearly independent.
- $|A| = |B|$.
- $\operatorname{span}(B) \subseteq \operatorname{span}(A)$.

Prove $\operatorname{span}(A) = \operatorname{span}(B)$.

Solution: We start with 2 lemmas.

Lemma 1. *Let V be an F-vector space. Given a subset $\mathcal{A} \subseteq V$, $\operatorname{span}(A)$ is a subspace of V; and if $\mathcal{A}$ is linearly independent, then $\mathcal{A}$ forms a basis of $\operatorname{span}(A)$.*

Proof. Clearly, given $v_1, v_2 \in \operatorname{span}(A)$, we have $v_1 + v_2 \in \operatorname{span}(A)$ (since for each component $a_i\alpha_i$ of v_1 and corresponding $a_i'\alpha_i$ of v_2, $(a_i + a_i') \in F$ by closure of addition in fields, so $v_1 + v_2$ is still in $\operatorname{span}(A)$ by definition); and for any $c \in F$, $cv \in \operatorname{span}(A)$ as well (like above, for each component $a_i\alpha_i$ of v, $(ca_i) \in F$, so $cv \in \operatorname{span}(A)$ by definition). Finally, setting each a_i coefficient to 0 shows that $\mathbf{0} \in \operatorname{span}(A)$. Hence $\operatorname{span}(A)$ is a subspace of V.

If $\mathcal{A}$ is linearly independent, clearly $\mathcal{A}$ forms a basis for $\operatorname{span}(A)$ (since $\mathcal{A}$ is both linearly independent and spans $\operatorname{span}(A)$. Wow!) ☐

Lemma 2. *Let a set $\mathcal{A}$ span a vector space V. If a linearly independent set $\mathcal{B}$ has the same number of elements as $\mathcal{A}$, $\mathcal{A}$ is also linearly independent, and is thus a basis for V.*

Proof. Suppose $\mathcal{A}$ is linearly dependent. Then for some $\alpha_i \in \mathcal{A}$, $\alpha_i \in \operatorname{span}(\mathcal{A} \setminus \{\alpha_i\})$, so $\mathcal{A} \setminus \{\alpha_i\}$ spans V as well. But then $|\mathcal{A} \setminus \{\alpha_I\}| < |\mathcal{A}| = |\mathcal{B}|$ a linearly independent set in V, a contradiction of Lemma 4.24. Thus $\mathcal{A}$ is linearly independent in V as well, and thus is a basis for V. ☐

Lemma 3. *Let $\mathcal{A}$ be a basis for a vector space V. If a linearly independent set $\mathcal{B}$ has the same number of elements as $\mathcal{A}$, then $\mathcal{B}$ is a basis for V as well.*

Proof. Suppose $\mathcal{B}$ is not a basis for V. Then for some $v \in V$ where $v \notin \operatorname{span}(\mathcal{B})$, $\mathcal{B}' = \mathcal{B} \cup \{v\}$ is linearly independent. However, Lemma 4.24 tells us that the size of any linearly independent set in V is less than or equal to the length of any spanning set in V, and since $\mathcal{A}$ spans V and $|\mathcal{A}| = |\mathcal{B}| < |\mathcal{B}'|$, this is a contradiction. Hence $\mathcal{B}$ must span V as well, and so $\mathcal{B}$ is a basis for V. ☐

Lemma 1 tells us that $\operatorname{span}(\mathcal{A})$ is a subspace of V (and thus also a vector space, allowing us to apply theorems about bases and dimensions of vector spaces). Since $\operatorname{span}(\mathcal{B}) \subseteq \operatorname{span}(\mathcal{A})$, clearly $\mathcal{B} \subseteq \operatorname{span}(\mathcal{A})$. Thus $\mathcal{B}$ is a linearly independent set in $\operatorname{span}(\mathcal{A})$. Since $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements, Lemma 2 tells us that $\mathcal{A}$ is linearly independent in $\operatorname{span}(\mathcal{A})$ as well, and thus is a basis. Lemma 3 then tells us that $\mathcal{B}$ is a basis for $\operatorname{span}(\mathcal{A})$ too; thus $\operatorname{span}(\mathcal{A}) = \operatorname{span}(\mathcal{B})$, as required.

Problem §4 Suppose V, W are finite-dimensional F-vector spaces. Let $L : V \to W$ be a linear transformation.

- If L injective, prove $\dim V \leq \dim W$.
- If L surjective, prove $\dim V \geq \dim W$.

Solution: Let $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_m\}$ be basis for V and W respectively.

- Suppose $L : V \to W$ is injective. For any $v \in V$, we can write

$$v = \sum_{i=1}^{n} a_i v_i, \text{ where } a_i \in F.$$

Clearly, $\mathcal{B} = \{L(v_1), \ldots, L(v_n)\}$ spans $\operatorname{range} L$, since for any $L(v) \in \operatorname{range} L$, we have

$$L(v) = L(a_1 v_1 + \ldots + a_n v_n) = a_1 L(v_1) + \ldots + a_n L(v_n).$$

We claim that $\mathcal{B}$ is a basis for $\operatorname{range} L$.

Suppose $L(v),\ L(v') \in \operatorname{range} L$ are different ways of representing a vector in $\operatorname{range} L$; in other words, $L(v) = L(v')$ and

$$L(v) = \sum_{i=1}^{n} a_i L(v_i),\ L(v') = \sum_{i=1}^{n} a_i' L(v_i),\ a_i \neq a_i'.$$

By linearity,

$$L(v) = \sum_{i=1}^{n} a_i L(v_i) = \sum_{i=1}^{n} L(a_i v_i)$$

and

$$L(v') = \sum_{i=1}^{n} a_i' L(v_i) = \sum_{i=1}^{n} L(a_i' v_i).$$

L injective then means

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i' v_i,$$

and since $\{v_1, \ldots, v_n\}$ is a basis for V (and thus is linearly independent), we necessarily have

$$\sum_{i=1}^{n} (a_i - a_i') v_i = 0,\ a_i - a_i' = 0.$$

and hence $a_i = a_i'$. Equivalently, $L(v)$ and $L(v')$ are the same, and thus every $L(v) \in \operatorname{range} L$ can be represented uniquely as

$$L(v) = \sum_{i=1}^{n} a_i L(v_i).$$

In other words, $\mathcal{B}$ is a basis for $\operatorname{range} L$.

Since $\operatorname{range} L \subseteq W$ and $\mathcal{B}$ is linearly independent in $\operatorname{range} L$ (and thus in W as well), by Lemma 4.24 any spanning set must have at least as many elements as $\mathcal{B}$. Hence any basis of W must have at least as many elements as $\mathcal{B}$; and since $|\mathcal{B}| = \dim V$, we get $\dim V \leq \dim W$, as required.

- Suppose $L : V \to W$ is surjective. From before, we know that $\mathcal{B} = \{L(v_1), \ldots, L(v_n)\}$ spans $\operatorname{range} L$; since L is surjective (and so $\operatorname{range} L = W$), $\mathcal{B}$ spans W as well. By Lemma 4.24, the size of any linearly independent set in W is less than or equal to the size of any spanning set of W. Since $\{w_1, \ldots, w_m\}$ is linearly independent, we thus have $\dim W \leq \operatorname{range} L = \dim V$. Therefore $\dim V \geq \dim W$, as required.

Problem §5 (4.18) Let V be a finite-dimensional F-vector space, and let $U \subseteq V$ be a vector subspace.

(a) Prove that U is finite-dimensional.

(b) Prove that $\dim_F U \leq \dim_F V$.

(c) Prove that

$$U = V \iff \dim_F U = \dim_F V.$$

Solution:

(a) If $U = \{0\}$, U is clearly finite-dimensional, so suppose $U \neq \{0\}$. Let $u_1 \in U$ be a non-zero vector. If $U = \operatorname{span}(\{u_1\})$, then we are done; otherwise, continue adding non-zero vectors $u_j \in U$ such that

$$u_j \notin \operatorname{span}(\{u_1, \ldots, u_{j-1}\}),$$

until $\{u_1, \ldots, u_j\}$ forms a spanning set of U. With each addition, $\{u_1, \ldots, u_j\}$ is a linearly independent set by construction (since each added vector was not in the span of the previous vectors). Moreover, every linearly independent set $\{u_1, \ldots, u_j\}$ is in V, since each $u_i \in U \subseteq V$.

Let $n = \dim V$. Since any basis of V is spanning, and by Lemma 4.24, the number of elements in any linearly independent set in V must be less than or equal to the length of any spanning set in V, the length of $\{u_1, \ldots, u_j\}$ must be less than or equal to n. Thus the above process will eventually terminate (it cannot repeat infinitely — or past $j = n$ — since the number of elements must be less than or equal to n), and so we are left with a finite linearly independent spanning set $\{u_1, \ldots, u_j\}$ of U. Thus U is finite-dimensional.

(b) From above, we see that a basis $\{u_1, \ldots, u_j\}$ of U cannot have more elements than $n = \dim V$. Hence $\dim_F U \leq \dim_F V$.

Alternatively, let $\{u_1, \ldots, u_m\}$ be a basis for U. Then $\{u_1, \ldots, u_m\} \subseteq U \subseteq V$ is a linearly independent set of vectors in V. By Lemma 4.24, any linearly independent set of vectors in V cannot have more elements than any spanning set of V. Since a basis $\{v_1, \ldots, v_n\}$ of V is the smallest spanning set of V and has $n = \dim V$ elements, any linearly independent set cannot have more than n elements. Thus $|\{u_1, \ldots, u_m\}| = \dim_F U \leq \dim_F V = n = |\{v_1, \ldots, v_n\}|$.

(c) Suppose $U = V$. Then a basis $\{v_1, \ldots, v_n\}$ of U is also a basis of V, and so $\dim_F U = \dim_F V$.

Conversely, suppose $\dim_F U = \dim_F V = n$, and let $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$ be bases for U and V respectively. Since U is a subspace of V, we know that $\{u_1, \ldots, u_n\}$ is a linearly independent set of vectors in V. From Lemma 3 (of Problem §3), since $\{u_1, \ldots, u_n\}$ is a linearly independent set of vectors with the same number of elements as a basis $\{v_1, \ldots, v_n\}$ of V, $\{u_1, \ldots, u_n\}$ is a basis of V as well. Thus

$$U = \operatorname{span}(\{u_1, \ldots, u_n\}) = V,$$

and so $U = V$, as required.