Problem §1 (Axler 3.B.30) Suppose φ_1 and φ_2 are linear maps from V to $\mathbb{F}$ that have the same null space. Show that there exists a constant $c \in \mathbb{F}$ such that $\varphi_1 = c\varphi_2$.

Solution: We start with a lemma (which is actually Axler 3.B.29, but proved to be very helpful):

Lemma 1. *Let $\varphi \in \mathcal{L}(V, \mathbb{F})$, and $u \in V$ not in $\operatorname{null}\varphi$. Then*

$$V = \operatorname{null}\varphi \oplus \{au \mid a \in \mathbb{F}\}.$$

Proof. Suppose $au \in \operatorname{null}\varphi$. Then

$$0 = \varphi(au) = a\varphi(u).$$

Since $u \notin \operatorname{null}\varphi$, $\varphi(u) \neq 0$, so $a = 0$; hence $\operatorname{null}\varphi \cap \{au \mid a \in \mathbb{F}\} = \{0\}$, and by Proposition 1.45, $\operatorname{null}\varphi \oplus \{au \mid a \in \mathbb{F}\}$ is a direct sum.

Now, for any $v \in V$, suppose $\varphi(v) = k,\ \varphi(u) = r$ for some $k, r \in \mathbb{F}$. Observe that

$$\varphi(v - \frac{k}{r}u) = \varphi(v) - \frac{k}{r}\varphi(u) = k - \frac{k}{r} \cdot r = 0,$$

so $v - \frac{k}{r}u \in \operatorname{null}\varphi$. Since $\frac{k}{r} \in \mathbb{F}$, $\frac{k}{r}u \in \{au \mid a \in \mathbb{F}\}$, and so any $v \in V$ can be written as

$$v = (v - \frac{k}{r}u) + \frac{k}{r}u.$$

Hence $V = \operatorname{null}\varphi \oplus \{au \mid a \in \mathbb{F}\}$, as required. $\square$

If $\operatorname{null}\varphi_1 = \operatorname{null}\varphi_2 = V$, then any $c \in \mathbb{F}$ works, so assume $\operatorname{null}\varphi \neq V$.

By the Lemma, any $v \in V$ can be represented as

$$v = w + au,\ w \in \operatorname{null}\varphi,\ u \notin \operatorname{null}\varphi,\ a \in \mathbb{F}.$$

Suppose $v \in V$, $v \notin \operatorname{null}\varphi$ (since $0 = c0$ is unenlightening), and let $\varphi_1(v) = k,\ \varphi_2(v) = r$. Then

$$c\varphi_1(v) = c\varphi_1(w + au) = c(\varphi_1(w) + a\varphi_1(u)) = cak.$$

Setting $c = \frac{r}{k} \in \mathbb{F}$, we get

$$\begin{aligned}\frac{r}{k}\varphi_1(v) = \frac{r}{k}ak = ar = 0 + a\varphi_2(u) &= \varphi_2(w) + \varphi_2(au) \\ &= \varphi_2(w + au) \\ &= \varphi_2(v).\end{aligned}$$

Hence $c\varphi_1(v) = \varphi_2(v)$ for some $c \in \mathbb{F}$, as required.

Problem §2 A square matrix $A \in \mathbb{F}^{m,n}$ is called *upper triangular* if $A_{j,k} = 0$ whenever $j > k$. Let A, C be $n \times n$ upper triangular matrices.

(a) Prove that AC is upper triangular.

(b) Also, prove that $(AC)_{j,j} = A_{j,j}C_{j,j}$ for each $j = 1, \ldots, n$.

Solution:

(a) Let $A, C \in \mathbb{F}^{m,n}$ be upper triangular matrices. Then for any $(AC)_{j,k}$ where $j > k$, we have

$$(AC)_{j,k} = \sum_{i=1}^{n} A_{j,i}C_{i,k} = \sum_{i=1}^{j-1} A_{j,i}C_{i,k} + \sum_{i=j}^{n} A_{j,i}C_{i,k}.$$

But for every $1 \leq i < j$, $A_{j,i} = 0$, and for any $j \leq i < n$, $j > k$, so $C_{i,k} = 0$. Hence $(AC_{j,k}) = 0 + 0 = 0$, and so AC is necessarily upper triangular as well.

(b) For any $(AC)_{j,j}$, we have

$$(AC)_{j,j} = \left(\sum_{i=1}^{j-1} A_{j,i}C_{i,j}\right) + A_{j,j}C_{j,j} + \left(\sum_{i=j+1}^{n} A_{j,i}C_{i,j}\right).$$

For any $1 \leq i < j$, $A_{j,i} = 0$ (since $j > i$); and for any $j < i < n$, $C_{i,j} = 0$ (since $i > j$). Hence $(AC)_{i,j}$ becomes

$$(AC)_{i,j} = 0 + A_{j,j}C_{j,j} + 0 = A_{j,j}C_{j,j},$$

as required.

Problem §3 Players $1, 2, 3, 4, 5, 6$ are seated around a circle, in that order. One of them is holding a hot potato. At each time $t = 1, 2, \ldots$ seconds from the beginning of the game, whoever is holding the potato passes it either to the person to their immediate right or to the person to their immediate left, with equal probability.

(a) Let $A \in \mathbb{R}^{6,6}$ be the matrix in which $(A)_{j,k}$ is the probability that, supposing player j has the potato at time 0, player k has the potato at time 1. Write down A.

(b) Give an argument that $(A^2)_{j,k}$ is the probability that, supposing player j has the potato at time 0, player k has the potato at time 2.

(c) Guess the matrices A^{10}, A^{11}. Once you have the best possible guess, compute the matrices and give an interpretation.

(d) Formulate a conjecture about locations of zero entries in A^n for all integers $n > 1$. Prove your conjecture by using the definition of matrix product, or by analyzing the rules of the game.

Solution:

(a) Since every player passes it following a second, at time $t = 1$, $A_{j,j} = 0$. So

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

(b) For each $(A^2)_{j,k}$, we have

$$(A^2)_{j,k} = \sum_{i=1}^{6} A_{j,i}A_{i,k} = A_{j,1}A_{1,k} + A_{j,2} + A_{2,k} + \ldots + A_{j,6}A_{6,k}.$$

For every multiple $A_{j,i}A_{i,k}$, it represents the probability of both:

- Given person j started with the hot potato, person i had the hot potato at time 1. In other words, the probability that j passed the hot potato to i.
- Given person i **has** the hot potato, person k received the hot potato.

Note the important difference in wording for the second probability: instead of interpreting $A_{i,k}$ as person i starting with the hot potato and then giving it to k after one second, we can instead think of it as, **given that person i had the hot potato at time n, what is the probability that person k had the hot potato at time $n+1$?**

Thus, the sum $(A^2)_{j,k}$ is just the sum of all the probabilities of, with person j starting with the hot potato, person j passing it to person i, and then person i passing it to person k. In other words, $(A^2)_{j,k}$ represents the probability that, supposing j started with the potato, k would have the potato after 2 passes (or 2 seconds).

(c)

$$A^{10} = \frac{1}{1024}\begin{pmatrix} 342 & 0 & 341 & 0 & 341 & 0 \\ 0 & 342 & 0 & 341 & 0 & 341 \\ 341 & 0 & 342 & 0 & 341 & 0 \\ 0 & 341 & 0 & 342 & 0 & 341 \\ 341 & 0 & 341 & 0 & 342 & 0 \\ 0 & 341 & 0 & 341 & 0 & 342 \end{pmatrix}.$$

$$A^{11} = \frac{1}{2048}\begin{pmatrix} 0 & 683 & 0 & 682 & 0 & 683 \\ 683 & 0 & 683 & 0 & 682 & 0 \\ 0 & 683 & 0 & 683 & 0 & 682 \\ 682 & 0 & 683 & 0 & 683 & 0 \\ 0 & 682 & 0 & 683 & 0 & 683 \\ 683 & 0 & 682 & 0 & 683 & 0 \end{pmatrix}.$$

From part (b), it makes sense to interpret $(A^{10})_{j,k}$ as the probability that, given person j started with the hot potato, person k has the hot potato at time 10 seconds; similarly, $(A^{11})_{j,k}$ is the probability that, given person j started with the hot potato, person k has the hot potato at time 11 seconds.

(d) Inspecting $n = 2, 3, \ldots, 10, 11$, we see that when n is even, any $(A)_{j,k}$ where j, k are either both even or both odd are non-zero, while any $(A)_{j,k}$ where j, k are of different parities are zero; and when n is odd, the opposite is true. Thus, it seems that any A^n with n even has zero entries whenever j, k have opposite parities, while any A^n with n odd has zero entries whenever j, k have identical parities.

From the rules, if any player j has the hot potato at any time n, at time $n + 1$, the player j must pass it to either $j + 1$ or $j - 1$ (with modular arithmetic at the edges). Thus, the receiving player k must have an opposite parity from j. Hence, if n is even (e.g. the hot potato has been passed an even number of times), then the receiving player k **must** have the same parity as j, since the hot potato has swapped parities n times (and since n is even, the parity of k is the same as the parity of j). Similarly, if n is odd, then the receiving player k **must** have a different parity from j, since the hot potato has swapped parities an odd number of times (so even $\rightarrow$ odd, odd $\rightarrow$ even). Therefore, we see that if n is even, then any $(A)_{j,k}$ with differing parities of j, k is not possible (and therefore 0), while if n is odd, then any $(A)_{j,k}$ with identical parities of j, k is not possible (and therefore 0).