Problem §1 Are the following subsets actually subspaces? If yes, just write yes. If no, show briefly how the subset fails to be a subspace.

1. Let V be any vector space. Is $\varnothing$ a subspace of V?
2. Let V be any vector space. Is V a subspace of V?
3. Is $W = \{(\alpha, i\alpha) \mid \alpha \in \mathbb{C}\}$ a subspace of $\mathbb{C}^2$?
4. Is $W = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \vee y = 0 \vee z = 0\}$ a subspace of $\mathbb{R}^3$?
5. Let V be a vector space over $\mathbb{F}$, and let $v \in V$ be any vector in V. Is
$$W = \{av \mid a \in \mathbb{F}\}$$
a subspace of V?
6. Given $V_{\mathbb{F}}$, and $v, w \in V$, is
$$W = \{av + bw \mid a, b \in \mathbb{F}\}$$
a subspace of V?
7. Given $V_{\mathbb{F}}$, and let U, W be subspaces of V. Is
$$X = \{u + w \mid u \in U, w \in W\}$$
a subspace of V?

Solution:

1. No. $\mathbf{0} \notin \varnothing$.
2. Yes
3. Yes
4. No. Choose $(1, 0, 1), (0, 1, 0) \in W$. Then $(1, 0, 1) + (0, 1, 0) = (1, 1, 1) \notin W$.
5. Yes
6. Yes (for 5 and 6, aren't these just another way of describing V?)
7. Yes

Problem §2 Let V be the real vector space of functions $f : \mathbb{R} \to \mathbb{R}$.

1. Let W be the set of functions $f : \mathbb{R} \to \mathbb{R}$ that are non-decreasing (that is, $f(b) \geq f(a)$ whenever $b \geq a$). Is W a subspace of V?
2. Let W be the set of functions $f : \mathbb{R} \to \mathbb{R}$ that have bounded support (that is, $\exists N \in \mathbb{R}_{\geq 0}$ s.t. $|x| > N$ implies $f(x) = 0$). Is W a subspace of V?
3. Let W be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that the range of f has exactly one element. Is W a subspace of V?
4. Let W be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that the range of f has exactly two elements. Is W a subspace of V?
5. Let W be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that the range of f has finitely many elements. Is W a subspace of V?

Solution:

1. No; consider $-1 \cdot f$ for some $f \in W$ (say, $f(x) = x$). Then if $a \leq b$, $-1 \cdot f(b) \leq -1 \cdot f(a)$, and so $-1 \cdot f$ is decreasing and thus not in W.

2. Yes

3. Yes

4. No; given two functions $f : \mathbb{R} \to \{a, b\} \subseteq \mathbb{R}$, $g : \mathbb{R} \to \{c, d\} \subseteq \mathbb{R}$, the range of $f + g$ has four possible values: $\{a + c, a + d, b + c, b + d\}$.

5. Yes