**Problem §1** Choose any three problems from 1.2(a-i) as a warmup. Then complete Exercise 1.3.

- 1.2: Use truth tables to prove:
  - 1.2.a: $P \iff \neg(\neg P)$.
  - 1.2.c: $(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)$.
  - 1.2.d: $(P \Rightarrow Q) \iff (\neg P) \vee Q$.
- 1.3: Let P and Q be statements.
  (a) Prove that
  $$P \veebar \neg P$$
  is true, and explain why this justifies the Law of the Excluded Middle (which states that exactly one of $P$ and $\neg P$ is true).
  (b) Prove that
  $$(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$$
  is true, and explain why this justifies the method of Proof by Contradiction (which states that in order to prove that $P$ is true, it suffices to show that $\neg P$ is false).

*Solution:*

- 1.2.a

| P | $\neg P$ | $\neg(\neg P)$ |
|---|---|---|
| T | F | T |
| F | T | F |

- 1.2.c

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q \Rightarrow \neg P$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | T | T |
| F | F | T | T |

- 1.2.d

| $P$ | $Q$ | $\neg P$ | $P \Rightarrow Q$ | $\neg P \vee Q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

- 

1.3.a

| $P$ | $\neg P$ | $P \veebar \neg P$ |
|---|---|---|
| T | F | T |
| F | T | T |

- Since the statement is true regardless of $P$, $P \veebar \neg P$ is true. This also justifies the Law of the Excluded Middle, as an XOR statement is true only when one, but not both, of the statements are true; hence only one of $P$ and $\neg P$ may be true at once in order for $P \veebar \neg P$ to be true.

1.3.b

| $P$ | $Q$ | $\neg P$ | $\neg Q$ | $P \Rightarrow Q$ | $\neg Q \Rightarrow \neg P$ | $(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

- Since the statement is true regardless of $P$ or $Q$, $(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$ is true. This also serves to justify the Proof by contradiction:

  Let $S$ be a statement $P \Rightarrow Q$. We wish to show that the negation of S being false implies that S is true, or equivalently, $\neg(\neg S) \Rightarrow S$. Using results from 1.2 (a, b, d), we get

$$\begin{aligned}\neg(\neg S) &= \neg(\neg\,(P \Rightarrow Q)) \\ &= \neg(\neg\,(\neg P \vee Q)) \\ &= \neg\,(P \wedge \neg Q) \\ &= \neg P \vee \neg\,(\neg Q) \\ &= \neg(\neg Q) \vee \neg P \\ &= \neg Q \Rightarrow \neg P.\end{aligned}$$

  Thus $(\neg(\neg S) \implies S) = ((\neg Q \Rightarrow \neg P) \implies (P \Rightarrow Q))$, as required.

**Problem §2** Complete Exercise 1.7: *Prove each of the following formulas:*

(a) $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$

(b) $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$

(c) Suppose $S, T \subseteq U$. Then

$$(S \cup T)^c = S^c \cap T^c \text{ and } (S \cap T)^c = S^c \cup T^c.$$

(d) $S \Delta T = (S \cup T) \setminus (S \cap T) = (S \setminus T) \cup (T \setminus S)$

*Solution:*

(a) *Proof.* Let $e \in S \cap (T \cup U)$. Then

$$\begin{aligned}&(e \in S) \wedge (e \in T \vee e \in U) \\ =&\,(e \in S \wedge e \in T) \vee (e \in S \wedge e \in U)\,.\end{aligned}$$

Thus $e \in (S \cap T) \cup (S \cap U)$, and so $S \cap (T \cup U) \subseteq (S \cap T) \cup (S \cap U)$.
Conversely, let $e \in (S \cap T) \cup (S \cap U)$. Then

$$\begin{aligned} &(e \in S \wedge e \in T) \vee (e \in S \wedge e \in U) \\ =\ &e \in S \wedge (e \in T \vee e \in U)\,. \end{aligned}$$

Thus, $e \in S \cap (T \cup U)$, and so $(S \cap T) \cup (S \cap U) \subseteq S \cap (T \cup U)$.
Since both are subsets of each other, $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$. □

(b) *Proof.* Let $e \in S \cup (T \cap U)$. Then

$$\begin{aligned} &e \in S \vee (e \in T \wedge e \in U) \\ =\ &(e \in S \vee e \in T) \wedge (e \in S \vee e \in U)\,. \end{aligned}$$

Thus $e \in (S \cup T) \cap (S \cup U)$, and so $S \cup (T \cap U) \subseteq (S \cup T) \cap (S \cup U)$.
Conversely, let $e \in (S \cup T) \cap (S \cup U)$. Then

$$\begin{aligned} &(e \in S \vee e \in T) \wedge (e \in S \vee e \in U) \\ =\ &e \in S \vee (e \in T \wedge e \in U)\,. \end{aligned}$$

Thus, $e \in S \cup (T \cap U)$, and so $(S \cup T) \cap (S \cup U) \subseteq S \cup (T \cap U)$.
Since both are subsets of each other, $s \cup (t \cap u) = (s \cup t) \cap (s \cup u)\,.$ □

(c) *Proof.* Let $e \in (S \cup T)^c$. Then

$$\begin{aligned} &e \in U \wedge \neg (e \in S \vee e \in T) \\ =\ &e \in U \wedge (e \notin S \wedge e \notin T) \\ =\ &(e \in U \wedge e \notin S) \wedge (e \in U \wedge e \notin T)\,. \end{aligned}$$

Thus, $e \in S^c \cap T^c$, and so $(S \cup T)^c \subseteq S^c \cap T^c$.
Conversely, let $e \in S^c \cap T^c$. Then

$$\begin{aligned} &(e \in U \wedge e \notin S) \wedge (e \in U \wedge e \notin T) \\ =\ &e \in U \wedge (e \notin S \wedge e \notin T) \\ =\ &e \in U \wedge \neg (e \in S \vee e \in T)\,. \end{aligned}$$

Thus, $e \in (S \cup T)^c$, and so $S^c \cap T^c \subseteq (S \cup T)^c$.
Since both subsets are equal, $(S \cup T)^c = S^c \cap T^c$.

Now, let $e \in (S \cap T)^c$. Then

$$\begin{aligned} &e \in U \wedge \neg (e \in S \wedge e \in T) \\ =\ &e \in U \wedge (e \notin S \vee e \notin T) \\ =\ &(e \in U \wedge e \notin S) \vee (e \in U \wedge e \notin T)\,. \end{aligned}$$

Thus, $e \in S^c \cup T^c$, and so $(S \cap T)^c \subseteq S^c \cup T^c$.
Conversely, let $e \in S^c \cup T^c$. Then

$$\begin{aligned} &(e \in U \wedge e \notin S) \vee (e \in U \wedge e \notin T) \\ =\ &e \in U \wedge (e \notin S \vee e \notin T) \\ =\ &e \in U \wedge \neg (e \in S \wedge e \in T)\,. \end{aligned}$$

Thus, $e \in (S \cap T)^c$, and so $S^c \cup T^c \subseteq (S \cap T)^c$.
Since both are subsets of each other, $(S \cap T)^c = S^c \cup T^c$. □

(d) *Proof.* Let $e \in (S \cup T) \setminus (S \cap T)$. Then

$$\begin{aligned}
&(e \in S \vee e \in T) \wedge \neg(e \in S \wedge e \in T) \\
=\,&(e \in S \vee e \in T) \wedge (e \notin S \vee e \notin T) \\
=\,&((e \in S \vee e \in T) \wedge e \notin S) \vee ((e \in S \vee e \in T) \wedge e \notin T) \\
=\,&((e \in S \wedge e \notin S) \vee (e \in T \wedge e \notin S)) \vee ((e \in S \wedge e \notin T) \vee (e \in T \wedge e \notin T)) \\
=\,&(e \in T \wedge e \notin S) \vee (e \in S \wedge e \notin T).
\end{aligned}$$

Thus, $e \in (S \setminus T) \cup (T \setminus S)$, and so $(S \cup T) \setminus (S \cap T) \subseteq (S \setminus T) \cup (T \setminus S)$.
Conversely, let $e \in (S \setminus T) \cup (T \setminus S)$. Then

$$\begin{aligned}
&(e \in T \wedge e \notin S) \vee (e \in S \wedge e \notin T) \\
=\,&(e \in S \wedge e \notin T) \vee (e \in T \wedge e \notin S) \\
=\,&((e \in S \wedge e \notin S) \vee (e \in T \wedge e \notin S)) \vee ((e \in S \wedge e \notin T) \vee (e \in T \wedge e \notin T)) \\
=\,&((e \in S \vee e \in T) \wedge e \notin S) \vee ((e \in S \vee e \in T) \wedge e \notin T) \\
=\,&(e \in S \vee e \in T) \wedge (e \notin S \vee e \notin T) \\
=\,&(e \in S \vee e \in T) \wedge \neg(e \in S \wedge e \in T).
\end{aligned}$$

Thus, we get the statement $e \in (S \cup T) \setminus (S \cap T)$, and so $(S \setminus T) \cup (T \setminus S) \subseteq (S \cup T) \setminus (S \cap T)$.
Since both are subsets of each other, $(S \cup T) \setminus (S \cap T) = (S \setminus T) \cup (T \setminus S)$. $\square$

From these problems, we observe that sets and logical statements are quite similar. A set is analogous to a logical statement, and the operators union and intersection resemble the logical "or" and "and" respectively (specifically, given sets $S, T$, $e \in S \cup T$ is equivalent to $e \in S \vee e \in T$, and $e \in S \cap T$ is equivalent to $e \in S \wedge e \in T$). Given a well defined complement of $S$, the complement $S^c$ is analogous to the logical "not" (just as only one of $P$ and $\neg P$ may be true, only one of $e \in S^c$ and $e \in S$ may be true). The symmetric difference is analogous to the "xor" operator in the sense that an element $e$ being in $S \Delta T$ meaning $e$ is in $S$ or $T$, but not both, is similar in structure to notion that $P \veebar Q$ means that in order to be true, either P or Q could be true, but not both.

**Problem §3** Complete Exercise 1.16:

- Let $S, T$ be finite sets with $|S| = |T|$, and let $f : S \to T$ be a function from $S$ to $T$. Prove the following are equivalent:
  - $f$ is injective.
  - $f$ is surjective.
  - $f$ is bijective.

and Exercise 1.17:

- Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is injective, but not surjective.
- Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective, but not injective.

*Solution:*

- (1.16)

  *Proof.* Let $n = |S| = |T|$. We start by showing $f$ injective implies $f$ surjective. Let $f$ be an injective function, and suppose that $f$ is not surjective. Then $\exists t \in T$ such that $\forall s \in S, f(s) \neq t$; and so $|\operatorname{im} S| < n$. By the definition of a function, every $s \in S$ is mapped to some element $f(s) \in T$; and since $|S| = n$ and $|\operatorname{im} S| < n$, at least one $e \in \operatorname{im} S$ is mapped to by at least two distinct elements $s, s' \in S$ (analogously, imagine each $e \in \operatorname{im} S$ represents a "hole", and each $s \in S$ a pigeon; since there are at most $n-1$ holes, and $n$ pigeons, by the PHP, at least one hole must have at least two distinct pigeons).

But this implies that $e = f(s) = f(s'), s \neq s'$, a contradiction to injectivity. Thus, if $f$ is injective, then $f$ must be surjective as well.

Now, we show that $f$ surjective implies $f$ injective. Let $f$ be a surjective function, and suppose that $f$ is not injective. Then $\exists s, s' \in S$ such that $f(s) = f(s'), s \neq s'$. By definition of a function, each $s \in S$ is mapped to one and only one $f(s) \in \operatorname{im} S$. But since $f$ is not injective, at least one $f(s) \in \operatorname{im} S$ is mapped to by at least two distinct $s, s' \in S$ (i.e. $\exists s, s' \in S, \exists f(s), f(s') \in \operatorname{im} S, f(s) = f(s'), s \neq s'$), which implies that $|\operatorname{im} S| < n$ (equivalently, at least one $t \in T$ is not mapped to by any $s \in S$), a contradiction to surjectivity. Thus, if $f$ is surjective, then $f$ must be injective as well.
Since $f$ injective implies $f$ surjective, and $f$ surjective implies $f$ injective, if $f$ is either injective or surjective, it is bijective as well; and trivially, $f$ bijective implies both injective and surjective. Thus the three statements are equivalent. $\square$

- (1.17)
  - Let

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto f(n) = n + 1. \end{aligned}$$

    $f$ is injective, as no two $n_1, n_2 \in \mathbb{N}$ share a $succ(n)$ unless $n_1 = n_2$ (equivalently, $n_1 + 1 = n_2 + 1$ implies $n_1 = n_2$). $f$ is also not surjective, as $1 \notin \operatorname{im} f$.
  - Let

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto f(n) = \left\lceil \frac{n}{2} \right\rceil. \end{aligned}$$

    $f$ is surjective, as for any $k \in \mathbb{N}$, take $n = 2k \in \mathbb{N}$; then we get $f(n) = \left\lceil \frac{2k}{2} \right\rceil = k$. On the other hand, $f$ is not injective. Let $n_1, n_2 \in \mathbb{N}, n_1 = 1, n_2 = 2$. Then $f(n_1) = f(n_2) = 1$, but $n_1 \neq n_2$.

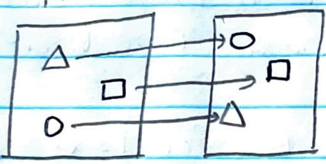
2.1 Define permutations $\sigma_1, \sigma_2$ on $\{\triangle, \square, \circ\}$

by:

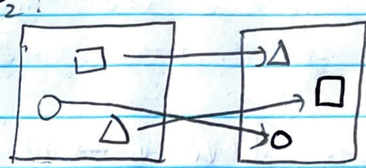
$\sigma_1(\triangle) = \circ \quad \sigma_1(\square) = \square \quad \sigma_1(\circ) = \triangle$

$\sigma_2(\triangle) = \square \quad \sigma_2(\square) = \triangle \quad \sigma(\circ) = \circ$

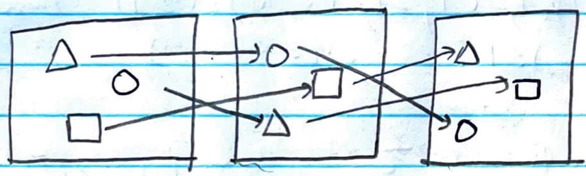
a) $\sigma_1$:



$\sigma_2$:



b)

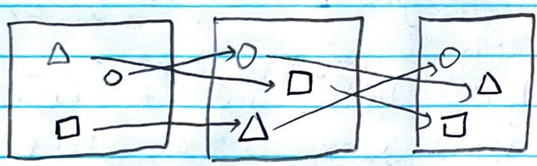


$\sigma_2 \circ \sigma_1$

First do $\sigma_1$. Then $\sigma_2$.

$\sigma_2 \circ \sigma_1(\triangle) = \circ \quad \sigma_2 \circ \sigma_1(\square) = \triangle \quad \sigma_2 \circ \sigma_1(\circ) = \square$

c)

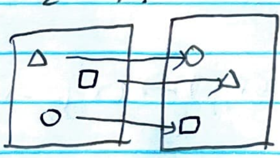


$\sigma_1 \circ \sigma_2$

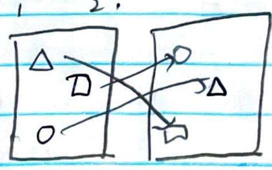
First do $\sigma_2$. Then $\sigma_1$.

$\sigma_1 \circ \sigma_2(\triangle) = \square \quad \sigma_1 \circ \sigma_2(\square) = \circ \quad \sigma_1 \circ \sigma_2(\circ) = \triangle$.

d) $\sigma_2 \circ \sigma_1$:



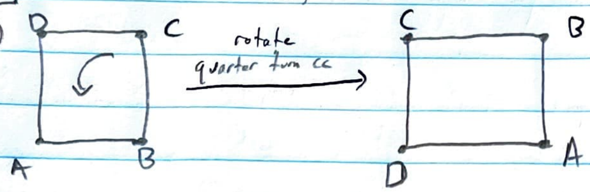
$\sigma_1 \circ \sigma_2$:



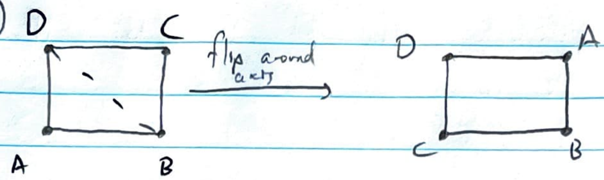
No, $\sigma_2 \circ \sigma_1 \neq \sigma_1 \circ \sigma_2$.

2.4) Fill in the blanks,

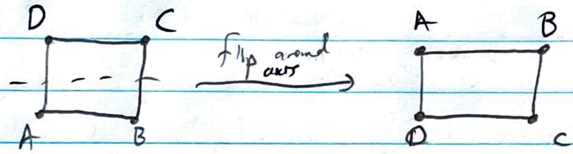
a)



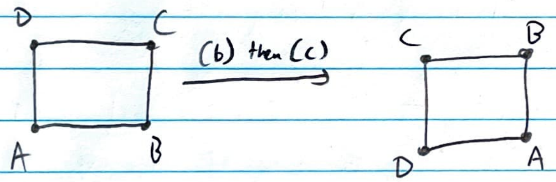

b)




c)




d)




e)