Problem §1 Let $v_1, \ldots, v_n$ be a basis for V, and let $w_1, \ldots, w_n$ be another basis for V.

(a) Prove that for any $j \in \{1, \ldots, n\}$, there exists an $i \in \{1, \ldots, n\}$ such that

$$v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$$

is a basis.

(b) Prove that for any $i \in \{1, \ldots, n\}$, there exists a $j \in \{1, \ldots, n\}$ such that

$$v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$$

is a basis.

Solution:

(a) Let w_j be any basis vector in the basis $w_1, \ldots, w_n$. Since $v_1, \ldots, v_n$ is a basis for V, we know that

$$w_j \in \operatorname{span}(v_1, \ldots, v_n),$$

and so there exists a unique linear combination

$$a_1 v_1 + \ldots + a_n v_n = w_j$$

where $a_1, \ldots, a_n$ not all zero. Let a_i be any non-zero coefficient. Then we have

$$\begin{aligned} -a_i v_i &= a_1 v_1 + \ldots + a_{i-1} v_{i-1} + a_{i+1} v_{i+1} + \ldots + a_n v_n - w_j \\ v_i &= \alpha_1 v_1 + \ldots + \alpha_n v_n - \frac{w_j}{a_i}, \end{aligned}$$

where $\alpha_j = -\frac{a_j}{a_i}$. Thus, $v_i \in \operatorname{span}(v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j)$, and so any linear combination

$$v = a_1 v_1 + \ldots + a_i v_i + \ldots + a_n v_n$$

can be replaced by

$$v = a_1 v_1 + \ldots + (b_1 v_1 + \ldots + b_{i-1} v_{i-1} + b_{i+1} v_{i+1} + \ldots + b_n v_n + b_j w_j) + \ldots + a_n v_n.$$

From this, we see that $\operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j)$, and since every spanning list of length $\dim V$ is a basis for V, we have that

$$v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$$

is a basis for V.

(b) Let v_i be any vector in $v_1, \ldots, v_n$. Then

$$v_1, \ldots, \hat{v}_i, \ldots, v_n$$

is a linearly independent list that doesn't span all of V. Thus, from the basis $w_1, \ldots, w_n$, there exists some w_j such that

$$w_j \notin \operatorname{span}(v_1, \ldots, \hat{v}_i, \ldots, v_n),$$

since otherwise $\operatorname{span}(v_1, \ldots, \hat{v}_i, \ldots, v_n) = \operatorname{span}(w_1, \ldots, w_n)$, a contradiction of $v_1, \ldots, \hat{v}_i, \ldots, v_n$ not spanning V. Thus the list

$$v_1, ..., \hat{v}_i, \ldots, v_n, w_j$$

is a linearly independent list. Since every linearly independent list of length $\dim V$ is a basis for V, we have that $v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$ is a basis for V.

Problem §2 Let V, W be vector spaces. Suppose $v_1, \ldots, v_m$ are linearly independent in V and suppose $w_1, \ldots, w_m$ are any vectors in W. Prove that there exists a linear map $T : V \to W$ such that

$$T(v_1) = w_1, \ldots, T(v_m) = w_m.$$

Solution: Let $v_1, \ldots, v_m$ be linearly independent in V, and extend the list to a basis $v_1, \ldots, v_m, u_1, \ldots, u_n$. Define a linear map

$$T(a_1v_1 + \ldots + a_mv_m + b_1u_1 + \ldots + b_nu_n) = a_1w_1 + \ldots + a_mw_m.$$

(All of the u_i's are sent to 0). Because $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis, T is a function, as each element of V can be uniquely written in the form $v = a_1v_1 + \ldots + a_mv_m + b_1u_1 + \ldots + b_nu_n$. By taking $a_i = 1$ and the other a's as zero, we have that

$$T(v_i) = w_i.$$

Now, take any two vectors $u, v \in V$ and any two scalars $\lambda_1, \lambda_2 \in \mathbb{F}$. We have

$$\begin{aligned}
T(\lambda_1u + \lambda_2v) &= T((\lambda_1a_1v_1 + \ldots + \lambda_1a_mv_m + \lambda_1b_1u_1 + \ldots + \lambda_1b_nu_n) + (\lambda_2c_1v_1 + \ldots + \lambda_2c_mv_m + \lambda_2d_1u_1 + \ldots + \lambda_2d_nu_n)) \\
&= (\lambda_1a_1w_1 + \ldots + \lambda_1a_mw_m) + (\lambda_2c_1w_1 + \ldots + \lambda_2c_mw_m) \\
&= \lambda_1(a_1w_1 + \ldots + a_mw_m) + \lambda_2(c_1w_1 + \ldots + c_mw_m) \\
&= \lambda_1T(a_1v_1 + \ldots + a_mv_m + b_1u_1 + \ldots + b_nu_n) + \lambda_2T(c_1v_1 + \ldots + c_mv_m + d_1u_1 + \ldots + d_nu_n) \\
&= \lambda_1T(u) + \lambda_2T(v).
\end{aligned}$$

Thus T preserves linearity and homogeneity, and so T is a linear map (note that T is very much not injective! Going from the 2nd last step to the 3rd last step is guaranteed, but the reverse is very much not guaranteed.)

Problem §3 Let V, W be vector spaces over $\mathbb{F}$, and suppose V is finite-dimensional with $\dim V > 0$. Let $w \in W$ be any vector. Prove that there exists a linear map $T : V \to W$ such that

$$\operatorname{range}(T) = \operatorname{span}(w).$$

Solution: Let $n = \dim V$. Since $n > 0$, there exists a length-n basis $v_1, \ldots, v_n$ of V. Define a linear map

$$T(a_1v_1 + \ldots + a_nv_n) = a_1w \qquad [\text{ all of the } v_j,\ j > 1 \text{ are mapped to } 0\ ]$$

Since $v_1, \ldots, v_n$ is a basis of V, each $v \in V$ has a unique representation, and so T is a valid function. Moreover, we see that

$$\begin{aligned}
\operatorname{range}(T) &= \{T(v) \mid v \in V, v = a_1v_1 + \ldots + a_nv_n,\ a_1, ..., a_n \in \mathbb{F}, v_1, \ldots, v_n \in V\} \\
&= \{a_1w \mid a_1 \in \mathbb{F}\} \\
&= \operatorname{span}(w),
\end{aligned}$$

as required. Now, take any two vectors $u, v \in V$ and any two scalars $\lambda_1, \lambda_2 \in \mathbb{F}$. We have

$$\begin{aligned}
T(\lambda_1u + \lambda_2v) &= T(\lambda_1a_1v_1 + \ldots + \lambda_na_nv_n + \lambda_2b_1v_1 + \ldots + \lambda_2b_nv_n) \\
&= \lambda_1a_1w + \lambda_2b_1w \\
&= \lambda_1T(a_1v_1 + \ldots + a_nv_n) + \lambda_2T(b_1v_1 + \ldots + b_nv_n) \\
&= \lambda_1T(u) + \lambda_2T(v).
\end{aligned}$$

Thus T preserves linearity and homogeneity, and so T is a linear map (much like problem 2, T is very much not injective).