MATH 1010 Homework 6

1) (11.4) $w_n = (-2)^n$, $x_n = 5^{(-1)^n}$, $y_n = 1 + (-1)^n$, $z_n = n\cos\left(\frac{n\pi}{4}\right)$.

a. $w_{n_k} = 4, 16, 64, \ldots = (-2)^{2n}$

$x_{n_k} = 5, 5, \ldots = 5^{(-1)^{2n}} = 5$

$y_{n_k} = 2, 2, \ldots, 2 = 1 + (-1)^{2n} = 2$

$z_{n_k} = 8, 16, \ldots = 8n\cos(2n\pi) = 8n$

b. $S_{w_n} = \{-\infty, \infty\}$

$S_{x_n} = \{\frac{1}{5}, 5\}$

$S_{y_n} = \{0, 2\}$

$S_{z_n} = \{-\infty, 0, \infty\}$

c. $\lim\sup w_n = +\infty$, $\lim\inf w_n = -\infty$

$\lim\sup x_n = 5$, $\lim\inf x_n = \frac{1}{5}$

$\lim\sup y_n = 2$, $\lim\inf y_n = 0$

$\lim\sup z_n = +\infty$, $\lim\inf z_n = -\infty$

d. None of the sequences' limits exist.

e. x_n and y_n are bounded above and below.

2) (11.8) Prove $\lim\inf s_n = -\lim\sup(-s_n)$

ex 5.4: $\inf S = -\sup(-S)$ for any $S \subset \mathbb{R}$.

defn 10.6: $\lim\sup s_n = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$, $\lim\inf s_n = \lim_{N\to\infty} \inf\{s_n \mid n > N\}$

Let S = subsequential limits of s_n; then $\inf S = \lim_{N\to\infty} \inf\{s_n \mid n > N\}$.

For $-s_n$, by Theorem 9.2 $\lim -s_n = -s$; thus any $\lim s_{n_k} = s_k$ has corresponding $\lim -s_{n_k} = -s_k$, and so subseq. lim. of $-s_n = -S$. By 5.4, $\inf S = -\sup(-S)$; but $\sup(-S) = \lim\sup(-s_n)$, and $\inf(S) = \lim\inf s_n$; thus

$\lim\inf s_n = -\lim\sup(-s_n)$.

3)(11.10) a. From the figure, we can select subsequences $s_{n_k} = \frac{1}{k}$; thus $S = \{\frac{1}{k} \mid k \in \mathbb{N}\} \cup \{0\}$. (0 is needed for the subsequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$)

b. As $k \to +\infty$, $\frac{1}{k} \to 0$; thus $\liminf s_n = 0$. Clearly, $\limsup s_n = 1$.

4) (12.4) Show $\limsup (s_n + t_n) \le \limsup s_n + \limsup t_n$ for bounded (s_n), (t_n).

Consider $s_n + t_n$. Since both bounded, $v_s = \sup\{s_n \mid n > N_0\}$, $v_t = \sup\{t_n \mid n > N_1\}$ exist. Thus any $s_n + t_n \le v_s + v_t = \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$ by definition of supremum. Thus $\sup\{s_n + t_n \mid n > N\} \le \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$, since $v_s + v_t$ bounds $s_n + t_n$ from above.

Since s_n, t_n bounded, $s_n + t_n$ is bounded as well; thus, $\limsup s_n$, $\limsup t_n$, and $\limsup (s_n + t_n)$ all exist. Thus, from 9.9c, we have

$$\limsup (s_n + t_n) \le \limsup s_n + \limsup t_n.$$

5)(12.8) Prove that if (s_n), (t_n) are bounded sequences of non-negative numbers, then $\limsup s_n t_n \le (\limsup s_n)(\limsup t_n)$.

Since (s_n) is bounded, $\sup\{s_n \mid n > N\}$ exists and is monotone; thus, Theorem 10.2 tells us $\limsup s_n$ converges. The same argument follows for $\limsup t_n$. Moreover, (s_n), (t_n) bounded, so $(s_n t_n)$ bounded, and $\limsup s_n t_n$ converges as well.

Since $0 \le s_n \le \sup s_n$, $0 \le t_n \le \sup t_n$, we have $0 \le s_n t_n \le \sup s_n \sup t_n$. Moreover, (s_n), (t_n) bounded means $(s_n t_n)$ bounded as well. Since $\sup s_n \sup t_n$ is an upper bound, $\sup (s_n t_n) \le \sup s_n \sup t_n$. Since both $\sup(s_n t_n)$ and $\sup s_n \sup t_n$ are monotone & bounded, their limit exists; thus, by 9.9c we have

$$\limsup s_n t_n \le (\limsup s_n)(\limsup t_n).$$

6)(12.10) Prove (S_n) bounded $\iff$ $\lim\sup|S_n| < +\infty$.

Suppose s_n bounded; then $\sup\{s_n | n > N\}$ exists by definition. Moreover, the sequence is monotonic & bounded, and so by Theorem 10.2 $\lim\sup s_n$ converges (i.e. is not $+\infty$).

Conversely, suppose $v_s = \lim\sup|s_n| < +\infty$. Then for some N, any $n > N$ satisfies $|\sup|s_n| - v_s| < \varepsilon$ for any $\varepsilon > 0$. Thus

$$v_s - \varepsilon < \sup|s_n| < v_s + \varepsilon.$$

Let $\varepsilon = 1$; then $\sup|s_n| < v_s + 1$. Let $M = \max\{v_s+1, |s_1|, \ldots, |s_N|\}$ (since any $|s_n|$ for $n > N$ satisfies $|s_n| \leq \sup|s_n| < v_s + 1$). Then for any $n \in \mathbb{N}$, $|s_n| \leq M$, so $-M \leq s_n \leq M$. Thus s_n is bounded.

7)(14.2)

a. $\sum \frac{n-1}{n^2} = \sum \frac{n}{n^2} - \sum \frac{1}{n^2}$, $\sum \frac{n}{n^2}$ diverges, while $\sum \frac{1}{n^2}$ converges, so $\sum \frac{n-1}{n^2}$ diverges.

b. $\sum (-1)^n = \sum(-1)(-1)^{n-1}$, $a = -1$, $|r| = 1 \to \sum(-1)^n$ does not converge.

c. $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2}$, $\sum \frac{1}{n^2}$ converges, so $3\sum\frac{1}{n^2} = \sum\frac{3}{n^2}$ does as well.

d. $\sum \frac{n^3}{3^n} \Rightarrow \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{1}{3} \cdot \frac{(n+1)^3}{n^3}$, $\lim \frac{(n+1)^3}{n^3} = 1$, so $\lim\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$, and so converges.

e. $\sum \frac{n^2}{n!} \Rightarrow \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} = \frac{(n+1)^2}{n^2(n+1)} \Rightarrow \lim \frac{(n+1)^2}{n^3+n^2} = 0$, so $\lim\left|\frac{a_{n+1}}{a_n}\right| = 0$, and so converges.

f. $\sum \frac{1}{n^n}$. $\frac{1}{n^n} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$; since $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{1}{n^n}$.

g. $\sum \frac{n}{2^n} \Rightarrow \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \Rightarrow \lim\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}$, and so $\sum \frac{n}{2^n}$ converges.

8)(14.6) a. Prove that if $\sum|a_n|$ converges and (b_n) bounded, $\sum a_n b_n$ converges.

Let $s_n = \sum|a_n|$. Then $\lim s_n = s$, so $|s_n - s| < \varepsilon$ for $n > N$, $\varepsilon > 0$. Since b_n bounded, for some $M > 0$, $|b_n| \leq M \;\forall n \in \mathbb{N}$. Choose $\frac{\varepsilon}{M}$; we have $|Ms_n - Ms| < M\frac{\varepsilon}{M} = \varepsilon$, so Ms_n converges; thus

Let $s_n = \sum|a_n|$. Since $\lim s_n = s$ (converges), s_n satisfies the Cauchy criterion; that is, for some N, $|s_n - s_m| < \varepsilon$ for any $\varepsilon > 0$. Consider $\sum a_k b_k$. Since $|b_k| < M$ for some $M > 0$, we have

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \sum|a_k b_k| \leq \sum|a_k M| \leq M|\sum a_k| = Ms_n.$$

Thus we have, for $\frac{\varepsilon}{M}$, $M|s_n - s| < M \cdot \frac{\varepsilon}{M}$, so $|Ms_n - Ms_m| < \varepsilon$. Then we have Ms_n satisfies the Cauchy criterion, and thus is convergent. It follows that $Ms_n \geq \sum a_k b_k$ converges as well.

b. Corollary 14.7 states that $\sum a_n$ converges implies $\lim a_n = 0$. We can choose a bounded sequence $(b_n) = 1$; hence Corollary 14.7 specifically looks at $\sum a_n b_n = \sum a_n$.