For each of the linear operators T in questions 1-7:

(a) I claim that $p(T) = 0$ for some non-zero polynomial $p(z) \in \mathcal{P}_2(\mathbb{R})$ of degree at most 2. Find such a polynomial $p(z)$.

(b) Factor $p(z)$ into linear factors over $\mathbb{R}$ *if possible*.

(c) If $p(z)$ does factor into linear factors $p(z) = c(z - \lambda_1)(z - \lambda_2)$ for $c, \lambda_1, \lambda_2 \in \mathbb{R}$, then we argued in class that one of λ_1, λ_2 must be an eigenvalue of T. Identify which one(s) of λ_1, λ_2 are in fact eigenvalues of T, in accordance with your answers in Review Sheet 12.

Problem §1 $T : R^2 \to R^2$ reflection across the line $y = x$.

Solution: Let $p(z) = z^2 - 1$. Then $p(T) = T^2 - I = 0$. We can factor $p(z) = (z + 1)(z - 1)$, where both $\lambda_1 = 1, \lambda_2 = -1$ are eigenvalues of T.

Problem §2 $T : R^2 \to R^2$ a 180° rotation about the origin.

Solution: Let $p(z) = z + 1$. Then $p(T) = T + I = 0$. $p(z)$ trivially factors into $z + 1$, where $\lambda_1 = -1$ is an eigenvalue of T.

Problem §3 $T : \mathbb{R}^2 \to \mathbb{R}^2$ a 60° counter-clockwise rotation about the origin.

Solution: Let $p(z) = z^2 - z + 1$. Then $p(T) = T^2 - T + I = 0$ (since flipping a vector rotated 60° gives a vector rotated $-120°$, which will then cancel out the vector rotated 120°). $p(x)$ has no real roots, which aligns with our answer that T has no eigenvalues.

Problem §4 $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + y, x + y)$.

Solution: Let $p(z) = z^2 - 2z$. Then $p(T) = T^2 - 2T = 0$. $p(z)$ factors into $(x - 0)(x - 2)$, where $\lambda_1 = 0,\ \lambda_2 = 2$ are both eigenvalues of T.

Problem §5 $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x, x)$.

Solution: Let $p(z) = z^2 - z$. Then $p(T) = T^2 - T = 0$. $p(z)$ factors into $(z - 0)(z - 1)$, where $\lambda_1 = 0,\ \lambda_2 = 1$ are eigenvalues of T.

Problem §6 $T : \mathbb{R}^3 \to \mathbb{R}^3$ projection to xy-plane; $T(x, y, z) = (x, y, 0)$.

Solution: Let $p(z) = z^2 - z$. Then $p(T) = T^2 - T = 0$. $p(z)$ factors into $(z - 0)(z - 1)$, where $\lambda_1 = 0,\ \lambda_2 = 1$ are eigenvalues of T.

Problem §7 $T : \mathbb{R}^3 \to \mathbb{R}^3$ rotation by 180° about the x-axis.

Solution: Let $p(z) = z^2 - 1$. Then $p(T) = T^2 - I = 0$. $p(z)$ factors into $(z - 1)(z + 1)$, where $\lambda_1 = 1,\ \lambda_2 = -1$ are eigenvalues of T.