Problem §1 Let $a \in \mathbb{F}$, and let $v \in V_{\mathbb{F}}$ be a non-zero vector. Prove that

$$av = \mathbf{0} \text{ only if } a = 0.$$

Solution: Suppose $a \neq 0$, yet $av = 0$. Then

$$\begin{aligned} av &= \mathbf{0} \\ &= av + (-av) \\ &= 0 + (-av) \\ &= -av. \end{aligned}$$

Hence $av = -av \implies av \cdot v^{-1} = -av \cdot v^{-1} \implies a = -a$. But $a = -a$ only when $a = 0$, a contradiction. Thus $av = 0$ only when $a = 0$.

Problem §2 Let $v, w \in V$, and suppose $v \neq 0$. Prove that there exists at most one $a \in \mathbb{F}$ such that

$$av = w.$$

Solution: Suppose there exists a $b \in \mathbb{F},\ b \neq a$ such that $bv = w$. Then

$$av + bv = w + w = 2w = w + w = av + av.$$

Adding $-av$ to both sides,

$$bv = av \implies a = b,$$

a contradiction.
Hence if $bv = w$ for some $b \in \mathbb{F}$, then $b = a$.

Problem §3

(a) (1.C.12) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

(b) (1.C.13, extra credit) Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution:

(a) Let $U_1, U_2 \in V$ be two subspaces of V.

Suppose that one of the subspaces is contained in the other; that is, suppose without loss of generality that $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2 \subseteq V$ is a subspace of V (since U_2 is a subspace of V).

Now, suppose that $U_1 \cup U_2 \subseteq V$ is a subspace of V. Let $v_1, v_2 \in U_1 \cup U_2$. Then

$$v_1 \in U_1 \text{ or } v_1 \in U_2, \text{ and } v_2 \in U_1 \text{ or } v_2 \in U_2.$$

$v_1, v_2 \in U_1$ or $v_1, v_2 \in U_2$ tell us nothing new about the relationship between U_1 and U_2 (we already know, by definition of a subspace, that $0,\ \lambda v_1,\ \lambda v_2,\ v_1 + v_2 \in U_i$ for some $\lambda \in \mathbb{F},\ i \in \{1, 2\}$), so suppose without loss of generality that $v_1 \in U_1, v_2 \in U_2$. We know (since $U_1 \cup U_2$ is a subspace of V) that $v_1 + v_2 \in U_1 \cup U_2$, so $v_1 + v_2 \in U_1$ or $v_1 + v_2 \in U_2$. If $v_1 + v_2 \in U_1$, by closure of addition in U_1, $v_2 \in U_1$ as well, and so since for any arbitrary $v_2 \in U_2,\ v_2 \in U_1$, we have $U_2 \subseteq U_1$. Similarly, if $v_1 + v_2 \in U_2$, by closure of addition, $v_1 \in U_2$ as well, and so $U_1 \subseteq U_2$.
Therefore, if $U_1 \cup U_2$ is a subspace of V, then one of the subspaces is contained in the other.

Thus, the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Problem §4 Solved in Review Sheet 5.

Problem §5 Let V, W be vector spaces over $\mathbb{F}$, and let $T : V \to W$ be a linear map. Suppose V is finite-dimensional and T is surjective. Prove that W is finite-dimensional.

Solution: We begin with a lemma.

Lemma 1. *Let V, W be vector spaces over a field $\mathbb{F}$. If V is an n-dimensional vector space, and $T : V \to W$ is a linear map, then* $\dim(\operatorname{range} T) \leq \dim V$.

Proof. If $\dim V = n$, then V has a basis $B = \{v_1, \ldots, v_n\}$ with n linearly independent vectors in V that span V. Then, for any $v \in V$,

$$v = a_1 v_1 + \ldots + a_n v_n, \quad \text{for } a_i \in \mathbb{F}.$$

Applying T to both sides, we have

$$T(v) = T(a_1 v_1 + \ldots + a_n v_n) = a_1 T(v_1) + \ldots + a_n T(v_n),$$

and since v was an arbitrary $v \in V$, $T(v_1), \ldots, T(v_n)$ span $\operatorname{range} T$. Hence $\dim(\operatorname{range} T) \leq n = \dim V$.

(Note that there is no guarantee that $T(v_1), \ldots, T(v_n)$ are linearly independent and thus form a basis for $\operatorname{range} T$; $\dim(\operatorname{range} T)$ could very much be less than n. For example, take $T = \mathbf{0}$ the zero map. Then $T(v_i) = \mathbf{0}$ for any $v_i \in B$, and so $\dim(\operatorname{range} T) < n$.) □

From this lemma, we get that if V is a finite-dimensional vector space (say, with $\dim V = n$), and $T : V \to W$ a linear map, then $\dim(\operatorname{range} T) \leq \dim V$. Moreover, since T is surjective, by definition we have $\operatorname{range} T = W$. Thus $\dim W \leq \dim V$, and so W is finite-dimensional as well.