HW 7

1) (15.2)

a. $\sum (\sin \frac{n\pi}{6})^n$ does not converge. Consider the subsequence $S_k = 12n + 3$, that is, all multiples of 12 offset by 3. Then $\sin(\frac{12\pi n}{6} + \frac{3\pi}{6}) = \sin(2\pi n + \frac{\pi}{2}) = 1$ for all n; thus, the sum of this subsequence $\sum S_k$ does not converge, $\sum (\sin \frac{\pi n}{6})^n$ doesn't converge either.

b. $\sum (\sin \frac{\pi n}{7})^n$ converges. For any $n = 7k$ (multiple of 7), $\sin(\frac{7\pi n}{7}) = 0$, and any n not a multiple of 7 is strictly less than 1 (since $\sin(\frac{n\pi}{7}) \neq \sin(\frac{\pi}{2}) = 1$ for any number $n \in \mathbb{N}$). Thus we can find an $0 < \alpha < 1$ such that for all $n \in \mathbb{N}$, $0 \leq |\sin(\frac{n\pi}{7})| < \alpha < 1$. Thus

$$\sum |(\sin \frac{\pi n}{7})^n| < \sum r^n < \alpha \quad \text{for some } 0 < r < 1,$$

and so $\sum (\sin \frac{\pi n}{7})^n$ is absolutely convergent. By Theorem 14.7, $\sum (\sin \frac{\pi n}{7})^n$ is convergent as well.

2) (15.6)

a. Let $a_n = \frac{1}{n}$. $\sum \frac{1}{n}$ diverges, while $\sum \frac{1}{n^2}$ converges.

b. Since $\sum a_n$ converges, & $\forall n \in \mathbb{N}$, $a_n \geq 0$, by Corollary 14.5 we have $\lim a_n = 0$. Thus, for some sufficiently large $N \in \mathbb{N}$, $n > N$ means $a_n^2 \leq a_n$ (since when $a_n \leq 1$, $a_n^2 \leq a_n$). Thus, by the Comparison Test, $\sum a_n^2$ converges as well.

c. First, observe that $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ $\forall n \in \mathbb{N}$. Moreover, clearly $\lim \frac{1}{\sqrt{n}} = 0$. Thus, the alternating series $\sum (-1)^n \frac{1}{\sqrt{n}}$ converges. However $((-1)^n \frac{1}{\sqrt{n}})^2 = \frac{1}{n}$, and we know $\sum \frac{1}{n}$ diverges.

3) (17.2) $f(x) = \begin{cases} 4, & x \geq 0 \\ 0, & x < 0 \end{cases}$ $\quad g(x) = x^2$

a. $(f+g)(x) = \begin{cases} x^2+4, & x \geq 0 \\ x^2, & x < 0 \end{cases}$, where $\operatorname{dom}(f+g) = \mathbb{R}$

$(fg)(x) = \begin{cases} 4x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $\operatorname{dom}(fg) = \mathbb{R}$

$(f \circ g)(x) = 4$, $\operatorname{dom}(f \circ g) = \mathbb{R}$

$(g \circ f)(x) = \begin{cases} 16, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

b. $g(x)$, $f \circ g(x)$, & $fg(x)$ are continuous, while the rest are discontinuous due to discontinuous jumps.

4) (17.10)

a. Consider $a_n = \frac{1}{n}$; $\lim a_n = 0$. But for any $n \in \mathbb{N}$, $\frac{1}{n} > 0$, so

$$f(a_n) = f(\tfrac{1}{n}) = 1 \neq 0 = f(0),$$

and so $f(x)$ is discontinuous.

b. Consider $a_n = \frac{2}{\pi(4n+1)}$; $\lim a_n = 0$. But then $\frac{2}{\pi(n+1)} > 0$ $\forall n \in \mathbb{N}$, so

$$g(a_n) = \sin(\tfrac{4\pi}{2}n + \tfrac{\pi}{2}) = \sin(2\pi n + \tfrac{\pi}{2}) = 1 \neq 0 = g(0),$$

and so $g(x)$ is discontinuous.

c. Consider $a_n = \frac{1}{n}$, $\lim a_n = 0$ again. Then

$$\operatorname{sgn}(a_n) = \operatorname{sgn}(\tfrac{1}{n}) = 1 \neq 0 = \operatorname{sgn}(0),$$

and so $\operatorname{sgn}(x)$ is discontinuous.

5) (17.12)

a. Let $x \in (a,b)$, $x \in \mathbb{Q}$ implies $f(x) = 0$, so suppose $x \notin \mathbb{Q}$. Denseness of $\mathbb{Q}$ lets us find a sequence of rationals r_n where $\lim r_n = x$, so we have

$$f(r_n) = f(x) = 0.$$

b. Consider $h(x) = f(x) - g(x)$. Since $f(r) = g(r)$ $\forall r \in \mathbb{Q}$, $h(r) = f(r) - g(r) = 0$. By (a), $h(x) = f(x) - g(x) = 0$ $\forall x \in (a,b)$, so $f(x) = g(x)$ $\forall x \in (a,b)$.

6) (18.2)

Let f be a continuous real-valued function on (a,b). Suppose f is not bounded on (a,b). Then to each $n \in \mathbb{N}$, some $x_n \in (a,b)$ satisfies $|f(x_n)| \geq n$. By Bolzano-Weierstrauss, (x_n) has a subsequence (x_{n_k}) converging to some $x_0 \in [a,b]$. However, this is where the Theorem breaks: we don't know for sure that $x_0 \in (a,b)$ (it could be $x_0 = a$ or b).

7) (18.4) Let $x_0 \notin S$, and let $(x_n) \subseteq S$ be some sequence such that $\lim(x_n) = x_0$. Let $f(x) = |x - x_0|$ be some function w/ $\text{dom}(f) = S$. By definition, we see that $f(x)$ is continuous (and positive) on S (since if a sequence $s_n \subseteq S$ converges to x_0, then $\lim f(s_n) = |s_n - x_0| < \varepsilon \ \forall \varepsilon > 0$, so $\lim f(s_n) = f(x_0)$). Moreover, $f(x) \neq 0$ for all $x \in S$, since $x_0 \notin S$.

Hence $g(x) = \frac{1}{f(x)}$ is well defined; moreover, $\lim g(x_n) = \frac{1}{f(x_n)} = \frac{1}{\lim |x_n - x_0|} = \infty$, so $g(x)$ is a continuous, unbounded function w/ domain S.