MATH 540 HONORS LINEAR ALGEBRA SUMMER 2021, Melody Chan
PROBLEM SET D

Due Monday June 14 at 11:59pm Eastern

Submit all of the following on Gradescope, and don't forget to tag each answer to its page. We have implemented a course policy whereby failing to tag results in half credit. I put a copy of this problem set in the Overleaf folder too.

1. (3+3 points) Let $v_1, \ldots, v_n$ be a basis of V, and let $w_1, \ldots, w_n$ be another basis of V.
   (a) Prove that for any j with $1 \le j \le n$, there exists i with $1 \le i \le n$ such that
   $$v_1, \ldots, \hat{v_i}, \ldots, v_n, w_j$$
   is a basis.
   (b) Prove that for any i with $1 \le i \le n$, there exists j with $1 \le j \le n$ such that
   $$v_1, \ldots, \hat{v_i}, \ldots, v_n, w_j$$
   is a basis.

   (These are not the same statement! The first one says "you can swap in any given w_j." The second one says "you can swap out any given v_i.")
2. (3 points) Let V and W be vector spaces. Suppose $v_1, \ldots, v_m$ are linearly independent in V and suppose $w_1, \ldots, w_m$ are any vectors in W. Prove that there exists a linear map $T : V \to W$ such that
   $$T(v_1) = w_1,\ \ldots,\ T(v_m) = w_m.$$
3. (3 points) Let V and W be vector spaces over $\mathbb{F}$, and suppose V is finite-dimensional with $\dim V > 0$. Let $w \in W$ be any vector. Prove that there exists a linear map $T\colon V \to W$ with
   $$\mathrm{range}(T) = \mathrm{span}(w).$$

Extra credit on the next page.

Extra credit, for those who yearn for infinity. (2 points) First, study from Wikipedia or any other source the definition of a *partially ordered set*, or *poset* for short, and the statement of Zorn's Lemma.

Let V be a vector space, let $\mathcal{I}$ be any set, and let $(v_i)_{i\in\mathcal{I}}$ be a list of vectors in V indexed by $\mathcal{I}$. (The setting from class is the case that $\mathcal{I} = \{1, \ldots, m\}$.)

The usual definitions then go through: a *linear combination of vectors* in $(v_i)_{i\in\mathcal{I}}$ is a *finite* sum of the form

$$a_1 v_{i_1} + \cdots + a_m v_{i_m}, \qquad i_1, \ldots, i_m \in \mathcal{I},\ a_1, \ldots, a_m \in \mathbb{F}.$$

The *span* of a list of vectors is the set of all linear combinations, and a list of vectors is *linearly independent* if the only linear combination of them that is zero has all scalars zero. A *basis* is a list of vectors that both spans V and is linearly independent.

Prove, assuming Zorn's lemma, that every vector space has a basis.[1]

[1] Remarkably, the statement that every vector space has a basis is *equivalent* to Zorn's lemma. This is not at all obvious. It's a theorem, proved by Andreas Blass in 1984. Blass proves that the statement that every vector space has a basis is equivalent to the *Axiom of Choice*, which is also equivalent to Zorn's lemma. See Andreas Blass, Existence of bases implies the axiom of choice. Contemporary Mathematics vol. 31 pp. 31-33, 1984. I have never studied this proof myself.