Problem §1 Apply the Gram-Schmidt Algorithm to the vectors $(2i, 0, 0), (1, 1, i), (1, 1, -i) \in \mathbb{C}^3$ with the Euclidean inner product.

Solution: Let $e_1 = \frac{(2i,0,0)}{\|(2i,0,0)\|} = \frac{(2i,0,0)}{2} = (i, 0, 0)$. Then for $v_2 = (1, 1, i)$, we get

$$\begin{aligned} e_2 &= \frac{(1,1,i) - \langle (1,1,i),(i,0,0)\rangle (i,0,0)}{\|(1,1,i) - \langle (1,1,i),(i,0,0)\rangle (i,0,0)\|} \\ &= \frac{(0,1,i)}{\|(0,1,i)\|} \\ &= \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right). \end{aligned}$$

For $v_3 = (1, 1, -i)$, we get

$$\begin{aligned} e_3 &= \frac{(1,1,-i) - \langle (1,1,-i),(i,0,0)\rangle (i,0,0) - \left\langle (1,1,-i), \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)\right\rangle \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)}{\|(1,1,-i) - \langle (1,1,-i),(i,0,0)\rangle (i,0,0) - \left\langle (1,1,-i), \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)\right\rangle \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)\|} \\ &= \frac{(0,1,-i)}{\|(0,1,-i)\|} \\ &= \left(0, \frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right). \end{aligned}$$

Thus, with starting vectors $(2i, 0, 0), (1, 1, i), (1, 1, -i) \in \mathbb{C}^3$, we get an three orthonormal vectors

$$(i,0,0),\ \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right),\ \text{and}\ \left(0, \frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right).$$

Problem §2 Find an orthonormal basis for $\mathcal{P}_1(\mathbb{R})$ with respect to the following inner product:

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

Solution: We start with the basis $1, x$, and use Gram-Schmidt. Let $e_1 = \frac{1}{\|1\|} = 1$. Then for $v_2 = x$, we get

$$\begin{aligned} e_2 &= \frac{x - \langle x, 1\rangle 1}{\|x - \langle x, 1\rangle 1\|} \\ &= \frac{x - \int_0^1 xdx}{\|x - \int_0^1 xdx\|} \\ &= \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} \\ &= \frac{x - \frac{1}{2}}{\sqrt{\left(\int_0^1 (x - \frac{1}{2})^2 dx\right)}} \\ &= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}} \\ &= 2x\sqrt{3} - \sqrt{3}. \end{aligned}$$

Thus we have an orthonormal basis

$$1, 2x\sqrt{3} - \sqrt{3}.$$

Problem §3 (6.A.10) Find vectors $u, v \in \mathbb{R}^2$ such that u is a scalar multiple of $(1,3)$, v is orthogonal to $(1,3)$, and $(1,2) = u + v$.

Solution: We decompose $(1,2)$ into a scalar multiple of $(1,3)$, and a vector orthogonal to $(1,3)$:

$$(1,2) = \frac{\langle (1,2),(1,3)\rangle}{\langle (1,3),(1,3)\rangle}(1,3) + \left(u - \frac{\langle (1,2),(1,3)\rangle}{\langle (1,3),(1,3)\rangle}(1,3)\right).$$

The left vector becomes

$$\frac{1+6}{1+9}(1,3) = \left(\frac{7}{10}, \frac{21}{10}\right);$$

this is the scalar multiple of $(1,3)$ we're looking for; in other words, $u = \left(\frac{7}{10}, \frac{21}{10}\right)$. The right vector then becomes

$$(1,2) - u = \left(\frac{3}{10}, -\frac{1}{10}\right);$$

but this is the vector orthogonal to $(1,3)$ that we were looking for.

Thus

$$u = \left(\frac{7}{10}, \frac{21}{10}\right) \text{ and } v = \left(\frac{3}{10}, -\frac{1}{10}\right).$$