

Problem §1 Let $v = (2 + 2i, -1 + i)$.

- (a) Compute $\frac{1}{2} \cdot v$.
- (b) Compute $(3i) \cdot v$.
- (c) Compute $(-1 + i) \cdot v$.

Solution:

- (a) $\frac{1}{2} \cdot v = (1 + i, -\frac{1}{2} + \frac{1}{2}i)$.
- (b) $(3i)v = (-6 + 6i, -3 - 3i)$.
- (c) $(-1 + i)v = ((-1 + i)(2 + 2i), (-1 + i)^2) = (-2 - 2i + 2i - 2, 1 - 2i - 1) = (-4, -2i)$.

Problem §2 For each of the following statements, write down its negation, then assert whether the original statement or the negation is true. Finally, prove your assertion.

- (a) There is some $\alpha \in \mathbb{C}$ such that $\alpha \cdot (1 + i, 1 - i) = (1, i)$.
- (b) For all $s \in \mathbb{C}$, there exists $t, u \in \mathbb{C}$ such that $(1, s) = t(u, -i)$.
- (c) In any field \mathbb{F} and given any $a, b \in \mathbb{F}$ such that $a \neq 0$, there is some $x \in \mathbb{F}$ such that $ax = b$.
- (d) For every $a, b \in \mathbb{Q}$ not both 0, there exist $c, d \in \mathbb{Q}$ such that the following equation holds true in \mathbb{R} :

$$(a + b\sqrt{3})(c + d\sqrt{3}) = 1.$$

Solution:

- (a) Negation: For all $\alpha \in \mathbb{C}$, $\alpha \cdot (1 + i, 1 - i) \neq (1, i)$. The **negation** is true.

Proof. Let $\alpha \in \mathbb{C}$. Then

$$\alpha + \alpha i = 1$$

$$\alpha - \alpha i = i.$$

Plugging the second equation into the first equation, we get

$$\alpha + \alpha^2 i + \alpha^2 = 1$$

$$(\alpha^2 + \alpha) + \alpha^2 i = 1 + 0i.$$

From this, we observe that $\alpha^2 = 0$, and so $\alpha = 0$ as well. Then $\alpha^2 + \alpha = 0 \neq 1$, and so the negation is true. \square

- (b) Negation: There exists an $s \in \mathbb{C}$ such that for any $t, u \in \mathbb{C}$, $(1, s) \neq t \cdot (u, -i)$. The **negation** is true.

Proof. Let $s = 0$. We observe that $s = 0 = -ti$, which holds true only when $t = 0$; thus $t = 0$ as well. Then $1 \neq t \cdot u = 0$. \square

- (c) Negation: There exists a field \mathbb{F} and some $a, b \in \mathbb{F}, a \neq 0$ such that for any $x \in \mathbb{F}$, $ax \neq b$. The **negation** is true.

Proof. Choose $\mathbb{F} = \mathbb{Z}, a = 2, b = 1$. Then $x = \frac{b}{a} = \frac{1}{2} \notin \mathbb{Z}$. \square

(d) Negation: There exists some $a, b \in \mathbb{Q}$, not both zero, such that for any $c, d \in \mathbb{Q}$,

$$(a + b\sqrt{3})(c + d\sqrt{3}) \neq 1.$$

The **original statement** is true.

Proof. Let $c = \frac{a}{a^2 - 3b^2}$, $d = -\frac{b}{a^2 - 3b^2}$. Then

$$\begin{aligned} & (a + b\sqrt{3}) \left(\frac{a}{a^2 - 3b^2} - \frac{b}{a^2 - 3b^2} \sqrt{3} \right) \\ &= \frac{a^2}{a^2 - 3b^2} - \frac{3b^2}{a^2 - 3b^2} \\ &= \frac{a^2 - 3b^2}{a^2 - 3b^2} \\ &= 1. \end{aligned}$$

(Because we know that a and b are not both 0, the denominator is nonzero.) \square