Problem §1 Suppose $u, w \in V$ are vectors with norm 1. Let $U = \operatorname{span}(u)$ and $W = \operatorname{span}(w)$. Prove that

$$\mathcal{P}_W \mathcal{P}_U w = |\langle u, w\rangle|^2 w.$$

Solution: Since $\|u\| = 1$ and $\|w\| = 1$ and they clearly form a basis for U and W respectively, u and w are orthonormal bases for U and W respectively. Recall that for any $v \in V$ and any subspace U of V,

$$\mathcal{P}_U v = \langle v, e_1\rangle e_1 + \ldots + \langle v, e_m\rangle e_m,$$

where $e_1, \ldots, e_m$ is an orthonormal basis for U. Then

$$\mathcal{P}_U w = \langle w, u\rangle u,$$

so

$$\mathcal{P}_W \mathcal{P}_U w = \langle \mathcal{P}_U w, w\rangle w = \langle \langle w, u\rangle u, w\rangle w = \langle w, u\rangle \langle u, w\rangle w = \overline{\langle u, w\rangle} \langle u, w\rangle w = |\langle u, w\rangle|^2 w,$$

as required.

Problem §2 Prove that for any polynomial $f \in \mathcal{P}(\mathbb{R})$,

$$\frac{1}{3}\int_0^1 (f(x))^2 dx \geq \left(\int_0^1 x f(x)\right)^2.$$

When is the above inequality an equality?

Solution: We begin with a few observations:

- From class and previous homeworks,

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx$$

  is an inner product on $\mathcal{P}(\mathbb{R})$.

- For any $p, q \in \mathcal{P}(\mathbb{R})$, the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx \in \mathbb{R}.$$

  That is, the inner product is a real number. In other words, the integral of a real function must be a real number. Geometrically, one can verify this: the "area under the curve" of any real function must be a real number.

Let $x, f(x) \in \mathcal{P}(\mathbb{R})$. Then

$$\langle x, f(x)\rangle = \int_0^1 x f(x) dx.$$

By the Cauchy-Schwarz inequality,

$$|\langle x, f(x)\rangle| \leq \|x\| \, \|f(x)\|.$$

Since $\langle \cdot, \cdot\rangle \in \mathbb{R}$, we have $|\langle x, f(x)\rangle| = \langle x, f(x)\rangle$. Now, square both sides in the above inequality:

$$\begin{aligned}
\langle x, f(x)\rangle^2 &\leq \|x\|^2 \|f(x)\|^2 \\
&= \langle x, x\rangle \langle f(x), f(x)\rangle \\
&= \int_0^1 x^2 dx \cdot \int_0^1 (f(x))^2 dx \\
&= \frac{1}{3} x^3 |_0^1 \int_0^1 (f(x))^2 dx \\
&= \frac{1}{3} \int_0^1 (f(x))^2 dx,
\end{aligned}$$

as required.

From Cauchy-Schwarz, inequality becomes an equality if and only if $f(x)$ is a scalar multiple of x. Thus, equality holds when $f(x) = ax$ for some $a \in \mathbb{R}$.

Problem §3

(a) Let $T \in \mathcal{L}(\mathbb{R}^2)$ be given by

$$T(x,y) = (\frac{7}{2}x + \frac{1}{2}y, \frac{7}{2}x + \frac{1}{2}y).$$

Compute the singular values of T.

(b) Find a non-zero vector v such that $\|Tv\| = 5\|v\|$.

Solution:

(a) We first find the adjoint operator $T^* \in \mathcal{L}(\mathbb{R}^2)$. Let $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$; then

$$\begin{aligned}
\langle (x_1,y_1), T^*(x_2,y_2)\rangle &= \langle T(x_1,y_1),(x_2,y_2)\rangle \\
&= \left\langle (\frac{7}{2}x_1 + \frac{1}{2}y_1, \frac{7}{2}x_1 + \frac{1}{2}y_1),(x_2,y_2)\right\rangle \\
&= \frac{7}{2}x_1x_2 + \frac{1}{2}x_2y_1 + \frac{7}{2}x_1y_2 + \frac{1}{2}y_1y_2 \\
&= \left\langle (x_1,y_1),(\frac{7}{2}x_2 + \frac{7}{2}y_2, \frac{1}{2}x_2 + \frac{1}{2}y_2)\right\rangle.
\end{aligned}$$

Thus we need

$$T^*(x,y) = (\frac{7}{2}x + \frac{7}{2}y, \frac{1}{2}x + \frac{1}{2}y),$$

and so

$$T^*T(x,y) = (\frac{49}{2}x + \frac{7}{2}y, \frac{7}{2}x + \frac{1}{2}y).$$

T^*T has the following matrix with respect to the standard basis:

$$\mathcal{M}(T^*T) = \begin{pmatrix} \frac{49}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{1}{2} \end{pmatrix}.$$

Computing the determinant of $T^*T - \lambda I$, we get

$$\begin{aligned}
\begin{vmatrix} \frac{49}{2} - \lambda & \frac{7}{2} \\ \frac{7}{2} & \frac{1}{2} - \lambda \end{vmatrix} &= \frac{49}{4} - \frac{50}{2}\lambda + \lambda_2 - \frac{49}{4} \\
&= \lambda^2 - 25\lambda = 0 \\
&= \lambda(\lambda - 25).
\end{aligned}$$

Thus T^*T has eigenvalues $\lambda = 0$ and 25, so $\sqrt{T^*T}$ has eigenvalues $\lambda = 0$ and 5. Therefore T has singular values 0 and 5.

(b) We want

$$\sqrt{T^*T}(x,y) = 5(x,y),$$

since $\|av\| = |a|\|v\|$ for any scalar $a \in \mathbb{F}$, and $\|Tv\| = \|\sqrt{T^*T}v\|$. Thus,

$$\begin{aligned}
\sqrt{T^*T}(x,y) &= 5(x,y) \\
\sqrt{(\frac{49}{2}x + \frac{7}{2}y, \frac{7}{2}x + \frac{1}{2}y)} &= (5x, 5y) \\
\frac{49}{2}x + \frac{7}{2}y &= 25x^2 \\
\frac{7}{2}y &= 25x^2 - \frac{49}{2}x \\
y &= \frac{50}{7}x^2 - \frac{49}{7}x.
\end{aligned}$$

Inspection reveals $x = 1,\ y = \frac{1}{7}$. Therefore, the vector $v = (1, \frac{1}{7}) \in \mathbb{R}^2$ satsfies

$$\|Tv\| = 5\|v\|.$$

Problem §4

(a) Use cofactors to compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix}.$$

(b) Let $n \geq 2$, and let A be an $n \times n$ matrix over a field $\mathbb{F}$. Let C be its matrix of cofactors. Find and prove a formula for $\det C$, by relating $\det C$ to $\det A$.

Solution:

(a) First, observe that $\det A = 1(2-0) = 2 \neq 0$, so a matrix exists. Let C be the cofactor matrix of A; then

$$C = \begin{pmatrix} 0 & 0 & 2 \\ 1 & -3 & -1 \\ 0 & 2 & 0 \end{pmatrix}.$$

In class, we showed that if $\det A \neq 0$, then $A^{-1} = \frac{1}{\det A} C^T$ (Corollary .28). Thus

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 2 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}.$$

(b) In class, we proved that for any square matrix A and its matrix of cofactors C, we have

$$AC^T = (\det A)I = \begin{pmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{pmatrix}$$

(Proposition 8.26). Additionally, we showed that $\det AB = \det A \det B$ (Proposition 8.21). Taking the determinant of both sides, we get

$$\begin{aligned}
\det AC^T &= \det \begin{pmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{pmatrix} \\
(\det A)(\det C^T) &= (\det A)^n \\
\det C^T &= \frac{(\det A)^n}{\det A} = (\det A)^{n-1}.
\end{aligned}$$

Recall additionally that for any square matrix A,

$$\det A^T = \det A$$

(Proposiiton 8.20). Thus the determinant of the cofactor matrix of A is given by

$$\det C = (\det A)^{n-1}.$$

Problem §5 Classify the self-adjoint isometries of $\mathbb{R}^3$ and describe each one geometrically.

Alternative Problem:
Suppose $T \in \mathcal{L}(V)$ is a positive operator and an isometry. Does it follow that $T = I$?

Solution: We start with the **alternative problem**: Suppose $T \in \mathcal{L}(V)$ is a positive operator and an isometry. Then T is self-adjoint, and must have non-negative eigenvalues; moreover, for any $v \in V$, we must have

$$\|Tv\| = \|v\|.$$

By the Spectral Theorem, since T is self-adjoint, T has an orthonormal basis composed of eigenvectors, say $e_1, \ldots, e_m$ (let $m = \dim V$) and corresponding eigenvectors $\lambda_1, \ldots, \lambda_m$ (not necessarily distinct). By the properties of isometries (Proposition 10.23), T sends the orthonormal basis $e_1, \ldots, e_m$ to another orthonormal basis $Te_1, \ldots, Te_m$. Since T is an isometry, we need

$$\|Te_i\| = |\lambda_i|\|e_i\| = \|e_i\|$$

for any $1 \leq i \leq m$. In other words, we need $|\lambda_i| = 1$, or equivalently

$$\lambda_i = 1, -1, i, \text{ or } -i;$$

however, T is positive, so all λ_i must be non-negative. Additionally, T is self-adjoint, so all eigenvalues must be real. Hence only $\lambda_i = 1$ works. Therefore, every eigenvalue λ_i of T must equal $\lambda_i = 1$. Since T has a basis of eigenvectors $e_1, \ldots, e_m$, every corresponding eigenvalue $\lambda_1, \ldots, \lambda_m$ must also equal 1; therefore, for any vector $v \in V$, we have

$$Tv = T(\langle v, e_1 \rangle e_1) + \ldots + T(\langle v, e_m \rangle e_m) = 1 \cdot \langle v, e_1 \rangle e_1 + \ldots + 1 \cdot \langle v, e_m \rangle e_m = v.$$

That is, $T = I$.

Now, we attempt the **main problem**. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ is an isometry. Then by isometry properties, we have $T^*T = I$; but T is self-adjoint, so $T^* = T$, so $T^2 = I$. In other words, T is the square root of the identity matrix. Since T is self-adjoint, it is diagonalizable by the Spectral Theorem, and additionally must have all real eigenvalues. Therefore, T is of the form:

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

For all positive and all negative 1s on the diagonal, this is the **identity operator and the** $180°$ **rotation about the origin** respectively. We observe that the cases of 2 +1s and 1 −1 along the diagonal are the same as the cases of 1 +1 and 2 −1s along the diagonal (since we can swap the basis vectors associated with 1, achieving the same result). There are three separate classes of operators that have this property (of 2 +1s and 1 −1 along the diagonal): a **reflection across the** xy**-plane**, a **reflection across the** xz**-plane**, and a **reflection across the** yz**-plane**, given by $T(x,y,z) = (x,y,-z)$, $T(x,y,z) = (x,-y,z)$, and $T(x,y,z) = (-x,y,z)$ respectively (this also shows the property of eigenvalues above). These are thus all four self-adjoint isometries of $\mathbb{R}^3$.