Problem §1 Let $a \in \mathbb{F}$, and let $v \in V_{\mathbb{F}}$ be a non-zero vector. Prove that

$$av = \mathbf{0} \text{ only if } a = 0.$$

Solution: Let $v \in V_{\mathbb{F}}$ be a non-zero vector, and let $a \in \mathbb{F}$ such that $av = \mathbf{0}$. Suppose $a \neq \mathbf{0}$. Then

$$\begin{aligned} av &= \mathbf{0} \\ a^{-1}av &= a^{-1}\mathbf{0} && [\text{since } a \in \mathbb{F} \text{ and } a \neq 0, \text{ there exists } a^{-1} \in \mathbb{F} \text{ such that } aa^{-1} = 1] \\ v &= \mathbf{0}. \end{aligned}$$

But this is a contradiction, as we assume v non-zero. Hence a must be zero.

Problem §2 Let $v, w \in V$, and suppose $v \neq 0$. Prove that there exists at most one $a \in \mathbb{F}$ such that

$$av = w.$$

Solution: Let $v, w \in V_{\mathbb{F}}$, and let $a \in \mathbb{F}$ such that $av = w$. Suppose there exists a $b \in \mathbb{F},\ b \neq a$ such that $bv = w$. Then

$$\begin{aligned} av &= w = bv \\ av &= bv \\ av - bv &= bv - bv \\ (a-b)v &= \mathbf{0} && [\text{since } -bv = (-b)v]. \end{aligned}$$

From problem 1, we get that if v is a non-zero vector, then $(a-b)v = 0$ only when $a - b = 0$. Hence $a = b$; but we assume $b \neq a$, a contradiction. Thus if $\exists b \in \mathbb{F}$ such that $bv = w$, then $b = a$.

Problem §3

(a) (1.C.12) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

(b) (1.C.13, extra credit) Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution:

(a) Let $U_1, U_2 \in V$ be two subspaces of V.

Suppose that one of the subspaces is contained in the other; that is, suppose without loss of generality that $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2 \subseteq V$ is a subspace of V (since U_2 is a subspace of V).

Now, suppose that $U_1 \cup U_2 \subseteq V$ is a subspace of V. Let $v_1, v_2 \in U_1 \cup U_2$. Then

$$v_1 \in U_1,\ v_1 \in U_2,\ \text{or } v_1 \in U_1 \cap U_2,\ \text{and } v_2 \in U_1,\ v_2 \in U_2,\ \text{or } v_2 \in U_1 \cap U_2.$$

$v_1, v_2 \in U_1$ or $v_1, v_2 \in U_2$ tell us nothing new about the relationship between U_1 and U_2 (we already know, by definition of a subspace, that $0,\ \lambda v_1,\ \lambda v_2,\ v_1 + v_2 \in U_i$ for some $\lambda \in \mathbb{F}$, $i \in \{1, 2\}$; and if $v_i \in U_1 \cap U_2$, then v_i is in each individual subspace as well), so suppose without loss of generality that $v_1 \in U_1, v_2 \in U_2$ (if either are in the intersection of the two subspaces, then they are also in each individual subspace). We know (since $U_1 \cup U_2$ is a subspace of V) that $v_1 + v_2 \in U_1 \cup U_2$, so $v_1 + v_2 \in U_1$, $v_1 + v_2 \in U_2$, or $v_1 + v_2 \in U_1 \cap U_2$.

Suppose $v_1 + v_2 \in U_1$. Then since $v_1 \in U_1$, we have $v_1^{-1} \in U_1$, and by closure of addition $v_1^{-1} + v_1 + v_2 \in U_1$; thus $v_2 \in U_1$, and so since for any arbitrary $v_2 \in U_2,\ v_2 \in U_1$, we have $U_2 \subseteq U_1$.

Similarly, suppose $v_1 + v_2 \in U_2$. Then $v_2^{-1} \in U_2$, and by closure of addition $v_1 + v_2 + v_2^{-1} \in U_2$; thus $v_1 \in U_2$ as well, and so $U_1 \subseteq U_2$.

The case of $v_1 + v_2 \in U_1 \cap U_2$ follows the same logic; since being in the intersection implies being in each individual subspace, using the steps above, we have $v_2 \in U_1$, and $v_1 \in U_2$. Thus, we have $U_1 \subseteq U_2$, and $U_2 \subseteq U_1$, and so the subspaces are contained in each other (indeed, they are equivalent).

Thus, the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Problem §4 Solved in Review Sheet 5.

Problem §5 Let V, W be vector spaces over $\mathbb{F}$, and let $T : V \to W$ be a linear map. Suppose V is finite-dimensional and T is surjective. Prove that W is finite-dimensional.

Solution: We begin with a lemma.

Lemma 1. *Let V, W be vector spaces over a field $\mathbb{F}$. If V is an n-dimensional vector space, and $T : V \to W$ is a linear map, then* $\dim(\operatorname{range} T) \leq \dim V$.

Proof. If $\dim V = n$, then V has a basis $B = \{v_1, \dots, v_n\}$ with n linearly independent vectors in V that span V. Then, for any $v \in V$,

$$v = a_1 v_1 + \ldots + a_n v_n, \quad \text{for } a_i \in \mathbb{F}.$$

Applying T to both sides, we have

$$T(v) = T(a_1 v_1 + \ldots + a_n v_n) = a_1 T(v_1) + \ldots + a_n T(v_n),$$

and since v was an arbitrary $v \in V$, $T(v_1), \dots, T(v_n)$ span $\operatorname{range} T$. Hence $\dim(\operatorname{range} T) \leq n = \dim V$.

(Note that there is no guarantee that $T(v_1), \dots, T(v_n)$ are linearly independent and thus form a basis for $\operatorname{range} T$; $\dim(\operatorname{range} T)$ could very much be less than n. For example, take $T = \mathbf{0}$ the zero map. Then $T(v_i) = \mathbf{0}$ for any $v_i \in B$, and so $\dim(\operatorname{range} T) < n$.) $\square$

From this lemma, we get that if V is a finite-dimensional vector space (say, with $\dim V = n$), and $T : V \to W$ a linear map, then $\dim(\operatorname{range} T) \leq \dim V$. Moreover, since T is surjective, by definition we have $\operatorname{range} T = W$. Thus $\dim W \leq \dim V$, and so W is finite-dimensional as well.