Problem §1 Let $v = (2+2i, -1+i)$.

(a) Compute $\frac{1}{2} \cdot v$.

(b) Compute $(3i) \cdot v$.

(c) Compute $(-1+i) \cdot v$.

Solution:

(a) $\frac{1}{2} \cdot v = \left(1+i, -\frac{1}{2} + \frac{1}{2}i\right)$.

(b) $(3i)v = (-6+6i, -3-3i)$.

(c) $(-1+i)v = \left((-1+i)(2+2i), (-1+i)^2\right) = (-2-2i+2i-2, 1-2i-1) = (-4, -2i)$.

Problem §2 For each of the following statements, write down its negation, then assert whether the original statement or the negation is true. Finally, prove your assertion.

(a) There is some $\alpha \in \mathbb{C}$ such that $\alpha \cdot (1+i, 1-i) = (1, i)$.

(b) For all $s \in \mathbb{C}$, there exists $t, u \in C$ such that $(1, s) = t(u, -i)$.

(c) In any field $\mathbb{F}$ and given any $a, b \in \mathbb{F}$ such that $a \neq 0$, there is some $x \in \mathbb{F}$ such that $ax = b$.

(d) For every $a, b \in \mathbb{Q}$ not both 0, there exist $c, d \in \mathbb{Q}$ such that the following equation holds true in $\mathbb{R}$:

$$\left(a + b\sqrt{3}\right)\left(c + d\sqrt{3}\right) = 1.$$

Solution:

(a) Negation: For all $\alpha \in \mathbb{C}$, $\alpha \cdot (1+i, 1-i) \neq (1, i)$. The **negation** is true.

Proof. Let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} \alpha + \alpha i &= 1 \\ \alpha - \alpha i &= i. \end{aligned}$$

Plugging the second equation into the first equation, we get

$$\begin{aligned} \alpha + \alpha^2 i + \alpha^2 &= 1 \\ \left(\alpha^2 + \alpha\right) + \alpha^2 i &= 1 + 0i. \end{aligned}$$

From this, we observe that $\alpha^2 = 0$, and so $\alpha = 0$ as well. Then $\alpha^2 + \alpha = 0 \neq 1$, and so the negation is true. $\square$

(b) Negation: There exists an $s \in \mathbb{C}$ such that for any $t, u \in \mathbb{C}, (1, s) \neq t \cdot (u, -i)$. The **negation** is true.

Proof. Let $s = 0$. We observe that $s = 0 = -ti$, which holds true only when $t = 0$; thus $t = 0$ as well. Then $1 \neq t \cdot u = 0$. $\square$

(c) Negation: There exists a field $\mathbb{F}$ and some $a, b \in \mathbb{F}, a \neq 0$ such that for any $x \in \mathbb{F}$, $ax \neq b$. The **original statement** is true.

Proof. Observe that

$$\begin{aligned} ax &= b \\ a^{-1}ax &= a^{-1}b \\ x &= a^{-1}b. \end{aligned}$$

By definition of a field, multiplication is closed; and since $a^{-1}, b \in \mathbb{F}$, $a^{-1}b \in \mathbb{F}$, and we choose $x = a^{-1}b$. □

(d) Negation: There exists some $a, b \in \mathbb{Q}$, not both zero, such that for any $c, d \in \mathbb{Q}$,

$$\left(a + b\sqrt{3}\right)\left(c + d\sqrt{3}\right) \neq 1.$$

The **original statement** is true.

Proof. Let $c = \frac{a}{a^2-3b^2}, d = -\frac{b}{a^2-3b^2}$. Then

$$\begin{aligned} &\left(a + b\sqrt{3}\right)\left(\frac{a}{a^2 - 3b^2} - \frac{b}{a^2 - 3b^2}\sqrt{3}\right) \\ &= \frac{a^2}{a^2 - 3b^2} - \frac{3b^2}{a^2 - 3b^2} \\ &= \frac{a^2 - 3b^2}{a^2 - 3b^2} \\ &= 1. \end{aligned}$$

(We can do the final step, as we know that $a^2 - 3b^2 \neq 0$. If it were 0, then

$$\begin{aligned} a^2 - 3b^2 &= 0 \\ a^2 &= 3b^2 \\ a &= b\sqrt{3}, \end{aligned}$$

which is not possible since $a \in \mathbb{Q}$.) □