Problem §1 (1.6) Prove that 7 divides $11^n - 4^n$ for all $n \in \mathbb{Z}^+$.

Proof. Let P_n: "7 divides $11^n - 4^n$ for some $n \in \mathbb{Z}^+$ ".
P_1 is true because $11 - 4 = 7$.
Now, assume P_n is true. Then

$$11^n - 4^n = 7k, \ k \in \mathbb{Z}^+.$$

To prove P_{n+1} from P_n, we have

$$\begin{aligned}
11^{n+1} - 4^{n+1} &= 11^n \cdot 11 - 4^n \cdot 4 \\
&= 11^n \cdot 11 - 4^n \cdot 11 + 4^n \cdot 11 - 4^n \cdot 4 \\
&= (11^n \cdot 11 - 4^n \cdot 11) + (4^n \cdot 11 - 4^n \cdot 11) \\
&= 11 \cdot 7k + 7 \cdot 4^n \\
&= 7\,(11k + 4^n)\,.
\end{aligned}$$

Hence P_{n+1} is true whenever P_n is true, and thus by mathematical induction the statement is true. □

Problem §2 (1.11) Let P_n: " $n^2 + 5n + 1$ is even."

(a) Show that P_{n+1} is true whenever P_n is true.

(b) For which P_n is the statement actually true? What is the moral of this exercise?

Solution:

(a) Assume P_n is true. Then

$$n^2 + 5n + 1 = 2k.$$

To prove P_{n+1} from P_n, we have

$$\begin{aligned}
(n+1)^2 + 5(n+1) + 1 &= n^2 + 2n + 1 + 5n + 5 + 1 \\
&= (n^2 + 5n + 1) + 2n + 6 \\
&= 2k + 2(n+3) \\
&= 2(k + n + 6).
\end{aligned}$$

Hence P_{n+1} is true whenever P_n is true, and thus by mathematical induction the statement is true.

(b) None! $P_1 = 7, \ P_2 = 15, \ldots$ In general, P_n is always odd:

- If $n = 2k$ (even): $4k^2 + 10k + 1 \implies$ odd
- If $n = 2k+1$ (odd): $4k^2 + 4k + 1 + 10k + 5 + 1 = 2(2k^2 + 7k + 3) + 1 \implies$ odd

Thus, in order for mathematical induction to be valid, there must be a base case.

Problem §3 Prove that

$$\left(1 - \frac{1}{\sqrt{2}}\right) \ldots \left(1 - \frac{1}{\sqrt{n}}\right) < \frac{2}{n^2}.$$

for all $n \geq 2$.

Proof. Let the above statement denote P_n.
P_2 is true:

$$\begin{aligned}
1 - \frac{1}{\sqrt{2}} = 1 - \frac{\sqrt{2}}{2} &< \frac{1}{2} \\
\frac{1}{2} &< \frac{\sqrt{2}}{2} \\
\frac{1}{4} &< \frac{2}{4}.
\end{aligned}$$

Now, assume P_n is true. Then

$$\left(1-\frac{1}{\sqrt{2}}\right)\dots\left(1-\frac{1}{\sqrt{n}}\right)<\frac{2}{n^2}.$$

Let $a_n = 1-\frac{1}{\sqrt{n}}$. Then $a_{n+1}=1-\frac{1}{\sqrt{n+1}}=\frac{\sqrt{n+1}-1}{\sqrt{n+1}}=\frac{n}{n+1+\sqrt{n+1}}$. We can rewrite P_n using logarithms:

$$\log(a_1)+\ldots+\log(a_n)<\log(2)-2\log(n).$$

To prove P_{n+1} from P_n, we have

$$\begin{aligned}\log(a_1)+\ldots+\log(a_n)+\log(a_{n+1}) &< \log(2)-\log(n)+\log(a_{n+1})\\ &=\log(2)-\log(n)+\log(n)-\log\left((n+1+\sqrt{n+1})\right)\\ &=\log(2)-\log\left(n+1+\sqrt{n+1}\right).\end{aligned}$$

We now show that $\log\left(n+1+\sqrt{n+1}\right)$ is greater than $2\log(n+1)$ (and so its reciprocal is less), which completes the proof.

$$\begin{aligned}\log\left(n+1+\sqrt{n+1}\right) &> 2\log(n+1)\\ n+1+\sqrt{n+1} &> (n+1)^2=n^2+2n+1\\ n\sqrt{n+1} &> n+1\\ n^2(n+1) &> n^2+2n+1\\ n^3+n^2 &> n^2+2n+1\\ n^3 &> 2n+1\\ n^3-2n &> 1,\end{aligned}$$

which is clearly true for all $n\geq 2$, and so $n+1+\sqrt{n+1}>(n+1)^2=n^2+2n+1$. From this we get (after removing logs)

$$\frac{2}{n+1+\sqrt{n+1}}<\frac{2}{(n+1)^2},$$

completing the proof.
Hence P_{n+1} is true whenever P_n is true, and thus by mathematical induction the statement is true. □

Problem §4 Prove that for all $n\geq 3$, there exist different natural numbers $a1, a_2,\ldots,a_n$ such that

$$1=\frac{1}{a_1}+\ldots+\frac{1}{a_n}.$$

Proof. We begin by observing $n=3,4,5$.

- For $n=3$: $a_1=2, a_2=3, a_3=6$
- For $n=4$: $a_1=2, a_2=4, a_3=6, a_4=12$
- For $n=5$: $a_1=2, a_2=4, a_3=8, a_4=12, a_5=24$

From this, we get a pattern: for an a_{n-1}, a_n, we have $a_n=2a_{n-1}$. Moreover, when we add a new block, a_{n-1} is updated, and $a_{n+1}=2a_n$. Formally, we define

P_n : There exists different natural numbers $a_1,\ldots,a_n$, with $a_n=2a_{n-1}$ such that $1=\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_{n-1}}+\frac{1}{a_n}$.

From above, we see that P_3 is true.
Suppose P_n is true. Then

$$1=\frac{1}{a_1}+\ldots+\frac{1}{a_{n-1}}+\frac{1}{a_n}.$$

To prove P_{n+1} from P_n, we have

$$1 = \frac{1}{b_1} + \ldots + \frac{1}{b_{n-1}} + \frac{1}{b_n} + \frac{1}{b_{n+1}}.$$

Choose $a_i = b_i$ for all $1 \le i < n-1$ and $i = n$. Let $b_{n-1} = \frac{2a_n}{3}, b_{n+1} = 2a_n$. Then

$$\begin{aligned}
1 &= \frac{1}{b_1} + \ldots + \frac{1}{b_{n-1}} + \frac{1}{b_n} + \frac{1}{b_{n+1}} \\
1 &= \frac{1}{a_1} + \ldots + \frac{3}{2a_n} + \frac{1}{a_n} + \frac{1}{2a_n} \\
1 &= \frac{1}{a_1} + \ldots + \frac{2}{a_n} + \frac{1}{a_n} \\
1 &= \frac{1}{a_1} + \ldots + \frac{1}{a_{n-1}} + \frac{1}{a_n}.
\end{aligned}$$

Hence P_{n+1} is true whenever P_n is true, and thus by mathematical induction the statement is true. □

Problem §5 (2.4) Show that $\sqrt[3]{5-\sqrt{3}} \notin \mathbb{Q}$.

Solution: Let $a^3 = \sqrt[3]{5-\sqrt{3}}$. Then

$$\begin{aligned}
a^3 &= 5 - \sqrt{3} \\
\sqrt{3} &= 5 - a^3 \\
3 &= 25 - 10a^3 + a^6 \\
0 &= a^6 - 10a^3 + 22.
\end{aligned}$$

By the Rational Roots Theorem, we see that the only possible rational solutions are $\pm1, \pm2, \pm11, \pm22$. Simple inspection by plugging in each possible rational solution indicates that none of them work, and so $\sqrt[3]{5-\sqrt{3}}$ is not rational.

Problem §6 (2.7) Show that

(a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$

(b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$

are actually rational.

Solution: We observe that the insides of the large square roots are actually perfect squares.

(a)

$$\begin{aligned}
\sqrt{4+2\sqrt{3}} &= \sqrt{3 + 2\sqrt{3}\cdot 1 + 1} \\
&= \sqrt{\left(\sqrt{3}+1\right)^2} \\
&= \sqrt{3} + 1.
\end{aligned}$$

From this, we get $\sqrt{4+2\sqrt{3}} - \sqrt{3} = \sqrt{3} + 1 - \sqrt{3} = 1 \in \mathbb{Q}$.

(b)

$$\begin{aligned}
\sqrt{6+4\sqrt{2}} &= \sqrt{4 + 2\cdot 2\cdot \sqrt{2} + 2} \\
&= \sqrt{\left(2+\sqrt{2}\right)^2} \\
&= 2 + \sqrt{2}.
\end{aligned}$$

From this, we get $\sqrt{6+4\sqrt{2}} - \sqrt{2} = 2 + \sqrt{2} - \sqrt{2} = 2 \in \mathbb{Q}$.

Thus both are actually rational.

Problem §7 Find all rational solutions of the equation $3x^3 + x^2 - 8x + 4 = 0$.

Solution: By the Rational Root Theorem, the only possible rational solutions are of the form $\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}$, $\pm\, 4$, and $\pm\,\frac{4}{3}$. By plugging in each possible value, we observe that $1, -2, \frac{2}{3}$ satisfy the above equation, and thus are the three rational roots of $3x^3 + x^2 - 8x + 4 = 0$.