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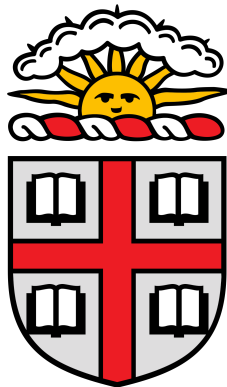
# HONORS LINEAR ALGEBRA

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MATH0540

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## Chapter 1

# Set Theory

Sets serve as a fundamental construct in higher-level mathematics. We start with a brief introduction to set theory.

### §1.1 Sets

#### Definition 1.1.1: Sets

A **set** is a collection of elements.

1.  $x \in X$  means  $x$  is an element of  $X$ .
2.  $x \notin X$  means  $x$  is not an element of  $X$ .
3.  $X \subset Y$  means  $X$  is a subset of  $Y$  (i.e.  $\forall x \in X, x \in Y$ .)
4.  $X = Y \iff X \subset Y \wedge Y \subset X$ .
5.  $A \cap B := \{x \mid x \in A \wedge x \in B\}$  means set intersection.
6.  $A \cup B := \{x \mid x \in A \vee x \in B\}$  means set union.
7.  $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$  means set difference.

**Example 1.** Let

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}.$$

denote the set of integers, and let

$$\mathbb{Z}^+ = \{0, 1, \dots\}.$$

denote the set of positive integers.

#### §1.1.1 Set Builder notation

Sets may be defined formally with set-builder notation:

$$X = \{ \text{expression} \mid \text{rule} \}.$$

**Example 2.** 1. Let  $E$  represent the set of all even numbers. This set is expressed

$$E = \{n \in \mathbb{Q} \mid \exists k \in \mathbb{Z} \text{ s.t. } n = 2k\}.$$

2. Let  $A$  represent the set of real numbers whose squares are rational numbers:

$$A = \{a \in \mathbb{R} \mid a^2 \in \mathbb{Q}\}.$$

### §1.1.2 Cartesian Products

#### Definition 1.1.2: Ordered Tuples

An **ordered pair** is defined  $(x, y)$ . An  **$n$ -ordered tuple** is an ordered list of  $n$  items

$$(x_1, \dots, x_n).$$

#### Definition 1.1.3: Cartesian Products

Let  $A, B$  be sets. The **cartesian product**  $A \times B$  is defined

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, define the  $n$ -fold cartesian product

$$A^n := A \times A \times \dots \times A.$$

**Example 3.**  $\mathbb{R}^2$  and  $\mathbb{R}^3$  are examples of commonly known Cartesian products, which represent the 2D- and 3D-plane respectively.

**Example 4.**  $\mathbb{R}^n$  is a first example of a **vector space**. Let  $n \in \mathbb{Z}^+ \cup \{0\}$ :

1. (Addition in  $\mathbb{R}^n$ ) We define an **addition operation** on  $\mathbb{R}^n$  by adding coordinate-wise

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

2. (Scaling) Given  $(x_1, \dots, x_n) \in \mathbb{R}^n, \lambda \in \mathbb{R}$ , we define

$$\lambda \cdot (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n).$$

**Remark 1.**  $\mathbb{R}_0 = \{0\}$ .

### §1.1.3 Functions

Let  $A, B$  be sets. Informally, a function  $f : A \rightarrow B$  deterministically returns an element  $b \in B$  for each  $a \in A$ . We write  $f(a) = b$ .

**Example 5.** The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$  maps  $\mathbb{R}$  to the subset

$$S \subset \mathbb{R} = \{(x, x^2) \mid x \in \mathbb{R}\}.$$

#### Definition 1.1.4: Functions

Let  $A, B$  be sets. A function  $f : A \rightarrow B$  is a subset  $G_f \subset A \times B$  such that  $\forall a \in A, \exists! b \in B$  s.t.  $(a, b) \in G_f$ . We write  $f(a) = b$  when  $(a, b) \in G_f$ .

#### Definition 1.1.5: Codomain

Given a function  $f : A \rightarrow B$ ,  $A$  is the **domain** of  $f$ , and  $B$  is the **codomain** or **target** of  $f$ . Let the **range** of  $f$  be defined as

$$\{b \in B \mid f(a) = b, a \in A\}.$$

The range is the subset of  $B$ .

#### Definition 1.1.6: Bijectivity

Let  $f : A \rightarrow B$  be a function.

1.  $f$  is **injective**, or an **injection**, if  $a_1, a_2 \in A$  and  $f(a_1) = f(a_2)$  implies  $a_1 = a_2$ .
2.  $f$  is **surjective**, or a **surjection**, if  $\forall b \in B, \exists a \in A$  s.t.  $f(a) = b$ . Equivalently, the range is the whole codomain.
3.  $f$  is **bijective**, or a **bijection**, if it is both injective and surjective. Equivalently,  $\forall b \in B$ , there is a unique  $a \in A$  such that  $f(a) = b$ .

## §1.2 Fields

Roughly speaking, a **field** is a set, together with operations addition and multiplication. Vector spaces may be defined *over* fields.

#### Definition 1.2.1: Fields

A **field** is a set  $\mathbb{F}$  containing elements named 0 and 1, together with binary operations  $+$  and  $\cdot$  satisfying:

- **commutativity:**  $a + b = b + a, a \cdot b = b \cdot a \forall a, b \in \mathbb{F}$ .
- **associativity:**  $a + (b + c) = (a + b) + c, a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall a, b, c \in \mathbb{F}$ .
- **identities:**  $0 + a = a, 1 \cdot a = a \forall a \in \mathbb{F}$ .

- **additive inverse:**  $\forall a \in \mathbb{F}, \exists b \in \mathbb{F} \text{ s.t. } a + b = 0.$
- **multiplicative inverse:**  $\forall a \in \mathbb{F} \setminus \{0\}, \exists c \in \mathbb{F} \text{ s.t. } ac = 1.$
- **distributivity:**  $a \cdot (b + c) = a \cdot b + a \cdot c \forall a, b, c \in \mathbb{F}.$

**Example 6.**  $\mathbb{R}^+ \setminus \{0\}$  is **not** a field under  $+, \cdot$ .

**Example 7.** (Finite Fields) Let  $p$  prime (e.g.  $p = 5$ ). Define

$$\mathbb{F}_p = \{0, \dots, p-1\},$$

with binary operations  $+_p, \cdot_p$  given by addition and multiplication modulo  $p$ . We claim (without proof) that  $\mathbb{F}_p$  is a field.

**Example 8.** Let  $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ . Elements of  $\mathbb{C}$  are called **complex numbers**. Formally, a complex number is an ordered pair  $(a, b), a, b \in \mathbb{R}$ . We define addition as

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and multiplication as

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Showing  $\mathbb{C}$  is a set is left as an exercise for the reader.

### Proposition 1.2.1: $\mathbb{C}$ Multiplicative Inverse

For every  $\alpha \in \mathbb{C} \setminus \{0\}$ , there exists  $\beta \in \mathbb{C}$  with  $\alpha \cdot \beta = 1$ .

**Proof.** Given  $\alpha \in \mathbb{C} \setminus \{0\}$ , let us write  $\alpha = a + bi$ . Then not both  $a, b = 0$ . Let  $\beta = \frac{a}{a^2+b^2} + -\frac{b}{a^2+b^2}i$ . Then  $\alpha\beta = (a + bi) \left( \frac{a}{a^2+b^2} + -\frac{b}{a^2+b^2}i \right) = 1$ .  
Thus  $\forall \alpha \in \mathbb{C} \setminus \{0\}, \exists \beta \in \mathbb{C} \text{ s.t. } \alpha \cdot \beta = 1$ . □

$\mathbb{R}^n$  and  $\mathbb{C}^n$  are specific examples of fields, but by no means the only ones (for instance,  $\mathbb{F}^2$  with addition and multiplication modulo 2 is a field). Fields serve as the underlying set of numbers and operations that vector spaces are built on. In this course, we focus primarily on  $\mathbb{R}$  and  $\mathbb{C}$ ; but many of the definitions, theorems, and proofs work interchangeably with abstract fields.

## §1.3 Vector Spaces

Vector spaces serve as the fundamental abstract structure of linear algebra. All future topics will build on vector spaces.

**Definition 1.3.1: Vector Spaces**

Given a field  $\mathbb{F}$ , A **vector space** over  $\mathbb{F}$ , denoted  $V_{\mathbb{F}}$ , is a set  $V$ , together with vector addition on  $V$

$$+ : V \times V \longrightarrow V$$

and scalar multiplication on  $V$

$$\cdot : \mathbb{F} \times V \longrightarrow V$$

satisfying the following properties:

- (additive associativity) For all  $u, v, w \in V$ ,  $u + (v + w) = (u + v) + w$ .
- (additive identity) There exists an element  $0 \in V$  such that  $v + 0 = 0 + v = 0$ .
- (additive inverse) For all  $v \in V$ , there exists  $w \in V$  such that  $v + w = w + v = 0$ . We denote  $w = -v$ .
- (commutativity) For all  $v, w \in V$ ,  $v + w = w + v$ .
- (scalar multiplicative associativity) For all  $\alpha, \beta \in \mathbb{F}, v \in V$ ,  $\alpha(\beta v) = (\alpha\beta)v$ .
- (scalar multiplicative identity) There exists an element  $1 \in \mathbb{F}$  such that  $1v = v$  for all  $v \in V$ .
- (Distributive Law I) For every  $\alpha \in \mathbb{F}, v, w \in V$ ,  $\alpha \cdot (v + w) = \alpha \cdot v + \alpha \cdot w$ .
- (Distributive Law II) For every  $\alpha, \beta \in \mathbb{F}, v \in V$ ,  $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$ .

We call elements of  $\mathbb{F}$  **scalars**, and elements of  $V$  **vectors**, or **points**.

**Example 9.** Let  $\mathbb{F}$  be a field.

1. For some integers  $n \geq 0$ ,  $\mathbb{F}^n = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{F}\}$  with vector addition defined

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$

and scalar multiplication defined

$$\lambda \cdot (v_1, v_2, \dots, v_n) = (\lambda v_1, \lambda v_2, \dots, \lambda v_n).$$

Note that  $\mathbb{F}^0 = \{0\}$ .

2.  $\mathbb{F}^\infty = P\{(a_1, a_2, a_3, \dots) \mid a_j \in \mathbb{F}, j \in \mathbb{N}\}$  with vector addition and scalar multiplication defined similarly.
3. Let  $S$  be any set; consider  $\{g : S \rightarrow \mathbb{F}\}$  be the set of functions from  $S$  to  $\mathbb{F}$ . Given  $f, g : S \rightarrow \mathbb{F}$ ,  $\lambda \in \mathbb{F}$ , define vector addition  $(f + g) : S \rightarrow \mathbb{F}$  as

$$(f + g)(x) = f(x) + g(x)$$

and scalar multiplication  $\lambda f : S \rightarrow \mathbb{F}$  as

$$(\lambda f)(x) = \lambda f(x).$$

Perhaps counterintuitively, example 3 subsumes example 1! For example, let  $S = \{1, 2, \dots, n\}$ , and let  $\mathbb{R}^{\{1, \dots, n\}}$  be the set of all functions from  $\{1, \dots, n\} \rightarrow \mathbb{R}$ . One such  $f$  may be

$$\begin{aligned} f : \{1, \dots, n\} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = x^2 - 3. \end{aligned}$$

But  $f$  can also be thought of as an  $n$ -tuple. For instance, with  $n = 3$ , we can define a function

$$f = (-2, 1, 6) \in \mathbb{R}^3.$$

This is equivalent to  $f(1) = -2, f(2) = 1, f(3) = 6$ . Similarly, if  $f(x) = e^x$ , then  $f \in \mathbb{R}^{\{1, 2, 3\}} = (e, e^2, e^3) \in \mathbb{R}^3$ , since  $f(1) = e, f(2) = e^2, f(3) = e^3$ .

In other words, every  $n$ -tuple  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  could be represented as a function  $f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}$ , where  $f(1) = x_1, f(2) = x_2, \dots, f(n) = x_n$ . The key insight here is that **the function  $f$  is the  $n$ -tuple**; the one function  $f(x) = e^x$  is equivalent to the  $n$ -tuple  $(e, e^2, \dots, e^n)$ .

From this, we get that the set of functions  $\mathbb{R}^{\{1, \dots, n\}} = \mathbb{R}^n$ , the set of  $n$ -tuples.

**Remark 2.** Reinterpret  $\mathbb{F}^0 = \{\text{functions } f : \emptyset \longrightarrow \mathbb{F}\}$ . How many functions are there from  $\emptyset \longrightarrow \mathbb{F}$ ?  
One function  $\emptyset = \emptyset \times \mathbb{F}$ .

**Example 10.** The set of continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  forms a vector space over  $\mathbb{R}$ . In particular, the sum of two continuous functions is continuous; and  $a \cdot f$  is continuous for any  $a \in \mathbb{R}$ , and  $f$  continuous.