Problem §1 Compute the following determinants, using properties of determinants or the definition given in class. Explain answers *very briefly.* (NB: determinants are omitted for timeliness).

Solution:

(a) $\det A = 0$. By the alternating property, since rows 1 and 2 are identical, the determinant is 0.

(b) $\det A = -1$. Determinants of upper triangular matrices are simply the result of multiplying the diagonal values.

(c) $\det A = -2$. Adding columns to other columns doesn't change the determinant value, so adding columns $-5+6$ gives us a diagonal matrix of all 1s, except the last column (with value -2).

Problem §2 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map with eigenvalues and corresponding eigenvectors:

- $\lambda_1 = 2,\ v_1 = (1, 1, 0)$
- $\lambda_2 = -1,\ v_2 = (1, 0, 0)$
- $\lambda_3 = 0,\ v_3 = (0, 0, 1)$

(a) Express $(3, 1, 4)$ as a linear combination of the three vectors above.

(b) Compute $T^{10}(3, 1, 4)$.

Solution:

(a) $(3, 1, 4) = 1(1, 1, 0) + 2(1, 0, 0) + 4(0, 0, 1) = v_1 + 2v_2 + 4v_3$.

(b)

$$\begin{aligned} T^{10}(3,1,4) = T^{10}(v_1 + 2v_2 + 4v_3) &= T^{10}(v_1) + 2T^{10}(v_2) + 4T^{10}(v_3) \\ &= \lambda_1^{10} v_1 + 2\lambda_2^{10} v_2 + 4\lambda_3^{10} v_3 \qquad \text{[since } T^n(v) = \lambda^n v\text{]} \\ &= 1024v_1 + 2v_2 \\ T^{10}(3,1,4) &= (1026, 1024, 0). \end{aligned}$$

Problem §3 Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear operator. Prove that if $T^3 = T^2$ and T injective, then $T = I$.

Solution: Suppose $T^3 = T^2$ and T injective. Then

$$\begin{aligned} T^3 = T^2 &\iff T^3 - T^2 = 0 \\ &\iff T^2(T - I) = 0. \end{aligned}$$

Let $p(z) = z^3 - z^2 = z^2(z-1)$. From above, for any $v \in V$, we have

$$p(T)(v) = (T^2(T - I))(v) = 0,$$

so either T^2 is not injective or $T - I$ is not injective (and so at least one of them is a root of $p(T)$). But T is injective, so T^2 must be injective as well; hence $T - I$ is not injective. In other words, since T^2 injective means it's not a root of $p(T)$, and $T - I$ not injective means it is a root of $p(T)$, and $p(T)v = 0$ for any $v \in V$; for any non-zero $v \in V$,

$$T^2(v) \neq 0 \text{ and } (T - I)(v) = 0;$$

or $\operatorname{null} T^2 = \{0\}$, and $\operatorname{null}(T - I) = V$. But this means that for any $v \in V$,

$$(T - I)(v) = T(v) - v = 0 \implies T(v) = v,$$

and so $T = I$.

Problem §4 Let V and W be vector spaces, and suppose W is finite-dimensional. Suppose $T : V \to W$ is a surjective linear map. Prove that there exists a linear map $S : W \to V$ such that $TS = I_W$.

Solution: Since T is surjective and W finite-dimensional, $\operatorname{range} T = W$, and so any $w_i \in W$ can be represented as

$$T(v_i) = w_i$$

for some (not necessarily unique) $v \in V$.

For any $w_i \in W$, let $T_{w_i} = \{v \in V \mid T(v) = w_i\}$. In other words, if $T(v_{i_1}) = \ldots = T(v_{i_n}) = w_i$, then

$$T_{w_i} = \{v_{i_1}, \ldots, v_{i_n}\}.$$

Define a map $S : W \to V$ as

$$S(w_i) = v_{i_1}, \text{ where } v_{i_1} \in T_{w_i}.$$

Surjectivity guarantees that T_{w_i} is non-empty (since at least one v_i is mapped to w_i), so v_{i_1} exists. Then by construction, we have $TS = I_W$:

$$T \circ S(w_i) = T(v_{i_1}) = w_i = I_W(w_i) \text{ for any } w_i \in W.$$

It remains to show that S is a linear map. For any $c_1, c_2 \in \mathbb{F},\ w_1, w_2 \in W$, we have

$$\begin{aligned}
S(c_1 w_1 + c_2 w_2) &= S(c_1 T(v_{1_1}) + c_2 T(v_{2_1})) && [T \text{ surjective means } v_{i_1} \in T_{w_i} \text{ exists}] \\
&= S(T(c_1 v_{1_1} + c_2 v_{2_1})) && [T \text{ is a linear map}] \\
&= c_1 v_{1_1} + c_2 v_{2_1} && [\text{by construction of } S^1] \\
&= c_1 S(w_1) + c_2 S(w_2).
\end{aligned}$$

Thus S is a linear map that satisfies $TS = I_W$. [1: since if $w_3 = v_{3_1} = c_1 v_{1_1} + c_2 v_{2_1} = w_1 + w_2$, then $S(w_3) = v_{3_1} = c_1 v_{1_1} + c_2 v_{2_1}$]

Problem §5 Let W be the subspace of $\mathcal{P}_6(\mathbb{R})$ consisting of polynomials $f \in \mathcal{P}_6(\mathbb{R})$ such that

$$f(7) = f(11) = f(15) = f(19) = 0.$$

What is the dimension of W?

Solution: We start with two observations:

1. $W = \{f \in \mathcal{P}_6(\mathbb{R}) \mid f(x) = (x-7)(x-11)(x-15)(x-19)a(x),\ a(x) \in \mathcal{P}_2(\mathbb{R})\}$; in other words, W consists of all polynomials with roots $7, 11, 15,$ and 19.
2. $\mathcal{P}_3(\mathbb{R})$ is a subspace of $\mathcal{P}_6(\mathbb{R})$.

Recall from Problem Set F, Problem 4 where we proved that for distinct a_0, a_1, a_2, a_3, the map $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$ given by

$$T(f) = (f(a_0), f(a_1), f(a_2), f(a_3))$$

is an isomorphism. In other words, $\mathcal{P}_3(\mathbb{R}) \cong \mathbb{R}^4$.

Consider the map $T : \mathcal{P}_6(\mathbb{R}) \to \mathbb{R}^4$ given by

$$T(f) = (f(7), f(11), f(15), f(19)).$$

Then any function $f \in \mathcal{P}_6(\mathbb{R})$ is in $\operatorname{null} T$ only if it has $7, 11, 15,$ and 19 as roots (since we need $f(7) = f(11) = f(15) = f(19) = 0$ in order to get $(0, 0, 0, 0)$); in other words,

$$\operatorname{null} T = W = \{f \in \mathcal{P}_6(\mathbb{R}) \mid f(x) = (x-7)(x-11)(x-15)(x-19)a(x),\ a(x) \in \mathcal{P}_2(\mathbb{R})\}.$$

Additionally, T is surjective, since by observation 2 and from above, $\mathcal{P}_3(\mathbb{R}) \subset \mathcal{P}_6(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$, so any $(b_0, b_1, b_2, b_3) \in \mathbb{R}^4$ can be represented by $T(f)$ for some $f \in \mathcal{P}_3(\mathbb{R}) \subset \mathcal{P}_6(\mathbb{R})$ (note that the chosen f is not unique; there could very well be some $f \in \mathcal{P}_6(\mathbb{R})$ that also satisfies $T(f) = (b_0, b_1, b_2, b_3)$). Thus

$$\operatorname{range} T = \mathbb{R}^4.$$

By the rank-nullity theorem,

$$\dim \mathcal{P}_6(\mathbb{R}) = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim W + \dim \mathbb{R}^4.$$

But we know that $\dim \mathcal{P}_6(\mathbb{R}) = 7$, and $\dim \mathbb{R}^4 = 4$; thus

$$\dim W = \dim \operatorname{null} T = \dim \mathcal{P}_6(\mathbb{R}) - \dim \mathbb{R}^4 = 7 - 4 = 3.$$

Thus the dimension of W is 3.