Problem §1 Compute the following matrix products, or explain why they're undefined:

(a)
$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \\ 11 & 13 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}.$$

(b)
$$\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 7 \\ 11 & 13 \end{pmatrix}.$$

(c)
$$\begin{pmatrix} 1+i \\ 0 \\ i \end{pmatrix} \begin{pmatrix} -i & 0 & i \end{pmatrix}.$$

(d)
$$\begin{pmatrix} -i & 0 & i \end{pmatrix} \begin{pmatrix} 1+i \\ 0 \\ i \end{pmatrix}.$$

Solution:

(a)
$$\begin{pmatrix} -3 & -4 \\ -4 & -6 \\ -2 & -22 \end{pmatrix}.$$

(b) Not defined; the matrices are not compatible (given 2×2 and 3×2, $2 \neq 3$)

(c)
$$\begin{pmatrix} -1-i & 0 & -1+i \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

(d)
$$\begin{pmatrix} -1-i-0-1 \end{pmatrix} = (-2-i).$$

Problem §2 Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by counterclockwise rotation by angle θ about the origin.

(a) Let $A = \mathcal{M}(T_\theta)$ w.r.t. the standard basis of $\mathbb{R}^2$. Compute A.

(b) Compute A^2 and A^3, then deduce formulas for $\cos(2\theta), \sin(2\theta), \cos(3\theta), \sin(3\theta)$.

Solution:

(a) For $(1,0)$, a CC rotation by θ sends to $(\cos\theta, \sin\theta)$.
For $(0,1)$, a CC rotation by θ sends to $(-\sin\theta, \cos\theta)$.
Thus,
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(b)
$$A^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix},$$

and

$$A^3 = \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^3\theta - 3\sin^2\theta\cos\theta & \sin^3\theta - 3\sin\theta\cos^2\theta \\ 3\sin\theta\cos^2\theta - \sin^3\theta & \cos^3\theta - 3\sin^2\theta\cos\theta \end{pmatrix}.$$

Thus,

$$\sin(2\theta) = 2\sin\theta\cos\theta, \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta, \qquad \cos(3\theta) = \cos^3\theta - 3\sin^2\theta\cos\theta.$$

Problem §3 Suppose $T \in \mathcal{L}(V, V)$, is a linear operator on V. Does $T^2 - T = 0$ imply that $T = 0$ or $T = 1$? Either prove the original statement, or provide a counter example.

Solution: Consider the map $T \in \mathcal{L}(V, V)$ given by

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = T,$$

and so $T^2 - T = 0$, yet $T \neq 0$ and $T \neq 1$. Thus the original statement is false.