Problem §1 (6.A.6) Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbb{F}$.

Solution: Suppose $\langle u, v\rangle = 0$. Observe that $\langle u, av\rangle = 0$ for all $a \in \mathbb{F}$; the Pythagorean Theorem then tells us that

$$\|u + av\|^2 = \|u\|^2 + \|av\|^2.$$

Since $\|av\|^2 \geq 0$ for any $a \in F,\ v \in V$, we have

$$\|u + av\|^2 = \|u\|^2 + \|av\|^2 \geq \|u\|^2.$$

Taking the square root of both sides, we get $\|u\| \leq \|u + av\|$.

Conversely, suppose $\|u\| \leq \|u + av\|$. Then

$$\|u\|^2 \leq \|u + av\|^2 \implies \|u + av\|^2 - \|u\|^2 \geq 0.$$

But $\|u + av\|^2 = \langle u + av, u + av\rangle$, and likewise for $\|u\|^2$, so

$$\|u + av\|^2 - \|u\|^2 = \langle u, u\rangle + \overline{a}\,\langle u, v\rangle + a\,\langle v, u\rangle + a\overline{a}\,\langle v, v\rangle - \langle u, u\rangle \geq 0.$$

This then becomes

$$|a|^2\,\|v\|^2 + \overline{a}\,\langle u, v\rangle + a\,\langle v, u\rangle \geq 0.$$

Observe that $\langle u, v\rangle\,\langle v, u\rangle = |\langle u, v\rangle|^2 = \langle u, v\rangle\,\langle u, v\rangle$. Since the above situation holds for any $a \in \mathbb{F}$, let

$$a = -\frac{\langle u, v\rangle}{\|v\|^2} = \overline{a}.$$

Then

$$\begin{aligned}
|a|^2\,\|v\|^2 + \overline{a}\,\langle u, v\rangle + a\,\langle v, u\rangle &= \frac{|\langle u, v\rangle|^2\,\|v^2\|}{(\|v\|^2)^2} - \frac{\langle u, v\rangle\,\langle u, v\rangle}{\|v\|^2} - \frac{\langle u, v\rangle\,\langle v, u\rangle}{\|v\|^2} \\
&= \frac{|\langle u, v\rangle|^2}{\|v\|^2} - 2\frac{|\langle u, v\rangle|^2}{\|v\|^2} \\
&= -\frac{|\langle u, v\rangle|^2}{\|v\|^2} \\
&= -\,|\langle u, v\rangle|^2 \\
&\geq 0.
\end{aligned}$$

However, $|\langle u, v\rangle|^2 \geq 0$ for any u, v, with equality occurring only when $\langle u, v\rangle = 0$. Thus u and v are orthogonal.

Problem §2 (6.A.13) Suppose u, v are non-zero vectors in $\mathbb{R}^2$. Prove that

$$\langle u, v\rangle = \|u\|\|v\|\cos\theta$$

where θ is the angle between u and v.

Solution: Consider the triangle formed by u, v, and $u - v$, with θ the angle between u and v. By the law of cosines,

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta.$$

Note that since $\mathbb{R}$ is a real inner product space, $\langle u, v\rangle = \langle v, u\rangle$; so

$$\|u - v\|^2 = \langle u - v, u - v\rangle = \langle u, u\rangle - 2\,\langle u, v\rangle + \langle v, v\rangle = \|u\|^2 - 2\,\langle u, v\rangle + \|v\|^2.$$

Then

$$\begin{aligned}\|u\|^2 - 2\langle u, v\rangle + \|v\|^2 &= \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta \\ -2\langle u, v\rangle &= -2\|u\|\|v\|\cos\theta \\ \langle u, v\rangle &= \|u\|\|v\|\cos\theta,\end{aligned}$$

as required.

> **Problem §3** (6.A.22) Show that the square of an average is less than or equal to the average of the squares; that is, for $a_1, \dots, a_n \in \mathbb{R}$, then
>
> $$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \ldots + a_n^2}{n}.$$
>
> Also: for which choices of $a_1, \dots, a_n \in \mathbb{R}$ does inequality become equality?

Solution: Let $u, v \in \mathbb{R}^n$, where $u = (a_1, \dots, a_n)$ and $v = (1, \dots, 1)$. Then

$$\begin{aligned}\|u\|^2 &= \langle u, u\rangle = a_1^2 + \ldots + a_n^2 \\ \|v\|^2 &= \langle v, v\rangle = 1 + \ldots + 1 = n \\ \langle u, v\rangle &= a_1 + \ldots + a_n.\end{aligned}$$

By the Cauchy-Schwarz Inequality,

$$|\langle u, v\rangle| \leq \|u\|\|v\| \implies |\langle u, v\rangle|^2 \leq \|u\|^2\|v\|^2.$$

Thus

$$(a_1 + \ldots + a_n)^2 \leq (a_1^2 + \ldots + a_n^2)n \implies \frac{(a_1 + \ldots + a_n)^2}{n} \leq a_1^2 + \ldots + a_n^2.$$

Dividing by n on both sides, we get

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \ldots + a_n^2}{n}.$$

In other words, the square of the average is less than or equal to the average of the squares.

By Cauchy-Schwarz, inequality holds only when one vector is a scalar multiple of the other. Thus

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 = \frac{a_1^2 + \ldots + a_n^2}{n}$$

only when $a_1 = \ldots = a_n$.