Real Analysis 1

MATH1010

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Contents

Chapter 1

Real Numbers

In this class, we operate on the set of real numbers. It is important to rigorously define it, as well as all underlying sets of numbers (natural numbers, integers, rationals, etc).

§1.1 Natural Numbers

Definition 1.1.1: Natural Numbers

The set $\mathbb{N} = 1, 2, \ldots$ is the set of natural numbers. Each integer n has a **successor**, $succ(n) = n + 1$. 1 is not the successor of any number.
The following properties constitute the Peano Axioms of $\mathbb{N}$:

1. $1 \in \mathbb{N}$.
2. $n \in \mathbb{N} \Rightarrow succ(n) = n + 1 \in \mathbb{N}$
3. $\nexists n$ s.t. $succ(n) = 1$
4. If $n, m \in \mathbb{N}, succ(n) = succ(m)$, then $n = m$.
5. A subset $A \subset \mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.

We accept only these 5 axioms to prove all other properties of $\mathbb{N}$.

(5) is the basis for the principle of induction.

Theorem 1.1.1: Principle of Mathematical Induction

Let $P_1, P_2, \ldots$ be a list of statements. Assume the following:

1. P_1 is true. [Basis of induction]
2. $\forall n \in N, n \geq 1$, if P_n is true, then P_{n+1} is true. [Inductive step]

Then all the statements $P_1, \ldots$ are true.

Proof. Let A be the set of integers n for which P_n is true. We want to prove $A = \mathbb{N}$. We use (5) to prove this.
Indeed, $1 \in A$ by assumption 1. Assuming that $n \in A$ for some n, we prove that $n + 1 \in A$. This is true by assumption 2: if $n \in A$, then P_n is true, hence P_{n+1} is true, hence $n + 1 \in A$. Thus, $A = \mathbb{N}$. $\square$

Example 1. *Prove that* $2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$.

Proof. Let P_n: "$2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$".
P_1 is true because $2^1 = 2^2 - 2$.
For the induction step, we assume P_n is true for some n and prove P_{n+1} is true. Since P_n is true,

$$2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2.$$

P_{n+1} states that

$$2^1 + \ldots + 2^n + 2^{n+1} = 2^{(n+1)+1} - 2.$$

Using P_n we have

$$2^1 + \ldots + 2^{n+2^{n+1}} = \left(2^{n+1} - 2\right) + 2^{n+1} = 2 \cdot 2^{n+1} - 2 = 2^{(n+1)+1} - 2,$$

Thus P_{n+1} is true. By the principle of induction, P_n is true for all $n \geq 1$. □

Example 2. *Let $x_1 = 1$ and define*

$$x_{n+1} = \frac{1}{2}x_n + 1.$$

Prove that $\forall x, x_n \leq x_{n+1}$ (or, x_n is inreasing).

Proof. Let P_n: "$x_n \leq x_{n+1}$".
P_1 is true because $x_1 = 1 \leq \frac{3}{2} = x_2$.
For the induction step, we assume P_n is true for some n and prove P_{n+1} is true. Since P_n is true,

$$x_n \leq x_{n+1}.$$

P_{n+1} states that

$$x_{n+1} \leq x_{n+2}.$$

Using P_n we have

$$x_{n+1} = \frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1 = x_{n+2} \text{ [by } P_n\text{, we know } x_n \leq x_{n+1}]$$

Thus P_{n+1} is true.
By the principle of induction, P_n is true for all $n \geq 1$. □

The principle of induction can be extended by allowing the first statement to begin at P_m instead of P_1 for some fixed integer m.

Theorem 1.1.2: Generalized Principle of Induction

Let m be an integer, and consider a list of statements $P_m, P_{m+1}, \ldots$. Then all the statements are true if the following two properties are true:

1. P_m is true
2. $\forall n \geq m$, if P_n is true, then P_{n+1} is true.

Example 3. *Prove that*

$$n! > n^2.$$

for all $n \geq 4$.

Proof. Recall $n! = 1 \cdot 2 \cdot \ldots \cdot n$. Let P_n: "$n! > n^2$". We prove P_n is true $\forall n \geq 4$.
P_4 is true because

$$4! = 24 > 16 = 4^2.$$

Assuming $n! > n^2$, we prove

$$(n+1)! > (n+1)^2 \,.$$

Using P_n we have

$$(n+1)! = n!\,(n+1) = (1 \cdot 2 \cdot \ldots \cdot n) \cdot (n+1) > n^2(n+1) > (n+1)^2 \,.$$

Thus P_{n+1} is true.
By the principle of induction, P_n is true for all $n \geq 4$. $\square$