Problem §4b (I'm only including part b, since it was noted that parts a, c, and d had correct proofs)
Let G be a group, and let the centralizer of $g \in G$ be denoted

$$Z_G(g) = \{g' \in G \mid gg' = g'g\}.$$

Compute the centralizer $Z_G(g)$ for the following elements and groups:

- $G = \mathcal{D}_4$ and g is rotation by $90°$.
- $G = \mathcal{D}_4$ and g is a flip fixing two vertices of a square.
- $G = \mathrm{GL}_2(\mathbb{R})$ and $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

Solution: Using the Dihedral group $\mathcal{D}_n$ and rotations as defined before (where $r_i(j) = j + i \mod n$, $f_i(j) = n - j + i \mod n$; compared to the book, $r_k = \rho_k,\ f_k = \phi_{k+1}$), and noting that a rotation by $90°$ is the same as r_i, we see that for any $r_k,\ k \in \{0, 1, \ldots, n-1\} = V_n$,

$$r_k \circ r_i(j) = r_k(j+i) = j + i + k = j + k + i = r_i(j+k) = r_i \circ r_k(j);$$

equivalently, r_1 commutes with any other rotation (for $\mathcal{D}_4$, this means $\{\rho_0 = e, \rho_1, \rho_2, \rho_3\}$). Moreover, for any flip f_k, $k \in V_n$, we have

$$f_k \circ r_1(j) = f_k(j+1) = n - j - 1 + k \qquad r_1 \circ f_k(j) = r_1(n - j + k) = n - j + 1 + k.$$

Clearly, these are not equivalent, so any flip f_k does not commute with r_1. Thus, $Z_{\mathcal{D}_4}(\rho_1) = \{e, \rho_1, \rho_2, \rho_3\}$.

Now, consider flips that fix two vertices of a square; for this, we see that these flips are f_0 and f_2. Trivially, the identity commutes. Consider any rotation f_i, $i \in V_4$. Then

$$f_i \circ f_0(j) = f_i(n-j) = n - n + j + i = j + i \quad f_0 \circ f_i(j) = f_0(n - j + i) = n - n + j - i = j - i$$
$$f_i \circ f_2(j) = f_i(n - j + 2) = n - n + j - 2 + i = j + i - 2 \qquad f_2 \circ f_i(j) = f_2(n - j + i) = n - n + j - i + 2.$$

For f_0, $j + i \equiv j - i \mod 4$ only when $i = 0, 2$; thus only $f_{0,2}$ commutes with f_0. The same is the case with f_2 (since $j + i - 2 \equiv j - i + 2 \mod 4$). Thus, out of the flips, only $f_{0,2}$ commute with flips that fix two vertices.

Finally, consider any rotation r_k, $k \in V_4$. Then

$$r_k \circ f_0(j) = r_k(n-j) = n - j + k \qquad f_0 \circ r_k(j) = f_0(j+k) = n - j - k$$
$$r_k \circ f_2(j) = r_k(n - j + 2) = n - j + k + 2 \qquad f_2 \circ r_k(j) = f_2(j+k) = n - j - k + 2.$$

In both situations, $r_k \circ f_{0,2} \equiv f_{0,2} \circ r_k$ only if $k \equiv -k \mod 4$; clearly, $k = 2$ is the only valid solution, so $r_2 = \rho_2$ is the only valid rotation. Thus $Z_{\mathcal{D}_4}(\phi_{1,3}) = \{e, \phi_1, \phi_3, \rho_2\}$.

Finally, consider elements $\alpha, \beta \in \mathrm{GL}_2(\mathbb{R})$ where $a = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $b = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then

$$\alpha\beta = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af \\ dg & dh \end{pmatrix},$$

and

$$\beta\alpha = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ae & df \\ ag & dh \end{pmatrix}.$$

In other words, the two commute when $af = df$ and $ag = dg$; equivalently, whenever $a = d$ (or $f = g = 0$, but that's included in $\mathrm{GL}_2(\mathbb{R})$). Thus, if $a = d$, then $Z_{\mathrm{GL}_2(\mathbb{R})} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \mathrm{GL}_2(\mathbb{R})$. If $a \neq d$, then $af = df$ and $ag = dg$ implies $f = g = 0$. Thus, if $a \neq d$, then $Z_{\mathrm{GL}_2(\mathbb{R})} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \left\{ \begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \mid e, h \in \mathbb{R} \setminus \{0\} \right\}$ (we exclude 0 since otherwise the matrix would not be invertible, and thus would not be in $\mathrm{GL}_2(\mathbb{R})$).

The main thing I was lacking in my original part b was no explanation; I had stated the centralizer of the specified group element given a group, but I didn't provide justification for why the centralizer was what it was. Thus, in this revision I sought to provide clear explanations behind how I got my results.