MATH 540 HONORS LINEAR ALGEBRA SUMMER 2021, Melody Chan
PROBLEM SET C

Due Tuesday June 1 at 11:59pm Eastern

Submit all of the following on Gradescope, and don't forget to tag each answer to its page. We have implemented a course policy whereby failing to tag results in half credit. I put a copy of this problem set in the Overleaf folder too.

1. (2 points; this is essentially Axler 1.B.2 page 17) Let v be a nonzero vector in a vector space V, and let $a \in \mathbb{F}$. Prove that
$$av = \mathbf{0} \text{ only if } a = 0.$$
In other words, prove that $av = \mathbf{0}$ implies $a = 0$. Possible hint below.[1]

2. (2 points) Let $v, w \in V$, and suppose $v \neq \mathbf{0}$. Prove that there exists at most one $a \in \mathbb{F}$ such that
$$av = w.$$

3. ("Unions of subspaces are pretty much never subspaces, except in silly ways.")
   (a) (3 points) Axler Exercise 1.C.12 page 25. Here V is a vector space over any[2] field $\mathbb{F}$.
   (b) (extra credit, 2 points)
   Axler Exercise 1.C.13 page 25, for V a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ has more than two elements.
   What about four subspaces? Do some examples and/or make the most general *conjecture* (informed guess) that you can. What fails in the Axler problem when $\mathbb{F} = \mathbb{F}_2$, the finite field with two elements?

4. (Ungraded) If you did not already do so on Review Sheet 5, prove for yourself that the range of a linear map $T: V \to W$ is a subspace of W. Check against the textbook's proof on page 62.

5. (4 points) Let V and W be vector spaces over $\mathbb{F}$, and let $T: V \to W$ be a linear map. Suppose V is finite-dimensional and T is surjective. Prove, carefully, that W is finite-dimensional.

[1] You could try a **proof by contradiction:** assume for a moment that the *negation* of the statement that you're trying to prove holds. Try to deduce an absurdity. That is, deduce a contradiction with something you know to be true. That would show that the aforementioned negation *can't* hold. For more, see Hammack's book p. 137.

[2] Note: Axler's convention is that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ only. But in our class, $\mathbb{F}$ denotes an arbitrary field.