Problem §1 (Treil 4.2) Let P be a **permutation matrix**, i.e. an $n \times n$ matrix consisting of zeroes and ones such that there is exactly one 1 in every row and every column.

(a) Can you describe the corresponding linear transformation? That will explain the name.

(b) Show that P is invertible. Can you describe P^{-1}?

(c) Show that for some $N > 0$

$$P^N := \underbrace{PP \dots P}_{N \text{ times}} = I.$$

Solution:

(a) It swaps the basis vectors around (e.g. it *permutes* them); for instance, e_i becomes $Te_i = e_j$ for some j not necessarily equal to i.

(b) One can clearly see that no column is a linear combination of other columns (since each row and column only has one 1 element); thus all columns are linearly independent, and so $\det A \neq 0$; i.e. P is invertible.

Alternatively, let $\sigma \in \mathcal{S}_n$ such that $P_{i,\sigma(i)} = 1$ for all $1 \leq i \leq n$. Since each row and column only has one j such that $\sigma(i) = j$ and $P_{i,j} = 1$, σ is unique (if another permutation φ does the same for all $1 \leq i \leq n$, then $\varphi = \sigma$). Additionally, for any other permutation $\tau \in \mathcal{S}_n$ with $\tau \neq \sigma$, we have $P_{i,\tau(i)} = 0$ (since otherwise, we must have $\sigma = \tau$). Thus,

$$\det P = \sum_{\phi \in \mathcal{S}_n} p_{1,\phi(1)} \dots p_{n,\phi(n)} \operatorname{sgn}(\phi)$$

becomes $p_{1,\sigma(1)} \dots p_{n,\sigma(n)} \operatorname{sgn}(\sigma)$, since all $\tau \in \mathcal{S}_n,\ \tau \neq \sigma$ has $p_{j,\tau(j)} = 0$ for some $1 \leq j \leq n$ (since that means $\tau(j) \neq \sigma(j)$ for some $1 \leq j \leq n$, and so $p_{j,\sigma(j)} \neq p_{j,\tau(j)} = 0$).

All $p_{i,\sigma(i)} = 1$, and $\operatorname{sgn}(\sigma) = \pm 1$, so

$$\det P = \pm 1 \neq 0,$$

so P is invertible.

P^{-1} essentially reverts any permutation that σ applies to i; thus, we get that all 1s are in the indices

$$P_{\sigma(i),i}$$

which is in contrast to $P_{i,\sigma(i)}$; this aligns with our intuition of "reverting" a permutation.

(c) Let $\sigma \in \mathcal{S}_n$ as defined above. $\sigma \in \mathcal{S}_n$ forms a cyclic subgroup in $\mathcal{S}_n$, and by Lagrange's Theorem, the order of $\langle \sigma \rangle \subseteq \mathcal{S}_n$ divides the order of $\mathcal{S}_n$ (which we know to be finite, specifically $n!$). Thus the order of σ is finite (i.e. $\sigma^k = e$ the identity permutation for some integer k).

Next, we observe that multiplying an $n \times n$ matrix A by P permutes the columns of A by σ:

$$(AP)_{i,\sigma(j)} = A_{i,j}$$

(one can check this: only $p_{j,\sigma(j)} = 1 \neq 0$, so we have $(AP)_{i,\sigma(j)} = \sum_{r=1}^{n} A_{i,r} P_{r,\sigma(j)} = A_{i,j} P_{j,\sigma(j)} = A_{i,j}$). Thus $P^2_{i,\sigma(j)} = 1$ only when $j = \sigma(i)$, and so we get $P^2_{i,\sigma^2(i)} = 1$. This repeats for powers, until we get

$$P^n_{i,\sigma(\sigma^{n-1}(1))} = P^n_{i,\sigma^n(i)} = P^n_{i,i} = 1.$$

In other words, for any row i, its (only) 1 entry is at $P^n_{i,i}$, and all $P^k_{i,j}$ with $i \neq j$ has value 0. Thus $P^k = I$, where k is the order of σ.

Alternatively, recall that $\mathcal{S}_n$ has a finite number of permutations, specifically $n!$. Raise P to the power $n! + 1$ (i.e. $P^{n!+1}$). Then by the Pigeonhole Principle, for some $1 \leq j, k \leq n! + 1$, $P^j = P^k$ (since we have $n! + 1$ different powers, and only $n!$ permutations). Assume without loss of generality that $j < k$ (we ignore the trivial $j = k$). Since P^{-1} exists from above, we have

$$P^k = P^j \iff P^k P^{-j} = P^j P^{-j} \iff P^{k-j} = I.$$

Since $j < k$, the integer $0 < k - j = N$ exists, and so

$$P^N = I$$

for some $N > 0$, as required.

Problem §2 Let n be a positive integer, and let $A \in \mathbb{R}^{n,n}$ be a square matrix whose diagonal entries are odd integers, and whose off-diagonal entries are even integers. Prove that A is invertible.

Solution: We use induction to prove that any matrix $A_n \in \mathbb{R}^{n,n}$ with the above property (odd on diagonal, even elsewhere) has an odd (and thus non-zero) determinant. This is clearly true for $n = 1$, so assume the property holds for $1, \ldots, n-1$.

Let $A_n \in R^{n \times n}$ be a matrix with the above property. By co-factor expansion,

$$\det A_n = \sum_{j=1}^{n} a_{1,j} C_{1,j}.$$

But note that $a_{1,1}$ is odd, and all $a_{1,j}$ for $1 < j \le n$ are even (since they're all off-diagonal), and that $C_{1,1} = A_{n-1}$ (since all of the diagonal entries of $C_{1,1}$ are also on the diagonal of A_n, which are all odd, and all of the off-diagonal entries of $C_{1,1}$ are also off the diagonal of A_n; thus $C_{1,1}$ has the same property as above).

By induction, we know $\det A_{n-1}$ is odd. Thus $a_{1,1}C_{1,1}$ is odd (since for two odds $2i+1,\ 2j+1,\ (2i+1)(2j+1) = (4ij+2i+2j)+1 = 2k+1$), and all $a_{1,j}C_{1,j}$ for $1 < j \le n$ are even (since for an even $2i$ and any integer, even $2j$ or odd $2k+1$, the resulting product, either $2i \cdot 2j = 4ij = 2(2ij)$ or $2i(2k+1) = 4ik+2i = 2(2ik+i)$, is even). Therefore the determinant

$$\det A_n = \text{ odd integer } + \sum_{j=2}^{n} \text{ even integer } = \text{ odd integer } \neq 0$$

is odd, and thus non-zero (clearly, odd + even is still odd).

Thus, by induction any $A_n \in \mathbb{R}^{n,n}$ with the above property has an odd (non-zero) determinant for all $n \in \mathbb{N}$, and so A_n is invertible.

Problem §3 Use Cramer's Rule to solve the following system of linear equations:

(a)

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 2x_1 + 3x_2 &= 4. \end{aligned}$$

(b)

$$\begin{aligned} -x_1 + 0x_2 + 2x_3 &= 0 \\ 0x_1 + 5x_2 + x_3 &= 4 \\ 3x_1 - x_2 - x_3 &= 7. \end{aligned}$$

Solution:

(a) $\det A = 3 \cdot 3 - 4 \cdot 2 = 1.$

(a) $B_1 = \begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$, so $\det B_1 = 5 \cdot 3 - 4 \cdot 4 = -1$, so $x_1 = -1$.

(b) $B_2 = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$, so $\det B_2 = 3 \cdot 4 - 5 \cdot 2 = 2$, so $x_2 = 2$.

(b) $\det A = -1(-5 - (-1)) + 2(-15) = -26$.

(a) $B_1 = \begin{pmatrix} 0 & 0 & 2 \\ 4 & 5 & 1 \\ 7 & -1 & -1 \end{pmatrix}$, so $\det B_1 = 2(-4 - 35) = -78$, so $x_1 = \frac{78}{26} = 3$.

(b) $B_2 = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 4 & 1 \\ 3 & 7 & -1 \end{pmatrix}$, so $\det B_2 = -(-4 - 7) + 2(-12) = -13$, so $x_2 = \frac{1}{2}$.

(c) $B_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 4 \\ 3 & -1 & 7 \end{pmatrix}$, so $\det B_3 = -(35 - (-4)) = -39$, so $x_3 = \frac{39}{26} = \frac{3}{2}$.

Problem §4 Prove the formula $x_i = \frac{\det B_i}{A}$ in Cramer's Rule.

Solution: Suppose $\det A \neq 0$ (since otherwise, Cramer's Rule would not hold). Then A is invertible, with $A^{-1} = \frac{1}{\det A} C^T$ (as proven in class). Thus

$$Ax = b \iff A^{-1}Ax = A^{-1}b \iff x = \frac{1}{\det A} C^T b.$$

Next, consider the i-th row of $C^T b$:

$$(C^T b)_{i,1} = \sum_{r=1}^{n} C^T_{i,r} b_{r,1} = b_{1,1} C^T_{i,1} + \ldots + b_{n,1} C^T_{i,n}.$$

But this is the same as

$$b_{1,1} C_{1,i} + \ldots + b_{n,1} C_{n,i},$$

which is exactly the formula for the cofactor expansion for the determinant along column i:

$$\det B_i = \sum_{r=1}^{n} b_{r,1} C_{r,i}.$$

Thus the i-th row of $C^T b = \det B_i = \det A x_i$ (since $x = \frac{1}{\det A} C^T b$, and the i-th row of x is x_i), and so we get

$$x_i = \frac{\det B_i}{\det A},$$

as required.