MATH 540 HONORS LINEAR ALGEBRA SUMMER 2021
EXAM 2

You can use your notes, the course notes, Canvas and Gradescope course pages, and the course textbooks. Other than that, you may use the Internet only at the level of generality of Wikipedia, though honestly I think you're better off sticking with class resources. You're welcome to use a calculator except for on Problem 1.

Under no circumstances may you communicate with anyone, either in the class or outside of the class, about any aspect of the exam, until the entire exam period is over. If you have any uncertainty about what is or is not allowed, it is on you to ask me for clarification.

You are welcome to cite statements that were proved either in class or on your homework. You can cite statements in the textbooks, provided they were covered in some form in class or on homework. Please just state clearly what you're citing.

Feel free to send me an email at melody_chan+linear@brown.edu if you think something on the exam needs to be clarified.

Best of luck! –Melody

5 questions, 2 hours and 15 minutes, (3+3+3)+(1+8)+12+15+15 points

1. Compute the following determinants, using either properties of determinants or the definition of the determinant proved in class. Explain your answers *very briefly*, e.g., half a sentence.[1]

(a)
$$\begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ -1 & -2 & 3 & 0 & 0 \\ 0 & 0 & -3 & 4 & -5 \\ 1 & 2 & 0 & -4 & 5 \end{vmatrix}$$

(b)
$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$

(c)
$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

[1] Just enough so that it is evident that you are thinking about properties of determinants rather than using a calculator.

2. Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map with eigenvectors $(1,1,0), (1,0,0), (0,0,1)$ corresponding to eigenvalues $2, -1$, and 0 respectively.

   (a) Express $(3,1,4)$ as a linear combination of the three vectors above.

   (b) Compute $T^{10}(3,1,4)$.

3. Let V be a finite-dimensional vector space, and let $T\colon V \to V$ be a linear operator. Prove that if $T^3 = T^2$ and T is injective, then $T = I$.

4. Let V and W be vector spaces, and suppose W is finite-dimensional. Suppose $T\colon V \to W$ is a surjective linear map. Prove that there exists a linear map $S\colon W \to V$ such that $TS = I$. Here I denotes the identity map on W.

5. Let W be the subspace of $\mathcal{P}_6(\mathbb{R})$ consisting of polynomials $f \in \mathcal{P}_6(\mathbb{R})$ such that

$$f(7) = f(11) = f(15) = f(19) = 0.$$

   What is the dimension of W? Prove your answer.

   More than one approach is possible. One possibility is to follow a strategy reminiscent of Problem 4, Problem Set F.