Problem §3 Suppose $T : V \to W$ is an injective linear map between finite-dimensional vector spaces. Prove that there exists a linear map $S : W \to V$ such that $ST = I_V$.

Solution: Let $T \in \mathcal{L}(V, W)$ be an injective linear map. Then every $v \in V$ maps to a unique $w \in W$; that is, if $T(v_1) = T(v_2)$, then $v_1 = v_2$. Define a linear map

$$S : W \longrightarrow V,\ S(w) = \text{the unique } v \text{ such that } T(v) = w; \text{ or, } \mathbf{0} \text{ if no such } v \text{ exists.}$$

In other words, if $w = T(v)$ for some $v \in V$, then $S(w) = v$; and if $w' \neq T(v')$ for any $v' \in V$, then $S(w') = \mathbf{0}$. Then for any $v \in V$ we have $S(T(v)) = v = I_V(v)$. (However, clearly under this construction $TS \neq I_W$, since if $w \neq T(v)$ for any $v \in V$, then $T(S(w)) = \mathbf{0} \neq I_W(w)$).

Problem §4 Prove that given distinct $a_0, \ldots, a_d$ and any real numbers $b_0, \ldots, b_d$, there exists a unique $f \in \mathcal{P}_d(\mathbb{R})$ such that

$$f(a_0) = b_0, \ldots, f(a_d) = b_d.$$

Solution: Let

$$T : \mathcal{P}_d(\mathbb{R}) \to \mathbb{R}^{d+1},\ T(f) = (f(a_0), \ldots, f(a_d)),$$

and recall the definition of function addition and scalar multiplication:

$$(f + g)(x) = f(x) + g(x),\ (cf)(x) = c(f(x)).$$

Clearly, T is a linear transformation. For $f, g \in \mathcal{P}_d(\mathbb{R}),\ c_1, c_2 \in \mathbb{R}$, we have

$$\begin{aligned} T(c_1 f + c_2 g) &= ((c_1 f + c_2 g)(a_0), \ldots, (c_1 f + c_2 g)(a_d)) \\ &= ((c_1 f)(a_0) + (c_2 g)(a_0), \ldots, (c_1 f)(a_d) + (c_2 g)(a_d)) \\ &= (c_1 f(a_0), \ldots, c_1 f(a_d)) + (c_2 g(a_0), \ldots, c_2 g(a_d)) \\ &= c_1(f(a_0), \ldots, f(a_d)) + c_2(g(a_0), \ldots, g(a_d)) \\ &= c_1 T(f) + c_2 T(g). \end{aligned}$$

Now, we consider the kernel of T. In order for a function $f \in \mathcal{P}_d(\mathbb{R})$ to have $T(f) = (0, \ldots, 0)$, since all of $a_i \in \mathbb{R}$ are unique, f must have each a_i as a root (i.e. $(x - a_i)a(x) = 0$, where $a(x) \in P_{d-1}(\mathbb{R})$). However, recall that a polynomial $a_0 + \ldots + a_d x^d \in \mathcal{P}_d(\mathbb{R})$ can have at max d distinct roots; thus, no polynomial in $\mathcal{P}_d(\mathbb{R})$ can have more than d roots. Since we have $d + 1$ unique a_i's, it necessarily follows that no non-zero function $f(x) \in \mathcal{P}_d(\mathbb{R})$ will satisfy $f(x) = 0$. Thus $\ker(T) = \{0\}$.

Next, observe that the dimensions of $\mathcal{P}_d(\mathbb{R})$ and $\mathbb{R}^{d+1}$ have the same dimension, and recall that a trivial kernel implies an injective linear transformation (and so T is injective). But we know that if two vector spaces have equal dimensions, and a map between the two is injective, then the map is also bijective.

Hence T is an isomorphism; and so for any $(b_0, \ldots, b_d) \in \mathbb{R}^{d+1}$, we can find a $f \in \mathcal{P}_d(\mathbb{R})$ such that $T(f) = (f(a_0), \ldots, f(a_d)) = (b_o, \ldots, b_d)$. Thus $f(a_0) = b_0, \ldots, f(a_d) = b_d$, as required.