Problem §2 Describe the following subsets of $\mathbb{R}^2$ (not shown) using set-builder notation.

Solution:

(a) $\{\lambda_1 \cdot (-1, 1) \mid \lambda_1 \in \mathbb{R}\}$

(b) $\{\lambda_1 \cdot (-1, 2) + \lambda_2 \cdot (-2, -2) \mid \lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}\}$

(c) $\{\lambda_1 \cdot (1, 0) + \lambda_2 \cdot (0, 1) \mid \lambda_1, \lambda_2 \in \mathbb{Z}\}$

Problem §3 Define the following functions below using $f(\cdot)$ or $\mapsto$ notation. Then, assert whether they are injective, surjective, both (bijective), or neither.

(a) The natural logarithm function (here denoted $\log x$) from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$.

(b) The binary operation of multiplication in the field of rational numbers (e.g. domain $\mathbb{Q} \times \mathbb{Q}$ and codomain $\mathbb{Q}$).

(c) The function from $\mathbb{R}^2$ to $\mathbb{R}^3$ sending (x, y) to $(x, x + y, y)$.

(d) The function from $\mathbb{R}^2$ to $\mathbb{R}^2$ sending (x, y) to $(x + y, -x - y)$.

Solution: [NB: proofs weren't necessary, but I wanted to practice proving injective/surjective/bijective.]

(a) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be given by $f : x \to \log x$. I claim that f is bijective.

Proof. Let $a, b \in \mathbb{R}^+$ such that $\log a = \log b$. From this, we have

$$e^{\log a} = e^{\log b} \Rightarrow a = b,$$

and thus f is injective.

Now let $a \in \mathbb{R}$. We wish to show that $\exists x \in \mathbb{R}^+$ such that $\log x = a$. Clearly, we choose $x = e^a$. Thus, since $\forall a \in \mathbb{R}, \exists x \in \mathbb{R}^+$ s.t. $f(x) = a$, f is surjective.

Since f is both injective and surjective, f is bijective. □

(b) Let $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be given by $f(x, y) = xy$. I claim f is surjective, but not injective.

Proof. Let $(x_0, y_0)\,(x_1, y_1) \in \mathbb{Q} \times \mathbb{Q}; (x_0, y_0) = (1, 2); (x_1, y_1) = (2, 1)$. Then

$$f(x_0, y_0) = 1 \cdot 2 = 2 = 2 \cdot 1 = f(x_1, y_1),$$

and so f is not injective.

Now let $a \in \mathbb{Q}$. Then $a = \frac{p}{q}$ for some $p, q \in \mathbb{R}, q \neq 0$. Let $x = \frac{p}{q}, y = 1$. Clearly, $\forall a \in \mathbb{Q}, \exists\,(x, y) \in \mathbb{Q} \times \mathbb{Q}$ such that $f(x, y) = a$. Thus, f is surjective. □

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x, y) = (x, x + y, y)$. I claim f is injective, but not surjective.

Proof. Let $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$such that $(x_0, x_0 + y_0, y_0) = (x_1, x_1 + y_1, y_1)$. From this, we see $x_0 = x_1, y_0 = y_1$, so $(x_0, y_0) = (x_1, y_1)$, and thus f is injective.

To show that f is not surjective, observe $(0, 10, 0) \in \mathbb{R}^3$. Clearly, for $f(x_0, y_0) = (0, 10, 0)$, $x_0 = y_0 = 0$; but then $x_0 + y_0 \neq 10$. Thus f is not surjective. □

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f : (x, y) \mapsto (x + y, -x - y)$. I claim f is neither injective nor surjective.

Proof. Let $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2; (x_0, y_0) = (0, 1), (x_1, y_1) = (1, 0)$. Then

$$f(x_0, y_0) = (0 + 1, 0 - 1) = (1, -1) = (1 + 0, -1 - 0) = f(x_1, y_1).$$

Thus, f is not injective.

Now, choose $(1, 1) \in \mathbb{R}^2$. Then $x + y = 1 = -(x + y)$, which implies that $1 = -1$, a contradiction. Thus, f is not surjective. □