Problem §1 Let $v_1, \ldots, v_n$ be a basis for V, and let $w_1, \ldots, w_n$ be another basis for V.

(a) Prove that for any $j \in \{1, \ldots, n\}$, there exists an $i \in \{1, \ldots, n\}$ such that

$$v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$$

is a basis.

(b) Prove that for any $i \in \{1, \ldots, n\}$, there exists a $j \in \{1, \ldots, n\}$ such that

$$v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$$

is a basis.

Solution:

(a) Let w_j be any basis vector in the basis $w_1, \ldots, w_n$, and create the list

$$w_j, v_1, \ldots, v_n.$$

Since $v_1, \ldots, v_n$ spans V, so does $w_j, v_1, \ldots, v_n$; additionally, the list is linearly dependent, since $w_j \in \operatorname{span}(v_1, \ldots, v_n)$. Consider

$$a_j w_j + a_1 v_1 + \ldots + a_n v_n = 0,$$

and let i be the largest value in $\{j, 1, \ldots, n\}$ such that $a_i \neq 0$.

We know that $i \neq j$, since otherwise the list would be linearly independent, a contradiction. Thus $i \in \{1, \ldots, n\}$. By the Linear Dependence Lemma, $v_i \in \operatorname{span}(w_j, v_1, \ldots, v_{i-1})$, and

$$\operatorname{span}(w_j, v_1, \ldots, \hat{v}_i, \ldots, v_n) = \operatorname{span}(w_j, v_1, \ldots, v_n).$$

Since every spanning set with length $n = \dim V$ is a basis for V, and $w_j, v_1, \ldots, \hat{v}_i, \ldots, v_n$ has length n, we have that

$$v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$$

is a basis, as required.

(b) Let v_i be any basis vector in the basis $v_1, \ldots, v_n$, and form the list

$$v_1, \ldots, \hat{v}_i, v_n, w_1, \ldots, w_n.$$

This list spans V (since $w_1, \ldots, w_n$ form a basis for V) and is linearly dependent. We then proceed with an iterative step to remove elements from the list: in order from $j = 1$ to $(n-1) + n$, for $w_j \in v_1, \ldots, \hat{v}_i, \ldots, v_n, w_1, \ldots, w_n$, if $w_j \in \operatorname{span}(v_1, \ldots, \hat{v}_i, \ldots, v_n, \ldots, w_{j-1})$, then remove it from the list. Since $v_1, \ldots, \hat{v}_i, \ldots, v_n$ is linearly independent, none of the v's are removed. Now, we have two options with w_j from $j = 1$ to n:

(a) If $w_j \in \operatorname{span}(v_1, \ldots, \hat{v}_i, \ldots, v_n)$, then we delete w_j from the list and proceed to w_{j+1} (the span is unchanged, by the Linear Dependence Lemma).

(b) If $w_j \notin \operatorname{span}(v_1, \ldots, \hat{v}_i, \ldots, v_n)$, then the list $v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$ is a linearly independent list of length n. Since every linearly independent list with length $n = \dim V$ is a basis, any w_k with $k > j$ is in the span of $v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$, and so we can remove every w_k.

Observe also that we cannot remove every w_j; at least (and at most, as shown above) one of the w_j's must not be in the span of $v_1, \ldots, \hat{v}_i, \ldots, v_n$. Otherwise, the final list $v_1, \ldots, \hat{v}_i, \ldots, v_n$ does not span V, a contradiction to the requirement of not changing the span. Hence, after removing any v_i, we are left with a basis $v_1, \ldots, \hat{v}_i, \ldots, v_n, w_j$ for some $w_j \in \{w_1, \ldots, w_n\}$, as required.

Problem §2 Let V, W be vector spaces. Suppose $v_1, \ldots, v_m$ are linearly independent in V and suppose $w_1, \ldots, w_m$ are any vectors in W. Prove that there exists a linear map $T : V \to W$ such that

$$T(v_1) = w_1, \ldots, T(v_m) = w_m.$$

Solution: Let $v_1, \ldots, v_m$ be linearly independent in V, and extend the list to a basis $v_1, \ldots, v_m, u_1, \ldots, u_n$. Define a linear map

$$T(a_1v_1 + \ldots + a_mv_m + b_1u_1 + \ldots + b_nu_n) = a_1w_1 + \ldots + a_mw_m.$$

(All of the u_i's are sent to 0). Because $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis, T is a function, as each element of V can be uniquely written in the form $v = a_1v_1 + \ldots + a_mv_m + b_1u_1 + \ldots + b_nu_n$. By taking $a_i = 1$ and the other a's as zero, we have that

$$T(v_i) = w_i.$$

Now, take any two vectors $u, v \in V$ and any two scalars $\lambda_1, \lambda_2 \in \mathbb{F}$. We have

$$\begin{aligned}
T(\lambda_1 u + \lambda_2 v) &= T((\lambda_1a_1v_1 + \ldots + \lambda_1a_mv_m + \lambda_1b_1u_1 + \ldots + \lambda_1b_nu_n) + (\lambda_2c_1v_1 + \ldots + \lambda_2c_mv_m + \lambda_2d_1u_1 + \ldots + \lambda_2d_nu_n)) \\
&= (\lambda_1a_1w_1 + \ldots + \lambda_1a_mw_m) + (\lambda_2c_1w_1 + \ldots + \lambda_2c_mw_m) \\
&= \lambda_1(a_1w_1 + \ldots + a_mw_m) + \lambda_2(c_1w_1 + \ldots + c_mw_m) \\
&= \lambda_1T(a_1v_1 + \ldots + a_mv_m + b_1u_1 + \ldots + b_nu_n) + \lambda_2T(c_1v_1 + \ldots + c_mv_m + d_1u_1 + \ldots + d_nu_n) \\
&= \lambda_1T(u) + \lambda_2T(v).
\end{aligned}$$

Thus T preserves linearity and homogeneity, and so T is a linear map (note that T is very much not injective! Going from the 2nd last step to the 3rd last step is guaranteed, but the reverse is very much not guaranteed.)

Problem §3 Let V, W be vector spaces over $\mathbb{F}$, and suppose V is finite-dimensional with $\dim V > 0$. Let $w \in W$ be any vector. Prove that there exists a linear map $T : V \to W$ such that

$$\operatorname{range}(T) = \operatorname{span}(w).$$

Solution: Let $n = \dim V$. Since $n > 0$, there exists a length-n basis $v_1, \ldots, v_n$ of V. Define a linear map

$$T(a_1v_1 + \ldots + a_nv_n) = a_1w \qquad [\text{ all of the } v_j,\ j > 1 \text{ are mapped to } 0\ ]$$

Since $v_1, \ldots, v_n$ is a basis of V, each $v \in V$ has a unique representation, and so T is a valid function. Moreover, we see that

$$\begin{aligned}
\operatorname{range}(T) &= \{T(v) \mid v \in V, v = a_1v_1 + \ldots + a_nv_n,\ a_1, ..., a_n \in \mathbb{F}, v_1, \ldots, v_n \in V\} \\
&= \{a_1w \mid a_1 \in \mathbb{F}\} \\
&= \operatorname{span}(w),
\end{aligned}$$

as required. Now, take any two vectors $u, v \in V$ and any two scalars $\lambda_1, \lambda_2 \in \mathbb{F}$. We have

$$\begin{aligned}
T(\lambda_1u + \lambda_2v) &= T(\lambda_1a_1v_1 + \ldots + \lambda_na_nv_n + \lambda_2b_1v_1 + \ldots + \lambda_2b_nv_n) \\
&= \lambda_1a_1w + \lambda_2b_1w \\
&= \lambda_1T(a_1v_1 + \ldots + a_nv_n) + \lambda_2T(b_1v_1 + \ldots + b_nv_n) \\
&= \lambda_1T(u) + \lambda_2T(v).
\end{aligned}$$

Thus T preserves linearity and homogeneity, and so T is a linear map (much like problem 2, T is very much not injective).