Problem §1 For each of the following linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$, write down the matrix of T with respect to the standard basis of $\mathbb{R}^2$ for both copies of $\mathbb{R}^2$.

(a) Dilation by a factor of 2 with respect to the origin.

(b) Reflection across the line $x = y$.

(c) Projection to the line $x = y$.

(d) The identity map.

Solution:

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: Under the matrix, any $(x, y) \in \mathbb{R}^2$ becomes

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 0y \\ 0x + 2y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: Under the matrix, any $(x, y) \in \mathbb{R}^2$ becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

(c) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$: Under the matrix, any $(x, y) \in \mathbb{R}^2$ becomes

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \end{pmatrix}.$$

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Trivially, this is the identity matrix.

Problem §2 Repeat for each of the linear maps, except under the basis $(1, 0), (1, 1)$ for both $\mathbb{R}^2$.

Solution: [Note: the solutions to (b) and (c) operate under the assumption that $y = x$ remains the same; that is, $y = x$ represents the line passing through the origin and (1,1) under the standard basis. Otherwise, the answers would be identical to the ones provided in Problem 1, which seemed rather uneducational.]

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: Under the matrix, any $(x, y) \in \mathbb{R}^2$ under the new basis becomes

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$: Under the matrix, any $(x, y) \in \mathbb{R}^2$ under the new basis becomes

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ x + y \end{pmatrix}.$$

(c) $\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$: Under the matrix, any $(x, y) \in \mathbb{R}^2$ becomes

$$\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}x + y \end{pmatrix}.$$

(d) The identity remains the same: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

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