Problem §1

(a) Prove that if α is a cut, then

$$-\alpha := \{c - b \mid c \in \mathbb{Q}, c < 0, b \in \mathbb{Q} \setminus \alpha\}$$

is a cut.

(b) Prove that for all cuts α, $\alpha \geq 0^*$ if and only if $-\alpha \leq 0^*$.

Solution:

(a) Clearly, $-\alpha \neq \varnothing$. Choose any $c \in \mathbb{Q}$, $c < 0$; since α is a cut, there exists a $b \in \mathbb{Q} \setminus \alpha$, and so $c - b \in \alpha \neq \varnothing$. Additionally, $-\alpha \neq \mathbb{Q}$, as one can choose any $a \in \alpha$; $c - \alpha \notin -\alpha$. Thus property (i) holds.

Observe that $c \in 0^*$. Suppose $r \in -\alpha$; then $r = c_1 - b_1$ for some $c_1 \in 0^*$, $b_1 \in \mathbb{Q} \setminus \alpha$. For any $s \in \mathbb{Q}$, if $s < r$, then either

- $s = c_2 - b_1,\ c_2 < c_1$. By property (ii) for 0^*, for any $c \in 0^*$, if $c' \in \mathbb{Q}$ and $c' < c$, then $c' \in 0^*$. Hence $s = c_2 - b_1 \in -\alpha$, as required.
- $s = c_1 - b_2,\ b_2 > b_1$. By property (ii), if $b_1 \in \mathbb{Q} \setminus \alpha$, and $b_2 > b_1$, then $b_2 \in \mathbb{Q} \setminus \alpha$ as well. Hence $s = c_1 - b_2 \in -\alpha$, as required.

Thus property (ii) holds.

For any $r \in -\alpha$, where $r = c_1 - b_1$, since $c_1 \in 0^*$, we can choose $c_2 \in 0^*$ such that $c_2 > c_1$ (by property (iii) of 0^*), and so there exists an $s \in -\alpha$, where $s = c_2 - b_1 > c_1 - b_1 = r$. Thus property (iii) holds.

Therefore all three properties hold, and so $-\alpha$ is a cut.

(b) Suppose $a \geq 0^*$. Then $0^* \subset \alpha$. Let $r \in -\alpha$. Then $r = c - b$ for some $c \in 0^*$, $b \in \mathbb{Q} \setminus \alpha$. Since $b \in \mathbb{Q} \setminus \alpha$, and $0 \in \alpha$, we have $0 < b$ by property (ii) (and so $-b < 0$); moreover, $c < 0$ for any $c \in 0^*$. Thus, we have

$$r = c - b < 0 - b = -b < 0,$$

and so $r < 0$; hence $r \in -\alpha$ implies that $r \in 0^*$, and so $-\alpha \subset 0^*$. Therefore $-\alpha \leq 0^*$.

Now, suppose that $-\alpha \leq 0^*$. Then $-\alpha \subset 0^*$; that is, for any $r \in -\alpha$, where $r = c - b$ for some $c \in 0^*$, $b \in \mathbb{Q} \setminus \alpha$, we have $r = c - b < 0$.

Since $b \in \mathbb{Q} \setminus \alpha$, any b satisfies $a < b$ for any $a \in \alpha$. Moreover, $r = c - b < 0$ implies that $c < b$ for all $c \in 0^*$. In other words, $b \geq 0$; and by the denseness of Q^*, there exists an $a \in \mathbb{Q}$ such that $a \leq b$ and $0 \leq a$. Since $a < b$, we have $a \in \alpha$; hence $0^* \subset \alpha$, and so $0^* \leq \alpha$.

Problem §2 Let α be a cut, $\alpha > 0^*$. Prove that

$$\alpha^{-1} := \{r \in \mathbb{Q} \mid r < 0\} \cup \{r \in \mathbb{Q} \mid 0 \leq r < t \text{ for some } t \in \mathbb{Q} \text{ such that } \frac{1}{t} \notin \alpha\ \}$$

is a cut and $\alpha^{-1} > 0^*$.

Solution: Clearly, $\alpha^{-1} \neq \varnothing$ (since $-1 \in \alpha^{-1}$) and $\alpha^{-1} \neq \mathbb{Q}$ (choose any $s > t$; $s \notin \alpha^{-1}$). Thus property (i) holds.

Suppose $r \in \alpha^{-1}$, and let $s \in \mathbb{Q}$ such that $s < r$. If $s < 0$, clearly $s \in \alpha^{-1}$, so choose $s \geq 0$. Since $0^* < \alpha$, there exists an $a \in \alpha$ that satisfies $a > 0$; and by properties of ordered fields, we know $a^{-1} = \frac{1}{a} > 0$. Hence if $\frac{1}{t} \notin \alpha$, then $\frac{1}{t} > a > 0$ and so $t > 0$. From this, we get $0 \leq s < r < t$, and so $s \in \alpha^{-1}$. Thus property (ii) holds.

Suppose $r \in \alpha^{-1}$, and choose $s = \frac{r+t}{2}$. Since $s = \frac{r+t}{2} < \frac{t+t}{2} = t$, $s < t$; additionally, $s = \frac{r+t}{2} > \frac{r+r}{2} = r$, so $r < s$. From this, we get that $0 \leq r < s < t$, and so $s \in \alpha^{-1}$. Hence for any $r \in \alpha^{-1}$, there exists an $s \in \alpha^{-1}$ such that $r < s$. Thus property (iii) holds, and so α^{-1} is a cut.

Since $\alpha > 0$, any $a \in \alpha$ satisfies $\alpha > 0$; thus any $t \notin \alpha$ satisfies $t > \alpha > 0$. Choose any $0 \leq s < t$; clearly $s \notin 0^*$, so we have $0^* \neq alpha^{-1}$. Since we also have $0^* \leq \alpha^{-1}$ (one can easily see $0^* = \{r \in \mathbb{Q} \mid r < 0\} \subset \alpha^{-1}$), we necessarily have $0^* < \alpha^{-1}$.

Problem §3 (8.2.a,e) Determine the limits of the following sequences, and then prove your claims:

a. $a_n = \frac{n}{n^2+1}$

e. $s_n = \frac{1}{n} \sin n$

(8.7.a) Show that $\cos\left(\frac{n\pi}{3}\right)$ does not converge.

Solution: (8.2)

a. Intuitively, the denominator increases faster than the numerator, so we hypothesize that $\lim a_n = 0$. For $n \geq 1$, we can drop the absolute value. Since $\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon$, we have $n > \frac{1}{\varepsilon}$.

Proof. Let $\varepsilon > 0$ and set $N = \frac{1}{\varepsilon}$. Then for any $n > N$, we have $n > \frac{1}{\varepsilon}$, hence $\varepsilon > \frac{1}{n} = \frac{n}{n^2} > \frac{n}{n^2+1}$, and so $\left|\frac{n}{n^2+1} - 0\right| < \varepsilon$, as desired. □

e. We propose $\lim s_n = 0$. We know that for any n, $|\sin n| \leq 1$, so $\left|\frac{1}{n}\sin n\right| \leq \left|\frac{1}{n}\right| < \varepsilon$. Dropping the absolute value for positive n, we get $n > \frac{1}{\varepsilon}$.

Proof. Let $\varepsilon > 0$, $N = \frac{1}{\varepsilon}$. Then $n > N$ implies $n > \frac{1}{\varepsilon}$, hence $\varepsilon > \frac{1}{n} = \left|\frac{1}{n}\right| \geq \left|\frac{1}{n}\sin n - 0\right|$, as desired. □

(8.7) Assume that $\lim \cos\left(\frac{n\pi}{3}\right) = a$ for some a. Setting $\varepsilon = 1$,

$$\left|\cos\frac{n\pi}{3} - a\right| < 1.$$

Considering multiples of 3, we see both

$$\left|\cos\frac{3\pi}{3} - a\right| = |-1-a| = |a+1| < 1,$$

and

$$\left|\cos\frac{6\pi}{3} - a\right| = |1-a| < 1.$$

By the Triangle Inequality, we have

$$2 = |1-(-1)| = |(1-a+a-(-1))| \leq |1-a| + |a-(-1)| < 1+1 = 2,$$

a contradiction. Hence $\lim \cos\left(\frac{n\pi}{3}\right)$ does not converge.

Problem §4 (8.4) Let (t_n) be a bounded sequence (i.e. there exists M such that for all n, $t_n \leq M$), and let (s_n) be a sequence such that $\lim s_n = 0$. Prove that $\lim s_n t_n = 0$.

Solution: Let $\varepsilon > 0$. Since $\lim s_n = 0$, there exists an N such that $n > N$ implies $|s_n| < \varepsilon_1$ for any $\varepsilon_1 > 0$. Moreover, since $|t_n| \leq M$, $n > N$ implies

$$|s_n t_n| < |\varepsilon_1 t_n| \leq |\varepsilon_1 M| = \varepsilon_1 |M| \qquad [\text{ since } \varepsilon_1 > 0 \,] \tag{1}$$

By the Archimedean property, since both $\varepsilon_1 |M|$ and ε are positive, there exists a k such that $k\varepsilon > \varepsilon_1 |M|$. Since Equation (1) holds for any $\varepsilon_1 > 0$, set $\varepsilon_1 = \frac{k}{|M|}\left(\frac{\varepsilon}{1+\varepsilon}\right)$. Then

$$\begin{aligned} k\varepsilon &> \varepsilon_1 |M| = \frac{k|M|}{|M|}\left(\frac{\varepsilon}{1+\varepsilon}\right) \\ k\varepsilon &> k\left(\frac{\varepsilon}{1+\varepsilon}\right) \\ \varepsilon &> \frac{\varepsilon}{1+\varepsilon}, \end{aligned}$$

which is true for any $\varepsilon > 0$. Hence

$$|s_n t_n - 0| < \varepsilon,$$

as required.

Problem §5 (8.6) Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

Solution:

(a) Suppose $\lim s_n = 0$. Then $|s_n - 0| = |s_n| < \varepsilon$; thus $||s_n| - 0| = |s_n| < \varepsilon$ as well.

Conversely, suppose $\lim |s_n| = 0$. Then $||s_n| - 0| < \varepsilon$; but $||s_n|| = |s_n|$ (repeatedly applying absolute values has the same effect as applying only once); hence $|s_n| = |s_n - 0| < \varepsilon$, so $\lim s_n = 0$.

(b) Observe that $|s_n| = |(-1)^n| = 1$. Clearly, $|1 - 1| = 0 < \varepsilon$ for any $\varepsilon > 0$, so $\lim |s_n|$ exists, and equals 1. From Example 4, however, one can clearly see that $\lim s_n$ does not exist.

Problem §6 (8.10) Let (s_n) be a convergent sequence, and suppose $\lim s_n > a$. Prove there exists a number N such that $n > N$ implies $s_n > a$.

Solution: Since $\lim s_n$ exists, let $s = \lim s_n$. Then for some $n > N$, we have

$$|s_n - s| < \varepsilon$$

for all $\varepsilon > 0$. Moreover, since $s > a$, there exists some $\delta > 0$ such that $s - \delta > a$.

Choose an N' such that $n > N'$ implies

$$|s_n - s| < \varepsilon$$

$$s_n - s < \varepsilon \implies s_n < s + \varepsilon \qquad\qquad s_n - s > -\varepsilon \implies s_n > s - \varepsilon.$$

Thus for any $\varepsilon > 0$, we have $s - \varepsilon < s_n < s + \varepsilon$. Set $\varepsilon = \delta$. Then we have $s_n > s - \varepsilon > s - \delta > a$, and so $s_n > a$, as required.

Problem §7 (9.1) Use limit Theorems 9.2-9.7 to prove:

(a) $\lim \frac{n+1}{n} = 1$

(b) $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$

(c) $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

Solution:

(a) Multiplying by $\frac{\frac{1}{n}}{\frac{1}{n}}$, we get $\frac{1+\frac{1}{n}}{1}$. Trivially, $\lim 1 = 1$. By Theorem 9.7(a), we get $\lim \frac{1}{n} = 0$; by Theorem 9.3 we get $\lim 1 + \frac{1}{n} = \lim 1 + \lim \frac{1}{n} = 1 + 0 = 1$; and by Theorem 9.6, we have $\lim \frac{n+1}{n} = \frac{1}{1} = 1$, as desired.

(b) Multiplying by $\frac{\frac{1}{n}}{\frac{1}{n}}$, we get $\frac{3+\frac{7}{n}}{6-\frac{5}{n}}$. Trivially, $\lim 3 = 3$, $\lim 6 = 6$, and by Theorems 9.2 and 9.7(a), we get $\lim \frac{7}{n} = \lim \frac{5}{n} = 0$. By Theorem 9.3 we get $\lim 3 + \frac{7}{n} = 3$, $\lim 6 - \frac{5}{n} = 6$, and so by Theorem 9.6 we get $\lim \frac{3n+7}{6n-5} = \frac{3}{6} = \frac{1}{2}$.

(c) Multiplying by $\frac{1}{n^5}$, we get $\frac{17+\frac{73}{n}-\frac{18}{n^3}+\frac{3}{n^5}}{23+\frac{13}{n^2}}$. By Theorems 9.2 and 9.7(a), we get any $\frac{a}{n^p} = 0$, for all $a \in \mathbb{Z}$ and $p > 0$. Trivially, $\lim \alpha = \alpha$ for $\alpha \in \mathbb{Z}$, so we get $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17+0-0+0}{23+0} = \frac{17}{23}$.

Problem §8 (9.6) Let $x_1 = 1$, $x_{n+1} = 3x_n^2$.

(a) Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.

(b) Does $\lim x_n$ exist? Explain.

(c) Discuss the apparent contradiction between (a) and (b).

Solution:

(a) Suppose that for all $n > N$, we have $\lim x_n = a$. If $a \neq 0$, then $\lim x_{n+1} = a = \lim 3x^2 = 3a^2$, and so $\frac{a^2}{a} = a = \frac{1}{3}$. However, if $a = 0$, then the above equality also holds (indeed, if $a = 0$, we cannot do $\frac{a^2}{a}$): $0 = 3 \cdot 0^2 = 0$. Hence $a = \frac{1}{3}$ or 0.

(b) The limit does not actually exist; Since $x_1 = 1 \geq 1$, then $x_n^2 \geq 1$ for any $n \geq 1$, and so $3x_n^2$ will always increase (and thus never converse to a value).

(c) Part (a) relies on the assumption that $\lim x_n$ exists in the first place; it assumes $\lim x_n = a$ for some a, and *then* proceeds with finding the value of a. If the limit did not exist in the first place, then such calculations would be meaningless; any result could be returned. That is also why we see that $\lim x_n$ approaches **2** values, a clear contradiction of what a limit is.