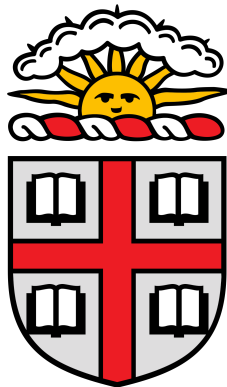

HONORS LINEAR ALGEBRA

MATH0540

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Chapter 1

Set Theory

Sets serve as a fundamental construct in higher-level mathematics. We start with a brief introduction to set theory.

§1.1 Sets

Definition 1.1.1: Sets

A **set** is a collection of elements.

1. $x \in X$ means x is an element of X .
2. $x \notin X$ means x is not an element of X .
3. $X \subset Y$ means X is a subset of Y (i.e. $\forall x \in X, x \in Y$.)
4. $X = Y \iff X \subset Y \wedge Y \subset X$.
5. $A \cap B := \{x \mid x \in A \wedge x \in B\}$ means set intersection.
6. $A \cup B := \{x \mid x \in A \vee x \in B\}$ means set union.
7. $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$ means set difference.

Example 1. Let

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}.$$

denote the set of integers, and let

$$\mathbb{Z}^+ = \{0, 1, \dots\}.$$

denote the set of positive integers.

§1.1.1 Set Builder notation

Sets may be defined formally with set-builder notation:

$$X = \{ \text{expression} \mid \text{rule} \}.$$

Example 2. 1. Let E represent the set of all even numbers. This set is expressed

$$E = \{n \in \mathbb{Q} \mid \exists k \in \mathbb{Z} \text{ s.t. } n = 2k\}.$$

2. Let A represent the set of real numbers whose squares are rational numbers:

$$A = \{a \in \mathbb{R} \mid a^2 \in \mathbb{Q}\}.$$

§1.1.2 Cartesian Products

Definition 1.1.2: Ordered Tuples

An **ordered pair** is defined (x, y) . An **n -ordered tuple** is an ordered list of n items

$$(x_1, \dots, x_n).$$

Definition 1.1.3: Cartesian Products

Let A, B be sets. The **cartesian product** $A \times B$ is defined

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, define the n -fold cartesian product

$$A^n := A \times A \times \dots \times A.$$

Example 3. \mathbb{R}^2 and \mathbb{R}^3 are examples of commonly known Cartesian products, which represent the 2D- and 3D-plane respectively.

Example 4. \mathbb{R}^n is a first example of a **vector space**. Let $n \in \mathbb{Z}^+ \cup \{0\}$:

1. (Addition in \mathbb{R}^n) We define an **addition operation** on \mathbb{R}^n by adding coordinate-wise

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

2. (Scaling) Given $(x_1, \dots, x_n) \in \mathbb{R}^n, \lambda \in \mathbb{R}$, we define

$$\lambda \cdot (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n).$$

Remark 1. $\mathbb{R}_0 = \{0\}$.

§1.1.3 Functions

Let A, B be sets. Informally, a function $f : A \rightarrow B$ deterministically returns an element $b \in B$ for each $a \in A$. We write $f(a) = b$.

Example 5. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ maps \mathbb{R} to the subset

$$S \subset \mathbb{R} = \{(x, x^2) \mid x \in \mathbb{R}\}.$$

Definition 1.1.4: Functions

Let A, B be sets. A function $f : A \rightarrow B$ is a subset $G_f \subset A \times B$ such that $\forall a \in A, \exists! b \in B$ s.t. $(a, b) \in G_f$. We write $f(a) = b$ when $(a, b) \in G_f$.

Definition 1.1.5: Codomain

Given a function $f : A \rightarrow B$, A is the **domain** of f , and B is the **codomain** or **target** of f . Let the **range** of f be defined as

$$\{b \in B \mid f(a) = b, a \in A\}.$$

The range is the subset of B .

Definition 1.1.6: Bijectivity

Let $f : A \rightarrow B$ be a function.

1. f is **injective**, or an **injection**, if $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ implies $a_1 = a_2$.
2. f is **surjective**, or a **surjection**, if $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$. Equivalently, the range is the whole codomain.
3. f is **bijective**, or a **bijection**, if it is both injective and surjective. Equivalently, $\forall b \in B$, there is a unique $a \in A$ such that $f(a) = b$.

§1.2 Fields

Roughly speaking, a **field** is a set, together with operations addition and multiplication. Vector spaces may be defined *over* fields.

Definition 1.2.1: Fields

A **field** is a set \mathbb{F} containing elements named 0 and 1, together with binary operations $+$ and \cdot satisfying:

- **commutativity:** $a + b = b + a, a \cdot b = b \cdot a \forall a, b \in \mathbb{F}$.
- **associativity:** $a + (b + c) = (a + b) + c \forall a, b, c \in \mathbb{F}$.
- **identities:** $0 + a = a, 1 \cdot a = a \forall a \in \mathbb{F}$.

- **additive inverse:** $\forall a \in \mathbb{F}, \exists b \in \mathbb{F} \text{ s.t. } a + b = 0.$
- **multiplicative inverse:** $\forall a \in \mathbb{F} \setminus \{0\}, \exists c \in \mathbb{F} \text{ s.t. } ac = 1.$
- **distributivity:** $a \cdot (b + c) = a \cdot b + a \cdot c \forall a, b, c \in \mathbb{F}.$

Example 6. $\mathbb{R}^+ \setminus \{0\}$ is **not** a field under $+, \cdot$.

Example 7. (Finite Fields) Let p prime (e.g. $p = 5$). Define

$$\mathbb{F}_p = \{0, \dots, p-1\},$$

with binary operations $+_p, \cdot_p$ given by addition and multiplication modulo p . We claim (without proof) that \mathbb{F}_p is a field.

Example 8. Let $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. Elements of \mathbb{C} are called **complex numbers**. Formally, a complex number is an ordered pair $(a, b), a, b \in \mathbb{R}$. We define addition as

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and multiplication as

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Showing \mathbb{C} is a set is left as an exercise for the reader.

Proposition 1.2.1: \mathbb{C} Multiplicative Inverse

For every $\alpha \in \mathbb{C} \setminus \{0\}$, there exists $\beta \in \mathbb{C}$ with $\alpha \cdot \beta = 1$.

Proof. Given $\alpha \in \mathbb{C} \setminus \{0\}$, let us write $\alpha = a + bi$. Then not both $a, b = 0$. Let $\beta = \frac{a}{a^2+b^2} + -\frac{b}{a^2+b^2}i$. Then $\alpha\beta = (a + bi) \left(\frac{a}{a^2+b^2} + -\frac{b}{a^2+b^2}i \right) = 1$.
Thus $\forall \alpha \in \mathbb{C} \setminus \{0\}, \exists \beta \in \mathbb{C} \text{ s.t. } \alpha \cdot \beta = 1$. □

\mathbb{R}^n and \mathbb{C}^n are specific examples of fields, but by no means the only ones (for instance, \mathbb{F}^2 with addition and multiplication modulo 2 is a field). Fields serve as the underlying set of numbers and operations that vector spaces are built on. In this course, we focus primarily on \mathbb{R} and \mathbb{C} ; but many of the definitions, theorems, and proofs work interchangeably with abstract fields.

§1.3 Vector Spaces

Vector spaces serve as the fundamental abstract structure of linear algebra. All future topics will build on vector spaces.

Definition 1.3.1: Vector Spaces

Given a field \mathbb{F} , A **vector space** over \mathbb{F} , denoted $V_{\mathbb{F}}$, is a set V , together with vector addition on V

$$+ : V \times V \longrightarrow V$$

and scalar multiplication on V

$$\cdot : \mathbb{F} \times V \longrightarrow V$$

satisfying the following properties:

- (additive associativity) For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$.
- (additive identity) There exists an element $0 \in V$ such that $v + 0 = 0 + v = 0$.
- (additive inverse) For all $v \in V$, there exists $w \in V$ such that $v + w = w + v = 0$. We denote $w = -v$.
- (commutativity) For all $v, w \in V$, $v + w = w + v$.
- (scalar multiplicative associativity) For all $\alpha, \beta \in \mathbb{F}, v \in V$, $\alpha(\beta v) = (\alpha\beta)v$.
- (scalar multiplicative identity) There exists an element $1 \in \mathbb{F}$ such that $1v = v$ for all $v \in V$.
- (Distributive Law I) For every $\alpha \in \mathbb{F}, v, w \in V$, $\alpha \cdot (v + w) = \alpha \cdot v + \alpha \cdot w$.
- (Distributive Law II) For every $\alpha, \beta \in \mathbb{F}, v \in V$, $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.

We call elements of \mathbb{F} **scalars**, and elements of V **vectors**, or **points**.

Example 9. Let \mathbb{F} be a field.

1. For some integers $n \geq 0$, $\mathbb{F}^n = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{F}\}$ with vector addition defined

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$

and scalar multiplication defined

$$\lambda \cdot (v_1, v_2, \dots, v_n) = (\lambda v_1, \lambda v_2, \dots, \lambda v_n).$$

Note that $\mathbb{F}^0 = \{0\}$.

2. $\mathbb{F}^\infty = P\{(a_1, a_2, a_3, \dots) \mid a_j \in \mathbb{F}, j \in \mathbb{N}\}$ with vector addition and scalar multiplication defined similarly.
3. Let S be any set; consider $\{g : S \rightarrow \mathbb{F}\}$ be the set of functions from S to \mathbb{F} . Given $f, g : S \rightarrow \mathbb{F}$, $\lambda \in \mathbb{F}$, define vector addition $(f + g) : S \rightarrow \mathbb{F}$ as

$$(f + g)(x) = f(x) + g(x)$$

and scalar multiplication $\lambda f : S \rightarrow \mathbb{F}$ as

$$(\lambda f)(x) = \lambda f(x).$$

Perhaps counterintuitively, example 3 subsumes example 1! For example, let $S = \{1, 2, \dots, n\}$, and let $\mathbb{R}^{\{1, \dots, n\}}$ be the set of all functions from $\{1, \dots, n\} \rightarrow \mathbb{R}$. One such f may be

$$\begin{aligned} f : \{1, \dots, n\} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = x^2 - 3. \end{aligned}$$

But f can also be thought of as an n -tuple. For instance, with $n = 3$, we can define a function

$$f = (-2, 1, 6) \in \mathbb{R}^3.$$

This is equivalent to $f(1) = -2, f(2) = 1, f(3) = 6$. Similarly, if $f(x) = e^x$, then $f \in \mathbb{R}^{\{1, 2, 3\}} = (e, e^2, e^3) \in \mathbb{R}^3$, since $f(1) = e, f(2) = e^2, f(3) = e^3$.

In other words, every n -tuple $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ could be represented as a function $f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}$, where $f(1) = x_1, f(2) = x_2, \dots, f(n) = x_n$. The key insight here is that **the function f is the n -tuple**; the one function $f(x) = e^x$ is equivalent to the n -tuple (e, e^2, \dots, e^n) .

From this, we get that the set of functions $\mathbb{R}^{\{1, \dots, n\}} = \mathbb{R}^n$, the set of n -tuples.

Remark 2. Reinterpret $\mathbb{F}^0 = \{\text{functions } f : \emptyset \longrightarrow \mathbb{F}\}$. How many functions are there from $\emptyset \longrightarrow \mathbb{F}$?
One function $\emptyset = \emptyset \times \mathbb{F}$.

Example 10. The set of continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ forms a vector space over \mathbb{R} . In particular, the sum of two continuous functions is continuous; and $a \cdot f$ is continuous for any $a \in \mathbb{R}$, and f continuous.