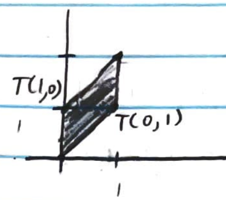
Review Sheet 5

(1) Let $Q = \{(a,b) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$ be the unit square in the first quadrant. Draw $T(Q)$, and show where $T(1,0)$, $T(0,1)$ land.

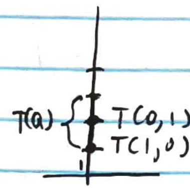
a. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (y, x+y)$



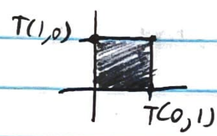
b. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (0, x+2y)$



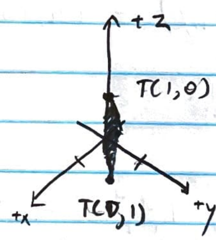
c. $T : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (y,x)$



d. $T : \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \longmapsto (y, y, x)$



(Hard to draw, but essentially a square with vertices $(0,0,0)$, $(1,1,0)$, $(1,1,1)$, and $(0,0,1)$)

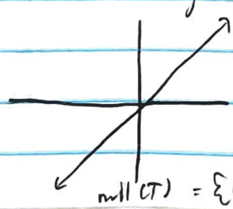
(2) Draw pictures for null(T) and range(T).

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (0,0)$

$\text{null}(T) = \mathbb{R}^2$

$\text{range}(T) = \{(0,0)\}$

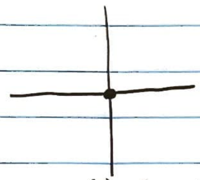
b. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (3x-3y, y-x)$

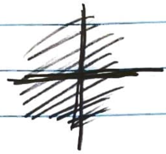


$\text{null}(T) = \{(x,y) \in \mathbb{R}^2 \mid x = y\}$

$\text{range}(T) = \{(x,y) \in \mathbb{R}^2 \mid y = -\frac{x}{3}\}$

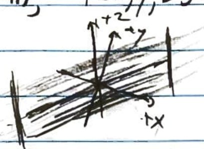
c. $T:\mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (y,x)$





$\text{null}(T) = \{(0,0)\}$ $\qquad$ $\text{range}(T) = \mathbb{R}^2$

d. $T:\mathbb{R}^3 \to \mathbb{R}$, $T(x,y,z) = x - y$



(hard to see, but essentially a plane along $y=x$, with $z \in \mathbb{R}$)

$\text{null}(T) = \{(x,y,z) \in \mathbb{R}^3 \mid x = y\}$

$\text{range}(T) = \mathbb{R}$

(3) Prove that given $T: V \to W$,

a. null T is a subspace of V.

$\text{null}\, T = \{v \in V \mid T(v) = 0\}$.

Let $v_1, v_2 \in \text{null}\, T$. Then

$$T(v_1 + v_2) = \underbrace{T(v_1) + T(v_2)}_{\text{by property of a linear map}} = 0 + 0 = 0.$$

Hence $v_1 + v_2 \in \text{null}\, T$.

Now, let $v \in \text{null}\, T$, $\lambda \in \mathbb{F}$ (the underlying field of V). Then

$$T(\lambda v) = \underbrace{\lambda T(v)}_{\text{by property of a linear map}} = \lambda \cdot 0 = 0$$

Hence $\lambda v \in \text{null}\, T$.

Finally, $0 \in \text{null}\, T$: $T(0) = T(0+0) = T(0) + T(0) \Rightarrow 0 = T(0)$

• range T is a subspace of W.

$\text{range}\, T = \{T(v) \in W \mid v \in V\}$

Let $w_1, w_2 \in \text{range}\, T$. Then $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2) \in \text{range}\, T$ ($v_1 + v_2 \in$).

Let $w \in \text{range}\, T$, $\lambda \in \mathbb{F}$. Then $\lambda w = \lambda T(v) = T(\lambda v) \in \text{range}\, T$ ($\lambda v \in$)

Finally, $0 \in \text{range}\, T$: $0 = T(0) \in \text{range}\, T$.

Hence, null T is a subspace of V, and range T is a subspace of W.