Problem §1 (9.8) Give the following when they exist; otherwise, assert "NOT EXIST".

(a) $\lim n^3$

(b) $\lim -n^3$

(c) $\lim (-n)^n$

(d) $\lim (1.01)^n$

(e) $\lim n^n$

Solution:

(a) $+\infty$

(b) $-\infty$

(c) NOT EXIST

(d) $+\infty$

(e) $+\infty$

Problem §2 (9.12) Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.

Solution:

(a) First, observe that L is positive (since $\left|\frac{s_{n+1}}{s_n}\right|$ is positive). Let $a \in \mathbb{R}$ such that $L < a < 1$.
We know that

$$\begin{aligned}\left|\frac{s_{n+1}}{s_n}\right| &= \left|\frac{s_{n+1}}{s_n} - a + a\right| \\ &\leq \left|\frac{s_{n+1}}{s_n} - a\right| + |a| \\ &< \varepsilon + |a| \\ &= \varepsilon + a,\end{aligned}$$

and so $\left|\frac{s_{n+1}}{s_n}\right| < \varepsilon + |a|$. Since $a > L > 0$, we have that $a - L > 0$, so let $\varepsilon = a - L$. Then

$$\frac{|s_{n+1}|}{|s_n|} < a - L + L = a,$$

and so

$$|s_{n+1}| < a\,|s_n|$$

for $n > \mathbb{N}$.
Let $|s_N|$ be s_n at N. Then

$$|s_n| = \left|s_{N+(n-N)}\right| = \left|s_{N+\underbrace{1+\ldots+1}_{n-N\text{ times}}}\right| < a\left|s_{N+\underbrace{1+\ldots+1}_{n-N-1\text{ times}}}\right| < \ldots < a^{n-N}\,|s_N|\,,$$

so $|s_n| < a^{n-N}\,|s_N|$. Since $|a| < 1$, $\lim \left|a^{n-N}\right| = \lim \left|a^k\right| = 0$, and so it necessarily follows that $\lim s_n = 0$ as well.

(b) Let $t_n = \frac{1}{|s_n|}$; then $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{\frac{1}{|s_{n+1}|}}{\frac{1}{|s_n|}}\right| = \left|\frac{s_n}{s_{n+1}}\right|$. By Lemma 9.5, since $L > 1 > 0$ and $\left|\frac{s_{n+1}}{s_n}\right|$ converges to L, we have that

$$\lim \left|\frac{1}{\left|\frac{s_{n+1}}{s_n}\right|}\right| = \lim \left|\frac{s_n}{s_{n+1}}\right| = \frac{1}{L},$$

and so $\lim \left|\frac{t_{n+1}}{t_n}\right| = \frac{1}{L}$. Since $L > 1$, $\frac{1}{L} < 1$; thus, by part (a), $\lim |t_n| = 0$. Theorem 9.10 tells us that if $\lim |t_n| = \lim \left|\frac{1}{s_n}\right| = 0$, then $\lim |s_n| = +\infty$, as required.

Problem §3 (9.14) Let $p > 0$. Show that

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if} \quad |a| \le 1 \\ +\infty & \text{if} \quad a > 1 \\ \text{does not exist} & \text{if} \quad a < -1 \end{cases}$$

Solution: For $|a| \le 1$, we have that $-\frac{1}{n^p} \le \frac{a^n}{n^p} \le \frac{1}{n^p}$, and $\lim \left|\frac{1}{n^p}\right| = 0$, so $0 \le \lim \frac{a^n}{n^p} \le 0$.
Let $s_n = \frac{a^n}{n^p}$. For $a > 1$,

$$\frac{\frac{a^{n+1}}{(n+1)^p}}{\frac{a^n}{n^p}} = \frac{a^{n+1}}{a^n}\frac{n^p}{(n+1)^p} = a\frac{n^p}{(n+1)^p}.$$

Since $\lim \frac{n^p}{(n+1)^p} = 1$, and $a > 1$, we have that $\lim \frac{s_{n+1}}{s_n} = a > 1$. By 9.12b, we have that $\lim |s_n| = \lim s_n = +\infty$ (since $s_n > 0$ for all n).

For $a < 1$, $s_n = \frac{a^n}{n^p} = \frac{(-1)^n|a|^n}{n^p}$; clearly, $\lim (-1)^n$ does not exist, so $\lim s_n$ does not exist either.

Problem §4 (10.1) Which of the following sequences are increasing? Decreasing? Bounded?

(a) $\frac{1}{n}$

(b) $\frac{(-1)^n}{n^2}$

(c) n^5

(d) $\sin\left(\frac{n\pi}{7}\right)$

(e) $(-2)^n$

(f) $\frac{n}{3^n}$

Solution: Only (c) is increasing. (a) and (f) are decreasing. (a), (b), (d), (f) are bounded.

Problem §5 (10.6)

(a) Let (s_n) be a sequence such that
$$|s_{n+1} - s_n| < 2^{-n}$$
for all $n \in \mathbb{N}$. Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in N$?

Solution:

(a) Suppose without loss of generality that $m > n$. Then

$$\begin{aligned} |s_m - s_n| &= |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \ldots + s_{n+1} - s_n| \\ &\le |s_m - sm - 1| + |s_{m-1} - s_{m-2}| \\ &= \frac{1}{2^{m-1}} + \ldots + \frac{1}{2^n}. \end{aligned}$$

Since $\sum_{i=n} \frac{1}{2^i}$, there is some N such that for $n > N$ and $\varepsilon > 0$, we have $\sum_{i=n} \frac{1}{2^i} < \varepsilon$. Thus, for $m, n > N$, we have

$$|s_m - s_n| < \sum_{i=n} \frac{1}{2^i} < \varepsilon,$$

and so s_n is a Cauchy sequence. By Theorem 10.11, s_n is a convergent sequence.

(b) Unfortunately, no; for some $n \in \mathbb{N}$, $\sum_n \frac{1}{n}$ diverges, and so it's not necessarily the case that $|s_m - s_n| < \varepsilon$, so convergence is not guaranteed.

Problem §6 (10.10) Let $s_1 = 1$, and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find $s_2,\ s_3,\ s_4$.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.

(c) Show (s_n) is a decreasing sequence.

(d) Show $\lim s_n$ exists and find $\lim s_n$.

Solution:

(a) $s_2 = \frac{2}{3},\ s_3 = \frac{5}{9},\ s_4 = \frac{14}{27}$.

(b) For s_2, $s_2 > \frac{1}{2}$, so the base case holds. Assume that $s_n > \frac{1}{2}$; then

$$s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2+1}) = \frac{\frac{3}{2}}{3} = \frac{1}{2},$$

and so $s_{n+1} > \frac{1}{2}$ as well.

(c)

$$s_{n+1} - s_n = \frac{1}{3}(s_n + 1) - s_n = \frac{1}{3} - \frac{2}{3}s_n < \frac{1}{3} - \frac{2}{3}\frac{1}{2} = 0.$$

Hence (s_n) is a decreasing sequence.

(d) Since $\frac{1}{2} < s_n \leq 1$ for all n, s_n is bounded and therefore convergent, and so $\lim s_n$ exists. Hence

$$\begin{aligned}
\lim s_n = s &= \lim s_{n+1} \\
&= \frac{1}{3}(s+1) \\
s &= \frac{s}{3} + \frac{1}{3} \\
\frac{2}{3}s &= \frac{1}{3} \\
s &= \frac{1}{2}.
\end{aligned}$$

Problem §7 (10.12) Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) \cdot t_n$ for $n \geq 1$.

(a) Show $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

(c) Use induction to show $t_n = \frac{n+1}{2n}$.

(d) Repeat part *b*.

Solution:

(a) or all $n \in N$, $0 < 1 - \frac{1}{(n+1)^2} < 1$, hence $0 < t_n \leq 1$ and so $\lim t_n$ exists (converges).

(b) As n becomes large, $1 - \frac{1}{(n+1)^2}$ approaches 1; moreover, $t_2 = \frac{3}{4},\ t_3 = \frac{2}{3},\ t_4 = \frac{5}{8}$. Thus, it appears that $\lim t_n$ would approach somewhere around $\frac{1}{2}$.

(c) Clearly, $t_1 = \frac{1+1}{2\cdot 1} = 1$. Suppose $t_n = \frac{n+1}{2n}$. Then

$$\begin{aligned} t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) \cdot \frac{n+1}{2n} &= \frac{n+1}{2n} - \frac{1}{2n(n+1)} \\ &= \frac{(n+1)^2 - 1}{2n(n+1)} \\ &= \frac{n^2 + 2n + 1 - 1}{2n^2 + 2n} \\ &= \frac{n(n+2)}{n(2n+2)} \\ &= \frac{(n+1)+1}{2(n+1)}. \end{aligned}$$

Hence if $t_n = \frac{n+1}{2n}$, then $t_{n+1} = \frac{(n+1)+1}{2(n+1)}$.

(d) If $t_n = \frac{n+1}{2n} = \frac{1+\frac{1}{n}}{2}$, then $\lim t_n = \frac{1}{2}$.

Problem §8 (11.6) Show that every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence.

Solution: Let $t_1 = s \circ \sigma_1$ be a subsequence of s, where $\sigma_1 : \mathbb{N} \to \mathbb{N}$ is an increasing function. If $t_2 = t_1 \circ \sigma_2$ is a subsequence of t_1, then $t_2 = t_1 \circ \sigma_2 = s \circ (\sigma_1 \circ \sigma_2) = s \circ \sigma'$ is a subsequence of s, since $\sigma' = \sigma_1 \circ \sigma_2$ is clearly an increasing function as well.