Problem §1 (3.3) Prove, from Theorem 3.1, that for all $a, b, c \in \mathbb{F}$,

(a) $(-a)(-b) = ab$

(b) If $ac = bc$, and $c \neq 0$, then $a = b$.

Solution:

(a) We have

$$\begin{aligned} -ab + (-a)(-b) &= (-a)b + (-a)(-b) && [\text{ Theorem 3.1.iii }] \\ &= (-a)(b + -b) && [\text{ DL }] \\ &= (-a)0 \\ &= 0 && [\text{ Theorem 3.1.ii }]. \end{aligned}$$

Hence $(-a)(-b)$ is an inverse of $-ab$, and so $(-a)(-b) = ab$.

(b) If $c \neq 0$, then by M4, $c^{-1} \in \mathbb{F}$ exists. Then

$$\begin{aligned} ac &= bc \\ acc^{-1} &= bcc^{-1} \\ a \cdot 1 &= b \cdot 1 \\ a &= b, \end{aligned}$$

as required.

Problem §2 (3.4) Prove, from Theorem 3.2, that for all a, b in an ordered field $\mathbb{F}$,

(a) $0 < 1$.

(b) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Solution:

(a) We start with a lemma.

Lemma 1 $(0 \neq 1)$**.** *Suppose* $0 = 1$*, and let* $a \in \mathbb{F}$ *be a non-zero element in an arbitrary field. Then*

$$\begin{aligned} a &= a \cdot 1 && [\ \mathit{M3}\ ] \\ &= a \cdot 0 && [\ \textit{by assumption}\ ] \\ &= 0 && [\ \textit{Theorem 3.1.ii}\ ], \end{aligned}$$

a contradiction of a non-zero. Hence $0 \neq 1$*.*

From Theorem 3.2.iv, we have $0 \leq a^2$ for all a. Then $0 \leq 1$; and by the lemma, $0 < 1$.

(b) By Theorem 3.2.vi, if $0 < a,\ 0 < b$, then $0 < a^{-1},\ 0 < b^{-1}$; and by Theorem 3.2.iii, we have $0 < a^{-1}b^{-1}$. Let $c = a^{-1}b^{-1}$. Then from O5, we have

$$\begin{aligned} ac &< bc \\ aa^{-1}b &< a^{-1}b^{-1}b \\ b^{-1} &< a^{-1}. \end{aligned}$$

Since both are greater than zero, we have $0 < b^{-1} < a^{-1}$.

Problem §3 (4.7) Let S, T be bounded subsets of $\mathbb{R}$.

(a) Prove that $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

(b) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Solution:

(a) We know, by definition, that $\inf S \leq \sup S$.

For any $s \in S$, we have $s \in T$; and by definition, we have $\inf T \leq s$. Thus $\inf T$ is a lower bound for S, and so $\inf T \leq \inf S$.

Similarly, by definition we have $s \leq \sup T$. Thus $\sup T$ is an upper bound for S, and so $\sup S \leq \sup T$.

(b) Suppose, without loss of generality, that $\sup S \geq \sup T$. We know that for any $a \in S \cup T$ that $a \in S$ or $a \in T$. $a \in S$ implies that $a \leq \sup S$ by definition, so let $a \in T$. Then $a \leq \sup T$; but $\sup T \leq \sup S$ by our assumption. Thus $\sup S$ is an upper bound for $S \cup T$. Additionally, since $\sup S$ is by definition the least upper bound of S, and any $t \in T$ is bounded above by $\sup T \leq \sup S$, any $a \in S \cup T$ has $\sup S$ as its least upper bound; and so $\sup(S \cup T) = \sup S = \max\{\sup S, \sup T\}$. An analogous argument follows if $\sup T \geq \sup S$.

Problem §4 (4.8) Let S, T be non-empty subsets of $\mathbb{R}$, and for all $s \in S, t \in T$, $s \leq t$.

(a) Observe that S is bounded above, and T is bounded below.

(b) Prove that $\sup S \leq \inf T$.

(c) Give an example of such S, T where $S \cap T$ is non-empty.

(d) Give an example of such S, T where $S \cap T = \varnothing$.

Solution:

(a) Any $t \in T$ bounds S from above, and any $s \in S$ bounds T from above.

(b) For any $s \in S$, $s \leq \sup S$, and for any $a \in \mathbb{R}$ that satisfies $s \leq a$, $\sup S \leq a$. But since any $t \in T$ satisfies $s \leq t$, we have $\sup S \leq t$.

Let $\mathbf{m}$ be the set of all $m \in \mathbb{R}$ such that $m \leq t$. By definition, for any $m \in \mathbf{m}$, $m \leq \inf T$; and since $\sup S \in \mathbf{m}$, we have $\sup S \leq \inf T$.

(c) Let $S = (-1, 0]$, $T = [0, 1)$. Then $\sup S = 0 \leq 0 = \inf T$, and $S \cap T = \{0\} \neq \varnothing$.

(d) Let $S = (-1, 0)$, $T = (0, 1)$. Then $\sup S = 0 = \inf T$, and $S \cap T = \varnothing$.

Problem §5 (4.12) Let $\mathbb{I}$ be the set of irrational numbers. Prove that if $a < b$, then there exists an $x \in \mathbb{I}$ such that $a < x < b$.

Solution: We start with two lemmas.

Lemma 2. *If $a \in \mathbb{I}$ and $b \in \mathbb{Q}$, then $a + b \in \mathbb{I}$ (in other words, irrational + rational = irrational).*

Proof. Let $a \in \mathbb{I}$, $b \in \mathbb{Q}$. Then $b = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Let $a + b = c$, and suppose c is rational. Then $c = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$. We have

$$\begin{aligned} a + b &= c \\ a + \frac{p}{q} &= \frac{m}{n} \\ a &= \frac{m}{n} - \frac{p}{q} \\ a &= \frac{mq - pn}{nq}; \end{aligned}$$

but a is irrational, a contradiction. Hence $c = a + b$ is irrational. $\square$

Lemma 3. *$\sqrt{2}$ is irrational.*

Proof. Suppose $\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$. Then

$$2 = \frac{p^2}{q^2}$$
$$2q^2 = p^2.$$

Thus p^2 is even, and so p is even, so let $p = 2k$. Then

$$2q^2 = 4k^2$$
$$q^2 = 2k^2.$$

Thus q^2 is even, and so q is even as well. Thus $\gcd(p,q) \geq 2$.
Bit $\gcd(p,q) = 1$, a contradiction. Hence $\sqrt{2}$ is irrational. $\square$

From these two lemmas, we get that for any $r \in \mathbb{Q}$, $r + \sqrt{2} \in \mathbb{I}$. Hence $\{r + \sqrt{2} \mid r \in \mathbb{Q}\} \subseteq \mathbb{I}$. Now, suppose $a, b \in \mathbb{R}$, with $a < b$. Then $a - \sqrt{2} < b - \sqrt{2}$ by O4. Since $a - \sqrt{2}, b - \sqrt{2} \in \mathbb{R}$, by the denseness of $\mathbb{Q}$, there exists some $r \in \mathbb{Q}$ such that

$$a - \sqrt{2} < r < b - \sqrt{2}.$$

Adding $\sqrt{2}$ to each side, we get

$$a < r + \sqrt{2} < b,$$

and since $r + \sqrt{2} \in \{r + \sqrt{2} \mid r \in \mathbb{Q}\} \subseteq \mathbb{I}$, there exists some $x \in \mathbb{I}$ such that $a < x < b$.

Problem §6 (4.14) Let A, B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$, $b \in B$.

(a) Prove that $\sup(A + B) = \sup A + \sup B$.

(b) Prove that $\inf(A + B) = \inf A + \inf B$.

Solution:

(a) We have, by definition, $a + b \leq \sup(A + B)$ for any $a \in A$, $b \in B$. Then

$$a + b - b \leq \sup(A + B) - b$$
$$a \leq \sup(A + B) - b,$$

and so $\sup(A + B) - b$ is an upper bound for A ; thus $\sup A \leq \sup(A + B) - b$. From this, we have

$$\sup A \leq \sup(A + B) - b$$
$$\sup A - \sup A + b \leq \sup(A + B) - b + b - supA$$
$$b \leq \sup(A + B) - \sup A,$$

and so $\sup(A + B) - \sup A$ is an upper bound for B; thus $\sup B \leq \sup(A + B) - \sup A$. Then $\sup A + \sup B \leq \sup(A + B)$. But since

$$a + b \leq \sup A + \sup B,$$

we have that $\sup A + \sup B$ is an upper bound for $A + B$ as well; but since $\sup(A + B)$ is the least upper bound, and $\sup A + \sup B \leq \sup(A + B)$, we necessarily have equality: $\sup A + \sup B = \sup(A + B)$.

(b) We have, by definition, $\inf(A+B) \leq a+b$ for any $a \in A$, $b \in B$. Then

$$\inf(A+B) - b \leq a,$$

and so $\inf(A+B) - b$ is a lower bound for A; thus $\inf(A+B) - b \leq \inf A$. From this, we have

$$\begin{aligned} \inf(A+B) - b &\leq \inf A \\ \inf(A+B) - \inf A &\leq b, \end{aligned}$$

and so $\inf(A+B) - \inf A$ is a lower bound for B; thus $\inf(A+B) - \inf A \leq \inf B$. Then $\inf(A+B) \leq \inf A + \inf B$. But since

$$\inf A + \inf B \leq a+b,$$

we have that $\inf A + \inf B$ is a lower bound for $A+B$ as well; but since $\inf(A+B)$ is the least lower bound, and $\inf(A+B) \leq \inf A + \inf B$, we necessarily have equality: $\inf A + \inf B = \inf(A+B)$.