Problem §1 (6.3) In the dihedral group $\mathcal{D}_n$, Let R be a clockwise rotation of $\frac{2\pi}{n}$ radians, and let F be a flip.

(a) Prove that the subgroup of rotations, $\{e, R, R^2, \ldots, R^{n-1}\}$ is a normal subgroup of $\mathcal{D}_n$.

(b) Prove that the subgroup $\{e, F\}$ is not a normal subgroup.

(6.5) Let G be a group, and $H \subseteq G$ a subgroup with index 2. Prove that H is a normal subgroup of G.

Solution:
(6.3) Recall our definition of a flip:

$$f_i(j) = n - j + i \mod n,$$

and our definition of a rotation:

$$r_i(j) = j + i \mod n,$$

where $j \in V_n = \{0, 1, \ldots, n-1\}$. One can easily verify that $f_i^{-1} = f_i$, and $r_i^{-1} = r_{n-i}$. We show three things:

- Rotation * Rotation * Rotation = Rotation; that is, three rotations is still a rotation. For any rotations r_i, r_j, r_k and any $m \in V_n$,

$$r_i \circ r_j \circ r_k(m) = r_i \circ r_j(m+k) = r_i(m+j+k) = m + (i+j+k) = r_{i'}(m),$$

where $i' \equiv i + j + k \mod n \in V_n$.

- Flip * Rotation * Flip = Rotation; that is, any flip, followed by a rotation, followed by the flip again, yields a rotation. For any rotation r_i any flip f_j, and any $k \in V_n$,

$$f_j \circ r_i \circ f_j(k) = f_j \circ r_i(n-k+j) = f_j(n-k+i+j) = n - (n-k+i+j) + j = k - i = r_{n-i}(k).$$

- For different flips f_i, f_j where $2j \not\equiv 0 \mod n$, $f_i \circ f_j \circ f_i \neq f_j$. Let $k \in V_n$; then

$$f_j \circ f_i \circ f_j(k) = f_j \circ f_i(n-k+j) = f_j(n-(n-k+j)+i) = n-(k-j+i)+j = n-k+2j+i \not\equiv n-k+i \mod n = f_i(k).$$

Thus $f_j \circ f_i \circ f_j \neq f_i$.

From the first two observations, (a) follows immediately: since every element in $\mathcal{D}_n$ is either a flip or a rotation, the inverse of a rotation is also a rotation, and the inverse of a flip is the same flip, we can see that for any $\sigma \in \mathcal{D}_n$ and any $R^k \in H = \{e, R, \ldots, R^{n-1}\}$,

$$\sigma^{-1} R^k \sigma \in H,$$

and so $\sigma^{-1} H \sigma \subseteq H$ for every $\sigma \in \mathcal{D}_n$; Proposition 6.10 then shows that H is a normal subgroup. The third observation also proves (b); for any flip $\phi_j \in \mathcal{D}_n$ where $\phi_j \neq F$ and $2j \not\equiv 0 \mod n$ (clearly, such a flip exists in any $n > 2$),

$$\phi_j^{-1} F \phi_j \neq F,$$

and so $H = \{e, F\}$ does not satisfy $\phi^{-1} H \phi$, and so is not a normal subgroup.

(6.5) Since H has index 2, there are only two cosets of H: H itself, and some $\mathcal{C}$ where $H \cap \mathcal{C} = \varnothing$, and $H \cup \mathcal{C} = G$ is a disjoint union (by Proposition 2.39). If $g \in H$, then clearly $H = gH = Hg$ (by subgroup closure).

Consider an element $g' \in G$, $g' \notin H$. Then the coset $g'H \neq H$, since $g' \notin H$ and $e \in H$ implies $g'e = g' \in g'H, g' \notin H$; similarly, $Hg' \neq H$, since $eg' = g' \in Hg', g' \notin H$. Thus we must have $\mathcal{C} = g'H = H'g$.

Therefore, for any $g \in G$, we have $gH = Hg$. Thus

$$g^{-1}gH = g^{-1}Hg \implies H = g^{-1}Hg,$$

and so H is a normal subgroup of G.

Problem §2 (6.6) Let G be a group, H, K subgroups of G, and assume K is a normal subgroup of G.

(a) Prove that $HK = \{hk \mid h \in H,\ k \in H\}$ is a subgroup of G.

(b) Prove that $H \cap K$ is a normal subgroup of H, and that K is a normal subgroup of HK.

(c) Prove that HK/K is isomorphic to $H/(H \cap K)$.

(d) Rather than assuming that K is a normal subgroup, suppose we only assume that $H \subset N(K)$, i.e. we assume that H is contained in the normalizer of K. Prove that (a), (b), and (c) are true.

Solution:

(a) We start with a lemma:

Lemma 1. *Let G be a group, H, K subgroups of G, and K a normal subgroup. Then $HK = KH$.*

Proof. For $HK \subseteq KH$: let $h \in H$, $k \in K$. K normal means that $k = h^{-1}k'h$ for some $k' \in K$; thus for any $hk \in HK$, we have

$$hk = hh^{-1}k'h = k'h \in KH.$$

Thus $HK \subseteq KH$.

For $KH \subseteq HK$: Let $k \in K$, $h \in H$. H subgroup means $h^{-1} \in H$ as well, and K normal means $k = (h^{-1})^{-1}k'h^{-1} = hk'h^{-1}$ for some $k' \in K$. Thus for any $kh \in KH$, we have

$$kh = hk'h^{-1}h = kh' \in HK.$$

Thus $KH \subseteq HK$, and so $HK = KH$. $\square$

To show closure, let $h_1, h_2 \in H$, $k_1, k_2 \in K$. From the lemma, $k_1h_2 \in KH \subseteq HK$, so $k_1h_2 = h'k' \in HK$ for some $h' \in H,\ k' \in K$. Then

$$h_1k_1h_2k_2 = \underbrace{h_1h'}_{\text{in } H}\underbrace{k'k_2}_{\text{in } K} \in HK.$$

$e \in H,\ e \in K$ by subgroup definition; thus $ee = e \in HK$.

For $h \in H,\ k \in K$, we have $k^{-1} \in K,\ h^{-1} \in H,\ k^{-1}h^{-1} \in KH \subseteq HK$, so $k^{-1}h^{-1} \in HK$. Thus

$$hkk^{-1}h^{-1} = e,\ k^{-1}h^{-1}hk = e.$$

Thus HK is a subgroup of G.

(b)
- First, we show that $H \cap K$ is a subgroup of H.
  - $e \in H,\ e \in K$ by subgroup definition, so $e \in H \cap K$.
  - Let $g_1, g_2 \in H \cap K$. Then $g_1 = h_1, g_2 = h_2$ for some $h_1, h_2 \in H$, and since H is closed, $g_1g_2 = h_1h_2 \in H$ as well; likewise for K ($g_1 = k_1, g_2 = k_2$ for some $k_1, k_2 \in K$), and so $g_1g_2 \in H \cap K$.
  - Finally, H, K subgroup means $h_1^{-1} \in H,\ k_1^{-1} \in K$. $h_1^{-1}g_1 = h_1^{-1}h_1 = e$ (and similarly for $g_1h_1^{-1}$), but $g_1 = h_1 = k_1$, so $h_1^{-1}h_1 = h_1^{-1}k_1 = e$; uniqueness of inverse means $h_1^{-1} = k_1^{-1}$, and so $h_1^{-1} \in H \cap K$. An analogous argument follows for k_1^{-1}, so $g_1^{-1} = h_1^{-1} = k_1^{-1} \in H \cap K$ for any $g_1 \in H \cap K$. Thus for any element $g \in H \cap K$, the inverse exists.

  Hence, $H \cap K$ is a subgroup of H. Now, we show that $H \cap K$ is a normal subgroup. For any $g \in H \cap K$, $g = h = k$ for some $h \in H,\ k \in K$. Additionally, K normal means that for any $h' \in H$, (since $h' \in H$ means $h'^{-1} \in H$) we have

$$k = (h'^{-1})^{-1}k'h'^{-1} \text{ for some } k' \in K.$$

Thus for any $h' \in H$, $g \in H \cap K$, we have

$$h'^{-1}gh' = h'^{-1}h'k'h'^{-1}h' = k' \in K;$$

but $g = h$ means $k' = h'^{-1}gh' = h'^{-1}hh' \in H$. Thus $h'^{-1}gh' \in H \cap K$. Since our choice of g, h was arbitrary, we thus have

$$h^{-1}(H \cap K)h \subseteq H \cap K,$$

and so $H \cap K$ is a normal subgroup of H by Proposition 6.10.

- K is clearly a subgroup of HK, so we only need to show that K is a normal subgroup of HK. Since K is a normal subgroup of G, $g^{-1}Kg = K$ for any $g \in G$; but HK is a subgroup of G (from (a)), so any $hk \in HK$ satisfies $hk = g' \in G$. Thus, for any $hk \in HK$,

$$(hk)^{-1}K(hk) = g'^{-1}Kg = K,$$

and so K is a normal subgroup of HK.

(c) Consider the map

$$\phi : H \longrightarrow HK/K,\ \phi(h) = hK.$$

This is a group homomorphism, as for any $h_1, h_2 \in H$, $\phi(h_1h_2) = h_1h_2K = h_1K \cdot h_2K = \phi(h_1)\phi(h_2)$ (coset multiplication is well-defined since K is a normal subgroup of HK). This map is also surjective; all (left) cosets of K in HK are of the form

$$hkK = hK$$

for some $h \in H,\ k \in K$ (since $kk' \in K$ for any $k' \in K$, so $kK = K$), so for any coset hK, we can simply choose $h \in H$ such that $\phi(h) = hK$.

Now, recall that a coset $K = hK$ if and only if $h \in K$, and that the identity element of HK/K is $e_{HK/K} = K$. Thus the kernel of ϕ is simply the elements in H that are also elements of K; equivalently,

$$\ker(\phi) = \{h \in H \mid h \in K\} = H \cap K.$$

Theorem 6.12 then tells us that the homomorphism

$$\lambda : H/\ker(\phi) = H/(H \cap K) \longrightarrow HK/K,\ \lambda(h(H \cap K)) = \phi(h)$$

is injective, and isomorphic to the range of λ; but since ϕ is surjective onto HK/K, so is λ, so λ is an isomorphism. Therefore $H/(H \cap K) \cong HK/K$.

(d) Suppose now that $H \subset N(K)$, where $N(K) = \{g \in G \mid g^{-1}Kg = K\}$. It turns out that many of the proofs remain unaffected by this change, since they only used the fact that K is normal over any element $h \in H$, and this remains true since $h \in H \subset N(K)$, so $h^{-1}Kh = K$ still!

(a) The lemma can be loosened to satisfy our current constraints:

Lemma 2. *Let G be a group, H, K subgroups of G, and $H \subset N(K)$. Then $HK = KH$.*

Proof. For $HK \subseteq KH$: let $h \in H$, $k \in K$. $h \in H \subset N(K)$ means that for any $h \in H$, $k = h^{-1}k'h$ for some $k' \in K$. The proof then follows analogously to the proof of the previous lemma.

For $KH \subseteq HK$: Let $k \in K,\ h \in H$. H subgroup means $h^{-1} \in H$ as well, and $h^{-1} \in H \subset N(K)$ means $k = (h^{-1})^{-1}k'h^{-1} = hk'h^{-1}$ for some $k' \in K$. The proof also follows analogously. Thus $HK = KH$. □

Closure and Inverse thus follows the same structure given in part (a), and Identity is trivial; thus HK is still a subgroup of G.

(b) The proof that $H \cap K$ is a subgroup of H remains unaffected by this change. For any $g \in H \cap K$, $g = h = k$ for some $h \in H,\ k \in K$. Additionally, $H \subset N(K)$ means that for any $h' \in H$, we have

$$k = (h'^{-1})^{-1} k' h'^{-1} \text{ for some } k' \in K.$$

Thus for any $h' \in H$, $g \in H \cap K$, we have

$$h'^{-1} g h' = h'^{-1} h' k' h'^{-1} h' = k' \in K,$$

and $g = h$ means $k' = h'^{-1} g h' = h'^{-1} h h' \in H$; thus $h'^{-1} g h' \in H \cap K$, and so $h^{-1}(H \cap K)h \subseteq H \cap K$. Proposition 6.10 then tells us that $H \cap K$ is a normal subgroup of H.

K is still a subgroup of HK, so we only need to show that K is a normal subgroup of HK. From the above lemma, we see that $hk \in HK$ satisfies $hk = k'h' \in KH$ for some $k' \in K, h' \in H$; additionally, $(k'h')^{-1} = h'^{-1} k'^{-1}$. Thus for any $hk \in HK$,

$$(hk)^{-1} K (hk) = (k'h')^{-1} K (k'h') = h'^{-1} k'^{-1} K k' h' = h'^{-1} K h' = K,$$

since any $h \in H \subset N(K)$, so $h^{-1} K h = K$ for any $h \in H$ (also, one can easily verify that if $k \in K$, then $kK = K = Kk$). Thus $(hk)^{-1} K (hk) \subseteq K$, so Proposition 6.10 tells us that K is a normal subgroup of HK.

(c) The map $\phi : H \to HK/K,\ \phi(h) = hK$ remains well-defined, since K is still a normal subgroup of HK (despite not being a normal subgroup of G); ϕ is also still surjective for the same reasons given above. The rest of the proof remains identical, so $HK/K \cong H/(H \cap K)$ is maintained.

Problem §3 (6.9) Let G be a group, let X be a set, and let $\mathcal{S}_n$ be the symmetry group of X. Let

$$\alpha : G \longrightarrow \mathcal{S}_X$$

be a function from G to $\mathcal{S}_X$, and for $g \in G$ and $x \in X$, let $g \cdot x = \alpha(g)(x)$. Prove that this defines a group action if and only if α is a group homomorphism.

Solution: Suppose that α defines a group action, and let $g_1, g_2 \in G$. Then for any $x \in X$,

$$\alpha(g_1 g_2)(x) = (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = g_1 \cdot (\alpha(g_2)(x)) = \alpha(g_1) \circ \alpha(g_2)(x)$$

by the associativity of group actions. Thus $\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2)$, and so α is a group homomorphism.

Conversely, suppose α is a group homomorphism. Then for any $g_1, g_2 \in G$, $\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2)$. Recall that for any homomorphism, $\alpha(e) = e_{\mathcal{S}_X}$, the identity element of $\mathcal{S}_X$ (here the identity permutation). Thus $e \cdot x = \alpha(e)(x) = x$, so the identity axiom holds. For any two $g_1, g_2 \in G$ and any $x \in X$,

$$(g_1 g_2) \cdot x = \alpha(g_1 g_2)(x) = \alpha(g_1)\alpha(g_2)(x) = \alpha(g_1)(g_2 \cdot x) = g_1 \cdot (g_2 \cdot x).$$

Hence the associative axiom holds, and so α is a group action.

Problem §4 (6.10)

(a) Prove that G acts transitively on X if and only if there is at least one $x \in X$ such that $Gx = X$.

(b) Prove that G acts transitively on X if and only if for every pair of elements $x, y \in X$ there exists a group element $g \in G$ such that $gx = y$.

(c) If G acts transitively on X, prove that $|X|$ divides $|G|$.

Solution:

(a) Suppose first that G acts transitively on X. By definition, for all $x \in X$, $Gx = X$; thus at least one $x \in X$ satisfies $Gx = X$.

Conversely, suppose that at least one $x \in X$ satisfies $Gx = X$. Then for every $y \in X$, there exists some $g \in G$ such that

$$y = gx.$$

Thus $x \sim y$ for every $y \in X$; equivalently, if $[x]$ is the equivalence class of an $x \in X$, then $Gx = [x]$. But for any $y \in X$, $[x] = [y]$, since $\sim$ is an equivalence relation; hence $Gy = [y] = [x] = Gx = X$, and so every $y \in X$ satisfies $Gy = X$. Therefore G acts transitively on X.

(b) Suppose first that G acts transitively on X. Then for any $x \in X$, $Gx = X$. Recall that $Gx = \{g \cdot x \mid g \in G\}$. $Gx = X$ thus means that every $y \in X$ has some $g \in G$ such that $g \cdot x = y$. Therefore, for any pair $x, y \in X$, there exists some $g \in G$ such that $g \cdot x = y$.

Conversely, suppose that for any pair $x, y \in X$, there exists some $g \in G$ such that $gx = y$. Then for any $x \in X$, $x \sim y$ for every $y \in X$; equivalently, $[x] = Gx = X$. Like in (a), this thus means that for any $y \in X$, $[x] = [y]$, so $Gy = [y] = [x] = Gx = X$, and so $Gx = X$ for every $x \in X$. Therefore G acts transitively on X.

(c) We know that for any $x \in X$, there is a well-defined bijection

$$\alpha : G/G_x \to Gx,$$

and

$$|Gx| = \frac{|G|}{|G_x|},$$

by Proposition 6.19. Note that $|G_x| \geq 1$ (is non-empty), since by definition $e \in G$ stabilizes x (and so $e \in G_x$). Since G acts transitively on X, for any $x \in X$, $Gx = X$. The above equation thus becomes

$$|X| = \frac{|G|}{|G_x|} \iff |G_x|\,|X| = |G|$$

for every $x \in X$. Thus $|G| = k\,|X|$ for some positive integer $k = |G_x|$, so $|X|$ divides $|G|$.

Problem §5 (6.11) Let G be a group that acts on a set X. We say that the action is **doubly transitive** if it has the following property:

For all $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists an element $g \in G$ of the group satisfying $gx_1 = y_1$ and $gx_2 = y_2$.

(a) Let Z be the following set of ordered pairs:

$$Z = \{(x, y) \in X \times X \mid z_1 \neq z_2\}.$$

Let G act on Z by the rule

$$(z_1, z_2) = (gz_1, gz_2).$$

Prove that the action of G on X is doubly transitive if and only if the action of G on Z is transitive.

(b) For each of the following groups and group actions, determine whether the action is transitive, and also whether the action is doubly transitive:

- The symmetric group $\mathcal{S}_n$ acting on the set $\{1, 2, \ldots, n\}$.
- The dihedral group $\mathcal{D}_n$ acting on the vertices of a regular n-gon.
- A group G that acts on itself via left-multiplication, i.e. take X to be another copy of G, and let $g \in G$ send $x \in X$ to gx.

- The group of invertible 2×2 matrices $\mathrm{GL}_2(\mathbb{R})$ acting on the set of non-zero vectors $X = \mathbb{R}^2 \setminus \{0, 0\}$.

Solution:

(a) Suppose first that the action of G on X is doubly transitive. Then for any $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq x_2,\ y_1 \neq y_2$, there exists some $g \in G$ such that $gx_1 = y_1$ and $gx_2 = y_2$.

Let $(x_1, x_2) \in Z$ be any pair of elements of X satisfying $x_1 \neq x_2$. Since G acts doubly transitively on X, for any pair $(y_1, y_2) \in Z$, there exists a $g \in G$ such that

$$gx_1 = y_1, \text{ and } gx_2 = y_2.$$

Equivalently, the orbit of any pair $(x_1, x_2) \in Z$ is equivalent to all of Z; thus for any $z = (z_1, z_2) \in Z$,

$$Gz = \{(y_1, y_2) \in X \times X \mid y_1 \neq y_2\} = Z.$$

Thus G acts transitively on Z.

Conversely, suppose that G acts transitively on Z. Then for any pair $(x_1, x_2) \in Z$, $G(x_1, x_2) = Z$. Equivalently, for any $(y_1, y_2) \in Z$, there exists some $g \in G$ such that

$$g(x_1, x_2) = (gx_1, gx_2) = (y_1, y_2).$$

But then $gx_1 = y_1,\ gx_2 = y_2$, so this is the same as the following statement:

For any two pairs $x_1, x_2 \in X$, $y_1, y_2 \in Z$ with $x_1 \neq x_2$ (a requirement for $(x_1, x_2) \in Z$) and $y_1 \neq y_2$, there exists some $g \in G$ such that

$$gx_1 = y_1 \text{ and } gx_2 = y_2.$$

In other words, if G acts transitively on Z, then G acts doubly transitively on X.

(b)

- The symmetric group $\mathcal{S}_n$ acting on $X = \{1, \ldots, n\}$ is transitive, as noted in Example 6.16 (for any pair $x, y \in X$, some permutation exists that swaps x and y). $\mathcal{S}_n$ is also doubly transitive on X, since for any $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq x_2,\ y_1 \neq y_2$, there exists a permutation $\pi \in \mathcal{S}_n$ with $\pi(x_1) = y_1,\ \pi(x_2) = y_2$, and $\pi(z) = z$ for all other $z \in X$. $x_1 \neq x_2$ and $y_1 \neq 2$ means π is still bijective; moreover, we can thus find a $\pi \in \mathcal{S}_n$ that sends any two $x_1, x_2 \in X$ with $x_1 \neq x_2$ to any other $y_1, y_2 \in X$ with $y_1 \neq y_2$.
- The dihedral group $\mathcal{D}_n$ acting on vertices of an n-gon is transitive, as noted in Example 6.17 (for any two vertices $x, y \in V_n$, since there's a rotation that sends x to y). However, $\mathcal{D}_n$ is not doubly transitive for any $n > 3$ (one can check that $\mathcal{D}_3$ is doubly transitive by permuting through all possible combinations of $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$; I won't list them here). Consider the combination $x_1 = i, x_2 = i + 1, y_1 = i, y_2 = i + 2 \in V_n$. All actions of $\mathcal{D}_n$ on V_n must "preserve geometric structure" in that for any vertex $i \in V_n$ and any transformation $\sigma \in \mathcal{D}_n$, we must have
$$\sigma(i \pm 1) = \sigma(i) \pm 1 \text{ or } \sigma(i \pm 1) = \sigma(i) \mp 1$$
(intuitively, any action on an n-gon must preserve every vertex's neighbors; otherwise the n-gon would be distorted). Thus, no such $\sigma \in \mathcal{D}_n$ can have both $\sigma(x_1) = \sigma(i) = i$ and $\sigma(y_1) = \sigma(i+1) = i + 2$, since i would lose $i + 1$ as its neighbor and thus fail the above condition of "preserving geometric structure."
- G acting on $X = G$ is transitive. Since $e \in X = G$, $Ge = X = G$; and by Problem 4(a), if at least one $x \in X$ satisfies $Gx = X$, then G acts transitively on X. However, G is not transitive for any G with $|G| > 2$ (one can easily see why groups with one or two elements are doubly transitive). Consider $x_1 = e, x_2 = g, y_1 = e, y_2 = g' \in X$ for some non-trivial $g, g' \in G$, $g \neq g'$ (g, g' exist since G has at least 3 elements). In order for $gx_1 = ge = e = y_1$, we need $g = e$; but then $gx_2 = eg = g \neq g'$. Hence it is not possible for this combination to work, and so G does not act doubly transitively on X.

- $\mathrm{GL}_2(\mathbb{R})$ acting on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ is transitive. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ be any vector in X where $x_1, x_2 \in \mathbb{R}$ and x_1, x_2 not both 0.

  Consider the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in X$; that is, the collection of all vectors in the form

$$\mathrm{GL}_2(\mathbb{R}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\{ M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid M \in \mathrm{GL}_2(\mathbb{R}) \right\}.$$

  If x_1, x_2 both non-zero, then let $M = \begin{pmatrix} x_1 & 0 \\ x_2 & \lambda \end{pmatrix}$, where $\lambda \in \mathbb{R} \setminus \{0\}$ (in order to preserve invertibility). M is invertible, and $M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x}$. If $x_1 = 0$, then $x_2 \neq 0$, so let $M = \begin{pmatrix} 0 & \lambda \\ x_2 & 0 \end{pmatrix}$. M is invertible, and $M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Alternatively, if $x_2 = 0$, then $x_1 \neq 0$, and use the first matrix: $M = \begin{pmatrix} x_1 & 0 \\ 0 & \lambda \end{pmatrix}$. M is invertible, and $M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Thus $\mathrm{GL}_2(\mathbb{R}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = X$, and so by Problem 4(a), since at least one $x \in X$ satisfies $\mathrm{GL}_2(\mathbb{R})x = X$, it follows that $\mathrm{GL}_2(\mathbb{R})$ acts transitively on X.

  However, $\mathrm{GL}_2(\mathbb{R})$ acting on X is *not* doubly transitive. Let $Z = X \times X$ be the set of all non-zero pairs of elements in X. Let $(\vec{v}, 2\vec{v}) \in Z$, where $\vec{v} \in X$. Then $2\vec{v}$ is linearly dependent on $\vec{v}$; and for any $M \in \mathrm{GL}_2(\mathbb{R})$,

$$M(\vec{v}, 2\vec{v}) = (M\vec{v}, 2M\vec{v}).$$

  Thus the resulting vector pair $(M\vec{v}, 2M\vec{v}) \in Z$ is linearly dependent as well, and so $(M\vec{v}, 2M\vec{v}) \neq (\vec{v}_1, \vec{v}_2)$, where $(\vec{v}_1, \vec{v}_2) \in Z$ and $\vec{v}_1, \vec{v}_2$ are linearly independent vectors (since it's impossible for $M(\vec{v}, 2\vec{v})$ to form a pair of linearly independent vectors). Thus $\mathrm{GL}_2(\mathbb{R})(\vec{v}, 2\vec{v}) \neq Z$, and so $\mathrm{GL}_2(\mathbb{R})$ is not doubly transitive on X.