

**Problem §2** Consider the following list of vectors in  $\mathbb{R}^3$ . From left to right, remove each vector if it is in the span of the vectors to the left of it. What list of vectors remains?

$(1, 1, 0), (-2, -2, 0), (0, 0, 0), (0, 0, 4), (1, 1, 1), (4, 0, 0), (0, 1, 1), (4, -5, 3), (7, 2, 2)$ .

*Solution:*  $(1, 1, 0)$  remains.  $(-2, -2, 0), (0, 0, 0)$  are removed.  $(0, 0, 4)$  remains.  $(1, 1, 1)$  is removed.  $(4, 0, 0)$  remains.  $(0, 1, 1), (4, -5, 3), (7, 2, 2)$  are removed. Thus, the final list is

$(1, 1, 0), (0, 0, 4), (4, 0, 0)$ .

**Problem §3** Let  $V$  be a finite-dimensional vector space over a field  $\mathbb{F}$ . Prove the following statements briefly.

- (a) If  $\dim V > 0$ , then  $V$  has a non-zero vector.
- (b) If  $\dim V = 0$ , then  $V = \{0\}$ .
- (c) If  $U$  is a subspace of  $V$ , then  $\dim U \leq \dim V$ .
- (d) If  $T : V \rightarrow W$  is injective, then  $\dim V = \dim \text{range } T$ . Then, conclude that if  $W$  is finite-dimensional,  $\dim V \leq \dim W$ .
- (e) If  $T : V \rightarrow W$  is surjective, then  $\dim V \geq \dim W$ .
- (f) If  $T : V \rightarrow W$  is bijective, then  $\dim V = \dim W$ .

*Solution:*

- (a) If  $\dim V = n > 0$ , then  $V$  has a length- $n$  basis with non-zero vectors  $v_1, \dots, v_n$  (if  $v_i = 0$ , then the list would be linearly dependent and thus not a basis). Thus there exists a non-zero  $v \in \text{span}(v_1, \dots, v_n) = V$ .
- (b) If  $\dim V = 0$ , then  $V$  only has basis  $()$  (the empty basis). Since the span of the empty basis is just  $\{0\}$ , we have  $V = \{0\}$ .
- (c) If  $U$  is a subspace of  $V$ , then the basis of  $U$  is a linearly independent (not necessarily spanning) list of vectors in  $V$ . Thus the basis of  $V$  has at least as many elements as the basis of  $U$ , and so  $\dim U \leq \dim V$ .
- (d) If  $T : V \rightarrow W$  is injective, then  $T$  has a trivial kernel (that is,  $\text{null } T = \{0\}$ , and so  $\dim \text{null } T = 0$ ); by the rank-nullity theorem, we have  $\dim V = \dim \text{null } T + \dim \text{range } T = 0 + \dim \text{range } T = \dim \text{range } T$ , as required. Moreover, since  $\text{range } T$  is a subspace of  $W$ , by part (c), we have  $\dim \text{range } T \leq \dim W$ .
- (e) If  $T : V \rightarrow W$  is surjective, then  $W = \text{range } T$ ; by the rank-nullity theorem, we have  $\dim V = \dim \text{null } T + \dim \text{range } T \geq \dim \text{range } T = \dim W$ , as required.
- (f) By part (c),  $\dim V = \dim \text{range } T$ ; and by surjectivity,  $\text{range } T = W$ . Hence  $\dim V = \dim \text{range } T = \dim W$ , as required.