Review Sheet 19

1) $x = (x_1, \dots, x_n) \in \mathbb{R}^n$; $U = \{(\lambda, \dots, \lambda) \mid \lambda \in \mathbb{R}\}$; find closest vector in U to x.

$u \in U$; Let $u = (1, \dots, 1)$.

$x = cu + w$; $x = \frac{\langle x, u\rangle}{\langle u, u\rangle} u + \left(x - \frac{\langle x, u\rangle}{\langle u, u\rangle} u\right)$

$\langle u, u\rangle = 1 + \dots + 1 = n$

$\langle x, u\rangle = x_1 + \dots + x_n = \sum_{i=1}^{n} x_i$

$cu \in U$, $w \in U^{\perp}$; thus the closest vector to x is

$$\underline{P_U x = cu = \frac{x_1 + \dots + x_n}{n}(1, \dots, 1)}$$

2) $p(x) \in P_1(\mathbb{R})$; approximate $\cos\left(\frac{\pi x}{2}\right)$ on $[0, 1]$; that is minimize
$\int_0^1 \left[\cos\left(\frac{\pi x}{2}\right) - p(x)\right]^2 dx$.

From RS 18, an orthonormal basis of $P_1(\mathbb{R})$ w.r.t. inner product
$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$

is

$1,\ 2x\sqrt{3} - \sqrt{3}$

Let $U = P_1(\mathbb{R})$. From 6.55(i)

$$P_U\left(\cos\left(\frac{\pi x}{2}\right)\right) = \left\langle \cos\left(\frac{\pi x}{2}\right), 1\right\rangle 1 + \left\langle \cos\left(\frac{\pi x}{2}\right), 2x\sqrt{3} - \sqrt{3}\right\rangle (2x\sqrt{3} - \sqrt{3})$$

From Wolfram Alpha:

$\left\langle \cos\left(\frac{\pi x}{2}\right), 1\right\rangle = \frac{2}{\pi}$

$\left\langle \cos\left(\frac{\pi}{2}x\right), 2x\sqrt{3} - \sqrt{3}\right\rangle = \int_0^1 \cos\left(\frac{\pi x}{2}\right)(2x\sqrt{3} - \sqrt{3})\,dx = \frac{2\sqrt{3}(\pi - 4)}{\pi^2}$

Thus the nearest polynomial $p(x) \in P_1(\mathbb{R})$ to $\cos\left(\frac{\pi x}{2}\right)$ is

$$\underline{P_U\left(\cos\left(\frac{\pi x}{2}\right)\right) = \frac{2}{\pi} + \left(\frac{2\sqrt{3}(\pi - 4)}{\pi^2}\right)(2x\sqrt{3} - \sqrt{3})}$$

Simplified, this becomes

$$P_U\left(\cos\left(\frac{\pi x}{2}\right)\right) = 1.1585 - 1.04370x$$

3) (6.C.6) $\mathcal{U}, W$ subspaces of V; prove $P_U P_W = 0$ iff $\langle u, w\rangle = 0$ $\forall u \in \mathcal{U}, w \in W$.

Suppose $P_{\mathcal{U}} P_W = 0$, and let $u \in \mathcal{U}$. Then $P_{\mathcal{U}} u = u$, so we need $P_W u = 0$; but this happens only when $u \in W^{\perp}$. In other words, in order for $P_{\mathcal{U}} P_W = 0$, any $u \in \mathcal{U}$ must be orthogonal to any $w \in W$ (since $u \in W^{\perp}$); thus

$$\langle u, w\rangle = 0 \quad \text{for every } u \in \mathcal{U}, w \in W.$$

Conversely, suppose $\langle u, w\rangle = 0$ for every $u \in \mathcal{U}$, $w \in W$. Observe that for any $w \in W$, $w \in \mathcal{U}^{\perp}$ as well (by definition of $\mathcal{U}^{\perp}$: $\{v \in V \mid \langle v, u\rangle = 0 \ \forall u \in \mathcal{U}\}$). So, consider $P_{\mathcal{U}} P_W \in \mathcal{L}(V)$. For any $v \in V$, $P_W v \in W$; but then $P_W v \in \mathcal{U}^{\perp}$, and $P_{\mathcal{U}} w = 0$ for any $w \in \mathcal{U}^{\perp}$. Thus

$$P_{\mathcal{U}} P_W = 0.$$