MATH0540 Exam 1

1) For any linearly dependent vectors $v, w \in \mathbb{R}^2$, there is some $c \in \mathbb{R}$ s.t. $cv = w$.

Negation: There exists some linearly dependent vectors $v, w \in \mathbb{R}$ such that for any $c \in \mathbb{R}$,

$cv \neq w$.

The original statement is true:

~~Let $v, w \in \mathbb{R}^2$ be linearly dependent vectors. Then for $a_1, a_2 \in \mathbb{R}$, not both zero,~~

~~$$a_1 v + a_2 w = 0$$~~

~~$$\frac{a_1}{a_2} v = w$$~~

~~Let $c = \frac{a_1}{a_2} \in \mathbb{R}$ Then $cv = w$.~~

Let $v, w \in \mathbb{R}^2$ be linearly dependent vectors. By the Linear Dependence Lemma, either $v \in \operatorname{span}(w)$ or $w \in \operatorname{span}(v)$. Suppose WLOG that $w \in \operatorname{span}(v)$. Then by the definition of span, for some $c \in \mathbb{R}$,

$$cv = w,$$

as required. $\square$

2) Let V be the vector space of functions $f: \mathbb{R} \to \mathbb{R}$. Let W be the set of $f: \mathbb{R} \to \mathbb{R}$ that are bounded above ($\exists M \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$, $f(x) \leq M$). Is W a subspace of V?

- Choose $0: \mathbb{R} \to \mathbb{R}$ the zero function. $0 \in W$ (since any $M \geq 0$ bounds 0 from above.)

  $x \mapsto 0$

- Let $f, g \in W$, and choose $M_1, M_2 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $f(x) \leq M_1$, $g(x) \leq M_2$.

For $f+g$,

$$f(x) + g(x) \leq M_1 + g(x) \leq M_1 + M_2;$$

thus $f+g \in W$ (since it is bounded above by $M \geq M_1 + M_2$)

- Let $f \in W$, and choose $M \in \mathbb{R}$ s.t. $f(x) \leq M$ for all x. Choose $\lambda \in \mathbb{R}$. Then $\lambda f(x) \leq \lambda M$ for all x, and so $\lambda f \in W$.

Thus W is a subspace of V. $\square$

3) Let $a \in \mathbb{R}$. Prove $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x,y) = (y, ax-y)$$

is injective iff $a \neq 0$.

Proof: Suppose T is injective. Then for any $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, if

$$T(x_1,y_1) = T(x_2,y_2),$$

then $(x_1,y_1) = (x_2,y_2)$.

$T(x_1,y_1) = T(x_2,y_2)$ implies

$$(y_1, ax_1-y_1) = (y_2, ax_2-y_2)$$

From the first coordinate, we get $y_1 = y_2$. Thus

$$ax_1 - y_1 = ax_2 - y_2$$

becomes

$$ax_1 = ax_2.$$

Suppose $a=0$. Then any $x_1, x_2 \in \mathbb{R}, x_1 \neq x_2$, satisfies $ax_1 = ax_2$; but this contradicts our assumption of injectivity. Thus if T is injective, $a \neq 0$.

Conversely, suppose $a \neq 0$, and choose $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ such that

$$T(x_1,y_1) = T(x_2,y_2), \text{ so } (y_1, ax_1-y_1) = (y_2, ax_2-y_2).$$

Then $y_1 = y_2$, and so

$$ax_1 - y_1 = ax_2 - y_2$$

$$ax_1 = ax_2.$$

Since $a \neq 0$, we can divide both sides by a, resulting in

$$x_1 = x_2.$$

Thus if $a \neq 0$, $T(x_1,y_1) = T(x_2,y_2)$ implies $(x_1,y_1) = (x_2,y_2)$, and so T is injective.

Therefore T is injective iff $a \neq 0$. $\square$

4) Let V, W be v.s. over a field $\mathbb{F}$, and let $T: V \to W$ be a linear map. Suppose $v_1, \dots, v_m \in V$ are vectors s.t. $T(v_1), \dots, T(v_m)$ are linearly independent. Prove that

$$\operatorname{null} T \cap \operatorname{span}(v_1, \dots, v_m) = \{0\}.$$

Proof: Suppose $T(v_1), \dots, T(v_m)$ are linearly independent. Then

$$a_1 T(v_1) + \dots + a_m T(v_m) = 0$$

only when $a_i = 0$ for all $i \in \{1, \dots, m\}$. By linearity, we have

$$a_1 T(v_1) + \dots + a_m T(v_m) = 0$$
$$T(a_1 v_1 + \dots + a_m v_m) = 0;$$

hence $a_1 v_1 + \dots + a_m v_m \in \operatorname{null} T$. But by linear independence, all $a_i = 0$, and so

$$a_1 v_1 + \dots + a_m v_m = 0;$$

thus $\operatorname{null} T = \{0\}$.

Clearly, $0 \in \operatorname{span}(v_1, \dots, v_m)$ (simply choose $a_i = 0$); and since 0 is the only element in $\operatorname{null} T$, we necessarily have

$$\operatorname{null} T \cap \operatorname{span}(v_1, \dots, v_m) = \{0\}. \qquad \square$$