Problem §1 Deduce directly from the Spectral Theorem that all eigenvalues of a self-adjoint operator $T \in \mathcal{L}(V)$ are real.

Solution: From the Spectral Theorem, a self-adjoint operator $T \in \mathcal{L}(V)$ has a diagonal matrix $\mathcal{M}(T)$ with respect to some orthonormal basis of V. Recall that

$$\mathcal{M}(T^*) = \overline{\mathcal{M}(T)}^T;$$

but transposing matrix preserves the diagonal entries (that is, $\mathcal{M}(T)_{j,j} = (\mathcal{M}(T))^T_{j,j}$). Let $n = \dim V$. Then for every $1 \le j \le n$,

$$\mathcal{M}(T)_{j,j} = \lambda_j = \overline{\lambda}_j = \overline{\mathcal{M}(T)}^T_{j,j} = \mathcal{M}(T^*)_{j,j}.$$

Thus every eigenvalue of T satisfies $\lambda_j = \overline{\lambda}_j$, which means that every eigenvalue of T is real.

Problem §2 Given any complex number $a \in \mathbb{C}$, consider the linear operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$T(x, y) = ((a - i)x + ay, -ax + y).$$

(a) For which $a \in \mathbb{C}$ is T self-adjoint?

(b) For any $a \in \mathbb{C}$ found in part (a), calculate the eigenvalues of T.

Solution:

(a) $T(1, 0) = (a - i, -a)$, and $T(0, 1) = (a, 1)$. Thus

$$\mathcal{M}(T) = \begin{pmatrix} a - i & a \\ -a & 1 \end{pmatrix}.$$

In order for T to be self-adjoint, we need

$$\mathcal{M}(T^*) = \overline{\mathcal{M}(T)}^T = \begin{pmatrix} \overline{a - i} & \overline{-a} \\ \overline{a} & 1 \end{pmatrix}.$$

Thus, we need $a - i = \overline{a - i}$, $\overline{a} = -a$, $\overline{-a} = a$. This occurs only when $a = i$ (and so $\mathcal{M}(T) = \begin{pmatrix} 0 & i \\ -i & 1 \end{pmatrix}$).

(b) $\det \mathcal{M}(T) - I = \begin{vmatrix} -\lambda & i \\ -i & 1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 1$. Thus the eigenvalues of T are

$$\lambda = \frac{1 \pm \sqrt{5}}{2},$$

which is consistent with Problem (1).

Problem §3 Let $T : V \to W$ be a linear map on finite-dimensional inner product spaces V and W.

(a) Prove that T^*T is self-adjoint.

(b) Prove that each eigenvalue of T^*T is non-negative.

(c) Prove that $T^*T + I$ is invertible.

Solution:

(a) Recall that $(T^*)^* = T$ (Axler 7.6c; I won't reproduce the proof here). Then for any $v \in V$, $w \in W$, we have

$$\langle T^*Tv, w \rangle = \langle Tv, (T^*)^*w \rangle = \langle v, T^*Tw \rangle.$$

Thus T^*T is self-adjoint.

(b) Let $\lambda \in \mathbb{F}$ be an eigenvalue. Then for any $v \in V$,

$$\lambda \|v\|^2 = \lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle T^*Tv, v \rangle = \langle Tv, (T^*)^* v \rangle = \langle Tv, Tv \rangle .$$

But $\langle Tv, Tv \rangle \geq 0$ and $\|v\|^2 \geq 0$ for every $v \in V$; thus we need $\lambda \geq 0$ as well.

(c) Recall that $\langle u, v \rangle = 0$ only if either $u = 0$ or $v = 0$, that a map $T : V \to V$ is injective if and only if its null space is trivial, and finally that an operator is invertible iff bijective iff injective. Thus, if

$$\langle (T^*T + I)v, v \rangle \neq 0$$

for every $v \in V \setminus \{0\}$, then $(T^*T + I)v \neq 0$ for all non-zero v (and so its null space is trivial), so $T^*T + I$ is injective and hence invertible.

We have, for all $v \in V \setminus \{0\}$,

$$\begin{aligned} \langle (T^*T + I)v, v \rangle &= \left\langle T^T v + v, v \right\rangle \\ &= \langle T^*Tv, v \rangle + \langle v, v \rangle \\ &= \langle Tv, Tv \rangle + \langle v, v \rangle \\ &> 0, \end{aligned}$$

since $\langle Tv, Tv \rangle \geq 0$ and $\langle v, v \rangle > 0$.

Therefore $(T^*T + I)v \neq 0$ for all non-zero v, and so is injective and hence invertible.