Problem §1 A square matrix is *skew-symmetric* if $A^T = -A$. Prove that if A is skew-symmetric and n odd, then $\det A = 0$. Is this true for even n?

Solution: Since $\det A = \det A^T$, $\det A = \det -A$. Since there are odd -1s, we get

$$\det A = (-1)^n \det A = -\det A.$$

Then $2 \det A = 0$, so $\det A = 0$, as required.

For even n, $(-1)^n = 1$, so we don't know anything else about $\det A$.

Problem §2 Suppose the permutation of σ takes $(1,2,3,4,5)$ to $(5,4,1,2,3)$.

(a) Find the sign of σ.

(b) What does σ^2 do to $(1,2,3,4,5)$.

(c) What does the inverse permutation σ^{-1} do to $(1,2,3,4,5)$?

(d) What is the sign of σ^{-1}.

Solution:

(a) $\operatorname{sgn}(\sigma) = -1$

(b) $\sigma^2(1,2,3,4,5) = (3,2,5,4,1)$

(c) $\sigma^{-1}(1,2,3,4,5) = (3,4,5,2,1)$

(d) $\operatorname{sgn}(\sigma^{-1}) = -1$.

Problem §3 Why is there an even number of permutations of $(1,2,\ldots,9)$ and why are exactly half of them old permutations?

Solution: The number of permutations of $(1,2,\ldots,9)$ is $9! = 362880$; thus there are an even number of permutations. Now, recall that

$$\det(A) = \sum_{\sigma \in \text{ permutations}} a_{\sigma(1),1} \ldots a_{\sigma(9),9} \operatorname{sgn}(\sigma).$$

Consider a matrix A filled entirely with 1s. Then the determinant becomes

$$\det A = \sum_{\sigma \in \text{ permutations}} \operatorname{sgn}(\sigma) = 0$$

(because if two rows are linearly dependent, then $\det A = 0$). In other words, the determinant is the number of even permutations, minus the number of odd permutations. Therefore, exactly half are even, and half are odd.

Problem §4 Evaluate the determinants:

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 5 \\ 1 & -3 & 0 \end{vmatrix}, \begin{vmatrix} 4 & -6 & -4 & 4 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ -2 & 2 & -3 & -5 \end{vmatrix}.$$

Solution:

(a) $\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 5 \\ 1 & -3 & 0 \end{vmatrix} = -5 \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = 25$

(b)

$$
\begin{aligned}
\begin{vmatrix} 4 & -6 & -4 & 4 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ -2 & 2 & -3 & -5 \end{vmatrix} &= -2\begin{vmatrix} -6 & 4 & 4 \\ -3 & 1 & 3 \\ 2 & -3 & -5 \end{vmatrix} + 1\begin{vmatrix} 4 & -4 & 4 \\ 0 & 1 & 3 \\ -22 & -3 & -5 \end{vmatrix} \\
&= -2\left(-6\begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} + 4\begin{vmatrix} -3 & 3 \\ 2 & -5 \end{vmatrix} + 4\begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix}\right) + \left(-1\begin{vmatrix} 4 & 4 \\ -2 & -5 \end{vmatrix} - 3\begin{vmatrix} 4 & -4 \\ -2 & -3 \end{vmatrix}\right) \\
&= 12(4) - 8(9) - 8(7) - 12 - 3(-20) \\
&= -32.
\end{aligned}
$$