Problem §1 (Treil 5.4) Use cofactor formula to compute the inverses of the following matrices:

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 19 & -17 \\ 3 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

Solution:

(a) $A^{-1} = \frac{1}{4-6}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

(b) $A^{-1} = \frac{1}{-38-(-51)}\begin{pmatrix} -2 & -3 \\ 17 & 19 \end{pmatrix}^T = \begin{pmatrix} -\frac{2}{13} & \frac{17}{13} \\ -\frac{3}{13} & \frac{19}{13} \end{pmatrix}$

(c) $A^{-1} = \frac{1}{5}\begin{pmatrix} 5 & -3 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}$

(d) $A^{-1} = \frac{1}{(1-2)-(2)}\begin{pmatrix} -1 & (-1)^3(2) & 2 \\ (-1)^3(1) & 1 & (-1)^5(1) \\ 2 & (-1)^5(2) & -1 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

Problem §2 (Axler Problem 5.C.16) The **Fibonacci Sequence** $F_1, F_2, \ldots$ is defined by

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-2} + F_{n-1},\ n \geq 3.$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

(a) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for all positive integers n.

(b) Find the eigenvalues of T.

(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

(d) Use the solution to part (c) to compute $T^n(0, 1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Solution:

(a) We begin by induction. Clearly, $T^1(0, 1) = (1, 1) = (F_1, F_2)$. Now, assume $T^n(0, 1) = (F_n, F_{n+1})$. Then

$$T^{n+1}(0, 1) = T(T^n(0, 1)) = T(F_n, F_{n+1}) = (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2}),$$

as required. Thus

$$T^n(0, 1) = (F_n, F_{n+1})$$

for all positive integers n.

(b) Let $A = \mathcal{M}(T)$ with respect to the standard basis. Then

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1.$$

Thus

$$\lambda_1 = \frac{1+\sqrt{5}}{2},\ \lambda_2 = \frac{1-\sqrt{5}}{2},$$

since those are the roots of $\lambda^2 - \lambda - 1$.

(c) We now find the eigenspaces associated with λ_1, λ_2:

$$E(\lambda_1, T) = \{(x,y) \mid y = \left(\frac{1+\sqrt{5}}{2}\right) x,\ x + y = \left(\frac{1+\sqrt{5}}{2}\right) y\}.$$

From the first condition, we get that any eigenvector corresponding to λ_1 is a scalar multiple of $\left(1, \left(\frac{1+\sqrt{5}}{2}\right)\right)$. Similarly, we get that any eigenvector corresponding to λ_2 is a scalar multiple of $\left(1, \left(\frac{1-\sqrt{5}}{2}\right)\right)$. Thus

$$\{\left(1, \frac{1+\sqrt{5}}{2}\right), \left(1, \frac{1-\sqrt{5}}{2}\right)\}$$

is a basis of eigenvectors in $\mathbb{R}^2$ (since eigenvalues of distinct eigenvalues are linearly independent, and thus we have a list of 2 linearly independent vectors in $\mathbb{R}^2$, thus forming a basis).

(d) Let $v_1 = \left(1, \frac{1+\sqrt{5}}{2}\right)$, $v_2 = \left(1, \frac{1-\sqrt{5}}{2}\right)$. Note that $(0,1) = \frac{v_1 - v_2}{\sqrt{5}}$. Thus

$$T^n(0,1) = \frac{1}{\sqrt{5}} T^n(v_1 - v_2) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n v_1 - \left(\frac{1-\sqrt{5}}{2}\right)^n v_2 \right].$$

Since the first component is F_n, we get

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right],$$

as required.