Problem §1 Let $c \in \mathbb{R}$ be any real number. Does there exist a polynomial $g \in \mathcal{P}_d(\mathbb{R})$ such that for all $f \in \mathcal{P}_d(\mathbb{R})$,

$$\int_0^1 f(x)g(x)dx = f(c)?$$

Solution: From Review Sheet 18, we know that

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx$$

is an inner product in the vector space $\mathcal{P}_d(\mathbb{R})$. Additionally, the evaluation map at c,

$$E_c : \mathcal{P}_d(\mathbb{R}) \to \mathbb{R},\ E_c(f) = f(c)$$

is trivially a linear functional on $\mathcal{P}_d(\mathbb{R})$. Then by the Reisz Representation Theorem, there exists some $g(x) \in \mathcal{P}_d(\mathbb{R})$ such that

$$E_c(f) = f(c) = \langle f, g\rangle = \int_0^1 f(x)g(x)dx,$$

as required.

Problem §2 Find such a polynomial above in the case $c = 0$, $d = 1$.

Solution: From previous review sheets, given the vector space $\mathcal{P}_1(\mathbb{R})$ together with the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx,$$

we found an orthonormal basis

$$1, 2x\sqrt{3} - \sqrt{3}.$$

Consider the evaluation map $E_0 \in \mathcal{L}(V, \mathbb{F})$ (which is a linear functional); then by the Reisz Representation Theorem, some $g \in \mathcal{P}_1(\mathbb{R})$ exists that satisfies

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx = E_0(f)$$

for all $f \in \mathcal{P}_1(\mathbb{R})$; in particular,

$$\begin{aligned} g(x) &= \overline{E_0(1)} \cdot 1 + \overline{E_0(2x\sqrt{3} - \sqrt{3})} \cdot (2x\sqrt{3} - \sqrt{3}) \\ &= 1 + (-\sqrt{3})(2x\sqrt{3} - \sqrt{3}) \\ &= 4 - 6x. \end{aligned}$$

Indeed, if we have any $f(x) = a_0 + a_1x \in \mathcal{P}_1(\mathbb{R})$, then

$$\begin{aligned} \int_0^1 (a_0 + a_1x)(4 - 6x)dx &= \int_0^1 4a_0 - 6a_0x + 4a_1x - 6a_1x^2 dx \\ &= 4a_0 - 3a_0 + 2a_1 - 2a_1 \\ &= a_0 = E_0(f), \end{aligned}$$

as required.

Problem §3 Consider $\mathbb{R}^2$ with the inner product

$$\langle (w_1, w_2), (z_1, z_2)\rangle = 2w_1z_1 + 3w_2z_2.$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(x_1, x_2) = (x_1 + 3x_2, -x_1).$$

Compute T^*.

Solution: Fix a point $(y_1, y_2) \in \mathbb{R}^2$. Then for any $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
\langle (x_1, x_2), T^*(y_1, y_2) \rangle &= \langle T(x_1, x_2), (y_1, y_2) \rangle \\
&= \langle (x_1 + 3x_2, -x_1), (y_1, y_2) \rangle \\
&= 2(x_1 y_1 + 3x_2 y_1) - 3x_1 y_2 \\
&= 2x_1 y_1 + 6x_2 y_1 - 3x_1 y_2 \\
&= 2y_1(x_1 + 3x_2) - 3x_1 y_2 \\
&= \left\langle (x_1, x_2), \left(y_1 - \frac{3}{2} y_2, 2y_1 \right) \right\rangle .
\end{aligned}$$

Hence $T^*(y_1, y_2) = (y_1 - \frac{3}{2} y_2, 2y_1)$