MATH 540 HONORS LINEAR ALGEBRA SUMMER 2021, Melody Chan
PROBLEM SET F

Due Monday June 28 at 11:59pm Eastern

Submit all of the following on Gradescope, unless otherwise indicated, and don't forget to tag each answer to its page.

1. (4 points) If you participated in the class discussion Tuesday June 22 and are happy with the response you submitted, you can be done.

   Otherwise, read the prompt here and watch the Introduction to the discussion, and browse the linked articles. Then

   (a) Enter your response in the Google Form. Please don't just B.S. it. I really want a sharp, specific, well-reasoned opinion in response to a hard-hitting question.

   (b) Discuss your response with at least one other person, ideally another person in this class.[1] Then briefly summarize an interesting point from your discussion. Identify as sharply as possible a difference or disagreement in your perspectives.

   I also encourage you to submit additional thoughts in the Google Form any time even if you already participated. Just leave your name on each one!

2. (Ungraded, 0 points) Study the statement, e.g., on p. 123 of the textbook, that a nonzero polynomial of degree at most m has at most m distinct roots. You don't have to prove this statement, but you should be willing to use it.

3. (4 points) Suppose $T\colon V \to W$ is an injective linear map between finite-dimensional vector spaces. Prove that there exists a linear map $S\colon W \to V$ such that $ST = I_V$.

4. (5 points) Informally, this problem says *you can always "thread" the graph of a unique degree $\leq d$ polynomial through $d+1$ points.* For example, you can draw a unique line through two points, a unique parabola through three points, etc.

   Prove that given distinct real numbers $a_0, \ldots, a_d$ and any real numbers $b_0, \ldots, b_d$, there exists a unique $f \in \mathcal{P}_d(\mathbb{R})$ such that

   $$f(a_0) = b_0,\ \ldots,\ f(a_d) = b_d.$$

   Possible hint below, if you're stuck.[2]

Extra credit on the next page.

[1] Not just a text-based discussion, please. In-person, Zoom, Discord video call, or the equivalent.

[2] First verify that the function $T\colon \mathcal{P}_d(\mathbb{R}) \to \mathbb{R}^{d+1}$ sending $T(f) = (f(a_0), \ldots, f(a_d))$ is a linear map.

Extra credit, for finite field enthusiasts. (2 points) In this class, we have studied only finite fields of prime order[3], i.e., $\mathbb{F}_p$. Several weeks ago in class I asserted that there exists a finite field of order q iff q is a *power* of a prime. In this problem, you prove half of this statement: if K is a field with exactly q elements, then q is a power of a prime.

Let K be a finite field. Consider the subset F of K of all elements of the form

$$\underbrace{1 + \cdots + 1}_{n}$$

for all $n = 0, 1, 2, \ldots$. Show that F is a subfield of K, and show that F has a prime number of elements. Recalling that K is then a F-vector space, deduce that the number of elements in any finite field is a power of a prime.

[3] "Finite field" just means a field with finitely many elements. "Order" means number of elements.