Problem §1 Let $a \in \mathbb{F}$, and let $v \in V_{\mathbb{F}}$ be a non-zero vector. Prove that

$$av = \mathbf{0} \text{ only if } a = 0.$$

Solution: Let $v \in V_{\mathbb{F}}$ be a non-zero vector, and let $a \in \mathbb{F}$ such that $av = \mathbf{0}$. Suppose $a \neq \mathbf{0}$. Then

$$
\begin{aligned}
av &= \mathbf{0} \\
a^{-1}av &= a^{-1}\mathbf{0} \qquad &&[\text{since } a \in \mathbb{F} \text{ and } a \neq 0, \text{ there exists } a^{-1} \in \mathbb{F} \text{ such that } aa^{-1} = 1] \\
v &= \mathbf{0}.
\end{aligned}
$$

But this is a contradiction, as we assume v non-zero. Hence a must be zero.

Problem §2 Let $v, w \in V$, and suppose $v \neq 0$. Prove that there exists at most one $a \in \mathbb{F}$ such that

$$av = w.$$

Solution: Let $v, w \in V_{\mathbb{F}}$, and let $a \in \mathbb{F}$ such that $av = w$. Suppose there exists a $b \in \mathbb{F}$, $b \neq a$ such that $bv = w$. Then

$$
\begin{aligned}
av &= w = bv \\
av &= bv \\
av - bv &= bv - bv \\
(a - b)v &= \mathbf{0} \qquad &&[\text{since } -bv = (-b)v].
\end{aligned}
$$

From problem 1, we get that if v is a non-zero vector, then $(a - b)v = 0$ only when $a - b = 0$. Hence $a = b$; but we assume $b \neq a$, a contradiction. Thus if $\exists b \in \mathbb{F}$ such that $bv = w$, then $b = a$.

Problem §3

   (a) (1.C.12) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

   (b) (1.C.13, extra credit) Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution:

   (a) Let $U_1, U_2 \in V$ be two subspaces of V.

       Suppose that one of the subspaces is contained in the other; that is, suppose without loss of generality that $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2 \subseteq V$ is a subspace of V (since U_2 is a subspace of V).

       Now, suppose that $U_1 \cup U_2 \subseteq V$ is a subspace of V. Let $v_1, v_2 \in U_1 \cup U_2$. Then

$$v_1 \in U_1 \text{ or } v_1 \in U_2, \text{ and } v_2 \in U_1 \text{ or } v_2 \in U_2.$$

       $v_1, v_2 \in U_1$ or $v_1, v_2 \in U_2$ tell us nothing new about the relationship between U_1 and U_2 (we already know, by definition of a subspace, that 0, λv_1, λv_2, $v_1 + v_2 \in U_i$ for some $\lambda \in \mathbb{F}$, $i \in \{1, 2\}$), so suppose without loss of generality that $v_1 \in U_1, v_2 \in U_2$. We know (since $U_1 \cup U_2$ is a subspace of V) that $v_1 + v_2 \in U_1 \cup U_2$, so $v_1 + v_2 \in U_1$ or $v_1 + v_2 \in U_2$. If $v_1 + v_2 \in U_1$, by closure of addition in U_1, $v_2 \in U_1$ as well, and so since for any arbitrary $v_2 \in U_2$, $v_2 \in U_1$, we have $U_2 \subseteq U_1$. Similarly, if $v_1 + v_2 \in U_2$, by closure of addition, $v_1 \in U_2$ as well, and so $U_1 \subseteq U_2$.
       Therefore, if $U_1 \cup U_2$ is a subspace of V, then one of the subspaces is contained in the other.

       Thus, the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

> **Problem §4** Solved in Review Sheet 5.

> **Problem §5** Let V, W be vector spaces over $\mathbb{F}$, and let $T : V \to W$ be a linear map. Suppose V is finite-dimensional and T is surjective. Prove that W is finite-dimensional.

    Solution: We begin with a lemma.

Lemma 1. *Let V, W be vector spaces over a field $\mathbb{F}$. If V is an n-dimensional vector space, and $T : V \to W$ is a linear map, then* $\dim(\operatorname{range} T) \leq \dim V$.

Proof. If $\dim V = n$, then V has a basis $B = \{v_1, \ldots, v_n\}$ with n linearly independent vectors in V that span V. Then, for any $v \in V$,

$$v = a_1 v_1 + \ldots + a_n v_n, \quad \text{for } a_i \in \mathbb{F}.$$

Applying T to both sides, we have

$$T(v) = T(a_1 v_1 + \ldots + a_n v_n) = a_1 T(v_1) + \ldots + a_n T(v_n),$$

and since v was an arbitrary $v \in V$, $T(v_1), \ldots, T(v_n)$ span $\operatorname{range} T$. Hence $\dim(\operatorname{range} T) \leq n = \dim V$.

(Note that there is no guarantee that $T(v_1), \ldots, T(v_n)$ are linearly independent and thus form a basis for $\operatorname{range} T$; $\dim(\operatorname{range} T)$ could very much be less than n. For example, take $T = \mathbf{0}$ the zero map. Then $T(v_i) = \mathbf{0}$ for any $v_i \in B$, and so $\dim(\operatorname{range} T) < n$.) $\quad \square$

    From this lemma, we get that if V is a finite-dimensional vector space (say, with $\dim V = n$), and $T : V \to W$ a linear map, then $\dim(\operatorname{range} T) \leq \dim V$. Moreover, since T is surjective, by definition we have $\operatorname{range} T = W$. Thus $\dim W \leq \dim V$, and so W is finite-dimensional as well.