1 May 8 notes

Suppose that the observations are a Hermitian matrix. If they are not, then start from the Gram matrix and take a matrix sqrt of this matrix to be the observation matrix.

We would like to analyze $||\Sigma||_{\text{op}}$.

$$\Sigma = D^T M^T M D$$

Where M is the Hermitian data matrix, and D is an n-by-k matrix: $\begin{pmatrix} e_{d_1} & e_{d_2} & ... & e_{d_k} \end{pmatrix}$, selecting k columns of M using multiplication on the right.

We introduced u, which can be initialized to be the left singular vector of MD for some well-performing D ($\sigma = |uMD|_2$ large). This is slightly useful, because inner products with u could be factored out to be computed using entries of Σ only, and discard/never synthesize an M.

Set $A = D^T M^T u u^T M D$ (large and rank-one), and $E = \Sigma - A$. Analyzing E for different permissible D should give us a smaller recursive problem, compared to analyzing Σ (shrinking the problem). However, we would like to produce a new upper-bound deterministically in polynomial time instead.

We found that E is similar to:

$$(I - \frac{1}{\sigma^2} M D u u^T D^T M) M^2$$

We hope that $||E||_{\text{op}}$ is small compared to $||\Sigma||_{\text{op}}$. Note that with a change-of-basis using the diagonalization of M, then the matrix in parentheses is still a rank-$n-1$ projection matrix. It could have a zero for the dominant eigenvalue of M^2, and that is the best case. We want a large sine of the angular comparison of the dominant eigenspaces of the projection matrix on the left and of M, and this sine can be bounded using the Davis-Kahan theorem.

Let $P = I - \frac{1}{\sigma^2} M D u u^T D^T M$. For Davis-Kahan, we should come up with our own upper-bound on $||P - \frac{1}{\sigma^2} M^2||_{\text{op}}$. Davis-Kahan bounds the dissimilarity in the largest eigenspaces of P and $\frac{1}{\sigma^2} M^2$, and if this dissimilarity obtains its maximum value, then the dominant eigenvalue would drop out of PM^2 and $||PM^2||_{\text{op}}$ would be driven by the second-largest eigenvalue of M^2 instead.