Analytical solution to bidiagonalize rank-one singular values matrix updahe

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1 Rank-one updated singular values matrix

We start with a singular values diagonal matrix Σ. We can apply different orthogonal matrices $(Q_1 \Sigma Q_2)$ to a singular values matrix, because:

$$(Q_1 \Sigma Q_2)^T (Q_1 \Sigma Q_2) = Q_2^T \Sigma^2 Q_2 = Q_2^{-1} \Sigma^2 Q_2$$

The resulting matrix is similar to the squared singular values matrix Σ^2.

1.1 Rank-one update (more background needed)

$$\Sigma' = \begin{pmatrix} v_1 & 0 & 0 & 0 \\ v_2 & d_1 & 0 & 0 \\ v_3 & 0 & d_2 & 0 \\ v_4 & 0 & 0 & d_3 \end{pmatrix}$$

2 Initialization routine: Move one entry to off-diagonal

At each step, we are going to compute one Givens rotation on the left, and one Givens rotation on the right (they are generally not each other's transpose). Set $A = Q_1 \Sigma' Q_2$. We will assume that A is mutable and the rotated A is being persisted after every rotation.

2.1 Givens rotation: Review

Try to set the entry v_4 to zero, by rotating $\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$. A Givens rotation on the left will affect the third and fourth entry of every column:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \Sigma'$$

We want the inverse rotation matrix to produce:

$$\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{v_3^2 + v_4^4} \end{pmatrix} = \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$$

We have $\sin\theta = \frac{v_3}{\sqrt{v_3^2+v_4^2}}, \cos\theta = \frac{v_4}{\sqrt{v_3^2+v_4^4}}$.

On the left column of Σ', we could add further Givens rotations moving up the matrix, but we would lose the useful sparsity when we spread the diagonal entries around. We have already created a 2x2 bottom-right block which we can assume is now all nonzero.

2.2 Sparsity invariant

We just updated Σ'_{31} and zeroed Σ'_{41}, say for this step we use the counter $i = 3$. Next, we are going to examine columns i and $i+1$. We assume that the diagonal entries are nonzero and remain not close to zero after each step. If $i + 1 < n$, then column $i + 1$ also has a lower off-diagonal entry $a_{i+2,i+1}$.

2.3 Right-hand side rotation

Apply a rotation affecting the third and fourth entries in every row (rotating the third column and the fourth column) of:

$$A = \begin{pmatrix} v_1 & 0 & 0 & 0 \\ v_2 & d_1 & 0 & 0 \\ v'_3 & 0 & d_2^* & x_{34} \\ 0 & 0 & f_3^* & d_3^* \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We use a rotation matrix so that d_2^* goes to some $d'_2 = \pm\sqrt{d_2^{*2} + x^2}$ (which is the final value that it will attain for this subroutine). In total (if $i + 2 \leq n$), then a 3x2 block of the matrix may be affected. Now (if we had hypothetically $n > 4$), we would be storing a single entry x_{53} which is outside of the expected entries in this matrix. However, if $i + 1 = n$, then rows $[i + 1, n]$ are in the bidiagonal format, and the subroutine terminates.

2.4 Initialization routine closed-form

The trigonometric parameter θ could be used, but we will avoid any parameterization of the rotation matrices, and we will simplify the expression later. So far, we have:

$$v'_3 = \sqrt{v_3^2 + v_4^2}$$

$$d_2^* = d_2 \cos\theta = \frac{d_2 v_4}{v'_3}$$

$$x_{34} = -d_3 \sin\theta = -\frac{d_3 v_3}{v_3'}$$

$$f_3^* = d_2 \sin\theta = \frac{d_2 v_3}{v_3'}$$

$$d_3^* = d_3 \cos\theta = \frac{d_3 v_4}{v_3'}$$

For now, we will continue with a Givens rotation solved so that we drive the result to a positive square root term. The cosine value comes from the matrix entry which we are pushing to zero. We have some angle: $\sin\phi \propto f_3^*, \cos\phi \propto d_3^*$, which can be normalized:

$$\sin\phi = \frac{1}{\sqrt{d_2^2 v_3^2 + d_3^2 v_4^2}} d_2 v_3, \cos\phi = \frac{1}{\sqrt{d_2^2 v_3^2 + d_3^2 v_4^2}} d_3 v_4$$

Apply a RHS rotation matrix block to the 3x2 A matrix block which could be affected.

$$\begin{pmatrix} d_{i-1}^* & x_{i,i+1} \\ f_i^* & d_i^* \\ 0 & f_{i+1} \end{pmatrix} = \frac{1}{v_3'} \begin{pmatrix} d_{i-1} v_{i+1} & -d_i v_i \\ d_{i-1} v_i & d_i v_{i+1} \\ 0 & f_{i+1} \end{pmatrix}$$

Apply:

$$\frac{1}{v_3'} \begin{pmatrix} d_{i-1} v_{i+1} & -d_i v_i \\ d_{i-1} v_i & d_i v_{i+1} \\ 0 & f_{i+1} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

We have a block which we are going to persist: