Triangularize Eigenvalues

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We have Σ, which is a diagonal singular values matrix. Our original, full-rank data matrix is augmented with a column of zeros, so Σ has rank $n-1$ because the last row and column are zero. For a new column of data, there is a $\mathbb{R}^n$ vector update to the last column of Σ, and new additional left and right singular vector, to reproduce the data.

We can readily compute the sparse matrix $\Sigma'^T\Sigma' = D'$:

$$D' = \begin{pmatrix} d_1 & 0 & 0 & w_1 \\ 0 & d_2 & 0 & w_2 \\ 0 & 0 & d_3 & w_3 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

We can use Givens rotations on both sides to efficiently rotate the entries of D' to upper triangular. We want $n-1$ matrices on the LHS which progressively push down the w entries in the last column. The last column of D' may be isolated, and the effect of the series of Givens rotations will be computed in parallel.

Now, suppose that we have applied a Givens rotation to zero out D'_{in}. We need to consider two progressive 2x2 rotations affecting certain diagonal entries: $d_i \to d'_i \to d''_i$

First, d_1 will only have one 2x2 rotation touching it, so initialize $d'_1 = d_1$.

Next, consider $d_i, i > 1$. We have an angle: $\sin\theta_i = \frac{w'_{i-1}}{w'_i}$. For w'_2, we formed the hypotenuse of the two entries, and by reversing the angle (which is in the first quadrant; both positive), we push $\sin \frac{w_1}{\sqrt{w_1^2+w_2^2}}$ to zero.

Apply the 2x2 rotation to the 2x2 block:

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} d_{i-1} & 0 \\ 0 & d_i \end{pmatrix} = \begin{pmatrix} \ell_1 & r_1 \\ \ell_2 & r_2 \end{pmatrix} = \begin{pmatrix} d'_{i-1}\cos\theta_i & -d_i\sin\theta_i \\ d'_{i-1}\sin\theta_i & d_i\cos\theta_i \end{pmatrix}$$

Before writing our updates (d''_{i-1} and d'_i), we need r_1 to go to zero. Apply a 2x2 rotation matrix to the adjoint:

$$\begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix} \begin{pmatrix} \ell_1 & r_1 \\ \ell_2 & r_2 \end{pmatrix}^T = \begin{pmatrix} d''_{i-1} & 0 \\ * & d'_i \end{pmatrix}^T$$

We expect: $d''_{i-1} = \sqrt{\ell_1^2 + r_1^2}$
Apply the reverse rotation matrix to the transposed entries:

$$\begin{pmatrix} \cos\phi_i & -\sin(-\phi_i) \\ \sin(-\phi_i) & \cos\phi_i \end{pmatrix} \begin{pmatrix} d''_{i-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \ell_1 \\ r_1 \end{pmatrix}$$

Therefore: $\cos\phi_i = \frac{\ell_1}{d''_{i-1}}$
Our last step is to solve for d'_i:

$$\begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix} \begin{pmatrix} \ell_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} * \\ d'_i \end{pmatrix}$$

Using the results of the two rotations:

$$d'_i = \ell_2 \sin\phi_i + r_2 \cos\phi_i = d'_{i-1} \sin\theta_i \sin\phi_i + d_i \cos\theta_i \cos\phi_i$$

Note: $\sin\phi_i = r_1/\sqrt{\ell_1^2 + r_1^2} = -d_i w_{i-1}/\sqrt{w_i'^2 (d_{i-1}'^2 \cos^2\theta_i + d_i^2 \sin^2\theta_i)}$

We need an approximation before continuing. Say that we will place the diagonal in ascending order of magnitude, and the value w'_{i-1} which we want to push towards zero keeps growing relative to the next w_i ($|\sin\theta_i|$ is growing, $|\sin\theta_i| > |\cos\theta_i|$). Therefore, it makes the most sense to disregard $d_{i-1}'^2 \cos^2\theta_i$.

$$\sin\phi_i \approx -\frac{d_i w_{i-1}}{w'_i d_i \sin\theta_i} = -\frac{w_{i-1}}{w'_{i-1}}$$

$$\cos\phi_i \approx \frac{d'_{i-1} \cos\theta_i}{w'_i d_i \sin\theta_i} = \frac{d'_{i-1} w_i}{d_i w'_i w'_{i-1}}$$

Plug this into the closed-form for d'_i:

$$d'_i = -\frac{d'_{i-1} w_{i-1}}{d_i w'_{i-1}} \frac{w'_{i-1}}{w'_i} + d_i \frac{w_i}{w'_i} \frac{d'_{i-1} w_i}{d_i w'_i w'_{i-1}} = -\frac{d'_{i-1} w_{i-1}}{d_i w'_i} + \frac{d'_{i-1} w_i^2}{w_i'^2 w'_{i-1}}$$

We have each term increasing by a ratio:

$$d'_i = \left(\frac{w_i^2}{w_i'^2 w'_{i-1}} - \frac{w_{i-1}}{d_i w'_i} \right) d'_{i-1}$$