Triangularize Eigenvalues

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We have A^2 which was a diagonal matrix of size $n-1$ ($A_i^2, i < n$), and is augmented by $v_i^2, 1 \leq i \leq n$ as last row/column symmetric entries.

In one iteration, we update v^2, but in the next iteration, this entry will go to zero.

$$v_2^{2\prime} = \pm\sqrt{\left(v_1^{2\prime}\right)^2 + \left(v_2^2\right)^2}$$

Calculate $\left(v_i^{2\prime}\right)^2$ using cumsum of $\left(v_i^2\right)^2$ (originally this matrix was formed by squaring another matrix with augmented vector v, now we are squaring again).

We wanted $\sin\theta_i$ to drive v_i^2 to zero. Then: $\cos\theta_i = \frac{v_2^{2\prime}}{v_2^2}$. θ_i should have appropriate sign (modulus of π) so that $\left|v_2^{2\prime}\right|$ grows instead of shrinking.

Consider A_i^2 when we update v_i^2. There is an intermediate step where A_i^2 goes to $A_i^2\cos\theta_i$, and there is an upper off-diagonal in the row above: $A_i^2\sin\theta_i$ (upper off-diagonal must be avoided). We have a 2x2 block:

$$M\begin{pmatrix} A_{i-1}^{2'} & 0 \\ 0 & A_i^2 \end{pmatrix} = \begin{pmatrix} A_{i-1}^{2'}\sin\theta_i & A_i^2\sin\theta_i \\ A_{i-1}^{2'}\cos\theta_i & A_i^2\cos\theta_i \end{pmatrix}$$

Next, we need the top-right (upper off-diagonal) to go to zero. The angles would be precomputed in parallel after the first computation, by lining up $A_{i-1}^{2'}$ and A_i^2. Applying a Givens rotation on the right (rotation of two elements in the row vector, broadcasted against each row of the matrix), the result is:

$$\begin{pmatrix} A_{i-1}^{2'}\sin\theta_i & A_i^2\sin\theta_i \\ A_{i-1}^{2'}\cos\theta_i & A_i^2\cos\theta_i \end{pmatrix}\begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix}$$

The first row is discarded

try again

We have Σ, which is a diagonal singular values matrix. Our original, full-rank data matrix is augmented with a column of zeros, so Σ has rank $n-1$ because the last row and column are zero. For a new column of data, there is a $\mathbb{R}^n$ vector update to the last column of Σ, and new additional left and right singular vector, to reproduce the data.

We can readily compute the sparse matrix $\Sigma'^T\Sigma' = D'$:

$$D' = \begin{pmatrix} d_1 & 0 & 0 & w_1 \\ 0 & d_2 & 0 & w_2 \\ 0 & 0 & d_3 & w_3 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

We can use Givens rotations on both sides to efficiently rotate the entries of D' to upper triangular. We want $n-1$ matrices on the LHS which progressively push down the w entries in the last column. The last column of D' may be isolated, and the effect of the series of Givens rotations will be computed in parallel.

Now, suppose that we have applied a Givens rotation to zero out D'_{in}. We need to consider two progressive 2x2 rotations affecting certain diagonal entries: $d_i \to d'_i \to d''_i$

First, d_1 will only have one 2x2 rotation touching it, so initialize $d'_1 = d_1$.

Next, consider $d_i, i > 1$. We have an angle: $\sin\theta_i = \frac{w_1}{w'_2}$. For w'_2, we formed the hypotenuse of the two entries, and by reversing the angle (which is in the first quadrant; both positive), we push $\sin\frac{w_1}{\sqrt{w_1^2+w_2^2}}$ to zero.

Apply the 2x2 rotation to the 2x2 block:

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} d_{i-1} & 0 \\ 0 & d_i \end{pmatrix} = \begin{pmatrix} \ell_1 & r_1 \\ \ell_2 & r_2 \end{pmatrix} = \begin{pmatrix} d'_{i-1}\cos\theta_i & -d_i\sin\theta_i \\ d'_{i-1}\sin\theta_i & d_i\cos\theta_i \end{pmatrix}$$

Before writing our updates (d''_{i-1} and d'_i), we need r_1 to go to zero. Apply a 2x2 rotation matrix to the adjoint:

$$\begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix} \begin{pmatrix} \ell_1 & r_1 \\ \ell_2 & r_2 \end{pmatrix}^T = \begin{pmatrix} d''_{i-1} & 0 \\ * & d'_i \end{pmatrix}^T$$

We expect: $d''_{i-1} = \sqrt{\ell_1^2 + r_1^2}$-

Apply the reverse rotation matrix to the transposed entries:

$$\begin{pmatrix} \cos\phi_i & -\sin(-\phi_i) \\ \sin(-\phi_i) & \cos\phi_i \end{pmatrix} \begin{pmatrix} d''_{i-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \ell_1 \\ r_1 \end{pmatrix}$$

Therefore: $\cos\phi_i = \frac{\ell_1}{d''_{i-1}}$

Our last step is to solve for d'_i:

$$\begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix} \begin{pmatrix} \ell_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} * \\ d'_i \end{pmatrix}$$

Using the results of the two rotations:

$$d'_i = d'_{i-1}\sin\theta_i\sin\phi_i + d_i\cos\theta_i\cos\phi_i$$

$$d'_i = d'_{i-1} \frac{w_{i-1}}{w'_i} \frac{(-d_i w_{i-1}/w'_i)}{\sqrt{d'^2_{i-1} \cos^2 \theta_i + d_i^2 \sin^2 \theta_i}} + d_i \frac{w_i}{w'_i} \frac{d'^2_{i-1} w_i / w'_i}{\sqrt{d'^2_{i-1} \cos^2 \theta_i + d_i^2 \sin^2 \theta_i}}$$

$$d'_i = \frac{d'^2_{i-1} d_i w_i^2 - d'_{i-1} d_i w_{i-1}^2}{}$$