Boolean Algebra

1. Switching circuits are also called - Logic circuits
   Gate circuits
   Digital circuits

2. Switching algebra is also called Boolean Algebra

3. In the middle of 19th century, George Boole developed rules for manipulations of binary variables, known as Boolean Algebra.

4. ~~It~~ It is an algebraic system consisting of the set of elements (0,1), two binary operators called OR and AND and one unary operator called NOT.

5. Boolean algebra differs from both the ordinary algebra and the binary number system.

In Boolean algebra, $A + A = A$ $\quad A \cdot A = A$

In ordinary algebra, $A + A = 2A$ $\quad A \cdot A = A^2$

In Boolean algebra, $1 + 1 = 1$

In binary no. system, $1 + 1 = 10$

In ordinary algebra, $1 + 1 = 2$

6. There is no subtraction or division in Boolean algebra.

7. There is no negative or fractional nos. in " "

8. Multiplication & addition are also only logical.

Axioms and Laws of Boolean Algebra -

These are sets of logical expressions which we accept without proof.

AND operation	OR operation	NOT operation
$0 \cdot 0 = 0$	$0 + 0 = 0$	$\bar{1} = 0$
$0 \cdot 1 = 0$	$0 + 1 = 1$	$\bar{0} = 1$
$1 \cdot 0 = 0$	$1 + 0 = 1$	
$1 \cdot 1 = 1$	$1 + 1 = 1$	

Complementation Laws -

$\bar{0} = 1$

$\bar{1} = 0$

If $A = 0$, then $\bar{A} = 1$

If $A = 1$, then $\bar{A} = 0$

$\bar{\bar{A}} = A$ Double complementation law

AND Laws -

$A \cdot 0 = 0$ (Null Law)

$A \cdot 1 = A$ (Identity Law)

$A \cdot A = A$

$A \cdot \bar{A} = 0$

OR Laws -

$A + 0 = A$ (Null Law)

$A + 1 = 1$ (Identity law)

$A + A = A$

$A + \bar{A} = 1$

Commutative Laws - These laws allow change in position of AND or OR variables.

Law 1: $A + B = B + A$ (Proofs by Truth Table)

Law 2: $A \cdot B = B \cdot A$

Associative Laws - These laws allow grouping of variables.

Law 1: $(A+B)+C = A+(B+C)$

Law 2: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

Distributive Laws - These laws allow factoring or multiplying out of expressions.

Law 1: $A(B+C) = AB + AC$

Law 2: $A + BC = (A+B)(A+C)$

Proof:- RHS $= (A+B)(A+C)$

$= AA + AC + BA + BC$

$= A + AC + AB + BC$ Using commutative law

$= A(1 + C + B) + BC$ Using Identity law

$= A \cdot 1 + BC$ Using Identity law

$= A + BC$

$=$ LHS

Redundant Literal Rule (RLR) -

Law 1: $A + \bar{A}B = A + B$

Proof: $A + \bar{A}B = (A+\bar{A})(A+B)$
$= 1.(A+B)$
$= A+B$

Law 2: $A(\bar{A}+B) = AB$

Proof: $A(\bar{A}+B) = A.\bar{A} + AB$
$= 0 + AB$
$= AB$

Idempotence Laws -

Law 1: $A.A = A$ — If $A=0$, then, $A.A = 0.0 = 0 = A$; If $A=1$, then, $A.A = 1.1 = 1 = A$

Law 2: $A + A = A$ — If $A=0$, then, $0+0 = 0 = A$; If $A=1$, then, $1+1 = 1 = A$

Absorption Laws -

Law 1: $A + A.B = A$ — $A+A.B = A(1+B) = A.1 = A$

Law 2: $A(A+B) = A$ — $A.A + A.B = A + AB = A(1+B) = A.1 = A$

Consensus Theorem - (Included factor Theorem)

Theorem 1: $AB + \bar{A}C + BC = AB + \bar{A}C$

Proof:- $AB + \bar{A}C + BC$
$= AB + \bar{A}C + BC(A+\bar{A})$
$= AB + \bar{A}C + BCA + BC\bar{A}$
$= AB(1+C) + \bar{A}C(1+B)$
$= AB(1) + \bar{A}C(1)$
$= AB + \bar{A}C$
$=$ RHS

Theorem 2: $(A+B)(\bar{A}+C)(B+C) = (A+B)(\bar{A}+C)$

Proof: $LHS = (A+B)(\bar{A}+C)(B+C)$

$= (A\bar{A}+AC+B\bar{A}+BC)(B+C)$

$= (AC+BC+\bar{A}B)(B+C)$

$= ABC+BC+\bar{A}B+AC+BC+\bar{A}BC$

$= AC+BC+\bar{A}B$

$RHS = (A+B)(\bar{A}+C)$

$= \underbrace{A\bar{A}}_{=0}+AC+B\bar{A}+BC$

$= AC+BC+\bar{A}B \quad = LHS$

Transposition Theorem -

Theorem: $AB+\bar{A}C = (A+C)(\bar{A}+B)$

Proof: $RHS = (A+C)(\bar{A}+B)$

$= A\bar{A}+AB+\bar{A}C+BC$

$= 0+\bar{A}C+AB+BC$

$= \bar{A}C+AB+BC(A+\bar{A})$

$= \bar{A}C+AB+ABC+\bar{A}BC$

$= \bar{A}C(1+B)+AB(1+C)$

$= \bar{A}C(1)+AB(1)$

$= AB+\bar{A}C$

$= LHS$

V. Imp.

De Morgan's Theorem -

Law 1: This law states that the complement of a sum of variables is equal to the product of their individual complements.

$$\overline{A+B} = \bar{A}.\bar{B}$$

Law 2: This law states that the complement of the product of variables is equal to the sum of their individual complements.

$$\overline{AB} = \bar{A}+\bar{B}$$

Proof of Law 1 (by Truth Table) :-

A	B	A+B	$\overline{A+B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

=

A	B	$\overline{A}$	$\overline{B}$	$\overline{A}.\overline{B}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

Proof of Law 2

A	B	AB	$\overline{AB}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

=

A	B	$\overline{A}$	$\overline{B}$	$\overline{A}+\overline{B}$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

The transformations $\overline{A+B} = \overline{A}.\overline{B}$

$\overline{AB} = \overline{A}+\overline{B}$

can be extended to complicated expressions by the following ~~four~~ steps -

1. Complement the entire given function.
2. Change all the ANDs to ORs and all the ORs to ANDs.
3. Complement each of the individual variables.
4. Change all 0s to 1s and 1s to 0s.

This procedure is called demorganization or complementation of switching expressions.

$$f(A,B,C,\dots,0,1,+,.)_c = f(\overline{A},\overline{B},\overline{C},\dots,1,0,.,+)$$

Ques.1. Demorganize $f = \overline{(A+\overline{B})(C+\overline{D})}$

$f = \overline{(A+\overline{B})(C+\overline{D})}$

$= (\overline{A}.B)+(\overline{C}.D)$

1. $(A+\overline{B})(C+\overline{D})$
2. $(A.\overline{B})+(C.\overline{D})$
3. $\overline{A}.B+\overline{C}.D$ ← Ans.
4. NA

Ques. 2. Apply Demorgan's theorem to the expression

$$f = \overline{\overline{AB}\,(CD+\overline{E}F)\,(\overline{AB}+\overline{CD})}$$

Break the line, change the sign

$$f = \overline{\overline{AB}} + \overline{(CD+\overline{E}F)} + \overline{(\overline{AB}+\overline{CD})}$$

$$= AB + (\overline{CD}\,.\,\overline{\overline{E}F}) + (\overline{\overline{AB}}\,.\,\overline{\overline{CD}})$$

$$= AB + (\overline{C}+\overline{D})\,.\,(E+\overline{F}) + ABCD$$

Ques. 3. Reduce the expression $f = \overline{\overline{AB}+\overline{A}+AB}$.

$$f = \overline{\overline{AB}+\overline{A}+AB}$$

$$= \overline{\overline{AB}}\,.\,\overline{\overline{A}}\,.\,\overline{AB}$$

$$= AB\,.\,A\,.\,(\overline{A}+\overline{B})$$

$$= AB\,(\overline{AB})$$

$$= 0$$

Duality –

There are two logic systems < +ve logic system, –ve logic system

an OR gate in +ve logic system = an AND gate in –ve logic system

& vice-versa.

Positive & negative logics thus give rise to a basic duality in all Boolean identities.

Given a Boolean identity, we can produce a dual identity by changing all '+' signs to '.' signs, all '.' signs to '+' signs and complementing all 0s and 1s. The variables are not complemented in this process.

$$[f(A, B, C, \ldots, 0, 1, +, .)]_d = f(A, B, C, \ldots, 1, 0, ., +)$$

This characteristic of Boolean Algebra is called Principle of Duality & the expressions satisfying this property is called dual expression.

Given expressions	Duals
1. $\bar{0} = 1$	$\bar{1} = 0$
2. $0 \cdot 1 = 0$	$1 + 0 = 1$
3. $0 \cdot 0 = 0$	$1 + 1 = 1$
4. $1 \cdot 1 = 1$	$0 + 0 = 0$
5. $A \cdot 0 = 0$	$A + 1 = 1$
6. $A \cdot 1 = A$	$A + 0 = A$
7. $A \cdot A = A$	$A + A = A$
8. $A \cdot \bar{A} = 0$	$A + \bar{A} = 1$
9. $A \cdot B = B \cdot A$	$A + B = B + A$
10. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$	$A + (B + C) = (A + B) + C$
11. $A \cdot (B + C) = AB + AC$	$A + BC = (A + B)(A + C)$
12. $A(A + B) = A$	$A + AB = A$
13. $A \cdot (A \cdot B) = A \cdot B$	$A + A + B = A + B$
14. $\overline{AB} = \bar{A} + \bar{B}$	$\overline{A + B} = \bar{A} \cdot \bar{B}$
15. $(A+B)(\bar{A}+C)(B+C) = (A+B)(\bar{A}+C)$	$AB + \bar{A}C + BC = AB + \bar{A}C$

Reducing Boolean Expressions -

Simplification of every boolean expression is necessary because every logic operation in the expression represents a corresponding element of hardware.

Realization of a digital circuit with minimal expression, ∴, results in reduction of cost & complexity and the corresponding increase in reliability.

The procedure to reduce the Boolean expression is as follows :-

1) Multiply all variables necessary to remove parentheses.

2) Look for identical terms. Only one of those terms be retained & all others dropped.

ex:- $AB + AB + AB + AB = AB$

3) Look for a variable & its negation in the same term. This term can be dropped.

ex:- $A \cdot B\bar{B} = A \cdot 0 = 0$

$ABC\bar{C} = AB \cdot 0 = 0$

4.) Look for pairs of terms that are identical except for one variable which may be missing in one of the terms. The larger term can be dropped.

ex: $AB\bar{C}\bar{D} + AB\bar{C} = AB\bar{C}(\bar{D}+1) = AB\bar{C} \cdot 1 = AB\bar{C}$

5.) Look for pairs of terms which have the same variables, with one or more variables complemented.

ex: $AB\bar{C}\bar{D} + AB\bar{C}D = AB\bar{C}(\bar{D}+D) = AB\bar{C}(1) = AB\bar{C}$

ex: 1. Reduce the expression $f = (B+BC)(B+\bar{B}C)(B+D)$

$$f = (B+BC)(B+\bar{B}C)(B+D)$$
$$= (BB + B\bar{B}C + BCB + BC\bar{B}C)(B+D)$$
$$= (B + 0 + BC + 0)(B+D)$$
$$= BB + BD + BBC + BDC$$
$$= B + BD + BC + BDC$$
$$= B(1+D) + BC(1+D)$$
$$= B \cdot 1 + BC \cdot 1$$
$$= B(1+C)$$
$$= B \cdot 1$$
$$= B$$

ex:-2. Show that $A\bar{B}C + B + B\bar{D} + AB\bar{D} + \bar{A}C = B + C$

$$A\bar{B}C + B + B\bar{D} + AB\bar{D} + \bar{A}C = A\bar{B}C + \bar{A}C + B(1 + \bar{D} + A\bar{D})$$
$$= C(\bar{A} + A\bar{B}) + B.1$$
$$= C(\underbrace{\bar{A}+A}_{1})(\bar{A}+\bar{B}) + B$$
$$= C(\bar{A}+\bar{B}) + B$$
$$= C\bar{A} + \underbrace{C\bar{B}+B}$$
$$= C\bar{A} + (B+C)(\underbrace{B+\bar{B}}_{=1})$$
$$= C\bar{A} + B + C$$
$$= C(1+\bar{A}) + B$$
$$= B + C.1 = B + C$$

ex:-3 Simplify the function

$$f(A,B,C) = (A+B)(A+\bar{C}) + \bar{A}\bar{B} + \bar{A}\bar{C}$$
$$= AA + A\bar{C} + AB + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$$
$$= A + A\bar{C} + AB + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$$
$$= A(\underbrace{1+B+\bar{C}}_{=1}) + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$$
$$= A + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$$
$$= \underbrace{A+\bar{A}\bar{B}}_{\text{Applying}} + \underbrace{A+\bar{A}\bar{C}}_{\text{RLR}} + B\bar{C}$$
$$= A + \bar{B} + A + \bar{C} + B\bar{C}$$
$$= A + \bar{C} + \underbrace{\bar{B}+B\bar{C}}_{\text{Applying RLR}}$$
$$= A + \bar{C} + \bar{B} + \bar{C}$$
$$= A + \bar{B} + \bar{C}$$

Note:- OR, AND and NOT (+, ., -) form a functionally complete set in the sense that any function can be realized using these operators in SOP & POS form. Also, further using only NAND op. or NOR op., it is possible to produce all the Boolean operations. Hence, each one of them forms a functionally complete single element set.

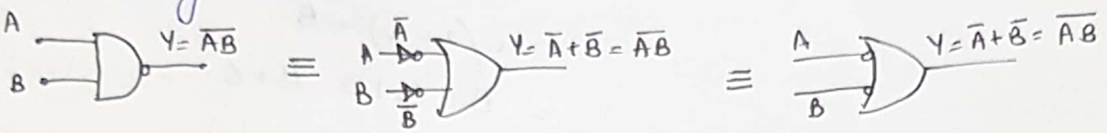
Conversion to NAND-NAND and NOR-NOR Gate n/ws -

1. From De-Morgan's theorem, $\overline{A+B} = \overline{A}.\overline{B}$ and $\overline{AB} = \overline{A} + \overline{B}$

These expressions can also be written as :- $A+B = \overline{\overline{A}.\overline{B}}$ and $AB = \overline{\overline{A}+\overline{B}}$

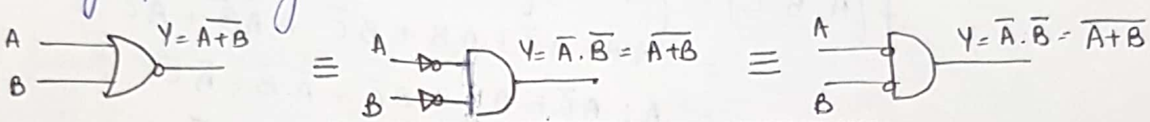
⇒ OR gate is equivalent to a NAND gate with bubbles at its i/ps & an AND gate is equivalent to a NOR gate with bubbles at its i/ps.

2. Also, for NAND gate, $\overline{AB} = \overline{A} + \overline{B}$



$\boxed{\text{NAND gate} \equiv \text{Bubbled OR gate}}$

Similarly, for NOR gate, $\overline{A+B} = \overline{A}.\overline{B}$



$\boxed{\text{NOR gate} = \text{Bubbled AND gate}}$

Converting AND/OR/INVERT (AOI) Logic to NAND/NOR Logic -

The procedure to convert AOI logic into either NAND logic or NOR logic is given as follows:-

i) Draw the circuit in AOI logic.

ii) If NAND h/w is chosen, add a circle at the o/p of each AND gate and at the i/ps to all the OR gates.

iii) If NOR h/w is chosen, add a circle at the o/p of each OR gate and at the i/p to all the AND gates.

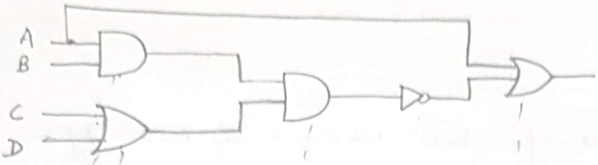
iv) Add or subtract an inverter on each line that received a circle in steps 2 or 3 so that the polarity of signals on those lines remains unchanged from that of the original diagram.

v) Replace bubbled OR by NAND & bubbled AND by NOR.

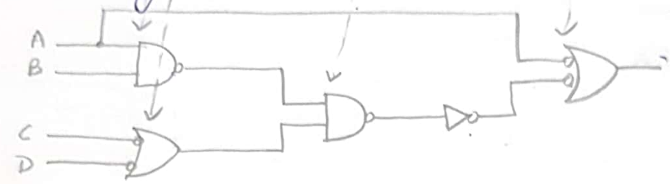
vi) Eliminate double inversions.

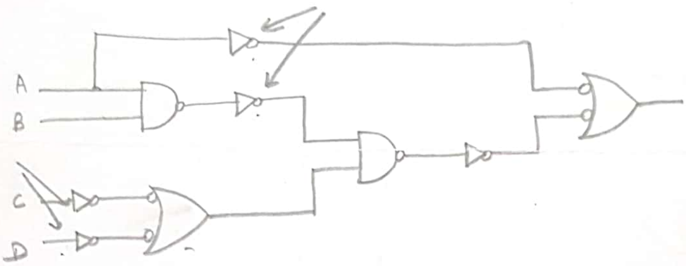
Example :- $\boxed{f = A + \overline{AB(C+D)}}$

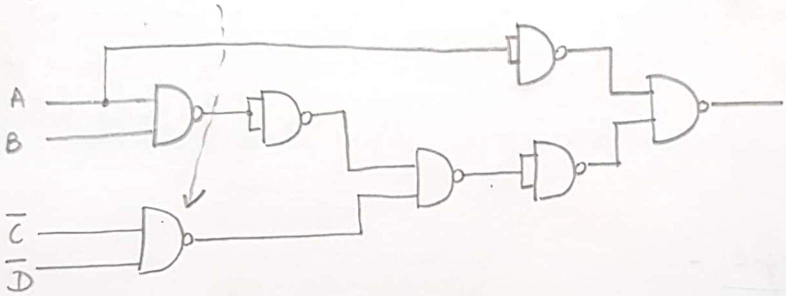
Convert the following AOI logic circuit to -
a) NAND Logic
b) NOR Logic



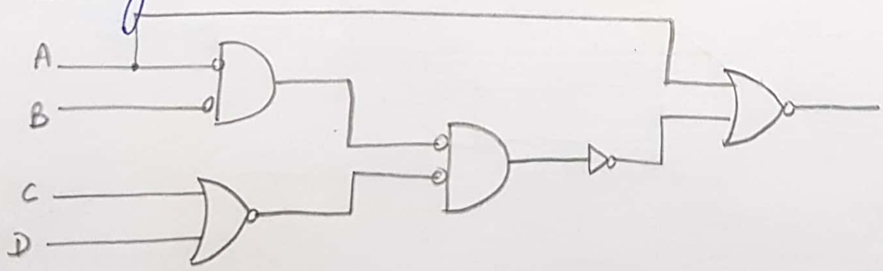
Sol.^{n} - NAND Logic -

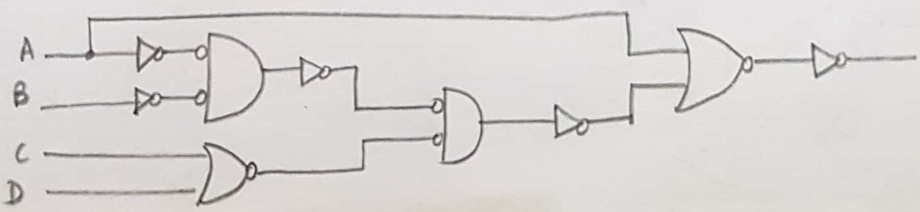


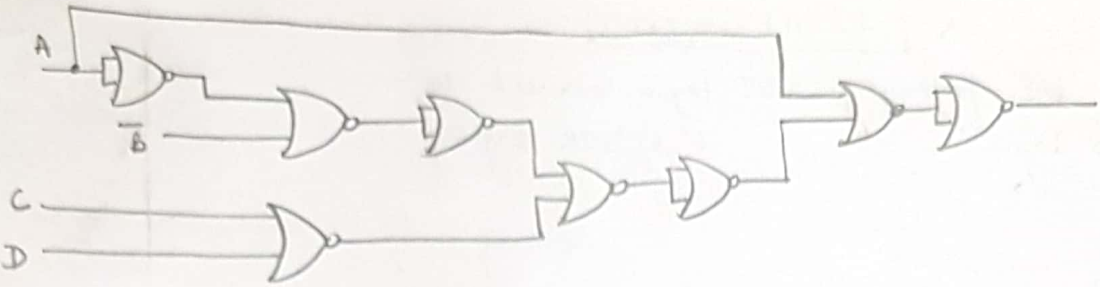




NOR Logic -

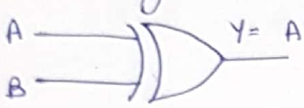


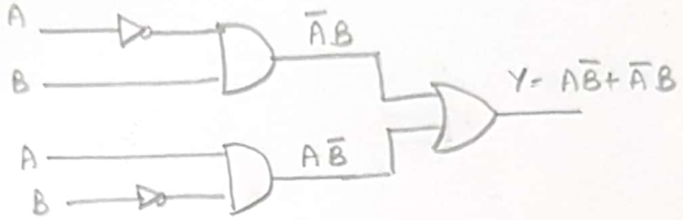




Imp. Ques. Realize the X-OR function using – a) AOI Logic b) NAND Logic and c) NOR Logic.

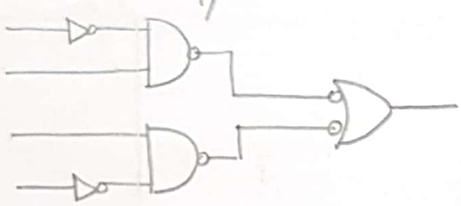
Sol.n Using AOI Logic –





Using NAND Logic –

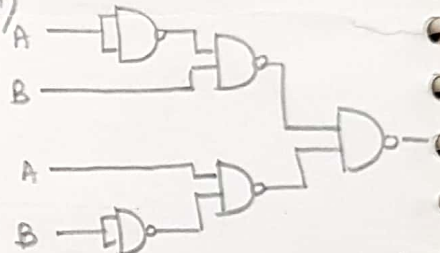
i)



ii)

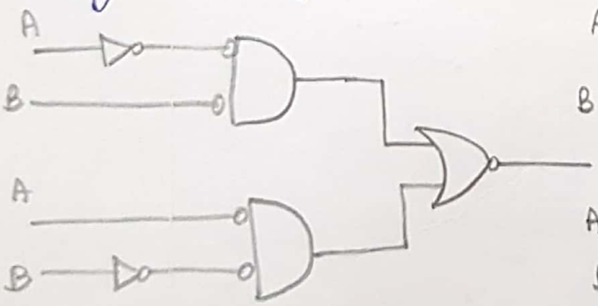
circuit is same. (No inverter is required)

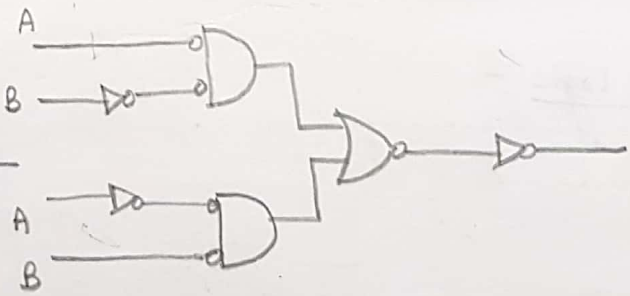
iii)

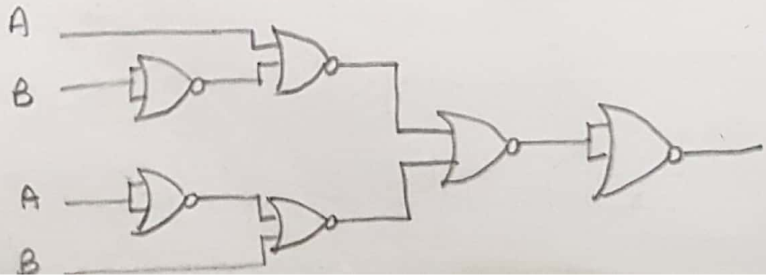


(5 2-i/p NAND gates)

Using NOR Logic –





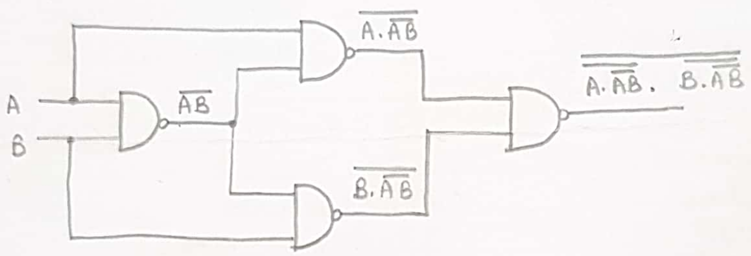


Ques. Realize the X-OR function using 4 2i/p NAND gates only and 4 2-i/p NOR gates only.

Or, Derive the Boolean expression for a 2-i/p EX-OR gate to realize with 2-i/p NAND gates & NOR-gates without using complemented variables and draw the circuit.

Sol.^{n} With 2-i/p NAND gates w/t using complemented variables :-

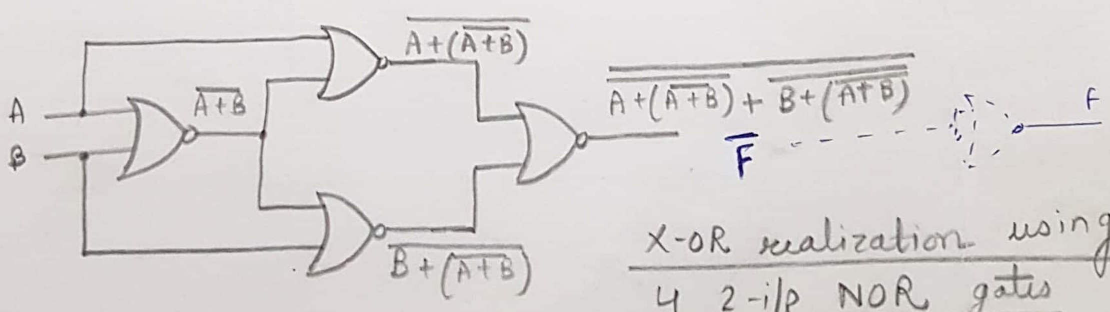
$$f = A\bar{B} + \bar{A}B$$
$$= A\bar{B} + A\bar{A} + \bar{A}B + B\bar{B}$$
$$= A(\bar{A}+\bar{B}) + B(\bar{A}+\bar{B})$$
$$= A\cdot(\overline{AB}) + B(\overline{AB})$$
$$= \overline{\overline{A.\overline{AB}}\,.\,\overline{B.\overline{AB}}}$$



X-OR realization using 4 2-i/p NAND gates

with 2-i/p NOR gates w/t using complemented variables :-

$$\bar{F} = \overline{A\bar{B}+\bar{A}B}$$
$$= \overline{A\bar{B} + B\bar{B} + \bar{A}B + A\bar{A}}$$
$$= \overline{\bar{B}(A+B) + \bar{A}(A+B)}$$
$$= \overline{\overline{\overline{\bar{B}(A+B)}} + \overline{\overline{\bar{A}(A+B)}}}$$
$$= \overline{\overline{B+\overline{(A+B)}} + \overline{A+\overline{(A+B)}}}$$



X-OR realization using 4 2-i/p NOR gates

Date

Ex. Reduce the Expression

(1) $A[B + \bar{C}(\overline{AB + A\bar{C}})]$

$A[B + \bar{C}(\overline{AB}\cdot\overline{A\bar{C}})]$ Demorganize

$A[B + \bar{C}(\bar{A}+\bar{B})(\bar{A}+C)]$

$A[B + \bar{C}(\bar{A}\bar{A} + \bar{A}C + \bar{A}\bar{B} + \bar{B}C)]$

$A[B + \bar{C}\bar{A} + \underbrace{\bar{A}\bar{C}C}_{=0} + \bar{A}\bar{B}\bar{C} + \underbrace{\bar{B}C\bar{C}}_{=0}]$

$AB + \underbrace{A\bar{C}\bar{A}}_{=0} + \underbrace{A\cdot\bar{A}\bar{B}\bar{C}}_{=0}$

$= AB$

(4) $[A\bar{B}(C+BD) + \bar{A}\bar{B}]C$

$[A\bar{B}C + \underbrace{A\bar{B}BD}_{0} + \bar{A}\bar{B}]C$

$A\bar{B}C + \bar{A}\bar{B}C$

$\bar{B}C(A+\bar{A}) = \bar{B}C$

(2) $(\overline{A + \overline{BC}})(A\bar{B} + ABC)$ Demorganize

$(\bar{A}\overline{\overline{BC}})(A\bar{B} + ABC)$

$(\bar{A}BC)(A\bar{B} + ABC)$

$\bar{A}BCA\bar{B} + \bar{A}BCABC$

$= 0$

(5) $(A+B)[\overline{\bar{A}(\bar{B}+\bar{C})}] + \bar{A}\bar{B} + \bar{A}\bar{C}$

$(A+B)[\bar{\bar{A}} + \overline{(\bar{B}+\bar{C})}] + \bar{A}\bar{B} + \bar{A}\bar{C}$

$(A+B)(A+BC) + \bar{A}\bar{B} + \bar{A}\bar{C}$

$A + AB(C+1) + BC + \bar{A}\bar{B} + \bar{A}\bar{C}$

$A(1+B) + BC + \bar{A}\bar{B} + \bar{A}\bar{C}$

$A + BC + \bar{A}\bar{B} + \bar{A}\bar{C}$ Absorption & RLR

$A + BC + \bar{B} + \bar{C}$

$A + C + \bar{B} + \bar{C}$

$A + \bar{B} + 1 = 1$

(3) $(A+B)(A+\bar{C}) + \bar{A}\bar{B} + \bar{A}\bar{C}$

$= AA + A\bar{C} + AB + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$

$= A(\underbrace{1+\bar{C}}_{=1}) + AB + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$

$= A(1+B) + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$

$= A + B\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{C}$ RLR RLR

$= A + \bar{B} + \bar{C}A + B\bar{C}$

$= A + \bar{B} + \bar{C} + B\bar{C}$ RLR

$= A + \bar{C} + \bar{B} + \bar{C}$

$= A + \bar{B} + \bar{C}$

(7) $\overline{\overline{A\bar{B} + ABC} + A(B + \overline{AB})}$

$\overline{\overline{A(\bar{B}+BC)} + A(B + \bar{A}\bar{B})}$

$\overline{\overline{A(\bar{B}+C)} + AB + A\bar{A}\bar{B}}$

$\overline{\overline{A\bar{B} + AC} + AB + AA}$

$\overline{\overline{A\bar{B} + AC} + A(1+B)}$

$\overline{\overline{A\bar{B} + AC} + A}$

$\overline{(\bar{A}+B)(\bar{A}+\bar{C}) + A}$

$\overline{\bar{A}\bar{A} + \bar{A}\bar{C} + B\bar{A} + B\bar{C} + A}$

$\overline{\bar{A} + \bar{A}\bar{C} + B\bar{A} + B\bar{C} + A}$

$\overline{\bar{A}(1+\bar{C}) + \bar{A}B + B\bar{C} + A}$

$\overline{\bar{A} + \bar{A}B + B\bar{C} + A}$

$\overline{\bar{A}(1+B) + B\bar{C} + A}$

$\overline{\bar{A} + B\bar{C} + A} = \overline{1 + B\bar{C}}$

$\overline{1 + \bar{B} + C}$ $\quad \overline{(1+C) + \bar{B}}$

$\overline{(1+\bar{B})} = \bar{1} = 0$

(6) $\bar{Y}\bar{Z} + \bar{W}\bar{X}\bar{Z} + \bar{W}XY\bar{Z} + WY\bar{Z} = \bar{Z}$

$\bar{Y}\bar{Z} + \bar{W}\bar{X}\bar{Z}(1+Y) + \bar{W}XY\bar{Z} + WY\bar{Z}$

$\bar{Y}\bar{Z} + \bar{W}\bar{X}\bar{Z} + \bar{W}\bar{X}\bar{Z}Y + \bar{W}XY\bar{Z} + WY\bar{Z}$

$\bar{Y}\bar{Z} + \bar{W}\bar{X}\bar{Z} + \bar{W}Y\bar{Z}(X+\bar{X}) + WY\bar{Z}$

$\bar{Y}\bar{Z} + \bar{W}\bar{X}\bar{Z} + Y\bar{Z}(\bar{W}+W)$

$\bar{Z}(Y+\bar{Y}) + \bar{W}\bar{X}\bar{Z}$

$\bar{Z}(1+\bar{W}\bar{X})$

$\bar{Z}(1) = \bar{Z}$