Numerical Integration – The process of evaluating a definite Integral from a set of tabulated values of the integrand $f(x)$ is called numerical Integration. This process when applied to a function of a single variable, is known as quadrature.

The problem of numerical Integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. First we study the Newton Cotes quadrature formula and then use it to to study ① Trapezoidal rule ② Simpson's one third rule (iii) Simpson's three eighth rule.

Newton Cotes Quadrature formula –

$$\text{Let} \quad I = \int_a^b f(x)\,dx$$

where $f(x)$ takes the values $y_0, y_1, y_2 \ldots y_n$ for $x = x_0, x_1, \ldots x_n$.

Let us divide the interval (a,b) into n subintervals of width h so that $x_0 = a$, $x_1 = a+h$, $x_2 = x_0 + 2h$, ... $x_0 + nh = x_n$

Then

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

Putting $x = x_0 + ph$ & $dx = h\,dp$.

$$= h\int_0^n f(x_0 + rh)\,dr$$

$(r^2 - r)$ $\qquad$ $r(r^2-3r+2) = r^3-3r^2+2r$

$$= h\int_0^n \Big[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \dots \Big] dr$$

(by Newton's forward Interpolation).

Integrating term by term, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\Big[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \dots \Big] \quad \text{--- (1)}$$

This is known as Newton Cotes quadrature formula. From this general formula we deduce the following quadrature rules by takes different values of n.

(1) Trapezoidal Rule –

Putting $n=1$ in (1) & taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line i.e a polynomial of first order so that the differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$ $\qquad \Delta y_0 = y_1 - y_0$

Similarly

$$\int_{x_0+h}^{x_0+2h} f(x)dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

$\vdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}(y_{n-1} + y_n).$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots+y_{n-1})\right] \quad — ②$$

This is known as Trapezoidal Rule.

② Simpson's One third Rule–

Putting $n=2$ in ① above & taking the curve through (x_0, y_0), (x_1, y_1) & (x_2, y_2) as a parabola i.e. a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

$$= 2h\left(y_0 + y_1 - y_0 + \frac{1}{6}(\Delta y_1 - \Delta y_0)\right)$$

$$= 2h\left(y_0 + y_1 - y_0 + \frac{1}{6}(y_2 - y_1 - y_1 + y_0)\right) = 2h\left[\frac{y_0}{6} + \frac{2}{3}y_1 + \frac{y_2}{6}\right]$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$\vdots$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \; ; \; n \text{ being even.}$$

Adding all these integrals, we have when n is even

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\dots+y_{n-1}) + 2(y_2+y_4+\dots+y_{n-2})\right] \quad — ③$$

This is known as Simpson's One-third Rule or simply Simpson's rule & is most commonly used. While applying this rule the given interval must be divided into even number of equal sub-intervals, since we find the area of two stripes at a time.

3. Simpson's three Eighth rule –

Putting $n=3$ in ① & taking the curve through (x_0, y_0) (x_1, y_1), (x_2, y_2) & (x_3, y_3) as a polynomial of third order so that differences above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3}{8}h\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right] \text{ \& so on.}$$

Adding all such expressions from x_0 to x_0+nh, where n is a multiple of 3, we obtain.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\dots+y_{n-1}) + 2(y_3+y_6+\dots+y_{n-3})\right] \quad \text{— ④}$$

which is known as Simpson's three Eighth rule. While applying ④ the number of sub-intervals should be taken as a multiple of 3.

e.g.: Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using ① Trapezoidal rule taking $h=1/4$.

② Simpson's One third rule taking $h=1/4$.

③ Simpson's three eighth rule taking $h=1/6$

Solu: ① $h = 1/4$ ∴ $x_0 = 0$ $x_1 = 0.25$ $x_2 = 0.50$ $x_3 = 0.75$;
$x_4 = 1.$

x	0	0.25	0.50	0.75	1
$f(x)$	1	0.941	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4.

By Trapezoidal Rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{1}{4}\times\frac{1}{2}\left[(1+0.5) + 2(0.941+0.8+0.64)\right]$$

$$= \frac{1}{8}\left[6.262\right] = 0.7827.$$

② $h = 1/4$ & by simpson's 1/3 rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2y_2\right]$$

$$= \frac{1}{4}\times\frac{1}{3}\left[(1+0.5) + 4(0.941+0.64) + 2\times 0.8\right].$$

$$= \frac{1}{12}\left[9.424\right] = 0.7853$$

③ By simpson's 3/8 rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_4) + 3(y_1+y_2) + 2(y_3)\right].$$

$$= \frac{3}{8}\times\frac{1}{4}\left[(1+0.5) + 3(0.941+0.8) + 2\times 0.64\right].$$

$$= 0.7502$$

Errors in quadrature formula—

If y_p is the polynomial representing the function $y=f(x)$ in the interval $[a,b]$ then error in the quadrature formula is given by

$$E = \int_a^b f(x)\,dx - \int_a^b y_p\,dx$$

① Error in Trapezoidal rule—

Expanding $y=f(x)$ in the neighbourhood of $x=x_0$ by Taylor's series, we get

$$y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \dots \quad ①$$

$$\therefore \int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_0+h}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' - \dots\right]dx$$

$$= hy_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \dots \quad ②$$

Also A_1 = area of first trapezium in the interval $[x_0, x_1]$

$$= \frac{1}{2}h(y_0+y_1) \quad \text{(by Trapezoidal rule)} \quad ③$$

Putting $x = x_0+h$ & $y=y_1$ in ① we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \dots$$

$$\therefore A_1 = \frac{1}{2}h\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' \dots\right] \quad \text{by putting in ③}$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2.2!}y_0'' + \dots \quad ④$$

$\therefore$ Error in the interval $[x_0, x_1]$

$$= \int_{x_0}^{x_1} y\,dx - A_1 \quad [\text{by ② − ④}]$$

$$= \left(hy_0 + \frac{h^2}{2!}y_0' + \dots\right) - \left(hy_0 + \frac{h^2}{2!}y_0'' \dots\right)]$$

$$= \left(\frac{1}{3!} - \frac{1}{2.2!}\right) h^3 y_0'' + \dots = -\frac{h^3}{12} y_0'' + \dots$$

i.e Principal part of error in $[x_0, x_1] = -\frac{h^3}{12} y_0''$

Similarly " " " " " $[x_1, x_2] = -\frac{h^3}{12} y_1''$ & so on

Hence total error is

$$E = -\frac{h^3}{12}\left[y_0'' + y_1'' + \dots + y_{n-1}''\right]$$

Assuming $y''(x)$ is the largest of n quantities $[y_0'', y_1'' \dots y_{n-1}'']$ we obtain

$$E < -\frac{nh^3}{12} y''(x) = -\frac{(b-a)h^2}{12} y''(x) \quad -⑤$$

$\because (b-a = nh)$

$\therefore$ error in trapezoidal rule is of order h^2.

② Error in Simpson 1/3 rule —

Expanding $y = f(x)$ around $x = x_0$ by Taylor's series we get ①

$\therefore$ over the first double strip, we get

$$\int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_0+2h} \left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \dots\right] dx$$

$$= 2hy_0 + \frac{4h^2}{2!} y_0' + \frac{8h^3}{3!} y_0'' + \frac{16h^4}{4!} y_0''' + \dots \quad -⑥$$

Also area of first double strip by Simpson's 1/3 rule

$$A_1 = \frac{1}{3} h\left[y_0 + 4y_1 + y_2\right] \quad -⑦$$

Putting $x = x_0 + h$ & $y = y_1$ in ①, we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

Putting $x = x_0 + 2h$ & $y = y_2$ in ①, we get

$$y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!}y_0'' + \frac{8h^3}{3!}y_0''' + \dots$$

Substituting these values of y_1 & y_2 in ⑦ we get

$$A_1 = \frac{1}{3}h\Big[y_0 + 4\Big(y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots\Big) +$$

$$\Big(y_0 + 2hy_0' + \frac{4h^2}{2!}y_0'' + \frac{8h^3}{3!}y_0''' - \dots\Big)\Big]$$

$$= 2hy_0 + 2h^2y_0' + \frac{4h^3}{3}y_0'' + \frac{2h^2}{3}y_0''' + \frac{5h^5}{18}y_0^{iv} \dots \quad ⑧$$

∴ Error in interval $[x_0, x_2]$

$$= \int_{x_0}^{x_2} y\,dx - A_1 = \Big[2hy_0 + \frac{4h^2}{2!}y_0' + \frac{8h^3}{3!}y_0'' + \frac{16h^4}{4!}y_0''' \dots\Big]$$

$$- \Big[2hy_0 + 2h^2y_0' + \frac{4h^3}{3}y_0'' \dots\Big]$$

(by ⑥ - ⑧)

$$= \Big(\frac{4}{15} - \frac{5}{18}\Big)h^5 y_0^{iv} + \dots$$

∴ Principal part of error in $[x_0, x_2] = -\frac{h^5}{90}y_0^{iv}$

Similarly " " " " " $[x_2, x_4] = -\frac{h^5}{90}y_2^{iv}$ & so on

Hence total error

$$E = -\frac{h^5}{90}\Big[y_0^{iv} + y_2^{iv} + \dots + y_{2(n-1)}^{iv}\Big]$$

Assuming $y^{iv}(x)$ as the largest of $y_0^{iv}, y_2^{iv}, y_4^{iv} \dots$, we get

$$E < -\frac{nh^5}{90}y_0^{iv}(x) = -\frac{(b-a)h^4}{180}y^{iv}(x)$$

So error in Simpson's ⅓ rule is of order h^4.

③ Error in Simpson's $\frac{3}{8}$th rule — The principal part of error in the interval $[x_0, x_3] = -\frac{3h^5}{80} y_0^{iv}$

Gaussian quadrature formula —

In Newton cotes quadrature formula, for the evaluation of $\int_0^b f(x)dx$, we require the values of the function at equally spaced points of the interval. Gauss derived a formula which uses the same no. of functional values but with different spacing & yields better accuracy.

Gauss formula is expressed as

$$\int_{-1}^{1} f(x)dx = \omega_1 f(x_1) + \omega_2 f(x_2) + \dots + \omega_n f(x_n) \quad (1)$$
$$= \sum_{i=1}^{n} \omega_i f(x_i)$$

where ω_i & x_i are called the weights & abscissae respectively. The abscissae and weights are symmetrical with respect to the middle point of the interval. There being $2n$ unknowns in (1), $2n$ relations between them are necessary so that the formula is exact for all polynomials of degree not exceeding $(2n-1)$.

Thus, we consider

$$f(x) = C_0 + C_1 x + C_2 x^2 + \dots + C_{2n-1} x^{2n-1} \quad (2)$$

Then (1) gives

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (C_0 + C_1 x + C_2 x^2 + \dots + C_{2n-1} x^{2n-1}) dx$$

$$= 2C_0 + \frac{2C_2}{3} + \frac{2C_4}{5} + \dots, \quad (3)$$

Putting $x = x_i$ in (2), we get

$$f(x_i) = C_0 + C_1 x_i + C_2 x_i^2 + C_3 x_i^3 + \cdots + C_{2n-1} x_i^{2n-1}$$

Substituting these values on RHS of (1) we get

$$\int_{-1}^{1} f(x)dx = w_1 \left[C_0 + C_1 x_1 + C_2 x_1^2 + \cdots + C_{2n-1} x_1^{2n-1}\right] +$$
$$w_2 \left[C_0 + C_1 x_2 + C_2 x_2^2 + \cdots + C_{2n-1} x_2^{2n-1}\right] +$$
$$\vdots$$
$$+ w_n \left[C_0 + C_1 x_n + C_2 x_n^2 + \cdots + C_{2n-1} x_n^{2n-1}\right]$$

$$= C_0(w_1 + w_2 + \cdots + w_n) + C_1(w_1 x_1 + w_2 x_2 \cdots + w_n x_n) + C_2(w_1 x_1^2 + w_2 x_2^2 + \cdots + w_n x_n^2) + \cdots + C_n(w_1 x_1^{2n-1} + w_2 x_2^{2n-1} \cdots + w_n x_n^{2n-1}) \quad \text{--- (4)}$$

Since (3) & (4) are same for all values of C $\therefore$ comparing the coefficients of C_i, we obtain $2n$ eq^{n}s in $2n$ unknowns w_i & x_i

$$\left.\begin{aligned} & w_1 + w_2 + \cdots + w_n = 2 \\ & w_1 x_1 + w_2 x_2 + \cdots + w_n x_n = 0 \\ & w_1 x_1^2 + w_2 x_2^2 + \cdots + w_n x_n^2 = \frac{2}{3} \\ & \vdots \\ & w_1 x_1^{2n-1} + w_2 x_2^{2n-1} + \cdots + w_n x_n^{2n-1} = 0 \end{aligned}\right] \quad (5)$$

Solution of these eq^{n}s is out of scope. Howeree, the values x_i comes out to be zeros of $(n+1)$th Legendre polynomial.

Gauss formula for $n=2$ is

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

Then (5) becomes

$$w_1 + w_2 = 2$$
$$w_1 x_1 + w_2 x_2 = 0$$
$$w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}$$
$$w_1 x_1^3 + w_2 x_2^3 = 0$$

Solving, we get

$$w_1 = w_2 = 1, \quad x_1 = \frac{-1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}}$$

∴ Thus the gauss formula for $n=2$ is

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

which gives the correct value of Integral of $f(x)$ in $(-1,1)$.

Note: The abscissae x_i & weights w_i in (1) are tabulated for different values of n.

n	x_i	w_i
2	−0.57735	1
	0.57735	1
3	−0.77460	0.5555
	0.0000	0.8888
	0.77460	0.5555
4	−0.86114	0.34785
	−0.33998	0.65214
	0.33998	0.65214
	0.86114	0.34785

& ctd.

Note: Gauss formula imposes a restriction on the limits of integration to be from -1 to 1. In general, the limits of integration $\int_a^b f(x)\,dx$ are changed to -1 to 1 by means of transformation

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using gauss formula for $n=2$ & $n=3$

Solu: Gauss formula for $n=2$ is

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = \omega_1 f(x_1) + \omega_2 f(x_2)$$

where $f(x_1) = f(-1/\sqrt{3})$; $\omega_1 = 1$

$f(x_2) = f(1/\sqrt{3})$; $\omega_2 = 1$

$$I = 1.f\left(\frac{-1}{\sqrt{3}}\right) + 1.f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{1+(-1/\sqrt{3})^2} + \frac{1}{1+(1/\sqrt{3})^2} = \frac{3}{4} + \frac{3}{4} = 1.5$$

Gauss formula for $n=3$

$$I = \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3)$$

$\omega_1 = 0.55555$; $\omega_2 = 0.88889$

$\omega_3 = 0.55555$

$f(x_1) = \frac{1}{1+(-0.77460)^2}$ $\qquad f(x_2) = \frac{1}{1+(0.0)^2}$

$f(x_3) = \frac{1}{1+(0.77460)^2}$

$$I = 0.55555 \times \frac{1}{1+(-0.77460)^2} + 0.88889 \times \frac{1}{1+(0.0)^2} + 0.55555 \times \frac{1}{1+(0.7746)^2}$$

$$= 1.5833$$

Evaluate $\int_0^1 \frac{dx}{1+x}$ using $n=3$

Solu: We first change limits $(0,1)$ to $(-1,1)$

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

$$= \frac{1}{2}(1-0)u + \frac{1}{2}(1+0) = \frac{u}{2} + \frac{1}{2} = \frac{u+1}{2}$$

$$\therefore\ I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{\cancel{2}}{(u+3)} \frac{du}{\cancel{2}} = \int_{-1}^{1} \frac{du}{u+3}$$

$$= \omega_1 f(u_1) + \omega_2 f(u_2) + \omega_3 f(u_3)$$

$\omega_1 = 0.55555$ $\quad f(-0.77460) = \frac{1}{(-0.77460)+3}$

$\omega_2 = 0.88889$ $\quad f(0) = \frac{1}{0+3}$

$\omega_3 = 0.55555$ $\quad f(0.77460) = \frac{1}{0.77460+3}$

$I = 0.6931$

Romberg's Algorithm

Romberg's Algorithm provides a simple modification to the quadrature formulae for finding their bettee approximation. Let us here improve upon the value of integral

$$I = \int_a^b f(x)dx$$

by Trapezoidal rule. If I_1 & I_2 be the values of I with subintervals of width h_1, h_2 & E_1, E_2 be their corresponding errors resp., then

$$E_1 = -\frac{(b-a)h_1^2}{12} y''(x)$$

$$E_2 = -\frac{(b-a)^2 h_2^2}{12} y''(\bar{x})$$

Since $y''(\bar{x})$ is also the largest value of $y''(x)$, we can reasonably assume that $y''(x)$ & $y''(\bar{x})$ are very nearly equal.

$$\frac{E_1}{E_2} = \frac{h_1^2}{h_2^2} \quad \text{or} \quad \frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^2 - h_1^2} \qquad -①$$

Now since $I = I_1 + E_1 = I_2 + E_2$

$$\therefore \quad E_2 - E_1 = I_1 - I_2 \qquad -②$$

from ① & ②, we have

$$E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

Hence $I = I_1 + E_1 = I_1 + \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$

$$\therefore I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \qquad -③$$

which is better approximation of I

To evaluate I systematically, we take $h_1 = h$ & $h_2 = \frac{1}{2}h$

so ③ gives

$$I = \frac{I_1\left(\frac{h}{2}\right)^2 - I_2(h)^2}{\left(\frac{h}{2}\right)^2 - h^2} = \frac{4I_2 - I_1}{3}$$

$$I(h, h/2) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] \qquad -④$$

Now we use the trapezoidal rule several times successively halving h & apply ④ to each pair of values as per the following scheme

$$
\begin{array}{llll}
I(h) & & & \\
 & I(h, h/2) & & \\
I(h/2) & & I\left(h, \frac{h}{2}, \frac{h}{4}\right) & \\
 & I(h/2, h/4) & & I\left(h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right) \\
I(h/4) & & I\left(\frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right) & \\
 & I(h/4, h/8) & & \\
I(h/8) & & &
\end{array}
$$

The computation is continued till successive values are close to each other. An obvious advantage of this method is that the accuracy of the computed value is known at each step.