$= 22.84057797.$

7.7 HERMITE INTERPOLATION FORMULA

The Hermite interpolating polynomial interpolates not only the function $f(x)$ but also its (certain order) derivatives at a given set of tabular points. The simple interpolating conditions are given in (1). We now give an explicit expression for the interpolating polynomial satisfying which (1), that is

$$\left.\begin{aligned} H(x_i) &= f(x_i) \\ H'(x_i)) &= f'(x_i),\ i = 0,1,.....,n \end{aligned}\right\} \quad ...(i)$$

Since there are $2n + 2$ conditions to be satisfied, $H(x)$ must be a polynomial of degree $\leq 2n + 1$. The required polynomial may be written as

$$H(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f'(x_i) \quad ...(ii)$$

where $A_i(x)$ and $B_i(x)$ are polynomials of degree $\leq 2n + 1$ and satisfy

$$\left.\begin{aligned} &(i)\ A_i(x_j) = \begin{vmatrix} 0, & i \neq j \\ 1, & i = j \end{vmatrix} \\ &(ii)\ A'\ i(x_j) = 0 \text{ for all } i \text{ and } j \\ &(iii)\ B_i(x_j) = 0 \text{ for all i and j} \\ &(iv)\ B'_i\ (x_j) = \begin{vmatrix} 0, & i \neq j \\ 1, & i = j \end{vmatrix} \end{aligned}\right\} \quad ...(iii)$$

Using the Lagrange fundamental polynomial $li(x)$, we write

$$A_i(x) = \gamma_i(x)\ l_i^2(x)$$

Substituting ...

$A_i(x) = [\ldots$

$B_i(x) = (x - x_i)\ l_i\ldots$

Now substituting $A_i(x)$ and $B_i(x)$ in ...

$H(x) = \sum_{i=0}^{n} [1 - 2(\ldots$

which is known as the Hermite I...

Working Rules For Solving Proble...

Step I. From the given data, ide... $f'(x_1)$, ... etc.

Step II. Write the Hermite inter...

$H(f : x) = \sum_{i=0}^{n} A_i(x\ldots$

where $A_i(x) = [1 - 2(\ldots$

Step III. Compute $l_i(x)$ and l'_i (...

Step IV. Substituting the values ...

Step V. Estimate the value(s) o... interpolating polynomial obtained i...

$$B_i(x) = \delta_i(x)\ l_i^2(x). \qquad ...(iv)$$

where $$l_i(x) = \frac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$$

Since $l_i^2(x)$ is a polynomial of degree $2n$, $\gamma_i(x)$, and $\delta_i(x)$ must be linear polynomials.

Let $$\left.\begin{aligned} \gamma_i(x) &= a_i x + b_i \\ \delta_i(x) &= c_i\ (x) + d_i \end{aligned}\right\} \qquad ...(v)$$

where a_i, b_i, and d_i are the constants to be determined.

Using the conditions (3) and (4), we obtain

$$\left.\begin{aligned} a_i x + b_i = 1, \quad & c_i x + d_i = 0 \\ \text{and} \quad a_i + 2l'_i(x_i) = 0 \quad & c_i = 1 \end{aligned}\right\} \qquad ...(vi)$$

On solving these equations, we have

$$a_i x = -2l'_i\ (x_i),\ b_i = 1 + 2x_i l'_i\ (x_i)$$

$$c_i = 1 \text{ and } d_i = -x_i,$$

Putting these values in (v), we get

$$\gamma_i\ (x) = -2xl'_i\ (x_i) + 1 + 2x_i l'_i\ (x_i)$$

$$= 1 - 2(x - x_i)\ l'_i\ (x_i)$$

$$\delta_i(x) = (x - x_i)$$

Substituting these values in (iv), we get

$$A_i(x) = [l - 2(x - x_i)l'_i(x_i)](l^2_i(x)$$

$$B_i(x) = (x - x_i)\ l_i^2(x)$$

Now substituting $A_i(x)$ and $B_i(x)$ in (ii), we get

$$H(x) = \sum_{i=0}^{n}[1-2(x-x_i)l'_i(x_i)]l_i^2(x)f(x_i) + \sum_{i=0}^{n}(x-x_i)l_i^2(x)f'(x_i) \qquad ...(vii)$$

which is known as the **Hermite Interpolating polynomial.**

Working Rules For Solving Problems

Step I. From the given data, identify the values of n, x_0, x_1, ..., $f(x_0)$, $f\ (x_1)$, ... and $f'(x_1)$, ... etc.

Step II. Write the Hermite interpolating polynomial as

$$H(f : x) = \sum_{i=0}^{n} A_i(x)f(x_i) + \sum_{i=0}^{n} B_i(x)f'(x_i)$$

where $A_i(x) = [1 - 2(x - x_i)\ l'_i\ (x_i)]\ l^2_i\ (x)$ and $B_i(x) = (x - x)\ l^i_2(x)$.

Step III. Compute $l_i(x)$ and $l'_i\ (x)$. Use these to find $A_i(x)\ B_i(x)$.

Step IV. Substituting the values obtained in step III into the formula given in step II.

Step V. Estimate the value(s) of $f(x)$ if required in the question using the Hermite interpolating polynomial obtained in step IV.

SOLVED EXAMPLES

Example 7.44. For the following data

x	$f(x)$	$f'(x)$
0.5	4	–16
1	1	–2

find the Hermite interpolating polynomial, fitting the data.

Solution: The Hermite interpolating polynomial is given by

$$H(f\ ;\ x) = \sum_{i=0}^{1} A_i(x) f(x_i) + \sum_{i=0}^{1} B_i(x) f'(x_i)$$

where $A_i(x) = [1 - 2(x - x_i)\ l_i'^2(x)$ and $B_i(x) = (x - x_i)\ l_i^2(x)$

Now $A_0(x) = [1 - 2(x - 0.5)\ l_0'(0.5)] l_0^2\ (x)$.

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-1}{(0.5-1)} = -2(x-1),\ l'_0 = -2$$

$$A_1(x) = [1 - 2(x - 1)\ l_1'(x)]\ l_1^2(x),\ l_1(x)$$

$$= \frac{x - x_1}{x_1 - x_0} = \frac{x - 0.5}{1 - 0.5} = 2x - 1,\ l'_1 = 2$$

$$B_0(x) = (x - 0.5)\ l_0^2(x),\ B_1(x) = (x - 1)\ l_1^2(x)$$

∴ The Hemite polynomial is

$$H(f{:}\ x) = 4A_0 + A_1 - 16B0 - 2B_1$$

Now
$$A_0 = \left[1 - 2\left(x - \frac{1}{2}\right)(-2)\right][-2(x-1)]^2$$

$$= [4x - 1] \times [2(x - 1)]^2 = 4(4x - 1)\ (x^2 - 2x + 1)$$

$$= 4(x^2 - 9x^2 + 6x - 1)$$

$$A_i = [1 - 2(x - 1)\ 2](2x - 1)^2$$

$$= (5 - 4x)\ (2x - 1)^2 = -16x^3 + 36x^2 - 24x + 5$$

$$B_0 = \left(x - \frac{1}{2}\right)[-2-(x-1)\,]^2 = \frac{1}{2}(2x-1)\ 4(x-1)^2 = 2(2x - 1)\ (x - 1)^2$$

$$= 2(2x^3 - 5x^2 + 4x - 1)$$

$$B_1 = (x - 1)\ (2x - 1)^2 = 4x^3 - 8x^2 + 5x - 1$$

So
$$H(f\ ;\ x) = 4 \times 4\ (4x^3 - 9x^2 + 6x - 1) - 16x^3 + 36x^2 - 24x + 5$$

$$- 16 \times 2(2x^2 - 5x^2 + 4x - 1) - 2(4x^3 - 8x^2 + 5x - 1)$$

$$= 64x^3 - 144x^2 + 96x - 16 - 16x^3 + 36x^2 - 24x + 5$$

$$-64x^3 + 160x^2 - 128x + 32 - 8x^3 + 16x^2 - 10x + 2$$

$$= -\ 24x^3 + 68x^2 - 66x + 23.$$

		[illegible]	1.3863
$y' = \frac{1}{x}$:	0.3333	0.2857	0.2500

(**Ans:** 1.1631)

7.8 SPLINE INTERPOLATION

When computers were not available, the draftsman used a device to draw a smooth curve through a given set of points such that the slope and the curvature are also continuous along the curve, that is, $f(x)$, $f'(x)$ and $f''(x)$ are continuous on the curve. Such a device was called a spline and plotting of the curve was called **spline fitting**.

The given interval $[a, b]$ is subdivided into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, ... $[x_{n-1}, x_n]$ where $a = x_0 < x_1 < x_2 < ..... < x_n = b$. The nodes (knots) $x_1, x_2, ..., x_{n-1}$ are called internal nodes.

We now define a spline function.

Definition 4.2. A spline function of degree n with knots (nodes) x_i, $i = 0, 1, ..., n$ is a function $F(x)$ satisfying the properties.

(*i*) $F(x_i) = f(x_i)$, $i = 0, 1$ .... n.

(*ii*) On each subinterval $[x_i - 1|, x_i]$, $1 \le i \le n$, $F(x)$ is a polynomial of degree n.

(*iii*) $F(x)$ and its first $(n - 1)$ derivatives are continuous on (a, b).

7.8.1 Linear Spline Interpolation

Linear spline interpolation is same as linear piecewise interpolation.

7.8.2 Quadratic Spline Interpolation

A quadratic spline satisfies the following properties:

(i) $F(x_i) = f(x_i)$, $i = 0, 1, \ldots, n$

(ii) On each subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$, $F(x)$ is a second degree polynomial, except in the first or the last interval.

(iii) $F(x)$ and $F'(x_i)$ are continuous on (a, b).

We denote $F''(xi) = M_i$, if the second derivative exists. On each subinterval $[x_{i-1}, x_i]$, we approximate $f(x)$ by a second degree polynomial as

$$F(x) = P_i(x) = a_i x^2 + b_i x + c_i, \ i = 1, 2, \ldots, n.$$

There are 3n unknown to be determined which are $a_1, b_1, c_1, a_2, b_2, c_2$; ...... ; a_n, b_n, c_n.

Since $F(x)$ is continuous at the internal nodes $x_1, x_2, \ldots, x_{n-1}$, we obtain the equations

on $\quad [x_{i-1}, x_i]$: $P_i(x_i) = f_i = a_i x_i^2 + b_i x_i + c_i$ ...(1)

on $\quad [x_i, x_{i+1}]$: $P_{i+1}(x_i) = f_i = a_{i+1} x_i^2 + b_{i+1} x_i + c_{i+1}$ ...(2)

$$i = 1, 2, \ldots, n-1.$$

From this set, we have $2n - 2$ equations. Since $F'(x)$ is also continuous at the internal nodes, we obtain the equations continuity at x_i:

$$p';(xi) = P'_{i+1}(x_i)$$

or
$$2a_i x_i + b_i = 2a_{i+1} x_i + b_{i+1} \quad \text{...(3)}$$
$$i = 1, 2, \ldots, n-1.$$

From this set, we have $n - 1$ equations. At the end points x_0, x_n, interpolatory conditions give the equations

$$f_0 = a_1 x_0^2 + b_1 x_0 + c_1 \quad \text{...(4)}$$

and
$$f_n = a_n x_n^2 + b_n x_n + c_n \quad \text{...(5)}$$

Now, we have a total of $(2n - 2) + (n - 1) + 2 = 3n - 1$ equations to determine the $3n$ unknowns. We need one more equation to determine the polynomials uniquely. This extra condition can be provided in a number of ways.

For example. Suppose we have 3 subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$. Then, from (1) to (5), we have the equations

$$a_1 x_1^2 + b_1 x_i + c_1 = f_1, \ a_2 x_1^2 + b_2 x_1 + c_2 = f_1 \quad (6, a, b)$$
$$a_2 x_2^2 + b_2 x_2 + c_2 = f_2, \ a_3 x_2^2 + b_3 x_2 + c_3 = f_2 \quad (7, a, b)$$
$$2a_1 x_1 + b_1 = 2a_2 x_1 + b_2, = 2a_2 x_2 + b_2 = 2a_3 x_2 + b_3 \quad (8, a, b)$$
$$a_1 x_0^2 + b_1 x_0 + c_1 = f_0, \ a_3 x_3^2 + b_3 x_3 + c_3 = f_3, \quad (9, a, b)$$

Let us choose $M_0 = f''(x_0) = 0$ as the extra condition. This gives $a_1 = 0$. Using the equations (9a), (6a) ; (6 b), (7 a), (8 a) ; (7, b), 8 (b), (9 b), we write them in the following order.

$$\left.\begin{aligned} b_1 x_0 + c_1 &= f_0, \\ b_1 x_1 + c_1 &= f_1 \end{aligned}\right\} \quad \text{...(10)}$$

$$\left.\begin{aligned} a_2 x_i^2 + b_2 x_1 + c_2 &= f_1 \\ a_2 x_2^2 + b_2 x_2 + c_2 &= f_2 \\ 2a_2 x_1 + b_2 &= 2a_1 x_1 + b_1 \end{aligned}\right\} \quad \text{...(11)}$$

and
$$\left.\begin{aligned} a_3x_2^2 + b_3x_2 + c_3 &= f_2 \\ 2a_3x_2 + b_3 &= 2a_2x_2 + b_2 \\ a_3x_3^2 + b_3x_3 + c_3 &= f_3 \end{aligned}\right\} \quad ...(12)$$

The system of equation (10) is solved for b_1, c_1. Using these solutions, the system of equation (11) is solved. Finally, the system of equation (12) is solved. The systems of equations are solved in the forward direction. If $M_3 = f''(x_3) = 0$ is prescribed, then we rearrange the equations so that the solution is obtained in the backward direction, that is, we solve for b_3, c_3 first, then for a_2, b_2, c_2, etc.

Quadratic splines have two disadvantages. They are

(*i*) a straight line connects the first two or the last two points.

(*ii*) the spline for the last interval or the first interval may swing high in the above cases.

For these reasons, quadratic splines are not often used.

SOLVED EXAMPLES

Example 7.47. Given the data

x	0	1	2	3
$f(x)$	1	2	33	244

Fit quadratic splines with $M(0) = f''(0) = 0$. Hence, find an estimate of $f(2.5)$.

Solution: We write the spline approximation as

$$P_1(x) = a_1x_2 + b_1x + c_1,\ 0 \le x \le 1$$
$$P_2(x) = a_2x_2 + b_2x + c_2,\ 1 \le x \le 2$$
$$P_3(x) = a_3y_2 + b_3x + c_3,\ 2 \le x \le 3.$$

Since $M(0) = f''(0) = 0$, we get $a_1 = 0$. Substituting $x_0 = 0$, $x_1 = 2$, $x_2 = 2$, $x_3 = 3$, $f_0 = 1$, $f_1 = 2$, $f_2 = 33$, $f_3 = 244$ in equations (10), (11) and (12), we obtain

$$\left.\begin{aligned} b_1(0) + c_1 &= 1, \\ b_i + c_i &= 2, \end{aligned}\right\}$$

$$\left.\begin{aligned} a_2 + b_2 + c_2 &= 2 \\ 4a_2 + 2b_2 + c_2 &= 33, \\ 2a_2 + b_2 &= 2a_1 + b_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} 4a_3 + 2b_2 + c_2 &= 2 \\ 4a_2 + 3b_3 + c_3 &= 244, \\ 4a_3 + b_3 &= 4a_2 + b_2 \end{aligned}\right\}$$

Solving the first system, we get $c_1 = 1$, $b_1 = 1$.

The second system becomes

$$a_2 + b_2 + c_2 = 2$$
$$4a^2 + 2b_2 + c_2 = 33$$
$$2a_2 + b_2 = 1$$

The solution to this system is $a_2 = 30$, $b_2 = -59$, $c_2 = 31$.

The third system becomes

$$4a_3 + 2b_3 + c_3 = 33$$
$$9a_3 + 3b_3 + c_3 = 244$$
$$4a_3 + b_3 = 61$$

The solution to this system is $a_3 = 150$, $b_3 = -539$, $c_3 = 511$. Therefore, the quadratic splines in the corresponding intervals can be written as

$$P_1(x) = x + 1,\ 0 \le x < 1$$
$$P_2(x) = 30x^2 - 59x + 31,\ 1 \le x \le 2$$
$$P_3(x) = 150x^2 - 539x + 511,\ 2 \le x \le 3.$$

An estimate at 2.5 is

$$f(2.5) = P_3\ (2.5) = 150(2.5)^2 - 539(2.5) + 511 = 101.$$

7.8.3 Cubic Spline Interpolation

A cubic spline satisfies the following properties:

(*i*) $F(x_i) = f_i$, $i = 0, 1, \ldots, n$.

(*ii*) On each subinterval $[x_i - 1, x_i]$, $1 \le i \le n$, $F(x)$ is a third degree polynomial.

(*iii*) $F(x)$, $F'(x)$ and $F''(x)$ are continuous on (a, b).

Let $\quad F'(x_i) = m_i$, and $F''(x_i) = M_i$.

Cubic splines do not have the disadvantages of the quadratic splines.

On each subinterval $[x_{i-1}, x_i]$, we approximate $f(x)$ by a cubic polynomial as

$$F(x) = P_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i\ \ i = 1, 2 \ldots\ldots, n.$$

We have 4n unknowns a_i, b_i, c_i d_i, $i = 1, 2, \ldots., n$ to be determined.

Using the continuity of $F(x)$, $F'(x)$ and $F''(x)$ we have the following equations.

(*a*) Continuity of $F(x)$:

On $\quad [x_{i-1}, x_i]$; $P_i(x_i) = f_i = a_i x_i^3 + b_i x_i^2 + c_i x_i + d$

On $\quad [x_i, x_{i+1}]$: $P_{i+1}(x_i) = f_i = (a_{i+1} x_i^3 + b_{i+1} x_i^2 + c_{i+1} xi + d_{i+1}$ ...(13)

$$i = 1, 2, \ldots, n - 1$$

(*b*) Continuity of $F'(x)$:

$$3a_i x_i^2 + 2b_i x_i + c_i = 3a_{i+1}\, x_1^2 + 2b_{i+1}\, x_i + C_{i+1},\ i = 1, 2, \ldots, n - 1. \quad ...(14)$$

(*c*) Continuity of F"(x):

$$6a_i x_i + 2b_i = (6a_{i+1}\, x_i + 2b_{i+1},\ i = 1, 2, \ldots, n - 1. \quad ...(15)$$

At the end points x_0 and x_n, we have the interpolatory conditions

$$f_0 = a_1\, x_0^3 + b_1 x_0^2 + c_1\, x_0 + d_1 \quad ...(16)$$
$$f_n = a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n \quad ...(17)$$

We have $2(n - 1)$ equations from (13), $2(n - 1)$ equations from (14), (15) and 2 equations from (16), (17), that is, a total of $4n - 2$ equations. We need two more equations to obtain the polynomials uniquely. In most cases, we prescribe $F''(x)$ at the two end points, that is,

$$F''(x_0) = M_0 = p \text{ and } F''(x_n) = M_n = q.$$

The end conditions, $M_0 = 0$, $M_n = 0$, lead to a **natural spline**. It is called a natural spline since the drafting spline always behaves in this fashion. However, we can use the conditions as $p \ne 0$ or/ and $q \ne 0$. If the above two conditions are imposed, then we have 4n equations in 4n unknowns. These equations can be written in matrix form and solutions can be obtained.

Working Steps:

(i) $M_0 = M_n = 0$ (natural spline)

(ii) $M_0 = M_n$, $M_1 = M_{n+1} f_0 = f_n$, $f_1 = f_{n+1}$, $h_1 = h_{n+1}$.

(A spline satisfying these conditions is called a **Periodic spline**)

(iii) For a **non-periodic spline**, we use the conditions

$$F'(a) = f'(a) = f_0' \text{ and } F'(b) = f'(b) = f'n.$$

and using equations

$$2M_0 + M_1 = \frac{6}{h_i}\left(\frac{f_i - f_0}{h_1} f'_0\right)$$

$$M_{n-1} + 2M_n = \frac{6}{h_n}\left(f'_n - \frac{f_n - f_{n-1}}{h_n}\right)$$

(iv) For equispaced knots $h_i = h$ for all i,

$$F(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + 4(x - x_{i-1})^3 M_1]$$

$$+\frac{1}{h}(x_i - x)\left(f_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(f_i - \frac{h^2}{6}M_i\right) \quad ...(18)$$

and $$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f_{i+1} - 2f_i + f_{i-1}) \quad ...(19)$$

Splines usually provide a better approximation of the behaviour of functions that have abrupt local changes. Further, splines perform better than higher order polynomial approximations.

SOLVED EXAMPLES

Example 7.48. Obtain the cubic spline approximation for the function defined by the data

x	0	1	2	3
$f(x)$	1	2	33	244

with $M(0) = 0$, $M(3) = 0$. Hence find an estimate of $f(2,5)$.

Solution: Since the points are equispaced with $h = 1$, we obtain from (19)

$$M_{i-1} + 4M_i + M_{i+1} = 6\,(f_{i+1} - 2f_i + f_{i-1}),\ i = 1, 2.$$

Therefore, $M_0 + 4M_1 + M_2 = 6\,(f_2 - 2f_1 + f_0)$ $\quad M_1 + 4M_2 + M_3 = 6\,(f_3 - 2f_2 + f_1)$

Using $M_0 = 0$, $M_3 = 0$ and the given function values, we get

$$4M_1 + M_2 = 6(33 - 4 + 1) = 180$$

$$M_1 + 4M_2 = 6(244 - 66 + 2) = 1080$$

ch gives $$M_1 = -24,\ M_2 = 276.$$

hus, using (18) the cubic splines in the corresponding intervals are obtained as follows:

On [0,1]: $$F(x) = \frac{1}{6}[(1-x)^3 M_0 + (x-0)^3 M_1] + (1-x)\left(f_0 - \frac{1}{6}M_0\right)$$

$$+(x-0)\left(f_1 - \frac{1}{6}M_1\right)$$

$$\ldots + \frac{1}{h}(x - x_{i-1})\left(f_i - \frac{h^2}{6}M_i\right) \quad \ldots(18)$$

and
$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f_{i+1} - 2f_i + f_{i-1}) \quad \ldots(19)$$

Splines usually provide a better approximation of the behaviour of functions that have abrupt local changes. Further, splines perform better than higher order polynomial approximations.

SOLVED EXAMPLES

Example 7.48. **Obtain the cubic spline approximation for the function defined by the data**

x	0	1	2	3
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with $M(0) = 0$, $M(3) = 0$. Hence find an estimate of $f(2,5)$.

Solution: Since the points are equispaced with $h = 1$, we obtain from (19)

$$M_{i-1} + 4M_i + M_{i+1} = 6\,(f_{i+1} - 2f_i + f_{i-1}),\ i = 1, 2.$$

Therefore, $M_0 + 4M_1 + M_2 = 6\,(f_2 - 2f_1 + f_0)$ $M_1 + 4M_2 + M_3 = 6\,(f_3 - 2f_2 + f_1)$

Using $M_0 = 0$, $M_3 = 0$ and the given function values, we get

$$4M_1 + M_2 = 6(33 - 4 + 1) = 180$$
$$M_1 + 4M_2 = 6(244 - 66 + 2) = 1080$$

which gives
$$M_1 = -24,\ M_2 = 276.$$

Thus, using (18) the cubic splines in the corresponding intervals are obtained as follows:

On $[0,1]$:
$$F(x) = \frac{1}{6}[(1-x)^3 M_0 + (x-0)^3 M_1] + (1-x)\left(f_0 - \frac{1}{6}M_0\right) + (x-0)\left(f_1 - \frac{1}{6}M_1\right)$$

$$= \frac{1}{6}x^3(-24) + (1-x) + x\left[2 - \frac{1}{6}(-24)\right] = -4x^3 + 5x + 1.$$

On [1, 2]:

$$F(x) = \frac{1}{x}[(2-x)^3 M_1 + (x-1)^3 M_2] + (2-x)\left(f_1 - \frac{1}{6}M_1\right) + (x-1)\left(f_2 - \frac{1}{6}M_2\right)$$

$$= \frac{1}{6}[(2-x)^3(-24) + (x-1)^3(276)] + (2-x)\left[2 - \frac{1}{6}(-24)\right] + (x-1)\left[33 - \frac{1}{6}(276)\right]$$

$$= \frac{1}{6}[(8 - 12x + 6x^2 - x^3)(-24) + (x^3 - 3x^2 + 3x - 1)(276)] + 6(2-x) - 13(x-1)$$

$$= 50x^3 - 162x^2 + 167x - 53.$$

On [2, 3]:

$$F(x) = \frac{1}{6}[(3-x)^3 M_2 + (x-2)^2 M_3] + (3-x)\left(f_2 - \frac{1}{6}M_2\right) + (x-2)\left(f_3 - \frac{1}{6}M_3\right)$$

$$= \frac{1}{6}[(27 - 27x + 9x^2 - x^3)(276)] + (3-x)\left[33 - \frac{1}{6}(276)\right] + (x-2)[244]$$

$$= -46x^3 + 414x^2 - 985 + 715$$

As estimate at 2.5 is

$$f(2.5) = P_3(2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715$$
$$= 121.25$$

Example 7.49. Obtain cubic spline for every sub-interval for the following set of data. It is given that

x	1	2	3	4
y	1	5	11	8

Also find $P(1.6)$

Solution: In this problem, $h = 1$, $n = 3$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $y_0 = 1$, $y_1 = 5$, $y_2 = 11$, $y_3 = 8$. By the system of equations

$$M_{j-1} + 4M_j + M_{j+1} = \frac{6}{h^2}(y_{j+1} - 2y_j + y_{j-1}), \text{ where } j = 1, 2$$

$j = 1$, we get

$$M_0 + 4M_1 + M_2 = 6(y - 2y_1 + y_0)$$
$$4M_1 + M_2 = 6(11 - 2 \times 5 + 1) = 12 \quad \ldots(1)\ (\because M_0 = 0)$$

Similarly, when $j = 2$, we get

$$M_1 + 4M_2 + M_3 = 6(y_3 - 2y_2 + y_1)$$