Random Variable and Probability Distributions

A random variable is often described as a Variable whose values are determined by Chance.. A random variable is typically denoted using capital letter 'x'.

The values taken by the random variable 'x' may be countable or uncountable, based on which it is classified as discrete or continuous.

Discrete Probability Distributions

A discrete random variable has countable values. The discrete probability distribution 'P(x)' describes the probability of occurance of each variable 'x', such that $\Sigma P(x) = 1$

Q Find the probability distribution of the number of aces, when two cards are drawn at random, with replacement from a well shuffled pack of 52 cards.

Sol Let X be a random variable showing number of aces. Clearly, X can take values 0, 1 or 2. If S denotes success, i.e, getting an ace and F denote failure, i.e getting a non-ace ~~card~~ Card, then

$$P(S) = \frac{4}{52} = \frac{1}{13} \text{ and } P(F) = \frac{12}{13}$$

X	Event	$P(X)$
0	FF	$\frac{12}{13} \cdot \frac{13}{13} = \frac{144}{169}$
1	SF or FS	$\left(\frac{1}{13} \cdot \frac{12}{13}\right) + \left(\frac{12}{13} \cdot \frac{1}{13}\right) = \frac{24}{169}$
2	SS	$\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$

Q 3 Bad articles are mixed with 7 good ones. Find the probability distribution of number of bad articles, if 3 are drawn at random without replacement from the lot.

Sol. Let X be the random variable showing the number of bad articles. Clearly, X can take values 0, 1, 2 or 3.

X	Event	P(X)
0	0 Bad 3 Good	$\frac{{}^7C_3}{{}^{10}C_3} = \frac{210}{720}$
1	1 Bad 2 Good	$\frac{{}^3C_1 \times {}^7C_2}{{}^{10}C_3} = \frac{378}{720}$
2	2 Bad 1 Good	$\frac{{}^3C_2 \times {}^7C_1}{{}^{10}C_3} = \frac{126}{720}$
3	3 Bad 0 Good	$\frac{{}^3C_3}{{}^{10}C_3} = \frac{6}{720}$

Note: We use combination since the items are drawn without replacement.

Mean and Variance of a Random Variable

If X be a random variable which can assume any one of the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$; then the mathematical expectation of X (~~mean~~), denoted by E(X) is defined as:

$$E(X) = p_1x_1 + p_2x_2 + \text{---} + p_nx_n = \Sigma p_i x_i \; ; \text{ where } \Sigma p_i = 1$$

The mathematical expectation of a random ~~variable~~ Variable 'X' is nothing but its arithmetic mean.

Hence, Mean $(\bar{x}) = E(X) = \Sigma p_i x_i$

Also, Variance $(\sigma^2) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2$

$$= E(X^2) - (E(X))^2$$

Q What is the expected <u>number of heads appearing</u>, when a fair coin is tossed 3 times.

Sol Let X be a random variable showing number of heads. Clearly X can take values 0, 1, 2, 3.

x_i	Event	p_i	$p_i x_i$
0	TTT	$\frac{1}{8}$	0
1	HTT, THT, TTH	$\frac{3}{8}$	$\frac{3}{8}$
2	HHT, HTH, THH	$\frac{3}{8}$	$\frac{6}{8}$
3	HHH	$\frac{1}{8}$	$\frac{3}{8}$

$$\Sigma p_i x_i = \frac{12}{8} = 1.5$$

$\therefore E(X) = 1.5$

Q A man draws 2 balls from a bag containing 3 white and 5 black balls. If he receives Rs. 70/- for every white ball he draws and Rs 35 for every black ball, what is his expectation?

Sol

Event	Probability (p_i)	Amount (x_i)	$p_i x_i$
2 Black	$\frac{5C_2}{8C_2} = \frac{10}{28}$	$35+35=70$	$\frac{10}{28} \times 70$
1 White, 1 black	$\frac{3C_1 \times 5C_1}{8C_2} = \frac{15}{28}$	$70+35=105$	$\frac{15}{28} \times 105$
2 White	$\frac{3C_2}{8C_2} = \frac{3}{28}$	$70+70=140$	$\frac{3}{28} \times 140$

$$\text{Expectation (mean)} = \sum p_i x_i = \frac{700+1575+420}{28} = 96.25$$

Q For a random variable X, the probability mass function is:

$$f(x) = Kx \quad \text{for } x = 1, 2, \ldots n$$
$$= 0, \quad \text{otherwise}$$

Find the expectation of X.

Sol Since $f(x)$ denotes probability mass function,

$$\sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} Kx = K\sum_{x=1}^{n} x = 1$$

$$\Rightarrow \frac{K n(n+1)}{2} = 1 \Rightarrow K = \frac{2}{n(n+1)}$$

$$\therefore E(X) = \sum_{x=1}^{n} x f(x) = \sum_{x=1}^{n} x.Kx$$

$$= \sum_{x=1}^{n} x^2 . \frac{2}{n(n+1)} = \frac{2}{n(n+1)} \sum_{x=1}^{n} x^2$$

$$= \frac{2}{n(n+1)} (1^2 + 2^2 + \ldots n^2) = \frac{2}{\cancel{n(n+1)}} \frac{\cancel{n(n+1)}(2n+1)}{6}$$

$$= \frac{2n+1}{3}$$

Q A random variable X has the following probability function:

x	-2	-1	0	1	2	3
$P(x)$	K	0.1	0.3	$2K$	0.2	K

Calculate mean and variance

Sol $\sum p_i = 1 \Rightarrow K + 0.1 + 0.3 + 2K + 0.2 + K = 1 \Rightarrow K = 0.1$

x_i	p_i	$p_i x_i$	$p_i x_i^2$
-2	0.1	-0.2	0.4
-1	0.1	-0.1	0.1
0	0.3	0	0
1	0.2	0.2	0.2
2	0.2	0.4	0.8
3	0.1	0.3	0.9
		$\sum p_i x_i = 0.6$	$\sum p_i x_i^2 = 2.4$

$\therefore$ mean $= \sum p_i x_i = 0.6$

Variance $(\sigma^2) = \sum p_i x_i^2 - (\sum p_i x_i)^2$

$= 2.4 - 0.36 = 2.04$

Continuous Probability Distributions

The Probability distribution $P(X)$ associated with a continuous random variable X, is called a continuous distribution.

A continuous random variable is having a set of infinite and uncountable values, for example the set of real numbers in the interval $(0,1)$ is uncountable.

If X be a continuous random variable taking values in the interval $[a, b]$, the function $f(x)$ is said to be the Probability Density Function (PDF) of X, if it satisfies the following properties:

i. $f(x) \geq 0 \ \forall x \in X$ in $[a, b]$

ii. Total area under the probability curve is one, i.e., $P(a \leq x \leq b) = 1$

the value of constant k

For 2 distinct points c and d in the interval $[a, b]$; $P(c \le x \le d)$ is given by area under the probability curve between the ordinates $x = c$ and $x = d$, i.e.,

$$P(c \le x \le d) = \int_c^d f(x)\,dx$$

Q. Find whether the following is a probability density function: $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2x, & 1 < x \le 2 \end{cases}$

Sol For $f(x)$ to be a probability density function, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^{\infty} f(x)\,dx$

$$= 0 + \int_0^1 x\,dx + \int_1^2 2x\,dx + 0$$

$$= \frac{1}{2}[x^2]_0^1 + [x^2]_1^2$$

$$= \frac{1}{2}(1) + (4-1) = \frac{7}{2} \ne 1$$

Hence $f(x)$ is not a probability density function.

Q Let X be a random variable with PDF given by: $f(x) = Kx^4, \ |x| \le 1$
$= 0$, otherwise

(i) Find the value of constant K.

(ii) Find $E(X)$ and $Var(X)$

(iii) Find $P(X) \geq \frac{1}{2}$

Sol (i) For $f(x)$ to be PDF, $\int_{-\infty}^{\infty} f(x)dx = 1$

$\Rightarrow \int_{-1}^{1} Kx^4 dx = 1$

$\Rightarrow \frac{K}{5}[x^5]_{-1}^{1} = 1 \Rightarrow \frac{K}{5}(2) = 1 \Rightarrow K = \frac{5}{2}$

(ii) $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^{1} x Kx^4 dx$

$= K\int_{-1}^{1} x^5 dx = 0 \quad \because x^5$ is an odd function.

Also $Var(X) = E(X^2) - (E(X))^2$

$= \int_{-\infty}^{\infty} x^2 f(x) dx - 0$

$= K\int_{-1}^{1} x^6 dx = \frac{5}{2}\left[\frac{x^7}{7}\right]_{-1}^{1} = \frac{5}{7}$

(iii) $P(X) \geq \frac{1}{2} = \int_{+\frac{1}{2}}^{1} Kx^4 dx = K\left[\frac{x^5}{5}\right]_{\frac{1}{2}}^{1} = \frac{31}{64}$