Probability

The numerical measure of certainty of an event is called probability. The probability of any event lies between 0 and 1.

Sample Space: The set of all possible outcomes associated with an experiment is called sample space.

Ex. 1 While tossing a single coin, the sample space $S = \{H, T\}$.

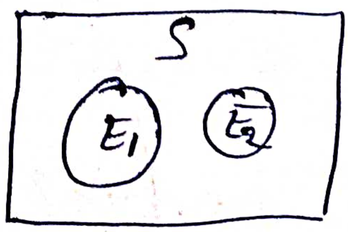
2. For tossing two coins simultaneously

$$S = \{HH, HT, TH, TT\}$$

Event: An event is the subset of a sample space. For example, while rolling a die; getting an odd number is the event $E = \{1, 3, 5\}$

Mutually Exclusive Events: $E_1, E_2, \ldots, E_n$ are said to be mutually exclusive if they have no point in common

i.e., $E_1 \cap E_2 \cap \ldots \cap E_n = \phi$



Example: While tossing a die

let $E_1 = \{1, 2\}$, $E_2 = \{4, 5\}$

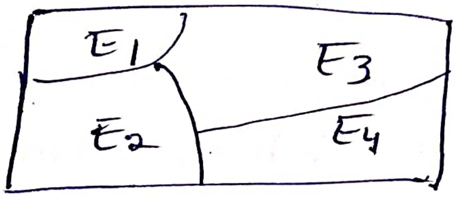
E_1 and E_2 are mutually exclusive.

Exhaustive Events: Two or more events are said to be exhaustive, if at least one of them occurs, when an experiment is performed.

Mutually Exclusive and Exhaustive Events:

If $E_1, E_2, \ldots E_n$ are mutually exclusive and exhaustive, then

$$P(E_1) + P(E_2) + \cdots + P(E_n) = 1$$



Mathematical Definition of Probability

$$P(E) = \frac{\text{Favourable no. of cases}}{\text{Exhaustive no. of Cases}} = \frac{n(E)}{n(S)} = \frac{m}{n}$$

Counting Techniques

I. Multiplication and Addition Rules: ~~Fundamental Counting Principle~~:

If an event can occur in m different ways and another event can occur in n ways, then the two events can occur together in m.n ways. Also event 1 ~~and~~ or event 2 can occur in m+n ways.

And(&) → Multiply

OR(+) → Add

Counting Techniques

I Multiplication and Addition Rules

If an event1 can occur in 'm' different ways and another event2 can occur in 'n' different ways, then

(1) Event 1 and Event 2 can occur in $m \times n$ ways.

(2) Event1 or Event2 can occur in $m+n$ ways.

And → multiply ($\times$)

or → Add (+)

II Repetition / Replacement and Non-Repetition / Non-Replacement Rules.

Ex. We have 3 alphabets A, B, C. How many 3 letter words can be formed
(1) with replacement (2) without Replacement.

Sol (1) with replacement

$\underline{3} \times \underline{3} \times \underline{3} = 27$

(2) without Replacement

$\underline{3} \times \underline{2} \times \underline{1} = 6 = 3!$

Ex 5 Buses are running in between 2 cities.
(1) In how many ways can a person go by one bus and comes back by a

different bus.

(2) In how many ways can a person go by a bus and comes back by same bus.

Sol ~~(a)~~ E_1: Going by a bus

E_2: Coming back by a different bus

E_3: " " " same bus

(1) $E_1 \underset{\text{and}}{\times} E_2 = 5 \times 4 = 20$

(2) $E_1 \times E_3 = 5 \times 1 = 5$

III Permutation and Combination Rules

Permutation	vs	Combination
Arrangement		Selection
$^nP_r = \dfrac{n!}{(n-r)!}$		$^nC_r = \dfrac{n!}{r!\,(n-r)!}$

$^nC_r = {}^nC_{n-r}$

eg. $^8C_5 = {}^8C_3$

$= \dfrac{8 \cdot 7 \cdot \cancel{6}}{\cancel{3!}} = 56$

Eg: 1. In how many ways the letters of the word BAT arranged, taken 2 at a time. (Rep. not allowed)

$^3P_2 = \dfrac{3!}{(3-2)!} = 6$ or $3 \times 2 = 6$

2. In how many ways the letters of BAT selected taken 2 at a time. (Rep. not allowed)

$^3C_2 = {}^3C_1 = 3$

Q In how many ways the letters of the word APPLE be arranged.

Sol $\frac{5!}{2!} = \frac{5\times4\times3\times\cancel{2}}{\cancel{2}} = 60$

Q In how many ways, the letters of the word SUCCESS, be arranges, such that all S are together.

Sol (SSS) UCCE $\frac{5!}{2!} = 60$

Q How many 3 digit numbers are possible with digits 0, 1, 2, 3, 4 (Rep not allowed)

$\underline{4} \times \underline{4} \times \underline{3} = 48$

Q In how many ways can you select a group of 3 students out of 8?

Sol $^8C_3 = \frac{8\cdot7\cdot\cancel{6}}{\cancel{3!}} = 56$

Q How many chords can be drawn through 6 points on a circle.

Sol $^6C_2 = \frac{6\cdot5}{2!} = 15$

Note: order does not matter, then use combination.

Probability Theory

E → Event, S → Sample space

$$P(E) = \frac{n(E)}{n(S)}$$

Qns. of Coins

1 Coin, $S = \{H, T\}$

2 Coins,

H T

$S = \{HH, HT, TH, TT\}$

3 Coins,

H T

HH HT TH TT

$S = \{HHH\ HHT\ HTH\ HTT\ THH\ THT\ TTH\ TTT\}$

Q Find the probability of at least 2 heads, when we toss 3 coins together.

Sol Let E : At least 2 heads

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$E = \{HHH, HHT, HTH, THH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

Qns on die / dice

1 die $\Rightarrow S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$

2 dice - $S = \left\{ \begin{matrix} (1,1), (1,2) \text{ --- }, (1,6) \\ (2,1), (2,2) \text{ --- }, (2,6) \\ \vdots \\ (6,1), (6,2) \text{ --- } (6,6) \end{matrix} \right\}$ $n(S) = 36$

$P(\text{Sum } 2) = \frac{1}{36}$, $\because E = \{(1,1)\}$

$P(\text{Sum } 3) = \frac{2}{36}$, $\because E = \{(1,2), (2,1)\}$

$P(\text{Sum } 4) = \frac{3}{36}$, $\because E = \{(1,3), (2,2), (3,1)\}$

$\vdots$

$P(\text{Sum } 7) = \frac{6}{36} = \frac{1}{6}$ $\rightarrow$ highest Probable no. when we throw a pair of dice.

$P(\text{Sum } 8) = \frac{5}{36}$

$\vdots$

$P(\text{Sum } 12) = \frac{1}{36}$

Qns on balls in an urn/box

Q An urn contains 5 white, 6 red and 4 black balls. Two balls are drawn at random.

1) Find the probability that both are red
2) Find the Probability of one white and one black ball.

Sol

W	R	B
5	6	4

1) Let E: Both red balls

P(E) = P(red ball) and P(red ball)

$\Rightarrow P(E) = \frac{6}{15} \times \frac{5}{14} = \frac{1}{7}$

or

$P(E) = \frac{{}^6C_2}{{}^{15}C_2} = \frac{6\times 5}{2!} \times \frac{2!}{15 \times 14} = \frac{1}{7}$

2) F: One white and one black ball

$P(F) = (P(W) \text{ and } P(B)) \text{ or } (P(B) \text{ and } P(W))$

$= \frac{5}{15} \times \frac{4}{14} + \frac{4}{15} \times \frac{5}{14} = \frac{4}{21}$

or

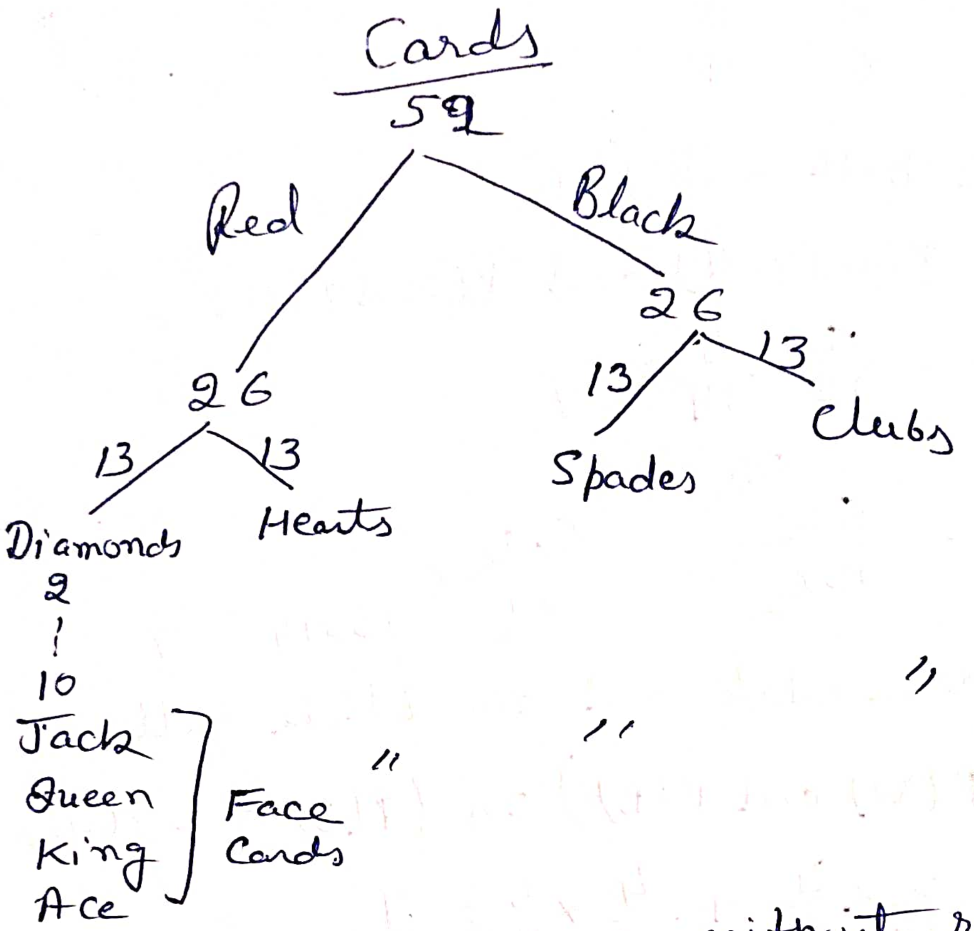
$P(F) = \frac{{}^5C_1 \times {}^4C_1}{{}^{15}C_2} = \frac{5\times 4 \times 2}{15 \times 14} = \frac{4}{21}$

Note: Use combination if and only if balls are drawn without replacement

In case balls are drawn with replacement

$P(E) = \frac{6}{15} \times \frac{6}{15}$

$P(F) = \frac{5}{15} \times \frac{4}{15} + \frac{4}{15} \times \frac{5}{15}$



Q 4 cards are drawn without replacement from a well shuffled pack of 52 cards. Find the Probability that

(i) All are spades

(ii) 2 spades and 2 hearts

(iii) All are black

Sol E_1: All cards are spades

E_2: There are 2 spades and 2 hearts

E_3: All are black

(i) $P(E_1) = \dfrac{13C_4}{52C_4}$

$= \dfrac{13\cdot 12\cdot 11\cdot 10}{52\cdot 51\cdot 50\cdot 49}$

$= \dfrac{11}{4165}$

(ii) $P(E_2) = \dfrac{{}^{13}C_2 \times {}^{13}C_2}{52C_4}$

$= \dfrac{13\cdot 12}{2!}\cdot\dfrac{13\cdot 12}{2!}\cdot\dfrac{4!}{52\cdot 51\cdot 50\cdot 49}$

$= \dfrac{468}{20825}$

(iii) $\dfrac{26C_4}{52C_4} = \dfrac{46}{833}$

Q A problem in mathematics is given to 3 students A, B, C whose chances of solving are $\frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ resp. What is the probability that the problem is solved.

Sol Prob. of A solving the problem $P(A) = \frac{2}{3}$, $P(\bar{A}) = \frac{1}{3}$

" " B " " " $P(B) = \frac{1}{2}$, $P(\bar{B}) = \frac{1}{2}$

" " C " " " $P(C) = \frac{1}{3}$, $P(\bar{C}) = \frac{2}{3}$

∴ The prob. that the problem is solved

= 1 − (Prob. that problem is not solved)

$= 1 - \left(\frac{1}{3} \times \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}\right) = 1 - \frac{1}{9} = \frac{8}{9}$

OR

$P(A) \times P(\bar{B}) \times P(\bar{C}) + P(\bar{A}) \times P(\bar{B}) \times P(C) + P(\bar{A}) \times P(B) \times P(\bar{C})$
$+ P(A) \times P(B) \times P(\bar{C}) + \text{———} \cdot \text{———} + P(A) \times P(B) \times P(C)$

Q Only 3 events A, B, C can happen. Given that chance of A is one-third of B and odds against C are 2:1, find odds in favour of A.

Sol Given $P(A) + P(B) + P(C) = 1$ — (1)

Also $P(A) = \frac{1}{3} P(B) \Rightarrow P(B) = 3P(A)$ — (2)

And $P(C) = \frac{1}{3}$ — (3)

Using (2), (3) in (1)

$P(A) + 3P(A) + \frac{1}{3} = 1 \Rightarrow P(A) = \frac{1}{6}$ and $P(\bar{A}) = \frac{5}{6}$

∴ Odds in favour of A are 1 : 5

Q A bag contains 50 tickets numbered from 1 to 50, out of which 5 are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4 < t_5$). Find the probability that t_4 carries the number 45. II-6 4

Sol Exhaustive no. of cases $= {}^{50}C_5$.

$$\underbrace{t_1 \quad t_2 \quad t_3}_{1 \text{ to } 44}, \quad \underbrace{t_4}_{45}, \quad \underbrace{t_5}_{46-50}$$

$\therefore$ Req. Probability $= \dfrac{{}^{44}C_3 \times {}^{1}C_1 \times {}^{5}C_1}{{}^{50}C_5}$

$= \dfrac{44.43.42}{3!} \times 1 \times 5 \times \dfrac{5!}{50.49.48.47.46}$

$= 0.03$

Q A has 2 shares in lottery in which there are 2 prizes and 3 blanks. B has 3 shares in another lottery in which there are 3 prizes and 6 blanks. Compare the ratio of A's success to that of B' success.

Sol Prob. of A not getting a prize in 2 shares $P(\bar{A}) = \dfrac{{}^{3}C_2}{{}^{5}C_2} = \dfrac{3\times2}{5\times4} = \dfrac{3}{10}$ $\therefore P(A) = \dfrac{7}{10}$

Also $P(\bar{B}) = \dfrac{{}^{6}C_3}{{}^{9}C_3} = \dfrac{6\times5\times4}{9\times8\times7} = \dfrac{5}{21}$, $\therefore P(B) = \dfrac{16}{21}$

$\therefore P(A) : P(B) = \dfrac{7}{10} : \dfrac{16}{21}$

$= 147 : 160$