Worksheet 1: SMP

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1 Build intuition through examples.

From this point onwards, we will read **Prefers**(x, y) as x prefers y over some other option.

1. Small and trivial instances of the problem, consider these instances:
   (a) Only one student s and one employer e:
   - Students: s_1
   - Employers: e_1
   - Preferences:
     A. **Prefers**(s_1, e_1)
     B. **Prefers**(e_1, s_1)
   - Triviality: Only one student and one employer, so there is only one match.

   (b) Equal preferences for all students and employers:
      i. Students: s_1, s_2
      ii. Employers: e_1, e_2
      iii. Preferences:
         A. **Prefers**$(s_1, e_1 \vee e_2)$
         B. **Prefers**$(s_2, e_1 \vee e_2)$
         C. **Prefers**$(e_1, s_1 \vee s_2)$
         D. **Prefers**$(e_2, s_1 \vee s_2)$
      iv. Triviality: All students and employers have indifferent preferences, so there are multiple stable matchings.

(c) Perfectly matched preferences between students and employers:

   i. Students: s_1, s_2
   ii. Employers: e_1, e_2
   iii. Preferences:
      A. **Prefers**(s_1, e_1)
      B. **Prefers**(s_2, e_2)
      C. **Prefers**(e_1, s_1)
      D. **Prefers**(e_2, s_2)
   iv. Triviality: All are perfectly matched, therefore there is only one stable matching.

2. Potential solutions to these instances:

   (a) Only one student s and one employer e:
      - The only stable matching is (s_1, e_1).
      - This solution is trivial but is optimal as well.

   (b) Equal preferences for all students and employers:
      - There are multiple stable matchings: $(s_1, e_1), (s_2, e_2)$ and $(s_1, e_2), (s_2, e_1)$.
      - This solution is optimal, but not unique.

   (c) Perfectly matched preferences between students and employers:
      - The only stable matching is $(s_1, e_1), (s_2, e_2)$.
      - This solution is optimal and unique.

   There are many ways to conclude if a solution is better. When we consider fairness and satisfaction, we may value **Instance b** more than others. However, when we consider uniqueness and optimality, we may value **Instance c** more than others.

2 Developing a Formal Problem Specification

1. Notation for describing the problem instance.

   (a) Let $S = \{s_1, s_2, \ldots, s_n\}$ be the set of students.
   (b) Let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of employers.

(c) Student's preference list $P(s_i)$, which is a ranked list of employers from most preferred to least preferred.

(d) Employer's preference list $P(e_j)$, which is a ranked list of students from most preferred to least preferred.

(e) A matching is a bijection $M : S \to E$, or a set of pairs $M = \{(s, e)\}$ where s and e are matched uniquely.

(f) A blocking pair (s_i, e_j) is a pair of student-employer that prefer each other over their current match.

    i. s_i prefers e_j over $M(s_i)$.

    ii. e_j prefers s_i over $M(e_j)$.

2. Notation for describing a potential solution

    (a) A set of potential matches $M = \{(s_1, e_1), (s_2, e_2)\}$ of student-employer pairs.

    (b) Valid if and only if every student and employer is assigned exactly one (uniqueness).

3. Good solutions

    (a) Optimality

        i. A solution is student-optimal if it provides the best possible match for every student.

        ii. A solution is employer-optimal if it provides the best possible match for every employer.

    (b) Uniqueness

        i. If a unique stable matching exists, it is the only correct solution.

        ii. If multiple stable matchings exist, we will pick the one that maximizes a criterion (e.g., student happiness, fairness).

    (c) Stability

        i. No blocking pairs exist in the matching.

        ii. A matching M is stable if there exists no student-employer pair (s_i, e_j) such that

A. s_i prefers e_j over $M(s_i)$, the employer currently matched with s_i.
B. e_j prefers s_i over $M(e_j)$, the student currently matched with e_j.