DATA315 Assignment 1

Rin Meng 51940633

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1. Suppose $Y = 3X+Z$ where Z and X are independent random variables. Suppose X has mean 5 and variance 1 and Z has mean 0 and variance 9.

   (a) $E[Y] = E[3Y+Z] = E[3Y]+E[Z] = 3E[Y]+E[Z] = 3\cdot5+0 = 15$

   (b) $V(2X) = 2^2V(X) = 4 \cdot 1$

   (c) $E[X^2] = V(X) + E[X]^2 = 1 + 5^2 = 26$

   (d) $E[XY] = E[3X^2 + XZ] = 3E[X^2] + E[XZ] = 3 \cdot 26 + 0 = 78$

   (e) $Cov(X, Y) = E[XY] - E[X]E[Y] = 78 - 5 \cdot 15 = 3$

   (f) $E[Y|Z = 2] = E[3X+2|Z = 2] = 3E[X|Z = 2]+2 = 3\cdot5+2 = 17$

2. Suppose X is a normal random variable with mean 2.0 and standard deviation 0.8, and suppose $Y = 5X + 7$. Find the mean and standard deviation of Y.

   (a) $E[Y] = E[5X + 7] = 5E[X] + 7 = 5 \cdot 2 + 7 = 17$

   (b) $V(Y) = V(5X + 7) = 5^2V(X) = 5^2 \cdot 0.8$

   (c) $SD(Y)^2 = V(Y) \Rightarrow SD(Y) = \sqrt{V(Y)} = \sqrt{5^2 \cdot 0.8} = 2\sqrt{5}$

3. Calculate the sample average and sample standard deviation of the level observations in `LakeHuron`. Use R for this question and submit the code together with your answers.

```
> data(LakeHuron)
> sample_average <- mean(LakeHuron)
> sample_sd <- sd(LakeHuron)
```

```
> print(paste("Sample Average:", sample_average))
[1] "Sample Average: 579.004081632653"
> print(paste("Sample Standard Deviation:", sample_sd))
[1] "Sample Standard Deviation: 1.31829852597076"
```

4. Suppose $Z1, Z2$ and $Z3$ are independent standard normal random variables. Write down the joint pdf for $Z1, Z2$ and $Z3$.

$$\begin{aligned} f(z_1, z_2, z_3) &= f(z_1)f(z_2)f(z_3) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{z_2^2}{2}} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{z_3^2}{2}} \\ &= \frac{1}{(2\pi)^{3/2}}e^{-\frac{z_1^2+z_2^2+z_3^2}{2}} \end{aligned}$$

5. Suppose $Z1, Z2, \ldots, Zn$ are time series with white noise model

$$z_t = \varepsilon_t$$

where ε_t is independent random variable with exponiential distribution $exp(\lambda)$. Find the maximum likelihood estimator for λ.

$$\begin{aligned} f(z_t) &= \lambda e^{-\lambda z_t} \\ L(\lambda) &= \prod_{t=1}^{n} f(z_t) \\ &= \prod_{t=1}^{n} \lambda e^{-\lambda z_t} \\ &= \lambda^n e^{-\lambda \sum_{t=1}^{n} z_t} \\ \log L(\lambda) &= n \log \lambda - \lambda \sum_{t=1}^{n} z_t \\ \frac{d}{d\lambda} \log L(\lambda) &= \frac{n}{\lambda} - \sum_{t=1}^{n} z_t \end{aligned}$$

$$\frac{n}{\lambda} - \sum_{t=1}^{n} z_t = 0$$

$$\hat{\lambda} = \frac{n}{\sum_{t=1}^{n} z_t}$$

$$\because \bar{z} = \frac{1}{n}\sum_{t=1}^{n} z_t$$

$$\hat{\lambda} = \frac{n}{n\bar{z}} = \frac{1}{\bar{z}}$$

6. Suppose $Z_1, Z_2, \ldots, Z_n$ is a time series. Let

$$L(\rho)\sum_{t=2}^{n}(Z_t - \rho Z_{t-1})^2$$

(a) Using calculus, find a formula for $\hat{\rho}$, the value of ρ which minimizes $L(\rho)$. (This is the least-squares estimator for ρ, the so-called lag 1 autocorrelation.)

$$\begin{aligned}
\frac{d}{d\rho}L(\rho) &= \frac{d}{d\rho}\sum_{t=2}^{n}(Z_t - \rho Z_{t-1})^2 \\
&= \sum_{t=2}^{n} 2(Z_t - \rho Z_{t-1}) \cdot (-Z_{t-1}) \\
&= -2\sum_{t=2}^{n} Z_{t-1}Z_t + 2\rho\sum_{t=2}^{n} Z_{t-1}^2 \\
&= 0 \\
\sum_{t=2}^{n} Z_{t-1}Z_t &= \rho\sum_{t=2}^{n} Z_{t-1}^2 \\
\hat{\rho} &= \frac{\sum_{t=2}^{n} Z_{t-1}Z_t}{\sum_{t=2}^{n} Z_{t-1}^2}
\end{aligned}$$

(b) Suppose, in addition, that $Z_1, Z_2, \ldots, Z_n$ are independent normally distributed random variables with mean 0 and standard deviation σ. Find $E\left[\sum_{t=2}^{n} Z_t Z_{t-1}\right]$ and $E\left[\sum_{t=1}^{n-1} Z_t^2\right]$.

i.

$$\begin{aligned} E\left[\sum_{t=2}^{n} Z_t Z_{t-1}\right] &= \sum_{t=2}^{n} E[Z_t Z_{t-1}] \\ &= \sum_{t=2}^{n} E[Z_t]E[Z_{t-1}] \\ &= \sum_{t=2}^{n} 0 \cdot 0 \\ &= 0 \end{aligned}$$

ii.

$$\begin{aligned} E\left[\sum_{t=1}^{n-1} Z_t^2\right] &= \sum_{t=1}^{n-1} E[Z_t^2] \\ &= \sum_{t=1}^{n-1} \mathrm{Var}(Z_t) \\ &= \sum_{t=1}^{n-1} \sigma^2 \\ &= (n-1)\sigma^2 \end{aligned}$$

(c) Let

$$X_t = Z_{t-1}, \quad t = 2, \ldots, n$$

and

$$Y_t = Z_t, \quad t = 1, \ldots, n-1.$$

Write down the formula for the sample correlation between X and Y, expressed in terms of the Z's, and compare the formula you obtained with $\hat{\rho}$ obtained in part (a).

$$\begin{aligned} \hat{\rho}_{XY} &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \\ \mathrm{Cov}(X,Y) &= \frac{1}{n-1}\sum_{t=1}^{n-1}(X_t - \bar{X})(Y_t - \bar{Y}) \\ &\Rightarrow \sum_{t=1}^{n-1}(Z_{t-1} - \bar{Z}_{t-1})(Z_t - \bar{Z}_t) \end{aligned}$$

$$\mathrm{Var}(X) = \frac{1}{n-1}\sum_{t=1}^{n-1}(X_t - \bar{X})^2$$

$$\Rightarrow \frac{1}{n-1}\sum_{t=1}^{n-1}(Z_{t-1} - \bar{X})^2$$

$$\mathrm{Var}(Y) = \frac{1}{n-1}\sum_{t=1}^{n-1}(Y_t - \bar{Y})^2$$

$$\Rightarrow \frac{1}{n-1}\sum_{t=1}^{n-1}(Z_t - \bar{Y})^2$$

$$\bar{X} = \frac{1}{n-1}\sum_{t=1}^{n-1} Z_{t-1}$$

$$\bar{Y} = \frac{1}{n-1}\sum_{t=1}^{n-1} Z_t$$

$$\hat{\rho}_{XY} = \frac{\sum_{t=1}^{n-1}(Z_{t-1} - \bar{Z}_{t-1})(Z_t - \bar{Z}_t)}{\sqrt{\sum_{t=1}^{n-1}(Z_{t-1} - \bar{Z}_{t-1})^2 \sum_{t=1}^{n-1}(Z_t - \bar{Z}_t)^2}}$$

$$\hat{\rho}_{XY} = \frac{\sum_{t=2}^{n}(Z_{t-1} - \bar{X})(Z_t - \bar{Y})}{\sqrt{\sum_{t=2}^{n}(Z_{t-1} - \bar{X})^2 \sum_{t=2}^{n}(Z_t - \bar{Y})^2}}$$

If $\hat{\rho}$ from part (a) is the sample autocorrelation of Z then at lag 1, then:

$$\hat{\rho} = \frac{\sum_{t=2}^{n} Z_{t-1}Z_t}{\sum_{t=2}^{n} Z_{t-1}^2}$$

The formula for $\hat{\rho}_{XY}$ is similar but includes centering with the sample means, $\bar{X}$ and $\bar{Y}$. If Z_t has mean of 0, then $\bar{X} = \bar{Y} = 0$ then it is true that $\hat{\rho} = \hat{\rho}_{XY}$.

$$\hat{\rho}_{XY} = \frac{\sum_{t=2}^{n} Z_{t-1}Z_t}{\sqrt{\sum_{t=2}^{n} Z_{t-1}^2 \sum_{t=2}^{n} Z_t^2}}$$

$$\hat{\rho}_{XY} = \frac{\sum_{t=2}^{n} Z_{t-1}Z_t}{\sum_{t=2}^{n} Z_{t-1}^2}$$

7. Let Z_t be the time series where $t = 1, 2, \ldots, n$. Consider a time series with $n = 4$.

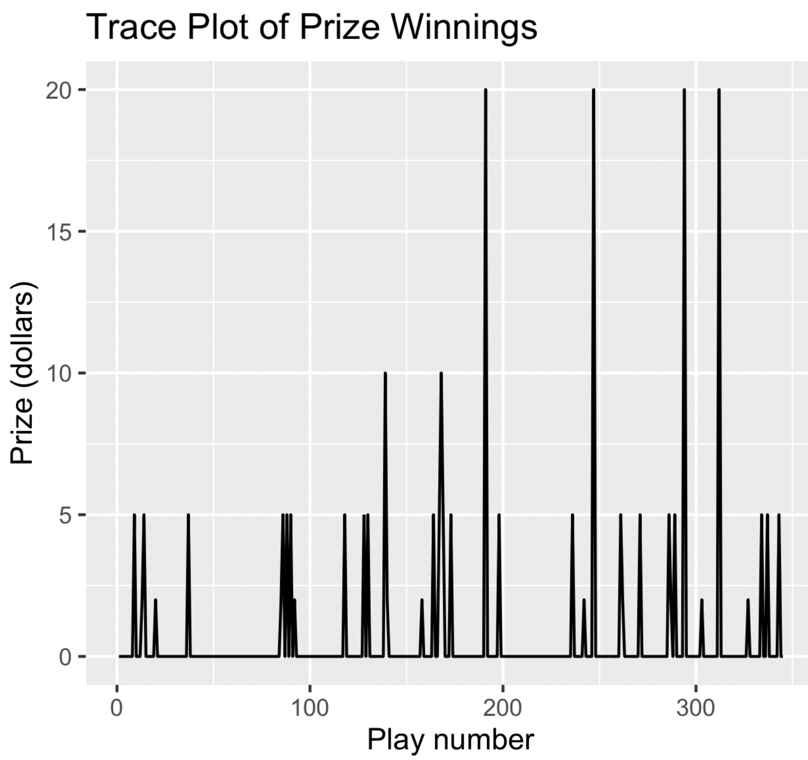
   (a) Find the distribution of the statistic τ for the Mann-Kendall test. Since it is a time series wiht $n = 4$, then we would have

   (b) Verify in these cases that the variance of τ is $\frac{2(2n+5)9}{n(n-1)}$.

   (c) Suppose z_1, z_2, z_3, z_4 are $0.2, 0.25, 0.22, 0.3$. Perform the Mann-Kendall test using the exact distribution you have derived.

8. The data in `DAAG::vlt$prize` consist of a time series of observations of prize winnings (in dollars) taken on a video lottery terminal in a sequence of 345 plays. Obtain a trace plot of the series and conduct a trend test. Is there strong evidence of a decrease or increase in the prize winnings over time?

```
> library(DAAG)
> library(ggplot2)
> data(vlt)
> prize <- vlt$prize
> ggplot(data = data.frame(play = 1:345,
prize = prize), aes(x = play, y = prize)) +
geom_line() + labs(x = "Play number",
y = "Prize (dollars)",
title = "Trace Plot of Prize Winnings")
```



Here, we can see that there is a random fluctuations in the prize winnings over time. Now we use the Mann-Kendall test, to test for a trend.

```
> library(Kendall)
> MannKendall(prize)
tau = 0.0367, 2-sided pvalue = 0.3973
```

This suggests that there is an upward trend, but the p-value is not significant enough to reject the null hypothesis that there is no trend. $\therefore$ There is no evidence of a systematic increase or decrease in prize winnings over the 345 plays.

9. Consider the time series: $13, 11, 14, 17, 16, 15$. Conduct the Mann-Kendall trend test by hand, using the following steps.

   (a) Calculate the value of τ showing the steps that you are using.

(b) Calculate the variance of τ under the null hypothesis.

(c) Under the approximate normal assumption, calculate the test statistic using τ and the standard error. What would you conclude from this test?