DATA315 Assignment 1

Rin Meng 51940633

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1. Suppose $Y = 3X + Z$ where Z and X are independent random variables. Suppose X has mean 5 and variance 1 and Z has mean 0 and variance 9.

   (a) $E[Y] = E[3Y + Z] = E[3Y] + E[Z] = 3E[Y] + E[Z] = 3 \cdot 5 + 0 = 15$

   (b) $V(2X) = 2^2V(X) = 4 \cdot 1$

   (c) $E[X^2] = V(X) + E[X]^2 = 1 + 5^2 = 26$

   (d) $E[XY] = E[3X^2 + XZ] = 3E[X^2] + E[XZ] = 3 \cdot 26 + 0 = 78$

   (e) $Cov(X, Y) = E[XY] - E[X]E[Y] = 78 - 5 \cdot 15 = 3$

   (f) $E[Y|Z = 2] = E[3X + 2|Z = 2] = 3E[X|Z = 2] + 2 = 3 \cdot 5 + 2 = 17$

2. Suppose X is a normal random variable with mean 2.0 and standard deviation 0.8, and suppose $Y = 5X + 7$. Find the mean and standard deviation of Y.

   (a) $E[Y] = E[5X + 7] = 5E[X] + 7 = 5 \cdot 2 + 7 = 17$

   (b) $V(Y) = V(5X + 7) = 5^2V(X) = 5^2 \cdot 0.8$

   (c) $SD(Y)^2 = V(Y) \Rightarrow SD(Y) = \sqrt{V(Y)} = \sqrt{5^2 \cdot 0.8} = 2\sqrt{5}$

3. Calculate the sample average and sample standard deviation of the level observations in `LakeHuron`. Use R for this question and submit the code together with your answers.

```
> data(LakeHuron)
> sample_average <- mean(LakeHuron)
> sample_sd <- sd(LakeHuron)
> print(paste("Sample Average:", sample_average))
[1] "Sample Average: 579.004081632653"
> print(paste("Sample Standard Deviation:", sample_sd))
[1] "Sample Standard Deviation: 1.31829852597076"
```

4. Suppose Z_1, Z_2 and Z_3 are independent standard normal random variables. Write down the joint pdf for Z_1, Z_2 and Z_3.

$$\begin{aligned} f(z_1, z_2, z_3) &= f(z_1)f(z_2)f(z_3) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{z_2^2}{2}} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{z_3^2}{2}} \\ &= \frac{1}{(2\pi)^{3/2}}e^{-\frac{z_1^2+z_2^2+z_3^2}{2}} \end{aligned}$$

5. Suppose $Z_1, Z_2, \ldots, Z_n$ are time series with white noise model

$$z_t = \varepsilon_t$$

where ε_t is independent random variable with exponiential distribution $exp(\lambda)$. Find the maximum likelihood estimator for λ.

$$\begin{aligned}
f(z_t) &= \lambda e^{-\lambda z_t} \\
L(\lambda) &= \prod_{t=1}^{n} f(z_t) \\
&= \prod_{t=1}^{n} \lambda e^{-\lambda z_t} \\
&= \lambda^n e^{-\lambda \sum_{t=1}^{n} z_t} \\
\log L(\lambda) &= n \log \lambda - \lambda \sum_{t=1}^{n} z_t \\
\frac{d}{d\lambda} \log L(\lambda) &= \frac{n}{\lambda} - \sum_{t=1}^{n} z_t \\
\frac{n}{\lambda} - \sum_{t=1}^{n} z_t &= 0 \\
\hat{\lambda} &= \frac{n}{\sum_{t=1}^{n} z_t} \\
\because \bar{z} &= \frac{1}{n} \sum_{t=1}^{n} z_t \\
\hat{\lambda} &= \frac{n}{n\bar{z}} = \frac{1}{\bar{z}}
\end{aligned}$$

6. Suppose $Z_1, Z_2, \ldots, Z_n$ is a time series. Let

$$L(\rho) \sum_{t=2}^{n} (Z_t - \rho Z_{t-1})^2$$

(a) Using calculus, find a formula for $\hat{\rho}$, the value of ρ which minimizes $L(\rho)$. (This is the least-squares estimator for ρ, the so-called lag 1 autocorrelation.)

$$\begin{aligned}
\frac{d}{d\rho} L(\rho) &= \frac{d}{d\rho} \sum_{t=2}^{n} (Z_t - \rho Z_{t-1})^2 \\
&= \sum_{t=2}^{n} 2(Z_t - \rho Z_{t-1}) \cdot (-Z_{t-1}) \\
&= -2 \sum_{t=2}^{n} Z_{t-1} Z_t + 2\rho \sum_{t=2}^{n} Z_{t-1}^2 \\
&= 0 \\
\sum_{t=2}^{n} Z_{t-1} Z_t &= \rho \sum_{t=2}^{n} Z_{t-1}^2 \\
\hat{\rho} &= \frac{\sum_{t=2}^{n} Z_{t-1} Z_t}{\sum_{t=2}^{n} Z_{t-1}^2}
\end{aligned}$$

(b) Suppose, in addition, that $Z_1, Z_2, \ldots, Z_n$ are independent normally distributed random variables with mean 0 and standard deviation σ. Find $E\left[\sum_{t=2}^{n} Z_t Z_{t-1}\right]$ and $E\left[\sum_{t=1}^{n-1} Z_t^2\right]$.

i.

$$\begin{aligned} E\left[\sum_{t=2}^{n} Z_t Z_{t-1}\right] &= \sum_{t=2}^{n} E[Z_t Z_{t-1}] \\ &= \sum_{t=2}^{n} E[Z_t]E[Z_{t-1}] \\ &= \sum_{t=2}^{n} 0 \cdot 0 \\ &= 0 \end{aligned}$$

ii.

$$\begin{aligned} E\left[\sum_{t=1}^{n-1} Z_t^2\right] &= \sum_{t=1}^{n-1} E[Z_t^2] \\ &= \sum_{t=1}^{n-1} \mathrm{Var}(Z_t) \\ &= \sum_{t=1}^{n-1} \sigma^2 \\ &= (n-1)\sigma^2 \end{aligned}$$

(c) Let

$$X_t = Z_{t-1}, \quad t = 2, \ldots, n$$

and

$$Y_t = Z_t, \quad t = 1, \ldots, n-1.$$

Write down the formula for the sample correlation between X and Y, expressed in terms of the Z's, and compare the formula you obtained with $\hat{\rho}$ obtained in part (a).

$$\begin{aligned} \hat{\rho}_{XY} &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \\ \mathrm{Cov}(X,Y) &= \frac{1}{n-1}\sum_{t=1}^{n-1}(X_t - \bar{X})(Y_t - \bar{Y}) \\ &\Rightarrow \sum_{t=1}^{n-1}(Z_{t-1} - \bar{Z}_{t-1})(Z_t - \bar{Z}_t) \\ \mathrm{Var}(X) &= \frac{1}{n-1}\sum_{t=1}^{n-1}(X_t - \bar{X})^2 \\ &\Rightarrow \frac{1}{n-1}\sum_{t=1}^{n-1}(Z_{t-1} - \bar{X})^2 \end{aligned}$$

$$\text{Var}(Y) = \frac{1}{n-1}\sum_{t=1}^{n-1}(Y_t - \bar{Y})^2$$
$$\Rightarrow \frac{1}{n-1}\sum_{t=1}^{n-1}(Z_t - \bar{Y})^2$$
$$\bar{X} = \frac{1}{n-1}\sum_{t=1}^{n-1} Z_{t-1}$$
$$\bar{Y} = \frac{1}{n-1}\sum_{t=1}^{n-1} Z_t$$
$$\hat{\rho}_{XY} = \frac{\sum_{t=1}^{n-1}(Z_{t-1} - \bar{Z}_{t-1})(Z_t - \bar{Z}_t)}{\sqrt{\sum_{t=1}^{n-1}(Z_{t-1} - \bar{Z}_{t-1})^2 \sum_{t=1}^{n-1}(Z_t - \bar{Z}_t)^2}}$$
$$\hat{\rho}_{XY} = \frac{\sum_{t=2}^{n}(Z_{t-1} - \bar{X})(Z_t - \bar{Y})}{\sqrt{\sum_{t=2}^{n}(Z_{t-1} - \bar{X})^2 \sum_{t=2}^{n}(Z_t - \bar{Y})^2}}$$

If $\hat{\rho}$ from part (a) is the sample autocorrelation of Z then at lag 1, then:

$$\hat{\rho} = \frac{\sum_{t=2}^{n} Z_{t-1}Z_t}{\sum_{t=2}^{n} Z_{t-1}^2}$$

The formula for $\hat{\rho}_{XY}$ is similar but includes centering with the sample means, $\bar{X}$ and $\bar{Y}$. If Z_t has mean of 0, then $\bar{X} = \bar{Y} = 0$ then it is true that $\hat{\rho} = \hat{\rho}_{XY}$.

$$\hat{\rho}_{XY} = \frac{\sum_{t=2}^{n} Z_{t-1}Z_t}{\sqrt{\sum_{t=2}^{n} Z_{t-1}^2 \sum_{t=2}^{n} Z_t^2}}$$
$$\hat{\rho}_{XY} = \frac{\sum_{t=2}^{n} Z_{t-1}Z_t}{\sum_{t=2}^{n} Z_{t-1}^2}$$

7. Let Z_t be the time series where $t = 1, 2, \ldots, n$. Consider a time series with $n = 4$.

   (a) Find the distribution of the statistic τ for the Mann-Kendall test.
   Since it is a time series with $n = 4$, then we would have $\frac{4(4-1)}{2} = 6$ pairs (x_i, x_j) where $i < j$. Under H_0 (no trend), the probability of $x_i > x_j$ is the same as $x_i < x_j$. The possible values of S and their probabilities are derived from all $4 = 24$ possible permutations of the data. The statistic $\tau = \frac{S}{6}$ scales S to the

S	-6	-4	-2	0	2	4	6
Count	1	3	5	6	5	3	1
$P(\tau)$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{5}{24}$	$\frac{6}{24}$	$\frac{5}{24}$	$\frac{3}{24}$	$\frac{1}{24}$

   range $[-1, 1]$. The distribution of τ is:

$$\tau = \{-1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1\}$$
$$P(\tau) = \{\frac{1}{24}, \frac{4}{24}, \frac{6}{24}, \frac{1}{24}, \frac{6}{24}, \frac{4}{24}, \frac{1}{24}\}$$

(b) Verify in these cases that the variance of τ is $\frac{2(2n+5)/9}{n(n-1)}$.

Variance of S (no tied values) is given by:

$$\text{Var}(S) = \frac{n(n-1)(2n+5)}{18} = \frac{4 \cdot 3 \cdot 13}{18} = 8.67$$

Variance of τ is given by:

$$\text{Var}(\tau) = \frac{Var(S)}{\frac{n(n-1)}{2}^2} = \frac{8.67}{6^2} = 0.24$$

Given formula:

$$\frac{2(2n+5)/9}{n(n-1)} = \frac{2(2 \cdot 4 + 5)/9}{4(4-1)} = \frac{18/9}{12} = 0.24$$

$\therefore$ This formula holds for $n = 4$.

(c) Suppose z_1, z_2, z_3, z_4 are $0.2, 0.25, 0.22, 0.3$. Perform the Mann-Kendall test using the exact distribution you have derived.

Calculate S:

$$\begin{aligned} \text{For i} = 1\text{: } & S = 1 + 1 + 1 = 3 \\ \text{For i} = 2\text{: } & S = -1 + 1 = 0 \\ \text{For i} = 3\text{: } & S = 1 = 1 \\ \text{Total } & S = 3 + 0 + 1 = 4 \end{aligned}$$

$$\tau = \frac{S}{\frac{n(n-1)}{2}} = \frac{4}{6} = \frac{2}{3} = 0.6667$$

From part (a) the distribution of S for $n = 4$ is:

S	-6	-4	-2	0	2	4	6
Count	1	3	5	6	5	3	1
$P(\tau)$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{5}{24}$	$\frac{6}{24}$	$\frac{5}{24}$	$\frac{3}{24}$	$\frac{1}{24}$

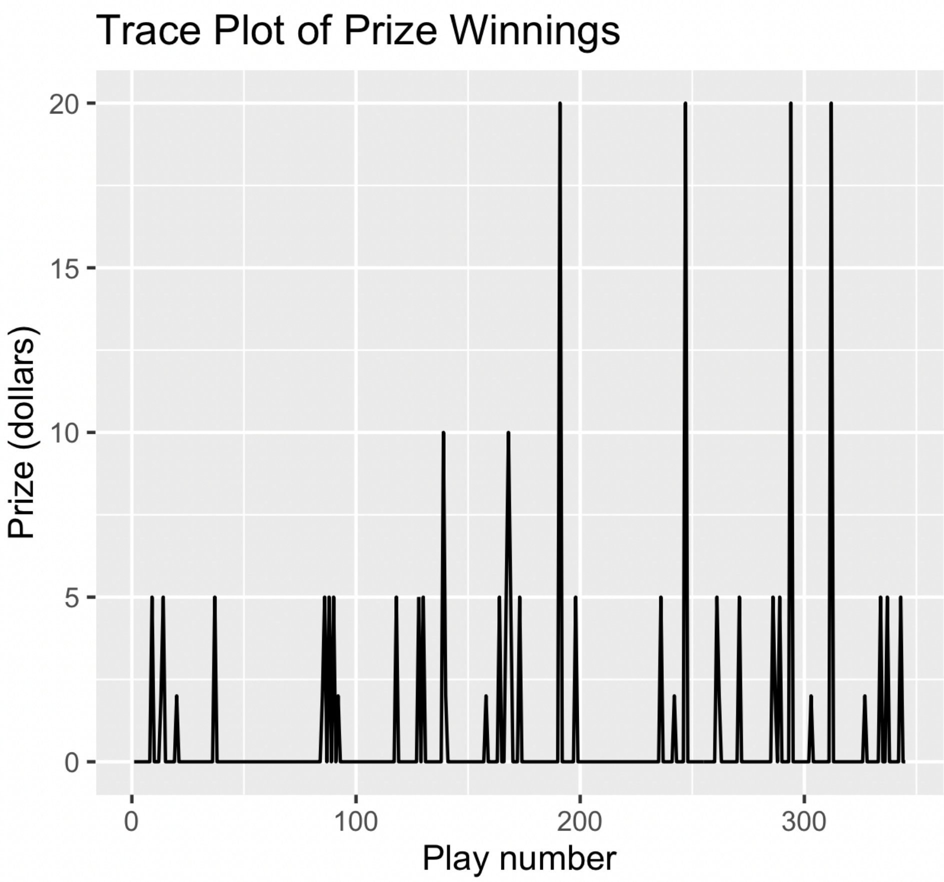
Observed $S = 4$, probability $= \frac{3}{24}$. Two-tailed p-value:

$$P(|S|) \geq 4 = \text{p-value} = \frac{3+1+3+1}{24} = \frac{8}{24} = 0.3333$$

Since $0.3333 > 0.05$, we fail to reject the null hypothesis. There is no evidence of a trend in the data.

8. The data in `DAAG::vlt$prize` consist of a time series of observations of prize winnings (in dollars) taken on a video lottery terminal in a sequence of 345 plays. Obtain a trace plot of the series and conduct a trend test. Is there strong evidence of a decrease or increase in the prize winnings over time?

```
> library(DAAG)
> library(ggplot2)
> data(vlt)
> prize <- vlt$prize
> ggplot(data = data.frame(play = 1:345,
prize = prize), aes(x = play, y = prize)) +
geom_line() + labs(x = "Play number",
y = "Prize (dollars)",
title = "Trace Plot of Prize Winnings")
```



Here, we can see that there is a random fluctuations in the prize winnings over time. Now we use the Mann-Kendall test, to test for a trend.

```
> library(Kendall)
> MannKendall(prize)
tau = 0.0367, 2-sided pvalue = 0.3973
```

This suggests that there is an upward trend, but the p-value is not significant enough to reject the null hypothesis that there is no trend. $\therefore$ There is no evidence of a systematic increase or decrease in prize winnings over the 345 plays.

9. Consider the time series: $13, 11, 14, 17, 16, 15$. Conduct the Mann-Kendall trend test by hand, using the following steps.

(a) Calculate the value of τ showing the steps that you are using.

$$\tau = \frac{S}{\frac{n(n-1)}{2}}$$

$$S = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{sign}(x_j - x_i)$$

Using the formula, we have:

$$\begin{aligned}
\text{For i} = 1:&\ S = -1 + 1 + 1 + 1 + 1 = 3 \\
\text{For i} = 2:&\ S = 1 + 1 + 1 + 1 = 4 \\
\text{For i} = 3:&\ S = 1 + 1 + 1 = 3 \\
\text{For i} = 4:&\ S = -1 - 1 = -2 \\
\text{For i} = 5:&\ S = -1 \\
\text{Total}&\ S = 3 + 4 + 3 - 2 - 1 = 7
\end{aligned}$$

$$\tau = \frac{7}{\frac{6 \cdot 5}{2}} = \frac{7}{15} = 0.4667$$

(b) Calculate the variance of τ under the null hypothesis. For $n = 6$, the variance of τ (no tied values) is given by:

$$\text{Var}(S) = \frac{n(n-1)(2n+5)}{18} = \frac{6 \cdot 5 \cdot 17}{18} = 28.33$$

$$\text{Var}(\tau) = \frac{Var(S)}{\frac{n(n-1)}{2}^2} = \frac{28.33}{15^2} = 0.126$$

(c) Under the approximate normal assumption, calculate the test statistic using τ and the standard error. What would you conclude from this test? Test statistic Z (using τ and standard error) is given by:

$$Z = \frac{\tau}{\sqrt{\text{Var}(\tau)}} = \frac{0.4667}{\sqrt{0.126}} = 1.315$$

The critical value for a two-tailed test at $\alpha = 0.05$ is 1.96. Since $Z = 1.315 < 1.96$, we fail to reject the null hypothesis. There is insufficient evidence to suggest that there is a trend in the data.