

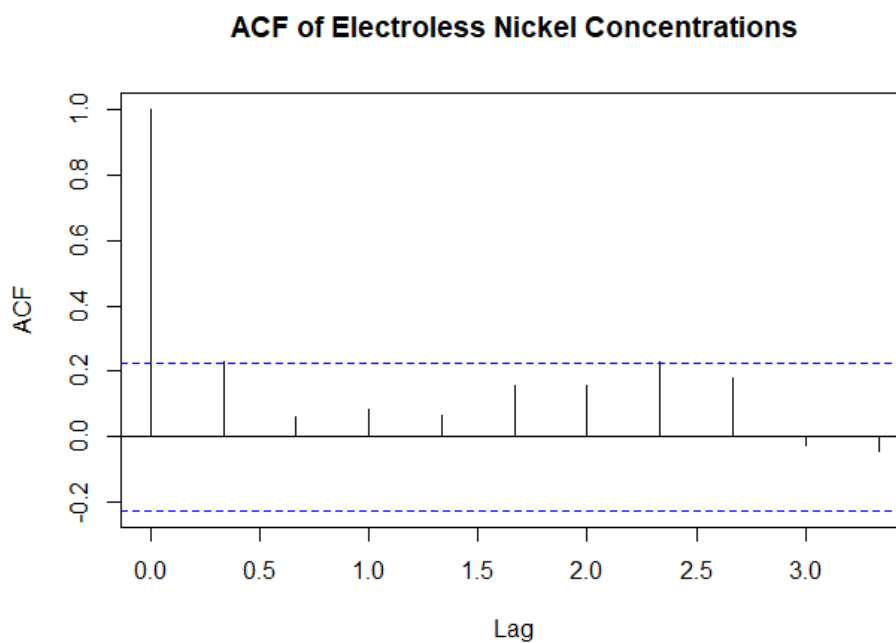
# DATA315 Assignment 3

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1. (a) 

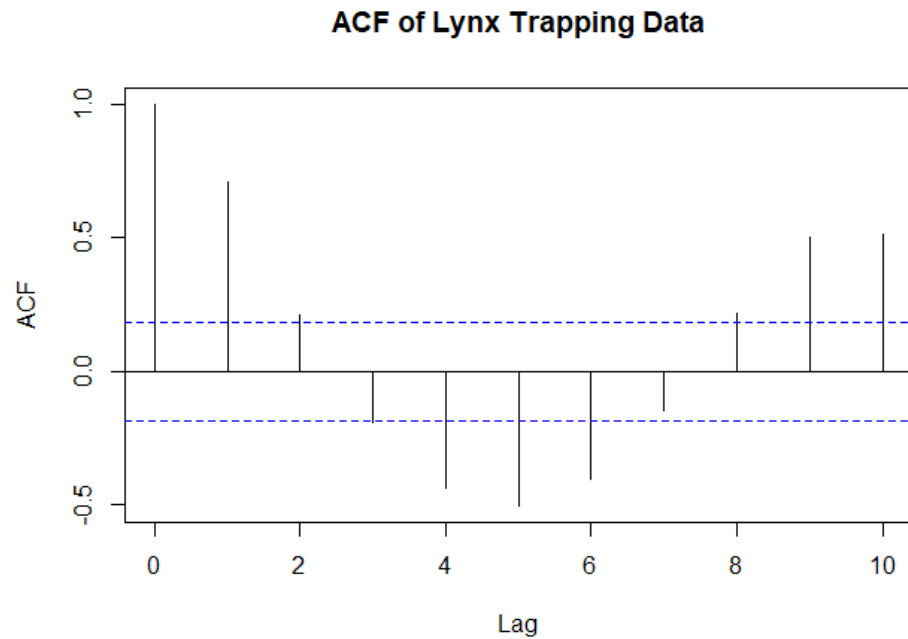
```
source("nickel.R")  
acf(nickel, lag.max = 10,  
main = "ACF of Electroless Nickel Concentrations")
```



The ACF plot seems to follow an MA(1) process, as significant correlation at lag 1 followed by immediate drop to near zero.

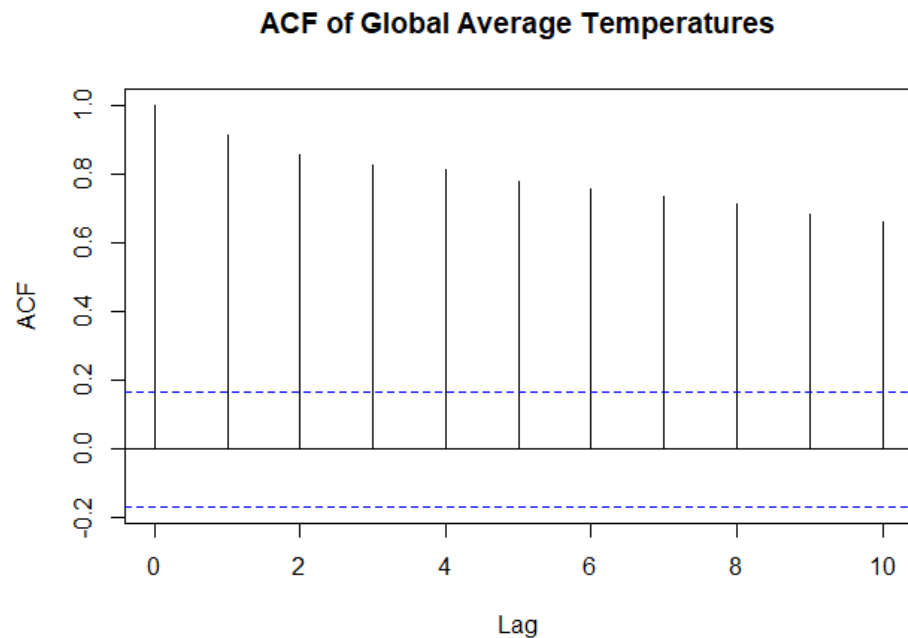
- (b) 

```
data(lynx)  
acf(lynx, lag.max = 10, main = "ACF of Lynx Trapping Data")
```



So far, there are no models that fit, because the plot shows a cyclic pattern between predator and prey populations.

```
(c) source("Globaltemps.R")
    temps <- ts(temps, start = 1880, end = 2016)
    acf(temps, lag.max = 10,
        main = "ACF of Global Average Temperatures")
```



The ACF plot seems to follow an AR(1) process, as significant correlation at lag 1 followed by gradual decay.

```
(d) data("EuStockMarkets")
```

```

dax <- EuStockMarkets[, 1]

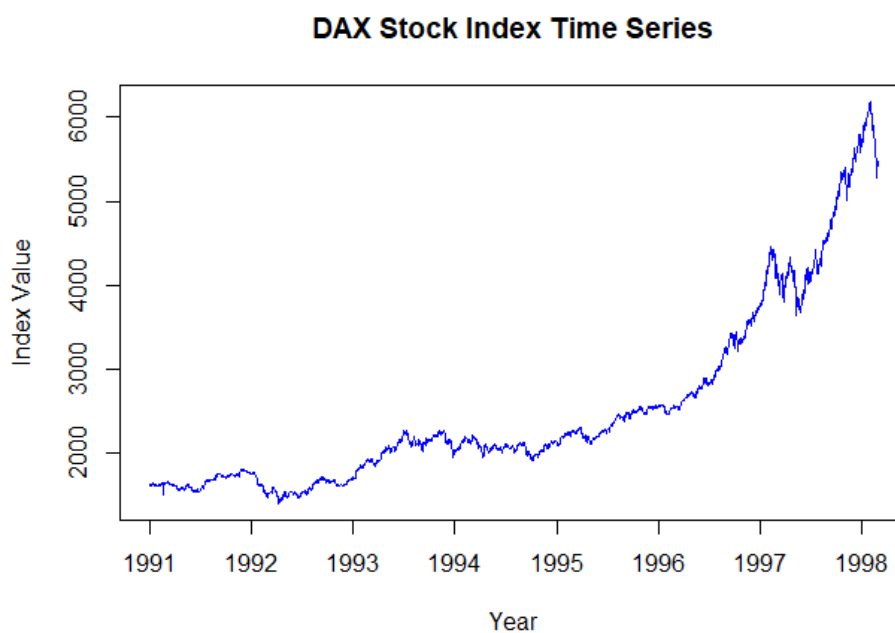
# 260 trading days per year
dax_ts <- ts(dax, start = c(1991, 1), frequency = 260)

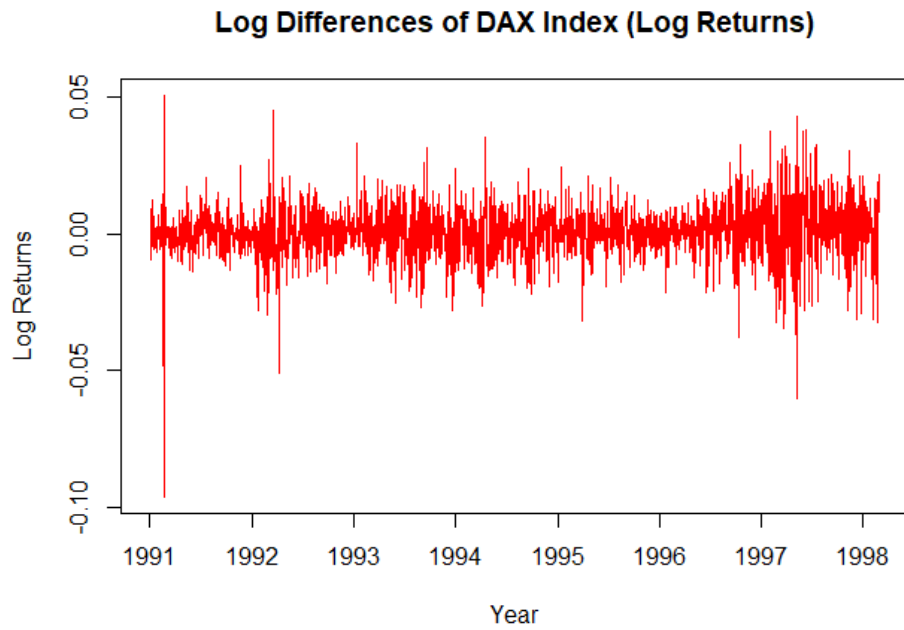
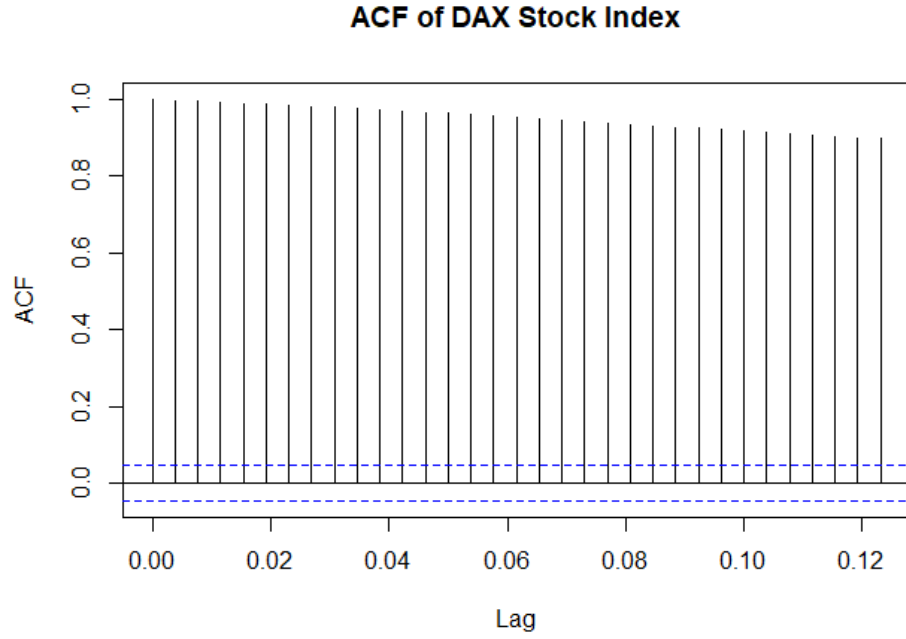
# Time series plot
plot(dax_ts,
     main = "DAX Stock Index Time Series",
     ylab = "Index Value", xlab = "Year",
     col = "blue", type = "l")

# ACF plot
acf(dax_ts, main = "ACF of DAX Stock Index")

# Take the natural log
log_dax <- log(dax_ts)
# Compute first differences (log returns)
diff_log_dax <- diff(log_dax)
acf(diff_log_dax, main = "ACF of Log Returns of DAX")

```





Some observations for the DAX stock index time series plot; visually, we can see that there is a general upward trend with some fluctuations. The ACF plot shows that there is a significant correlation at lag 1, followed by a gradual, slow decay, which means that it may be following other process we have not covered yet. The log returns plot shows that the data revolves around 0, which is a good sign for stationarity.

2. (a) Given that the MA(1) process is defined as

$$X_t = \mu_t + \epsilon_t + \theta\epsilon_{t-1}$$

Now we have to test whether the MA parameter is equal to 0.

```

# Print the model summary
summary(ma1_model)

Call:
arima(x = nickel, order = c(0, 0, 1))

Coefficients:
            ma1  intercept
            0.2260      4.6223
s.e.    0.1099      0.0277

sigma^2 estimated as 0.03857:
log likelihood = 15.63,  aic = -25.26

# Extract MA(1) coefficient and its standard error
theta_hat <- ma1_model$coef["ma1"]
se_theta <- sqrt(ma1_model$var.coef["ma1", "ma1"])

# Compute t-statistic
t_value <- theta_hat / se_theta

# Compute p-value (two-tailed test)
p_value <- 2 * (1 - pnorm(abs(t_value)))

# Print results
t_value
      ma1
2.05681
p_value
      ma1
0.03970446

```

The fitted model is  $X_t = 4.6223 + \epsilon_t + 0.2260\epsilon_{t-1}$ . Since the p-value is less than 0.05, we reject the null hypothesis that the MA parameter is equal to 0. Forecasting the 2nd and 3rd values after the end of the series, we can use

$$X_{t+1} = 4.6223 + \epsilon_{t+1} + 0.2260\epsilon_t$$

$$X_{t+2} = 4.6223 + \epsilon_{t+2} + 0.2260\epsilon_{t+1}$$

- (b) The portmanteau test checks whether the residuals from our fitted MA(1) model behave like white noise, meaning they are uncorrelated. We can use the Box-Ljung test to check this.

```

# Perform the Box-Ljung test
Box.test(ma1_model$residuals, lag = 10, type = "Ljung-Box")

library(forecast)
checkresiduals(ma1_model)

```

Ljung-Box test

```
data: Residuals from ARIMA(0,0,1) with non-zero mean
Q* = 2.5221, df = 5, p-value =
0.7732
```

```
Model df: 1. Total lags used: 6
```

Since the p-value is greater than 0.05, we fail to reject the null hypothesis that the residuals are uncorrelated.

- (c) If the 75th value is missing, we can forecast it using the fitted MA(1) model.

$$X_{75} = 4.6223 + \epsilon_{75} + 0.2260\epsilon_{74}$$

Now we can extract the last residuals

```
# Extract last residual
epsilon_74 <- residuals(ma1_model)[74]

# Compute forecast
X_75_hat <- 4.6223 + (0.2260 * epsilon_74)
X_75_hat
[1] 4.545646
```

The forecasted value for  $X_{75}$  is 4.545646. The standard deviation of the forecast is given by

$$\sigma_{\text{forecast}} = \sqrt{\hat{\sigma}^2} \sqrt{1 + \theta^2}$$

From the model, we have

$$\hat{\sigma}^2 = 0.03857$$

$$\theta = 0.2260$$

So we can calculate the standard deviation of the forecast.

$$\sigma_{\text{forecast}} = \sqrt{0.03857} \sqrt{1 + 0.2260^2} = 0.2013455$$

The error in terms of standard deviations is given by

$$Z = \frac{X_{75} - X_{75}^{\text{forecast}}}{\sigma_{\text{forecast}}} = \frac{4.3 - 4.545646}{0.2013455} = -1.220022$$

Since  $|Z| < 2$ , the forecast is within the 95% confidence interval.

- (d) First we fit an AR(1) model to the data.

```
ar1_model <- arima(nickel, order = c(1, 0, 0))

# Print model summary
summary(ar1_model)

Call:
arima(x = nickel, order = c(1, 0, 0))
```

Coefficients:

```
          ar1  intercept
      0.2363    4.6221
s.e.  0.1139    0.0295
```

```
sigma^2 estimated as 0.03845:
log likelihood = 15.74,  aic = -25.47
```

Then we forecast for the 2nd and 3rd values after the end of the series.

```
# Forecast 2 steps ahead
ar1_forecast <- predict(ar1_model, n.ahead = 3)
```

```
# Print forecasted values
ar1_forecast$pred
Time Series:
Start = c(26, 1)
End = c(26, 3)
Frequency = 3
[1] 4.545956 4.604083 4.617820
```

Now we check if the residuals are white noise.

```
# Perform Ljung-Box test on AR(1) residuals
Box.test(ar1_model$residuals, lag = 10, type = "Ljung-Box")
```

Box-Ljung test

```
data:  ar1_model$residuals
X-squared = 7.1631, df = 10, p-value = 0.71
```

Since the p-value is greater than 0.05, we fail to reject the null hypothesis that the residuals are white noise.

3. The given time series model is

$$y_t = \mu + \phi(y_{t-1} - \mu) + \varepsilon_t$$

The expected value of  $y_t$  is

$$\begin{aligned} E(y_t) &= \mu + \phi(E(y_{t-1}) - \mu) \\ &= \mu + \phi(\mu - \mu) = \mu \end{aligned}$$

$\mu$  can be estimated by the sample mean:

$$\hat{\mu} = \frac{1}{n} \sum_{t=1}^n y_t$$

for the given data  $\{3.2, 3.2, 2.2, 2.3, 1.8, 1.3, 2.2, 2.7\}$

$$\hat{\mu} = \frac{1}{8}(3.2 + 3.2 + 2.2 + 2.3 + 1.8 + 1.3 + 2.2 + 2.7) = 2.3625$$

$$\hat{\mu} = 2.3625$$

Now, we will estimate  $\phi$ . The autocovariance at lag 1 is given by

$$\gamma_1 = E[(y_t - \mu)(y_{t-1} - \mu)]$$

which can be estimated by

$$\hat{\gamma}_1 = \frac{1}{n} \sum_{t=2}^n (y_t - \hat{\mu})(y_{t-1} - \hat{\mu})$$

similarly, the variance is

$$\gamma_0 = E[(y_t - \mu)^2]$$

which can be estimated by

$$\hat{\gamma}_0 = \frac{1}{n-1} \sum_{t=1}^n (y_t - \hat{\mu})^2$$

Since for this process, the autocorrelation at lag 1 is given by we can estimate  $\phi$  by

$$\hat{\phi} = \frac{\hat{\gamma}_1}{\hat{\gamma}_0}$$

Computing the estimates for  $\gamma_1$  and  $\gamma_0$ :

$$\hat{\gamma}_0 = \frac{1}{8} \sum_{t=1}^8 (y_t - 2.3625)^2 = 0.3773438$$

$$\hat{\gamma}_1 = \frac{1}{8} \sum_{t=2}^8 (y_t - 2.3625)(y_{t-1} - 2.3625) = 0.189442$$

So the estimate for  $\phi$  is

$$\hat{\phi} = \frac{0.189442}{0.3773438} = 0.502$$

Now we can estimate  $\sigma$  by

$$\hat{\sigma}^2 = \hat{\gamma}_0(1 - \hat{\phi}^2) = 0.282$$

$$\hat{\sigma} = \sqrt{0.282} = 0.531$$

4. To be done

5. Given the TS

$$x_t = 0.5x_{t-1}$$

for  $t = 1, 2, \dots, n$  and  $x_0 = 0$ .

(a)  $x_1$  and  $x_2$  can be found by,

$$x_1 = 0.5x_0 = 0.5(0) = 0$$

$$x_2 = 0.5x_1 = 0.5(0) = 0$$



(b) A formula for  $x_t$  in terms of  $t$  is

$$x_t = 0$$

(c) The  $\lim_{t \rightarrow \infty} x_t$  is

$$\lim_{t \rightarrow \infty} x_t = 0$$

(d) Repeating (a), (b), and (c) for where  $x_0 = 1$ .

$$x_1 = 0.5x_0 = 0.5(1) = 0.5$$

$$x_2 = 0.5x_1 = 0.5(0.5) = 0.25$$

$$\lim_{t \rightarrow \infty} x_t = 0$$

because of geometric sequence convergence.

6. Given that  $x_0 = 10$  and

$$x_t = 0.8x_{t-1}$$

for  $t = 1, 2, 3, \dots, n$ .

(a)  $x_1, x_2, x_3, x_4$  can be found,

$$x_1 = 0.8x_0 = 0.8(10) = 8$$

$$x_2 = 0.8x_1 = 0.8(8) = 6.4$$

$$x_3 = 0.8x_2 = 0.8(6.4) = 5.12$$

$$x_4 = 0.8x_3 = 0.8(5.12) = 4.096$$

(b) A formula for  $x_t$  in terms of  $t$  is

$$x_t = 10(0.8)^t$$

(c) The  $\lim_{t \rightarrow \infty} x_t$  is

$$\lim_{t \rightarrow \infty} x_t = 0$$

(d) Sketching the plot of  $x_t$  vs.  $x_{t-1}$  for  $t = 1, 2, 3, \dots, 10$

```
# Initial value
x0 <- 10
```

```
# Part (a): Compute the sequence xt = 0.8 * xt-1
t_values <- 0:10
x_values <- numeric(length(t_values))
x_values[1] <- x0
```

```
for (t in 2:length(t_values)) {
  x_values[t] <- 0.8 * x_values[t-1]
}
```

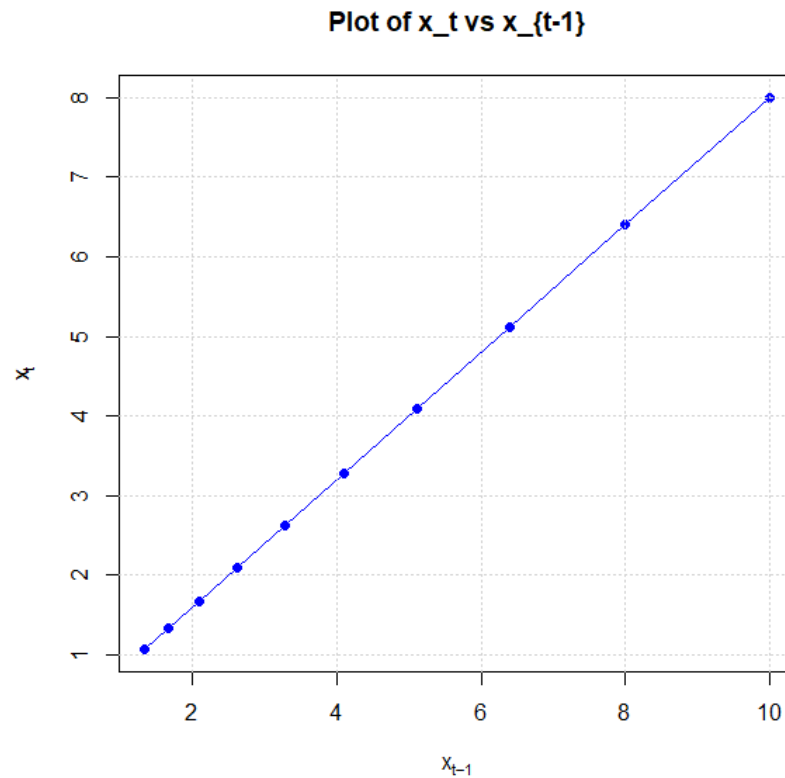
```
# Extract x_t and x_{t-1}
```

```

x_t <- x_values[-1] # Remove x0
x_t_minus_1 <- x_values[-length(x_values)] # Remove last value

# Plot x_t vs. x_{t-1}
plot(x_t_minus_1, x_t, type="b", col="blue", pch=16,
     xlab="x_{t-1}", ylab="x_t", main="x_t vs. x_{t-1}")

```



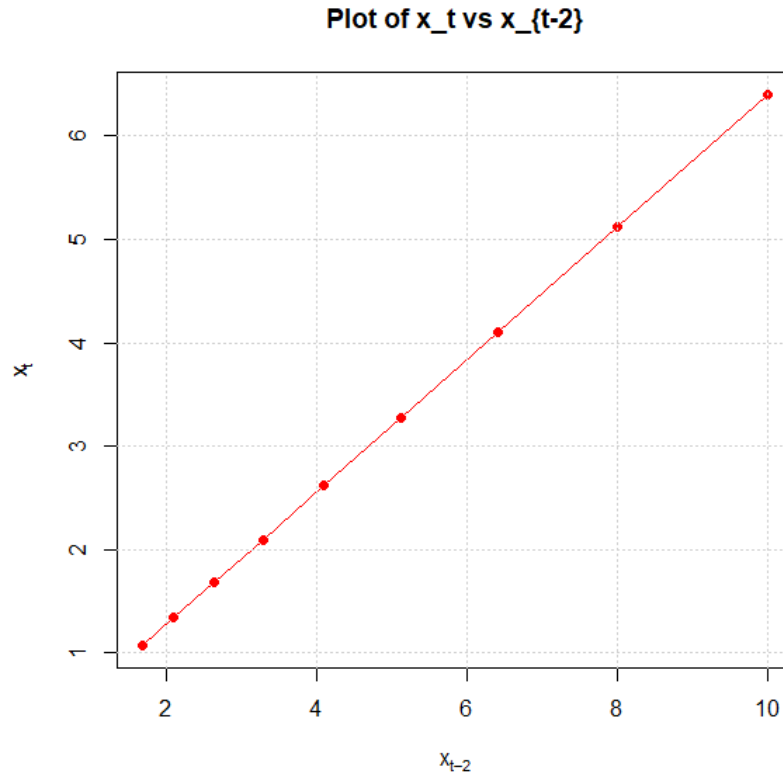
(e) Sketching the plot of  $x_t$  vs.  $x_{t-2}$  for  $t = 2, 3, 4, \dots, 10$

```

# Part (e): Plot x_t vs. x_{t-2}
x_t_minus_2 <- x_values[-c(length(x_values),
length(x_values)-1)] # Remove last two
x_t_2 <- x_values[-c(1,2)] # Remove first two

plot(x_t_minus_2, x_t_2, type="b", col="red", pch=16,
     xlab="x_{t-2}", ylab="x_t", main="x_t vs. x_{t-2}")

```



(f) Repeating (a), (b), (c), (d), and (e) for

$$x_t = 0.8x_{t-1} + z_t$$

where  $z_t, \dots, z_n$  take on the values

$$\{-1.2, 0.2, -1.0, 0.5, 1.7, -0.5, -2.1, 1.0, 0.8, -0.1\}$$

i.  $x_1, x_2, x_3, x_4$  can be found,

$$x_1 = 0.8x_0 + z_1 = 0.8(10) - 1.2 = 8 - 1.2 = 6.8$$

$$x_2 = 0.8x_1 + z_2 = 0.8(6.8) + 0.2 = 5.44 + 0.2 = 5.64$$

$$x_3 = 0.8x_2 + z_3 = 0.8(5.64) - 1.0 = 4.512 - 1.0 = 3.512$$

$$x_4 = 0.8x_3 + z_4 = 0.8(3.512) + 0.5 = 2.8096 + 0.5 = 3.3096$$

ii. A formula for  $x_t$  in terms of  $t$  is

$$x_t = 10(0.8)^t + \sum_{i=1}^t z_i$$

iii. The  $\lim_{t \rightarrow \infty} x_t$  is

$$\lim_{t \rightarrow \infty} x_t = 0$$

iv. Sketching the plot of  $x_t$  vs.  $x_{t-1}$  for  $t = 1, 2, 3, \dots, 10$

```

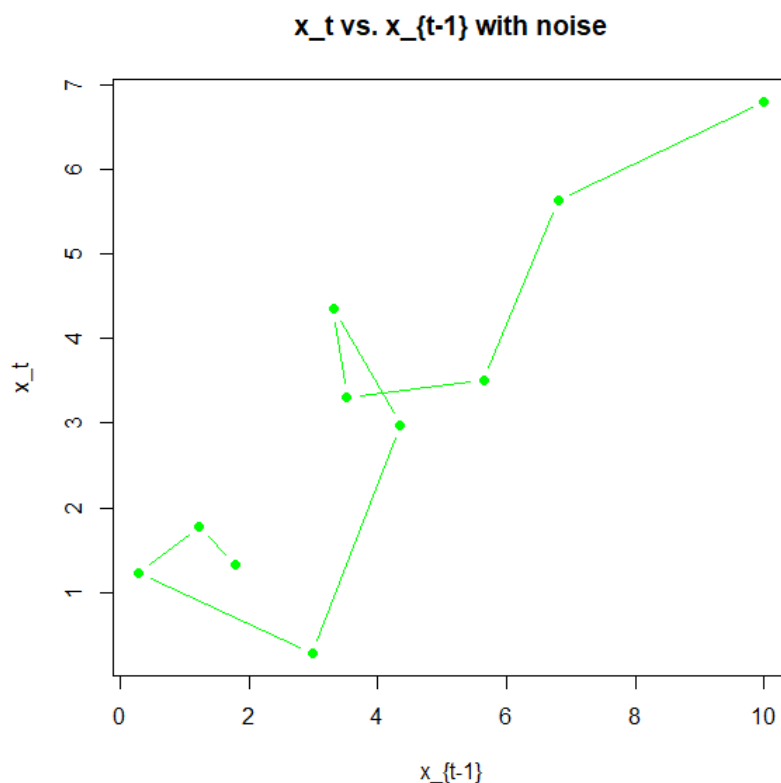
# Part (f): Compute the noisy sequence  $x_t = 0.8 * x_{t-1} + z_t$ 
z_values <- c(-1.2, 0.2, -1.0, 0.5, 1.7,
-0.5, -2.1, 1.0, 0.8, -0.1)
x_values_noise <- numeric(length(t_values))
x_values_noise[1] <- x0

for (t in 2:length(t_values)) {
  x_values_noise[t] <- 0.8 * x_values_noise[t-1]
  + z_values[t-1]
}

# Extract  $x_t$  and  $x_{t-1}$  for noisy data
x_t_noise <- x_values_noise[-1]
x_t_minus_1_noise <- x_values_noise[-length(x_values_noise)]

plot(x_t_minus_1_noise, x_t_noise, type="b",
col="green", pch=16,
xlab="x_{t-1}", ylab="x_t",
main="x_t vs. x_{t-1} with noise")

```



- v. Sketching the plot of  $x_t$  vs.  $x_{t-2}$  for  $t = 2, 3, 4, \dots, 10$

```

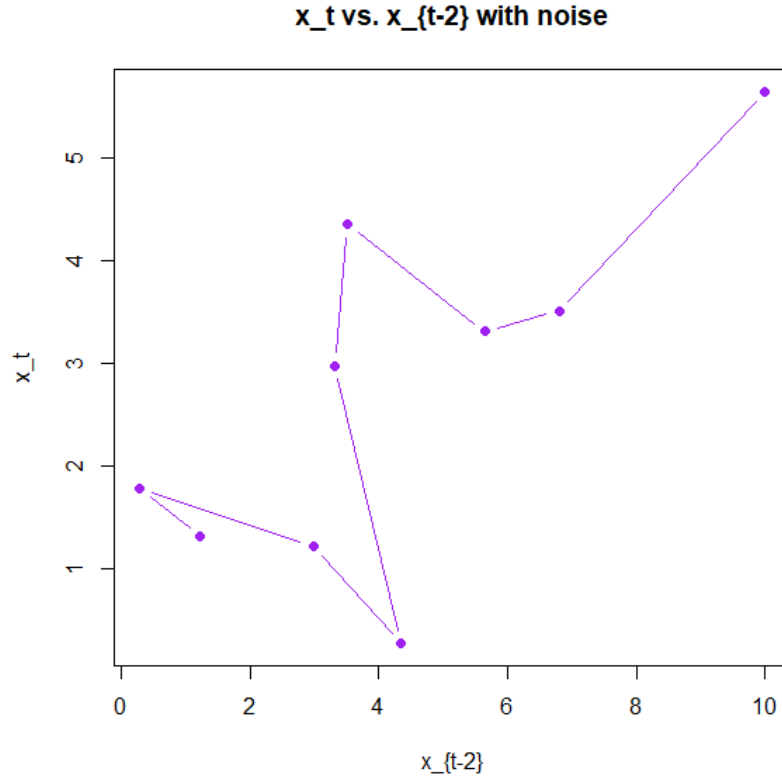
# Part (e) with noise:  $x_t$  vs.  $x_{t-2}$ 
x_t_minus_2_noise <-
x_values_noise[-c(length(x_values_noise),
length(x_values_noise)-1)]
x_t_2_noise <- x_values_noise[-c(1,2)]

```

```

plot(x_t_minus_2_noise, x_t_2_noise, type="b",
col="purple", pch=16,
xlab="x_{t-2}", ylab="x_t",
main="x_t vs. x_{t-2} with noise")

```



7. Given that  $x_0 = 2$  and  $x_1 = 1$  for

$$x_t = 0.8x_{t-1} - 0.7x_{t-2}$$

for  $t = 2, 3, 4, \dots, n$

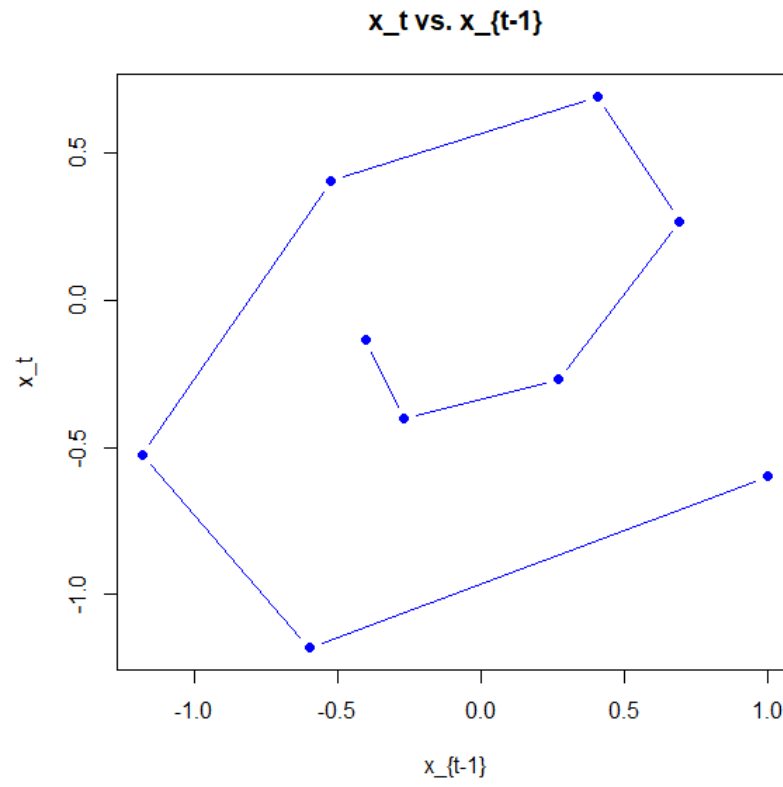
(a)  $x_2, x_3, x_4$  can be found,

$$x_2 = 0.8x_1 - 0.7x_0 = 0.8(1) - 0.7(2) = 0.8 - 1.4 = -0.6$$

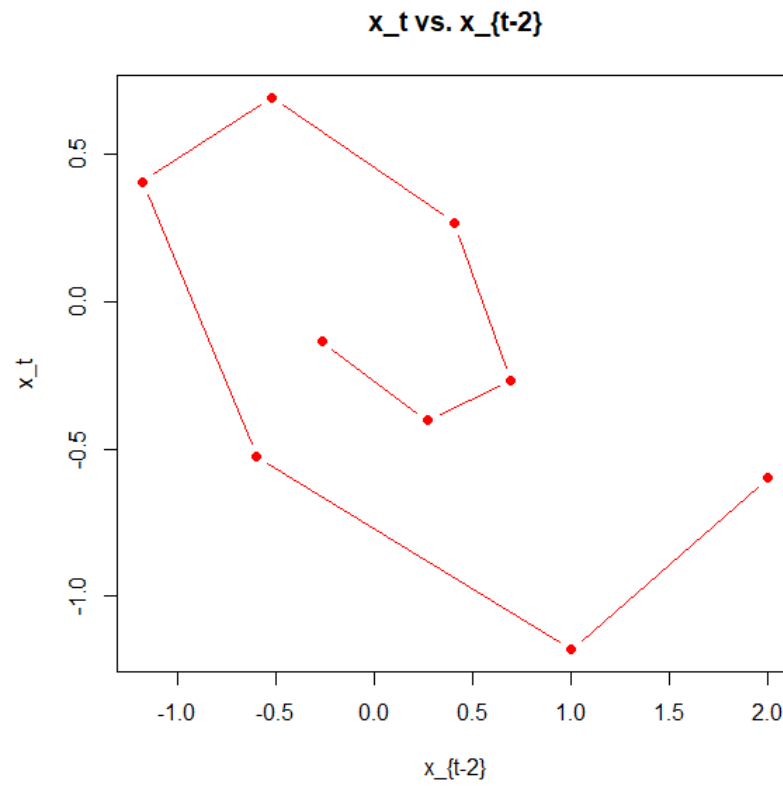
$$x_3 = 0.8x_2 - 0.7x_1 = 0.8(-0.6) - 0.7(1) = -0.48 - 0.7 = -1.18$$

$$x_4 = 0.8x_3 - 0.7x_2 = 0.8(-1.18) - 0.7(-0.6) = -0.944 - (-0.42) = -0.524$$

(b) The sketched plot of  $x_t$  vs.  $x_{t-1}$  is for  $t = 2, 3, \dots, 10$  is shown below.



(c) The sketched plot of  $x_t$  vs.  $x_{t-2}$  is for  $t = 2, 3, \dots, 10$  is shown below.



(d) Repeat (a), (b), and (c) for

$$x_t = 0.8x_{t-1} - 0.7x_{t-2} + z_t$$

where  $z_2, \dots, z_{11}$  take on the values

$$\{-1.2, 0.2, -1.0, 0.5, 1.7, -0.5, -2.1, 1.0, 0.8, -0.1\}$$

i.  $x_2, x_3, x_4$  can be found,

$$x_2 = 0.8x_1 - 0.7x_0 + z_2 = 0.8(1) - 0.7(2) + 0.2 = 0.8 - 1.4 + 0.2 = -0.4$$

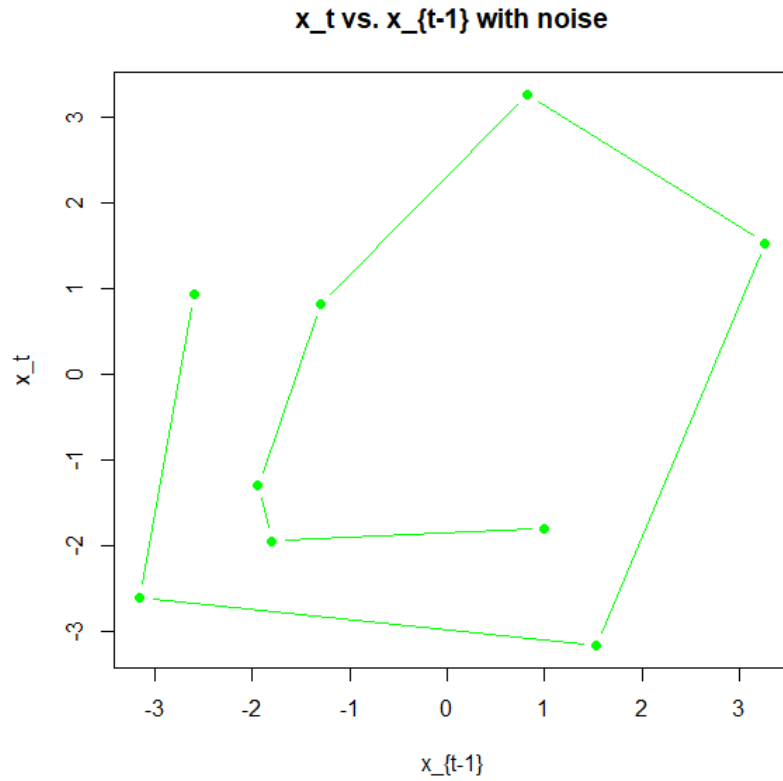
$$x_3 = 0.8x_2 - 0.7x_1 + z_3$$

$$= 0.8(-0.4) - 0.7(1) - 1.0 = -0.32 - 0.7 - 1.0 = -2.02$$

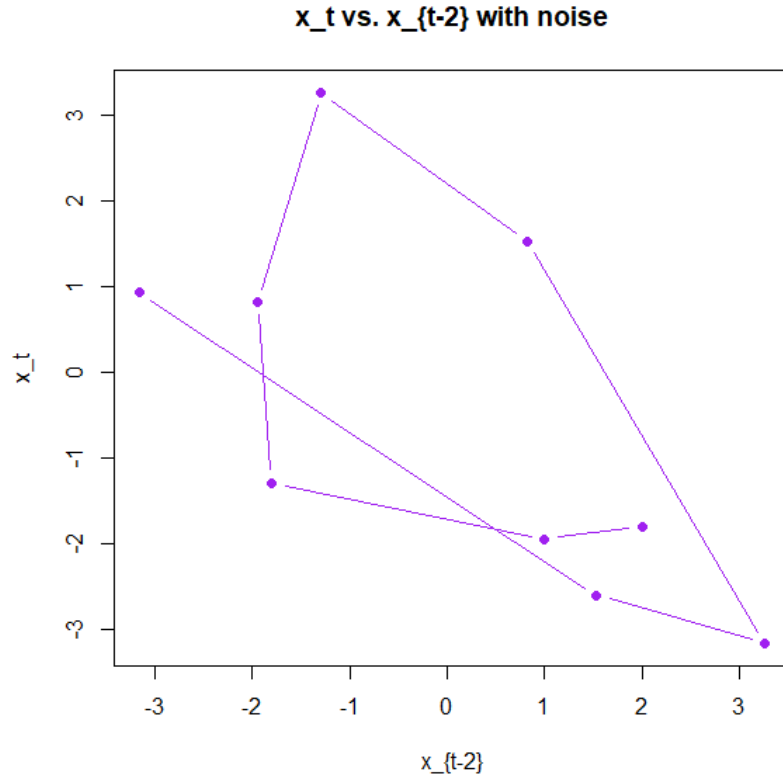
$$x_4 = 0.8x_3 - 0.7x_2 + z_4$$

$$= 0.8(-2.02) - 0.7(-0.4) + 0.5 = -1.616 - 0.28 + 0.5 = -1.396$$

ii. The sketched plot of  $x_t$  vs.  $x_{t-1}$  is for  $t = 2, 3, \dots, 10$  is shown below.



iii. The sketched plot of  $x_t$  vs.  $x_{t-2}$  is for  $t = 2, 3, \dots, 10$  is shown below.



8. The given TS

$$x_t = 0.8x_{t-1} + z_t$$

$z_t \sim N(0, 1)$  and  $x_0 = 0$ .

(a) The distribution of  $x_1$  its

$$x_1 = 0.8x_0 + z_1 = 0.8(0) + z_1 = z_1$$

$$E[x_1] = 0$$

$$Var[x_1] = 1$$

Since  $z_1 \sim N(0, 1)$ ,  $x_1 \sim N(0, 1)$ .

(b) The distribution of  $x_2$  is

$$x_2 = 0.8x_1 + z_2 = 0.8z_1 + z_2$$

$$E[x_2] = 0$$

$$Var[x_2] = Var(0.8x_1) + Var(z_2) = 0.8^2(1) + 1 = 1.64$$

Since  $z_1, z_2 \sim N(0, 1)$ ,  $x_2 \sim N(0, 1.64)$ .

(c) The distribution of  $x_3$  is

$$x_3 = 0.8x_2 + z_3 = 0.8(0.8z_1 + z_2) + z_3$$

$$= 0.64z_1 + 0.8z_2 + z_3$$

$$E[x_3] = 0$$

$$Var[x_3] = Var(0.8z_2) + Var(z_3) = 0.8^2(1.64) + 1 = 2.0496$$

Since  $z_1, z_2, z_3 \sim N(0, 1)$ ,  $x_3 \sim N(0, 2.0496)$ .



(d) If  $x_2$  takes the value 3, the point prediction for  $x_3$  is

$$E[x_3|x_2 = 3] = 0.64(0) + 0.8(3) = 2.4$$

(e) The distribution of the prediction error is

$$\text{Error} = x_3 - E[x_3|x_2]$$

Since  $x_3 = 0.8x_2 + z_3$  and  $E[x_3|x_2] = 0.8x_2$ ,

$$\text{Error} = (0.8x_2 + z_3) - 0.8x_2 = z_3$$

then it must be true that

$$\text{Error} \sim N(0, 1)$$

9. The AR(1) model with mean 0 is given by

$$x_t = \phi x_{t-1} + z_t$$

where  $z_t \sim N(0, \sigma^2)$ . Then it must be true that the matrix form looks something like this

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & \cdots & 0 \\ 1 & \phi & 0 & \cdots & 0 \\ 0 & 1 & \phi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

The backshift operator  $B$  is defined as

$$Bx_t = x_{t-1}$$

then, the AR(1) model can be written as

$$e_t = x_t - \phi x_{t-1}$$

$$(1 - \phi B)x_t = e_t$$

Yes, any process can be written in terms of  $\epsilon_t$ . An AR(1) is a process that can be expressed as a sum of past shocks  $\epsilon_t$ , if we write it recursively,

$$x_t = \phi x_{t-1} + \epsilon_t$$

Substituting  $x_{t-1} = \phi x_{t-2} + \epsilon_{t-1}$ ,

$$x_t = \phi(\phi x_{t-2} + \epsilon_{t-1}) + \epsilon_t$$

$$= \phi^2 x_{t-2} + \phi \epsilon_{t-1} + \epsilon_t$$

Repeating this, we can get

$$x_t = \sum_{k=0}^{\infty} \phi^k \epsilon_{t-k}$$

and this sum converges if  $|\phi| < 1$ , but if  $|\phi| \geq 1$ , then the process is stationary, and the sum can be fully written in terms of  $\epsilon_t$ . If  $|\phi| \geq 1$ , then the process is non-stationary, and the sum does not converge, which cannot be fully written in terms of  $\epsilon_t$ .