DATA315 Assignment 3

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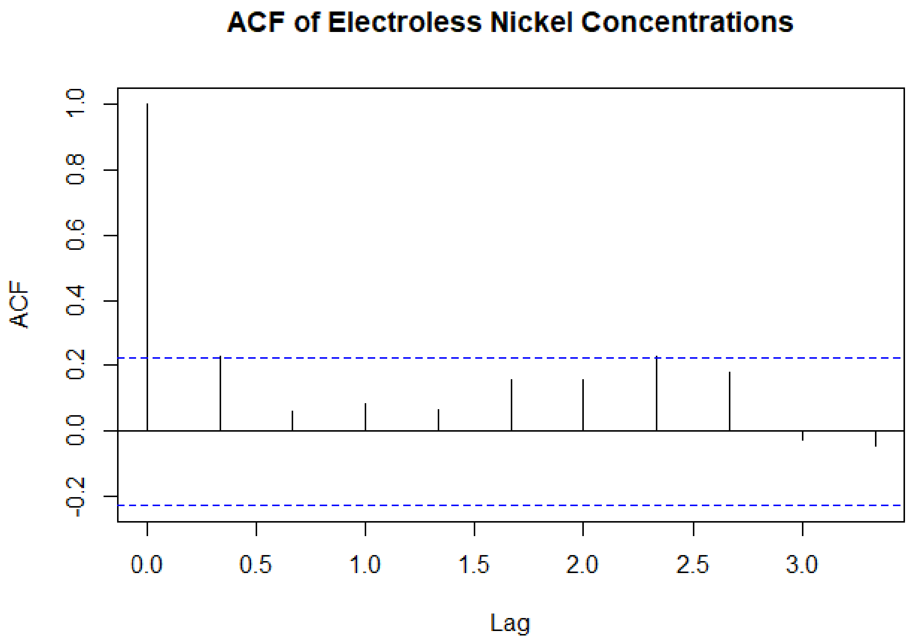
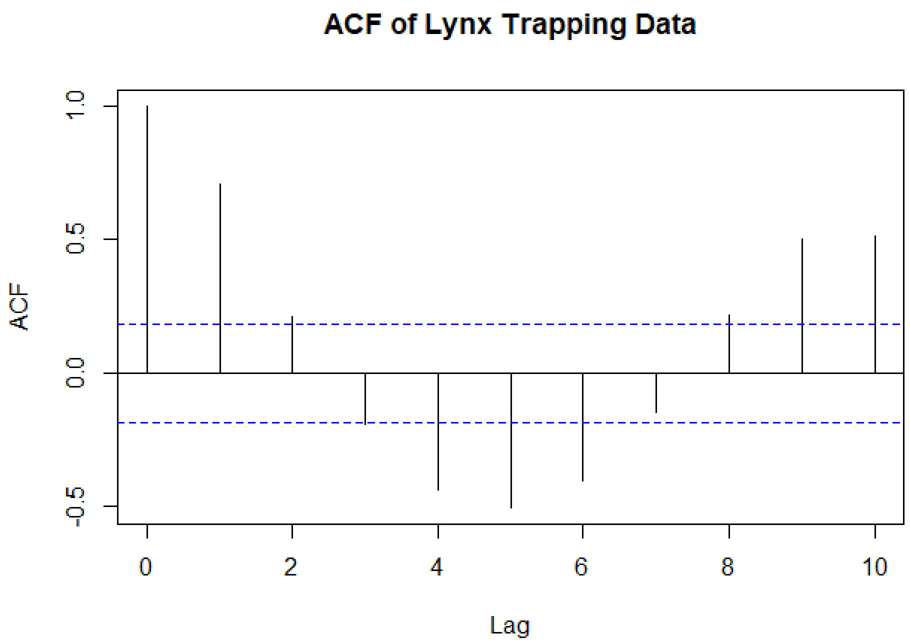
1. (a)
```
source("nickel.R")
acf(nickel, lag.max = 10,
main = "ACF of Electroless Nickel Concentrations")
```



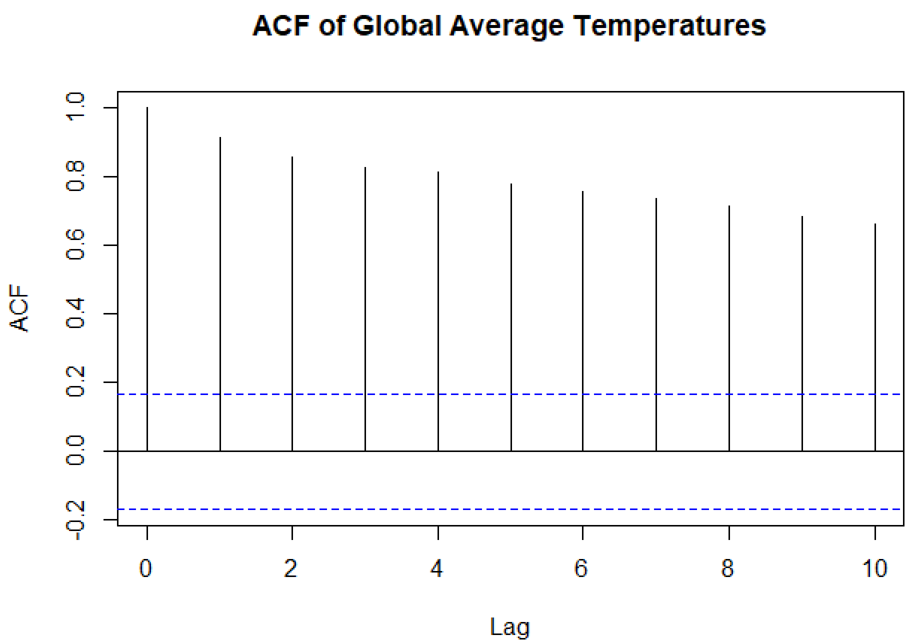
The ACF plot seems to follow an MA(1) process, as significant correlation at lag 1 followed by immediate drop to near zero.

(b)
```
data(lynx)
acf(lynx, lag.max = 10, main = "ACF of Lynx Trapping Data")
```



So far, there are no models that fit, because the plot shows a cyclic pattern between predator and prey populations.

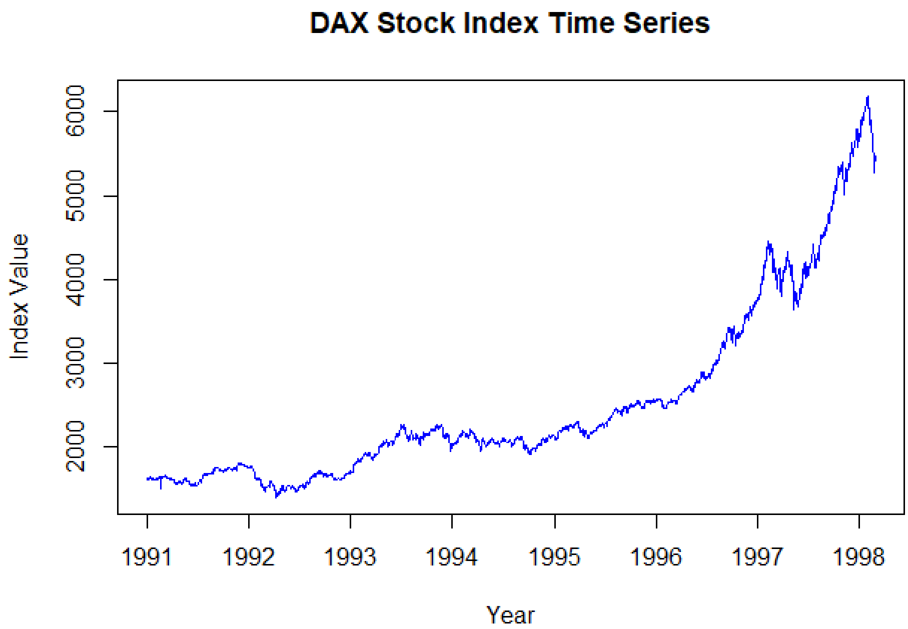
(c)
```
source("Globaltemps.R")
temps <- ts(temps, start = 1880, end = 2016)
acf(temps, lag.max = 10,
main = "ACF of Global Average Temperatures")
```

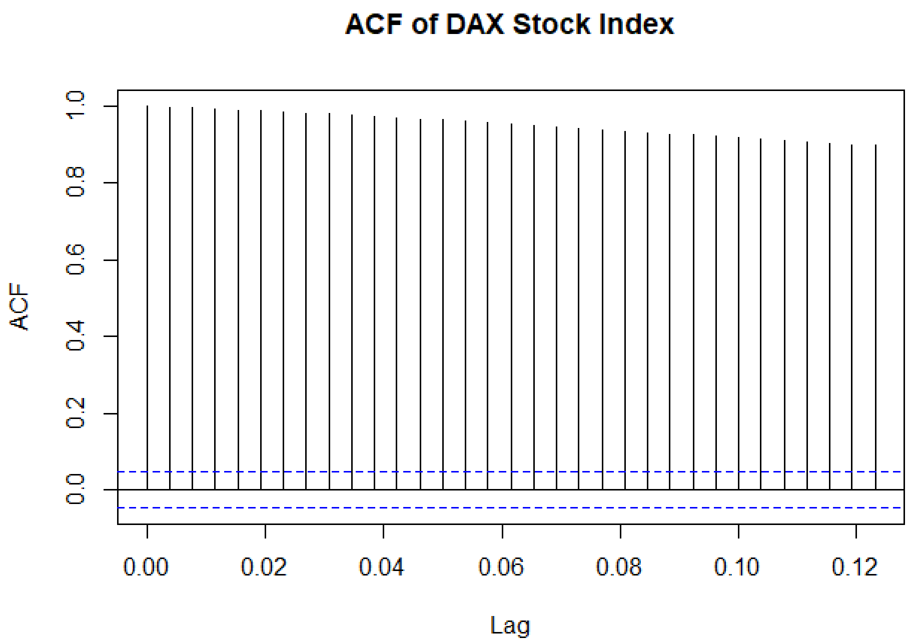


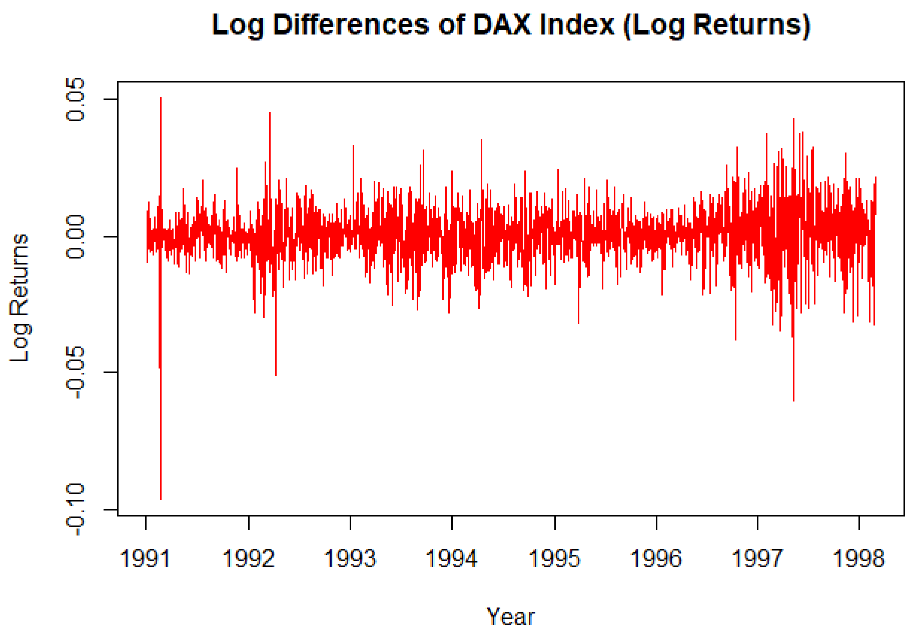
The ACF plot seems to follow an AR(1) process, as significant correlation at lag 1 followed by gradual decay.

(d)
```
data("EuStockMarkets")
```

```r
dax <- EuStockMarkets[, 1]

# 260 trading days per year
dax_ts <- ts(dax, start = c(1991, 1), frequency = 260)

# Time series plot
plot(dax_ts,
main = "DAX Stock Index Time Series",
ylab = "Index Value", xlab = "Year",
col = "blue", type = "l")

# ACF plot
acf(dax_ts, main = "ACF of DAX Stock Index")

# Take the natural log
log_dax <- log(dax_ts)
# Compute first differences (log returns)
diff_log_dax <- diff(log_dax)
acf(diff_log_dax, main = "ACF of Log Returns of DAX")
```







Some observations for the DAX stock index time series plot; visually, we can see that there is a general upward trend with some fluctuations. The ACF plot shows that there is a significant correlation at lag 1, followed by a gradual, slow decay, which means that it may be following other process we have not covered yet. The log returns plot shows that the data revolves around 0, which is a good sign for stationarity.

2. (a) Given that the MA(1) process is defined as

$$X_t = \mu_t + \epsilon_t + \theta\epsilon_{t-1}$$

Now we have to test whether the MA parameter is equal to 0.

```
# Print the model summary
summary(ma1_model)

Call:
arima(x = nickel, order = c(0, 0, 1))

Coefficients:
            ma1  intercept
         0.2260     4.6223
s.e.  0.1099     0.0277

sigma^2 estimated as 0.03857:
log likelihood = 15.63,  aic = -25.26

# Extract MA(1) coefficient and its standard error
theta_hat <- ma1_model$coef["ma1"]
se_theta <- sqrt(ma1_model$var.coef["ma1", "ma1"])

# Compute t-statistic
t_value <- theta_hat / se_theta

# Compute p-value (two-tailed test)
p_value <- 2 * (1 - pnorm(abs(t_value)))

# Print results
t_value
    ma1
2.05681
p_value
    ma1
0.03970446
```

The fitted model is $X_t = 4.6223 + \epsilon_t + 0.2260\epsilon_{t-1}$. Since the p-value is less than 0.05, we reject the null hypothesis that the MA parameter is equal to 0. Forecasting the 2nd and 3rd values after the end of the series, we can use

$$X_{t+1} = 4.6223 + \epsilon_{t+1} + 0.2260\epsilon_t$$

$$X_{t+2} = 4.6223 + \epsilon_{t+2} + 0.2260\epsilon_{t+1}$$

(b) The portmanteau test checks whether the residuals from our fitted MA(1) model behave like white noise, meaning they are uncorrelated. We can use the Box-Ljung test to check this.

```
# Perform the Box-Ljung test
Box.test(ma1_model$residuals, lag = 10, type = "Ljung-Box")

library(forecast)
checkresiduals(ma1_model)
```

```
Ljung-Box test

data:  Residuals from ARIMA(0,0,1) with non-zero mean
Q* = 2.5221, df = 5, p-value =
0.7732

Model df: 1.   Total lags used: 6
```

Since the p-value is greater than 0.05, we fail to reject the null hypothesis that the residuals are uncorrelated.

(c) If the 75th value is missing, we can forecast it using the fitted MA(1) model.

$$X_{75} = 4.6223 + \epsilon_{75} + 0.2260\epsilon_{74}$$

Now we can extract the last residuals

```
# Extract last residual
epsilon_74 <- residuals(ma1_model)[74]

# Compute forecast
X_75_hat <- 4.6223 + (0.2260 * epsilon_74)
X_75_hat
[1] 4.545646
```

The forecasted value for X_{75} is 4.545646. The standard deviation of the forecast is given by

$$\sigma_{\text{forecast}} = \sqrt{\hat{\sigma}^2}\sqrt{1+\theta^2}$$

From the model, we have

$$\hat{\sigma}^2 = 0.03857$$

$$\theta = 0.2260$$

So we can calculate the standard deviation of the forecast.

$$\sigma_{\text{forecast}} = \sqrt{0.03857}\sqrt{1+0.2260^2} = 0.2013455$$

The error in terms of standard deviations is given by

$$Z = \frac{X_{75} - X_{75}^{\text{forecast}}}{\sigma_{\text{forecast}}} = \frac{4.3 - 4.545646}{0.2013455} = -1.220022$$

Since $|Z| < 2$, the forecast is within the 95% confidence interval.

(d) First we fit an AR(1) model to the data.

```
ar1_model <- arima(nickel, order = c(1, 0, 0))

# Print model summary
summary(ar1_model)

Call:
arima(x = nickel, order = c(1, 0, 0))
```

```
Coefficients:
            ar1  intercept
        0.2363     4.6221
s.e.  0.1139     0.0295

sigma^2 estimated as 0.03845:
log likelihood = 15.74,  aic = -25.47
```

Then we forecast for the 2nd and 3rd values after the end of the series.

```
# Forecast 2 steps ahead
ar1_forecast <- predict(ar1_model, n.ahead = 3)

# Print forecasted values
ar1_forecast$pred
Time Series:
Start = c(26, 1)
End = c(26, 3)
Frequency = 3
[1] 4.545956 4.604083 4.617820
```

Now we check if the residuals are white noise.

```
# Perform Ljung-Box test on AR(1) residuals
Box.test(ar1_model$residuals, lag = 10, type = "Ljung-Box")

Box-Ljung test

data:  ar1_model$residuals
X-squared = 7.1631, df = 10, p-value = 0.71
```

Since the p-value is greater than 0.05, we fail to reject the null hypothesis that the residuals are white noise.

3. The given time series model is

$$y_t = \mu + \phi(y_{t-1} - \mu) + \varepsilon_t$$

The exepected value of y_t is

$$E(y_t) = \mu + \phi(E(y_{t-1}) - \mu)$$

$$= \mu + \phi(\mu - \mu) = \mu$$

μ can be estimated by the sample mean:

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} y_t$$

for the given data $\{3.2, 3.2, 2.2, 2.3, 1.8, 1.3, 2.2, 2.7\}$

$$\hat{\mu} = \frac{1}{8}(3.2 + 3.2 + 2.2 + 2.3 + 1.8 + 1.3 + 2.2 + 2.7) = 2.3625$$

$$\hat{\mu} = 2.3625$$

Now, we will estimate ϕ. The autocovariance at lag 1 is given by

$$\gamma_1 = E[(y_t - \mu)(y_{t-1} - \mu)]$$

which can be estimated by

$$\hat{\gamma}_1 = \frac{1}{n}\sum_{t=2}^{n}(y_t - \hat{\mu})(y_{t-1} - \hat{\mu})$$

similarly, the vairance its

$$\gamma_0 = E[(y_t - \mu)^2]$$

which can be estimated by

$$\hat{\gamma}_0 = \frac{1}{n-1}\sum_{t=1}^{n}(y_t - \hat{\mu})^2$$

Sine for this process, the autocorrealtion at lag 1 is given by we can estimate ϕ by

$$\hat{\phi} = \frac{\hat{\gamma}_1}{\hat{\gamma}_0}$$

Computing the estimates for γ_1 and γ_0:

$$\hat{\gamma}_0 = \frac{1}{8}\sum_{t=1}^{8}(y_t - 2.3625)^2 = 0.3773438$$

$$\hat{\gamma}_1 = \frac{1}{8}\sum_{t=2}^{8}(y_t - 2.3625)(y_{t-1} - 2.3625) = 0.189442$$

So the estimate for ϕ is

$$\hat{\phi} = \frac{0.189442}{0.3773438} = 0.502$$

Now we can estimate σ by

$$\hat{\sigma}^2 = \hat{\gamma}_0(1 - \hat{\phi}^2) = 0.282$$

$$\hat{\sigma} = \sqrt{0.282} = 0.531$$

4. To be done

5. Given the TS

$$x_t = 0.5x_{t-1}$$

for $t = 1, 2, \ldots, n$ and $x_0 = 0$.

(a) x_1 and x_2 can be found by,

$$x_1 = 0.5x_0 = 0.5(0) = 0$$

$$x_2 = 0.5x_1 = 0.5(0) = 0$$

(b) A formula for x_t in terms of t is

$$x_t = 0$$

(c) The $\lim_{t\to\infty} x_t$ is

$$\lim_{t\to\infty} x_t = 0$$

(d) Repeating (a), (b), and (c) for where $x_0 = 1$.

$$x_1 = 0.5x_0 = 0.5(1) = 0.5$$

$$x_2 = 0.5x_1 = 0.5(0.5) = 0.25$$

$$\lim_{t\to\infty} x_t = 0$$

because of geometric sequence convergence.

6. Given that $x_0 = 10$ and

$$x_t = 0.8x_{t-1}$$

for $t = 1, 2, 3, \ldots, n$.

(a) x_1, x_2, x_3, x_4 can be found,

$$x_1 = 0.8x_0 = 0.8(10) = 8$$

$$x_2 = 0.8x_1 = 0.8(8) = 6.4$$

$$x_3 = 0.8x_2 = 0.8(6.4) = 5.12$$

$$x_4 = 0.8x_3 = 0.8(5.12) = 4.096$$

(b) A formula for x_t in terms of t is

$$x_t = 10(0.8)^t$$

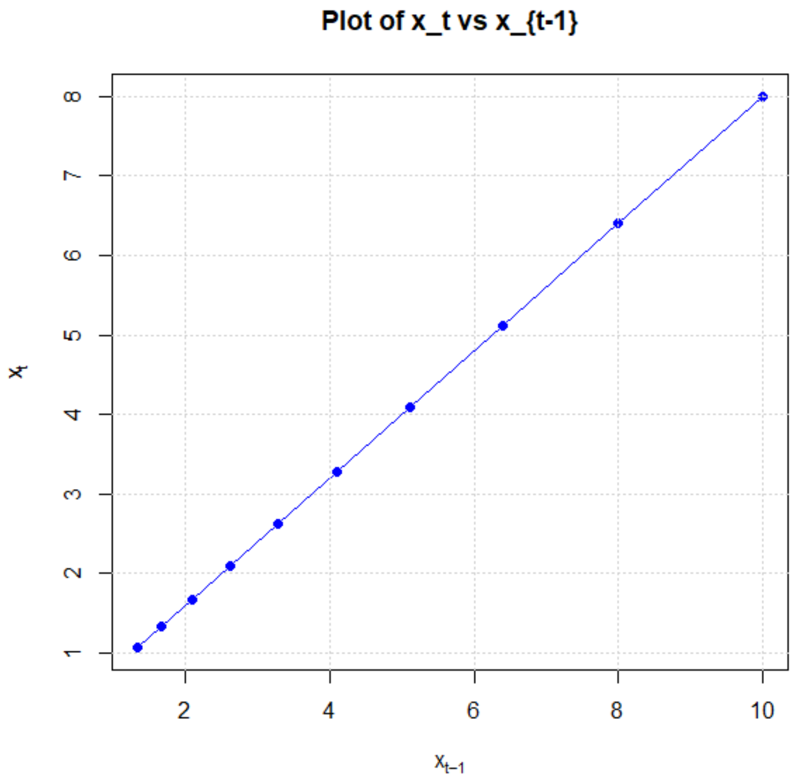
(c) The $\lim_{t\to\infty} x_t$ is

$$\lim_{t\to\infty} x_t = 0$$

(d) Sketching the plot of x_t vs. x_{t-1} for $t = 1, 2, 3, \ldots, 10$

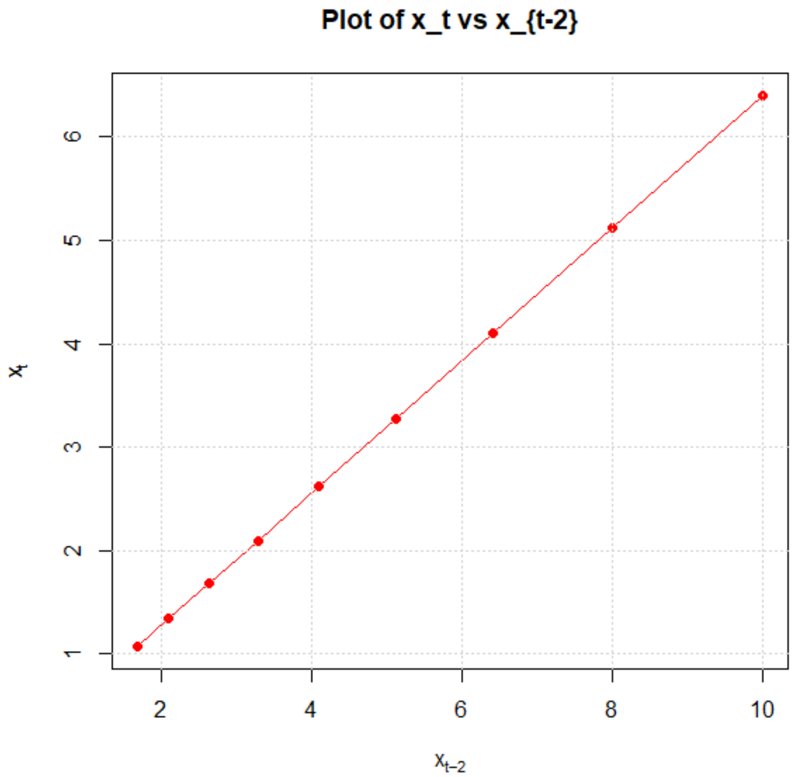
```
# Initial value
x0 <- 10

# Part (a): Compute the sequence xt = 0.8 * xt-1
t_values <- 0:10
x_values <- numeric(length(t_values))
x_values[1] <- x0

for (t in 2:length(t_values)) {
    x_values[t] <- 0.8 * x_values[t-1]
}

# Extract x_t and x_{t-1}
```

```
x_t <- x_values[-1]  # Remove x0
x_t_minus_1 <- x_values[-length(x_values)]  # Remove last value

# Plot x_t vs. x_{t-1}
plot(x_t_minus_1, x_t, type="b", col="blue", pch=16,
xlab="x_{t-1}", ylab="x_t", main="x_t vs. x_{t-1}")
```



(e) Sketching the plot of x_t vs. x_{t-2} for $t = 2, 3, 4, \dots, 10$

```
# Part (e): Plot x_t vs. x_{t-2}
x_t_minus_2 <- x_values[-c(length(x_values),
length(x_values)-1)]  # Remove last two
x_t_2 <- x_values[-c(1,2)]  # Remove first two

plot(x_t_minus_2, x_t_2, type="b", col="red", pch=16,
xlab="x_{t-2}", ylab="x_t", main="x_t vs. x_{t-2}")
```



(f) Repeating (a), (b), (c), (d), and (e) for

$$x_t = 0.8x_{t-1} + z_t$$

where $z_t, \ldots, z_n$ take on the values

$$\{-1.2, 0.2, -1.0, 0.5, 1.7, -0.5, -2.1, 1.0, 0.8, -0.1\}$$

i. x_1, x_2, x_3, x_4 can be found,

$$x_1 = 0.8x_0 + z_1 = 0.8(10) - 1.2 = 8 - 1.2 = 6.8$$

$$x_2 = 0.8x_1 + z_2 = 0.8(6.8) + 0.2 = 5.44 + 0.2 = 5.64$$
$$x_3 = 0.8x_2 + z_3 = 0.8(5.64) - 1.0 = 4.512 - 1.0 = 3.512$$
$$x_4 = 0.8x_3 + z_4 = 0.8(3.512) + 0.5 = 2.8096 + 0.5 = 3.3096$$

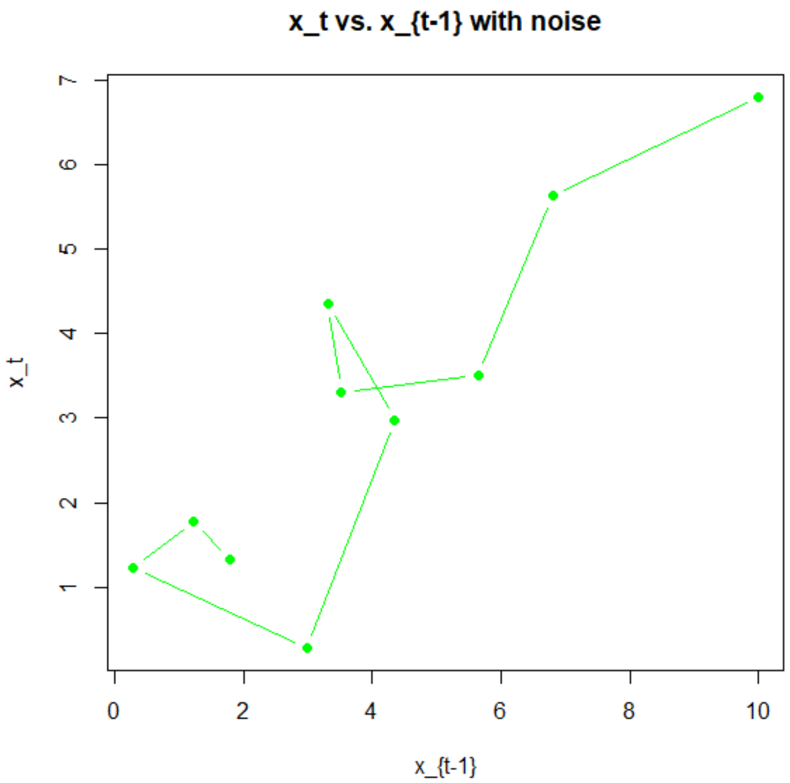
ii. A formula for x_t in terms of t is

$$x_t = 10(0.8)^t + \sum_{i=1}^{t} z_i$$

iii. The $\lim_{t\to\infty} x_t$ is
$$\lim_{t\to\infty} x_t = 0$$

iv. Sketching the plot of x_t vs. x_{t-1} for $t = 1, 2, 3, \ldots, 10$

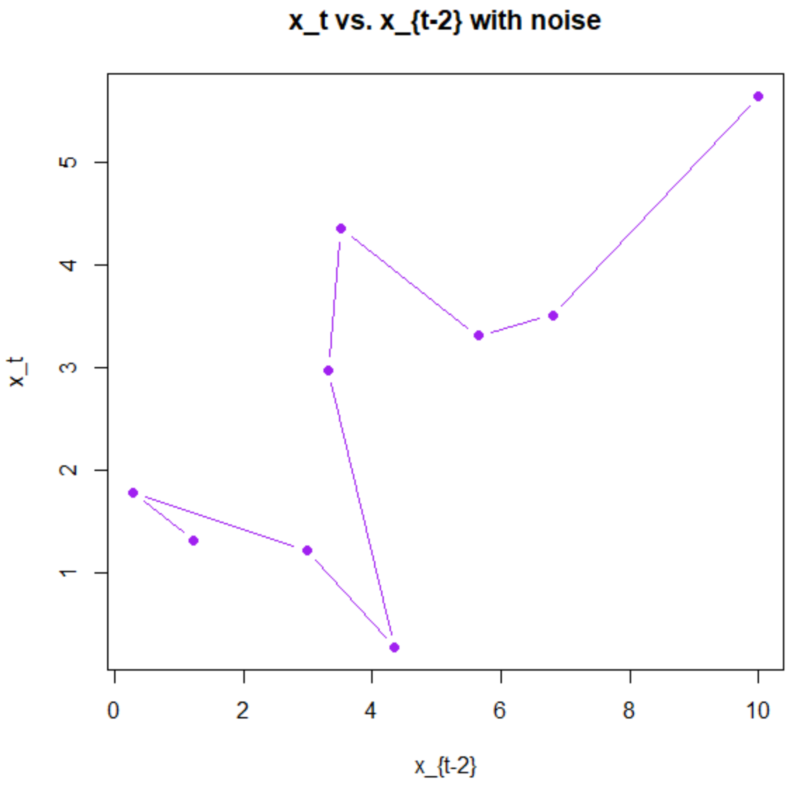
```
# Part (f): Compute the noisy sequence xt = 0.8 * xt-1 + zt
z_values <- c(-1.2, 0.2, -1.0, 0.5, 1.7,
-0.5, -2.1, 1.0, 0.8, -0.1)
x_values_noise <- numeric(length(t_values))
x_values_noise[1] <- x0

for (t in 2:length(t_values)) {
x_values_noise[t] <- 0.8 * x_values_noise[t-1]
+ z_values[t-1]
}

# Extract x_t and x_{t-1} for noisy data
x_t_noise <- x_values_noise[-1]
x_t_minus_1_noise <- x_values_noise[-length(x_values_noise)]

plot(x_t_minus_1_noise, x_t_noise, type="b",
col="green", pch=16,
xlab="x_{t-1}", ylab="x_t",
main="x_t vs. x_{t-1} with noise")
```



v. Sketching the plot of x_t vs. x_{t-2} for $t = 2, 3, 4, \ldots, 10$

```
# Part (e) with noise: x_t vs. x_{t-2}
x_t_minus_2_noise <-
x_values_noise[-c(length(x_values_noise),
 length(x_values_noise)-1)]
x_t_2_noise <- x_values_noise[-c(1,2)]
```

```
plot(x_t_minus_2_noise, x_t_2_noise, type="b",
col="purple", pch=16,
xlab="x_{t-2}", ylab="x_t",
main="x_t vs. x_{t-2} with noise")
```



7. Given that $x_0 = 2$ and $x_1 = 1$ for

$$x_t = 0.8x_{t-1} - 0.7x_{t-2}$$

for $t = 2, 3, 4, \ldots, n$

(a) x_2, x_3, x_4 can be found,

$$x_2 = 0.8x_1 - 0.7x_0 = 0.8(1) - 0.7(2) = 0.8 - 1.4 = -0.6$$

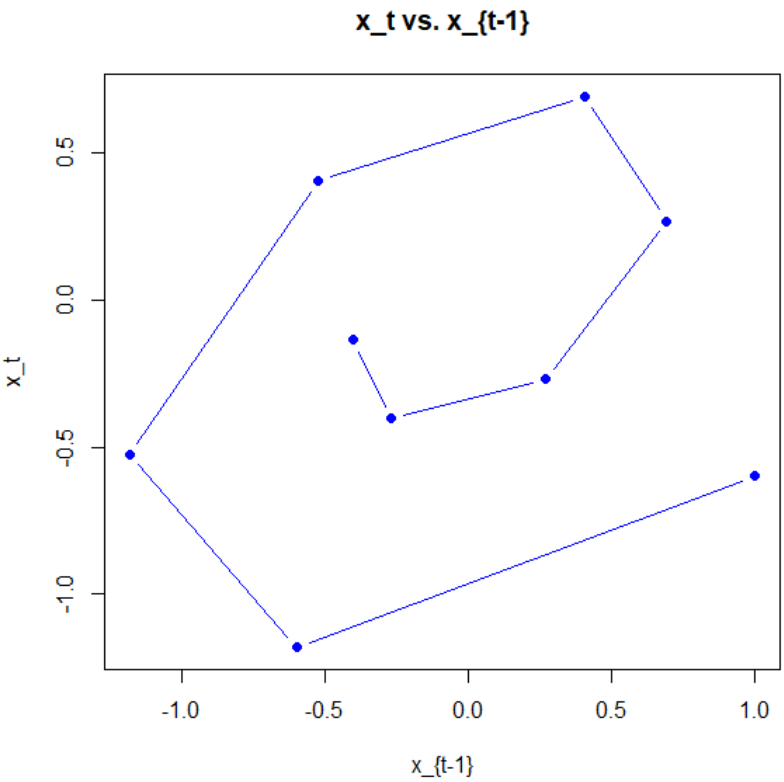
$$x_3 = 0.8x_2 - 0.7x_1 = 0.8(-0.6) - 0.7(1) = -0.48 - 0.7 = -1.18$$

$$x_4 = 0.8x_3 - 0.7x_2 = 0.8(-1.18) - 0.7(-0.6) = -0.944 - (-0.42) = -0.524$$

(b) The sketched plot of x_t vs. x_{t-1} is for $t = 2, 3, \ldots, 10$ is shown below.

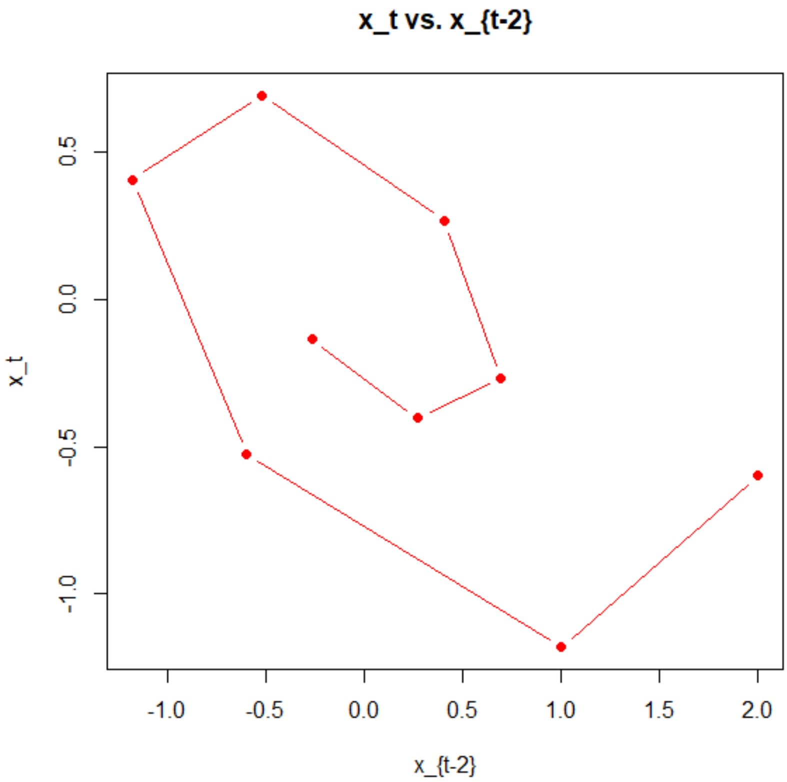
```
# Initial conditions
x_values <- numeric(11)
x_values[1] <- 2  # x0
x_values[2] <- 1  # x1

# Compute xt for t = 2 to 10 using
# recurrence relation xt = 0.8*xt-1 - 0.7*xt-2
for (t in 3:11) {
```

```
x_values[t] <- 0.8 * x_values[t-1] - 0.7 * x_values[t-2]
}

# Extract values for plotting
x_t <- x_values[3:11]      # From t=2 to 10
x_t_minus_1 <- x_values[2:10]  # From t=1 to 9
x_t_minus_2 <- x_values[1:9]   # From t=0 to 8

# Plot x_t vs. x_{t-1}
plot(x_t_minus_1, x_t, type="b", col="blue", pch=16,
xlab="x_{t-1}", ylab="x_t", main="x_t vs. x_{t-1}")
```



(c) The sketched plot of x_t vs. x_{t-2} is for $t = 2, 3, \ldots, 10$ is shown below.

```
# Plot x_t vs. x_{t-2}
plot(x_t_minus_2, x_t, type="b", col="red", pch=16,
xlab="x_{t-2}", ylab="x_t", main="x_t vs. x_{t-2}")
```



(d) Repeat (a), (b), and (c) for

$$x_t = 0.8x_{t-1} - 0.7x_{t-2} + z_t$$

where $z_2, \ldots, z_{11}$ take on the values

$$\{-1.2, 0.2, -1.0, 0.5, 1.7, -0.5, -2.1, 1.0, 0.8, -0.1\}$$

i. x_2, x_3, x_4 can be found,

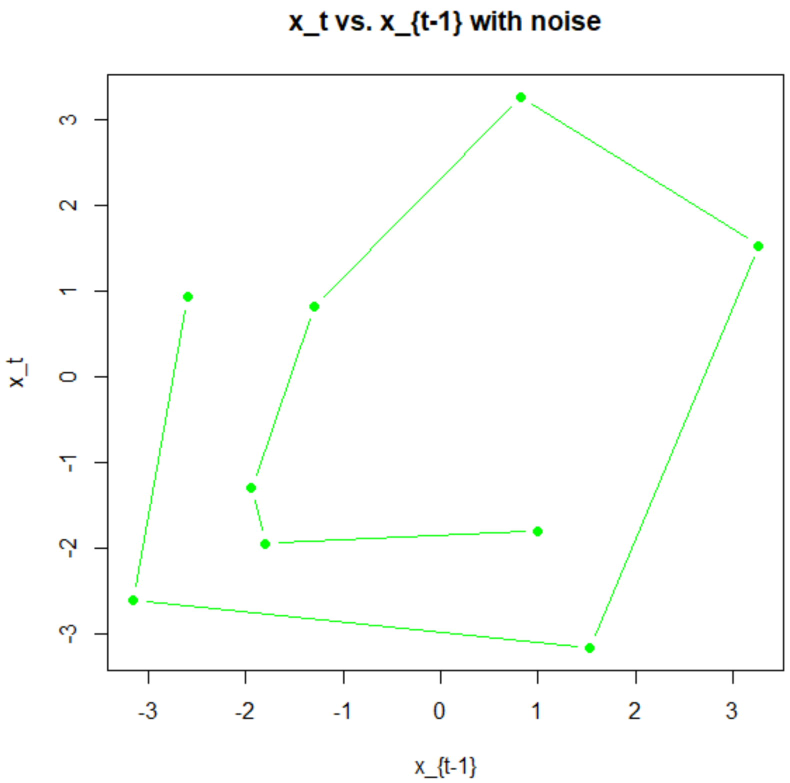
$$x_2 = 0.8x_1 - 0.7x_0 + z_2 = 0.8(1) - 0.7(2) + 0.2 = 0.8 - 1.4 + 0.2 = -0.4$$

$$x_3 = 0.8x_2 - 0.7x_1 + z_3$$
$$= 0.8(-0.4) - 0.7(1) - 1.0 = -0.32 - 0.7 - 1.0 = -2.02$$
$$x_4 = 0.8x_3 - 0.7x_2 + z_4$$
$$= 0.8(-2.02) - 0.7(-0.4) + 0.5 = -1.616 - 0.28 + 0.5 = -1.396$$

ii. The sketched plot of x_t vs. x_{t-1} is for $t = 2, 3, \ldots, 10$ is shown below.

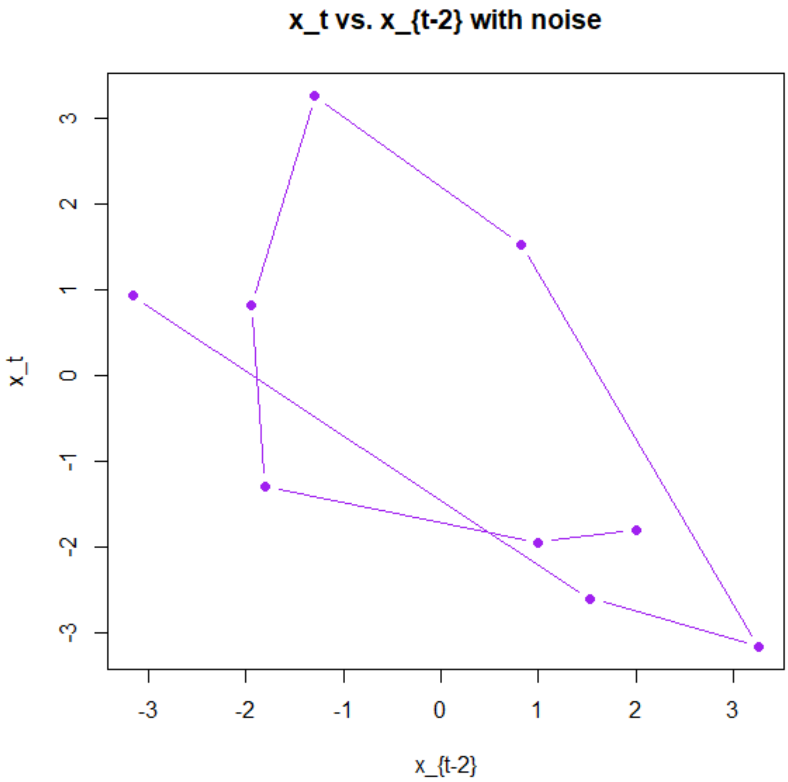
```
# Part (d): Compute with noise
z_values <- c(-1.2, 0.2, -1.0, 0.5, 1.7,
-0.5, -2.1, 1.0, 0.8, -0.1)
x_values_noise <- numeric(11)
x_values_noise[1] <- 2
x_values_noise[2] <- 1

for (t in 3:11) {
```

```
x_values_noise[t] <- 0.8 * x_values_noise[t-1]
- 0.7 * x_values_noise[t-2] + z_values[t-2]
}

# Extract noisy values
x_t_noise <- x_values_noise[3:11]
x_t_minus_1_noise <- x_values_noise[2:10]
x_t_minus_2_noise <- x_values_noise[1:9]

# Plot x_t vs. x_{t-1} with noise
plot(x_t_minus_1_noise, x_t_noise, type="b",
col="green", pch=16,
xlab="x_{t-1}", ylab="x_t",
main="x_t vs. x_{t-1} with noise")
```



iii. The sketched plot of x_t vs. x_{t-2} is for $t = 2, 3, \ldots, 10$ is shown below.

```
# Plot x_t vs. x_{t-2} with noise
plot(x_t_minus_2_noise, x_t_noise, type="b", col="purple", pch=16,
xlab="x_{t-2}", ylab="x_t", main="x_t vs. x_{t-2} with noise")
```



8. The given TS

$$x_t = 0.8x_{t-1} + z_t$$

$z_t \sim N(0, 1)$ and $x_0 = 0$.

(a) The distribution of x_1 its

$$x_1 = 0.8x_0 + z_1 = 0.8(0) + z_1 = z_1$$
$$E[x_1] = 0$$
$$Var[x_1] = 1$$

Since $z_1 \sim N(0, 1)$, $x_1 \sim N(0, 1)$.

(b) The distribution of x_2 is

$$x_2 = 0.8x_1 + z_2 = 0.8z_1 + z_2$$
$$E[x_2] = 0$$
$$Var[x_2] = Var(0.8x_1) + Var(z_2) = 0.8^2(1) + 1 = 1.64$$

Since $z_1, z_2 \sim N(0, 1)$, $x_2 \sim N(0, 1.64)$.

(c) The distribution of x_3 is

$$x_3 = 0.8x_2 + z_3 = 0.8(0.8z_1 + z_2) + z_3$$
$$= 0.64z_1 + 0.8z_2 + z_3$$
$$E[x_3] = 0$$
$$Var[x_3] = Var(0.8z_2) + Var(z_3) = 0.8^2(1.64) + 1 = 2.0496$$

Since $z_1, z_2, z_3 \sim N(0, 1)$, $x_3 \sim N(0, 2.0496)$.

(d) If x_2 takes the value 3, the point prediction for x_3 is

$$E[x_3|x_2 = 3] = 0.64(0) + 0.8(3) = 2.4$$

(e) The distribution of the prediction error its

$$\text{Error} = x_3 - E[x_3|x_2]$$

Since $x_3 = 0.8x_2 + z_3$ and $E[x_3|x_2] = 0.8x_2$,

$$\text{Error} = (0.8x_2 + z_3) - 0.8x_2 = z_3$$

then it must be true that

$$\text{Error} \sim N(0, 1)$$

9. The AR(1) model with mean 0 is given by

$$x_t = \phi x_{t-1} + z_t$$

where $z_t \sim N(0, \sigma^2)$. Then it must be true that the matrix form looks something like this

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & \cdots & 0 \\ 1 & \phi & 0 & \cdots & 0 \\ 0 & 1 & \phi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

The backshift operator B is defined as

$$Bx_t = x_{t-1}$$

then, the AR(1) model can be written as

$$e_t = x_t - \phi x_{t-1}$$
$$(1 - \phi B)x_t = e_t$$

Yes, any process can be written in terms of ϵ_t. An AR(1) is a process that can be express as a sum of past shocks ϵ_t, if we write it recursively,

$$x_t = \phi x_{t-1} + \epsilon_t$$

Substituting $x_{t-1} = \phi x_{t-2} + \epsilon_{t-1}$,

$$\begin{aligned} x_t &= \phi(\phi x_{t-2} + \epsilon_{t-1}) + \epsilon_t \\ &= \phi^2 x_{t-2} + \phi\epsilon_{t-1} + \epsilon_t \end{aligned}$$

Repeating this, we can get

$$x_t = \sum_{k=0}^{\infty} \phi^k \epsilon_{t-k}$$

and this sum converges if $|\phi| < 1$, but if $|\phi| < 1$, then the process is stationary, and the sum can be fully written in terms of ϵ_t. If $|\phi| \geq 1$, then the process is non-stationary, and the sum does not converge, whcih cannot be fully written in terms of ϵ_t.