1. Say whether the following is true or false and support your answer by a proof. $(\exists m\in\mathcal{N})(\exists n\in\mathcal{N})(3m+5n=12)$

The statement is True

Proof: list all the possible pairs of m and n

For pairs contains $m\geqslant5$ or $n\geqslant3$. there is no possible way to produce the equation $3m+5n=12$.

The reason is that:

If $m\geqslant5$, then $3m+5n>12$, because n is a natural number.

Similarly, If $n\geqslant3$, then $3m+5n>12$. because m is a natural number.

m must be one of 0,1, 2, 3,4 and n must be one of 0,1, 2

15 pairs of m and n can be formed, parts of them are:

m=4 and n=0, 3m+5n=12

m=1 and n=1, 3m+5n=8

m=1 and n=2, 3m+5n=13

m=2 and n=1, 3m+5n=11

m=2 and n=2, 3m+5n=16

m=3 and n=1, 3m+5n=14

m=3 and n=2, 3m+5n=19

So when m=4 and n=0, the equation is True.

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

The statement is true

Proof: By given arbitrary five consecutive integers

List these five numbers for smallest one to biggest one. Suppose the third a of the list is a.

Then the list can be represented as following:

a-2, a-1, a, a+1, a+2

As they are conserve integers

Their sum is $(a-2) + (a-1) + a + (a+1) + (a+2)=5a$

since a is a integers. The sum is divisible by 5.

3. Say whether the following is true or false and support your answer by a proof: For any integer *n*, the number n^2 + n + 1 is odd.

The statement is true

Proof: By assuming n is odd or even

Suppose n is odd, n^2 is odd. Because the product of two odd numbers is still odd
n^2 + n is even. Because the sum of two odd numbers is even
n^2 + n +1 is odd. Because even number + 1 is odd

Suppose n is even
n^2 is even. Because the product of two even numbers is still even
n^2+n is even. Because the sum of two even numbers is even
n^2+n+1 is odd. Because the sum of one odd number and one even number is odd.
That is either n is even or odd, n^2+n+1 is odd
Thus for any integer n, the number n^2+n+1 is odd

4. Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

When n=0, 4n+1=1, 4n+3=3
For any arbitrary n,n+1: 4n+3-(4n+1)=2, 4(n+1)+1-(4n+3)=2, that means when n plus 1, the forms 4n+1 and 4n+3 will increase by 2.
As it start with 1(when n=0), so 4n+1 and 4n+3 can cover all odd number

5. Prove that for any integer n, at least one of the integers n, $n+2$, $n+4$ is divisible by 3.

Solution:
We take out n, if n divided by 3, there are three conditions:
The remainder is 0, so we can prove that directly
The remainder is 1, so we can know that (n+2) can be divided by 3
The remainder is 2, so we can know that (n+1+3)=(n+4) can be divided by 3
So the statement is true

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of "twin primes", pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Solution:

From question 5 we know that in the three numbers (n, n+2, n+4), at least one number can be divided by 3.
If n>3 so n+2>3, n+4>3, that means at least one number can be divided by 3 and itself, so this number is not a prime.
So we just can let n=0, n=1, n=2, n=3:
If n=0, 0 is not prime
If n=1, 1 is not prime
If n=2, n+2=4, 4 is not prime
If n=3, n+2=5, n+4=7 3,5,7 are primes
So we can prove that

7. Prove that for any natural number n, 2 + 2^2 + 2^3 + … + 2^n = 2^{n+1} – 2

Solution:

2 + 2ˆ2 + 2ˆ3 + … + 2ˆn = 2ˆ{n+1} - 2=2ˆn *2 -2 =2ˆn+2ˆn-2

2 + 2ˆ2 + 2ˆ3 + … + 2ˆn-1 = 2ˆn - 2=2ˆ(n-1)+2ˆ(n-1)-2

2 + 2ˆ2 + 2ˆ3 + … + 2ˆn-2 = 2ˆ(n-1) - 2=2ˆ(n-2)+2ˆ(n-2)-2

……

2=2+2-2 which means 2=2

So the statement is true

8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}^{\infty}_{n=1}$ tends to limit L as $n\rightarrow\infty$, then for any fixed number M>0, the sequence $\{Man\}^{\infty}_{n=1}$ tends to the limit ML.

Proof: Take an arbitrary real number E

Because the sequence $\{a_n\}^{\infty}_{n=1}$ tends to limit L as $n\rightarrow\infty$. For the number E/M, there always exist an n_0 that when $m>n_0$, $|a_m-L|<$ E/M which, by algebra, can be transformed to

$|Ma_m - ML|<E$

Combining above two points, that is for an arbitrary real number E, there always exist an n_0 that when $m>n_0$, $|Ma_m - ML|<E$. This is necessary and sufficient for the sequence $\{Ma_n\}^{\infty}_{n=1}$ tends to the limit ML. As required.