TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 3 2016

Section 1 Calculator Free

Vectors in Two & Three Dimensions and Systems of Equations

STUDENT'S NAME: SOLUTIONS

DATE: Thursday 28th April **TIME:** 20 minutes **MARKS**: 23

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (7 marks)

Solve the following system of equations, explaining what the equations and their solution represent in space:

$$\begin{cases} 3x+2y-z=19 & R_1 \\ 4x-y+2z=4 & R_2 \\ 2x+4y-5z=32 & R_3 \end{cases}$$

$$\sim \begin{cases} 11x+3z=27 & R_1 \leftarrow R_1+2R_2 \\ 4x-y+2z=4 & R_2 \\ 18x+3z=48 & R_3 \leftarrow R_3+4R_2 \end{cases} \checkmark$$

$$\sim \begin{cases} 11x+3z=27 & R_1 \\ 4x-y+2z=4 & R_2 \\ 7x=21 & R_3 \leftarrow R_3-R_1 \end{cases} \checkmark$$

$\therefore \underline{x=3}$ ✓, $3z=27-33 \Rightarrow 3z=-6 \therefore \underline{z=-2}$ ✓, $y=4(3)+2(-2)-4 = 12-4-4 = \underline{4}$ ✓

$\therefore x=3, y=4, z=-2$

i.e. $(3,4,-2)$ represents the unique point of intersection ✓ of three planes ✓

2. (7 marks)

By considering the value(s) of *lambda*, λ, determine the number of points of intersection of the line, $\boldsymbol{r} = (1-3\lambda)\boldsymbol{i} + (4+9\lambda)\boldsymbol{j}$, with the circle, $|\boldsymbol{r} - (8\boldsymbol{i} + 3\boldsymbol{j})| = \sqrt{50}$. Hence state the coordinates of any point(s) of intersection.

$$\left| \begin{pmatrix} 1-3\lambda \\ 4+9\lambda \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \end{pmatrix} \right| = \sqrt{50} \quad \checkmark$$

$$\Rightarrow \left| \begin{pmatrix} -3\lambda - 7 \\ 9\lambda + 1 \end{pmatrix} \right| = \sqrt{50}$$

$$\Rightarrow (-3\lambda-7)^2 + (9\lambda+1)^2 = 50 \quad \checkmark$$

$$\Rightarrow 9\lambda^2 + 42\lambda + 49 + 81\lambda + 18\lambda + 1 = 50$$

$$\Rightarrow 90\lambda^2 + 60\lambda = 0$$

$$\Rightarrow 30\lambda(3\lambda + 2) = 0$$

$$\therefore \underline{\underline{\lambda = 0}} \text{ or } \underline{\underline{\lambda = -\tfrac{2}{3}}} \quad \checkmark \checkmark$$

ie. there are two distinct values of λ

$\therefore$ Two points of intersection ✓

When $\lambda = 0$, $r = \underset{\sim}{i} + 4\underset{\sim}{j}$ ie. point $\underline{\underline{(1, 4)}}$ ✓

When $\lambda = -\frac{2}{3}$ $r = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ ie. point $\underline{\underline{(3, -2)}}$ ✓

3. (9 marks)

Given the three points: $P(0,-2,1)$, $Q(4,1,3)$ and $R(-1,0,2)$

(a) (i) State the vector equation of the line through points P and Q in terms of λ. [2]

$$\underset{\sim}{r} = (0,-2,1) + \lambda(4,3,2)$$

$$= -2\underset{\sim}{j} + \underset{\sim}{k} + \lambda(4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k})$$

✓ ✓

(ii) What will be the impact of restricting *lambda* such that $0 \le \lambda \le 1$? [1]

$\underset{\sim}{r}$ will be points on the line segment PQ. ✓

(iii) Hence determine the Cartesian form of the equation of the line stated in part (i).

[2]

$$\frac{x}{4} = \frac{y+2}{3} = \frac{z-1}{2}$$

✓✓

(b) (i) Calculate the normal $n = \mathbf{PR} \times \mathbf{PQ}$ [2]

Hint: $\boldsymbol{a} \times \boldsymbol{b} = (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$

$$\underset{\sim}{n} = (-1,2,1) \times (4,3,2)$$ ✓

$$= (1, 6, -11)$$ ✓

$$= \underset{\sim}{i} + 6\underset{\sim}{j} - 11\underset{\sim}{k}$$

(ii) Hence determine the Cartesian equation of the plane that contains the three points P, Q and R.

[2]

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = c$$

$$\Rightarrow \underset{\sim}{r} \cdot \underset{\sim}{n} = (0,-2,1) \cdot \underset{\sim}{n}$$

$$\Rightarrow (x,y,z) \cdot (1,6,-11) = -12 - 11$$ ✓

$$\Rightarrow x + 6y - 11z = -23$$ ✓

$$\therefore x + 6y - 11z + 23 = 0$$

End of Questions

TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 3 2016

Section 2 Calculator Assumed

Vectors in Two & Three Dimensions and Systems of Equations

STUDENT'S NAME: ______________________________

DATE: Thursday 28th April **TIME:** 25 minutes **MARKS**: 27

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment).

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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4. (4 marks)

Determine the vector equation of the sphere which has Cartesian equation:

$$x^2+y^2+z^2+2x-4y+6z-11=0$$

$\Rightarrow x^2+2x+1-1+y^2-4y+4-4+z^2+6z+9-9=11$ ✓

$\Rightarrow (x+1)^2-1+(y-2)^2-4+(z+3)^2-9=11$

$\Rightarrow (x+1)^2+(y-2)^2+(z+3)^2=11+1+4+9$ ✓

$\Rightarrow (x+1)^2+(y-2)^2+(z+3)^2=5^2$ ✓

$\therefore \left|\underset{\sim}{r}-(-1, 2, -3)\right|=5$ ✓

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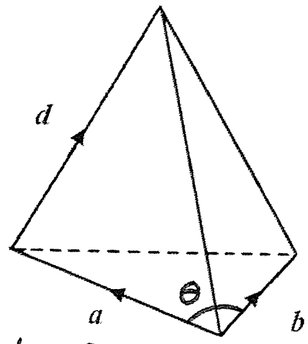
5. (7 marks)

The diagram shows a tetrahedron with three edges described by vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{d}$.



(a) Prove the area of the bottom face is given by:

$$A = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$$

[2]

Let the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be θ

Hence, area $= \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$ ✓

$= \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$ ✓ QED.

(b) Prove that the volume of the tetrahedron is given by:

$$V = \frac{1}{6}|\boldsymbol{d} \bullet (\boldsymbol{a} \times \boldsymbol{b})|$$

[5]

$V = \frac{1}{3} \times$ Area of Base $\times$ Height ✓

$= \frac{1}{3} \times \left| \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}| \times \underset{\sim}{d} \bullet \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{|\underset{\sim}{a} \times \underset{\sim}{b}|} \right|$ ✓ ✓✓

$= \frac{1}{6}|\underset{\sim}{d} \bullet (\underset{\sim}{a} \times \underset{\sim}{b})|$

✓ QED.

where the Height is the scalar projection of $\underset{\sim}{d}$ onto $\underset{\sim}{a} \times \underset{\sim}{b}$

(ie. the normal to the base)

6. (10 marks)

Consider the system of equations:

$x+y+z=3$, $x-2y+z=6$ and $x-y+kz=m$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 6 \\ 1 & -1 & k & m \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 0 & -3 \\ 0 & 2 & 1-k & 3-m \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & -3(1-k) & 3(m-5) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & k-1 & m-5 \end{bmatrix} \checkmark\checkmark$$

N.B. Could use ClassPad

$$\text{rref}\left(\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 6 \\ 1 & -1 & k & m \end{bmatrix}\right)$$

(a) Determine the value(s) of k and m so that the system has:

(i) a unique solution [3]

$k \neq 1$ and $m \in \mathbb{R}$ ✓

(ii) more than one solution [2]

$k = 1$ and $m = 5$ ✓ ✓

(iii) no solution [2]

$k = 1$ and $m \neq 5$, $m \in \mathbb{R}$

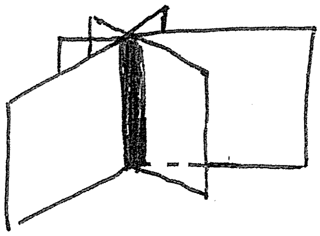
(b) For case (ii) above:

(i) describe the solution in words [1]

the three planes intersect along a common line ✓

(ii) illustrate the solution with a small sketch [1]

Not orientated accurately.



✓

(iii) state the solution in parametric form, with parameter, $t \in \mathbb{R}$. [1]

$y = -1$, $x - 1 + z = 3$

$\Rightarrow z = 4 - x$

$\therefore$ $x = t$, $y = -1$, $z = 4 - t$, $t \in \mathbb{R}$. ✓

7. (6 marks)

Consider two aircraft A and B, flying with constant velocities in m/s and initial positions as stated below:

A: $r_0=(5,-2,1)$ km $\qquad v_A=(-30,50,5)$ Beware units.

B: $r_0=(-8,-4,2.5)$ km $\qquad v_B=(40,70,15)$

State the closest distance these two aircraft come to each other and the time at which this happens.

Consider $\underset{\sim}{r} = \underset{\sim}{r}_0 + t\underset{\sim}{v}$ ✓

$$\underset{\sim}{r}_A = \begin{pmatrix} 5 - 0.03t \\ -2 + 0.05t \\ 1 + 0.005t \end{pmatrix} \qquad \underset{\sim}{r}_B = \begin{pmatrix} -8 + 0.04t \\ -4 + 0.07t \\ 2.5 + 0.015t \end{pmatrix}$$ ✓

<u>Seperation</u> Vector:

$$\underset{\sim}{r}_A - \underset{\sim}{r}_B = \begin{pmatrix} 13 - 0.07t \\ 2 - 0.02t \\ -1.5 - 0.01t \end{pmatrix}$$ ✓

$$\Rightarrow |\underset{\sim}{r}_A - \underset{\sim}{r}_B| = \sqrt{(13-0.07t)^2+(2-0.02t)^2+(-1.5-0.01t)^2}$$ ✓

Using ClassPad to find the <u>min. seperation</u>:

$\underline{\underline{t = 173\,s}}$ (nearest sec) ✓, Closest distance $\underline{\underline{3.65\text{ km}}}$ (2d.p.) ✓

<u>Note</u>: there are various methods!

End of Questions