SHENTON
COLLEGE

Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS
METHODS
UNIT 3

Section Two:
Calculator-assumed

Your name: SOLUTION

Teacher name (circle one): Ai Friday Smith

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available
Section One: Calculator-free	8	8	50	51
Section Two: Calculator-assumed	13	13	100	96

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen.
Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Markers use only		
Question	Maximum	Mark
9	5	
10	4	
11	5	
12	7	
13	4	
14	8	0
15	5	
16	5	
17	6	
18	9	
19	7	
20	7	
21	~~7~~ 9	
22	~~9~~ 7	
23	8	0
S2 Total	96	

Section Two: Calculator-assumed **(96 Marks)**

This section has **fifteen (15)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(5 marks)**

Fuel flows into a storage tank that is initially empty at a rate of $\sqrt{1+6t}$ litres per minute, where t is the time in minutes and $0 \le t \le 75$.

(a) Determine how much fuel is in the tank after 15 minutes. (2 marks)

$$V = \int_0^{15} \sqrt{1+6t}\, dt$$

$$= 96.34 \text{ L}$$

✓ writes integral

✓ evaluates integral

(b) If the tank is completely full after 75 minutes, determine the time required for the tank to become three-quarters full. (3 marks)

$$V = \int_0^{75} \sqrt{1+6t}\, dt$$

$$= 1064.09 \text{ L}$$

✓ calculates total volume

$$\int_0^{T} \sqrt{1+6t}\, dt = 0.75(1064.09)$$

$$= 61.88 \text{ minutes}$$

✓ writes integral and equates it to $\frac{3}{4}V$

✓ determines time.

Question 10 **(4 marks)**

Given that $\int_2^6 \frac{f(x)}{3}\,dx = 4$,

(a) evaluate $\int_2^6 f(x)\,dx$ (1 mark)

$= 3\int_2^6 \frac{f(x)}{3}\,dx$

$= 3(4)$

$= 12$

✓ determines correct $\int_2^6 f(x)\,dx$

(b) evaluate $\int_2^6 \frac{3f(x)-1}{2}\,dx$ (3 marks)

$= \frac{3}{2}\int_2^6 f(x)\,dx - \int_2^6 \frac{1}{2}\,dx$

$= \frac{3}{2}(12) - \left[\frac{1}{2}x\right]_2^6$

$= \frac{3}{2}(12) - (3-1)$

$= 16$

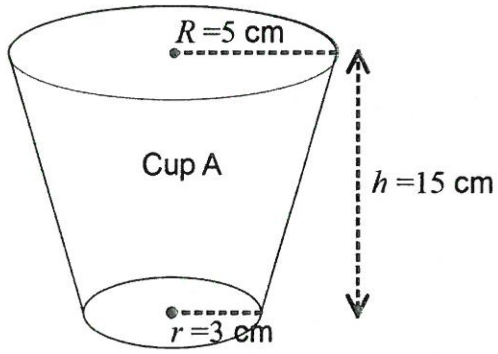
✓ shows use of linearity and additivity

✓ antidifferentiates $\frac{1}{2}$

✓ determines correct integral

Question 11 **(5 marks)**

A manufacturing company produces coffee cups in two different sizes, A and B. Both cups are in the shape of a truncated, right, circular cone as shown below. Both have a height, h, of 15 centimetres and have a base radius of 3 centimetres. Cup B is smaller than cup A.



The formula for the volume, V, of such a shape is given by $V = 5\pi(R^2 + 9 + 3R)$ where $R =$ upper radius of the cup.

(a) Calculate the volume, to the nearest cubic centimetre, of cup A which has an upper radius of 5 cm. (1 mark)

$$V = 5\pi(5^2 + 9 + 3(5))$$
$$= 770 \text{ cm}^3$$

✓ calculates volume to nearest cm^3

(b) Use the incremental formula to estimate the change in volume from cup A to cup B if cup B has an upper radius of 4.8 centimetres. (4 marks)

$$\delta V \approx \frac{dV}{dR}\Big|_{R=5} \cdot \delta R$$
$$\approx 65\pi(-0.2)$$
$$= -40.84 \text{ cm}^3$$

✓ demonstrates use of incremental formula using correct variables

$\delta R = -0.2$ cm ✓ δR correct

$$\frac{dV}{dR} = 10\pi R + 15\pi$$
$$\frac{dV}{dR}\Big|_{R=5} = 50\pi + 15\pi$$
$$= 65\pi$$

✓ $\frac{dV}{dR}$ evaluated when $R=5$

Volume decreases by 40.84 cm^3 when upper radius decreases by 0.2 cm

✓ states correct decrease in volume

Question 12 **(7 marks)**

X is a uniform discrete random variable where $x = \{2, 3, 5, 7, 11, 13\}$.

(a) Determine

(i) $P(X \geq 5)$. (1 mark)

$P(X \geq 5)$
$= \frac{4}{6}$
$= \frac{2}{3}$

✓ determines probability

(ii) $P(X < 12 \mid X \geq 3)$. (2 marks)

$P(X < 12 / X \geq 3) = \dfrac{P(X=3) + P(X=5) + P(X=7) + P(X=11)}{P(X \geq 3)}$

$= \dfrac{\frac{4}{5}}{\frac{5}{6}}$

✓ $P(X \geq 3)$ determined

$= \frac{4}{5}$

✓ determines probability

(b) Calculate the value of

(i) $\mathrm{E}(X)$. (2 marks)

$E(X) = \frac{1}{6}[2 + 3 + 5 + 7 + 11 + 13]$

$= \frac{41}{6} \quad (6 \cdot 8\dot{3})$

✓ expression for $E(X)$

✓ $E(X)$

(ii) $\mathrm{Var}(X)$. (2 marks)

$\sigma_x^2 = \frac{1}{6}(4 + 9 + 25 + 49 + 121 + 169) - \left(\frac{41}{6}\right)^2$

$= \frac{377}{6} - \frac{1681}{36}$

✓ expression for σ_x^2

$= \frac{581}{36}$

✓ $Var(X)$

$= 16 \cdot 13\dot{8}$

Question 13 **(4 marks)**

Use your classpad to sketch the graph of $f(x) = x^4 - x^3$. Use calculus to justify the type and location of any points of inflection and with justification state any intervals where the curve is concave down.

From calculator, Points of Inflection at $x = 0$ and $x = \frac{1}{2}$.

$f(x) = x^4 - x^3$

$f'(x) = 4x^3 - 3x^2$

$f''(x) = 12x^2 - 6x$ ✓ $f''(x) =$

$f''(x) = 0$ for Points of Inflection ✓ uses $f''(x) = 0$ to validate points of inflection at $x = 0$, $x = \frac{1}{2}$

$x = 0 \quad x = \frac{1}{2}$

$f'(x) = 0$

$x = 0$

$\therefore$ Horizontal point of Inflection at $x = 0$ ✓ Validates $x = 0$ is a horizontal point of inflections using $f'(x) = 0$

$x < 0 \quad f''(x) > 0$

$0 < x < \frac{1}{2} \quad f''(x) < 0$ ✓ tests concavity using $f''(x)$ to validate interval which is concave down.

$x > \frac{1}{2} \quad f''(x) > 0$

$\therefore$ Concave down $0 < x < \frac{1}{2}$

Question 14 **(8 marks)**

An insurance company offers a 'death and disability' policy that pays $50 000 if you die or $10 000 if you are permanently disabled. It makes no other payouts. The company charges a premium (the cost for this policy) of $1000 per year for this benefit. Ignore all other costs incurred by the company.

The death rate per year is estimated to be 1 in every 100 people and another 2 out of every 100 people suffer a permanent disability.

Let the random variable $\$X$ denote the amount of profit earned in a year by the insurance company from a typical policy. .

(a) Complete the table below

Outcome	Death	Permanent Disability	No payout
Profit earned (x)	$-\$49\,000$	$-\$9000$	$\$1000$
Probability $(X=x)$	$0\cdot01$	$0\cdot02$	$0\cdot97$

✓ 2 entries ✓ all correct ✓ all probs correct

(3 marks)

(b) How much profit can the insurance company expect to make, on average, from each policy that it sells? (2 marks)

$$E(X) = -49\,000(0\cdot01) + (-9000)(0\cdot02) + 1000(0\cdot97)$$
$$= \$300$$

✓ uses $E(X)$ formula correctly

✓ Calculates $E(X)$

(c) Determine the standard deviation of the company's yearly profit from a typical policy. (3 marks)

$$\sigma_x^2 = (-49\,000)^2(0\cdot01) + (-9000)^2(0\cdot02) + (1000)^2(0\cdot97) - (300)^2$$

✓ Demonstrates use of formula (3MK Qn)

$$= 26\,510\,000$$

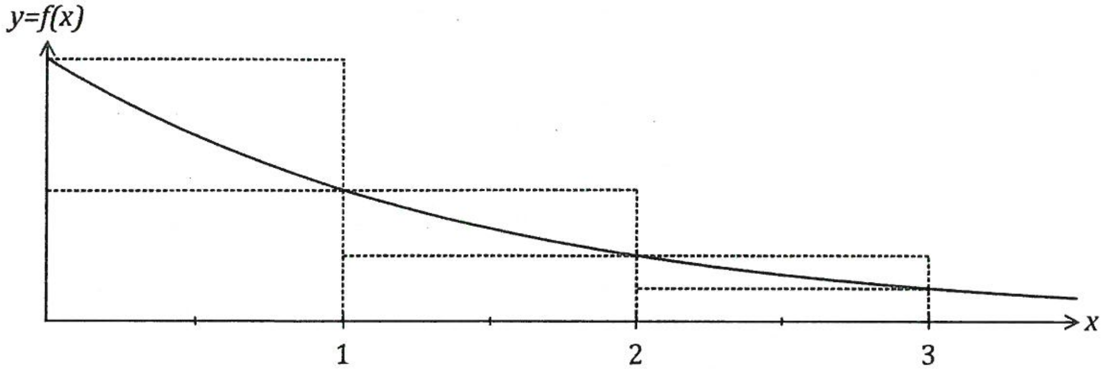
$$\sigma_x = \$5148\cdot79$$

✓ variance correct

✓ st. dev determined

Question 15 **(5 marks)**

The function $f(x) = \frac{1}{2^x}$ is shown below.



(a) Use the sum of the areas of the inscribed rectangles shown in the diagram to explain why $\int_0^3 f(x)\,dx > \frac{7}{8}$. (2 marks)

$$\text{Sum}_I = 1\left(f(1) + f(2) + f(3)\right)$$
$$= 1\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)$$
$$= \frac{7}{8}.$$

As sum inscribed rectangles $< \int_0^3 f(x)\,dx$, $\int_0^3 f(x)\,dx > \frac{7}{8}$

✓ calculates underestimate

✓ reason clear.

(b) Use the average of the sum of the areas of the inscribed rectangles and the sum of the areas of the circumscribed rectangles shown to determine an estimate for $\int_0^3 f(x)\,dx$. (2 marks)

$$\text{Sum}_C = 1\left(f(0) + f(1) + f(2)\right)$$
$$= 1\left(1 + \frac{1}{2} + \frac{1}{4}\right)$$
$$= \frac{7}{4}$$

✓ shows overestimate

$$\text{Average} = \frac{\frac{7}{8} + \frac{7}{4}}{2} = \frac{21}{16} \quad \therefore \int_0^3 f(x)\,dx \approx \frac{21}{16}.$$

✓ calculates average

(c) Suggest a modification to the method used in (b) to achieve a better estimate for $\int_0^3 f(x)\,dx$. (1 mark)

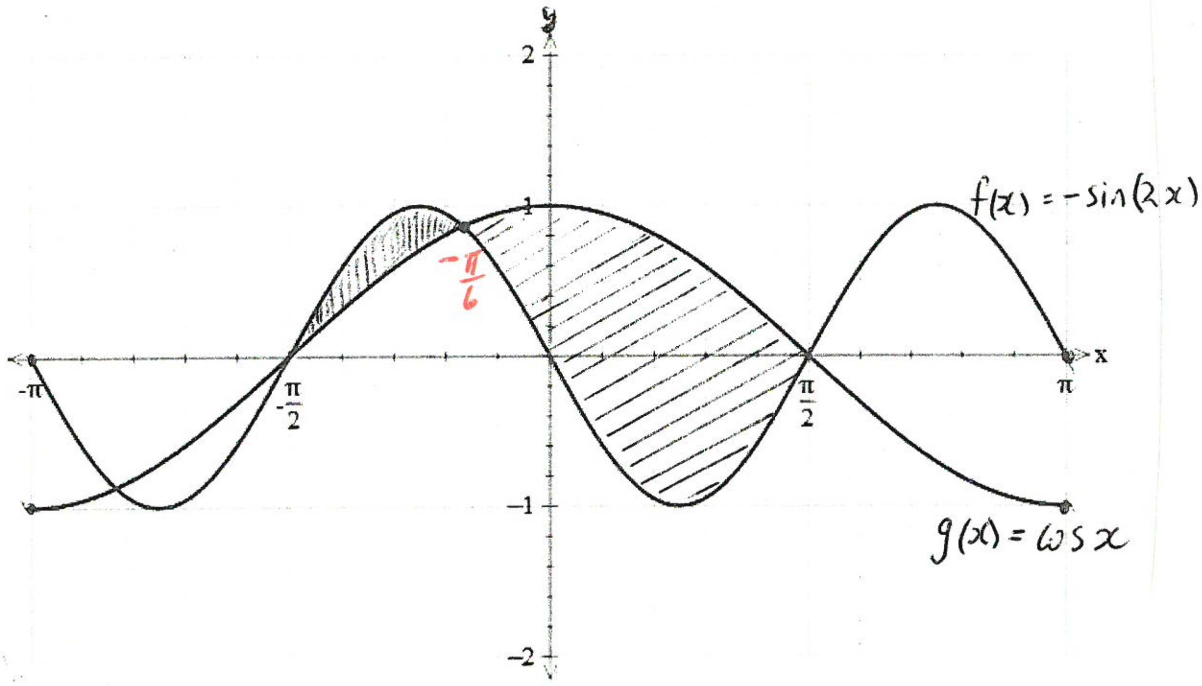
- Use larger number of narrower rectangles
- Use trapeziums
- Use integration

✓ reasonable response

Question 16 **(5 marks)**

Let $f(x) = -\sin(2x)$ and $g(x) = \cos x$

(a) Sketch the graph of $f(x)$ *and* $g(x)$ $-\pi \le x \le \pi$ on the axes below. (2 marks)



✓ $f(x)$
✓ $g(x)$ with accurate intersect at $x = -\frac{\pi}{6}$

(b) Without using absolute values, write an expression to determine the area of the region enclosed by the curves between $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$. (2 marks)

$$A = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} -\sin(2x) - \cos x \, dx + \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x + \sin(2x) \, dx$$

✓ $\int$ with correct bounds ✓ $\int$ with correct bounds

(c) Calculate the area of the region in (b). (1 mark)

$$A = \left[\frac{1}{2}\cos(2x) + \sin x\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + \left[\sin x + \frac{1}{2}\cos(2x)\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$= 2.5$ square units

✓ area determined

Question 17 **(6 marks)**

A local producer grows and sells figs. The costs involved amount to \$50 plus 80c per fig. The producer estimates that if she charges \$$r$ for each fig, she will sell n figs where $r = 3.5 - 0.01n$.

(a) If \$$P$ is the total profit from selling n figs, show that $P = 2.7n - 0.01n^2 - 50$. (2 marks)

Cost $= 50 + 0.8n$

Revenue $= (3.5 - 0.01n)n$

Profit $= 3.5n - 0.01n^2 - 50 - 0.8n$

$= 2.7n - 0.01n^2 - 50$

✓ provides cost and revenue expressions

✓ shows profit =

(b) If the producer charges \$2.50 per fig, how many will she sell **and** what will the marginal profit be?

(3 marks)

$3.5 - 0.01n = 2.5$

$n = 100$

Sells 100 figs.

✓ no of figs sold determined

$P = 2.7n - 0.01n^2 - 50$

$\frac{dP}{dn} = 2.7 - 0.02n$

✓ uses $\frac{dP}{dn}$ for Marginal Profit

$\left.\frac{dP}{dn}\right|_{n=100} = 0.7$

Marginal Profit = \$0.70

✓ determines marginal profit

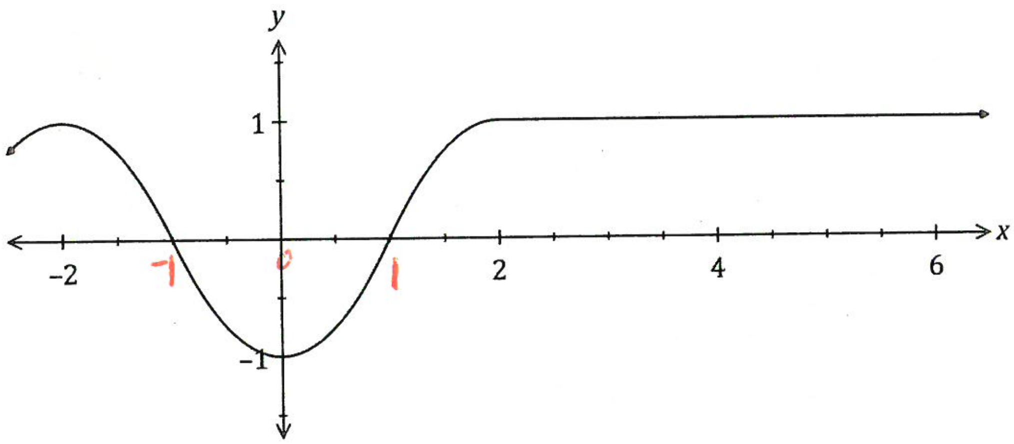
(c) Explain what the marginal profit means in this context. (1 mark)

The profit received when selling one extra fig after n figs have already been sold

✓ Suitable explanation

Question 18 **(9 marks)**

The graph of $y = f(x)$ is shown below.



Let $A(x)$ be defined by the integral $A(x) = \int_{-2}^{x} f(t)\,dt$ for $x \geq -2$.

(a) Use the graph of $y = f(x)$ to identify all the turning points of the graph of $y = A(x)$, stating the x-coordinate and nature of each point. (2 marks)

$x = -1$ Maximum T.P

$x = 1$ Minimum T.P

Identify location of

Max ✓

Min ✓

It is also known that $A(2) = 0$.

(b) Using the graph of $y = f(x)$ or otherwise, explain why $A(5) = 3$. (2 marks)

$$\int_0^5 f(x)\,dx = A(2) + \int_2^5 f(x)\,dx$$

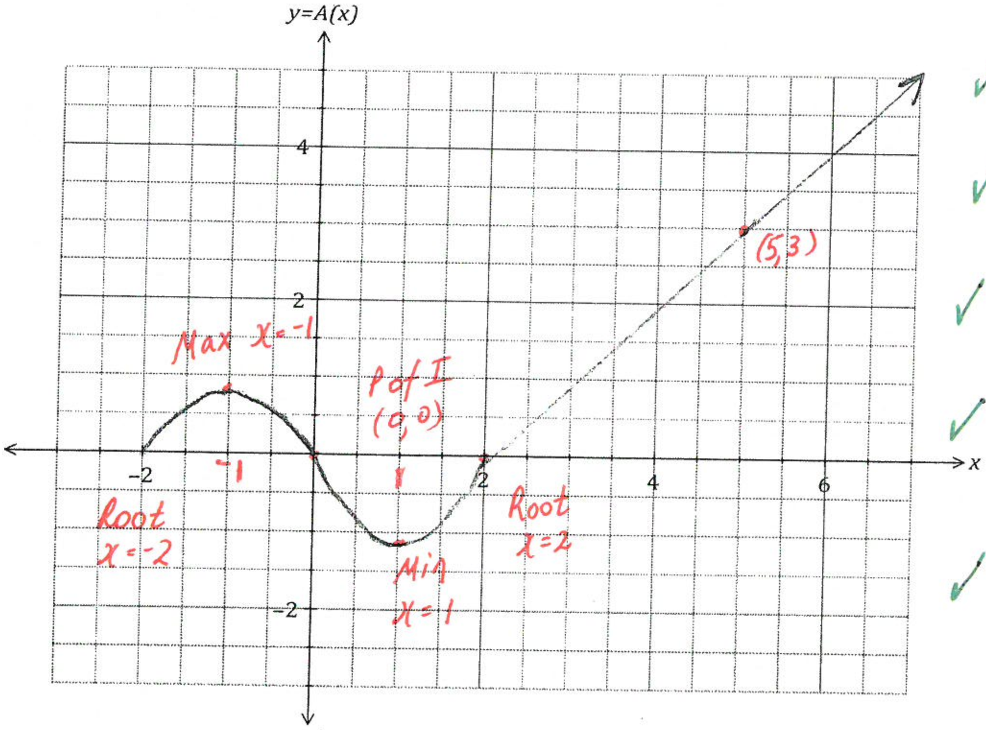
$$= 0 + (5-2) \times 1$$

$$= 3$$

✓ Uses A(2)

✓ clear explanation

(c) Sketch the graph of $y = A(x)$ on the axes below, indicating and labelling the location of all key features. (5 marks)



✓ labels p. of I

✓ labels roots.

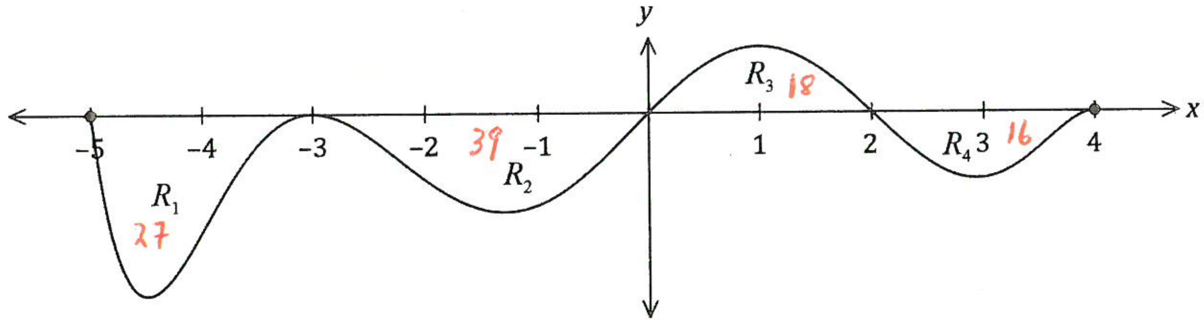
✓ $-2<x<0$ Max labelled

✓ $0<x<2$ Min labelled

✓ line indicated going close to (5,3)

Question 19 **(7 marks)**

The graph of $y = f(x)$ is shown below for $-5 \le x \le 4$.



The area trapped between the x-axis and the curve for regions R_1, R_2, R_3 and R_4 are $27, 39, 18$ and 16 square units respectively.

(a) Determine the value of

(i) $\int_2^4 f(x)\,dx$. $= -16$ (1 mark)

✓ determines $\int_2^4$

(ii) $\int_{-5}^2 f(x)\,dx$. (2 marks)

$= -27 - 39 + 18$

$= -48$

✓ shows sum of signed areas.

✓ determines $\int_{-5}^2$

(iii) $\int_{-3}^2 (f(x) - 4)\,dx$. (2 marks)

$= \int_{-3}^2 f(x)\,dx - \int 4\,dx$

$= -39 + 18 - [4x]_{-3}^{2}$

$= -39 + 18 - [8 + 12]$

$= -41$

✓ $\int 4$ determined

✓ correct definite integral

(iv) $\int_{-5}^0 f'(x)\,dx + \int_0^4 f(x)\,dx$. (2 marks)

$= f(-5) - f(0) + (18 - 16)$

$= (0 - 0) + 2$

$= 2$

✓ shows $\int_{-5}^{0} f'(x)\,dx = 0$

✓ determines correct value.

Question 20 **(7 marks)**

(a) For each of the situations below, decide whether the answer could be obtained using a binomial distribution, a Bernoulli distribution, a discrete uniform distribution or none of those distributions.

(i) A discrete random variable, X is such that $P(X = x) = k$ for $x = \{5,10,15,20\}$

Discrete Uniform distribution (1 mark)

✓ correct responses.

(ii) It is known that 40 students out of a group of 120 students are international. Two students, both international, are selected without replacement. Assume the selection is random. (1 mark)

None of the distributions ✓

(iii) A fair eight-sided die is rolled 100 times and the probability of obtaining more than 6 odd scores is calculated. (1 mark)

Binomial distribution ✓

(iv) In a very large population of students, 23% are known to be international students. 25 students are selected and the probability that exactly 9 students are international is calculated. Assume the selection is random.

Binomial distribution ✓ (1mark)

(b) Which of the following functions, $f(x)$ could represent a discrete probability function for the random variable, X? Justify your answer.

(i) $f(x) = \frac{x}{6}$, where $x = -1, 1, 2, 4$. (1 mark)

x	-1	1	2	4
$f(x)$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{4}{6}$

Not a probability function
$f(-1) < 0$

✓ No WITH reason

(ii) (2 marks)

x	4	6	8	10
$f(x)$	0.05	0.30	0.25	0.4

Probability function

$f(x) \geq 0$ for all values of x.

$\sum f(x) = 1$

✓ Yes with 1 reason

✓ 2 reasons supplied.

Question 21 (~~7~~ 9 marks)

Seeds were planted in rows of five and the number of seeds that germinated in each of the 120 rows are summarised below.

Number of germinating seeds	0	1	2	3	4	5
Number of rows	1	1	3	16	46	53

+1

(a) Use the results in the table to determine

(i) the probability that no more than 4 seeds germinated in a randomly selected row. (1 mark)

$P(X \leq 4) = \frac{67}{120}$

✓ correct

(ii) the mean number of seeds that germinated per row. (1 mark)

$\bar{X} = \frac{505}{120} \approx 4.2$

✓ Mean

(b) Another row of five seeds is planted. Determine the probability that no more than 4 seeds germinate in this row if the number that germinate per row is binomially distributed with the above mean. (2 marks)

$X \sim B(5, p)$

$np = 4.2$

$\therefore 5p = 4.2$

$p = 0.8417$

✓ calculates p

$\therefore Y \sim B(5, 0.84)$

$P(X \leq 4) = 0.5817$

✓ determines correct probability

MAY vary with accuracy of p

Suppose it is known that 66% of all seeds planted will germinate and that seeds are now planted in rows of 16.

(c) Assuming that seeds germinate independently of each other, determine

(i) the most likely number of seeds to germinate in a row. (1 mark)

$n=16 \quad p=0.66$

11 seeds (from GRAPH)

✓ correct number of seeds

(ii) the probability that at least 9 seeds germinate in a randomly chosen row. (2 marks)

$X \sim B(16, 0.66)$

$P(X \geq 9) = 0.8609$

✓ states distribution

✓ correct probability stated

(iii) the probability that in eight randomly chosen rows, exactly six rows have at least 9 seeds germinating in them. (2 marks)

$X \sim B(8, 0.8609)$

$P(X=6) = 0.2206$

✓ correct parameters

✓ correct probability

Question 22 (~~8~~ 7 marks)

A small body has displacement $x = 0$ when $t = 8$ and moves along the x-axis so that its velocity after t seconds is given by

$$v(t) = 10\sin\left(\frac{\pi t}{24}\right) \text{ cm/s}$$

(a) Determine an equation for $x(t)$, the displacement of the body after t seconds. (3 marks)

$$x(t) = \int v(t)\,dt$$

$$= -\frac{240}{\pi}\cos\left(\frac{\pi t}{24}\right) + c$$

$t = 8$
$x = 0$

$$0 = -\frac{240}{\pi}\cos\left(\frac{\pi}{3}\right) + c$$

$$c = \frac{120}{\pi}$$

$$x(t) = -\frac{240}{\pi}\cos\left(\frac{\pi t}{24}\right) + \frac{120}{\pi}$$

✓ determines indefinite integral

✓ shows a method to establish value of c

✓ correct equation

(b) Describe, with justification, how the speed of the body is changing when $t = 32$. (4 marks)

$$V(32) = 10\sin\left(\frac{32\pi}{24}\right)$$

$$= -5\sqrt{3} \text{ cm/s } (-8.66)$$

✓ determines $V(32)$

$$a(t) = \frac{5\pi}{12}\cos\left(\frac{\pi t}{24}\right)$$

✓ $a(t)$

$$a(32) = -\frac{5\pi}{24} \ (-0.6545) \text{ cm/s/s}$$

✓ $a(32)$

∴ Speed increasing as BOTH velocity and acceleration are negative when $t = 32$

✓ uses signs of $V(t)$ and $a(t)$ to explain speed increasing

Question 23 **(8 marks)**

An aquarium, with a volume of 80 000 cm^3, takes the shape of a rectangular prism with square ends of side x cm and no top. The glass for the base costs 0.05 cents per square cm and for the four vertical sides costs 0.08 cents per square cm. The cost of glue to join the edges of two adjacent pieces of glass is 0.6 cents per cm. Assume the glass has negligible thickness and ignore any other costs.

(a) Show that $C = \frac{x^2}{625} + \frac{9x}{250} + \frac{168}{x} + \frac{960}{x^2}$, where C is the cost, in dollars, to make the aquarium. (4 marks)

$x^2 y = 80\,000$

$y = \frac{80\,000}{x^2}$

✓ expression for other side (y)

Cost (C) $= 0.05xy + 0.08(2x^2) + 0.08(2xy) + 6x(0.6) + 2y(0.6)$

$= 0.05\left(\frac{80\,000}{x}\right) + 0.16x^2 + 0.08\,2\left(\frac{80\,000}{x}\right) + 3.6x + \frac{96\,000}{x^2}$

✓ Cost of Glass shown

$= \frac{4000}{x} + \frac{4}{25}x^2 + \frac{12800}{x} + \frac{18}{5}x + \frac{96\,000}{x^2}$

$= \frac{96\,000}{x^2} + \frac{16800}{x} + \frac{18}{5}x + \frac{4}{25}x^2$

✓ Cost of edges shown

Cost (\$) $= \frac{960}{x^2} + \frac{168}{x} + \frac{9x}{250} + \frac{x^2}{625}$

(not simplified OK)

✓ Shows cost in \$. Clear How they get this.

(b) Show use of a calculus method to determine the minimum cost of making the aquarium. (4 marks)

$\frac{dC}{dx} = \frac{4x^4 + 45x^3 - 210\,000x - 2\,400\,000}{1250x^3}$

✓ Shows $\frac{dC}{dx}$

$\frac{dC}{dx} = 0$ when $x = 37.49$ cm

✓ $\frac{dC}{dx} = 0$ $x =$

$\left.\frac{d^2C}{dx^2}\right|_{x=37.49} > 0$ ∴ minimum cost at $x = 37.49$ cm

✓ checks $\left.\frac{d^2C}{dx^2}\right|_{x=37.49} > 0$

$C(37.49) = \$8.76$

✓ Min Cost to nearest cent

End of Examination

Supplementary page

Question number: __________