Government of **Western Australia**
School Curriculum and Standards Authority

WACE
Western Australian Certificate of Education

MATHEMATICS SPECIALIST

Calculator-free

Sample WACE Examination 2016

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

2014/10411

Question 1 **(5 marks)**

Find all the values, real and complex, of x for which $H(x) = 0$ if

$$H(x) = -5x^3 + 25x^2 - 20x + 100.$$

Solution

$$H(x) = -5x^3 + 25x^2 - 20x + 100$$
$$= -5(x^3 - 5x^2 + 4x - 20)$$

$H(1) \neq 0$

$H(-1) \neq 0$

$H(2) \neq 0$

$H(-2) \neq 0$

$H(5) = 0$

ie $x - 5$ is a factor

$$\begin{array}{r|l} & x^2 \quad + \quad 4 \\ \hline x-5 & x^3 - 5x^2 + 4x - 20 \\ & \underline{x^3 - 5x^2} \\ & \qquad +4x - 20 \\ & \qquad \underline{+4x - 20} \end{array}$$

$$H(x) = -5(x-5)(x^2+4) = 0$$
$$x = 5 \text{ or } x^2 + 4 = 0$$
$$x^2 = -4$$
$$x = \pm 2i$$
$$\therefore x = 5 \text{ or } \pm 2i$$

Specific behaviours

✓ uses the factor theorem to find $x - 5$ is a factor
✓ uses long division to determine the factor
✓ evaluates the correct factor $(x^2 + 4)$
✓ solves $x^2 + 4 = 0$ to give complex roots
✓ acknowledges $x = 5$ is a root

Question 2 **(8 marks)**

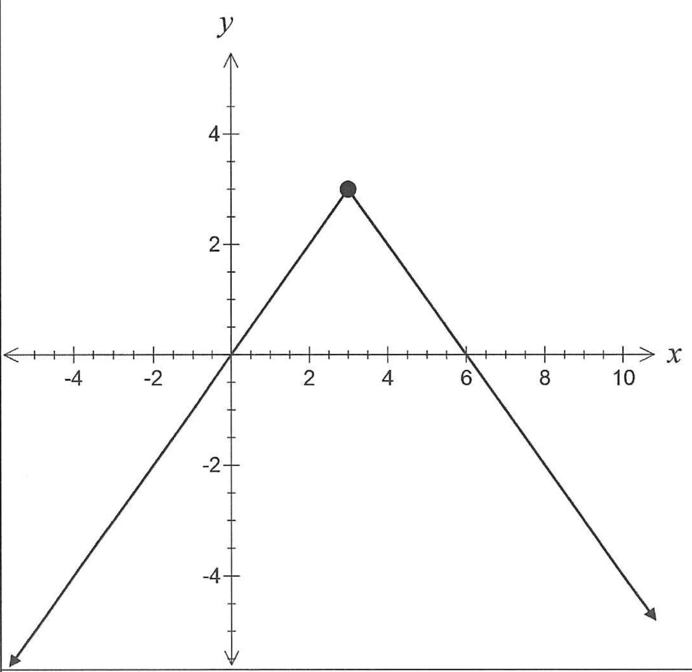
The functions f and g are given by

$$f(x) = 3 - \sqrt{x} \text{ and } g(x) = (3-x)^2.$$

(a) Show that the function defined by $y = f(g(x))$ is defined for all real values of x. (3 marks)

Solution
$f(g(x)) = f((3-x)^2)$ $= 3 - \sqrt{(3-x)^2}$ $= 3 - \lvert 3-x \rvert$ $= \begin{cases} 3-(3-x), & x<3 \\ 3-(x-3), & x \geq 3 \end{cases}$ $= \begin{cases} x, & x<3 \\ 6-x, & x \geq 3 \end{cases}$
Specific behaviours
✓ composes the functions in the correct order   ✓ shows $\sqrt{(3-x)^2} = \lvert 3-x \rvert$ ✓ defines the domain of the piecewise function correctly

(b) On the axes below, sketch the composite function $y = f(g(x))$. (2 marks)

Solution

Specific behaviours
✓ graphs the composite function in two parts correctly   ✓ shows that the function changes at $x = 3$ and $f(g(3)) = 3$

(c) How should the domain of $g(x)$ be changed so that $f(x)$ and $g(x)$ are inverse functions of each other? (3 marks)

Solution
Since $\quad f(x) = 3 - \sqrt{x}\ \ for\ \ x \geq 0$ $\qquad\qquad\qquad\qquad y \leq 3$ $f^{-1}(x) = (3-x)^2\ \ for\ \ x \leq 3$ $\qquad\qquad\qquad\qquad y \geq 0$ Hence domain of $g(x) = \{x : x \leq 3\}$
Specific behaviours
✓ states correct domain $x \leq 3$ ✓ states $g^{-1}(x)$ needs to be one-to-one   ✓ states the range of $f(x) \Rightarrow$ domain of $g(x)$

Question 3 **(9 marks)**

(a) Let y represent the income of a small nation, a the amount of the income spent on necessities and b the percentage of the remaining income spent on luxuries. The economic model that relates these three quantities is

$\frac{dy}{dt} = k(1-b)(y-a)$, where t is the time in years.

Given that b is 65%, express y in terms of a, k and t, where k is a constant. (4 marks)

Solution
When $b = 65\%$, $(1-b) = 0.35$. Therefore $\frac{dy}{dt} = k(0.35)(y-a)$ $\frac{1}{y-a}dy = 0.35k\,dt$ $\int\frac{1}{y-a}dy = \int 0.35k\,dt$ $\ln(y-a) = 0.35kt + c_1$ $y - a = Ce^{0.35kt}$ $y = a + Ce^{0.35kt}$
Specific behaviours
✓ substitutes for b and thus finds $(1-b)$ ✓ separates the variables ✓ integrates both sides ✓ expresses y in terms of a, k and t

(b) Solve the differential equation

$$xe^{x^2} + yy' = 0$$

subject to the initial condition that $y = 1$ when $x = 0$. (5 marks)

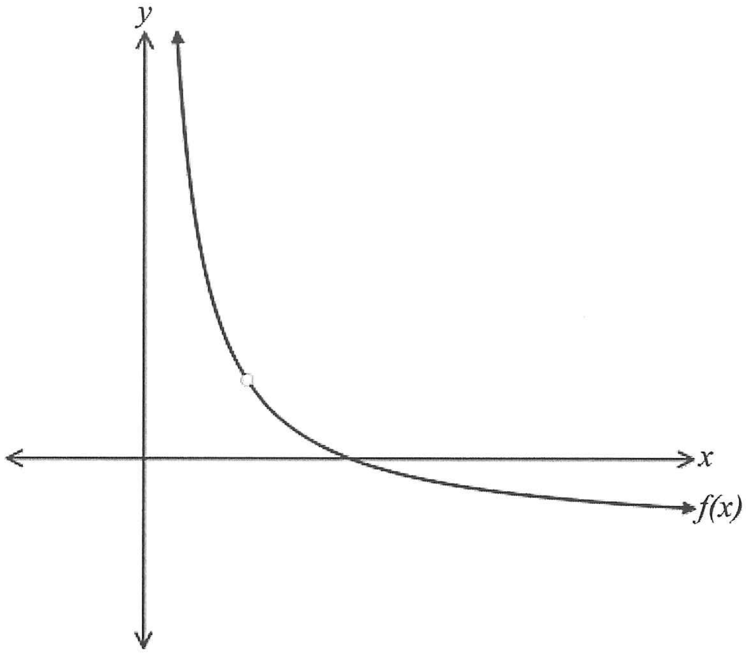
Solution
$xe^{x^2} + yy' = 0$ $y\frac{dy}{dx} = -xe^{x^2}$ $y\,dy = -xe^{x^2}\,dx$ $\int y\,dy = \int -xe^{x^2}\,dx$ $\frac{y^2}{2} = -\frac{1}{2}e^{x^2} + c_1$ $y^2 = -e^{x^2} + c$ To find the particular solution, substitute the initial condition: $(1^2) = -e^{0^2} + c$ This implies that $1 = -1 + c$, i.e. $c = 2$ Thus the particular solution that satisfies the original condition is:

$y^2 = -e^{x^2} + 2$
Specific behaviours
✓ separates the variables ✓ integrates LHS correctly ✓ integrates RHS correctly ✓ substitutes initial condition to determine c ✓ states the particular solution

Question 4 **(9 marks)**

The function $f(x)$ is defined for $x > 0$ by $f(x) = \dfrac{-2+3x-x^2}{x^2-x}$.

(a) Sketch the graph of $f(x)$ on the axes below. (4 marks)

Solution
 Note: $$f(x) = \frac{-2+3x-x^2}{x^2-x} = \frac{-(x^2-3x+2)}{x(x-1)} = \frac{-(x-1)(x-2)}{x(x-1)} = \frac{2-x}{x},\ x-1 \neq 0,\ x>0$$
Specific behaviours
✓ sketches the correct shape of the graph ✓ indicates the undefined point ✓ indicates the vertical asymptote at $x = 0$ ✓ indicates the horizontal asymptote at $y = -1$

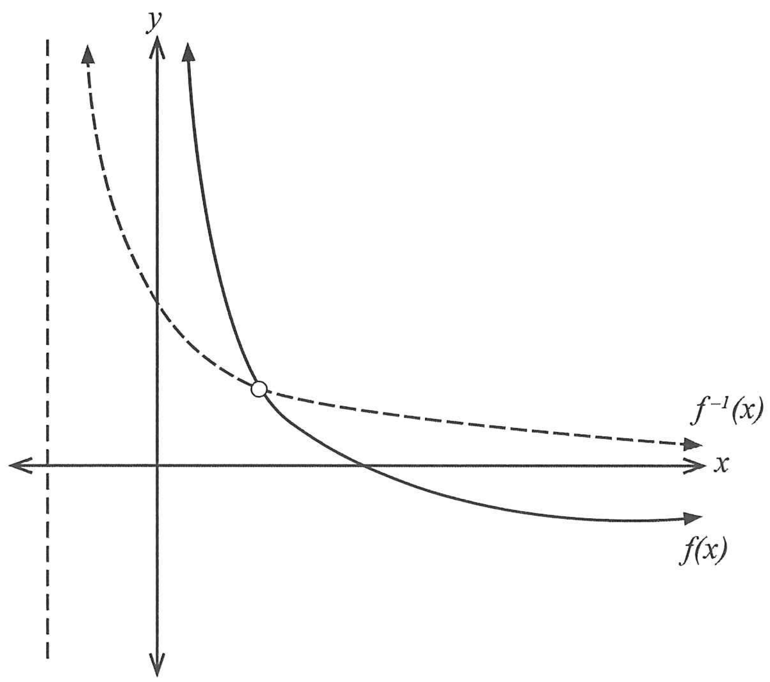
(b) What is the range of $f(x)$? (1 mark)

Solution
$y > -1,\ y \neq 1$
Specific behaviours
✓ defines the range correctly

(c) Show that $f^{-1}(x) = \dfrac{2}{x+1}$, $x > -1$, $x \neq 1$, and state the domain of $f^{-1}(x)$. (2 marks)

Solution
$y = \dfrac{2-x}{x}$, $x > 0$, $x \neq 1$ Inverse: $x = \dfrac{2-y}{y}$ $xy = 2 - y$ $xy + y = 2$ $y(x+1) = 2$ $y = \dfrac{2}{x+1}$ $f^{-1}(x) = \dfrac{2}{x+1}$, $x > -1$, $x \neq 1$
Specific behaviours
✓ interchanges x and y and rearranges equation successfully   ✓ indicates the correct domain

(d) Sketch the graph of $f^{-1}(x)$ on the same axes used for part (a). Label your sketch clearly. (2 marks)

Solution

$f^{-1}(x) = \dfrac{2}{x+1}$ $\quad x \neq 1$ $f(x) = \dfrac{2-x}{x}$ $\quad x \neq 1$
Specific behaviours
✓ sketches the correct shape and position of the graph, including the omission   ✓ shows the vertical asymptote at $x = -1$ and horizontal asymptote at $y = 0$

Question 5 **(8 marks)**

(a) Evaluate $V = \int_0^2 2\pi x(x^2 + 2)dx$ using substitution and express your answer in exact form. (4 marks)

Solution
Substitute $u = x^2 + 2$ then $du = 2x\,dx$ When $x = 0, u = 2; x = 2, u = 6$ hence $V = \int_0^2 \pi(x^2+2)(2x\,dx) = \pi\int_2^6 u\,du = \pi\left.\frac{u^2}{2}\right]_2^6$ $V = \pi[18-2] = 16\pi$
Specific behaviours
✓ uses substitution correctly ✓ determines the bounds correctly ✓ evaluates the integrand ✓ evaluates V

(b) Determine $\int(\sin^2 4x\cos 4x)\,dx$. (4 marks)

Solution
Let $u = \sin 4x \Rightarrow \frac{du}{dx} = 4\cos 4x$ $\int \sin^2 4x\cos 4x\;dx = \frac{1}{4}\int(\sin 4x)^2(4\cos 4x)\,dx$ $= \frac{1}{4}\int u^2\,du$ Hence $= \frac{1}{4}\frac{u^3}{3} + c$ $= \frac{1}{4}\frac{(\sin 4x)^3}{3} + c$ $= \frac{1}{12}\sin^3 4x + c$
Specific behaviours
✓ substitutes $u = \sin 4x$ and integrates for u or integrates by inspection ✓ expresses integrand in term of x ✓ simplifies integrand correctly ✓ includes the constant of integration

Question 6 **(7 marks)**

(a) Given z is a complex number with modulus r and argument θ, express the modulus and argument of each of the complex numbers z_1 and z_2 in terms of r and θ where

(i) $z_1 = \bar{z}$. (2 marks)

Solution
$z = r\cos\theta + r\sin\theta i$ $\bar{z} = r\cos\theta - r\sin\theta i = r\cos(-\theta) + r\sin(-\theta)$ $\therefore \lvert z_1 \rvert = r$ $\arg z_1 = -\theta$
Specific behaviours
✓ defines mod (z) ✓ defines $\arg z_1$

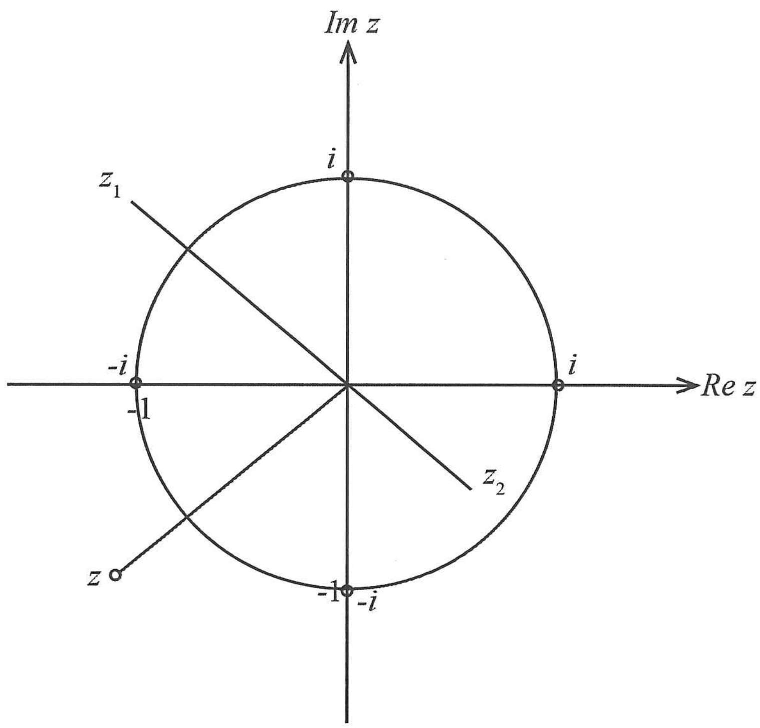
(ii) $z_2 = -z^{-1}$. (3 marks)

Solution
$z_2 = -z^{-1}$ $= -\dfrac{1}{r(\cos\theta + \sin\theta i)}$ $= -\dfrac{1}{r(\cos\theta + \sin\theta i)} \times \dfrac{(\cos\theta - \sin\theta i)}{(\cos\theta - \sin\theta i)}$ $= -\dfrac{\cos\theta - \sin\theta i}{r(\cos^2\theta + \sin^2\theta)}$ $= -\dfrac{1}{r}(\cos\theta - \sin\theta i)$ $= \dfrac{1}{r}(\cos(-\theta) + \sin\theta i)$ $\therefore \lvert z_2 \rvert = \dfrac{1}{r}$ $\arg z_2 = \pi - \theta$
Specific behaviours
✓ multiplies though by $\dfrac{\cos\theta - \sin\theta i}{\cos\theta - \sin\theta i}$ ✓ evaluates mod z_2 correctly ✓ determines $\arg z_2$ correctly

(b) The diagram below shows the circle in the complex plane and the position of the complex number z.

Given the approximate values for r and θ are 1.5 and $220°$ respectively, indicate the locations of the complex numbers z_1 and z_2 as defined in part (a) on the diagram above. (2 marks)

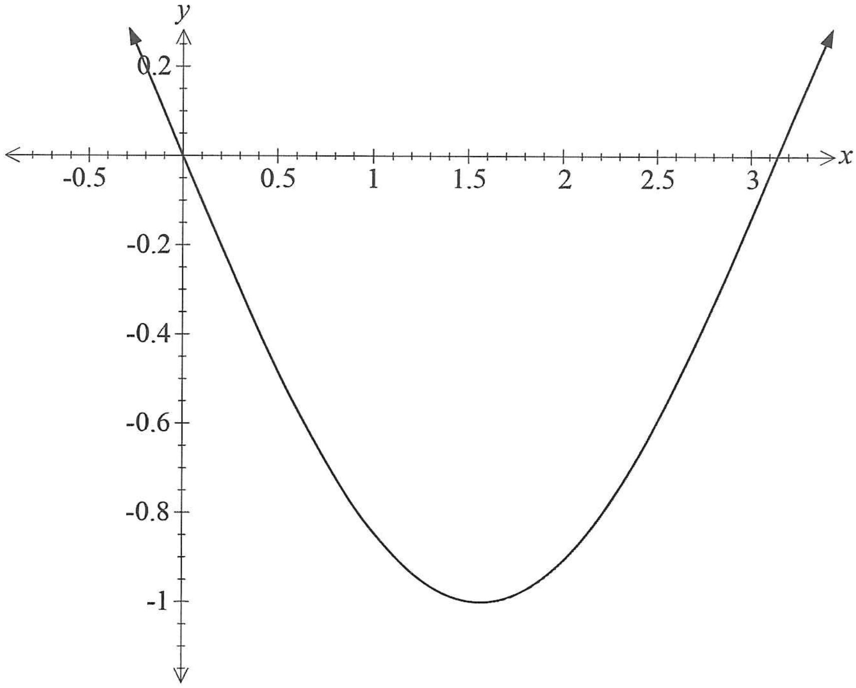
Solution



Specific behaviours
✓ correctly indicates location of z_1
✓ correctly indicates location of z_2

Question 7 **(8 marks)**

A region is bounded by the x-axis and one arc of the graph of $y = -\sin x$ as shown in the diagram below.



(a) Evaluate $\int_0^{\pi} (-\sin x)\,dx$. (3 marks)

Solution
$\int_0^{\pi} -\sin x \ dx = [\cos x]_0^{\pi}$ $= -1 - 1$ $= -2$
Specific behaviours
✓ correctly states the integral of $-\sin x$ ✓ determines correctly the magnitude of the integral ✓ determines correctly the sign of the integral

(b) Given that the curve is defined for all values of x, evaluate $\int_{-7\pi}^{8\pi} (-\sin x)\,dx$. (1 mark)

Solution
As this is simply an extension of the graph given in the diagram, the sum of the signed areas will result in one signed area being left on the right side. Hence $\int_{-7\pi}^{8\pi} (-\sin x)\,dx = 7(0) + 2 = 2$
Specific behaviours
✓ evaluates the correct integral

(c) Calculate the exact volume of the solid generated by rotating the region shown in the diagram around the x-axis. (4 marks)

Solution
$\text{Volume} = \pi \int \frac{1-\cos 2x}{2}\, dx$ $= \pi \left[\frac{x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi}$ $= \frac{\pi^2}{2}$ cubic units
Specific behaviours
✓ defines the correct integral ✓ uses double angle identity as substitution ✓ calculates the correct integrand ✓ evaluates the correct volume

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