MATHEMATICS SPECIALIST 3C/3D
MARKING KEY

176875_8DOC

MATHEMATICS: SPECIALIST
3CMAS/3DMAS
RESOURCE-FREE

Question 1 [4 marks]

Given $m = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $p = \mathbf{i} - \mathbf{j}$ **and** $n = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ **:**

(a) Find $2m - 3p$

2 marks	Description
1	$2m - 3p = 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$
1	$= \begin{pmatrix} 1 \\ 9 \\ 2 \end{pmatrix}$

(b) Find $\lvert n \rvert$**.**

2 marks	Description
1 + 1	$\lvert n \rvert = \sqrt{(7)^2 + 3^2 + 2^2} = \sqrt{62}$

Question 2 [6 marks]

(a) $\int \left(e^x + x^e\right)dx$

3 marks	Description
(1 + 1 + 1)	$I = e^x + \frac{x^{e+1}}{e+1} + c$

(b) Show that $\int_0^1 \frac{x}{1+x^2}\ dx = \ln\sqrt{2}.$

3 marks	Description
1 + 1	$\int_0^1 \frac{x}{1+x^2}dx = \left[\frac{1}{2}ln(1+x^2)\right]_0^1$
1	$= \frac{1}{2}\left[ln\,2 - ln\,1\right] = \frac{1}{2}ln(\,2\,) = ln\sqrt{2}$

Question 3 [7 marks]

(a) **Find an expression for** $\frac{dy}{dx}$ **and hence find the equation of the tangent to the curve at the point where:** $\theta = \pi$

5 marks	Description
1 1	$\frac{dx}{d\theta} = 2\cos\theta$ and $\frac{dy}{d\theta} = 2(3\sin\theta)\cos\theta$
1 1	$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$ $= 6\sin\theta\cos\theta \times \frac{1}{2\cos\theta}$ $= 3\sin\theta$
1	At $\theta = \pi$, $x = 0, y = 0$ and $\frac{dy}{dx} = 3\sin\pi = 0$. Hence the equation of the tangent is $y = 0$

(b) **Determine the Cartesian equation of the curve.**

2 marks	Description
1	From above $\sin\theta = \frac{x}{2}$ and $\sin^2\theta = \frac{y}{3}$ or $y = 3\left(\frac{x}{2}\right)^2$
1	So $\left(\frac{x}{2}\right)^2 = \left(\frac{y}{3}\right)$ or, directly $y = \frac{3x^2}{4}$

Question 4 [10 marks]

(a) **Find the solution to** $\frac{dy}{dx} = 4y - 2yx$ **satisfying** $y(0) = 3$**.**

5 marks	Description
1	Factorising: $\frac{dy}{dx} = (4 - 2x)y$
1	separating variables: $\int \frac{dy}{y} = \int (4 - 2x)dx$
1 1	$\ln\lvert y\rvert = 4x - x^2 + c$ $\therefore y(x) = Be^{4x - x^2}$
1	From $y(0) = 3$ we find B=3, so that $y(x) = 3e^{4x - x^2}$

(b) Use algebra and explain your reasoning, solve *exactly* the inequality |sin *2x*| $\leq$ sin *x* over the interval $0 \leq x \leq \pi$.

5 marks	Description
1 1	If $\lvert \sin 2x \rvert \leq \sin x$ then $2\sin x \lvert \cos x \rvert \leq \sin x$ as $\sin x \geq 0$ across the whole of the interval
1	Thus need $\lvert \cos x \rvert \leq 1/2$ or $x = 0$ or $x = \pi$
2	implying that $\frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$ or $x = 0$ or $x = \pi$ (-1 for each incorrect solution or inequality written as 1.05<*x*<2.09).

Question 5 [6 marks]

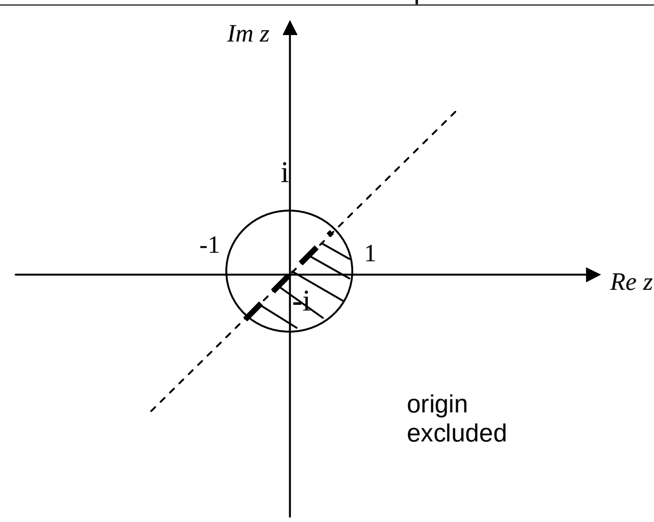
(a) Verify that:

$[z - \text{cis}(\theta)][z - \text{cis}(-\theta)] = z^2 - 2z\cos\theta + 1$.

3 marks	Description
1	$[z - \text{cis}(\theta)][z - \text{cis}(-\theta)] = z^2 - z[\text{cis}(\theta) + \text{cis}(-\theta)] + \text{cis}(\theta)\,\text{cis}(-\theta)$.
1	Now cis(θ)+cis($-\theta$) = $(\cos\theta + i\sin\theta) + [\cos(-\theta) + i\sin(-\theta)] = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$ and
1	cis(θ)cis($-\theta$)= $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$
	so that $[z - \text{cis}(\theta)][z - \text{cis}(-\theta)] = z^2 - 2z\cos\theta + 1$ as required.

(b) Sketch the following subset of the complex plane:

$\{\, z : \lvert z \rvert \leq 1 \quad \text{and} \quad Re\, z > Im\, z \,\}$

3 marks	Description
1 circle 1 shading 1 boundary	

Question 7 [7 marks]
The sum of the products of two consecutive positive integers seems to have the form,

$$1\times 2+2\times 3+3\times 4+\ldots=\frac{n^3+3n^2+2n}{3}$$

Prove that this conjecture is true for all positive integers, *n*.

7 marks	Description	
1	Firstly, show that the conjecture is true for $n = 1$ $LHS = 1(1+1) = 2$ $RHS = \frac{1\times 2\times 3}{3} = 2$ - as required Now, assume the conjecture is true for n terms, and consider the expression for n+1 terms	Firstly, show that the conjecture is true for n = 1 $LHS = 1(1+1) = 2$ $RHS = \frac{1^3+3\times 1^2+2\times 1}{3} = 2$ - as required Now, assume the conjecture is true for n terms, and consider the expression for n+1 terms
1 1 1 1 1	$1\times 2+2\times 3+3\times 4+\ldots+n(n+1)+(n+1)(n+2$ $=\frac{n(n+1)(n+2)}{3}+(n+1)(n+2)$ $=\frac{n(n+1)(n+2)+3(n+1)(n+2)}{3}$ $=\frac{(n+1)(n+2)(n+3)}{3}$ which has the same form as for the n^{th} term and so the statement is true for all n.	$1\times 2+2\times 3+3\times 4+\ldots+n(n+1)+(n+1)(n+2)$ $=\frac{n^3+3n^2+2n}{3}+(n+1)(n+2)$ $=\frac{n^3+3n^2+2n+3(n^2+3n+2)}{3}$ $=\frac{n^3+3n^2+2n+3n^2+9n+6)}{3}$ $=\frac{(n^3+3n^2+3n+1)+(3n^2+8n+5)}{3}$ $=\frac{(n+1)^3+(3n^2+6n+3)+2n+2}{3}$ $=\frac{(n+1)^3+3(n+1)^2+2(n+1)}{3}$ which has the same form as for the n^{th} term and so the statement is true for all n.

Question 1 [5 marks]

If $y = \frac{1+cos(2x)}{1-cos(2x)}$**, show that** $\frac{dy}{dx} = -\frac{2\cos x}{sin^3 x}$**.**

5 marks	Description
1 1	$\frac{dy}{dx} = \frac{-2\sin(2x)(1-\cos(2x))-(1+\cos(2x))(2\sin(2x))}{(1-\cos(2x))^2}$
1 1	$= \frac{-4\sin(2x)}{(2\sin^2 x)^2}$
1	$= \frac{-8\sin x\cos x}{4\sin^4 x} = -\frac{2\cos x}{\sin^3 x}$

Question 2 [6 marks]

Given the following expression $\frac{2x^3-3x^2-29x+60}{40+6x-x^2}$

(a) Simplify the expression.

2 marks	Description
1 1	factorise as $\frac{(x+4)(x-3)(2x-5)}{(10-x)(4+x)}$ or use CAS to give $\frac{-(x-3)(2x-5)}{x-10}$ or equivalent

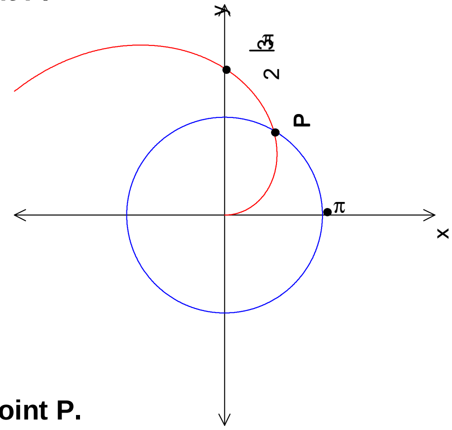
(b) For which value(s) of *x*, if any, is the original expression not defined.

2 marks	Description
1	$x = 10$
1	$x = -4$

(c) Compare the original expression and its simplified version. Are there any values of *x*, where these two expressions are unequal? Explain your answer.

2 marks	Description
1	yes
1	the simplified expression has a value at $x = -4$, whereas the original expression is undefined at that value

Question 3 [5 marks]
The diagram below shows the polar graphs of $r = k$ and $r = n\theta$ (n an integer). Also, the point of intersection of these two graphs is point P.



(a) Write down the values of k and n.

2 marks	Description
1	$k = \pi$
1	and since $r = \frac{3\pi}{2}$ when $\theta = \frac{\pi}{2}$, $n = 3$

(b) Determine, EXACTLY, the Cartesian coordinates of point P.

3 marks	Description
1	$r = \pi$ and $r = 3\theta \Rightarrow \pi = 3\theta$ i.e. $\theta = \frac{\pi}{3}$ so P$(\pi, \frac{\pi}{3})$.
1 1	Now convert to Cartesian – $x = \pi\cos(\frac{\pi}{3}) = \frac{\pi}{2}$ and $y = \pi\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}\pi$ $\Rightarrow$ P$(\frac{\pi}{2}, \frac{\sqrt{3}}{2}\pi)$.

Question 4 [8 marks]
(a) By comparing:
(i) the areas of sector OPQ and triangle OPB
and
(ii) the lengths of PB and arc AB,

or otherwise, establish the inequalities $x\cos x < \sin x < x$.

4 marks	Description
1	$OP = \cos x$ so the area of sector OPQ $= \frac{1}{2}x.\cos^2 x$
1	$BP = \sin x$ so the area of ΔOPB $= \frac{1}{2}\sin x.\cos x$
1	$\therefore \frac{1}{2}x.\cos^2 x < \frac{1}{2}\sin x.\cos x \Rightarrow x.\cos x < \sin x$
1	Now, the length of arc AB $= x$ so PB $<$ AB $\Rightarrow \sin x < x$ $\Rightarrow x.\cos x < \sin x < x$

(b) Explain why the above result shows that $\frac{\sin x}{x} \to 1$ as $x \to 0$ and hence, determine: $\lim_{x \to 0} \frac{2x}{\tan x}$.

4 marks	Description
1	$x.\cos x < \sin x < x \quad \Rightarrow \quad \cos x < \frac{\sin x}{x} < 1$
1	as $x \to 0, \quad \cos x \to 1$ so, by the sandwich principle $\frac{\sin x}{x} \to 1$
1	$\lim_{x \to 0} \frac{2x}{\tan x} = \lim_{x \to 0} 2.\left(\frac{x}{\sin x}\right).\cos x$
1	$= 2.$

Question 5 [7 marks]
Calculate the minimum distance they are apart and the time, after the aircraft leave their initial positions, when this distance is reached after the aircraft leave their initial positions.

7 marks	Description
1 1	The displacement between the two toys at any time t is given by: $\mathbf{d} = (\mathbf{r}_A + \mathbf{v}_A t) - (\mathbf{r}_B + \mathbf{v}_B t)$ $= \begin{pmatrix} 5-t \\ -1-2t \\ 2+4t \end{pmatrix} - \begin{pmatrix} -3 \\ -3t \\ 4+6t \end{pmatrix}$ $= \begin{pmatrix} 8-t \\ -1+t \\ -2-2t \end{pmatrix}$ where t represents time in seconds
1	Hence $\lvert d \rvert^2 = (8-t)^2 + (-1+t)^2 + (-2-2t)^2$ $= 6t^2 - 10t + 69$
2 2	Now by completing the square, using calculus or a graphics calculator (most likely) we find that the minimum value of $\lvert d \rvert^2$ is when $t = \frac{5}{6}$ seconds at which time $\lvert d \rvert^2 = 64\frac{5}{6}$. Hence the minimum distance they are apart is 8.05 m (to 2 dec pls) and the time taken is $\frac{5}{6}$ or $0.8\dot{3}$ seconds.

Question 6 [9 marks]

(a) Show that $M(t) = Ce^{-kt}$ satisfies the differential equation for any constant *C*.

2 marks	Description
2	If $M(t) = C\exp(-kt)$ then $\dfrac{dM}{dt} = -kC\exp(-kt) = -kM$ and so satisfies the DE.

(b) Find *k* and *C*.

3 marks	Description
1	If $M = 150$ when $t = 2$ then $150 = C\exp(-2k)$ and if $M = 100$ when $t = 3.5$ then $100 = C\exp(-3.5k)$ or
1	Dividing the equations gives $3/2 = \exp(1.5k)$ so $k = (2/3)\ln(3/2) \approx 0.27031$. an alternative solution method for *k* $\int_{100}^{150} \dfrac{dM}{M} = \int_{3.5}^{2} -kdt$ solving for *k* gives 0.27031
1	$C \approx 257.56$.

(c) How long will it take for the mass of the substance to reduce to 40 grams?

2 marks	Description
1	When $M = 40$ have $40 = C\exp(-kt) \Rightarrow -kt = \ln(40/C)$ so that $t \approx 6.890$.
1	Hence substance reduces to 40 grams after approximately 6.89 years.

(d) Find the radioactive half-life of the substance.

2 marks	Description
1	Initial mass of substance is *C*; half-life is defined to be time taken for mass to drop to $C/2$.
1	$C/2 = C\exp(-kt) \Rightarrow -kt = \ln(1/2) \Rightarrow t = k^{-1}\ln 2 \approx 2.564$ years.

Question 7 [7 marks]

Consider the curve $x^2y-4y=b$ **where *b* is a real value.**

(a) Determine the equation of the tangent line to this curve at the point (x_1, y_1).

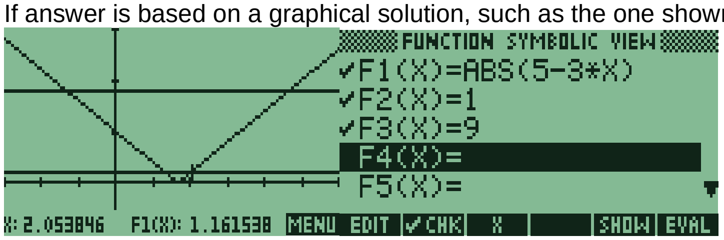
4 marks	Description
	Differentiating implicitly and re-arranging to find the derivative $x^2y-4y=b$
1	$2xy+x^2\dfrac{dy}{dx}-4\dfrac{dy}{dx}=0$
1	$\dfrac{dy}{dx}\left(x^2-4\right)=-2xy$
1	$\dfrac{dy}{dx}=\dfrac{-2xy}{x^2-4}=\dfrac{2xy}{4-x^2}$
OR	**OR** rearranging first :
1	$y=\dfrac{b}{x^2-4}$
1	$y'=(-1)b(x^2-4)^{-2}.2x=\dfrac{-2bx}{(x^2-4)^2}$
1	thus, at *(x_1,y_1)*, gradient is $\dfrac{2x_1y_1}{4-x_1{}^2}$
1	then the equation of the tangent to the curve at *(x_1,y_1)* is $y-y_1=\dfrac{2x_1y_1}{4-x_1{}^2}\left(x-x_1\right)$

(b) What is the restriction on *b* so that the curve has a vertical tangent line at the point *(x_1,y_1)*?

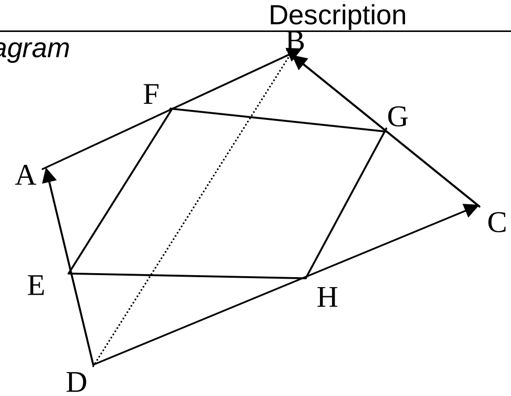
3 marks	Description
1	There will be a vertical tangent when $x^2-4=0$
2	so $4y-4y=b$ and $b=0$

Question 8
Solve the inequality $1 < |5 - 3x| < 9$ algebraically.

Solving $1 < |5-3x| < 9$

5 Marks	Description
1 1 1 1	If $1 < \lvert 5-3x \rvert < 9$ then: (i) either 5-3*x* > 0, *x* < 5/3 and $1 < 5-3x$ and $5-3x < 9$ $\therefore\ x < 4/3$ and $x > -4/3$ the intersection is $-4/3 < x < 4/3$ or $5-3x < 0$, x> 5/3 and $1 < -(5-3x)$ and $-(5-3x) < 9$ $\therefore\ 2 < x$ and $x < 14/3$ the intersection is $2 < x < 14/3$
or	
1 set up 1 for the modulus graph 1 for lines 1	If answer is based on a graphical solution, such as the one shown below:  so $-4/3 < x < 4/3$ or $2 < x < 14/3$
1	Correct intervals $-\frac{4}{3} < x < \frac{4}{3}$ **or** $2 < x < \frac{14}{3}$, stated exactly.

Question 9 [6 marks]
Use a vector method to prove that the mid-points of the sides of a quadrilateral form a parallelogram.

6 marks	Description
1	*Appropriate diagram* 
1	$\text{ABCD is a quadrilateral}$ *(symbolic set-up)* $\text{E, F, G and H are mid-points of the sides of the quadrilateral}$. $\text{let } \overrightarrow{DA} = \underset{\sim}{u}, \text{ let } \overrightarrow{AB} = \underset{\sim}{v}, \text{ let } \overrightarrow{DC} = \underset{\sim}{w} \text{ and } \overrightarrow{CB} = \underset{\sim}{x}$
4	*(clear, logical proof)* $\overrightarrow{HG} = \frac{1}{2}\underset{\sim}{w} + \frac{1}{2}\underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{w} + \underset{\sim}{x}\right)$ and $\overrightarrow{EF} = \frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v} = \frac{1}{2}\left(\underset{\sim}{u} + \underset{\sim}{v}\right)$ $\overrightarrow{DB} = \underset{\sim}{w} + \underset{\sim}{x} = \underset{\sim}{u} + \underset{\sim}{v}$ $\overrightarrow{HG} = \frac{1}{2}\overrightarrow{DB}$ *and* $\overrightarrow{EF} = \frac{1}{2}\overrightarrow{DB}$ So $\overrightarrow{HG} = \overrightarrow{EF}$ and EFGH is a parallelogram.

Question 10 [12 marks]

(a) Suppose that *z* is a complex number with modulus *r* and argument θ.
Express in terms of *r* and θ the modulus and argument of each of the complex numbers z_1, z_2, z_3 and z_4, where:

(i) $z_1 = \bar{z}$

2 marks	Description
1	$\lvert z_1 \rvert = r$
1	$\arg z_1 = -\theta$

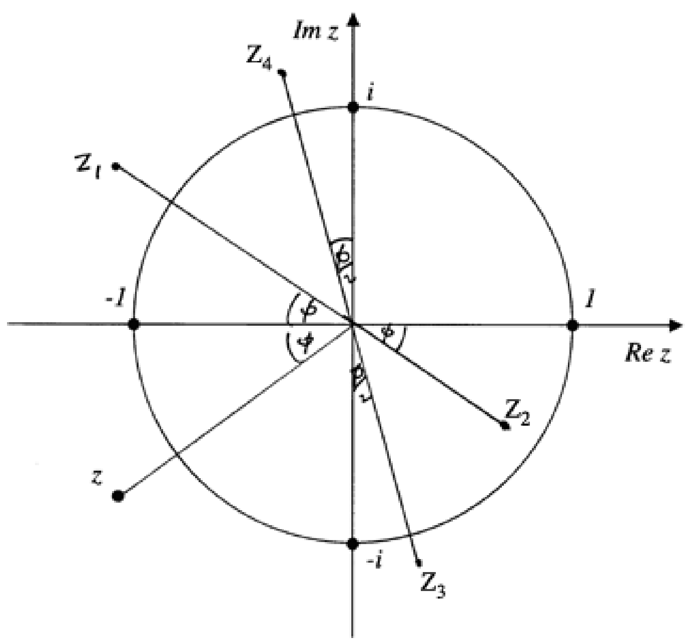
(ii) $z_2 = -z^{-1}$

3 marks	Description
1	$\lvert z_2 \rvert = r^{-1}$
1+1	$arg\ z_2 = \pi - \theta$

(iii) z_3 **and** z_4 **are the square roots of *z*.**

3 marks	Description
1	$\lvert z_3 \rvert = \lvert z_4 \rvert = \sqrt{r}$
1	$arg\ z_3 = \dfrac{\theta}{2}$
1	$arg\ z_4 = \dfrac{\theta}{2} + \pi$

(b) **Indicate as precisely as you can on the diagram above the locations of the complex numbers** z_1, z_2, z_3 **and** z_4**, as defined in part (a).**

5 marks	Description
(*1 mark - for each point correctly located and labelled*) 4	

Question 11 [10 marks]

(a) If initially there are 150 kids, 200 yearlings, 200 adults and 80 old goats what is the distribution after 20 years?

3 marks	Description
1 for 20th power 1 layout + multiply 1 interpret solution	$L^{20}\begin{bmatrix}150\\200\\200\\80\end{bmatrix}=\begin{bmatrix}4.4357\times10^{-2} & 5.2449\times10^{-2} & 8.4967\times10^{-2} & 2.7992\times10^{-2}\\3.6389\times10^{-2} & 4.4357\times10^{-2} & 4.8701\times10^{-2} & 1.4451\times10^{-2}\\2.0232\times10^{-2} & 3.9189\times10^{-2} & 4.4356\times10^{-2} & 1.2266\times10^{-2}\\1.9626\times10^{-2} & 2.4901\times10^{-2} & 4.4787\times10^{-2} & 1.4919\times10^{-2}\end{bmatrix}\begin{bmatrix}150\\200\\200\\80\end{bmatrix}=\begin{bmatrix}36.376\\25.226\\20.725\\18.075\end{bmatrix}$ *2 for knowing what to do, 1 for the result which can be done by calculator*

(b) Determine the net reproduction rate of the population of female goats.

3 marks	Description
1 1 1	$R=a_1+a_2b_1+a_3b_1b_2+a_4b_1b_2b_3$ $=0+0+1.2\times.65\times.7+.5\times.65\times.7\times.8$ $=0.826$

(c) Assuming there are enough male goats to fulfil their part, will the population of female goats become stable over time (that is, survive in the long run)?

4 marks	Description
1	using previous result in (a),
1 1	and finding a higher power of L in the recursive relationship, indicates that this population appears to be increasing without limit, which is an unrealistic expectation.
1	However, based on the assumption of ample food and no external threats, this population would survive in the ‘long term’.