



MATHEMATICS SPECIALIST

Calculator-assumed

Sample WACE Examination 2016

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

Question 8

(6 marks)

The system of linear equations given below can be reduced in three stages to a form where it can be solved easily.

$$\begin{array}{rcl} x + y + z = 4 & \dots R_1 \\ 2x + 3y + z = 8 & \dots R_2 \\ 3x + (3 - p)y + 2z = 13 - p^2 & \dots R_3 \end{array}$$

- (a) Two of the stages are given below.

In the space provided at the side of each stage, write the operation(s) that have been performed in terms of the rows of the previous system. (2 marks)

$$\begin{array}{rcl} x + y + z = 4 & \dots R_1 \\ 2x + 3y + z = 8 & \dots R_2 \\ py + z = p^2 - 1 & \dots R_3 \end{array}$$

$$\begin{array}{rcl} x + y + z = 4 & \dots R_1 \\ y - z = 0 & \dots R_2 \\ py + z = p^2 - 1 & \dots R_3 \end{array}$$

Solution		
$x + y + z = 4$	$\dots R_1$	no change
$2x + 3y + z = 8$	$\dots R_2$	no change
$py + z = p^2 - 1$	$\dots R_3$	$R_3 \rightarrow 3R_1 - R_3$
$x + y + z = 4$	$\dots R_1$	no change
$y - z = 0$	$\dots R_2$	$R_2 \rightarrow R_2 - 2R_1$
$py + z = p^2 - 1$	$\dots R_3$	no change
Specific behaviours		
✓ identifies the change in row 3 in the first set of equations		
✓ identifies the change in row 2 in the second set of equations		

- (b) Perform one further row operation so that the coefficient of z in R_3 is 0. (1 mark)

Solution		
$x + y + z = 4$	$\dots R_1$	
$y - z = 0$	$\dots R_2$	
$(p+1)y = p^2 - 1$	$\dots R_3$	$R_3 = R_2 + R_3$
Specific behaviours		
✓ adds rows 2 and 3 successfully		

- (c) For each of $p = 1$ and $p = -1$ indicate how many solutions there are to the system of equations. If there is a unique solution, give that solution. If there is an infinite number of solutions, give the resulting solution when $z = -1$. (3 marks)

Solution
$x + y + z = 4$ $y - z = 0$ $(p+1)y = p^2 - 1$ If $p = 1$, $2y = 0$ $y = 0$ giving a unique solution $x = 4, y = 0, z = 0$ If $p = -1$, $0 \times y = 0$ i.e. $0 = 0$, giving infinitely many solutions $z = -1, \therefore y = -1, x = 6$
Specific behaviours
<ul style="list-style-type: none"> ✓ identifies the possibility of the unique solution for $p = 1$ and states that solution ✓ identifies the possibility of infinitely many solutions for $p = -1$ ✓ describes the full solution for $z = -1$

Question 9



(8 marks)

The standard deviation of the durability of Performance Racing tyres is 410 kilometres. Racing experts plan to estimate μ , the mean lifetime of these tyres, using the mean lifetime of a random sample of the tyres.

- (a) The experts would like to be 95% confident that the mean lifetime of tyres in the sample is within 50 kilometres of μ . How large a sample should they take? (3 marks)

Solution
$50 = 1.96 \left(\frac{410}{\sqrt{n}} \right)$ $n = 258.309184$ <p>They should take a sample of at least 259 tyres.</p>
Specific behaviours
<ul style="list-style-type: none">✓ uses the correct value for z✓ uses $\frac{410}{\sqrt{n}}$ as the standard deviation✓ calculates the correct sample size, rounded up to 259

- (b) Suppose that a random sample of 80 tyres is taken, and the mean lifetime of these tyres is 1245 kilometres. Based on this sample, determine a 90% confidence interval for μ . (3 marks)

Solution
<p>The 90% interval is $1245 \pm 1.645 \left(\frac{410}{\sqrt{80}} \right)$</p> <p>$= (1169.6, 1320.4)$</p> <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"><div>C-Level <input type="text" value="0.90"/></div><div>σ <input type="text" value="410"/></div><div>\bar{x} <input type="text" value="1245"/></div><div>n <input type="text" value="80"/></div><div style="display: flex; justify-content: space-between; margin-top: 5px;"><< Back<input type="checkbox"/> HelpNext >></div></div> <p>OneSampleZInt </p> <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"><div>Lower <input type="text" value="1169.6009"/></div><div>Upper <input type="text" value="1320.3991"/></div><div>\bar{x} <input type="text" value="1245"/></div><div>n <input type="text" value="80"/></div><div style="display: flex; justify-content: space-between; margin-top: 5px;"><< Back<input type="checkbox"/> Help</div></div> <p>OneSampleZInt </p>

Accept lower end point from 1169–1170, upper end point from 1320–1321
Specific behaviours
✓ identifies that a 90% confidence interval is within 1.645 standard deviations of the mean
✓ uses $\frac{410}{\sqrt{80}}$ as the standard deviation
✓ calculates the correct interval

- (c) The manufacturer claims that the mean lifetime of Performance Racing tyres is at least 1250 kilometres. Does the sample in part (b) provide a strong reason to doubt this claim? Justify your answer. (2 marks)

Solution
No. 1250 lies within the 90% confidence interval. We would generally doubt the claim only if 1250 lay outside the confidence interval – and a 90% interval is a narrower interval than others that could be used (say 95%, 99%) to test the claim. 1250 would lie well within these intervals.
Specific behaviours
✓ states correct conclusion
✓ states that the value lies within the confidence interval

Question 10

(7 marks)

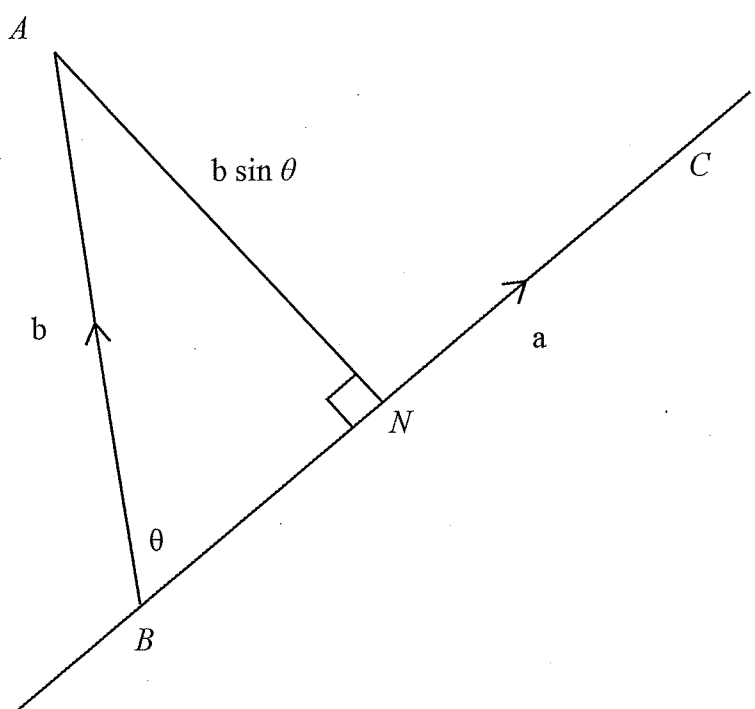
- (a) Let A be a point **not** on the line L that passes through the points B and C .

Given $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$ show that the distance d from the point A to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \overrightarrow{BC}$ and $\mathbf{b} = \overrightarrow{BA}$.

(3 marks)

Solution
 <p>$d = \mathbf{b} \sin \theta$ $= \frac{ \mathbf{a} \mathbf{b} \sin \theta}{ \mathbf{a} }$ $= \frac{ \mathbf{a} \times \mathbf{b} }{ \mathbf{a} }$</p>
Specific behaviours
<ul style="list-style-type: none">✓ draws a diagram to represent the situation✓ expresses d in terms of the sides of triangle ABN✓ expresses d in terms of the cross product

- (b) Use the formula in part (a) to find the distance from the point $A(1, 1, 1)$ to the line through $B(0, 6, 8)$ and $C(-1, 4, 7)$. (4 marks)

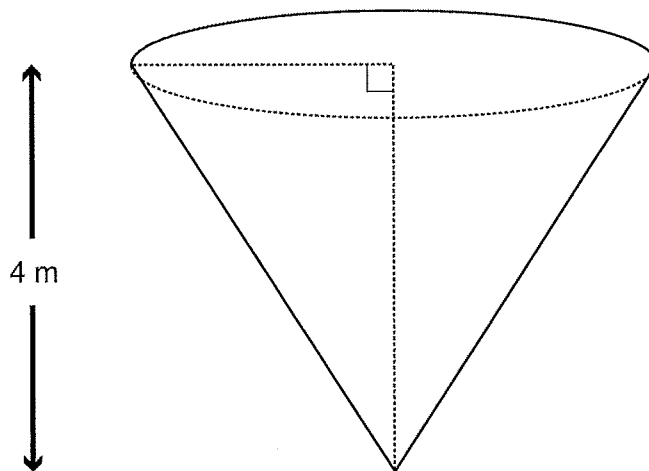
Solution
$\overrightarrow{BC} = \langle -1, 4, 7 \rangle - \langle 0, 6, 8 \rangle$ $\mathbf{a} = \langle -1, -2, -1 \rangle$ $ \mathbf{a} = \sqrt{6}$ $\overrightarrow{BA} = \langle 1, 1, 1 \rangle - \langle 0, 6, 8 \rangle$ $\mathbf{b} = \langle 1, -5, -7 \rangle$ $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix} = 9\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$ $ \mathbf{a} \times \mathbf{b} = \sqrt{9^2 + 8^2 + 7^2} = \sqrt{194}$ $d = \frac{\sqrt{194}}{\sqrt{6}} = 5.69 \text{ (2 dp)}$
Specific behaviours
<ul style="list-style-type: none"> ✓ defines the vector \mathbf{a} ✓ defines the vector \mathbf{b} ✓ evaluates $\mathbf{a} \times \mathbf{b}$ ✓ applies the distance formula correctly

Question 11

(9 marks)

- (a) The ethanol produced by a chemical factory is poured into a 4 m high conical container, with an upper diameter of 4 m, at a constant rate of 3 m^3 per minute. At what rate is the ethanol level rising in the container when the depth of the ethanol is exactly 2.5 m?

(5 marks)

**Solution**

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{12}\pi h^3, \text{ since } r = \frac{h}{2}$$

$$\frac{dV}{dh} = \frac{1}{4}\pi h^2$$

To find $\frac{dh}{dt}$ when $h = 2.5$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$= \frac{4}{\pi h^2} \cdot 3$$

$$= \frac{12}{\pi h^2}, h \neq 0$$

$$= \frac{48}{25\pi}, \text{ when } h = 2.5$$

The depth is increasing at the rate of $\frac{48}{25\pi}$ m/min when the depth of the ethanol is 2.5 m.

Specific behaviours

- ✓ expresses r in terms of h
- ✓ calculates $\frac{dV}{dh}$ correctly
- ✓ establishes the correct chain rule
- ✓ substitutes correctly for h and $\frac{dV}{dt}$
- ✓ determines the correct rate

- (b) Determine the equation of the graph that has the following characteristics:
- at each point (x, y) on the graph of the function, the gradient of the tangent at that point is given by $\frac{-x}{2y}$
 - the graph passes through the point $(2, 1)$. (4 marks)

Solution
$\frac{dy}{dx} = -\frac{x}{2y}$ $2y \, dy = -x \, dx$ $\int 2y \, dy = \int -x \, dx$ $y^2 = -\frac{x^2}{2} + c$ $2y^2 = -x^2 + c$ $x^2 + 2y^2 = c$ <p>Applying the given initial condition:</p> $(2)^2 + 2(1)^2 = c$ $c = 6$ <p>Thus:</p> $x^2 + 2y^2 = 6$
Specific behaviours
<ul style="list-style-type: none">✓ separates the variables✓ integrates correctly✓ substitutes initial condition to determine c✓ determines the equation of the graph

Question 12

(7 marks)

The equation of a sphere is given by $4x^2 + 4y^2 + 4z^2 + 16y - 24x + 32z = 612$.

- (a) Determine the vector equation of the sphere.

(3 marks)

Solution
$4x^2 + 4y^2 + 4z^2 + 16y - 24x + 32z = 612$ $x^2 - 6x + 9 + y^2 + 4y + 4 + z^2 + 8z + 16 = 153 + 9 + 4 + 16$ $(x-3)^2 + (y+2)^2 + (z+4)^2 = 182$ $ \mathbf{r} - \langle 3, -2, -4 \rangle = \sqrt{182}$
Specific behaviours
<ul style="list-style-type: none">✓ completes the square as necessary (after recognising the need to divide by 4)✓ determines the equation in factored form✓ determines the equation of the sphere in correct vector form (in terms of the centre and the radius)

- (b) Determine the position vector(s) of the points of intersection between the sphere and the line $\mathbf{r} = -3\mathbf{i} + 5\mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

(4 marks)

Solution
$\mathbf{r} = \langle -3 - 2\lambda, 5 + \lambda, 1 - 2\lambda \rangle$ $ \langle -3 - 2\lambda, 5 + \lambda, 1 - 2\lambda \rangle - \langle 3, -2, -4 \rangle = \sqrt{182}$ $\sqrt{(-6 - 2\lambda)^2 + (7 + \lambda)^2 + (5 - 2\lambda)^2} = \sqrt{182}$ $\lambda = -4, 2$ <p>i.e. position vectors of the points of intersection are</p> $\langle 5, 1, 9 \rangle \text{ or } \langle -7, 7, -3 \rangle$
Specific behaviours
<ul style="list-style-type: none">✓ constructs the vector equation using the line and the sphere✓ converts this equation into distances✓ solves for λ✓ substitutes the values for λ to define the required position vectors

Question 13

(8 marks)

- (a) A research scientist wished to estimate, with a 95% confidence interval, the mean amount of moisture absorbed through the skin of a particular species of large animal in a laboratory experiment. The scientist believed that the values were normally distributed, and from past experience, felt that the population variance was 4 grams.

A random sample of 25 of this particular species yielded a mean rate of 16.5 grams of moisture being absorbed. Determine the 95% confidence interval for this experiment and explain your findings. (4 marks)

Solution
$C(16.5 - 1.96 \frac{2}{\sqrt{25}} \leq \mu \leq 16.5 + 1.96 \frac{2}{\sqrt{25}}) = 0.95$ $C(16.5 - 0.784 \leq \mu \leq 16.5 + 0.784) = 0.95$ $C(15.7 \leq \mu \leq 17.3) = 0.95$ <p>The scientist can be 95% confident that the species would absorb somewhere between 15.7 and 17.3 grams because it is known that in repeated sampling 95% of the intervals constructed would contain the unknown population mean μ.</p>
Specific behaviours
<ul style="list-style-type: none"> ✓ uses the z value of 1.96 ✓ determines the lower value of 15.7 ✓ determines the upper value of 17.3 ✓ interprets correctly the meaning of the interval

- (b) The same scientist decided that she needed further evidence of the true size of the population mean of moisture absorbed for this particular species. She wished to be within 0.7 grams of the true mean with 99% confidence.

What size sample would the scientist need? (4 marks)

Solution
<p>Using $n = \frac{z^2 \sigma^2}{d^2}$, $d = 0.7$, $z = 2.58$, $\sigma = \sqrt{4}$</p> $n = \frac{(2.58)^2 (2)^2}{(0.7)^2} \approx 54.34$ <p>The scientist will need to take a sample size of 55 of the species of large animal to achieve the desired confidence and interval width.</p>
Specific behaviours
<ul style="list-style-type: none"> ✓ uses the correct formula to find n ✓ substitutes all the correct values ✓ calculates the correct value for n ✓ rounds correctly to give the sample size of 55

Question 14

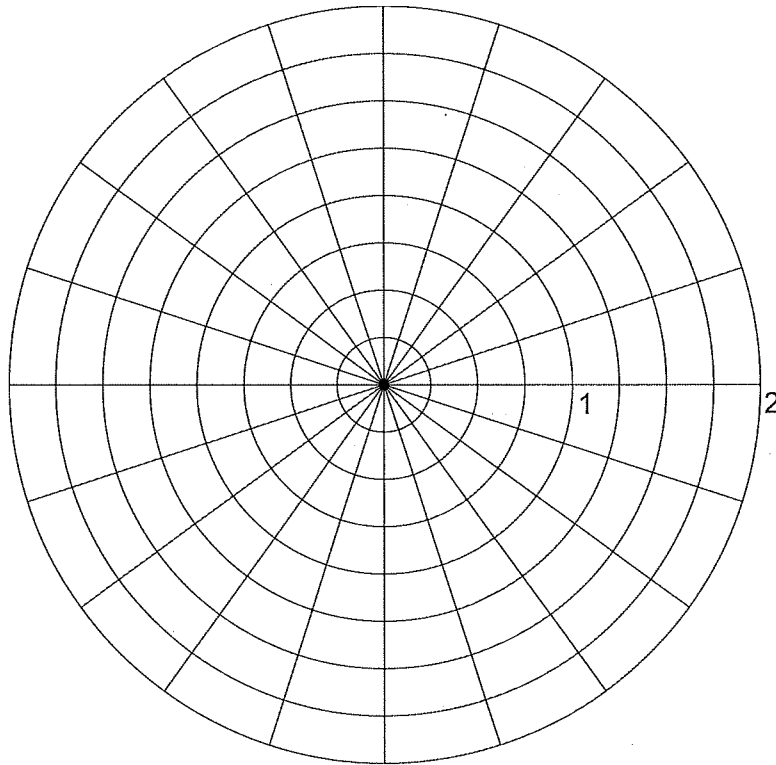
(10 marks)

- (a) Determine all of the roots of the equation $z^6 = \sqrt{3} + i$, expressing them in polar form $r \operatorname{cis} \theta$ where $r \geq 0$ and $-\pi < \theta \leq \pi$. (5 marks)

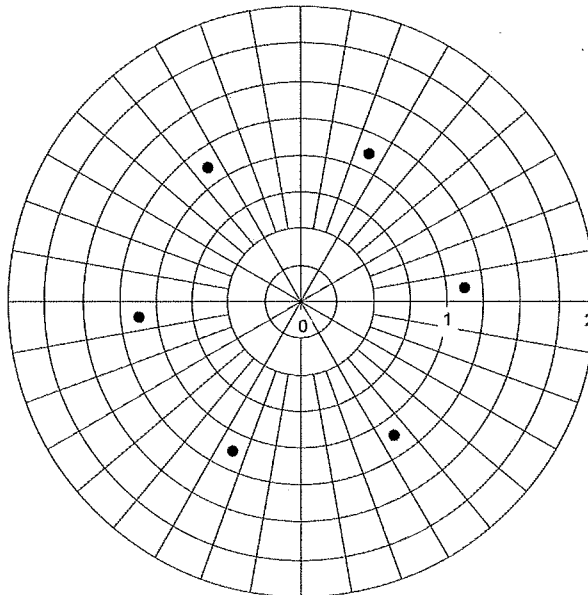
Solution	
$z^6 = 2 \operatorname{cis} \left(\frac{\pi}{6} + 2n\pi \right)$	
$z = 2^{\frac{1}{6}} \operatorname{cis} \left(\frac{\pi}{36} \pm \frac{2n\pi}{6} \right)$	
$z_1 = 2^{\frac{1}{6}} \operatorname{cis} \frac{\pi}{36}$	
$z_2 = 2^{\frac{1}{6}} \operatorname{cis} \frac{13\pi}{36}$	
$z_3 = 2^{\frac{1}{6}} \operatorname{cis} \frac{25\pi}{36}$	
$z_4 = 2^{\frac{1}{6}} \operatorname{cis} \left(-\frac{35\pi}{36} \right)$	
$z_5 = 2^{\frac{1}{6}} \operatorname{cis} \left(-\frac{23\pi}{36} \right)$	
$z_6 = 2^{\frac{1}{6}} \operatorname{cis} \left(-\frac{11\pi}{36} \right)$	
Specific behaviours	
<ul style="list-style-type: none">✓ expresses z^6 in polar form✓ expresses modulus correctly✓ expresses argument correctly✓ determines the roots are $\frac{12\pi}{36} = \frac{\pi}{3}$ radians apart✓ determines the remaining four roots within the range specified	

(b) Plot the roots found in part (a) on the diagram below.

(3 marks)



Solution

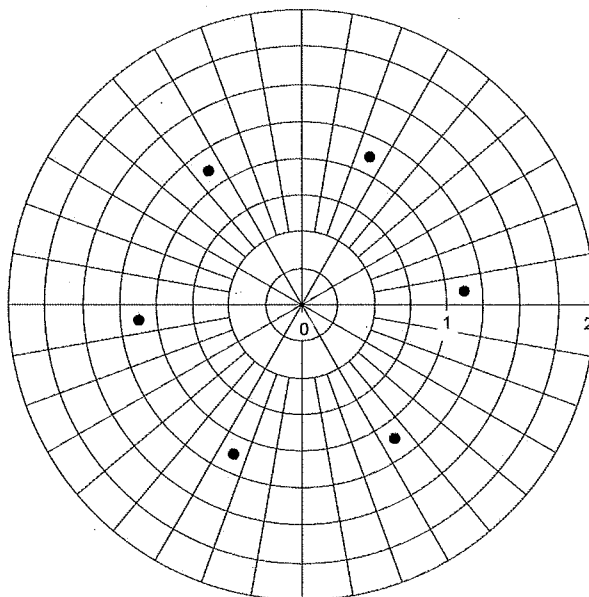


Six points on circle of radius $2^{\frac{1}{6}}$ and equally spaced in units of $\frac{\pi}{3}$ radians.

Specific behaviours

- ✓ shows each root has magnitude $2^{\frac{1}{6}} \approx 1.12$
- ✓ shows that first root has argument $\frac{\pi}{36}$
- ✓ spaces accurately the other roots at intervals of $\frac{\pi}{3}$ radians

- (c) The roots form the vertices of a hexagon. Determine the exact value for the perimeter of the hexagon. (2 marks)

Solution

Central angle for each triangle $\frac{\pi}{3}$ radians. Therefore the triangles determined by the roots and the origin are equilateral.

Equilateral triangles have side lengths of $2^{\frac{1}{6}}$ units

\therefore Perimeter is $6 \times 2^{\frac{1}{6}}$ units

Specific behaviours

- ✓ determines the side lengths of the hexagon
- ✓ evaluates the perimeter of the hexagon

Question 15

(6 marks)

An engine piston undergoes simple harmonic motion which can be described by the differential equation $\frac{d^2x}{dt^2} = -9x$, where x m is the displacement of the piston from its mean position at t seconds.

- (a) Determine the period of the motion. (1 mark)

Solution
$n^2 = 9$, where n is the angular velocity Hence the period of motion is $\frac{2\pi}{3}$ seconds.
Specific behaviours
✓ defines the period correctly

- (b) If the maximum speed of the piston is 5 m/s, calculate the amplitude of the motion. (2 marks)

Solution
$v_{\max} = An$ where A is the amplitude Hence $A = \frac{5}{3}$ metres
Specific behaviours
✓ uses the equation $v_{\max} = An$ or $v^2 = n^2(A^2 - x^2)$ at $x = 0$ ✓ correctly solves for A

- (c) The amplitude and period of the motion are now changed, but the piston still undergoes simple harmonic motion. These new readings are taken:

when $x = 1$ m, speed $= \sqrt{60}$ m/s; when $x = 3$ m, speed $= \sqrt{28}$ m/s.

Find the new exact values for the:

- (i) period
(ii) amplitude. (3 marks)

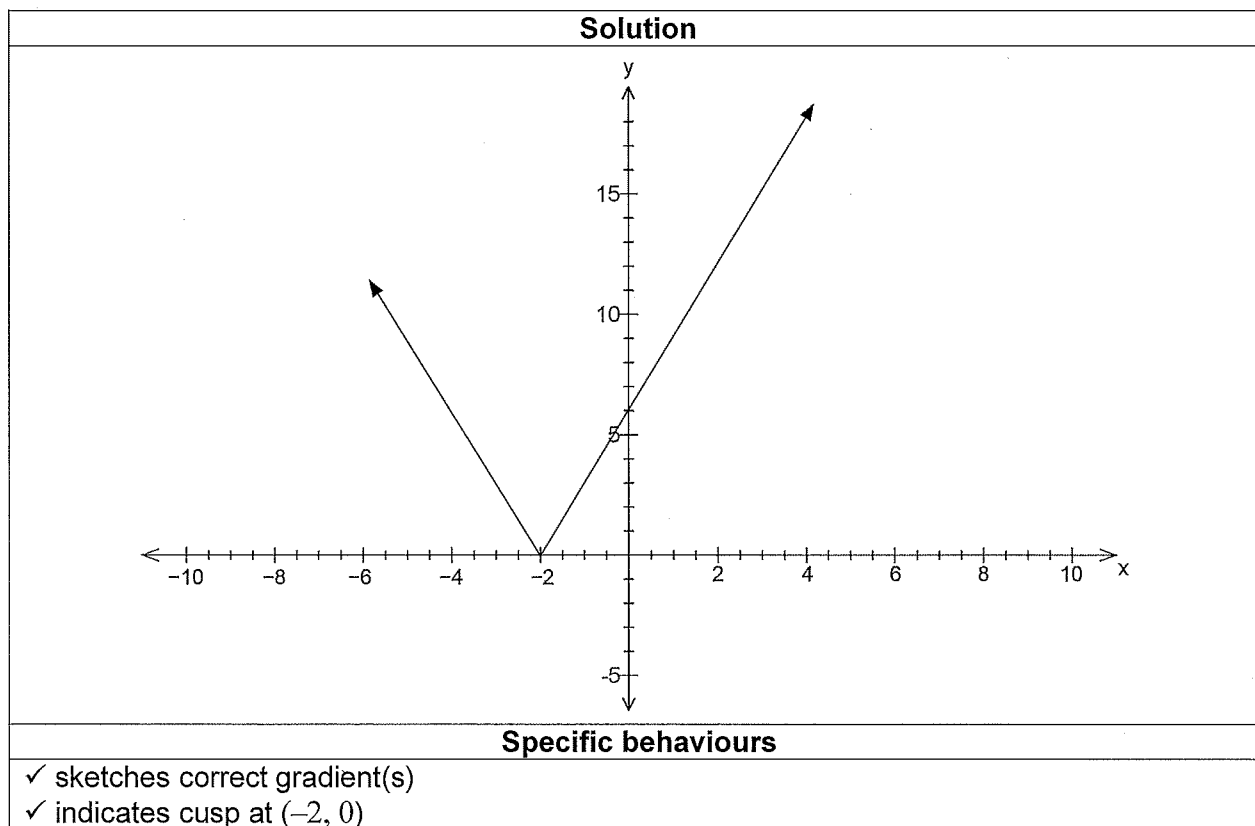
Solution
$v^2 = n^2(A^2 - x^2)$ Hence: $60 = n^2(A^2 - 1)$ and $28 = n^2(A^2 - 9)$ Solving gives $n = 2$ and $A = 4$ Hence: (i) period $= \pi$ seconds (ii) amplitude $= 4$ metres
Specific behaviours
✓ uses the equation $v^2 = n^2(A^2 - x^2)$ correctly ✓ solves for n correctly ✓ solves for A correctly

Question 16

(5 marks)

- (a) Sketch the graph of $y = |3x + 6|$ on the grid below.

(2 marks)



- (b) Use the graph to determine the values of the real constants p , q and r if the equation

$$|3x + 6| = p|x + q| + r$$

is satisfied for all $x \in [-2, 3]$ but no other real values.

(3 marks)

Solution
From the graph, when $x = 3, y = 15 \Rightarrow r = 15$ Also, $q = \pm 3$, since it is parallel to $y = 3x + 6 $. As cusp is at $x = 3$, then $q = -3$. Hence, $y = p x - 3 + 15$ Substituting $(-2, 0)$ into the equation $0 = p -2 - 3 + 15 \Rightarrow p = -3$
Specific behaviours
✓ evaluates r using the y value of the upper limit ✓ evaluates q using the gradient ✓ evaluates p by substituting the coordinates of the cusp into the equation

Question 17

(6 marks)

A designer of applications ('apps') for a major manufacturer of tablet computers is trying to establish the best price to charge for a new app that has been developed. On the basis of previous sales of similar apps, it has been established that the demand function for the product is modelled by

$$p = \frac{3}{0.000\,001x^3 + 0.01x + 1}$$

where p is measured in dollars and x is measured in hundreds of units.

- (a) Find the rate of change of the demand x with respect to the price p . (3 marks)

Solution	
$p = \frac{3}{0.000\,001x^3 + 0.01x + 1}$	
$0.000\,001x^3 + 0.01x + 1 = \frac{3}{p}$	
$0.000\,003x^2 \frac{dx}{dp} + 0.01 \frac{dx}{dp} = -\frac{3}{p^2}$	
$(0.000\,003x^2 + 0.01) \frac{dx}{dp} = -\frac{3}{p^2}$	
$\frac{dx}{dp} = -\frac{3}{p^2(0.000\,003x^2 + 0.01)}$	
Specific behaviours	
<ul style="list-style-type: none"> ✓ rearranges the original equation ✓ differentiates implicitly correctly ✓ determines the general rate of change 	

- (b) Find the rate of change of the demand x with respect to the price p when $x = 100$ and explain clearly the significance of your answer with respect to the designer. (3 marks)

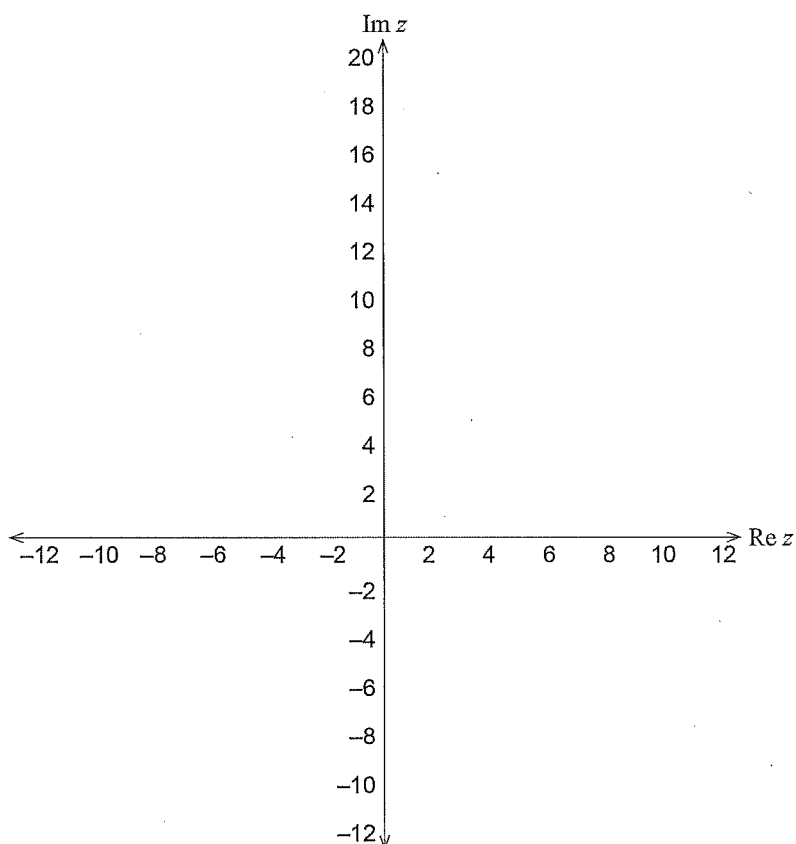
Solution	
When $x = 100$, the price is	
$p = \frac{3}{0.000\,001(100)^3 + 0.01(100) + 1} = \1	
Thus, when $x = 100$ and $p = 1$, the rate of change of the demand with respect to the price is	
$\frac{dx}{dp} = -\frac{3}{(1^2)[0.000\,003(100)^2 + 0.01]} = -75$	
This implies that when $x = 100$, the demand is dropping at the rate of 7500 units for each dollar increase in price.	
Specific behaviours	
<ul style="list-style-type: none"> ✓ uses the correct equation and correctly substitutes to evaluate p ✓ determines the correct value for $\frac{dx}{dp}$ ✓ describes the significance of the answer 	

Question 18

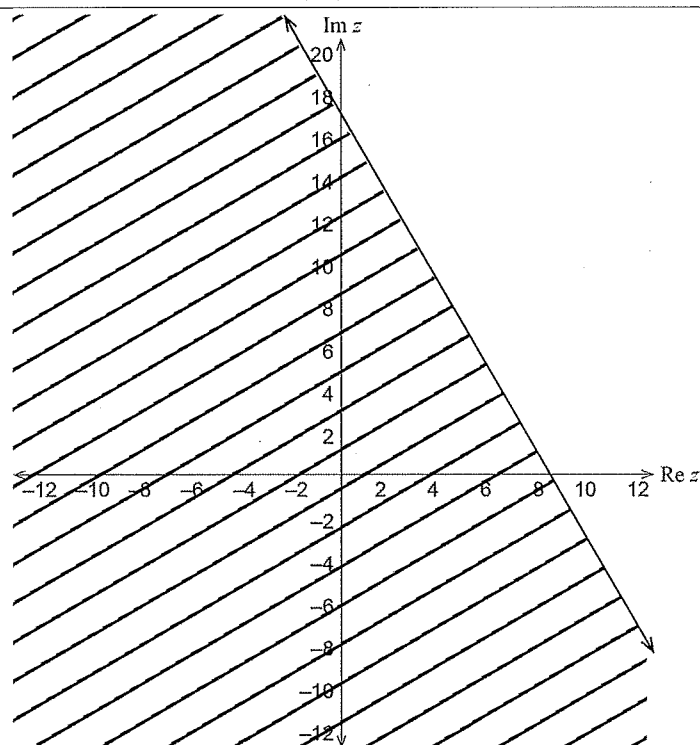
(7 marks)

- (a) On the Argand diagram below, sketch the inequality defined by $\text{Im } z \leq -2\text{Re } z + 17$.

(2 marks)



Solution



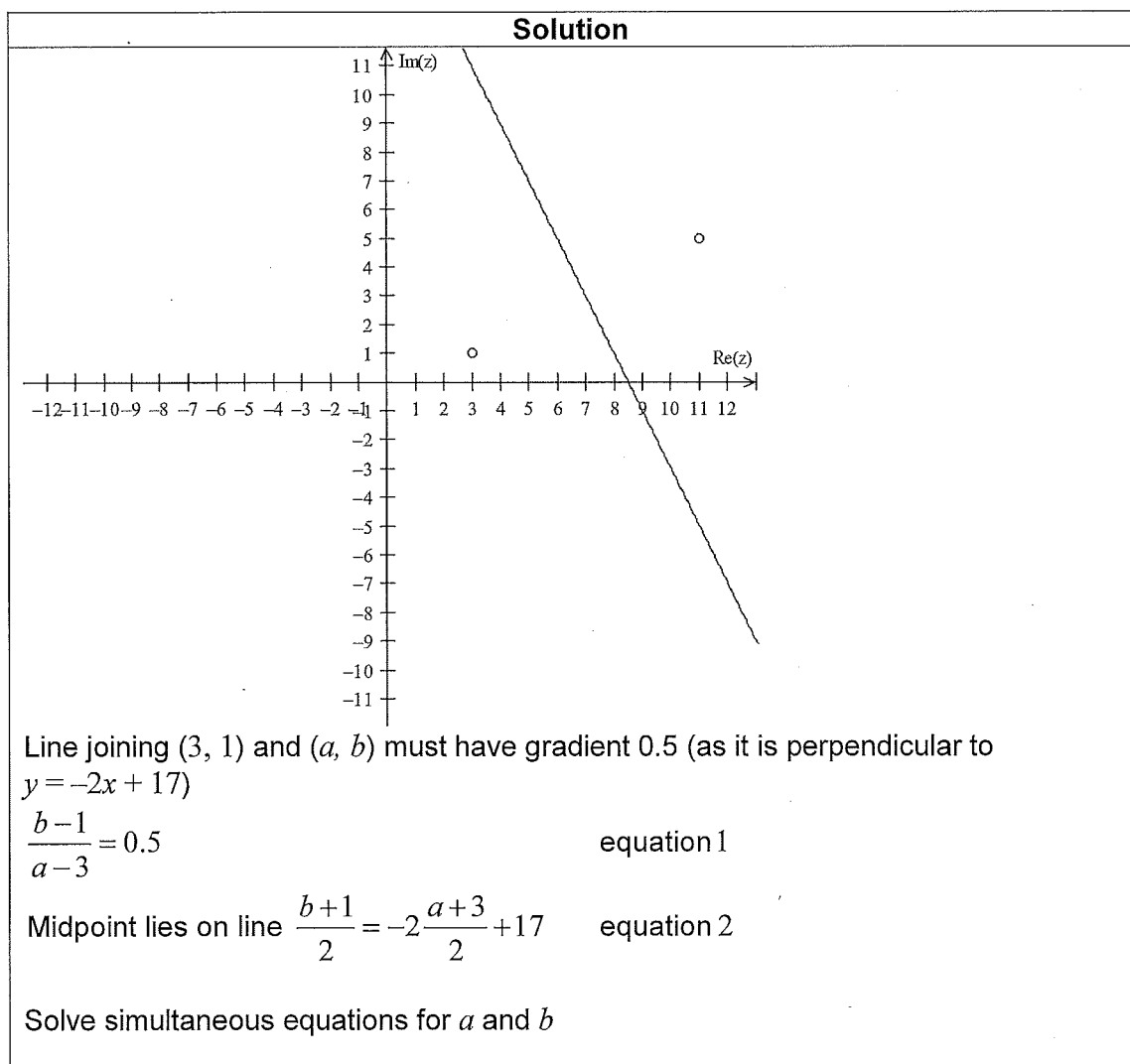
Specific behaviours

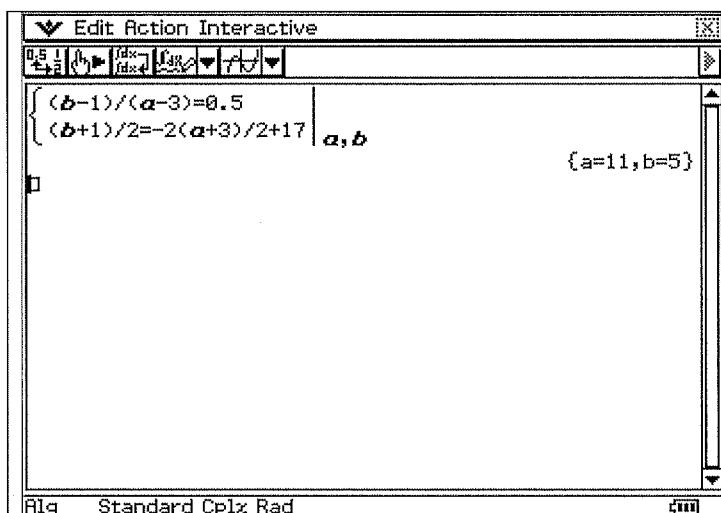
- ✓ sketches correct line
- ✓ shades correct half plane

- (b) Show that the point $(3, i)$ satisfies the inequality from part (a). (1 mark)

Solution
Substitute $(3, i)$ into the inequality: $1 \leq 2.3 + 17$ $1 \leq 23$ \therefore As this is true $(3, i)$ must satisfy the inequality
Specific behaviours
✓ substitutes correctly into inequality to show that the point satisfies the inequality

- (c) The set of points in the complex plane that satisfy $|z - 3 - i| = |z - a - bi|$, where a and b are certain real constants, can alternatively be defined by the property that they lie on the line $\text{Im } z = -2\text{Re } z + 17$. Determine the values of a and b . (4 marks)





Specific behaviours

- ✓ states line joining (a, b) and $(3, 1)$ is perpendicular to $y = -2x + 17$
- ✓ uses the gradient of the line joining (a, b) and $(3, 1)$ to obtain equation 1 in a and b
- ✓ uses the midpoint of the line joining (a, b) and $(3, 1)$ to obtain equation 2 in a and b
- ✓ solves the simultaneous equations to evaluate a and b

Question 19

(10 marks)

The velocity vector of a moving object at time t seconds is

$$\mathbf{v}(t) = -5 \sin\left(\frac{t}{2}\right) \mathbf{i} + 4 \cos\left(\frac{t}{2}\right) \mathbf{j}.$$

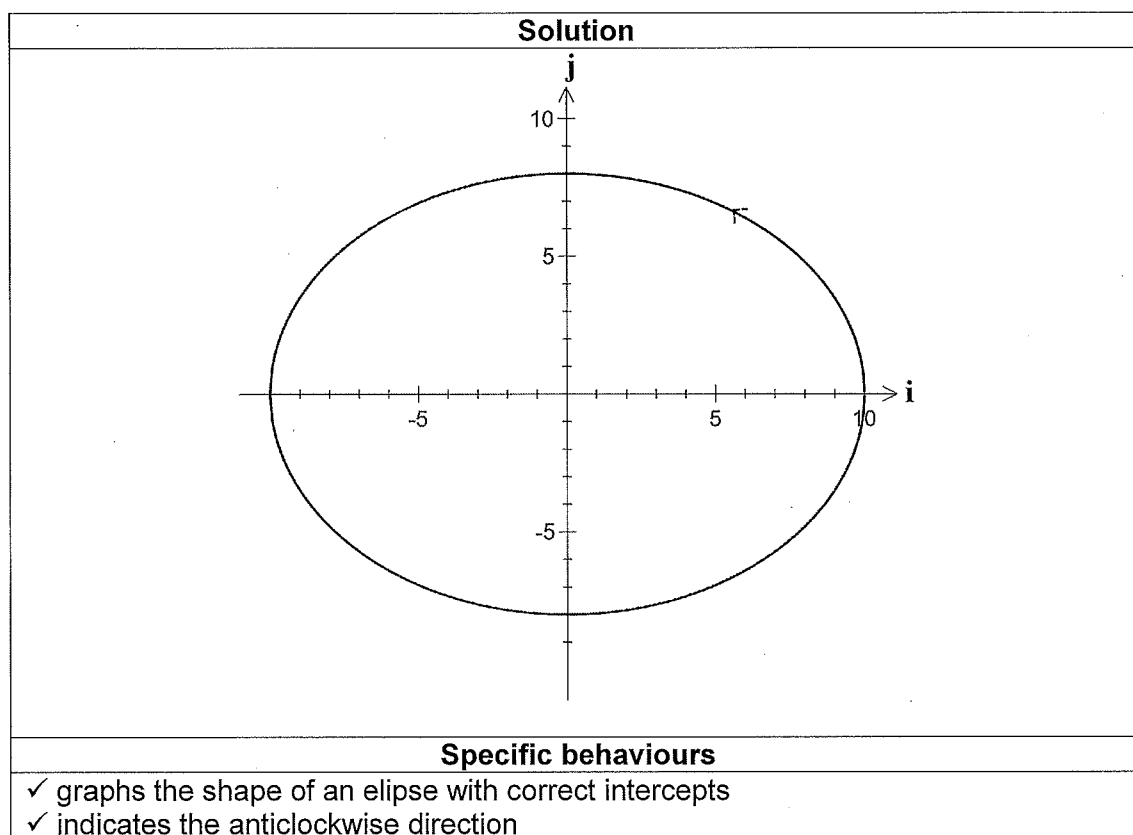
- (a) Determine the position vector $\mathbf{r}(t)$ of the object, given that initially, $\mathbf{r} = 10\mathbf{i}$. (2 marks)

Solution
$\mathbf{r}(t) = \int \left(-5 \sin \frac{t}{2} \mathbf{i} + 4 \cos \frac{t}{2} \mathbf{j} \right) dt$ $= 10 \cos \frac{t}{2} \mathbf{i} + 8 \sin \frac{t}{2} \mathbf{j} + \mathbf{c}$ but $\mathbf{r}(0) = 10\mathbf{i}$, hence $\mathbf{c} = 0$ so $\mathbf{r}(t) = 10 \cos \frac{t}{2} \mathbf{i} + 8 \sin \frac{t}{2} \mathbf{j}$
Specific behaviours
✓ finds the antiderivative of the \mathbf{i} and \mathbf{j} components ✓ evaluates the vector constant

- (b) Show that the acceleration vector is always parallel to the position vector. (2 marks)

Solution
$\mathbf{a}(t) = -\frac{5}{2} \cos \frac{t}{2} \mathbf{i} - 2 \sin \frac{t}{2} \mathbf{j}$ $= -\frac{1}{4} \left(10 \cos \frac{t}{2} \mathbf{i} + 8 \sin \frac{t}{2} \mathbf{j} \right)$ $= -\frac{1}{4} \mathbf{r}(t)$ ie $\mathbf{a}(t)$ is parallel to $\mathbf{r}(t)$
Specific behaviours
✓ differentiates \mathbf{i} and \mathbf{j} components of the velocity vector ✓ demonstrates $\mathbf{a}(t)$ is a constant times $\mathbf{r}(t)$ hence they are parallel

- (c) Sketch the path taken by the object and indicate the direction of travel on the axes below. (2 marks)



- (d) Given that $\int_0^{4\pi} \mathbf{v}(t) dt = 0$, explain what this means in terms of the path taken by the object. (2 marks)

Solution
This indicates zero displacement as the object has returned to its starting point
Specific behaviours
✓ shows a grasp of the notion of displacement ✓ explains the object has returned to original position

- (e) Evaluate $\int_0^{4\pi} |\mathbf{v}(t)| dt$ and explain what this means in terms of the path taken by the object. (2 marks)

Solution
$\int_0^{4\pi} \mathbf{v}(t) dt = \int_0^{4\pi} \sqrt{25 \sin^2 \frac{t}{2} + 16 \cos^2 \frac{t}{2}} dt$ $= 56.723$ <p>This is the distance travelled by the object in one circuit – i.e. the perimeter of the ellipse.</p>
Specific behaviours
✓ integrates the expression to find the distance ✓ explains the notion of distance travelled

Question 20

(7 marks)

The present population, P , of snakes on a small island is 154. Due to favourable breeding conditions and the availability of a steady food supply, the growth rate of the population in the future is expected to be given by

$$\frac{dP}{dt} = 0.16P\left(1 - \frac{P}{500}\right), \text{ where } t \text{ is the time, in months, from today.}$$

- (a) Express P as a function of t . [Hint: Use partial fractions]

(5 marks)

Solution
$\frac{dP}{dt} = 0.16 \frac{P(500-P)}{500}$
$\text{i.e. } \frac{500}{P(500-P)} \frac{dP}{dt} = 0.16$
$\text{i.e. } \int \frac{500}{P(500-P)} \frac{dP}{dt} dt = \int 0.16 dt$
$\text{i.e. } \int \left(\frac{1}{P} + \frac{1}{500-P} \right) dP = \int 0.16 dt$
$\text{i.e. } \ln \left \frac{P}{500-P} \right = 0.16t + c$
When $t = 0$, $P = 154$ giving c as approximately -0.8095
$\ln \left \frac{P}{500-P} \right \approx 0.16t - 0.8095$
$\text{i.e. } \frac{P}{500-P} \approx e^{0.16t - 0.8095}$
$\text{i.e. } \frac{500-P}{P} \approx e^{-0.16t + 0.8095}$
$\text{i.e. } \frac{500}{P} \approx 1 + 2.247e^{-0.16t}$
$\text{i.e. } P \approx \frac{500}{1 + 2.247e^{-0.16t}}$
Specific behaviours
<ul style="list-style-type: none">✓ expands and simplifies with a common denominator✓ uses partial fractions to separate the fraction✓ integrates fractions to obtain log function✓ converts the equation to exponential form✓ rewrites the equation in terms of P

- (b) Calculate the approximate numbers of snakes on the island after two years. (1 marks)

Solution
Two years = 24 months Hence, $t = 24$ Therefore $P \approx 476$ snakes
Specific behaviours
✓ calculates correctly the approximate number of snakes

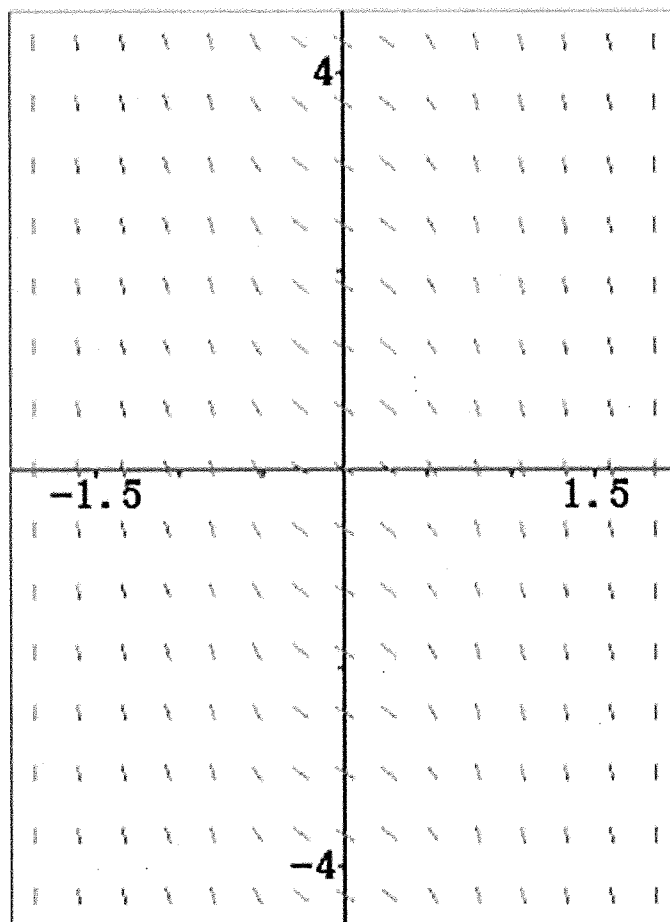
- (c) What is the limiting population size? (1 mark)

Solution
As $t \rightarrow \infty, e^{-0.16t} \rightarrow 0$ $\therefore P \rightarrow \frac{500}{1+0} = 500$ snakes
Specific behaviours
✓ determines correctly the limiting population size

Question 21

(6 marks)

A first-order differential equation has a slope field as shown in the diagram below.



- (a) Use the scale shown to determine a general differential equation that would result in this slope field. (2 marks)

Solution
$\frac{dy}{dx} = -bx^2 - c$
Specific behaviours
✓ determines that the coefficient is negative ✓ determines that the equation is quadratic (and that there is no x term)

- (b) Give **two** reasons for your answer in part (a). (2 marks)

Solution
Any two of the following: <ul style="list-style-type: none">• original equation must be cubic given that none of the isoclines are positive• slope field indicates gradient of original equation is negative• point of inflection appears to be on the y-axis
Specific behaviours
✓✓ one mark for each correct reason, to a maximum of 2 marks

- (c) Determine a possible general equation for y . (2 marks)

Solution
$y = -c_1x^3 + c_2x + c_3$
Specific behaviours
<ul style="list-style-type: none"> ✓ gives a negative value to c_1 ✓ expresses the cubic with no squared term

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