Particle Filters

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Background - Hidden Markov Models

Situations often arise when a Markov chain of interest $\{X_t\}$ can not be measured directly. We wish to make inference on our latent variables $\{X_t\}$ from a series of observations $\{Y_t\}$.
We assume the following distributions are known and that we can simulate from them:

$$y_t|x_{1:t}, y_{1:t-1} \sim p(y_t|x_t) \quad (1)$$
$$x_t|x_{1:t-1} \sim p(x_t|x_{t-1}) \quad (2)$$
$$x_1 \sim p(x_1) \quad (3)$$

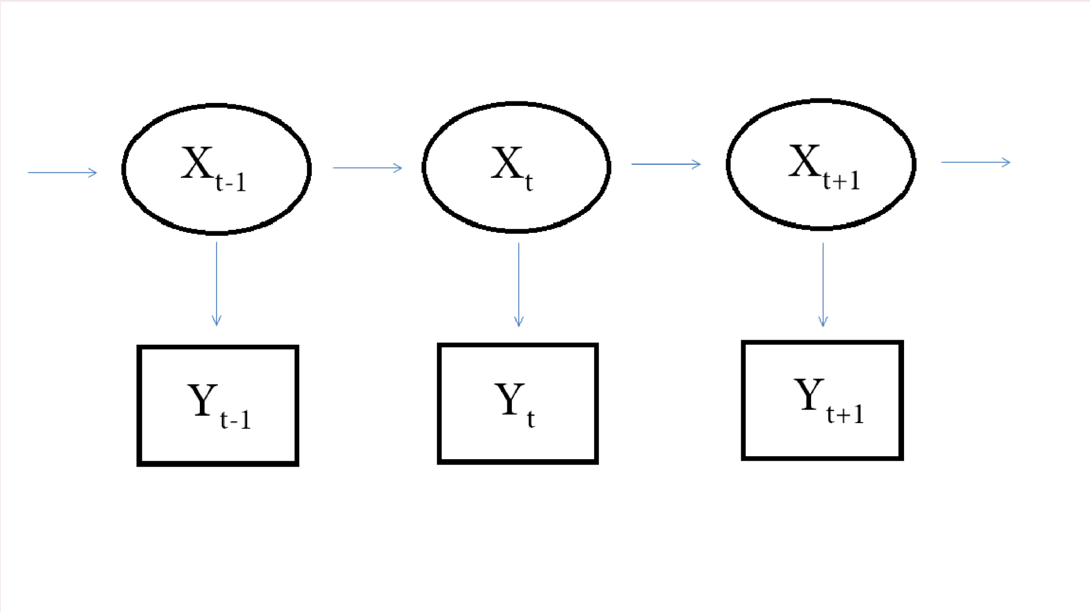

Figure 1: A hidden markov chain.

For example, we may be interested in tracking the position of a a moving object but can only measure its bearing.

Recursions

Using these conditional distributions one can easily derive the following recursions:

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})dx_{t-1} \quad (4)$$
$$p(x_t|y_{1:t}) = \frac{p(y_t|x_t)p(x_t|y_{1:t-1})}{p(y_t|y_{1:t-1})} \quad (5)$$

Equation (4) is known as the *predictive distribution* and equation (5) is known as the *filtering distribution*.

In the case where $\{X_n\}$ and $\{Y_n\}$ only take discrete values one can easily explicitly computer the above recursion formulae. Discrete space Hidden Markov Models have been used extensively in DNA sequencing. Another important solvable case is the *Linear Gaussian model*. Specifically, we have:

$$X_n = AX_{n-1} + BV_n$$
$$Y_n = CX_n + DW_n$$

where $V_n \sim \mathcal{N}(0, I_u)$, $W_n \sim \mathcal{N}(0, I_v)$ are i.i.d and A, B, C, D are matrices of appropriate dimensions. This case can be solved by *Kalman Filters*.

Kalman Filters and Particle Filter Methods

In general the above integral (4) is intractible and cannot be satisfactorily solved numerically so we must resort to other methods.

Particle filters provide a method for *simulating from a discrete approximation of the filtering distribution*. These algorithms typically consist of sampling from the known distributions with weighted resampling.

The Bootstrap Filter

The Bootstrap filter was the first particle filter method to be proposed by [Gordon et al., 1993].

1. Sample N observations $\{x_1^{(i)}\}_{i=1}^N$ from $p(x_1)$.
   Recursive Steps:
2. Propogate $\{x_{t-1}^{(i)}\}_{i=1}^N$ to $\{\tilde{x}_t^{(i)}\}_{i=1}^N$ by sampling via $p(x_t|x_{t-1})$.
3. Acquire $\{x_t^{(i)}\}_{i=1}^N$ by resampling from $\{\tilde{x}_t^{i}\}_{i=1}^N$ with weights $w_t^{(i)} \propto p(y_t|\tilde{x}_t^{(i)})$.

Comments on the Bootstrap Filter

- Low computational cost.
- Straight-forward to implement.
- Successive approximations causes accuracy of later simulations to degenerate
- Outliers lead to disproportionate weights necessitating extra draws in the sampling stage.

Auxiliary Particle Filter

The Auxiliary Particle Filter makes use of the predictive likelihood $p(y_t|x_t)$. It was first proposed in [Pitt, Shephard, 1999].

1. Sample N observations $\{x_1^{(i)}\}_{i=1}^N$ from $p(x_1)$.
   Recursive Steps:
2. Resample $\{x_{t-1}^{(i)}\}_{i=1}^N$ with weights $w_{t-1}^{(i)} \propto p(y_t|g(x_{t-1}))$ to get $\{\tilde{x}_{t-1}^{i}\}_{i=1}^N$.
3. Propogate $\{\tilde{x}_{t-1}^{i}\}_{i=1}^N$ to $\{\tilde{x}_t^{(i)}\}_{i=1}^N$ by sampling via $p(x_t|\tilde{x}_{t-1})$.
4. Resample $\{\tilde{x}_t^{(i)}\}_{i=1}^N$ with weights $w_t^{(i)} \propto p(y_t|\tilde{x}_t^{(i)})$ to get $\{x_t^{(i)}\}$.

Here, $g(x_{t-1})$ is some statistic of $\{x_{t-1}^{(i)}\}_{i=1}^N$ such as the mean or mode. The extra resampling stage ensures that we simulate from particles with large predictive likelihoods.

Tracking and Bearings Example

In this example taken from [Pitt, Shephard, 1999], one attempts to make inference on the position (x, y) of a ship based on noisy observations of the bearing $\theta = \arctan(\frac{y}{x})$. The problem is extremely non-linear so cannot be solved by a Kalman filter.

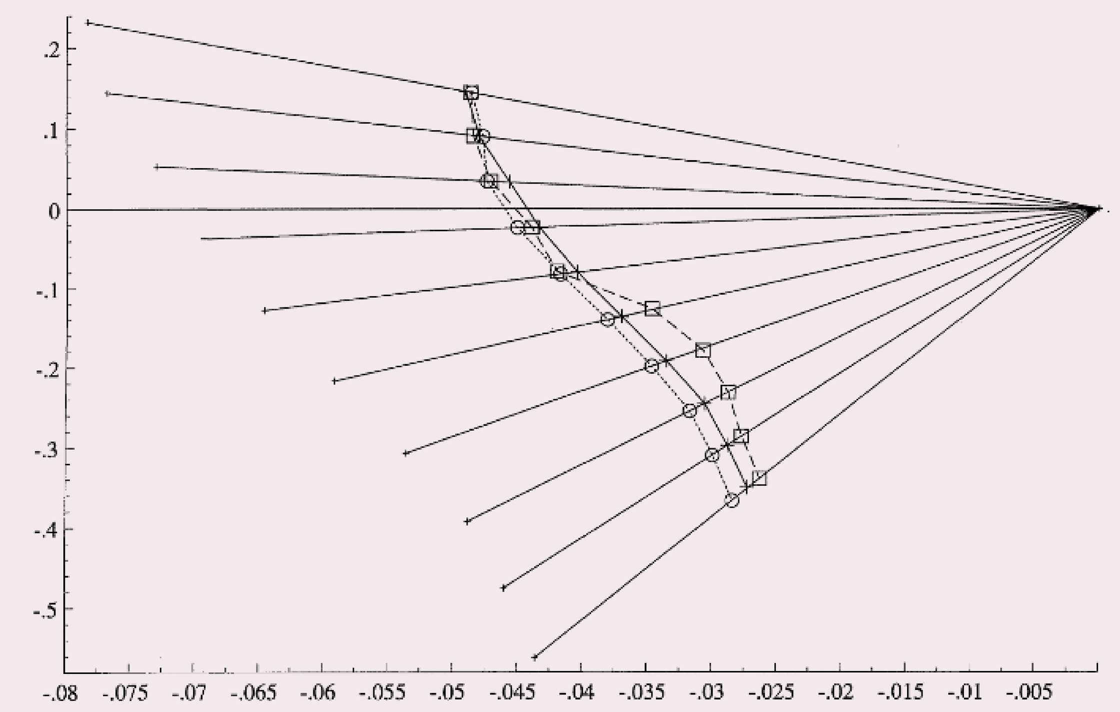

Figure 2: Plot of Angular Measurements from Origin, the true trajectory (solid line, crosses), the Bootstrap Filtered mean trajectory (dashed line, boxes), and the Auxiliary Particle mean trajectory (dotted line, circles). Ship moving southeast. T=10, M= 300, R=500. Figure taken from [Gordon et al., 1993].

Discussion

- Experimentation has shown that the Auxiliary Particle Filter outperforms the Bootstrap filter in terms of accuracy.
- Degeneracy, although still a problem in Auxiliary Particle, is reduced significantly. New methods attempt to further slow degeneracy such as Particle Learning techniques.
- Both the bootstrap filter and auxiliary filter outperform extended Kalman filter which is standardly used in non-linear Hidden Markov models.

References

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