Math 351 Assignment 3

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(1)

• Which of the following are metric functions on $(0, \infty)$? Write simply "metric" or "not metric".

a) $d(x, y) = \left| \frac{1}{x^4} - \frac{1}{y^4} \right|$

Metric. ✓

b) $d(x, y) = |x - 3y|$

Not metric. ✓

c) $d(x, y) = \sqrt{|x - y|} + \frac{|x-y|}{1+|x-y|}$

Metric. ✓

d) $d(x, y) = \tan^{-1}|x - y|$

Metric. ✓

e) $d(x, y) = \min\left\{|x - y|^{3/4}, 2\right\}$

Metric. ✓

• Which of the following are metric functions on $(0, \infty) \times (0, \infty)$? Write simply "metric" or "not metric".

f) $d((x, y), (w, z)) = \sqrt{\left|\frac{1}{x^4} - \frac{1}{w^4}\right|^2 + \left|\frac{1}{y^4} - \frac{1}{z^4}\right|^2}$

Metric. ✓

g) $d((x, y), (w, z)) = |x - 3w| + |y - z|$

Not **m**etric. ✓

h) $d((x, y), (w, z)) = \sqrt{|x - w|} + \frac{|y-z|}{1+|y-z|}$

Metric. ✓

i) $d((x, y), (w, z)) = \tan^{-1}\left(\sqrt{|x-w|^2 + |y-z|^2}\right)$

Metric. ✓

j) $d((x, y), (w, z)) = \min\left\{|x-w|^{3/4}, 2\right\} + \min\left\{|y-z|^{1/4}, 1\right\}$

Metric. ✓

(2) Let $M = (0, \infty)$ be supplied with the metric function $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ and let $\{n\}_{n=1}^{\infty}$ be a sequence of positive integers.

a) Is the sequence $\{n\}_{n=1}^{\infty}$ a Cauchy sequence in (M, d)? Justify your answer.

<u>Solution:</u>
The sequence $\{n\}_{n=1}^{\infty}$ is a Cauchy sequence in (M, d).
To see why, we pick any $\varepsilon > 0$. Then we let N be a positive integer such that if $n \geq N$, $\left|\frac{1}{n} - 0\right| \leq \frac{\varepsilon}{2}$.
Then for $m, n \geq N$,

$$d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \left|\frac{1}{m} - 0\right| + \left|\frac{1}{n} - 0\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, we have shown that $\{n\}_{n=1}^{\infty}$ is a Cauchy sequence. ✓

b) Does the sequence $\{n\}_{n=1}^{\infty}$ converge in (M, d)? Justify your answer.

<u>Solution:</u>
The sequence $\{n\}_{n=1}^{\infty}$ does not converge in (M, d). The reason why the sequence is not convergent is due to to the metric space M. In other words, the sequence $\{n\}_{n=1}^{\infty}$ with the given metric d converges to 0:

Since $M \subset \mathbb{R}$ is an ordered field, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$ iff $n > \frac{1}{\varepsilon}$. It follows from the Archimidean property of $\mathbb{R}$ (which M inherits) that $n > \frac{1}{\varepsilon}$ can be achieved for some sufficiently large integer N.
Thus, if $n \geq N$, $\frac{1}{n} < \varepsilon$.
Thus, we have shown that $\{n\}_{n=1}^{\infty}$ with the given metric d converges to 0. However, $0 \notin M = (0, \infty)$, thus the sequence is not convergent in (M, d). ✓

(3) True or false? $|\tan^{-1}|x| - \tan^{-1}|y|| \leq \tan^{-1}|x-y|$. Justify your answer. (Hint: Look at HW#3, problem 1)

Solution:

The statement is true.

We have previously shown in class that

$$\tan^{-1}|x-y| = \rho(x, y) = d(f(x), f(y))$$

is a metric with $d(x, y) = |x-y|$ and $f(t) = \tan^{-1}(t)$.

Then by HW#3 exercise 1 we have that

$$|\rho(x, 0) - \rho(y, 0)| \le \rho(x, y),$$

which in this case means that

$$|\tan^{-1}|x-0| - \tan^{-1}|y-0|| = |\tan^{-1}|x| - \tan^{-1}|y|| \le \tan^{-1}|x-y|. \checkmark$$

(4) Let $(\mathbb{R}, d)$ be a metric space with the metric function $d(x, y) = \frac{|x-y|}{1+|x-y|}$. Calculate $\text{diam}(0, \infty)$.

Solution:

The diameter of the set $(0, \infty)$ is given by $\sup\{d(a, b) : a, b \in (0, \infty)\}$. Now to compute this supremum, we have

$$\lim_{\substack{a \to 0 \\ b \to \infty}} d(a, b) = \lim_{\substack{a \to 0 \\ b \to \infty}} \frac{|a-b|}{1+|a-b|} = \lim_{b \to \infty} \frac{|0-b|}{1+|0-b|} = \lim_{b \to \infty} \frac{b}{1+b} = 1$$

Hence, we conclude that $\text{diam}(0, \infty) = 1$.