MATH 3101 HW # 6

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PROF. A. E. CLEMENT

1. Show that if G is nonabelian, then the factor group $G/Z(G)$ is not cyclic, where $Z(G)$ is a normal subgroup of G called the ***center*** of G.

Proof. We are going to show the contrapositive of this statement, namely, that if $G/Z(G)$ is cyclic, then G must necessarily be abelian. By *Theorem 15.9*, we know that a factor group of a cyclic group is cyclic, hence abelian. The proof of our statement thus follows by simply letting $G/Z(G)$ be cyclic and using the contrapositive of *Theorem 15.9*. However, we are now going to attempt to achieve this result without resorting to any previous theorem:

Suppose that $G/Z(G)$ is cyclic and that is generated by the coset $aZ(G)$ for $a \in G$. Let $x, y \in G$. Then, since $G/Z(G)$ is cyclic, we have $x \in a^s Z(G)$ and $y \in a^t Z(G)$ for some $s, t \in \mathbb{Z}$. We can thus write

$$x = a^s z_1 \quad \text{and} \quad y = a^t z_2,$$

where $z_1, z_2 \in Z(G)$. Since z_1 and z_2 are elements of the center of G, they commute with every element of G. Hence we have

$$\begin{aligned}
xy = (a^s z_1)(a^t z_2) &= a^s (z_1 a^t) z_2 && \text{(By associativity of } G\text{)} \\
&= a^s (a^t z_1) z_2 && \text{(By commutativity of } Z(G)\text{)} \\
&= (a^s a^t)(z_1 z_2) && \text{(By associativity of } G\text{)} \\
&= a^{s+t} z_1 z_2 && \\
&= a^{t+s} z_1 z_2 && \text{(By commutativity of } \mathbb{Z}\text{)} \\
&= (a^t a^s)(z_1 z_2) && \\
&= (a^t a^s)(z_2 z_1) && \text{(By commutativity of } Z(G)\text{)} \\
&= a^t (a^s z_2) z_1 && \text{(By associativity of } G\text{)} \\
&= a^t (z_2 a^s) z_1 && \text{(By commutativity of } Z(G)\text{)} \\
&= (a^t z_2)(a^s z_1) && \text{(By associativity of } G\text{)} \\
&= yx.
\end{aligned}$$

Hence we have shown that G is abelian, thus proving as desired that if $G/Z(G)$ is cyclic, then G must necessarily be abelian. Since this statement is equivalent to saying that if G is nonabelian, then the factor group $G/Z(G)$ is not cyclic, we have concluded our proof. ☐