MATH 725 HW#6

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Exercise (Exercise 1). *Let X and Y be nonempty sets. Show that $\mathcal{F}_{X\times Y} \cong \mathcal{F}_X \otimes F_Y$, where $\mathcal{F}_A$ is the free module (or vector space) on A.*

Proof. Take the tensor map $\tau\colon \mathcal{F}_X \times \mathcal{F}_Y \to \mathcal{F}_X \otimes \mathcal{F}_Y$, and define the bilinear map $\varphi\colon \mathcal{F}_X \times \mathcal{F}_Y \to \mathcal{F}_{X\times Y}$ by

$$\varphi\left(\sum_i \alpha_i x_i, \sum_j \beta_j y_j\right) = \sum_{i,j} \alpha_i \beta_j (x_i, y_j)$$

so that, by the universal property of tensor products, there exists a unique $\widetilde{\tau}\colon \mathcal{F}_X \otimes \mathcal{F}_Y \to \mathcal{F}_{X\times Y}$ that satisfies $\widetilde{\tau}\tau = \varphi$. Notice that $\widetilde{\tau}$ is injective because $\ker(\widetilde{\tau}) = \{0\}$.

Now if

$$\sum_{i,j} \alpha_i \beta_j (x_i, y_j) \in \mathcal{F}_{X\times Y},$$

then we have

$$\widetilde{\tau}\left(\sum_{i,j} \alpha_i \beta_j (x_i \otimes y_j)\right) = \sum_{i,j} \alpha_i \beta_j (x_i, y_j),$$

which shows that $\widetilde{\tau}$ is surjective.

Thus we have shown that $\mathcal{F}_{X\times Y} \cong \mathcal{F}_X \otimes F_Y$, as desired. $\square$

Exercise (Exercise 2). *Let S and T be subspaces of V_1 and V_2, respectively. Then show that*

$$(S \otimes V_2) \cap (V_1 \otimes T) \cong S \otimes T.$$

Proof. Let us choose a subspace $T' \subset V_2$ such that $V_2 = T \oplus T'$. Then we have

$$\begin{aligned}(S \otimes V_2) \cap (V_1 \otimes T) &= (S \otimes (T + T')) \cap (V_1 \otimes T) \\ &= ((S \otimes T) + (S \otimes T')) \cap (V_1 \otimes T) \\ &= (S \otimes T) + \underbrace{((S \otimes T') \cap (V_1 \otimes T))}_{W} \qquad \text{(By associativity)}\end{aligned}$$

But now notice that

$$\begin{aligned}W = (S \otimes T') \cap (V_1 \otimes T) &\subseteq (V_1 \otimes T') \cap (V_1 \otimes T) && \text{(Since } S \subseteq V_1\text{)} \\ &= V_1 \times (T' \cap T) && \text{(By exercise 2 of HW \# 5)} \\ &= \{0\}.\end{aligned}$$

Hence we have

$$(S \otimes V_2) \cap (V_1 \otimes T) = (S \otimes T) + \{0\} = S \otimes T.$$

$\square$

Exercise (Exercise 3)**.** *If $T \in \mathcal{L}(V)$ is a linear transformation of a finite-dimensional vector space such that $T^m = 0$ for some $m \geq 1$, then there is a basis of V of the form*

$$u_1, Tu_1, \ldots, T^{a_1-1}u_1, \ldots, u_k, Tu_k, \ldots, T^{a_k-1}u_k$$

where $T^{a_i}u_i = 0$ for $1 \leq i \leq k$.

Proof. We work by induction on $\dim(V)$. For the inductive step we may assume that $\dim(V) \geq 1$. Clearly $T(V)$ is properly contained in V, since otherwise $T^m(V) = \cdots = T(V) = V$, a contradiction. Moreover, if T is the zero map then the result is trivial. We may therefore assume that $0 \subset T(V) \subset V$. By applying the inductive hypothesis to the map induced by T on $T(V)$, we may find $v_1, \ldots, v_l \in T(V)$ so that

$$v_1, Tv_1, \ldots, T^{b_1-1}v_1, \ldots, v_l, Tv_l, \ldots, T^{b_l-1}v_l$$

is a basis for $T(V)$ and $T^{b_i}v_i = 0$ for $1 \leq i \leq l$.

For $1 \leq i \leq l$, choose $u_i \in V$ such that $Tu_i = v_i$. Clearly $\ker(T)$ contains the linearly independent vectors $T^{b_1-1}v_1, \ldots, T^{b_l-1}v_l$. Now we extend these to a basis of $\ker(T)$ by adjoining the vectors $w_1, \ldots, w_m$. We claim that

$$u_1, Tu_1, \ldots, T^{b_1}u_1, \ldots, u_l, Tu_l, \ldots, T^{b_l}u_l, w_1, \ldots, w_m$$

is a basis for V. Linear independence may easily be checked by applying T to a given linear relation between the vectors. To show that they span V, we use dimension counting:

We know that $\dim(\ker(T)) = l + m$ and that $\dim(T(V)) = b_1 + \cdots + b_l$. Hence, by the rank-nullity theorem,

$$\dim(V) = (b_1 + 1) + \cdots + (b_l + 1) + m,$$

which is the number of vectors in our claimed basis. We have therefore constructed a basis for V in which T is in Jordan normal form. $\square$