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Homework Assignment 1
General Relativity I

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Problem 1 (Exercise 1.10 (Schutz)). *For the pairs of events whose coordinates (t, x, y, z) in some frame are given below, classify their separations as timelike, spacelike, or null:*

(a) $(0,0,0,0)$ *and* $(-1,1,0,0)$;

(b) $(1,1,-1,0)$ *and* $(-1,1,0,2)$;

(c) $(6,0,1,0)$ *and* $(5,0,1,0)$;

(d) $(-1,1,-1,1)$ *and* $(4,1,-1,6)$.

Solution to (a). The separation between two events is given by the line element ds^2, in this case given by

$$\begin{aligned} ds^2 &= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= -(-1-0)^2 + (1-0)^2 + (0-0)^2 + (0-0)^2 \\ &= -1 + 1 = 0. \end{aligned}$$

Since $ds^2 = 0$ in this case, we conclude that the events are lightlike- (or null-)separated. □

Solution to (b). Proceeding as above,

$$\begin{aligned} ds^2 &= -(-1-1)^2 + (1-1)^2 + (0-(-1))^2 + (2-0)^2 \\ &= -4 + 1 + 4 = 1. \end{aligned}$$

Since $ds^2 = 1 > 0$, we conclude that these events are spacelike-separated. □

Solution to (c). Similarly,

$$\begin{aligned} ds^2 &= -(5-6)^2 + (0-0)^2 + (1-1))^2 + (0-0)^2 \\ &= -1. \end{aligned}$$

Since $ds^2 = -1 < 0$, we conclude that these events are timelike-separated. □

Solution to (d). Lastly,

$$\begin{aligned} ds^2 &= -(4-(-1))^2 + (1-1)^2 + (-1-(-1))^2 + (6-1)^2 \\ &= -25 + 25 = 0. \end{aligned}$$

Since $ds^2 = 0$, we conclude that these events are null-separated. □

Problem 2 (Exercise 1.17 (Schutz)). ***TODO*** *(Similar to barn problem from recitation)*

Solution. □

Problem 3 (Exercise 1.19 (Schutz)). (a) *Using the* velocity parameter u, *given by*

$$v = \tanh u, \tag{1}$$

show that the Lorentz transformation equations,

$$\boxed{\begin{aligned} \bar{t} &= \gamma(t - vx) && (2a) \\ \bar{x} &= \gamma(x - vt) && (2b) \\ \bar{y} &= y && (2c) \\ \bar{z} &= z && (2d) \end{aligned}}$$

(where $\gamma = (1-v^2)^{-1/2}$ is the Lorentz factor), can be put in the form

$$\boxed{\begin{aligned} \bar{t} &= t\cosh u - x\sinh u && (3a) \\ \bar{x} &= x\cosh u - t\sinh u && (3b) \\ \bar{y} &= y && (3c) \\ \bar{z} &= z && (3d) \end{aligned}}$$

(b) *Use the identity $\cosh^2 u - \sinh^2 u = 1$ to demonstrate the invariance of the interval from these equations.*

(c) *Draw as many parallels as you can between the geometry of spacetime and ordinary two-dimensional Euclidean geometry, where the coordinate transformation analogous to the Lorentz transformation is*

$$\bar{x} = x\cos\theta + y\sin\theta \tag{4a}$$
$$\bar{y} = y\cos\theta - x\sin\theta. \tag{4b}$$

What is the analog of the interval? Of the invariant hyperbolae?

Solution to (a). We plug (1) into γ:

$$\begin{aligned}\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\tanh^2 u}} &= \frac{1}{\sqrt{1-\frac{\sinh^2 u}{\cosh^2 u}}} \\ &= \frac{1}{\sqrt{\frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u}}} \times \frac{\cosh u}{\cosh u} \\ &= \cosh u,\end{aligned}$$

where on the last equality we used the hyperbolic identity $\cosh^2 u - \sinh^2 u = 1$. Now we can see that plugging $\gamma = \cosh u$ into (2a) we get (3a):

$$\bar{t} = \gamma(t - vx) = \cosh u(t - x\tanh u) = t\cosh u - x\sinh u. \qquad \surd$$

Similarly, inserting $\gamma = \cosh u$ into (2b) we get (3b):

$$\bar{x} = \gamma(x - vt) = \cosh u(x - t\tanh u) = x\cosh u - t\sinh u. \qquad \surd$$

□

Solution to (b). We now use Eqs. (3) to show the invariance of the line element:

$$\begin{aligned}d\bar{s}^2 &= -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \\ &= -d\left(t\cosh u - x\sinh u\right)^2 + d\left(x\cosh u - t\sinh u\right)^2 + dy^2 + dz^2 \\ &= -(\cosh u dt + t\sinh u \cancelto{0}{du} - \sinh u dx - x\cosh u \cancelto{0}{du})^2 \\ &\quad + (\cosh u dx + x\sinh u \cancelto{0}{du} - \sinh u dt - t\cosh u \cancelto{0}{du})^2 + dy^2 + dz^2 \\ &= -(\cosh u dt - \sinh u dx)^2 + (\cosh u dx - \sinh u dt)^2 + dy^2 + dz^2 \\ &= -\cosh^2 u dt^2 + 2\cosh u \sinh u \cancelto{0}{dxdt} - \sinh^2 u dx^2 + \cosh^2 u dx^2 \\ &\quad - 2\cosh u \sinh u \cancelto{0}{dxdt} + \sinh^2 u dt^2 + dy^2 + dz^2 \\ &= -(\cosh^2 u - \sinh^2 u)dt^2 + (\cosh^2 u - \sinh^2 u)dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= ds^2.\end{aligned}$$

Note a few things: on the third equality the differential form du vanishes because u is a constant (since v is a constant; i.e., there is no "acceleration" in *Special Relativity*). The cross-terms on the fifth equality also vanish, since $\mathcal{O}$ –the observer to which the "unbarred" measurements belong to– is stationary with respect to his own, inertial ("unbarred") frame. Lastly, on the second-to-last equality we used the hyperbolic identity $\cosh^2 u - \sinh^2 u = 1$. □

Solution to (c). In ordinary, two-dimensional Euclidean geometry, the coordinate transformations given by Eqs. (4) are responsible for rotations in the $x - y$ plane. If we write them in matrix form,

$$\Theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{5}$$

it is straightforward to check that these transformations satisfy

$$\Theta^T \Theta = \delta, \tag{6}$$

where Θ^T denotes the *transpose* of Θ, and δ is the identity (2×2 in this case) matrix. If we now write (6) as

$$\Theta^T \delta \Theta = \delta, \tag{7}$$

we have an expression that is suspiciously similar to the definition of the Lorentz transformation as a matrix $\mathbf{\Lambda}$ that satisfies [1]

$$\mathbf{\Lambda}^T \boldsymbol{\eta} \mathbf{\Lambda} = \boldsymbol{\eta}, \tag{8}$$

where $\boldsymbol{\eta} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric in Euclidean coordinates. Indeed, if we restrict our Lorentz transformations to spatial coordinates only, say on the $x - y$ plane, then we get

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0\cdots & \vdots \\ 0 & \Theta & 0 \\ \vdots & 0\cdots & 1 \end{pmatrix}. \tag{9}$$

Showing that (9), in fact, satisfies expression (8) is a straighforward calculation. Hence we have shown that Eqs. (4) can be thought of as "purely-spatial" Lorentz transformations. If, on the other hand, we have rotations between space and time coordinates (say, on the $t - x$ plane, then the Lorentz transformations take the form

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

which is expression (3) written in matrix notation. Again, showing that (10), satisfies expression (8) is a trivial exercise.

Let us now show the invariance of the $2D$ Euclidean spatial line element from Eqs. (4):

$$\begin{aligned} d\bar{s}^2 &= \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 \\ &= \mathrm{d}(x\cos\theta + y\sin\theta)^2 + \mathrm{d}(y\cos\theta - x\sin\theta)^2 \\ &= (\cos\theta \mathrm{d}x - x\sin\theta \cancelto{0}{\mathrm{d}\theta} + \sin\theta\mathrm{d}y + y\cos\theta\cancelto{0}{\mathrm{d}\theta})^2 + (\cos\theta\mathrm{d}y - y\sin\theta\cancelto{0}{\mathrm{d}\theta} - \sin\theta\mathrm{d}x - x\cos\theta\cancelto{0}{\mathrm{d}\theta})^2 \\ &= \cos^2\theta\mathrm{d}x^2 + 2\cos\theta\sin\theta\,\cancelto{0}{\mathrm{d}x\mathrm{d}y} + \sin^2\theta\mathrm{d}y^2 + \cos^2\theta\mathrm{d}y^2 - 2\cos\theta\sin\theta\,\cancelto{0}{\mathrm{d}y\mathrm{d}x} + \sin^2\theta\mathrm{d}x^2 \\ &= (\cos^2\theta + +\sin^2\theta)\mathrm{d}x^2 + (\sin^2\theta + \cos^2\theta)\mathrm{d}y^2 \\ &= \mathrm{d}x^2 + \mathrm{d}y^2 \\ &= ds^2. \end{aligned}$$

Here once again the cross-terms vanish because of the same argument we mentioned in the solution to part (b), and also on the third equality the differential forms $\mathrm{d}\theta$ vanish because the angle of rotation from one set of coordinates to another is set a priori by the observer.

As for the analog of the hyperbolae, of course, in Euclidean space we have the circle. Just as for the "unbarred" coordinates we have $x^2 + y^2 = 1$, for the "barred"coordinates we get

$$\begin{aligned} \bar{x}^2 + \bar{y}^2 &= (x\cos\theta + y\sin\theta)^2 + (y\cos\theta - x\sin\theta)^2 \\ &= x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta + y^2\cos^2\theta - 2xy\cos\theta\sin\theta + x^2\sin^2\theta \\ &= x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) \\ &= x^2 + y^2 = 1. \end{aligned}$$

□

Problem 4 (Exercise 1.21 (Schutz)). *For two events $\mathcal{A}$ and $\mathcal{B}$,*

(a) show that if the two events are timelike separated, there is a Lorentz frame in which they occur at the same point, i.e. at the same spatial coordinate values.

(b) Similarly, show that if the two events are spacelike separated, there is a Lorentz frame in which they are simultaneous.

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[1] Here we are using non-tensorial notation to stress the similarities between the rotation and Lorentz matrices. In the tensor notation we have developed in class, Eq. (8) takes the form

$$\Lambda_{\nu}{}^{\mu'}\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\mu} = \Lambda^{\nu'}{}_{\mu}\Lambda_{\nu}{}^{\mu'}\eta_{\mu'\nu'} = \eta_{\mu\nu}.$$

Solution to (a). $\mathcal{A}$ and $\mathcal{B}$ being timelike-separated means that one event is inside the lightcone of the other; say, WLOG, that $\mathcal{B}$ is in the future of $\mathcal{A}$, so that $\Delta t = t_{\mathcal{B}} - t_{\mathcal{A}}$ and $\Delta\vec{x} = \vec{x}_{\mathcal{B}} - \vec{x}_{\mathcal{A}}$. Here we use the convenient notation $\vec{x} = (x^1, x^2, x^3)$ [$= (x, y, z)$]. Then, since the separation is timelike,

$$ds^2 = -\Delta t + \Delta\vec{x} < 0 \quad \Longrightarrow \quad \Delta\vec{x} < \Delta t.$$

Now setting $\Delta\vec{x} = 0$ (same spatial coordinate values for both events) we have $\Delta t > 0$, which makes sense since, by construction, $\mathcal{B}$ is in the future of $\mathcal{A}$ and therefore $\Delta t = t_{\mathcal{B}} - t_{\mathcal{A}} > 0$. Thus we have shown that it is always possible to find a Lorentz frame where two timelike-separated events can have the same spatial coordinate values. □

Solution to (b). Similarly, if the separation is spacelike, we must have $\Delta\vec{x} > 0$. But for two spacelike-separated events,

$$ds^2 = -\Delta t + \Delta\vec{x} > 0 \quad \Longrightarrow \quad \Delta\vec{x} > \Delta t.$$

Now setting $\Delta t = 0$ (same time coordinate value for both events) we have $\Delta\vec{x} > 0$, which is the requirement we just mentioned above for two spacelike-separated events. Hence, we have shown that it is always possible to find a Lorentz frame where two spacelike-separated events can be seen as being simultaneous. □