MATH 710 HW # 1

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJIAN

Problem 1 (Problem 8-3)**.** *Let M be a nonempty positive-dimensional smooth manifold (with or without boundary). Show that $\mathfrak{X}(M)$ is infinite-dimensional.*

Proof. Let n be a positive integer and let $\{p_i\}_{i=1}^n$ be a set of distinct points in M. Let $\{U_i\}$ be a set of corresponding pairwise disjoint open neighborhoods, and for each i, let $v_i \in T_{p_i}M$ be nonzero. By Lemma 8.6,[1] there exist global smooth vector fields $\widetilde{X}_i$ on M such that $(\widetilde{X}_i)_{p_i} = v_i$ and $\operatorname{supp} \widetilde{X}_i \subseteq U_i$. Let $X = \sum_{i=1}^n a_i \widetilde{X}_i$ for some constants $a_i \in \mathbb{R}$. If $X = 0$, then $X_{p_i} = a_i(\widetilde{X}_i)_{p_i} = a_i v_i = 0$, and so $a_i = 0$ by construction. Hence we have that $\{\widetilde{X}_i\}_{i=1}^n$ is a linearly independent subset that spans $\mathfrak{X}(M)$. Since n was arbitrary, the result follows. □

Problem 2 (Problem 8-9)**.** *Show by finding a counterexample that Proposition 8.19*[2] *is false if we replace the assumption that F is a diffeomorphism by the weaker assumption that it is smooth and bijective.*

Solution. Let $M = [0, 1)$. Consider the smooth bijection $\varphi\colon M \to \mathbb{S}^1$ given by $s \mapsto e^{2\pi i s}$ and consider the smooth vector field $X\colon M \to TM$ given by $x \mapsto (1-2x)\, \mathrm{d}/\mathrm{d}t|_x$. Note that there is no way of defining a φ-related smooth vector field Y on $\mathbb{S}^1$, since the sign of Y_1 (that is, the direction of the vector field Y at the point $1 \in \mathbb{S}^1$) is ambiguous. □

Problem 3 (Problem 8-10)**.** *Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive, and let $F\colon M \to M$ be the map $F(x, y) = (xy, y/x)$. Show that F is a diffeomorphism, and compute F_*X and F_*Y, where*

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad \text{and} \qquad Y = y\frac{\partial}{\partial x}.$$

Proof. Clearly, F is a diffeomorphism on $M = \{(x, y) \mid x, y > 0\}$ since it is smooth and the inverse

$$F^{-1}(u, v) = \left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right)$$

is also smooth. Now to compute the pushforwards, we first find the differential

$$DF(x, y) = \begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix}.$$

[1] Here's Lemma 8.6, for reference:
(Extension Lemma for Vector Fields) Let M be a smooth manifold (with or without boundary), and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\widetilde{X}$ on M such that $\widetilde{X}|_A = X$ and $\operatorname{supp} \widetilde{X} \subseteq U$. (See proof of this lemma on HW set # 2.)

[2] Here's Proposition 8.19, for reference:
Suppose M and N are smooth manifolds (with or without boundary), and $F\colon M \to N$ is a diffeomorphism. Then for every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F-related to X.

so that

$$DF(F^{-1}(u,v)) = \begin{pmatrix} \sqrt{uv} & \sqrt{u/v} \\ -v\sqrt{v/u} & \sqrt{v/u} \end{pmatrix}.$$

Therefore the coordinates of F_*X are given by

$$\begin{pmatrix} \sqrt{uv} & \sqrt{u/v} \\ -v\sqrt{v/u} & \sqrt{v/u} \end{pmatrix} \begin{pmatrix} \sqrt{u/v} \\ \sqrt{uv} \end{pmatrix} = \begin{pmatrix} 2u \\ 0 \end{pmatrix},$$

while the coordinates of F_*Y are given by

$$\begin{pmatrix} \sqrt{uv} & \sqrt{u/v} \\ -v\sqrt{v/u} & \sqrt{v/u} \end{pmatrix} \begin{pmatrix} \sqrt{uv} \\ 0 \end{pmatrix} = \begin{pmatrix} uv \\ -v^2 \end{pmatrix}.$$

Thus we have

$$F_*X = 2u\frac{\partial}{\partial u} \qquad \text{and} \qquad F_*Y = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}.$$

□

Problem 4 (Problem 8-11)**.** *For each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half-plane* $\{(x,y) \mid x > 0\}$.

a) $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

b) $Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

c) $Z = (x^2+y^2)\frac{\partial}{\partial x}$.

Solution. Applying the standard cartesian-polar relations ($(x,y) \leftrightarrow (r\cos\theta, y\sin\theta)$) on the right half plane we have

$$\begin{aligned}
\frac{\partial\theta}{\partial x} &= \frac{\partial}{\partial x}\left(\arctan\frac{y}{x}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r};\\
\frac{\partial\theta}{\partial y} &= \frac{\partial}{\partial y}\left(\arctan\frac{y}{x}\right) = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r};\\
\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right) = \frac{x}{r} = \cos\theta;\\
\frac{\partial r}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) = \frac{y}{r} = \sin\theta.
\end{aligned}$$

Combining these results we get

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\\
&= -\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\\
&= \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r}.
\end{aligned}$$

Thus, substituting we have

$$
\begin{aligned}
X &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
&= r\cos\theta\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r}\right) + r\sin\theta\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r}\right) \\
&= r\frac{\partial}{\partial r}(\cos^2\theta + \sin^2\theta) \\
&= \boxed{r\tfrac{\partial}{\partial r}}. \\
Y &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \\
&= r\cos\theta\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r}\right) - r\sin\theta\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r}\right) \\
&= \boxed{r\cos 2\theta\tfrac{\partial}{\partial r} - \sin 2\theta\tfrac{\partial}{\partial\theta}}. \\
Z &= (x^2+y^2)\frac{\partial}{\partial x} \\
&= r^2\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r}\right) \\
&= \boxed{r^2\cos\theta\tfrac{\partial}{\partial r} - r\sin\theta\tfrac{\partial}{\partial\theta}}.
\end{aligned}
$$

□