MATH 709 HW # 3

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJAN

Problem 1 (Problem 3-1)**.** *Suppose M and N are smooth manifolds (with or without boundary), and $F\colon M \to N$ is a smooth map. Show that $dF_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ iff F is constant on each component of M.*

Proof. ($\Rightarrow$) Assume that $dF_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. Choose smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$. Now let $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ and $\widehat{p} = \varphi(p)$ denote the coordinate representation of F and p, respectively. Then, choosing any basis vector $\partial/\partial x^i|_p \in T_pM$, and using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we have

$$\begin{aligned} 0 = dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dF_p\left(d(\varphi^{-1})_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right) \\ &= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(d\widehat{F}_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right) \\ &= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\widehat{p})}\right) \\ &= \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \end{aligned}$$

Since $\{\partial/\partial y^j|_{F(p)}\}$ is a basis of $T_{F(p)}N$, we may conclude that all the coefficients $\partial\widehat{F}^j/\partial x^i(\widehat{p})$ are zero for all i and j, and so $\widehat{F}$ is constant on $\varphi\left(U \cap F^{-1}(V)\right)$, and consequently F is constant on $U \cap F^{-1}(V)$. Since this construction can be performed for any $p \in M$, we can conclude that F is constant on each component of M.

($\Leftarrow$) Conversely, assume that F is constant on each component of M and let $p \in M$. Then, clearly $f \circ F$ is also constant on a neighborhood of p (otherwise it wouldn't be a well-defined function), and for all $\nu \in T_pM$, we have $dF_p(\nu)(f) = \nu(f \circ F) = 0$. $\square$

Problem 2 (Problem 3-8)**.** *Let M be a smooth manifold (with or without boundary) and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi\colon \mathcal{V}_pM \to T_pM$ defined by $\Psi([\gamma]) = \gamma'(0)$ is well defined and bijective. (For $p \in \partial M$, we need to allow curves with domain $[0, \varepsilon)$ or $(-\varepsilon, 0]$ and to interpret the derivatives as one-sided derivatives.)*

Proof. If $\gamma_1, \gamma_2 \in [\gamma]$, then

$$\Psi([\gamma_1]) = \gamma_1'(0)(f) = (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) = \gamma_2'(0)(f) = \Psi([\gamma_2]),$$

which shows that Ψ is well-defined. Similarly, if $\Psi([\gamma_1]) = \Psi([\gamma_2])$, we must have

$$\gamma_1'(0)(f) = \gamma_2'(0)(f) \implies (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) \implies \gamma_1 \sim \gamma_2 \implies \gamma_1, \gamma_2 \in [\gamma],$$

which shows that Ψ is injective. Finally, by a previous proposition we know that if M is a smooth manifold (with or without boundary) and $p \in M$, then every $\nu \in T_pM$ is the velocity of some smooth curve in M. This is equivalent to saying that Ψ is surjective. $\square$