Math 746 Notes

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Measure Theory

Definition: Let X be equipped with a σ-algebra $\mathcal{A}$ (see def. of a σ-algebra below). A measure on $\mathcal{A}$ (or on $(X, \mathcal{A})$, or simply on X if $\mathcal{A}$ is understood) is a function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ such that:

(i) $\mu(\emptyset) = 0$

(ii) If $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{A}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$.

Property (ii) is called countable additivity. It implies finite additivity:
(iii) If $E_1, \ldots, E_n$ are disjoint sets in $\mathcal{A}$, then $\mu(\bigcup_{j=1}^{n} E_j) = \sum_{j=1}^{n} \mu(E_j)$ (because one can take $E_j = \emptyset$ for $j > n$).

A function μ that satisfies (i) and (iii) but not necessarily (ii) is called a finitely additive measure, but it's not a measure since it doesn't satisfy all three properties.

EXTERIOR MEASURE

Definition: If E is any subset of $\mathbb{R}^d$, the exterior measure (also known as outer measure) of E, denoted $m_*(E)$, is given by

$$m_*(E) = \inf \sum_{n=1}^{\infty} |Q_n| \,,$$

where the infimum is taken over all countable coverings $E \subset \bigcup_{n=1}^{\infty} Q_n$ by closed cubes. The exterior measure is always non-negative but could be infinite, so that in general we have $0 \leq m_*(E) \leq \infty$, and therefore takes values in the extended positive numbers.

Remark: The following is immediate from the definition of m_*:
For every $\varepsilon > 0$, there exists a covering $E \subset \bigcup_{n=1}^{\infty} Q_n$ with $\sum_{n=1}^{\infty} m_*(Q_n) \leq m_*(E) + \varepsilon$.

The relevant properties of exterior measure are now listed in a series of observations:

- Observation 1 (Monotonicity):

If $E_1 \subset E_2$, then $m_*(E_1) \le m_*(E_2)$.

- Observation 2 (Countable sub-additivity):

If $E = \bigcup_{n=1}^{\infty} E_n$, then $m_*(E) \le \sum_{n=1}^{\infty} m_*(E_n)$.

- Observation 3:

If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ containing E.

- Observation 4:

If $E = E_1 \bigcup E_2$ and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.

- Observation 5:

If a set E is the countable union of almost disjoint cubes $E = \bigcup_{n=1}^{\infty} Q_n$, then $m_*(E) = \sum_{n=1}^{\infty} |Q_n|$.

• Carathéodory's Theorem:

If m_* is an outer measure on X, the collection $\mathcal{A}$ of m_*-measurable sets is a σ-algebra, and the restriction of m_* to $\mathcal{A}$ is a complete measure.

Proof:

(See page 29, Folland's). ■

MEASURABLE SETS & LEBESGUE MEASURE

Definition: A subset $E \subset \mathbb{R}$ is said to have measure zero if for every $\varepsilon > 0$, there exists a countable family of open intervals $\{I_k\}_{k=1}^{\infty}$ such that

(i) $E \subset \bigcup_{k=1}^{\infty} I_k$

(ii) $\sum_{k=1}^{\infty} |I_k| < \varepsilon$, where $|I_k|$ denotes the length of the interval I_k.

The first condition says that the union of the intervals covers E, and the second that this union is arbitrarily small. It follows from this that any finite set of points has measure 0. As a matter of fact, it is true that a countable set of points has measure 0, even though the proof of this argument requires a more subtle treatment.

This last result is contained in the following lemma:

• Lemma: The union of countably many sets of measure zero has measure zero.

Definition: A subset E of $\mathbb{R}^d$ is said to be Lebesgue measurable (or simply measurable), if for any $\varepsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ that satisfies $m_*(\mathcal{O} \backslash E) \leq \varepsilon$.

‣ Alternatively, we can say that a set E is measurable if $\forall\, \varepsilon > 0$, there exists a closed set F, such that $F \subset E$ and $m_*(E \backslash F) \leq \varepsilon$.

‣ Yet another way to define a measurable set (this is actually a more widely used definition):
A set E is measurable if $\forall\, A \subset \mathbb{R}^d$, we have

$$m_*(A) = m_*(E \cap A) + m_*(E^c \cap A)\ .$$

Definition: If E is measurable, we define its Lebesgue measure (or simply measure) by $m(E) = m_*(E)$. That is, if E is measurable, then its measure is the same as its outer measure.

Clearly, the Lebesgue measure inherits all the features contained in Observations 1-5 of the exterior measure. Immediately from the defnition we find the following six properties:

‣ Property 1:
Every open set in $\mathbb{R}^d$ is measurable.

‣ Property 2:
If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.

‣ Property 3:
A countable union of measurable sets is measurable.

‣ Property 4:
Closed sets are measurable.

‣ Property 5:
The complement of a measurable set is measurable.

‣ Property 6:
A countable intersection of measurable sets is measurable.

• Theorem:
If $E_1,\ E_2,\ \ldots$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then $m(E) = \sum_{j=1}^{\infty} m(E_j)$.

Proof:

First, we assume further that each E_j is bounded. Then, for each j, by applying the definition of measurability to E_j^c, we can choose a closed subset $F_j \subset E_j$ with $m_*(E_j \setminus F_j) \leq \varepsilon / 2^j$. For each fixed N, the sets $F_1, \ldots, F_N$ are compact and disjoint, so that $m\left(\bigcup_{j=1}^N F_j\right) = \sum_{j=1}^N m(F_j)$.
Since $\bigcup_{j=1}^N F_j \subset E$, we must have

$$m(E) \geq \sum_{j=1}^{N} m(F_j) \geq \sum_{j=1}^{N} m(E_j) - \varepsilon .$$

Letting N tend to infinity, since ε was arbitrary, we find that

$$m(E) \geq \sum_{j=1}^{\infty} m(E_j) .$$

Since the reverse inequality always holds (by sub-additivity in Observation 2), this concludes the proof when each E_j is bounded.

In the general case, we select any sequence of cubes $\{Q_k\}_{k=1}^\infty$ that increases to $\mathbb{R}^d$, in the sense that $Q_k \subset Q_{k+1}$ for all $k \geq 1$ and $\bigcup_{k=1}^\infty Q_k = \mathbb{R}^d$. We then let $S_1 = Q_1$ and $S_k = Q_k - Q_{k-1}$ for $k \geq 2$. If we define measurable sets by $E_{j,k} = E_j \bigcap S_k$, then

$$E = \bigcup_{j,k} E_{j,k} .$$

This union is disjoint and every $E_{j,k}$ is bounded. Moreover $E_j = \bigcup_{k=1}^\infty E_{j,k}$ and this union is also disjoint. Putting these facts together, and using what has already been proved, we obtain

$$m(E) = \sum_{j,k} m(E_{j,k}) = \sum_{j} \sum_{k} m(E_{j,k}) = \sum_{j} m(E_j) ,$$

as claimed. ■

Remark: With this theorem, the countable additivity of the Lebesgue measure on measurable sets has been established. This result provides the necessary connection between the following:
- our primitive notion of volume given by the exterior measure
- the more refined idea of measurable sets
- the countably infinite operations allowed on these sets.

Definition: The symmetric difference between two sets E and F (denoted $E \triangle F$) consists of those points that belong to only one of the two sets E or F and it's defined by $E \triangle F = (E \setminus F) \bigcup (F \setminus E)$.

• Theorem:
Suppose E is a measurable subset of $\mathbb{R}^d$. Then, for every $\varepsilon > 0$:
(i) There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} \setminus E) \leq \varepsilon$.
(ii) There exists a closed set F with $F \subset E$ and $m(E \setminus F) \leq \varepsilon$.
(iii) If $m(E)$ is finite, there exists a compact set K with $K \subset E$ and $m(E \setminus K) \leq \varepsilon$.
(iv) If $m(E)$ is finite, there exists a finite union $F = \bigcup_{n=1}^{N} Q_n$ of closed cubes such that $m(E \triangle F) \leq \varepsilon$.

Some invariance properties of Lebesgue measure:

‣ A crucial property of Lebesgue measure in $\mathbb{R}^d$ is its translation-invariance, which can be stated as follows:
If E is a measurable set and $h \in \mathbb{R}^d$, then the set $E_h = E + h = \{x + h : x \in E\}$ is also measurable, and $m(E + h) = m(E)$.

‣ Similarly, we have the relative dilation-invariance of Lebesgue measure:
Suppose $\delta > 0$, and denote by δE the set $\{\delta x : x \in E\}$. We can then assert that δE is measurable whenever E is, and $m(\delta E) = \delta^d m(E)$ (where d is the dimension of $\mathbb{R}^d \supset E$).

‣ One can also easily see that the Lebesgue measure is reflection-invariant. That is, whenever E is measurable, so is $-E = \{-x : x \in E\}$ and $m(-E) = m(E)$.

ALGEBRAS, SIGMA ALGEBRAS, BOREL SETS

Definition: A nonempty system of sets $\mathfrak{R}$ is called a ring of sets if $A \triangle B \in \mathfrak{R}$ and $A \cap B \in \mathfrak{R}$ whenever $A, B \in \mathfrak{R}$. Since

$$A \cup B = (A \triangle B) \triangle (A \cap B) \qquad \text{and} \qquad A \setminus B = A \triangle (A \cap B),$$

we also have $A \cup B \in \mathfrak{R}$ and $A \setminus B \in \mathfrak{R}$ whenever $A, B \in \mathfrak{R}$.
Thus a ring of sets is a system closed under the operations of taking unions, intersections, differences, and symmetric differences. Clearly, a ring of sets is also closed under the operations of taking finite unions and intersections:

$$\bigcup_{k=1}^{n} A_k \qquad \text{and} \qquad \bigcap_{k=1}^{n} A_k.$$

Remark: A ring of sets must contain the empty set $\emptyset$, since $A \setminus A = \emptyset$.

Definition: A set E is called the unit (this is analogous to the unity of rings studied in abstract algebra) of a system of sets $\mathcal{S}$ if $E \in \mathcal{S}$ and

$$A \bigcap E = A$$

for every set $A \in \mathcal{S}$.
Clearly E is unique. Thus the unit of $\mathcal{S}$ is just the maximal set of $\mathcal{S}$, i.e. the set containing all other sets of $\mathcal{S}$.

Definition: A ring of sets with a unit is called an algebra of sets.

Definition: A ring of sets is called a σ-ring if it contains the union $\bigcup_{k=1}^{\infty} A_k$ whenever it contains the sets $A_1, A_2, \ldots, A_k, \ldots$. Furthermore, a σ-ring with a unit is called a σ-algebra.

In other words:

Definition: Let X be a set. An algebra is a collection $\mathcal{A}$ of subsets of X such that:
(i) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.
(ii) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.
(iii) if $A_1, \ldots, A_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$ are in $\mathcal{A}$.

We say that the algebra $\mathcal{A}$ is a σ-algebra if in addition:
(iv) whenever $A_1, A_2, \ldots \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are in $\mathcal{A}$ (both the union and intersection must be countable).

In other words, a σ-algebra is an algebra of sets, completed to include countably infinite operations:

Definition: A σ-algebra of sets is a collection of subsets that is closed under countable unions, countable intersections, and complements.

Definition: The pair $(X, \mathcal{A})$ is called a measurable space and the sets in $\mathcal{A}$ are called measurable sets, i.e. a set A is measurable (or $\mathcal{A}$-measurable) if $A \in \mathcal{A}$. If μ is a measure on $(X, \mathcal{A})$, then $(X, \mathcal{A}, \mu)$ is called a measure space.

Definition: A measure space X is said to be σ-finite if X can be written as the union of countably many measurable sets of finite measure.

Example:
- If X is any set, $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are σ-algebras.
- If X is uncountable, then

$$\mathcal{A} = \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\}$$

is a σ-algebra, called the σ-algebra of countable or co-countable sets. (The point here is that if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$, then $\bigcup_{j=1}^{\infty} E_j$ is countable if all E_j are countable and is co-countable otherwise.)

The collection of all subsets of $\mathbb{R}^d$ is of course a σ-algebra. A more interesting and relevant example consists of all measurable sets in $\mathbb{R}^d$, which also forms a σ-algebra. Another σ-algebra, which plays a vital role in analysis, is the Borel σ-algebra in $\mathbb{R}^d$, denoted by $\mathcal{B}_{\mathbb{R}^d}$, which by definition is the <u>smallest</u> σ-algebra that contains all open sets. Elements of this σ-algebra are called Borel sets.
Here the term "smallest" means that if S is any σ-algebra that contains all open sets in $\mathbb{R}^d$, then necessarily $\mathcal{B}_{\mathbb{R}^d} \subset S$. Since we observe that any intersection (not necessarily countable) of σ-algebras is again a σ-algebra, <u>we may define $\mathcal{B}_{\mathbb{R}^d}$ as the intersection of all σ-algebras that contain the open sets</u>. This shows the existence and uniqueness of the Borel σ-algebra.

More generally:

<u>Definition:</u> If X is any topological space, the σ-algebra generated by the family of open sets in X (or equivalently, by the closed sets in X) is called the Borel σ-algebra, denoted $\mathcal{B}_X$.

Let's try to list the Borel sets in order of their complexity:
- We start with the open and closed sets, which are the simplest Borel sets.
- Next in order would come countable intersections of open sets; such sets are called G_δ sets. Alternatively, one could consider their complements, the countable union of closed sets, called the F_σ sets.
- We can then consider a countable union of G_δ sets (called a $G_{\delta\sigma}$ set); a countable intersection of F_σ sets (called an $F_{\sigma\delta}$ set); and so forth...

Since open sets and closed sets are measurable, we conclude that the Borel σ-algebra is contained in the σ-algebra of measurable sets. Naturally, we may ask if this inclusion is strict: do there exist Lebesgue measurable sets which are not Borel sets? The answer is "yes". To see why we need to define what a complete measure is:

<u>Definition:</u> A complete measure is a measure whose domain includes all subsets of null sets.

In other words, from the point of view of the Borel sets, the Lebesgue sets arise as the completion of the σ-algebra of Borel sets, that is, by adjoining all subsets of Borel sets of measure zero. This is an immediate consequence of the corollary below.

• <u>Corollary:</u>
A subset E of $\mathbb{R}^d$ is measurable
(i) iff E differs from a G_δ by a set of measure zero,
(ii) iff E differs from an F_σ by a set of measure zero.

Now we present the theorem that justifies our discussion of complete measures:

• Theorem:
Suppose $(X, \mathcal{A}, \mu)$ is a measure space.
Let

$$\mathfrak{N} = \{N \in \mathcal{A} : \mu(N) = 0\} \qquad \text{and} \qquad \overline{\mathcal{A}} = \{E \bigcup F : E \in \mathcal{A} \text{ and } F \subset N \text{ for some } N \in \mathfrak{N}\}.$$

Then $\overline{\mathcal{A}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{A}}$.

Proof:
(See page 27, Folland's). ■

Now it is clear what was meant by "completion" before; the measure $\overline{\mu}$ is the completion of μ, and $\overline{\mathcal{A}}$ is the completion of $\mathcal{A}$ with respect to μ.

Definition: Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha \in A} X_\alpha$, and $\pi_\alpha : X \longrightarrow X_\alpha$ the coordinate maps. If M_α is a σ-algebra on X_α for each α, then the product σ-algebra on X is the σ-algebra generated by

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in M_\alpha,\ \alpha \in A\}.$$

We denote this σ-algebra by $\oplus_{\alpha \in A} M_\alpha$. (If $A = \{1, \ \ldots, \ n\}$ we also write $\oplus_{k=1}^{n} M_k$ or $M_1 \oplus \ldots \oplus M_n$.)

• Proposition:
If A is countable, then $\oplus_{\alpha \in A} M_\alpha$ is the σ-algebra generated by $\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in M_\alpha\}$.

• Proposition:
Suppose that M_α is generated by $\mathcal{E}_\alpha$, $\alpha \in A$. Then $\oplus_{\alpha \in A} M_\alpha$ is generated by $\mathcal{F}_1 = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha,\ \alpha \in A\}$. If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all α, then $\oplus_{\alpha \in A} M_\alpha$is generated by $\mathcal{F}_2 = \{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$.

• Proposition:
Let $X_1, \ \ldots, \ X_n$ be metric spaces and let $X = \prod_{k=1}^{n} X_k$, equipped with the product metric. Then $\oplus_{k=1}^{n} \mathcal{B}_{X_k} \subset \mathcal{B}_X$. If the X_k's are separable, then $\oplus_{k=1}^{n} \mathcal{B}_{X_k} = \mathcal{B}_X$.

• Corollary:
$\mathcal{B}_{\mathbb{R}^d} = \oplus_{k=1}^{n} \mathcal{B}_{\mathbb{R}}$.

CONSTRUCTION OF A NON – MEASURABLE SET

The construction of a non-measurable (Vitali) set $\mathcal{N}$ uses the axiom of choice, and rests on a simple equivalence relation among real numbers in $[0, 1]$:

$$\text{We write } x \sim y \text{ whenever } x - y \in \mathbb{Q}\,.$$

Note that this is an equivalence relation since it satisfies the reflexive, symmetric, and transitive properties.

Since equivalence classes partition a set into distinct cells, we know that two equivalence classes either are disjoint or coincide; thus the interval $[0, 1]$ is the disjoint union of all equivalence classes that live in this interval, that is

$$[0, 1] = \bigcup_{\alpha} \mathcal{E}_\alpha\,,$$

where each $\mathcal{E}_\alpha$ represents a unique equivalence class.

Now we construct the (Vitali) set $\mathcal{N}$ by choosing exactly one element x_α from each $\mathcal{E}_\alpha$ (this is justified by using the axiom of choice), and setting $\mathcal{N} = \{x_\alpha\}$.

Here's the important result:

• Theorem:
The Vitali set $\mathcal{N}$ constructed above is not measurable.

Proof:
Assume that $\mathcal{N}$ is measurable. Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of all the rationals in $[-1, 1]$, and consider the translates

$$\mathcal{N}_k = \mathcal{N} + r_k.$$

Note that the sets $\mathcal{N}_k$ are disjoint. To see why this is true, suppose that the intersection $\mathcal{N}_k \bigcap \mathcal{N}_{k'}$ is nonempty. Then there exist rationals $r_k \neq r_{k'}$ and α and β with

$$x_\alpha + r_k = x_\beta + r_{k'}$$

which implies that

$$x_\alpha - x_\beta = r_{k'} - r_k\ .$$

But this means that $\alpha \neq \beta$ and $x_\alpha - x_\beta$ is rational, which in turn implies that $x_\alpha \sim x_\beta$. This contradicts the fact that $\mathcal{N}$ contains only one representative of each equivalence class.

We also claim that

$$\text{(I)} \qquad [0,\ 1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \subset [-1,\ 2].$$

To see why, notice that if $x \in [0,\ 1]$, then $x \sim x_\alpha$ for some α, and therefore $x - x_\alpha = r_k$ for some k. Hence $x \in \mathcal{N}_k$ for some k and the first inclusion holds. The second inclusion above is straightforward since each $\mathcal{N}_k$ is contained in $[-1,\ 2]$ by construction.

Now we may conclude the proof of the theorem. If $\mathcal{N}$ were measurable, then so would be $\mathcal{N}_k$ for all k, and since the union $\bigcup_{k=1}^{\infty} \mathcal{N}_k$ is disjoint, the inclusions in (I) yield

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq 3.$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we must have $m(\mathcal{N}_k) = m(\mathcal{N})$ for all k.
Consequently,

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3.$$

This is the desired contradiction, since neither $m(\mathcal{N}) = 0$ nor $m(\mathcal{N}) > 0$ is possible. $(\Rightarrow\Leftarrow)$

In other words, $m(\mathcal{N}) = 0$ is not possible by the above inequality, and $m(\mathcal{N}) > 0$ is not possible either because we are trying to find the measure of a countable set, which would have measure zero if any.
■

MEASURABLE FUNCTIONS

Our starting point is the notion of a characteristic function of a set E, which is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}.$$

The next step is to pass to the functions that are the building blocks of integration theory. For the Riemann integral it is in effect the class of step functions, with each given as a finite sum

$$f = \sum_{k=1}^{\mathcal{N}} a_k\, \chi_{R_k}\ ;$$

where each R_k is a rectangle, and the a_k are constants.
However, for the Lebesgue integral we need a more general notion, as we shall see later on.

A simple function is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{E_k} \;;$$

where each E_k is a measurable set of finite measure, and the a_k are constants.

Definition: A function f defined on a measurable subset E of $\mathbb{R}^d$ is said to be a measurable function, if for all $a \in \mathbb{R}$, the set

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$$

is measurable. To simplify our notation, we shall often denote the set $\{x \in E : f(x) < a\}$ simply by $\{f < a\}$ whenever no confusion is possible.

In the same way, one can show that if f is finite-valued, then it is measurable iff the sets $\{a < f < b\}$ are measurable for every $a, b \in \mathbb{R}$. Similar conclusions hold for whichever combination of strict or weak inequalities one chooses. For example, if f is finite-valued, then it is measurable iff $\{a \leq f \leq b\}$ is measurable for all $a, b \in \mathbb{R}$. By the same arguments one sees the following:

‣ Property 1:
The finite-valued function f is measurable iff $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$, and iff $f^{-1}(F)$ is measurable for every closed set F.

Remark: Note that this property also applies to extended-valued functions, if we make the additional hypothesis that both $f^{-1}(-\infty)$ and $f^{-1}(\infty)$ are measurable sets.

‣ Property 2:
If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and Φ is continuous, then $\Phi \circ f$ is measurable.

Remark: In fact, Φ is continuous, so $\Phi^{-1}(-\infty, a)$ is an open set $\mathcal{O}$, and hence $(\Phi \circ f)^{-1}((-\infty, a)) = f^{-1}(\mathcal{O})$ is measurable. It should be noted, however, that in general it is not true that $f \circ \Phi$ is measurable whenever f is measurable and Φ is continuous (see Exercise 35,Chpt 2, Stein).

‣ Property 3:
Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions.
Then

$$\sup_n f_n(x) \;, \qquad \inf_n f_n(x) \;, \qquad \limsup_{n\to\infty} f_n(x) \;, \qquad \text{and} \qquad \liminf_{n\to\infty} f_n(x)$$

are measurable.

Remark: Proving that $\sup_n f_n$ is measurable requires noting that $\{\sup_n f_n > a\} = \bigcup_n \{f_n > a\}$. This also yields the result for $\inf_n f_n$, since this quantity equals $-\sup_n(-f_n(x))$.
The result for the limsup and liminf also follows from the two observations

$$\limsup_{n\to\infty} f_n = \inf_k \left\{\sup_{n\geq k} f_n\right\} \quad \text{and} \quad \liminf_{n\to\infty} f_n = \sup_k \left\{\inf_{n\geq k} f_n\right\}.$$

Property 4:

If $\{f_n\}_{n=1}^{\infty}$ is a collection of measurable functions, and $\lim_{n\to\infty} f_n(x) = f(x)$, then f is measurable.

Remark: Since $f(x) = \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$, this property is a consequence of property 3.

Property 5:

If f and g are measurable, then
(i) The integer powers f^k, for $k \geq 1$ are measurable.
(ii) $f+g$ and $f\,g$ are measurable if both f and g are finite-valued.

Remark: For (i) we simply note that if k is odd, then $\{f^k > a\} = \{f > a^{1/k}\}$, and if k is even and $a \geq 0$, then $\{f^k > a\} = \{f > a^{1/k}\} \bigcup \{f > -a^{1/k}\}$.
For (ii), we first see that $f+g$ is measurable because

$$\{f+g > a\} = \bigcup_{r\in\mathbb{Q}} \{f > a-r\} \bigcap \{g > r\}\,.$$

Finally, $f\,g$ is measurable because of the previous results and the fact that

$$f\,g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

Definition: We shall say that two functions f and g defined on a set E are equal almost everywhere, and write

$$f(x) = g(x) \quad \text{a.e.} \;\; x \in E$$

if the set $\{x \in E : f(x) \neq g(x)\}$ has measure zero (we sometimes abbreviate this by saying that $f = g$ a.e.). More generally, a property or statement is said to hold almost everywhere (a.e.) if it is true except on a set of measure zero.

One sees easily that if f is measurable and $f = g$ a.e., then g is measurable. This follows at once from the fact that $\{f < a\}$ and $\{g < a\}$ differ by a set of measure zero. Moreover, all the properties stated above this definition can be relaxed to conditions holding almost everywhere. For instance, if $\{f_n\}_{n=1}^{\infty}$is a collection of measurable functions, and

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e.}\,,$$

then f is measurable.
This observation gives us a sixth property of measurable functions:

▸ Property 6: Suppose f is measurable, and $f(x) = g(x)$ for a.e. x. Then g is also measurable.

APPROXIMATION BY SIMPLE OR STEP FUNCTIONS

The theorems in this section are all of the same nature and provide further insight in the structure of measurable functions. We begin by approximating pointwise, non-negative measurable functions by simple functions:

• Theorem:
Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that converges pointwise to f, namely,

$$\varphi_k(x) \le \varphi_{k+1}(x) \qquad \text{and} \qquad \lim_{k\to\infty} \varphi_k(x) = f(x)$$

for all x.

Proof:
We begin first with a truncation. For $N \ge 1$, let Q_N denote the cube centered at the origin and of side length N. Then we define

$$F_N(x) = \begin{cases} f(x) & \text{if } x \in Q_N \text{ and } f(x) \le N \\ N & \text{if } x \in Q_N \text{ and } f(x) > N \\ 0 & \text{otherwise} \end{cases}.$$

Then, $F_N(x) \to f(x)$ as N tends to infinity for all x. Now, we partition the range of F_N, namely $[0, N]$, as follows. For fixed N, $M \ge 1$, we define

$$E_{\ell,M} = \left\{x \in Q_N : \frac{\ell}{M} < F_N(x) \le \frac{\ell+1}{M}\right\}, \quad \text{for } 0 \le \ell < NM.$$

Then we may form

$$F_{N,M}(x) = \sum_{\ell} \frac{\ell}{M} \chi_{E_{\ell,M}}(x).$$

Each $F_{N,M}$ is a simple function that satisfies $0 \le F_N(x) - F_{N,M}(x) \le 1/M$ for all x. If we now choose $N = M = 2^k$ with $k \le 1$ integral, and let $\varphi_k = F_{2^k,2^k}$, then we see that $0 \le F_M(x) - \varphi_k(x) \le 1/2^k$ for all x, $\{\varphi_k\}$ is increasing, and this sequence satisfies all the desired properties. ■

Note that the result holds for non-negative functions that are extended-valued, if the limit $+\infty$ is allowed. We now drop the assumption that f is nonnegative, and also allow the extended limit $-\infty$:

• Theorem:
Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that satisfies

$$|\varphi_k(x)| \le |\varphi_{k+1}(x)| \qquad \text{and} \qquad \lim_{k\to\infty} \varphi_k(x) = f(x)$$

for all x.
In particular, we have $|\varphi_k(x)| \le |f(x)|$ for all x and k.

Proof:
We use the following decomposition of the function f:

$$f(x) = f^+(x) - f^-(x),$$

where

$$f^+(x) = \max(f(x),\ 0) \qquad \text{and} \qquad f^{-1}(x) = \max(-f(x),\ 0).$$

Since both f^+ and f^{-1} are non-negative, the previous theorem yields two increasing sequences of nonnegative simple functions $\left\{\varphi_k^{(1)}(x)\right\}_{k=1}^{\infty}$ and $\left\{\varphi_k^{(2)}(x)\right\}_{k=1}^{\infty}$ which converge pointwise to f^+ and f^{-1}, respectively. Then, if we let

$$\varphi_k(x) = \varphi_k^{(1)}(x) - \varphi_k^{(2)}(x)\ ,$$

we see that $\varphi_k(x)$ converges to $f\ (x)$ for all x. Finally, the sequence $\{|\varphi_k|\}$ is increasing because the definition of f^+, f^- and the properties of $\varphi_k^{(1)}$ and $\varphi_k^{(2)}$ imply that

$$|\varphi_k(x)| = \varphi_k^{(1)}(x) + \varphi_k^{(2)}(x)\ . \qquad \blacksquare$$

We may now go one step further, and approximate by step functions. Here, in general, the convergence may hold only almost everywhere:

• Theorem:
Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges pointwise to $f(x)$ for almost every x.

Proof:
By the previous result, it suffices to show that if E is a measurable set with finite measure, then $f = \chi_E$ can be approximated by step functions. To this end, we recall from a previous theorem that for every ε there exist cubes $Q_1,\ \ldots,\ Q_N$ such that

$$m\left(E \triangle \bigcup_{j=1}^{N} Q_j\right) \le \varepsilon.$$

By considering the grid formed by extending the sides of these cubes, we see that there exist almost disjoint rectangles $\tilde{R}_1,\ \ldots,\ \tilde{R}_M$ such that

$$\bigcup_{j=1}^{N} Q_j = \bigcup_{j=1}^{M} \tilde{R}_j\ .$$

By taking rectangles R_j contained in $\tilde{R}_j$, and slightly smaller in size, we find a collection of disjoint rectangles that satisfy

$$m\left(E \triangle \bigcup_{j=1}^{M} R_j\right) \leq 2\,\varepsilon$$

Therefore

$$f(x) = \sum_{j=1}^{M} \chi_{R_j}(x)\,,$$

except possibly on a set of measure $\leq 2\,\varepsilon$. Consequently, for every $k \geq 1$, there exists a step function $\psi_k(x)$ such that if

$$E_k = \{x : f(x) \neq \psi_k(x)\},$$

then $m(E_k) \leq 2^{-k}$. If we let $F_k = \bigcup_{j=K+1}^{\infty} E_j$ and $F = \bigcap_{K=1}^{\infty} F_K$, then $m(F) = 0$ since $m(F_K) \leq 2^{-K}$, and $\psi_k(x) \to f(x)$ for all x in the complement of F, which is the desired result. ■

Littlewood's three principles:

Although the notions of measurable sets and measurable functions represent new tools, we should not overlook their relation to the older concepts they replaced. Littlewood aptly summarized these connections in the form of three principles that provide a useful intuitive guide in the initial study of the theory:

(i) Every set is nearly a finite union of intervals.
(ii) Every function is nearly continuous.
(iii) Every convergent sequence is nearly uniformly convergent.

The sets and functions referred to above are of course assumed to be measurable. The catch is in the word "nearly," which has to be understood appropriately in each context.

• Egorov's Theorem:

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e on E. Given $\varepsilon > 0$, we can find a closed set $A_\varepsilon \subset E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ and $f_k \to f$ uniformly on A_ε.

Proof:

We may assume WLOG that $f_k(x) \to f(x)$ for every $x \in E$. For each pair of nonnegative integers n and k, let

$$E_k^n = \{x \in E : |f_j(x) - f(x)| < 1/n, \quad \text{for all } j > k\}\,.$$

Now fix n and note that $E_k^n \subset E_{k+1}^n$, and $E_k^n \nearrow E$ as k tends to infinity. By a previous corollary, we find that there exists k_n such that $m\left(E \setminus E_{k_n}^n\right) < 1/2^n$.

By construction, we then have

$$|f_j(x) - f(x)| < 1/n \quad \text{whenever} \quad j > k_n \quad \text{and} \quad x \in E^n_{k_n}.$$

We choose N so that $\sum_{n=N}^{\infty} 2^{-n} < \varepsilon/2$, and let

$$\tilde{A}_\varepsilon = \bigcap_{n \geq N} E^n_{k_n}.$$

We first observe that

$$m(E \setminus \tilde{A}_\varepsilon) \leq \sum_{n=N}^{\infty} m(E \setminus E^n_{k_n}) < \frac{\varepsilon}{2}.$$

Next, if $\delta > 0$, we choose $n \geq N$ such that $1/n < \delta$, and note that $x \in \tilde{A}_\varepsilon$ implies $x \in E^n_{k_n}$. We see therefore that $|f_j(x) - f(x)| < \delta$ whenever $j > k_n$. Hence f_k converges uniformly to f on $\tilde{A}_\varepsilon$. Finally, by a previous theorem we can choose a closed subset $A_\varepsilon \subset \tilde{A}_\varepsilon$ with $m(\tilde{A}_\varepsilon \setminus A_\varepsilon) < \varepsilon/2$. As a result, we have $m(E \setminus A_\varepsilon) < \varepsilon$ and the theorem is proved. ■

The next theorem attests to the validity of the second of Littlewood's principles:

• Lusin's Theorem:
Suppose f is measurable and finite valued on a set E of finite measure. Then for every $\varepsilon > 0$ there exists a closed set F_ε, with

$$F_\varepsilon \subset E \quad \text{and} \quad m(E \setminus F_\varepsilon) \leq \varepsilon$$

and such that $f|_{F_\varepsilon}$ is continuous.

Proof:
Let f_n be a sequence of step functions so that $f_n \to f$ a.e. . Then we may find sets E_n so that $m(E_n) < 1/2^n$ and f_n is continuous outside E_n. By Egorov's theorem, we may find a set $A_{\varepsilon/3}$ on which $f_n \to f$ uniformly and $m(E \setminus A_{\varepsilon/3}) \leq \varepsilon/3$. Then we consider

$$F' = A_{\varepsilon/3} \setminus \bigcup_{n \geq N} E_n$$

for N so large that $\sum_{n \geq N} 1/2^n < \varepsilon/3$. Now for every $n \geq N$ the function f_n is continuous on F'; thus f (being the uniform limit of $\{f_n\}$ by Egorov's theorem) is also continuous on F'.

To finish the proof, we merely need to approximate the set F' by a closed set $F_\varepsilon \subset F'$ such that $m(F' \setminus F_\varepsilon) < \varepsilon/3$. ■

Remark: The conclusion of the theorem states that if f is viewed as a function defined only on F_ε, then f is continuous. However, the theorem does not make the stronger assertion that the function f defined on E is continuous at the points of F_ε.