MATH 725 NOTES
PRELIMINARIES

MARIO L. GUTIERREZ ABED

Definition. *Let A be an $m \times n$ matrix with general entries a_{ij}. Then the* ***sup norm*** *of A is defined by*

$$|A| = \max\{|a_{ij}| \colon i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}. \qquad \bigstar$$

Theorem 1. *If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then*

$$|A \cdot B| \leq n|A||B|.$$

Definition. *Two matrices A and B are said to be* ***equivalent*** *if there exist invertible matrices P and Q for which*

$$A = PBQ.$$

In other words, A and B are equivalent if A can be reduced to B by performing a series of elementary row and/or column operations. ★

Definition. *Two matrices $A, B \in \mathcal{M}_n(\mathbb{F})$ (where $\mathcal{M}_n(\mathbb{F})$ is the space of all $n \times n$ matrices with coefficients in the field $\mathbb{F}$) are said to be* ***similar*** *if there exists an invertible matrix P such that*

$$A = PBP^{-1}. \qquad \bigstar$$

<u>*Remark*</u>: Similarity is easily seen to be an equivalence relation on $\mathcal{M}_n(\mathbb{F})$. As we will learn, two matrices are similar if and only if they represent the same linear operators on a given n-dimensional vector space V. Hence, similarity is extremely important for studying the structure of linear operators. One of the main goals of this course is to develop canonical forms for similarity.

Definition. *Two matrices $A, B \in \mathcal{M}_n(F)$ are said to be* ***congruent*** *if there exists an invertible matrix P for which*

$$A = PBP^T,$$

where P^T is the transpose of P. ★

<u>*Remark*</u>: This relation is easily seen to be an equivalence relation and we will devote some effort to finding canonical forms for congruence. For some base fields $\mathbb{F}$ (such as $\mathbb{R}$, $\mathbb{C}$, or a finite field), this is relatively easy to do, but for other base fields (such as $\mathbb{Q}$), it is extremely difficult.

Definition. *A* ***partially ordered set*** *(or* ***poset****) is a pair $(P, \leq)$ where P is a nonempty set and $\leq$ is a binary relation called a* ***partial order*** *with the following properties:*

i) **(Reflexivity)** *For all $a \in P$, we have $a \leq a$.*

ii) **(Antisymmetry)** *For all $a, b \in P$,*

$$a \leq b \quad \text{and} \quad b \leq a \quad \implies \quad a = b.$$

iii) **(Transitivity)** *For all $a, b, c \in P$,*

$$a \leq b \quad \text{and} \quad b \leq c \quad \Longrightarrow \quad a \leq c. \qquad \bigstar$$

<u>*Remark*</u>: Note that in a partially ordered set, it is possible that not all elements are comparable. In other words, it is possible to have $x, y \in P$ with the property that $x \not\leq y$ and $y \not\leq x$.

Definition. *A partially ordered set in which every pair of elements is comparable is called a* ***totally ordered set*** *(or* ***linearly ordered set****). Any totally ordered subset of a partially ordered set P is called a* ***chain*** *in P.* ★

Definition. *If a partially ordered set P has the property that every pair of elements has a least upper bound and greatest lower bound, then P is called a* ***lattice****. If P has a smallest element and a largest element and has the property that every collection of elements has a least upper bound and greatest lower bound, then P is called a* ***complete lattice****.* ★

Lemma 1 (Zorn's Lemma). *If P is a partially ordered set in which every chain has an upper bound, then P has a maximal element.*

<u>*Remark*</u>: Zorn's lemma is a result that is so fundamental that it cannot be proved or disproved in the context of ordinary set theory (it is a variation of the famous *Axiom of Choice*.) Therefore, Zorn's lemma (along with the *Axiom of Choice*) must either be accepted or rejected as an axiom of set theory.

Direct Sums & Direct Products

Definition. *Let $\mathcal{F} = \{V_i \mid i \in K\}$ be any family of vector spaces over some field of scalars $\mathbb{F}$. The* ***direct product*** *of $\mathcal{F}$ is the vector space*

$$\prod_{i \in K} V_i = \left\{ f\colon K \to \bigcup_{i \in K} V_i \mid f(i) \in V_i \right\},$$

thought of as a subspace of the vector space of all functions from K to $\cup_i V_i$. ★

It will prove more useful to restrict the set of functions to those with finite support:

Definition. *Let $\mathcal{F} = \{V_i \mid i \in K\}$ be a family of vector spaces over some field of scalars $\mathbb{F}$. The* ***support*** *of a function $f\colon K \to \cup_i V_i$ is the set*

$$\operatorname{supp}(f) = \{i \in K \mid f(i) \neq 0\}.$$

Thus, a function f is said to have ***finite support*** *if $f(i) = 0$ for all but a finite number of $i \in K$.* ★

Definition. *The* ***(external) direct sum*** *of the family of vector spaces $\mathcal{F} = \{V_i \mid i \in K\}$ is the vector space*

$$\bigoplus_{i \in K}^{\text{ext}} V_i = \left\{ f\colon K \to \bigcup_{i \in K} V_i \mid f(i) \in V_i,\ f \text{ has finite support} \right\},$$

thought of as a subspace of the vector space of all functions from K to $\cup V_i$. ★

<u>Remark</u>: An important special case occurs when $V_i = V$ for all $i \in K$. If we let V^K denote the set of all functions from K to V and $(V^K)_0$ denote the set of all functions in V^K that have finite support, then

$$\prod_{i \in K} V = V^K \qquad \text{and} \qquad \bigoplus_{i \in K}^{\text{ext}} V = (V^K)_0.$$

Note that the direct product and the external direct sum are the same for a finite family of vector spaces.

Definition. *Let V be a vector space. We say that V is the **(internal) direct sum** of a family $\mathcal{F} = \{S_i \mid i \in K\}$ of subspaces of V if every vector $v \in V$ can be written in a unique way (except for order), as a finite sum of vectors from the subspaces in $\mathcal{F}$; that is, if for all $v \in V$,*

$$v = u_1 + \cdots + u_n \qquad \textit{for } u_i \in S_i.$$

Furthermore, if

$$v = w_1 + \cdots + w_m \qquad \textit{for } w_i \in S_i,$$

then $m = n$ and (after reindexing if necessary) we also have that $w_i = u_i$ for all $i = 1, \ldots, n$.

If V is the internal direct sum of $\mathcal{F}$, we write

$$V = \bigoplus_{i \in K} S_i$$

*and refer to each S_i as a **direct summand** of V.* ★

<u>Remark</u>: It can be shown that the concepts of internal and external direct sums are essentially equivalent (isomorphic). For this reason, we often use the term "direct sum" to refer to either type.

Definition. *If $V = S \oplus T$, then T is called a **complement** of S in V.* ★

Theorem 2. *A vector space V is the direct sum of a family $\mathcal{F} = \{S_i \mid i \in K\}$ of subspaces if and only if the following two conditions are met:*

i) V is the sum of the S_i, i.e.

$$V = \sum_{i \in K} S_i.$$

ii) For each $i \in K$, we have

$$S_i \bigcap \left(\sum_{j \neq i} S_j \right) = \{0\}.$$

Basics of Linear Transformations

Definition. *Let V and W be vector spaces over a field $\mathbb{F}$. A function $\phi : V \to W$ is a* ***linear transformation*** *if*

$$\phi(\alpha u + \beta v) = \alpha\phi(u) + \beta\phi(v)$$

for all scalars $\alpha, \beta \in \mathbb{F}$ and vectors $u, v \in V$. A linear transformation $\phi : V \to V$ is called a ***linear operator*** *on V.* ★

Remark: The following terms are also employed (usually in categorical language):

- **homomorphism** for linear tranformation.
- **endomorphism** for linear operator.
- **monomorphism** (or **embedding**) for injective linear tranformation.
- **epimorphism** for surjective linear tranformation.
- **isomorphism** for bijective linear tranformation.
- **automorphism** for bijective linear operator.

Theorem 3. *Let $\phi \in \mathcal{L}(V, W)$ be an isomorphism. Let $S \subseteq V$. Then*

i) S spans $V \iff \phi(S)$ spans W.

ii) S is linearly independent in $V \iff \phi(S)$ is linearly independent in W.

iii) S is a basis for $V \iff \phi(S)$ is a basis for W.

Theorem 4. *Let A be an $m \times n$ matrix over F. Then*

i) The map $\phi_A : F^n \to F^m$ is injective $\iff$ rank$(A) = n$.

ii) The map $\phi_A : F^n \to F^m$ is surjective $\iff$ rank$(A) = m$.

Definition. *A linear transformation $T\colon V \to W$ (where V and W are normed vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively) is said to be* ***bounded*** *if there exists an $M \geq 0$ such that*

$$\|Tu\|_W \leq M\|u\|_V \quad \text{for each } u \in V.$$

If T is a bounded linear transformation, then its norm $\|T\|$ is the smallest M that satisfies the above inequality. That is,

$$\|T\| = \sup_{u \neq 0} \frac{\|Tu\|_W}{\|u\|_V}. \qquad ★$$

Theorem 5. *Let V and W be vector spaces over $\mathbb{F}$, with ordered bases $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{C} = (c_1, \ldots, c_m)$, respectively. Then,*

i) The map $\mu : \mathcal{L}(V, W) \to \mathcal{M}_{m,n}(\mathbb{F})$ defined by

$$\mu(\phi) = [\phi]_{\mathcal{B},\mathcal{C}}$$

is an isomorphism and so $\mathcal{L}(V, W) \cong \mathcal{M}_{m,n}(\mathbb{F})$.

ii) *If* $\sigma \in \mathcal{L}(U, V)$ *and* $\phi \in \mathcal{L}(V, W)$*, and if* $\mathcal{B}$*,* $\mathcal{C}$*, and* $\mathcal{D}$ *are ordered bases for* U*,* V*, and* W*, respectively, then*

$$[\phi\sigma]_{\mathcal{B},\mathcal{D}} = [\phi]_{\mathcal{C},\mathcal{D}}[\sigma]_{\mathcal{B},\mathcal{C}}.$$

Thus, the matrix of the product (composition) $\phi\sigma$ *is the product of the matrices of* ϕ *and* σ*. In fact, this is the primary motivation for the definition of matrix multiplication.*

Theorem 6. *Let* $\phi \in \mathcal{L}(V, W)$ *and let* $(\mathcal{B}, \mathcal{C})$ *and* $(\mathcal{B}', \mathcal{C}')$ *be pairs of ordered bases of* V *and* W*, respectively. Then,*

$$[\phi]_{\mathcal{B}',\mathcal{C}'} = M_{\mathcal{C},\mathcal{C}'}[\phi]_{\mathcal{B},\mathcal{C}}M_{\mathcal{B}',\mathcal{B}}.$$

Corollary 1. *Let* $\phi \in \mathcal{L}(V)$ *and let* $\mathcal{B}$ *and* $\mathcal{C}$ *be ordered bases for* V*. Then the matrix of* ϕ *with respect to* $\mathcal{C}$ *can be expressed in terms of the matrix of* ϕ *with respect to* $\mathcal{B}$ *as follows*

$$[\phi]_{\mathcal{C}} = M_{\mathcal{B},\mathcal{C}}[\phi]_{\mathcal{B}}M_{\mathcal{B},\mathcal{C}}^{-1}.$$

Theorem 7. *Let* V *be a real vector space of dimension* n*. There is a unique topology on* V*, called the* ***natural topology****, for which* V *is a topological vector space and for which all linear functionals on* V *are continuous. This topology is determined by the fact that the coordinate map* $\phi : V \to \mathbb{R}^n$ *is a homeomorphism.*