SOME RIEMANNIAN GEOMETRY PROBLEMS

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Remark: The space of all covariant k-tensors on a vector space V is denoted by $T^k(V)$, the space of contravariant l-tensors by $T_l(V)$, and the space of mixed $\binom{k}{l}$-tensors by $T_l^k(V)$. The ***rank*** of a tensor is the number of arguments (vectors and/or covectors) it takes. There are obvious identifications $T_0^k(V) = T^k(V)$, $T_l^0(V) = T_l(V)$, $T^1(V) = V^*$, $T_1(V) = V^{**} = V$, and $T^0(V) = \mathbb{R}$. A less obvious, but extremely important, identification is $T_1^1(V) = \text{End}(V)$. A more general version of this identification is expressed in the following lemma:

Exercise 1 (Lemma 2.1, [Lee's Riemannian Manifolds]). *Let V be a finite-dimensional vector space. Show that there is a natural (basis-independent) isomorphism between $T_{l+1}^k(V)$ and the space of multilinear maps*

$$\underbrace{V^* \times \cdots \times V^*}_{l} \times \underbrace{V \times \cdots \times V}_{k} \to V.$$

[Hint: In the special case $k = 1$, $l = 0$, consider the map $\Phi\colon \text{End}(V) \to T_1^1(V)$ by letting ΦA be the $\binom{1}{1}$-tensor defined by $\Phi A(\omega, X) = \omega(AX)$. The general case is similar.]

Proof. □

Exercise 2 (Exercise 2.2, [Lee's Riemannian Manifolds]). *Show that the trace on any pair of indices is a well-defined linear map from $T_{l+1}^{k+1}(V)$ to $T_l^k(V)$.*

Proof. □

Problem 1 (Problem 13-5, [Lee's Smooth Manifolds]). *Suppose (M, g) is a Riemannian manifold. A smooth curve $\gamma\colon J \to M$ is said to be a **unit-speed curve** if $\|\gamma'(t)\|_g \equiv 1$. Prove that every smooth curve with nowhere-vanishing velocity has a unit-speed reparametrization.*

Proof. We can assume that J is open. Choose any $t_0 \in J$ and define $\ell\colon J \to \mathbb{R}$ by

$$\ell(t) = \int_{t_0}^{t} \|\gamma'(x)\|_g \, \mathrm{d}x.$$

Then $x \mapsto \|\gamma'(x)\|_g$ is smooth since $v \mapsto \|v\|_g$ is smooth on $\{(x, v) \in TM \mid v \neq 0\}$ and $\gamma'(x) \neq 0$ for all $x \in J$. Therefore ℓ is smooth, and since $\ell'(t) \neq 0$ for all $t \in J$, the *Inverse Function Theorem* shows that $\ell^{-1}\colon \ell(J) \to J$ is smooth. Now define $\gamma_1\colon \ell(J) \to M$ by $\gamma_1 = \gamma \circ \ell^{-1}$. Then we have

$$\begin{aligned}\gamma_1' &= (\ell^{-1})'(x)\,\gamma'(\ell^{-1}(x))\\ &= \frac{\gamma'(\ell^{-1}(x))}{\ell'(\ell^{-1}(x))}\\ &= \frac{\gamma'(\ell^{-1}(x))}{\|\gamma'(\ell^{-1}(x))\|_g}.\end{aligned}$$

so γ_1 is a unit-speed reparametrization of γ. □

Problem 2 (Problem 13-6, [Lee's Smooth Manifolds]). *Prove that every Riemannian* 1*-manifold is flat. [Hint: use Problem 13-5. Note that this implies the round metric on* $\mathbb{S}^1$ *is flat!]*

Proof. Let (M, g) be a Riemannian 1-manifold. Suppose $x \in M$ and (U, φ) is a smooth chart containing x. By shrinking U, we can assume that U is connected. Therefore $\varphi^{-1} \colon \varphi(U) \to U$ is a smooth curve, and then the result of *Problem 13-5* above shows that there is a diffeomorphism $\psi \colon E \to \varphi(U)$ such that $\varphi^{-1} \circ \psi$ is a unit-speed curve. We want to show that $\varphi^{-1} \circ \psi \colon E \to U$ is an isometry. If $v, w \in T_p E$ are nonzero vectors, then $v = r\, \mathrm{d}/\mathrm{d}t|_p$ and $w = s\, \mathrm{d}/\mathrm{d}t|_p$ for some $r, s \in \mathbb{R}$. Thus,

$$\begin{aligned}
((\varphi^{-1} \circ \psi)^* g_p)(v, w) &= rs((\varphi^{-1} \circ \psi)^* g_p)\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_p, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_p\right) \\
&= rs\left\|\mathrm{d}(\varphi^{-1} \circ \psi)_p\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_p\right)\right\|_g^2 \\
&= rs\left\|(\varphi^{-1} \circ \psi)'(p)\right\|_g^2 \\
&= rs \\
&= \bar{g}_{(\varphi^{-1} \circ \psi)(p)}(v, w).
\end{aligned}$$

□

Problem 3 (Problem 13-7, [Lee's Smooth Manifolds]). *Show that a product of flat metrics is flat.*

Proof. If $(R_1, g_1), \ldots, (R_k, g_k)$ are Riemannian manifolds, then

$$(R_1 \times \cdots \times R_k, g_1 \oplus \cdots \oplus g_k)$$

is also a Riemannian manifold. Let $(x_1, \ldots, x_k) \in R_1 \times \cdots \times R_k$. For each $i = 1, \ldots, k$, let $F_i \colon U_i \to V_i$ be an isometry such that $x_i \in U_i$. Then $F_1 \times \cdots \times F_k$ is an isometry from the neighborhood $U_1 \times \cdots \times U_k$ to the open set $V_1 \times \cdots \times V_k$ since

$$\begin{aligned}
(F_1 \times \cdots \times F_k)^*(\bar{g} \oplus \cdots \oplus \bar{g}) &= F_1^* \bar{g} \oplus \cdots \oplus F_k^* \bar{g} \\
&= g_1 \oplus \cdots \oplus g_k.
\end{aligned}$$

□

Problem 4 (Problem 13-8, [Lee's Smooth Manifolds]). *Let* $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subseteq \mathbb{C}^n$*, and let* g *be the metric on* $\mathbb{T}^n$ *induced from the Euclidean metric on* $\mathbb{C}^n$ *(identified with* $\mathbb{R}^{2n}$*). Show that* g *is flat.*

Proof. □

Problem 5 (Problem 11-5, [Lee's Smooth Manifolds]). *For any smooth manifold* M*, prove that* T^*M *is a trivial vector bundle if and only if* TM *is trivial.*

Proof. If TM is trivial, then it admits a smooth global frame; Lemma 11.14[1] shows that its dual coframe is a smooth global coframe for M, so T^*M is trivial. The converse is similar. □

[1] Here's Lemma 11.14, for reference: Let M be a smooth manifold (with or without boundary). If (E_i) is a rough local frame over an open subset $U \subseteq M$ and (ε^i) is its dual coframe, then (E_i) is smooth if and only if (ε^i) is smooth.

Problem 6 (Problem 11-8, [Lee's Smooth Manifolds])**.** *Suppose $F\colon M \to N$ is a diffeomorphism, and let $\mathrm{d}F^*\colon T^*N \to T^*M$ be the map whose restriction to each cotangent space T_q^*N is equal to $\mathrm{d}F^*_{F^{-1}(q)}$. Then $\mathrm{d}F^*$ is a smooth bundle homomorphism.*

*In addition, let $\mathbf{Diff}_1$ be the category whose objects are smooth manifolds, but whose only morphisms are diffeomorphisms, and let $\mathbf{VB}$ be the category whose objects are smooth vector bundles and whose morphisms are smooth bundle homomorphisms. Show that the assignment $M \mapsto T^*M$, $F \mapsto \mathrm{d}F^*$ defines a contravariant functor from $\mathbf{Diff}_1$ to $\mathbf{VB}$, called the **cotangent functor**.*

Proof. It is clear that $\mathrm{d}F^*$ is a smooth bundle homomorphism covering F^{-1}. Now, since

$$\mathrm{d}(F \circ G)^*_p = (\mathrm{d}F_{G(p)} \circ \mathrm{d}G_p)^* = \mathrm{d}G^*_p \circ \mathrm{d}F^*_{G(p)},$$

we have $\mathrm{d}(F \circ G)^* = \mathrm{d}G^* \circ \mathrm{d}F^*$. Also, $\mathrm{d}(\mathrm{Id}_M)^* = \mathrm{Id}_{T^*M}$. This shows that $M \mapsto T^*M$, $F \mapsto \mathrm{d}F^*$ defines a contravariant functor, as desired. □

Problem 7 (Problem 11-10, [Lee's Smooth Manifolds])**.** *In each of the cases below, M is a smooth manifold and $f\colon M \to \mathbb{R}$ is a smooth function. Compute the coordinate representation for $\mathrm{d}f$, and determine the set of all points $p \in M$ at which $\mathrm{d}f_p = 0$.*

a) $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$; $f(x, y) = x/(x^2 + y^2)$. Use standard coordinates (x, y).

b) M and f are as in a); this time use polar coordinates (r, θ).

c) $M = \mathbb{S}^2 \subseteq \mathbb{R}^3$; $f(p) = z(p)$ (the z-coordinate of p as a point in $\mathbb{R}^3$). Use north and south stereographic coordinates.

d) $M = \mathbb{R}^n$; $f(x) = |x|^2$. Use standard coordinates.

Solution of a). We have

$$\mathrm{d}f = \frac{-x^2 + y^2}{(x^2 + y^2)^2}\,\mathrm{d}x + \frac{-2xy}{(x^2 + y^2)^2}\,\mathrm{d}y,$$

which is zero when $x^2 = y^2$ and $xy = 0$, i.e., never on M. □

Solution of b). We have

$$f(r, \theta) = \frac{\cos\theta}{r} \qquad \Longrightarrow \qquad \mathrm{d}f = -\frac{\cos\theta}{r^2}\,\mathrm{d}r - \frac{\sin\theta}{r}\,\mathrm{d}\theta,$$

which is zero when $\cos\theta = \sin\theta = 0$, i.e., never on M. □

Solution of c). We have

$$f(u, v) = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

on $\mathbb{S}^2 \smallsetminus N$ using the north stereographic projection. Hence,

$$\mathrm{d}f = \frac{4u}{(u^2 + v^2 + 1)^2}\,\mathrm{d}u + \frac{4v}{(u^2 + v^2 + 1)^2}\,\mathrm{d}v,$$

which is zero when $u = v = 0$, i.e., when p is the south pole $S = (0, 0, -1)$. A similar calculation using the south stereographic projection shows that $\mathrm{d}f_p = 0$ when p is the north pole $N = (0, 0, 1)$. □

Solution of d). We have

$$\mathrm{d}f = 2x^1\mathrm{d}x^1 + \cdots + 2x^n\mathrm{d}x^n,$$

which is zero when $p = 0$. □

Problem 8 (Problem 11-11, [Lee's Smooth Manifolds]). *Let M be a smooth manifold, and $C \subseteq M$ be an embedded submanifold. Let $f \in C^\infty(M)$, and suppose $p \in C$ is a point at which f attains a local maximum or minimum value among points in C. Given a smooth local defining function $\Phi\colon U \to \mathbb{R}^k$ for C on a neighborhood U of p in M, show that there are real numbers $\lambda_1, \dots, \lambda_k$ (called* ***Lagrange multipliers****) such that*

$$\mathrm{d}f_p = \lambda_1 \mathrm{d}\Phi^1|_p + \cdots + \lambda_k \mathrm{d}\Phi^k|_p.$$

Proof. Let n be the dimension of M so that C has dimension $n-k$. Let (V, φ) be a smooth slice chart for C in M centered at p such that $V \subseteq U$. Also, let

$$\widetilde{f} = f \circ \varphi^{-1}, \qquad \widetilde{\Phi} = \Phi \circ \varphi^{-1}, \qquad \widetilde{p} = \varphi(p).$$

It suffices to show that there are real numbers $\lambda_1, \dots, \lambda_k$ such that

$$D\widetilde{f}(\widetilde{p}) = \lambda_1 D\widetilde{\Phi}^1(\widetilde{p}) + \cdots + \lambda_k D\widetilde{\Phi}^k(\widetilde{p}).$$

Since $\Phi^{-1}(0) \subseteq V \cap C$, this follows from the method of Lagrange multipliers on $\mathbb{R}^n$. □

Problem 9 (Problem 11-13, [Lee's Smooth Manifolds]). *The* ***length*** *of a smooth curve segment $\gamma\colon [a,b] \to \mathbb{R}^n$ is defined to be the value of the (ordinary) integral*

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\,\mathrm{d}t.$$

Show that there is no smooth covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^n)$ with the property that $\int_\gamma \omega = L(\gamma)$ for every smooth curve γ.

Proof. Let $-\gamma$ denote the curve $t \mapsto \gamma(b - t + a)$. Then $L(-\gamma) = L(\gamma)$, but

$$\int_{-\gamma} \omega = -\int_\gamma \omega = -L(\gamma).$$ □

Problem 10 (Problem 11-14, [Lee's Smooth Manifolds]). *Consider the following two covector fields on $\mathbb{R}^3$:*

$$\omega = -\frac{4z\mathrm{d}x}{(x^2+1)^2} + \frac{2y\mathrm{d}y}{y^2+1} + \frac{2x\mathrm{d}z}{x^2+1},$$
$$\eta = -\frac{4xz\mathrm{d}x}{(x^2+1)^2} + \frac{2y\mathrm{d}y}{y^2+1} + \frac{2\mathrm{d}z}{x^2+1}.$$

a) *Set up and evaluate the line integral of each covector field along the straight line segment from $(0,0,0)$ to $(1,1,1)$.*

b) *Determine whether either of these covector fields is exact.*

c) *For each one that is exact, find a potential function and use it to recompute the line integral.*

Solution of a). Let $\gamma\colon [0,1] \to \mathbb{R}^3$ be given by $t \mapsto (t,t,t)$. Then

$$\begin{aligned}\int_\gamma \omega &= \int_0^1 \left(-\frac{4t}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2t}{t^2+1}\right) \mathrm{d}t \\ &= \int_0^1 \frac{4t^3}{(t^2+1)^2}\,\mathrm{d}t \\ &= 2\log 2 - 1,\end{aligned}$$

and

$$\begin{aligned}\int_\gamma \eta &= \int_0^1 \left(-\frac{4t^2}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2}{t^2+1}\right) \mathrm{d}t \\ &= \int_0^1 \frac{2(t^3-t^2+t+1)}{(t^2+1)^2}\,\mathrm{d}t \\ &= \log 2 + 1.\end{aligned}$$

□

Solution of b). It is easy to check that

$$\frac{\partial}{\partial z}\omega_1 \neq \frac{\partial}{\partial x}\omega_3,$$

so ω is not closed and thus not exact. It is similarly easy to check that η is in fact closed, which implies by Theorem 11.49[2] that η is exact. □

Solution of c). Let us suppose that f is a potential for η; then it must satisfy

$$\frac{\partial f}{\partial x} = -\frac{4xz}{(x^2+1)^2}, \qquad \frac{\partial f}{\partial y} = -\frac{2y}{y^2+1}, \qquad \frac{\partial f}{\partial z} = -\frac{2}{x^2+1}.$$

Integrating the first equation, we have

$$f(x,y,z) = \frac{2z}{x^2+1} + C_1(y,z).$$

Then,

$$\frac{\partial C_1}{\partial y} = \frac{2y}{y^2+1} \qquad \Longrightarrow \qquad C_1(y,z) = \log(y^2+1) + C_2(z).$$

Finally, we have

$$\frac{2}{x^2+1} + \frac{\partial C_2}{\partial z} = \frac{2}{x^2+1} \qquad \Longrightarrow \qquad C_2 \text{ is constant.}$$

It is now easy to check that for any $C \in \mathbb{R}$,

$$f(x,y,z) = \frac{2z}{x^2+1} + \log(y^2+1) + C$$

[2]Here's Theorem 11.49, for reference:

Theorem (Poincaré Lemma for Covector Fields). *If U is a star-shaped open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, then every closed covector field on U is exact.*

is a potential for η. Therefore

$$\int_\gamma \eta = \int_\gamma \mathrm{d}f = \frac{2}{2} + \log 2 - 0 = \log 2 + 1. \qquad \square$$

Problem 11 (Problem 11-15, [Lee's Smooth Manifolds]). `LINE INTEGRALS OF VECTOR FIELDS`*: Let X be a smooth vector field on an open subset $U \subseteq \mathbb{R}^n$. Given a piecewise smooth curve segment $\gamma\colon [a,b] \to U$, define the* ***line integral of*** *X* ***over*** *γ, denoted by $\int_\gamma X \cdot \mathrm{d}s$, as*

$$\int_\gamma X \cdot \mathrm{d}s = \int_a^b X_{\gamma(t)} \cdot \gamma'(t)\,\mathrm{d}t,$$

where the dot on the right-hand side denotes the Euclidean dot product between tangent vectors at $\gamma(t)$, identified with elements of $\mathbb{R}^n$. A ***conservative vector field*** *is one whose line integral around every piecewise smooth closed curve is zero.*

a) *Show that X is conservative if and only if there exists a smooth function $f \in C^\infty(U)$ such that $X = \operatorname{grad} f$. [Hint: consider the covector field ω defined by $\omega_x(v) = X_x \cdot v$.]*

b) *Suppose $n = 3$. Show that if X is conservative, then $\operatorname{curl} X = 0$, where*

$$(\clubsuit) \qquad \operatorname{curl} X = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right)\frac{\partial}{\partial x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.$$

c) *Show that if $U \subseteq \mathbb{R}^3$ is star-shaped, then X is conservative on U if and only if $\operatorname{curl} X = 0$.*

Proof. Take any $X \in \frac{X}{(}U)$ and define a smooth covector field ω by $\omega_x(v) = X_x \cdot v$. Then,

$$\int_\gamma X \cdot \mathrm{d}s = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t = \int_\gamma \omega.$$

Part a) then follows immediately from Theorem 11.42 (*Let M be a smooth manifold with or without boundary. Then a smooth covector field on M is conservative if and only if it is exact*).

Similarly, part b) follows from Proposition 11.44 (*Every exact covector field is closed*), while part c) follows from the *Poincaré Lemma for Covector Fields.* $\square$

Problem 12 (Problem 11-16, [Lee's Smooth Manifolds]). *Let M be a compact manifold of positive dimension. Show that every exact covector field on M vanishes at least at two points in each component of M.*

Proof. Let $\omega = \mathrm{d}f$ be an exact covector field. Let U be a component of M. Since U is compact, f attains a maximum at some $x \in U$ and a minimum at some $y \in U$. If $x = y$ then f is constant on U, so $\mathrm{d}f = 0$ on U. Otherwise, $\mathrm{d}f$ vanishes at the distinct points x and y. $\square$

Problem 13 (Problem 12-1, [Lee's Smooth Manifolds]). *Give an example of finite-dimensional vector spaces V and W and a specific element $\alpha \in V \otimes W$ that cannot be expressed as $v \otimes w$ for $v \in V$ and $w \in W$.*

Solution. Take $V = W = \mathbb{R}^2$, let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and let $\alpha = e_1 \otimes e_1 + e_2 \otimes e_2$. Now suppose $\alpha = v \otimes w$ for $v, w \in \mathbb{R}^2$, and write $v = v^i e_i$ and $w = w^j e_j$. (Here we are using Einstein's summation convention.) Then $\alpha = v^i w^j e_i \otimes e_j$, so

$$v^1 w^1 = 1, \quad v^1 w^2 = 0, \quad v^2 w^1 = 0, \quad \text{and} \quad v^2 w^2 = 1.$$

This implies that $w^1 \neq 0$, which in turn indicates that $v^2 = 0$, contradicting $v^2 w^2 = 1$. □

Problem 14 (Problem 12-2, [Lee's Smooth Manifolds]). *For any finite-dimensional real vector space V, prove that there are canonical isomorphisms $\mathbb{R} \otimes V \cong V \cong V \otimes \mathbb{R}$.*

Proof. We can define a multilinear map $f : \mathbb{R} \times V \to V$ by $(r, v) \mapsto rv$, which induces a linear map $\widetilde{f} : \mathbb{R} \otimes V \to V$ satisfying $\widetilde{f}(r \otimes v) = rv$. It is easy to check that the linear map $v \mapsto 1 \otimes v$ is an inverse to $\widetilde{f}$, so $\widetilde{f}$ is an isomorphism. A similar argument shows that $V \cong V \otimes \mathbb{R}$. □

Problem 15 (Problem 12-3, [Lee's Smooth Manifolds]). *Let V and W be finite-dimensional real vector spaces. Show that the tensor product space $V \otimes W$ is uniquely determined up to canonical isomorphism by its characteristic property. More precisely, suppose $\widetilde{\pi} : V \times W \to Z$ is a bilinear map into a vector space Z with the following property: for any bilinear map $B : V \times W \to Y$, there is a unique linear map $\widetilde{B} : Z \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} V \times W \arrow[r, "B"] \arrow[d, "\widetilde{\pi}"'] & Y \\ Z \arrow[ur, dashed, "\widetilde{B}"'] & \end{tikzcd}$$

Then there is a unique isomorphism $\Phi : V \otimes W \to Z$ such that $\widetilde{\pi} = \Phi \circ \pi$, where $\pi : V \times W \to V \otimes W$ is the canonical projection. [Remark: this shows that the details of the construction used to define the tensor product space are irrelevant, as long as the resulting space satisfies the characteristic property.]

Proof. Since $\widetilde{\pi}$ is bilinear, there exists a unique linear map $\Phi : V \otimes W \to Z$ such that $\widetilde{\pi} = \Phi \circ \pi$. Similarly, there exists a unique linear map $\Psi : Z \to V \otimes W$ such that $\pi = \Psi \circ \widetilde{\pi}$. We have

$$\Phi \circ \Psi \circ \widetilde{\pi} = \Phi \circ \pi = \widetilde{\pi},$$

so that $\Phi \circ \Psi = \mathrm{Id}_Z$ by uniqueness. Similarly,

$$\Psi \circ \Phi \circ \pi = \Psi \circ \widetilde{\pi} = \pi$$

implies that $\Psi \circ \Phi = \mathrm{Id}_{V \otimes W}$ by uniqueness. Therefore, Φ is an isomorphism. □

Problem 16 (Problem 13-21, [Lee's Smooth Manifolds]). *Let (M, g) be a Riemannian manifold, let $f \in C^\infty(M)$, and let $p \in M$ be a regular point of f.*

i) *Show that among all unit vectors $v \in T_pM$, the directional derivative vf is greatest when v points in the same direction as $\operatorname{grad} f|_p$ and that the length of $\operatorname{grad} f|_p$ is equal to the value of the directional derivative in that direction.*

ii) *Show that $\operatorname{grad} f|_p$ is normal to the level set of f through p.*

Solution of i). We have $\langle \operatorname{grad} f|_p, v\rangle = vf$, so the Cauchy-Schwarz inequality shows that

$$\|\langle \operatorname{grad} f|_p, v\rangle_g\|$$

is maximized when $v = \pm \operatorname{grad} f|_p$. □

Solution of ii). Let S be the level set of f through p, so that $T_pS = \ker \mathrm{d}f_p$ by Proposition 5.38, Lee's.[3] Clearly $\langle \operatorname{grad} f|_p, v\rangle = vf = \mathrm{d}f_p(v) = 0$ for every $v \in \ker \mathrm{d}f_p$. □

Problem 17 (Problem 13-23, [Lee's Smooth Manifolds])**.** *Is there a smooth covector field on $\mathbb{S}^2$ that vanishes at exactly one point?*

Solution. □

Exercise 3 (Exercise 14-28, [Lee's Smooth Manifolds])**.** *Recall from vector calculus the curl operator (only defined on $\mathbb{R}^3$) as well as the gradient of a function $f \in C^\infty(\mathbb{R}^n)$ and the divergence of a vector field $X \in \Gamma^\infty(T\mathbb{R}^n)$. These are given by*

$$\begin{aligned}
\operatorname{curl} X &= \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right)\frac{\partial}{\partial x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}\\
\operatorname{grad} f &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}\\
\operatorname{div} X &= \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}.
\end{aligned}$$

In addition, the Euclidean metric on $\mathbb{R}^3$ yields an index-lowering (musical) isomorphism $\flat\colon \Gamma^\infty(T\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$. Interior multiplication yields yet another map $\beta\colon \Gamma^\infty(T\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ as follows:

$$\beta(X) = X \lrcorner (\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z).$$

This map β is linear over $C^\infty(\mathbb{R}^3)$, so it corresponds to a smooth bundle homomorphism from TM to $\Lambda^2 T\mathbb{R}^3$. It is a bundle isomorphism because it is injective and both TM and $\Lambda^2 T\mathbb{R}^3$ are bundles of rank 3. Similarly, we define a smooth bundle isomorphism $\circledast\colon C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ by

$$\circledast(f) = f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

The relationships among all of these operators are summarized in the following diagram:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\operatorname{grad}"] \arrow[d, "\mathrm{Id}"'] & \Gamma^\infty(T\mathbb{R}^3) \arrow[r, "\operatorname{curl}"] \arrow[d, "\flat"] & \Gamma^\infty(T\mathbb{R}^3) \arrow[r, "\operatorname{div}"] \arrow[d, "\beta"'] & C^\infty(\mathbb{R}^3) \arrow[d, "\circledast"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "\mathrm{d}"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "\mathrm{d}"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "\mathrm{d}"'] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

This diagram commutes, so that $\operatorname{curl} \circ \operatorname{grad} \equiv 0$ and $\operatorname{div} \circ \operatorname{curl} \equiv 0$ on $\mathbb{R}^3$. Prove that also the analogues of the left-hand and right-hand squares commute when $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$ for any n.

[3]Here's the proposition, for reference:

Proposition. *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi\colon U \to N$ is any local defining map for S, then $T_pS = \ker \mathrm{d}\Phi_p\colon T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap U$.*

***Remark*:** The desire to generalize these vector calculus operators from $\mathbb{R}^3$ to higher dimensions was one of the main motivations for developing the theory of differential forms. The curl, in particular, makes sense as an operator on vector fields only in dimension 3, whereas the exterior derivative expresses the same information but makes sense in all dimensions.

Solution. In $\mathbb{R}^n$, we have

$$(\flat \circ \operatorname{grad})(f) = \flat\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}\right) = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i = \mathrm{d}f,$$

where on the next-to-last equality we are using Einstein's summation convention (thanks to the flat operator of course!). Also

$$\begin{aligned}
(\mathrm{d} \circ \beta) &= \mathrm{d}(X \lrcorner (\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)) \\
&= \mathrm{d}\left(\sum_{i=1}^n (-1)^{i-1}\,\mathrm{d}x^i(X)\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n\right) \\
&= \mathrm{d}\left(\sum_{i=1}^n (-1)^{i-1}\,X^i\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n\right) \\
&= \sum_{i=1}^n (-1)^{i-1}\frac{\partial X^i}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \qquad \text{(Einstein summation being used over } j) \\
&= \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \divideontimes\left(\sum_{i=1}^n \frac{\partial X^i}{\partial x^i}\right) \\
&= (\divideontimes \circ \operatorname{div})(X).
\end{aligned}$$

Now in $\mathbb{R}^3$ we have

$$\begin{aligned}
(\beta \circ \operatorname{curl})(X) &= \beta\left(\left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right)\frac{\partial}{\partial x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}\right) \\
&= \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \mathrm{d}(X^1\,\mathrm{d}x^1 + X^2\,\mathrm{d}x^2 + X^3\,\mathrm{d}x^3) \\
&= (\mathrm{d} \circ \flat)(X).
\end{aligned}$$

□

Exercise 4 (Exercise 4.8, [Lee's Riemannian Manifolds])**.** *Show that the geodesics on $\mathbb{R}^n$ with respect to the Euclidean connection*

$$\overline{\nabla}_X\left(Y^j\frac{\partial}{\partial x^j}\right) = \left(XY^j\right)\frac{\partial}{\partial x^j}$$

(where we use Einstein's summation convention) are exactly the straight lines with constant speed parametrizations.

Proof. □

Exercise 5 (Exercise 5.7, [Lee's Riemannian Manifolds])**.** *Define spherical coordinates (θ, φ) on the subset $\mathbb{S}^2_R \smallsetminus \{(x, y, z) \mid x \leq 0, y = 0\}$ of the sphere by*

$$(x, y, z) = (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R\cos\varphi), \quad \textit{for } -\pi < \theta < \pi,\ 0 < \varphi < \pi.$$

a) *Show that the round metric of radius R is $\mathring{g}_R = R^2 \mathrm{d}\varphi^2 + R^2 \sin^2\varphi \, \mathrm{d}\theta^2$ in spherical coordinates.*

b) *Compute the Christoffel symbols of $\mathring{g}_R$ in spherical coordinates.*

c) *Using the* geodesic equation

$$\ddot{x}^k(t) + \dot{x}^i(t)\,\dot{x}^j(t)\,\Gamma^k_{ij}(x(t)) = 0 \tag{1}$$

in spherical coordinates, verify that each meridian $(\theta(t), \varphi(t)) = (\theta_0, t)$ is a geodesic.

Solution of a). □

Solution of b). □

Solution of c). □

Exercise 6. *Prove that meridians on a surface of revolution are geodesics. Are all parallels geodesics too?*

Proof. It is easy to see that the sign of geodesic curvature (properly defined the same way as on the Euclidean plane) is reversed under the change of orientation of the surface. The reflection about the plane containing the meridian preserves the surface of revolution and leaves all points of the meridian fixed. Thus the signed geodesic curvature of the meridian at each point satisfies $k_g = -k_g$, i.e., $k_g = 0$. Moreover, the same is true for any fixed point curve of an isometric reflection on a Riemann surface.

Parallels are typically not geodesics (e.g., non-equatorial parallels on the sphere are not). □