Math 353 HW 11

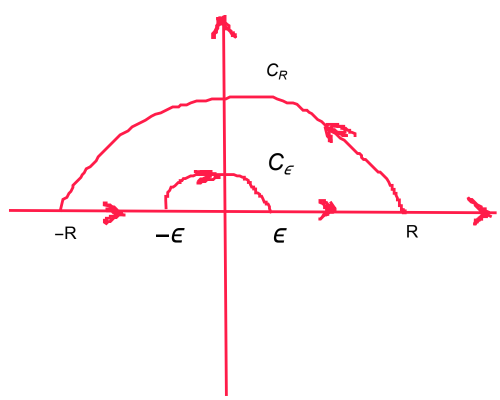
Mario L Gutierrez Abed

Section 4.3

(2) Show that $\int_0^\infty \frac{\sin x}{x(x^2+1)}\, dx = \frac{\pi}{2}\left(1 - \frac{1}{e}\right)$.

Solution:

We look at the integral $\oint_C \frac{e^{iz}}{z(z^2+1)}\, dz$, where the contour C is given by $C_R + [-R, -\varepsilon] + C_\varepsilon + [\varepsilon, R]$ (see figure below)



We have that

$$\oint_C \frac{e^{iz}}{z(z^2+1)}\, dz = \int_{C_R} \frac{e^{iz}}{z(z^2+1)}\, dz + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x(x^2+1)}\, dx + \int_{C_\epsilon} \frac{e^{iz}}{z(z^2+1)}\, dz + \int_\varepsilon^R \frac{e^{ix}}{x(x^2+1)}\, dx.$$

Now we take the limits

$$\underset{\substack{R\to\infty \\ \varepsilon\to 0}}{\text{limit}} \oint_C = \underset{\substack{R\to\infty \\ \varepsilon\to 0}}{\text{limit}} \left(\int_{C_R} + \int_{-R}^{-\varepsilon} + \int_{C_\epsilon} + \int_\varepsilon^R \right).$$

We see that $\underset{\substack{R\to\infty \\ \varepsilon\to 0}}{\text{limit}} \int_{C_R} \frac{e^{iz}}{z(z^2+1)}\, dz = 0$ by Jordan's lemma, since $f(z) = \frac{1}{z(z^2+1)}$, where the degree of the denominator is higher than the numerator's. Also, the terms $\underset{\substack{R\to\infty \\ \varepsilon\to 0}}{\text{limit}} \left(\int_{-R}^{-\varepsilon} + \int_\varepsilon^R \right)$ combine to $\int_{-\infty}^{\infty}$.

Now we evaluate $\int_{C_\epsilon}$:

We let $z = \varepsilon e^{i\theta}$; $dz = i\varepsilon e^{i\theta}\, d\theta$, then

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{e^{iz}}{z(z^2+1)}\,dz=\lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}(\varepsilon^2 e^{2i\theta}+1)}\,i\varepsilon e^{i\theta}\,d\theta$$
$$=\int_{\pi}^{0}\lim_{\varepsilon\to 0}\frac{e^{i\varepsilon e^{i\theta}}}{(\varepsilon^2 e^{2i\theta}+1)}\,i\,d\theta=\int_{\pi}^{0} i\,d\theta=-\pi\,i.$$

Now we're left with

$$\oint_C\frac{e^{iz}}{z(z^2+1)}\,dz=\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx-\pi i$$

or

$$2\pi i\cdot(\text{sum of residues})+\pi i=\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx\,.$$

The integrand on the left hand side has singularities at $z=0,\ \pm i$. However, only $z=i$ lies inside the enclosed region. Let's find the residue at this (simple) pole:

$$\operatorname{Res}(f(z);\,z=i)=\lim_{z\to i}\left((z-i)\,\frac{e^{iz}}{z(z+i)\,(z-i)}\right)=\lim_{z\to i}\left(\frac{e^{iz}}{z(z+i)}\right)=-\frac{1}{2e}$$

Thus,

$$2\pi i\cdot\left(-\frac{1}{2e}\right)+\pi i=\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx$$
$$\Longrightarrow \pi i\,\frac{e-1}{e}=\int_{-\infty}^{\infty}\frac{\cos x}{x(x^2+1)}\,dx+i\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+1)}\,dx$$
$$\Longrightarrow \int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+1)}\,dx=\pi\,\frac{e-1}{e}$$

and since $\frac{\sin x}{x(x^2+1)}$ is an even function, we have

$$\int_{0}^{\infty}\frac{\sin x}{x(x^2+1)}\,dx=\frac{\pi}{2}\,\frac{e-1}{e}=\frac{\pi}{2}\left(1-\frac{1}{e}\right).$$

Section 4.5

(1) Obtain the Fourier transforms of the following functions:

a) $e^{-|x|}$

Solution:

$$\hat{F}(k)=\int_{-\infty}^{\infty}e^{-ikx}\,e^{-|x|}\,dx=\int_{-\infty}^{0}e^{-ikx}\,e^{x}\,dx+\int_{0}^{\infty}e^{-ikx}\,e^{-x}\,dx$$
$$=\int_{-\infty}^{0}e^{(1-ik)x}\,dx+\int_{0}^{\infty}e^{-(1+ik)x}\,dx$$
$$=\frac{1}{1-ik}\,e^{(1-ik)x}\Big|_{-\infty}^{0}-\frac{1}{1+ik}\,e^{-(1+ik)x}\Big|_{0}^{\infty}$$
$$=\frac{1}{1-ik}-0-0+\frac{1}{1+ik}=\frac{1}{1-ik}+\frac{1}{1+ik}=\frac{1+ik+1-ik}{(1-ik)(1+ik)}=\frac{2}{1+k^2}.$$

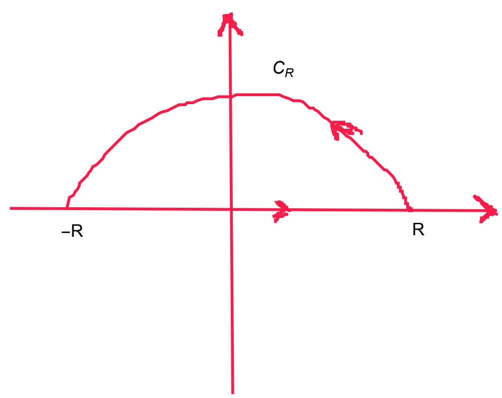
b) $\frac{1}{x^2+a^2}$; $a^2 > 0$

<u>Solution:</u>

We have $\hat{F}(k) = \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{-i k x} \, dx$. As in previous problems we are going to use complex contour integration to solve this real integral. This time however, we need to analyze the integral both when $k > 0$ and $k < 0$ separately, as we need to choose the upper or lower half plane accordingly in order to apply Jordan's lemma.

• Case 1: $k < 0$.

In order for Jordan's lemma to work in this case, we need to use the upper half plane:



From here we have

$$\lim_{R\to\infty} \oint_C \frac{1}{z^2+a^2} e^{-i k z} \, dz = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{x^2+a^2} e^{-i k x} \, dx + \int_{C_R} \frac{1}{z^2+a^2} e^{-i k z} \, dz \right)$$

Now, by Jordan's lemma, the term $\lim_{R\to\infty} \int_{C_R}$ goes to zero since $f(z) = \frac{1}{z^2+a^2}$, where the degree of the denominator exceeds the numerator's.

Thus, we're left with

$$\oint_C \frac{1}{z^2+a^2} e^{-i k z} \, dz = \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{-i k x} \, dx$$

or

$$2\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{-i k x} \, dx \, .$$

The function $\frac{1}{z^2+a^2} e^{-i k z}$ has singularities at $z = \pm a i$, however only $z = a i$ lies inside the region.

Let's find the residue at this (simple) pole:

$$\text{Res}(f(z); z = a i) = \lim_{z\to a i} (z - a i) \frac{e^{-i k z}}{(z+a i)(z-a i)} = \lim_{z\to a i} \frac{e^{-i k z}}{(z+a i)} = \frac{e^{a k}}{2 a i} \, .$$

Then,

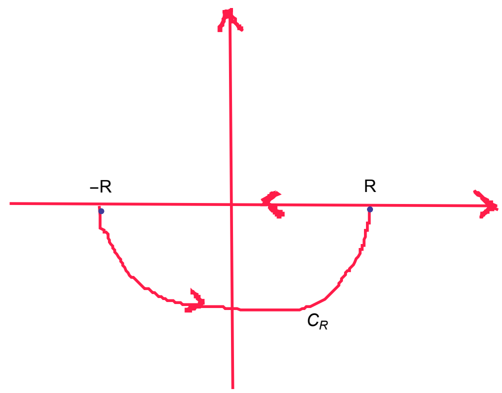
$$2\pi i \cdot \left(\frac{e^{a k}}{2 a i} \right) = \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{-i k x} \, dx$$

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\, e^{-i k x}\, dx = \frac{\pi e^{a k}}{a}.$$

Thus we have that $\hat{F}(k) = \frac{\pi e^{a k}}{a}$, for $k < 0$.

• Case 2: $k > 0$.

In this case, we need to use the lower half plane in order to apply Jordan's lemma:



From here we have

$$\underset{R\to\infty}{\text{limit}} \oint_C \frac{1}{z^2+a^2}\, e^{-i k z}\, dz = \underset{R\to\infty}{\text{limit}} \left(\int_R^{-R} \frac{1}{x^2+a^2}\, e^{-i k x}\, dx + \int_{C_R} \frac{1}{z^2+a^2}\, e^{-i k z}\, dz \right).$$

As before, by Jordan's lemma, the term $\underset{R\to\infty}{\text{limit}} \int_{C_R}$ goes to zero since $f(z) = \frac{1}{z^2+a^2}$, where the degree of the denominator exceeds the numerator's.

Now we're left with

$$\oint_C \frac{1}{z^2+a^2}\, e^{-i k z}\, dz = \int_{\infty}^{-\infty} \frac{1}{x^2+a^2}\, e^{-i k x}\, dz$$

or

$$2\pi i \cdot (\text{sum of residues}) = -\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\, e^{-i k x}\, dx\,.$$

The function $\frac{1}{z^2+a^2}\, e^{-i k z}$ has singularities at $z = \pm a\, i$, however this time it is $z = -a\, i$ the only singularity lying inside the region. Let's find the residue at this (simple) pole:

$$\text{Res}(f(z); z = -a\, i) = \underset{z\to -a i}{\text{limit}}\ (z + a\, i)\, \frac{e^{-i k z}}{(z+a\, i)(z-a\, i)} = \underset{z\to -a i}{\text{limit}}\ \frac{e^{-i k z}}{(z-a\, i)} = \frac{e^{-a k}}{-2\, a\, i}$$

Then,

$$2\pi i \cdot \left(-\frac{e^{-a k}}{2\, a\, i} \right) = -\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\, e^{-i k x}\, dx$$

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\, e^{-i k x}\, dx = \frac{\pi e^{-a k}}{a}.$$

Thus we have that $\hat{F}(k) = \frac{\pi e^{-a k}}{a}$, for $k > 0$.

Hence, we have found that $\hat{F}(k) = \frac{\pi e^{-a |k|}}{a}$, for $k \neq 0$.

Finally, when $k = 0$ we have

$$\hat{F}(0) = \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)\Big|_{-\infty}^{\infty} = \frac{1}{a}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{a}.$$

In conclusion, we have that $\hat{F}(k) = \frac{\pi e^{-a\,|k|}}{a}$, for $k \in \mathbb{R}$.

(2) Obtain the inverse Fourier transform of the following functions:

a) $\frac{1}{k^2+w^2}$; $w^2 > 0$

Solution:

The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\, e^{i x k}\, dk\,.$$

Here we need to consider when $x > 0$ and when $x < 0$, since depending on the sign of x, we choose the upper or lower half plane to apply Jordan's lemma:

• Case 1: $x > 0$

This time we take the upper half plane in order to be able to apply Jordan's lemma.

We have

$$\underset{R\to\infty}{\text{limit}} \oint_C \frac{1}{z^2+w^2}\, e^{i x z}\, dz = \underset{R\to\infty}{\text{limit}} \left(\int_{-R}^{R} \frac{1}{k^2+w^2}\, e^{i x k}\, dk + \int_{C_R} \frac{1}{z^2+w^2}\, e^{i x z}\, dz \right).$$

Here, by Jordan's lemma we have $\underset{R\to\infty}{\text{limit}} \int_{C_R} = 0$, since $f(z) = \frac{1}{z^2+w^2}$, where the degree of the denominator exceeds the numerator's.

So we're left with

$$\oint_C \frac{1}{z^2+w^2}\, e^{i x z}\, dz = \int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\, e^{i x k}\, dk$$

or

$$2\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\, e^{i x k}\, dk\,.$$

We can see that $\frac{e^{i x z}}{z^2+w^2}$ has singularities at $z = \pm w\, i$. However, only $z = w\, i$ lies inside the enclosed region. Let's find the residue at this (simple) pole:

$$\text{Res}(f(z);\, z = w\, i) = \underset{z\to w\, i}{\text{limit}}\, (z - w\, i)\, \frac{e^{i x z}}{(z+w\, i)\,(z-w\, i)} = \underset{z\to w\, i}{\text{limit}}\, \frac{e^{i x z}}{(z+w\, i)} = \frac{e^{-x w}}{2\, w\, i}\,.$$

Hence,

$$2\pi i \cdot \left(\frac{e^{-x w}}{2\, w\, i}\right) = \int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\, e^{i x k}\, dk$$

$$\Longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\, e^{i x k}\, dk = \frac{e^{-x w}}{2\, w} \qquad \text{for } x > 0\,.$$

• Case 2: $x < 0$

This time we take the lower half plane in order to be able to apply Jordan's lemma:

We have

$$\underset{R\to\infty}{\text{limit}} \oint_C \frac{1}{z^2+w^2}\ e^{i x z}\, dz = \underset{R\to\infty}{\text{limit}} \left(\int_R^{-R} \frac{1}{k^2+w^2}\ e^{i x k}\, dk + \int_{C_R} \frac{1}{z^2+w^2}\ e^{i x z}\, dz \right).$$

Here, by Jordan's lemma we have $\underset{R\to\infty}{\text{limit}} \int_{C_R} = 0$, since $f(z) = \frac{1}{z^2+w^2}$, where the degree of the denominator exceeds the numerator's.

So we're left with

$$\oint_C \frac{1}{z^2+w^2}\ e^{i x z}\, dz = \int_\infty^{-\infty} \frac{1}{k^2+w^2}\ e^{i x k}\, dk$$

or

$$2\pi i \cdot (\text{sum of residues}) = -\int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\ e^{i x k}\, dk\,.$$

We can see that $\frac{e^{i x z}}{z^2+w^2}$ has singularities at $z = \pm w\, i$. However, only $z = -w\, i$ lies inside the enclosed region. Let's find the residue at this (simple) pole:

$$\text{Res}(f(z);\, z = -w\, i) = \underset{z\to -w\, i}{\text{limit}}\ (z + w\, i)\ \frac{e^{i x z}}{(z+w\, i)\,(z-w\, i)} = \underset{z\to -w\, i}{\text{limit}}\ \frac{e^{i x z}}{(z-w\, i)} = \frac{e^{x w}}{-2\, w\, i}.$$

Hence,

$$2\pi i \cdot \left(-\frac{e^{x w}}{2\, w\, i}\right) = -\int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\ e^{i x k}\, dk$$

$$\Longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\ e^{i x k}\, dk = \frac{e^{x w}}{2\, w} \qquad \text{for } x < 0\,.$$

For the case when $x = 0$, we have

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k^2+w^2}\ dk = \frac{1}{2\pi}\ \frac{1}{w}\ \arctan \frac{k}{w}\Big|_{-\infty}^{\infty} = \frac{1}{2\pi w}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2\pi w} = \frac{1}{2\, w}.$$

Hence, we've found that the inverse Fourier transform is $f(x) = \frac{e^{-w\,|x|}}{2\, w}$ for $x \in \mathbb{R}$.

b) $\frac{1}{(k^2+w^2)^2}$; $w^2 > 0$

<u>Solution:</u>

• Case 1: $x > 0$

This time we take the upper half plane in order to be able to apply Jordan's lemma:

We have

$$\underset{R\to\infty}{\text{limit}} \oint_C \frac{1}{(z^2+w^2)^2}\ e^{i x z}\, dz = \underset{R\to\infty}{\text{limit}} \left(\int_{-R}^{R} \frac{1}{(k^2+w^2)^2}\ e^{i x k}\, dk + \int_{C_R} \frac{1}{(z^2+w^2)^2}\ e^{i x z}\, dz \right).$$

Here, by Jordan's lemma we have $\underset{R\to\infty}{\text{limit}} \int_{C_R} = 0$, since $f(z) = \frac{1}{(z^2+w^2)^2}$, where the degree of the denominator exceeds the numerator's.

So we're left with

$$\oint_C \frac{1}{(z^2+w^2)^2}\ e^{i\,x\,z}\,dz = \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\ e^{i\,x\,k}\,dk$$

or

$$2\,\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\ e^{i\,x\,k}\,dk\,.$$

We can see that $\frac{e^{i\,x\,z}}{(z^2+w^2)^2}$ has singularities at $z = \pm w\,i$. However, only $z = w\,i$ lies inside the enclosed region. Let's find the residue at this (double) pole:

$$\text{Res}(f(z);\, z = w\,i) = \frac{1}{1!}\,\frac{d}{dz}\left((z - w\,i)^2\,\frac{e^{i\,x\,z}}{(z+w\,i)^2\,(z-w\,i)^2}\right)\Bigg|_{z=w\,i}$$

$$= \frac{d}{dz}\,\frac{e^{i\,x\,z}}{(z+w\,i)^2}\Bigg|_{z=w\,i} = \frac{i\,x\,e^{i\,x\,z}(z+w\,i)^2 - e^{i\,x\,z}\,2\,(z+w\,i)}{(z+w\,i)^4}\Bigg|_{z=w\,i}$$

$$= \frac{e^{i\,x\,z}(z+w\,i)\,((z+w\,i)\,i\,x-2)}{(z+w\,i)^4}\Bigg|_{z=w\,i} = \frac{e^{-x\,w}(-2\,w\,x-2)}{(2\,w\,i)^3}$$

$$= \frac{-2\,e^{-x\,w}(\,w\,x+1)}{-8\,w^3\,i} = \frac{e^{-x\,w}(\,w\,x+1)}{4\,w^3\,i}.$$

Hence,

$$2\,\pi i \cdot \left(\frac{e^{-x\,w}(\,w\,x+1)}{4\,w^3\,i}\right) = \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\ e^{i\,x\,k}\,dk$$

$$\Longrightarrow \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\ e^{i\,x\,k}\,dk = \frac{e^{-x\,w}(\,w\,x+1)}{4\,w^3} \qquad \text{for } x > 0.$$

• Case 2: $x < 0$

This time we take the lower half plane in order to be able to apply Jordan's lemma:

We have

$$\underset{R\to\infty}{\text{limit}} \oint_C \frac{1}{(z^2+w^2)^2}\ e^{i\,x\,z}\,dz = \underset{R\to\infty}{\text{limit}}\left(\int_R^{-R} \frac{1}{\left(k^2+w^2\right)^2}\ e^{i\,x\,k}\,dk + \int_{C_R} \frac{1}{(z^2+w^2)^2}\ e^{i\,x\,z}\,dz\right).$$

Here, by Jordan's lemma we have $\underset{R\to\infty}{\text{limit}} \int_{C_R} = 0$, since $f(z) = \frac{1}{(z^2+w^2)^2}$, where the degree of the denominator exceeds the numerator's.

So we're left with

$$\oint_C \frac{1}{(z^2+w^2)^2}\ e^{i\,x\,z}\,dz = \int_{\infty}^{-\infty} \frac{1}{\left(k^2+w^2\right)^2}\ e^{i\,x\,k}\,dk$$

or

$$2\,\pi i \cdot (\text{sum of residues}) = -\int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\ e^{i\,x\,k}\,dk\,.$$

We can see that $\frac{e^{i\,x\,z}}{(z^2+w^2)^2}$ has singularities at $z = \pm w\,i$. However, only $z = -w\,i$ lies inside the enclosed region. Let's find the residue at this (double) pole:

$$\text{Res}(f(z);\, z = -w\,i) = \frac{1}{1!}\,\frac{d}{dz}\left((z + w\,i)^2\,\frac{e^{i\,x\,z}}{(z+w\,i)^2\,(z-w\,i)^2}\right)\Bigg|_{z=-w\,i}$$

$$= \frac{d}{dz} \frac{e^{i x z}}{(z-w\,i)^2}\Bigg|_{z=-w\,i} = \frac{i\,x\,e^{i x z}(z-w\,i)^2 - e^{i x z}\,2\,(z-w\,i)}{(z-w\,i)^4}\Bigg|_{z=-w\,i}$$

$$= \frac{e^{i x z}(z-w\,i)\,((z-w\,i)\,i\,x-2)}{(z-w\,i)^4}\Bigg|_{z=-w\,i} = \frac{e^{x w}(2\,w\,x-2)}{(-2\,w\,i)^3}$$

$$= \frac{2\,e^{x w}(\,w\,x-1)}{8\,w^3\,i} = \frac{e^{x w}(\,w\,x-1)}{4\,w^3\,i}.$$

Hence,

$$2\,\pi i \cdot \left(\frac{e^{x w}(\,w\,x-1)}{4\,w^3\,i}\right) = -\int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\, e^{i x k}\, dk$$

$$\Longrightarrow \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\, e^{i x k}\, dk = \frac{e^{x w}(\,1-w\,x)}{4\,w^3} \qquad \text{for } x < 0\,.$$

For the case when $x = 0$, we have

$$f(0) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \frac{1}{(k^2+w^2)^2}\, dk\,.$$

We use contour integration to solve this....

$$\lim_{R\to\infty} \oint_C \frac{1}{(z^2+w^2)^2}\, dz = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{\left(k^2+w^2\right)^2}\, dk + \int_{C_R} \frac{1}{(z^2+w^2)^2}\, dz\right)$$

where the term $\lim_{R\to\infty} \int_{C_R} = 0$, since $f(z) = \frac{1}{(z^2+w^2)^2}$, where the degree of the denominator is at least two higher than the numerator's.

Thus we're left with

$$\oint_C \frac{1}{(z^2+w^2)^2}\, dz = \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\, dk$$

or

$$2\,\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\, dk\,.$$

We can see that $\frac{1}{(z^2+w^2)^2}$ has singularities at $z = \pm w\,i$, however only $z = w\,i$ lies inside the enclosed region (we're using the upper half plane). Now let's find the residue at this (double) pole:

$$\text{Res}(f(z);\, z = w\,i) = \frac{1}{1!} \frac{d}{dz} \left((z - w\,i)^2 \frac{1}{(z+w\,i)^2\,(z-w\,i)^2}\right)\Bigg|_{z=w\,i}$$

$$= \frac{d}{dz} \frac{1}{(z+w\,i)^2}\Bigg|_{z=w\,i} = -\frac{2}{(z+w\,i)^3}\Bigg|_{z=w\,i}$$

$$= \frac{-2}{-8\,w^3\,i} = \frac{1}{4\,w^3\,i}.$$

Thus,

$$2\,\pi i \cdot \left(\frac{1}{4\,w^3\,i}\right) = \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\, dk$$

$$\Longrightarrow \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \frac{1}{\left(k^2+w^2\right)^2}\, dk = \frac{1}{4\,w^3}\,.$$

Hence, we've found that the inverse Fourier transform is $f(x) = \frac{e^{-w\,|x|}\,(\,1+w\,|x|\,)}{4\,w^3}$ for $x \in \mathbb{R}$.