MATH 710 HW # 7

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Exercise 1. *Consider the embedded submanifold $S \subset \mathbb{R}^3$ given by*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

Put the induced Riemannian metric and induced Levi-Civita connection on S.

a) Find all the geodesics on S.

b) Find $\mathrm{Isom}^+(S)$, the set of all orientation-preserving isometries of S (i.e., maps from S to itself which are isometries).

c) Of the geodesics in a), which ones come from the orbit of a point under a 1-parameter subgroup of $\mathrm{Isom}^+(S)$?

Solution. Note that the unit cylinder is a surface of revolution whose generating curve is the vertical line parametrized by $\gamma(t) = (1, t)$ for $t \in \mathbb{R}$. Now, for each local parametrization of the cylinder given by $X(\theta, t) = (\cos\theta, \sin\theta, t)$, the induced metric $X^*\bar{g}$ is the Euclidean metric on the (t, θ)-plane:

$$\begin{aligned} X^*\bar{g} &= \mathrm{d}(\cos\theta)^2 + \mathrm{d}(\sin\theta)^2 + \mathrm{d}(t)^2 \\ &= (-\sin\theta\,\mathrm{d}\theta)^2 + (\cos\theta\,\mathrm{d}\theta)^2 + \mathrm{d}t^2 \\ &= (\cos^2\theta + \sin^2\theta)\mathrm{d}\theta^2 + \mathrm{d}t^2 \\ &= \mathrm{d}\theta^2 + \mathrm{d}t^2. \end{aligned}$$

To put it another way, for any point p in the cylinder, a suitable restriction of X gives a Riemannian isometry between an open subset of $(\mathbb{R}^2, \bar{g})$ and a neighborhood of p in the cylinder with its induced metric (thus the induced metric on the cylinder is flat).

Now recall that for any geodesic γ in a given smooth chart (x^k), the geodesic equations must satisfy

$$(\heartsuit) \qquad \frac{\mathrm{d}^2(x^k \circ \gamma)}{\mathrm{d}t^2} + \Gamma^k_{ij}\,\frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}t}\,\frac{\mathrm{d}(x^j \circ \gamma)}{\mathrm{d}t} = 0,$$

where the Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\partial_i\, g_{jm} + \partial_j\, g_{im} - \partial_m\, g_{ij}\right).$$

But since the g_{ij}'s are all constant ($g_{11} = g_{22} = 1$ and $g_{12} = g_{21} = 0$), it follows that the Christoffel symbols all vanish and thus all that remains of ($\heartsuit$) is

$$\frac{\mathrm{d}^2(x^k \circ \gamma)}{\mathrm{d}t^2} = 0.$$

From this we get that the geodesics on the cylinder are straight lines, circles, and helices. Intuitively, if two surfaces are tangent along a curve that is a geodesic on one of them, then this curve will be a geodesic on the other. The reason is that at each point along the curve the normal to the curve coincides with the surfaces' normals. For an illustration of this fact see Figure 1 below. A straight plane strip of paper with a median line drawn on it is placed in different positions on a cylinder.

One can see from this example that a geodesic line on a cylinder can be one of three things: a straight line, a circle or a helix.

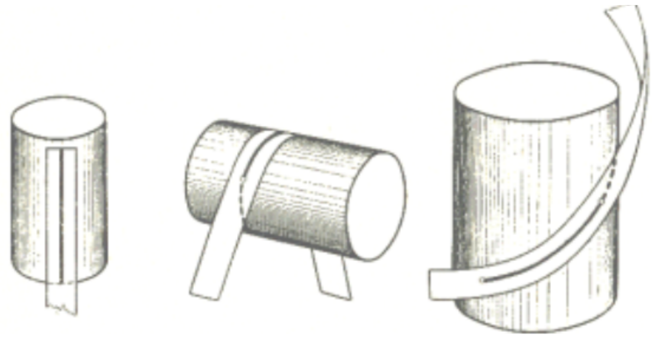

FIGURE 1. Geodesics on a cylinder.

Now for part b), using the local isometric relation between S and $\mathbb{R}^2$ described in part a), we are going to determine first $\mathrm{Isom}(\mathbb{R}^2)$. In fact, in the more general case considering $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric $\bar{g}$, we have the following two facts:

Fact 1 Suppose $U, V \subseteq \mathbb{R}^n$ are connected open sets. Let $\varphi, \psi \colon U \to V$ be Riemannian isometries that for some $p \in U$ satisfy $\varphi(p) = \psi(p)$ and $\mathrm{d}\varphi_p = \mathrm{d}\psi_p$. Then $\varphi = \psi$. To see why this holds, consider $\varphi \circ \psi^{-1} \colon V \to V$, it suffices to show that for any connected open set $U \subseteq \mathbb{R}^n$ and any Riemannian isometry $f \colon U \to U$ satisfying $f(p) = p$ and $\mathrm{d}f_p = \mathrm{Id}_{T_pU}$ for some $p \in U$, we have $f = \mathrm{Id}_U$. If $\gamma \colon I \to U$ is a smooth curve, then $L_{\bar{g}}(\gamma) = L_{\bar{g}}(f \circ \gamma)$, and so by a previous result we have that f takes straight line segments to straight line segments (of the same length). Now let

$$E = \{x \in U \mid f(x) = x, \mathrm{d}f_x = \mathrm{Id}_{T_xU}\};$$

then E is nonempty. Let $x \in E$ and let $B = B_r(x)$ be an open ball of radius r around x such that $\overline{B} \subseteq U$. For any $v \in \mathbb{S}^{n-1}$, let $\gamma \colon [-1, 1] \to U$ be the curve $t \mapsto x + trv$ and let $L = \gamma([-1, 1])$. Then $f \circ \gamma$ is a straight line segment of length $2r$ such that $(f \circ \gamma)(0) = x$. But $\mathrm{d}(f \circ \gamma)_0 = \mathrm{d}f_x \circ \mathrm{d}\gamma_0 = \mathrm{d}\gamma_0$, so $(f \circ \gamma)([-1, 1]) = L$. Using the fact that

$$\mathrm{d}_{\bar{g}}(x, y) = \mathrm{d}_{\bar{g}}(f(x), f(y)) = \mathrm{d}_{\bar{g}}(x, f(y)) \qquad \text{for all } y \in U,$$

we conclude that $f \circ \gamma = \gamma$. But v was arbitrary, so $f|_{\overline{B}}$ is the identity map on $\overline{B}(x)$. This shows that $B \subseteq E$, and therefore E is open. It is clear from continuity that E is also closed. Since U is connected, we must have $E = U$. This proves Fact 1.

Fact 2 The set of maps from $\mathbb{R}^n$ to itself given by the action of the Euclidean group[1] $E(n)$ on $\mathbb{R}^n$ is the full group of Riemannian isometries of $(\mathbb{R}^n, \bar{g})$. Let $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ be a Riemannian isometry. Then $\widetilde{\varphi}(x) = D\varphi(0)^{-1}(\varphi(x) - \varphi(0))$ defines a Riemannian isometry satisfying $D\widetilde{\varphi}(0) = I$ and $\widetilde{\varphi}(0) = 0$; thus Fact 1 shows that $\widetilde{\varphi} = \mathrm{Id}_{\mathbb{R}^n}$. Therefore $\varphi(x) = \varphi(0) + D\varphi(0)x$ is an affine transformation.

[1] If we consider $\mathbb{R}^n$ as a Lie group under addition, then the natural action of $O(n)$ on $\mathbb{R}^n$ is an action by automorphisms. The resulting semidirect product $E(n) = \mathbb{R}^n \rtimes O(n)$ is called the ***Euclidean group***; its multiplication is given by $(b, A)(b', A') = (b + Ab', AA')$. It acts on $\mathbb{R}^n$ via

$$(b, A) \cdot x = b + Ax.$$

This action preserves lines, distances, and angle measures, and thus all of the relationships of Euclidean geometry.

From this we get the induced isometries of S, which are products of reflections, and the the orientation-preserving isometries $\mathrm{Isom}^+(S)$, which are products of an even number of reflections.

Now for part c), note that we can describe the cylinder S as the quotient space (or "orbit" space) $\mathbb{R}^2/\Gamma$, where Γ is the group of integer horizontal translations of $\mathbb{R}^2$. In other words, a point $P \in S = \mathbb{R}^2/\Gamma$ is seen as a set of the form $\{(x+n, y) \mid n \in \mathbb{Z}\}$, called the Γ-***orbit*** of (x, y). (It is natural to use the abbreviation ΓP for the Γ-orbit of the point P because $\Gamma P = \{gP \mid g \in \Gamma\}$.) It follows that of the geodesics in a), the ones come from the orbit of a point under a 1-parameter subgroup of $\mathrm{Isom}^+(S)$ are the circles. □

Exercise 2 (Exercise 3-7 [DoCarmo] (Geodesic Frame))**.** *Let M be a Riemannian n-manifold and let $p \in M$. Show that there exists a neighborhood $U \subset M$ of p and n vector fields $E_1, \ldots, E_n \in \mathfrak{X}(U)$, orthonormal at each point of U, such that, at p, $\nabla_{E_i} E_j(p) = 0$. (Such a family E_i, for $i = 1, \ldots, n$ of vector fields is called a **(local) geodesic frame** at p.)*

Proof. Let $U = \exp_p(B_\varepsilon(0))$ be a normal neighborhood of p small enough, and let $(e_i)_{i=1}^n$ be an orthonormal basis of T_pM. For any $q \in U$, let γ be the radial geodesic joining p to q. Using parallel transport, we get $E_i \in \mathfrak{X}(U)$ for $i = 1, \ldots, n$, defined by

$$E_i(q) = P_{\gamma,p,q}(e_i).$$

Note that

- E_i is orthonormal, since parallel transport preserves the inner product;
- $\nabla_{E_i} E_j(p) = 0$, since $\nabla_v E_i = 0$ for all $v \in T_pM$. □