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Homework Assignment 6
Mathematical Modeling I

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1 The Muskingum Model of Flood Routing

The *Muskingum model* of flooding was introduced by G. T. McCarthy, presumably to address the problem of flooding on the Muskingum River (see Fig. 1). The Muskingum is a river in Ohio, a tributary of the Ohio River, into which it flows at Marietta.) Danacova and Szolgay present a modern account of the model along with a generalization of it and an application to flooding on the Danube.

Figure 1: Lowell, Ohio, is a town on the Muskingum River. This is a photo of Main Street during the 1907 flood.

The basic premise of the model is that any segment of a river has an inflow rate, that may vary with time, $Q^{\mathrm{in}}(t)$, an outflow rate that may also vary with time, $Q^{\mathrm{out}}(t)$, and a volume of water, $S(t)$, stored in it at any time. (This function, $S(t)$, is called the *storage* in the context of the Muskingum model.) Conservation of mass requires, of course, that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = Q^{\mathrm{in}}(t) - Q^{\mathrm{out}}(t). \tag{1}$$

The inflow rate is regarded as specified; it's a known external function (from the point of view, at least, of our particular segment of the river). The storage is then related to the inflow rate and the outflow rate by what we might call an equation of state,

$$S(t) = K[XQ^{\mathrm{in}}(t) + (1-X)Q^{\mathrm{out}}(t)]. \tag{2}$$

Here, K and X are parameters that are used to fit data for particular segments of particular rivers (generally X is constrained to $0 \le X \le 0.5$). By solving Eq. (2) for $Q^{\mathrm{out}}(t)$ and substituting that expression into Eq. (1), we obtain an ODE for $S(t)$:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{1-X}Q^{\mathrm{in}}(t) - \frac{1}{K(1-X)}S(t). \tag{3}$$

This should remind us of the *Windkessel Model*.

1.1 First Question

Please explain the analogy between the two models, and discuss whether it's reasonable that the two applications should have mathematically-identical models.

Solution. Indeed, there are clear similarities between the two models. In both cases the volume change is given –via conservation of mass– as the difference between inflow and outflow. It does make sense that the mathematics of these two models are similarly formulated since, in principle, they pose the same problem: the analogous of the heart from the Windkessel model in this case is the (section of the) river source, while the analogous to the artery's downstream node is the river's mouth. □

As frequently happens with models like this, as practitioners worked with the Muskingum model they found that the equation of state given in Eq. (2), with its two parameters, did a poor job fitting certain data, so they introduced a more general form with three parameters

$$S(t) = K[XQ^{\mathrm{in}}(t) + (1-X)Q^{\mathrm{out}}(t)]^m. \tag{4}$$

Of course, if $m = 1$ then this expression reduces to Eq. (2), so this new equation of state generalizes the old one. This means that any data that we could fit with the old data we can still fit with the new data –by setting m to 1– and the new parameter allows us to fit data better.

1.2 Second Question

Write down the analog of the ODE in Eq. (3) associated with the new equation of state, Eq. (4). You needn't solve the new ODE, but please comment on how you would solve it, and in particular, compare the problem of solving the new ODE with the problem of solving Eq. (3).

Solution. Using the new equation of state, Eq. (4), we solve for $Q^{\mathrm{out}}(t)$ as follows:

$$Q^{\mathrm{out}}(t) = \frac{\sqrt[m]{\frac{S}{K}} - XQ^{\mathrm{in}}(t)}{1-X}. \tag{5}$$

Plugging back into Eq. (1), we obtain a new ODE for $S(t)$:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= Q^{\mathrm{in}}(t) - \frac{\sqrt[m]{\frac{S}{K}} - XQ^{\mathrm{in}}(t)}{1-X} \\ &= (1-X)Q^{\mathrm{in}}(t) - \frac{\sqrt[m]{S}}{(1-X)\sqrt[m]{K}} - \frac{XQ^{\mathrm{in}}(t)}{1-X} \\ &= \frac{1}{1-X}Q^{\mathrm{in}}(t) - \frac{1}{\sqrt[m]{K}(1-X)}\sqrt[m]{S}. \end{aligned} \tag{6}$$

Personally, my approach to solve Eq. (6) will be identical to Eq. (3) because I would solve them both numerically. That being said, Eq. (3) seems to be doable analytically, by using separation of variables and an integrating factor. As for Eq. (6), on the other hand, these simple techniques would not work, and unless there is some ODE trick out there that I don't know of (which is very possible, given that my knowledge of closed-form solutions is admittedly quite limited), I reckon a numerical approach is required for this case. □

1.3 Third Question

In the Windkessel Model, the analog of Eq. (2) was an equation that related the volume in an elastic artery and the pressure in that artery. Did you find the argument that led to that linear relationship between volume and pressure convincing? Can you give a convincing argument for Eq. (2)? For a certain value of X, Eq. (2) has exactly the same form as the pressure-volume relation in the Windkessel Model. Can you give a convincing argument for Eq. (2) in that case? How about the equation of state in Eq. (4)? Can you give a good argument for it?

Solution. In the Windkessel Model, the equation that relates the volume in an elastic artery and the pressure in that artery is the linear relation

$$P(t) = KV(t). \tag{7}$$

The argument that an artery is an elastic reservoir that expands when the pressure inside it is high seems quite reasonable. On the other hand, a river doesn't quite have that "elastic" nature that an artery features. Hence Eq. (2) must instead relate the volume of water in a river to the incoming water at the river's source and the outgoing water at the river's mouth. Consider the extreme case when $X = 0$; in that case there is no water being sourced to the river, but the river does continue to run downstream whatever volume of water it has left; so the entire volume of water ends up at the river's mouth and the river runs dry. On the complete opposite end of the spectrum, if $X = 1$, there's no water exiting the river downstream or elsewhere. This is an unphysical scenario, because even if the river's mouth was somehow blocked, the river will still inundate and spill over along its bank, and this scenario is not covered in Eq. (2), which will instead simplify to $S(t) = KQ^{\mathrm{in}}(t)$ (an equation identical to relation (7)). Hence, for this particular case the model would fail; except it doesn't because, as we alluded to above, the parameter X in this model is constrained to $X \in [0, 0.5]$. All these arguments also apply to Eq. (4). The difference now is that, by having the extra parameter m, the model does offer more flexibility (i.e., the equation of state becomes "less linear"... As stated above, the introduction of this parameter was born out of necessity for data fitting). □