MATH 709 HW # 4

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Problem 1 (Problem 3-6)**.** *Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z \colon \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = (e^{it}z^1, e^{it}z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.*

Proof. First we show that γ_z is smooth. To this end, we are going to use the stereographic projection $\sigma \colon \mathbb{S}^3 \smallsetminus \{(0,0,0,1)\} \to \mathbb{R}^3$ and its inverse σ^{-1} given by

$$\sigma(x^1, x^2, x^3, x^4) = \frac{(x^1, x^2, x^3)}{1 - x^4},$$
$$\sigma^{-1}(x^1, x^2, x^3) = \frac{(2x^1, 2x^2, 2x^3, |x|^2 - 1)}{|x|^2 + 1} \qquad \text{(where } |x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2\text{)}.$$

We also use the stereographic projection on the south pole $\widetilde{\sigma} \colon \mathbb{S}^3 \smallsetminus \{(0,0,0,-1)\} \to \mathbb{R}^3$ given by $\widetilde{\sigma}(\mathbf{x}) = -\sigma(-\mathbf{x})$, and its inverse $\widetilde{\sigma}^{-1}(\mathbf{u}) = -\sigma^{-1}(-\mathbf{u})$. Our goal is to show that γ_z is indeed smooth by showing that $\sigma \circ \gamma_z$, $\widetilde{\sigma} \circ \gamma_z$, $(\sigma \circ \gamma_z)^{-1}$, and $(\widetilde{\sigma} \circ \gamma_z)^{-1}$ are all smooth.

First let us see what $\gamma_z(t)$ looks like in $\mathbb{R}^4$. Writing $z^1 = x^1 + iy^1$ and $e^{it} = \cos t + i \sin t$, note that

$$\begin{aligned} e^{it}z^1 &= (x^1 + iy^1)(\cos t + i \sin t) \\ &= x^1 \cos t - y^1 \sin t + i(x^1 \sin t + y^1 \cos t) \\ &\leftrightarrow (x^1 \cos t - y^1 \sin t, x^1 \sin t + y^1 \cos t) \in \mathbb{R}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \gamma_z(t) &= (e^{it}z^1, e^{it}z^2) \\ &\leftrightarrow (x^1 \cos t - y^1 \sin t, x^1 \sin t + y^1 \cos t, x^2 \cos t - y^2 \sin t, x^2 \sin t + y^2 \cos t) \in \mathbb{R}^4. \end{aligned}$$

Now we have everything required to compute:

$$\sigma \circ \gamma_z(t) = \frac{(x^1 \cos t - y^1 \sin t, x^1 \sin t + y^1 \cos t, x^2 \cos t - y^2 \sin t)}{1 - x^2 \sin t - y^2 \cos t},$$
$$\widetilde{\sigma} \circ \gamma_z(t) = -\frac{(x^1 \cos t + y^1 \sin t, y^1 \cos t - x^1 \sin t, x^2 \cos t + y^2 \sin t)}{1 + x^2 \sin t - y^2 \cos t}.$$

Also,

$$(\sigma \circ \gamma_z)^{-1}(x) = \gamma_z^{-1} \circ \sigma^{-1}\left(\left(\frac{x^1 \cos t - y^1 \sin t}{1 - x^2 \sin t - y^2 \cos t}, \frac{x^1 \sin t + y^1 \cos t}{1 - x^2 \sin t - y^2 \cos t}, \frac{x^2 \cos t - y^2 \sin t)}{1 - x^2 \sin t - y^2 \cos t}\right)\right)$$
$$(\widetilde{\sigma} \circ \gamma_z)^{-1}(x) = \gamma_z^{-1} \circ \widetilde{\sigma}^{-1}\left(\left(\frac{x^1 \cos t + y^1 \sin t}{y^2 \cos t - x^2 \sin t - 1}, \frac{y^1 \cos t - x^1 \sin t}{y^2 \cos t - x^2 \sin t - 1}, \frac{x^2 \cos t + y^2 \sin t)}{y^2 \cos t - x^2 \sin t - 1}\right)\right).$$

These are all rational smooth functions, thus we must have that γ_z is smooth, as desired.

Finally, to show that its velocity is never zero, note that by a straight computation we get

$$\sigma \circ \gamma_z'(t) = \frac{(x^1x^2 + y^1y^2 - x^1 \sin t - y^1 \cos t, x^2y^1 - x^1y^2 + x^1 \cos t - y^1 \sin t, (x^2)^2 + (y^2)^2 - y^2 \cos t - x^2 \sin t)}{(x^2 \sin t + y^2 \cos t - 1)^2}$$

$$\widetilde{\sigma} \circ \gamma_z'(t) = \dots$$

$$\dots = \frac{(-x^1x^2 - y^1y^2 - x^1 \sin t - y^1 \cos t, -x^2y^1 + x^1y^2 + x^1 \cos t - y^1 \sin t, -(x^2)^2 - (y^2)^2 - y^2 \cos t - x^2 \sin t)}{(x^2 \sin t + y^2 \cos t + 1)^2}.$$

Since the numerator can never equal zero on either expression, we conclude that γ_z has nonzero velocity for all $t \in \mathbb{R}$. □

Problem 2 (Exercise 4.4)**.** *Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.*

Proof. By a previous proposition (Proposition 3.6, Pg 55), we know that if $p \in M$, where M, N, and S are smooth manifolds, and $F: M \to N$ and $G: N \to S$ are smooth maps, then,

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}S.$$

It follows immediately from this statement that compositions of smooth immersions (or submersions) are smooth immersions (or submersions).

To see that a composition of maps of constant rank need not have constant rank, define for instance $f: \mathbb{R} \to \mathbb{R}^2$ and $g: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = (1, x) \qquad \text{and} \qquad g(x, y) = x + y^2.$$

Then we have

$$Df(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad Dg(x, y) = \begin{bmatrix} 1 & 2y \end{bmatrix},$$

which do have constant rank. However

$$D(g \circ f)(x) = 2x$$

depends on the point x, hence it is not of constant rank. □

Problem 3 (Problem 4-5)**.** *Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space. Then,*

a) Show that the quotient map $\pi: \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.

b) Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

Preliminaries for the proof of a). [1] A point in $\mathbb{C}^{n+1}$ is be represented by an $(n+1)$-tuple $(z^1, \dots, z^{n+1})$, not all 0. Two such points belong to the same line if and only if each $(n+1)$-tuple can be obtained from the other by multiplying each coordinate by a fixed (nonzero) scalar λ, i.e.

$$(z^1, \dots, z^{n+1}) \sim (\lambda z^1, \dots, \lambda z^{n+1})$$

This gives us an equivalence class $[z^1, \dots, z^{n+1}]$, so that we have $\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim$. Now we construct the charts. Let $\{U_i\}$ be the set of equivalence classes as above with $z^i \neq 0$. This

[1] Before proceeding to prove the statement, I will present some background on the construction of the smooth structure of $\mathbb{CP}^n$. The grader may skip to the actual proof of part a), which is presented below.

condition is independent of the choice of a representative. Each equivalence class in $\{U_i\}$ has a unique representative with $z^i = 1$, and this gives a bijection $\varphi_i \colon U_i \to \mathbb{C}^n$ for $i = 0, 1, \dots, n$:

$$\varphi_i([z^1, \dots, z^{n+1}]) = \left(\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i}\right),$$

with inverse

$$\varphi_i^{-1}(w^1, \dots, w^n) = [w^1, \dots, w^{i-1}, 1, w^i, \dots, w^n].$$

Then we have that the collection $\{(U_i, \varphi_i)\}_{i=1}^{n+1}$ is an atlas, as can be checked. $\square$

Proof of a). Equipped with the machinery from above, let $z = (z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \smallsetminus \{0\}$. Note that the map π sends z to the line it spans, i.e., $\pi(z) = [z] \in \mathbb{CP}^n$. Let $\widetilde{U}_{n+1} \subseteq \mathbb{C}^{n+1} \smallsetminus \{0\}$ be the set where $z^{n+1} \neq 0$ and let $U_{n+1} = \pi(\widetilde{U}_{n+1}) \subseteq \mathbb{CP}^n$. Then we have a coordinate map $\varphi_{n+1} \colon U_{n+1} \to \mathbb{C}^n$ given by

$$\varphi_{n+1}([z^1, \dots, z^{n+1}]) = \left(\frac{z^1}{z^{n+1}}, \dots, \frac{z^n}{z^{n+1}}\right).$$

Then a straight computation gives us

$$(\varphi_{n+1} \circ \pi)'(z^1, \dots, z^{n+1}) = \begin{bmatrix} (z^{n+1})^{-1} & 0 & \dots & 0 & -z^1(z^{n+1})^{-2} \\ 0 & (z^{n+1})^{-1} & \dots & 0 & -z^2(z^{n+1})^{-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (z^{n+1})^{-1} & -z^n(z^{n+1})^{-2} \end{bmatrix}.$$

This shows that the differential is surjective (note that the first n columns form an invertible matrix). Hence the quotient map $\pi \colon \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion, as desired. $\square$

Proof of b). By part a), we already know that the map $\pi \colon \mathbb{C}^2 \smallsetminus \{0\} \to \mathbb{CP}^1$ is a surjective smooth submersion. Now let

$$\widetilde{z}^1 = \frac{z^1}{\|(z^1, z^2)\|} \qquad \text{and} \qquad \widetilde{z}^2 = \frac{z^2}{\|(z^1, z^2)\|},$$

and define a map $\psi \colon \mathbb{C}^2 \smallsetminus \{0\} \to \mathbb{S}^2$ by

$$\psi(z^1, z^2) = \left(\widetilde{z}^1\overline{\widetilde{z}^2} + \overline{\widetilde{z}^1}\widetilde{z}^2, -i(\widetilde{z}^1\overline{\widetilde{z}^2} - \overline{\widetilde{z}^1}\widetilde{z}^2), \widetilde{z}^1\overline{\widetilde{z}^1} - \widetilde{z}^2\overline{\widetilde{z}^2}\right),$$

Then, by a previous theorem (Theorem 4.30, Pg 90), ψ descends to a unique smooth map $\widetilde{\psi} \colon \mathbb{CP}^1 \to \mathbb{S}^2$ satisfying $\widetilde{\psi} \circ \pi = \psi$. This map is the desired diffeomorphism. $\square$