$$\mathfrak{sl}_2$$

The aim of this workshop is to understand explicitly some of the representation theory of $\mathfrak{sl}_2$, since it turns out (see Chapter 10) that understanding this allows us to deduce many stronger results. Since $\mathfrak{sl}_2$ is only a three-dimensional algebra, how hard can its representation theory be?

Recall that $\mathfrak{sl}_2$ has basis $\{e, f, h\}$ subject to the relations

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

"Recall" (from a future lecture!) the following notions:
- A **representation** of a Lie algebra L is Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$ for some finite-dimensional complex vector space V. We also say that V is an L-**module**.

- A subspace $W \subseteq V$ is said to be a **submodule** (or a **subrepresentation**) if $\rho(\ell)w \in W$ for all $w \in W$ and $\ell \in L$.

- An L-module V is said to be **simple** if it has no proper submodules: so that if $W \subseteq V$ is a submodule, then either $W = 0$ or $W = V$.

Thus a representation of $\mathfrak{sl}_2$ is just a (finite-dimensional, complex) vector space V together with three linear maps $E, F, H \colon V \to V$ satisfying

$$EF - FE = H$$
$$HE - EH = 2E$$
$$HF - FH = -2F.$$

We will call the eigenvalues of $H \colon V \to V$ the **weights** of V. Further, if $a \in \mathbb{C}$ is a weight, then

$$V_a := \{v \in V \mid Hv = av\}$$

is called the **weight space of weight** a. Note that $0 \in V_a$.

We will see in the next couple of lectures (as a summary of Chapter 8 in the book) that simple representations of $\mathfrak{sl}_2$ are given by some explicit countable list. This is really nice, but their definition in the book is completely unmotivated, and it looks like magic! In this workshop, we will construct all simple representations of $\mathfrak{sl}_2$ using linear algebra; the construction should be natural and beautiful.

Throughout, we suppose that $V \neq 0$ is a finite dimensional representation of $\mathfrak{sl}_2$, we consider the linear maps E, F, H as above, and let $a \in \mathbb{C}$ be a weight.

1. Show that if $v \in V_a$, then $Ev \in V_{a+2}$, $Fv \in V_{a-2}$ and $Hv \in V_a$.

Solution: Throughout, let $v \in V_a$. Clearly $Hv = av \in V_a$. Further by the relations in the top blue box,
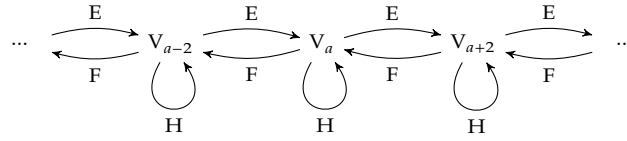
$$HEv = (EH + 2E)v = (a + 2)Ev,$$

which shows that $Ev \in V_{a+2}$. In a similar way

$$HFv = (FH - 2F)v = (a - 2)Fv,$$

which proves that $Fv \in V_{a-2}$.

In terms of maps, this means $E\colon V_a \to V_{a+2}$, $F\colon V_a \to V_{a-2}$ and $H\colon V_a \to V_a$. We summarise this in the diagram

2. Prove that there exists a weight $a \in \mathbb{C}$ such that $Ev = 0$ for all $v \in V_a$ (visually: far enough to the right, keep on applying E and we get zero).

> **Solution:** Since $H\colon V \to V$ has an eigenvalue (b say), there is some non-zero V_b. Further since $H\colon V \to V$ has only finitely many eigenvalues, V_{b+2i} must be zero for $i \geq 1$ sufficiently large. Combining these two facts, it follows that there must be some $j \in \mathbb{N}$ such that $V_{b+2j} \neq 0$ but $V_{b+2(j+1)} = 0$. Set $a := b + 2j$, then $Ev = 0$ for all $v \in V_a = V_{b+2j}$ by Q1.

Denote the complex number a from Q2 by m, and call any non-zero element $v \in V_m$ a **highest weight vector**. Now choose a highest weight vector, and denote it w_0. For each $i \geq 1$, define

$$w_i := \frac{1}{i!} F^i w_0.$$

3. By induction or otherwise, show that for all $i \geq 1$

   (a) $Fw_i = (i+1)w_{i+1}$.

   > **Solution:** $Fw_i = F(\frac{1}{i!}F^i w_0) = \frac{1}{i!}F^{i+1}w_0 = (i+1)\frac{1}{i+1!}F^{i+1}w_0 = (i+1)w_{i+1}$.

   (b) $Hw_i = (m - 2i)w_i$.

   > **Solution:** By Q1 (with now $a = m$), $F^i w_0 \in V_{m-2i}$. It follows that $w_i = \frac{1}{i!}F^i w_0 \in V_{m-2i}$ and so by definition, $Hw_i = (m - 2i)w_i$.

   (c) $Ew_i = (m - i + 1)w_{i-1}$.

   > **Solution:** We use induction. The base case is
   >
   > $$Ew_1 = EFw_0 = Hw_0 + FEw_0 = mw_0 + F0 = mw_0.$$
   >
   > Hence, assuming the result is true for some i, it follows that
   >
   > $$\begin{aligned}
   > (1 + i)Ew_{i+1} &= EFw_i \\
   > &= Hw_i + FEw_i \\
   > &= (m - 2i)w_i + (m - i + 1)Fw_{i-1} \\
   > &= (m - 2i)w_i + (m - i + 1)iw_i \\
   > &= (m - 2i + mi - i^2 + i)w_i \\
   > &= (1 + i)(m - i)w_i,
   > \end{aligned}$$
   >
   > and so $Ew_{i+1} = (m - i)w_i$, proving the induction step.

4. Observe that $w_i \in V_{m-2i}$, and further (by adapting Q2) prove that there exists some $j \in \mathbb{N}$ such that $w_j \neq 0$ but $w_{j+1} = 0$.

> **Solution:** The fact that $w_i \in V_{m-2i}$ has already been observed in the proof of Q3(b) above. Further, by exactly the same logic as in the proof of Q2 above, there exists $j \in \mathbb{N}$ such that $V_{m-2j} \neq 0$ but $V_{m-2(j+1)} = 0$. Thus some $w_{j+1} = 0$, and so taking the smallest one gives the result.

5. By combining your answers to Q3 and Q4, deduce that $m = j \in \mathbb{N}$. (This shows that V has only integer weights.)

> **Solution:** First observe that $0 = Ew_{j+1}$ since $w_{j+1} = 0$ by Q4. On the other hand, by Q3(c), we have $Ew_{j+1} = (m-j)w_j$. Combining, since $w_j \neq 0$ by Q4, it follows that $m - j = 0$, i.e. $m = j$. In particular m (which a priori is a complex number) is actually an integer.

Thus, inside any $\mathfrak{sl}_2$-module V, we can find vectors

$$\{w_0, \ldots, w_m\},$$

as above.

6. Prove that $w_0, \ldots, w_m$ are linearly independent, and hence

$$V(m) := \mathrm{Span}\{w_0, \ldots, w_m\}$$

is an $m + 1$ dimensional subspace of V.

> **Solution:** The fact that the w_i are linearly independent follows since they are all eigenvalues for $H \colon V \to V$ corresponding to different eigenvalues. The rest follows.

7. Using the formulae in Q3, prove that $V(m)$ is a subrepresentation (i.e., a submodule) of V, and further each $V(m)$ is simple.

> **Solution:** To show that $V(m)$ is a submodule, since it is by definition a subspace of V, we just need to show that it is closed under E, F and G, i.e. we need to show that $Ew, Fw, Hw \in V(m)$ for all $w \in V(m)$. But this holds by inspecting the formulae in Q3. To see that $V(m)$ is simple, consider a nonzero submodule W of $V(m)$. Being non-zero, it must contain some nonzero element
> $$w = \lambda_0 w_0 + \ldots + \lambda_n w_n$$
> say. Let j be maximal such that $\lambda_j \neq 0$. Using the formulae in Q3 $E^j w$ is clearly a non-zero multiple of w_0, and hence $w_0 \in W$. But then repeatedly applying F we see that $w_i \in W$ for all i, which forces $V = V(m)$.

8. Prove that if V is itself simple, then $V = V(m)$. This shows that every simple $\mathfrak{sl}_2$-module is isomorphic to one of the form $V(m)$.

> **Solution:** Suppose V is simple. All the above questions culminate in Q7, which shows that V contains a nonzero submodule $V(m)$. Since V is simple, $V = V(m)$ follows.

Please hand in your solution to Q6, Q7 and Q8 by the start of lecture on Monday 23 October.