MATH 710 HW # 8

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Exercise 1. *Consider the Riemannian manifold* $(\mathbb{R}^2, \bar{g})$*, where* $\bar{g}$ *is the usual Euclidean metric.*

a) Show that the positive y*-axis is a geodesic.*

b) Identify $\mathbb{R}^2$ *with* $\mathbb{C}$*. Then verify that the Möbius transformations are isometries of the upper-half plane* $\mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z > 0\}$*.*

c) Show that the geodesics in $\mathbb{H}$ *are exactly the vertical lines or semicircles that are perpendicular to* $\partial\mathbb{H}$*.*

Solution of a). See part c). $\square$

Solution of b). We use the Riemannian metric $\langle , \rangle = 1/y^2(\mathrm{d}x^2 + \mathrm{d}y^2)$. This means that for a curve $\gamma\colon [0,1] \to \mathbb{H}$, with $\gamma(t) = x(t) + iy(t)$ for x, y real valued, the length of γ is given by

$$\ell(\gamma) = \int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t.$$

Next we observe that Möbius transformations are defined as an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$:

$$\text{Let } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \quad \text{and } z \in \mathbb{H}. \qquad \text{Then, } g.z = \frac{az+b}{cz+d}.$$

The proof that this is a well-defined action follows from the fact that

$$\mathfrak{Im}(g.z) = \frac{\mathfrak{Im}\, z}{|cz+d|^2}.$$

We want to prove that this action is isometric with respect to the metric. Here the distance between two points $z, w \in \mathbb{H}$ is defined by

$$d(z,w) = \inf_\gamma \ell(\gamma),$$

where the infimum is taken over all curves γ which connect z and w. Note that with this hyperbolic distance between points z, w in $\mathbb{H}$, we have

$$\sinh\left(\frac{1}{2}\, d(z,w)\right) = \frac{|z-w|}{2\sqrt{\mathfrak{Im}\, z\, \mathfrak{Im}\, w}}.$$

We wish to prove that for $g \in \mathrm{SL}_2(\mathbb{R})$, we have $d(g.z, g.w) = d(z,w)$ for all $z, w \in \mathbb{H}$. We will verify this by showing that

$$\frac{|z-w|}{2\sqrt{\mathfrak{Im}\, z\, \mathfrak{Im}\, w}} = \frac{|g.z - g.w|}{2\sqrt{\mathfrak{Im}(g.z)\, \mathfrak{Im}(g.w)}}.$$

Our claim follows:

$$\frac{|g.z - g.w|^2}{4\,\mathfrak{Im}(g.z)\,\mathfrak{Im}(g.w)} = \frac{|cz+d|^2\,|cw+d|^2}{4\,\mathfrak{Im}\,z\,\mathfrak{Im}\,w} \cdot \left(\frac{|z-w|}{|cz+d|\,|cw+d|}\right)^2$$
$$= \frac{|z-w|^2}{4\,\mathfrak{Im}\,z\,\mathfrak{Im}\,w}.$$

Taking square roots we have the desired result. □

Solution of c). Let $z, w \in \mathbb{H}$ and suppose for now that $z = ia$ and $w = ib$ with $b > a > 0$. Let $\gamma\colon [0,1] \to \mathbb{H}$ with $\gamma(t) = x(t) + iy(t)$, x and y real valued, and $\gamma(0) = z$, $\gamma(1) = w$. Then,

$$\ell(\gamma) = \int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \geq \int_0^1 \frac{1}{y(t)} \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \int_a^b \frac{1}{s}\,\mathrm{d}s = \log\left(\frac{b}{a}\right).$$

On the other hand, if we take $\widetilde{\gamma}(t) = (1-t)ia + itb$, then we have $x(t) = 0$ and $y(t) = (1-t)a + tb$. Hence $\ell(\widetilde{\gamma}) = \log(b/a)$. So the imaginary axis is the geodesic for any $z, w \in i\mathbb{R}$. (This proves part a).)

In the general case $z, w \in \mathbb{H}$ we find a $g \in \mathrm{SL}_2(\mathbb{R})$ so that $g.z, g.w \in i\mathbb{R}^+$. Then, using the fact that distance is preserved under the action of $\mathrm{SL}_2(\mathbb{R})$ and that Möbius transformations map Euclidean circles to Euclidean circles we can conclude the truth of the theorem.

First consider $z, w \in \mathbb{H}$ with $\mathfrak{Re}\,z = \mathfrak{Re}\,w$. Then we may take g so that $g.u = u - \mathfrak{Re}\,z$. Otherwise, consider the circle C through z and w with center on the real axis. The circle will intersect the real axis at two points t_1 and t_2. Take $g \in \mathrm{SL}_2(\mathbb{R})$ with

$$g.u = \frac{1}{t_2 - t_1} - \frac{1}{u - t_1} \quad \text{for } u \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

Then g maps $t_1 \mapsto \infty$ and $t_2 \mapsto 0$. Since g maps circles to circles, g sends C to a circle through 0 and ∞. Noting that $\mathfrak{Im}(g.u) = \mathfrak{Im}\,u/|u - t_1|^2$, we see that g maps the circle C to the imaginary axis.

Now, applying the results from part b) and the first part of this proof, we conclude that the geodesic through z and w is the circle C. □

Exercise 2. *Show that for $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$, we have $R(X,Y)Z = 0$.*

Solution. Since $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{]X,Y]} Z$, it suffices to show that $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{]X,Y]} Z$ in order to prove the assertion that $R(X,Y)Z = 0$. If the vector field Z is given by $Z = (Z^1, \ldots, Z^n)$, with Z^i being the natural components of Z coming from the coordinates in $\mathbb{R}^n$, then we have

$$\nabla_Y Z = (Y(Z^1), \ldots, Y(Z^n)) \implies \nabla_X \nabla_Y Z = (XY(Z^1), \ldots, XY(Z^n)).$$

Similarly,

$$\nabla_Y \nabla_X Z = (YX(Z^1), \ldots, YX(Z^n)).$$

Thus,

$$\begin{aligned}
\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z &= (XY(Z^1),\dots,XY(Z^n)) - (YX(Z^1),\dots,YX(Z^n))\\
&= ((XY-YX)(Z^1),\dots,(XY-YX)(Z^n))\\
&= ([X,Y]Z^1,\dots,[X,Y]Z^n)\\
&= \nabla_{[X,Y]}(Z^1,\dots,Z^n)\\
&= \nabla_{[X,Y]}Z.
\end{aligned}$$

□

Exercise 3 (Exercise 3-5 [DoCarmo]). *Let M be a Riemannian manifold and $X \in \mathfrak{X}(M)$. Let $p \in M$ and let $U \subset M$ be a neighborhood of p. Let $\varphi\colon (-\varepsilon,\varepsilon)\times U \to M$ be a differentiable mapping such that for any $q \in U$, the curve $t \mapsto \varphi(t,q)$ is a trajectory of X passing through q at $t = 0$ (U and φ are given by the fundamental theorem of ODE, c.f.* Theorem 2.2*). The vector field X is called a **Killing field** (or an **infinitesimal isometry**) if, for each $t_0 \in (-\varepsilon,\varepsilon)$, the mapping $\varphi_{t_0}\colon U \subset M \to M$ is an isometry.*

a) A vector field v on $\mathbb{R}^n$ may be seen as a map $v\colon \mathbb{R}^n \to \mathbb{R}^n$; we say that the field is linear if v is a linear map. Prove that a linear field on $\mathbb{R}^n$, defined by a matrix A, is a Killing field if and only if A is anti-symmetric.

b) Let X be a Killing field on M, $p \in M$, and let U be a normal neighborhood of p on M. Assume that p is a unique point of U that satisfies $X(p) = 0$. Prove that in U, X is tangent to the geodesic spheres centered at p.

c) Let X be a differentiable vector field on M and let $f\colon M \to N$ be an isometry. Let Y be a vector field on N defined by $Y(f(p)) = \mathrm{d}f_p(X(p))$, for $p \in M$. Prove that Y is a Killing field if and only if X is also a Killing vector field.

*d) Prove that X is Killing $\iff$ $\langle\nabla_Y X, Z\rangle + \langle\nabla_Z X, Y\rangle = 0$ for all $Y, Z \in \mathfrak{X}(M)$ (this equation is called the **Killing equation**).*

e) Let X be a Killing field on M with $X(q) \neq 0$, $q \in M$. Prove that there exists a system of coordinates $(x^1,\dots,x^n)$ in a neighborhood of q, so that the coefficients g_{ij} of the metric in this system of coordinates do not depend on x^n.

Note: Before solving the problem, let us clarify the terminology and notation involved: a vector field X on M is a *Killing field* if for every point $p \in M$, the flow of X in a neighborhood of p gives an isometry for every time t. We say that X is a Killing field near p if this holds for the flow in a neighborhood of p. Moreover, if $\varphi\colon (-\varepsilon,\varepsilon)\times U \to M$ is the flow, for convenience of notation we denote by φ_t the map $q \mapsto \varphi(t,q)$, and by φ^q the trajectory $t \mapsto \varphi(t,q)$.

Proof of a). Let X be a linear vector field on $\mathbb{R}^n$, choose any $p \in \mathbb{R}^n$, and let φ be the local flow of X in a neighborhood U of p. This is characterized by the conditions

$$\varphi(0,q) = q \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\varphi^q(t) = X(\varphi^q(t)).$$

Thus we see that if

$$\varphi = (f^1(t,x^1,\dots,x^n),\dots,f^n(t,x^1,\dots,x^n)) \quad \text{for each } (x^1,\dots,x^n),$$

then the f^i are solutions to the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}f^i = \sum_j c_{ij}f^j \quad \text{with initial condition } f^i(0,x^1,\dots,x^n) = (x^1,\dots,x^n),$$

where $A = (c_{ij})$ is the matrix defining X. Thus we see that φ is uniquely determined as $\varphi = e^{At}$. Note that this is globally defined, so we don't need to restrict ourselves to p and U. Moreover, since e^{At} is always invertible, it always gives a diffeomorphism; thus if we can show that the map induced by the matrix e^{At} preserves inner products of tangent vectors for each t, then we would have that X is a Killing field. Furthermore, since this map is linear, its differential is canonically identified with itself, and so this is equivalent to asking that for each t, the linear map induced by e^{At} preserves inner products. Now, given $u, v \in \mathbb{R}^n$, we have

$$\langle u, v\rangle = \langle e^{At}u, e^{At}v\rangle \iff \langle u, v\rangle = \langle u, e^{A^T t}e^{At}v\rangle, \quad \text{where } A^T \text{ is the transpose of } A.$$

Thus, this holds for all u, v if and only if $e^{A^T t}e^{At} = I$, or equivalently, $e^{A^T t} = e^{-At}$. Differentiating with respect to t and then setting $t = 0$, we find this implies $A^T = -A$, while conversely if $A^T = -A$ we certainly have $e^{A^T t} = e^{-At}$. Thus we find that A is antisymmetric if and only if X is a Killing field, as desired. □

Quick digression

We need the following result to prove part b): *If X is a vector field, $p \in M$, and φ is the local flow of X near p, then we have $(\mathrm{d}\varphi_t)_q(X(q)) = X(\varphi^q(t))$. In particular, if $X(p) = 0$, then $\varphi^p(t) = p$ for all t.*

The proof of this is as follows: for all t, q, we know that $\mathrm{d}/\mathrm{d}t\, \varphi^q(t) = X(\varphi^q(t))$. Thus, for a particular t_0, we have

$$\begin{aligned}
(\mathrm{d}\varphi_{t_0})_q(X(q)) &= (\mathrm{d}\varphi_{t_0})_q\left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi^q(t)\Big|_{t=0}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_{t_0} \circ \varphi^q(t)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_{t_0} \circ \varphi(t, q)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(t_0 + t, q)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(t, q)\right)\Big|_{t=t_0} \\
&= X(\varphi^q(t_0)).
\end{aligned}$$

This proves the first statement. If $X(p) = 0$, we conclude that $\mathrm{d}/\mathrm{d}t\, \varphi^p(t) = 0$ for all t, so that $\varphi^p(t) = p$, as desired.

Proof of b). Suppose $X(p) = 0$ and X is a Killing field. Let U be a normal neighborhood of p, and $q \in U$. We claim that φ^q is contained in a geodesic sphere around p. First note that p, q are connected by the radial geodesic segment from p, which is compact. This means that there is a finite collection of neighborhoods $U_1, \ldots, U_m$ on which the local flow is defined for some times $\varepsilon_1, \ldots, \varepsilon_m$. If we set $\varepsilon = \min\{\varepsilon_i\}$, then by the uniqueness of the local flow we find that it is defined on $U := U_1 \cup \cdots \cup U_m$ for time ε. Now since X is a Killing field, this local flow is necessarily an isometry for all times t, and since it fixes p, the trajectory of q must stay at the same distance from p for all times t. Thus the trajectory is contained in a geodesic sphere, where its infinitesimal generator X is tangent, as desired. □

Proof of c). Since f is an isometry, it is in particular a diffeomorphism. If $\varphi\colon (-\varepsilon,\varepsilon)\times U \to M$ is a local flow for X, it is clear that

$$\varphi' := f\circ\varphi\circ(\mathrm{Id}\times f^{-1}|_{f(U)})\colon (-\varepsilon,\varepsilon)\times f(U)\to N$$

is a local flow for Y: Indeed, we have

$$\varphi'(0,q) = f(\varphi(0,f^1(q))) = f(f^{-1})(q) = q$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi'^q(t) = (\mathrm{d}\varphi')_{(t,q)}\left(\frac{\partial}{\partial t}\right) &= (\mathrm{d}f)_{\varphi^{f^{-1}(q)}(t)}\circ(\mathrm{d}\varphi)_{(t,f^{-1}(q))}\circ(\mathrm{d}(\mathrm{Id}\times f^{-1}|_{f(U)}))_{(t,q)}\left(\frac{\partial}{\partial t}\right)\\ &= (\mathrm{d}f)_{\varphi^{f^{-1}(q)}(t)}\circ(\mathrm{d}\varphi)_{(t,f^{-1}(q))}\left(\frac{\partial}{\partial t}\right)\\ &= (\mathrm{d}f)_{\varphi^{f^{-1}(q)}(t)}X(\varphi^{f^{-1}(q)}(t))\\ &= Y(f(\varphi^{f^{-1}(q)}(t)))\\ &= Y(\varphi'^q(t)).\end{aligned}$$

Now for each t, we have $\varphi'_t = f\circ\varphi_t\circ f^{-1}$. A composition of isometries is an isometry, so φ'_t is an isometry for all t if and only if φ_t is an isometry for all t. Thus Y is a Killing field if and only if X is also a Killing field. $\square$

Proof of d). Given X and $q\in M$ with $X(q)\neq 0$, choose a codimension 1 submanifold S of M in a neighborhood of q such that $X(q)$ is orthogonal to S: for instance, if we work in local coordinates, we can take S to be the hyperplane orthogonal to $X(q)$ (orthogonality is deduced by the inner product at the origin induced by the Riemannian metric on M).

Then choose coordinates $(x^1,\dots.x^{n-1},x^n=t)$ near q on M such that $t=0$ is the subset S, so that $x^1,\dots,x^{n-1}$ give coordinates on S, and such that $\partial/\partial t = X$. We can accomplish this as follows: let U be a neighborhood of q on which the local flow of X is defined, and restrict S to U. If we take the inclusion map $(-\varepsilon,\varepsilon)\times S\hookrightarrow(-\varepsilon,\varepsilon)\times U$ and compose with φ, because $X(q)\neq 0$, we get a local diffeomorphism $(-\varepsilon,\varepsilon)\times S\to M$ near $(0,q)$, and we can use this and any choice of coordinates $(x^1,\dots,x^{n-1})$ on S near q to get the desired coordinates on M.

Now write $X^i = \partial/\partial x^i$ for $i=1,\dots,n$. Then because partial derivatives commute in $\mathbb{R}^n$, we have $[X^i,X^j]=0$ for all i,j, and in particular $[X,X^i]=[X^n,X^i]=0$. Then by symmetry and compatibility with the metric, we have

$$\langle\nabla_{X^j}X,X^i\rangle+\langle\nabla_{X^i}X,X^j\rangle = X\langle X^i,X^j\rangle-\langle[X,X^i],X^j\rangle-\langle[X,X^j],X^i\rangle = X\langle X^i,X^j\rangle = \frac{\partial}{\partial t}\langle X^i,X^j\rangle.$$

It thus suffices to prove that X is a Killing field near q if and only if $\partial/\partial t\langle X^i,X^j\rangle = 0$ for all i,j. First note that local flows for any given time t always give local diffeomorphisms, since their inverse is provided by the local flow of the vector field $-X$. Thus, X being a Killing field near q is equivalent to having the property $\langle u,v\rangle = \langle(\mathrm{d}\varphi_t)_p u,(\mathrm{d}\varphi_t)_p v\rangle$ for all $u,v\in T_pM$, and all p near q. We claim, moreover, that for fixed $x^1,\dots,x^{n-1}$, the coefficients of $(\mathrm{d}\varphi_t)_p u$ and $(\mathrm{d}\varphi_t)_p v$ in terms of the X^i are constant as t varies. For this, by linearly of $\mathrm{d}\varphi_t$, it suffices to see that $(\mathrm{d}\varphi_t)_p\, X^i(p) = X^i(\varphi_t(p))$ for all i, t, p. If $i = n$, then $\varphi^p(s)$ is a curve through p with velocity $X^n(p)$ at $s = 0$; so by definition

$$\begin{aligned}(\mathrm{d}\varphi_t)_p\, X^n(p) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\varphi_t(\varphi^p(s))\right)|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\varphi(s+t, p)\right)|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\varphi_s(\varphi_t(p))\right)|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\varphi^{\varphi_t(p)}(s)\right)\Big|_{s=0} X^n(\varphi_t(p)).\end{aligned}$$

On the other hand, if $i < n$, we choose the curve $c(s)$ in S given by $x^j = 0$ for $j \neq i$, and $x^i = x^i(p) + s$. Thus, $c(0) = p$ and $c'(s) = X^i(c(s))$. Then,

$$(\mathrm{d}\varphi_t)_p\, X^i(p) = \frac{\mathrm{d}}{\mathrm{d}s}(\varphi_t(C(s)))|_{s=0} = X^i(\varphi_t(p)),$$

since our coordinates $x^1, \ldots, x^n$ are defined by flowing along X; so $\varphi_t(c(s))$ is given by

$$x^j = 0 \ \text{ for } j \neq= i, n, \quad x^i = x^i(p) + s, \quad x^n = t.$$

This proves our claim that, with respect to t, the coefficients of $(\mathrm{d}\varphi_t)_p u$ and $(\mathrm{d}\varphi_t)_p v$ in terms of the X^i are constant. It then follows that X is a Killing field if and only if $\langle(\mathrm{d}\varphi_t)_p u, (\mathrm{d}\varphi_t)_p v\rangle$ is constant with respect to t, if and only if $\partial/\partial t\langle X^i, X^j\rangle = 0$ for all i, j, as desired.

Finally, we have concluded the desired result for $X(q) \neq 0$. Now suppose X is Killing. We then conclude the Killing equation holds on the closure of the set where $X(q) \neq 0$, by continuity. Suppose q is not in this closure. Then $X = 0$ in a neighborhood of q, and it is clear that the desired equation is satisfied. Thus the Killing equation is satisfied everywhere. Conversely, if the Killing equation is satisfied, we have by the above that the local flow is an isometry at all points q with $X(q) \neq 0$, and clearly also at all points q with $X = 0$ in a neighborhood of q. Now suppose we have $q \in M$ with $X(q) = 0$, but in the closure of the subset of M where $X(q) \neq 0$. Then the local flow near q is an isometry on an open subset U with q in the closure of U. Since the condition of being an isometry is determined by equality of continuously varying quantities, we conclude it holds also at q. Thus the local flow is an isometry everywhere on M, and X is a Killing field. ☐

Proof of e). This follows immediately from the arguments in part d), using the same coordinates as there, since we proved that if X is Killing, we have

$$0 = \frac{\partial}{\partial t}\langle X^i, X^j\rangle = \frac{\partial}{\partial t}\, g_{ij} = \frac{\partial}{\partial x^n}\, g_{ij},$$

so that the g_{ij} do not depend on x^n. ☐