Geometry of General Relativity
Workshop 5 Hand-In

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Problem (WS5 Problem 6). *Consider the sphere $\mathbb{S}^2$ in polar coordinates (θ, ϕ) equipped with the unit round metric* $g = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

a) Show that the non-vanishing components of the Levi-Civita connection are $\Gamma^{\theta}_{\phi\phi} = -\cos\theta\sin\theta$ *and* $\Gamma^{\phi}_{\theta\phi} = \cot\theta$.

b) Show that $k = \sin^2\theta\,\dot{\phi}$ *is constant along a geodesic curve* $\gamma \colon I \to \mathbb{S}^2$ *that maps* $t \mapsto (\theta(t), \phi(t))$. *Deduce that* $\dot{\theta}^2 + k^2/\sin^2\theta$ *is also a constant.*

c) Compute $R_{\theta\phi\theta\phi}$ *and hence show that* $R_{ab} = g_{ab}$.

d) Verify that the vector fields X_1, X_2, X_3 *given in the lectures (end of sec* 4.4*) are Killing fields of* $(\mathbb{S}^2, g)$.

Solution to a). Recall that the Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2}\,g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

In our case at hand, we have

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix},$$

so that

$$g_{\theta\theta} = g^{\theta\theta} = 1, \quad g_{\phi\phi} = \sin^2\theta, \quad g^{\phi\phi} = \frac{1}{\sin^2\theta}, \quad \text{and} \quad g_{\theta\phi} = g_{\phi\theta} = g^{\theta\phi} = g^{\phi\theta} = 0.$$

Thus, the components of our Levi-Civita connection are

$$\begin{aligned}
\Gamma^{\theta}_{\theta\theta} &= \frac{1}{2}\,g^{\theta l}(\partial_\theta g_{\theta l} + \partial_\theta g_{\theta l} - \partial_l g_{\theta\theta}) \\
&= \frac{1}{2}\cdot 2\,g^{\theta l}\partial_\theta g_{\theta l} - \frac{1}{2}\,g^{\theta l}\partial_l g_{\theta\theta} \\
&= g^{\theta\theta}\partial_\theta g_{\theta\theta} + g^{\theta\phi}\partial_\theta g_{\theta\phi} - \frac{1}{2}(g^{\theta\theta}\partial_\theta g_{\theta\theta} + g^{\theta\theta}\partial_\phi g_{\theta\theta}) \\
&= 0 + 0 - 0 - 0 \\
&= 0;
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\phi}_{\theta\theta} &= \frac{1}{2}\,g^{\phi l}(\partial_\theta g_{\theta l} + \partial_\theta g_{\theta l} - \partial_l g_{\theta\theta}) \\
&= \frac{1}{2}\cdot 2\,g^{\phi l}\partial_\theta g_{\theta l} - \frac{1}{2}\,g^{\phi l}\partial_l g_{\theta\theta} \\
&= g^{\phi\theta}\partial_\theta g_{\theta\theta} + g^{\phi\phi}\partial_\theta g_{\theta\phi} - \frac{1}{2}(g^{\phi\theta}\partial_\theta g_{\theta\theta} + g^{\phi\phi}\partial_\phi g_{\theta\theta}) \\
&= 0 + 0 - 0 - 0 \\
&= 0;
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}\, g^{\theta l}(\underbrace{\partial_\phi g_{\phi l}}_{=0} + \underbrace{\partial_\phi g_{\phi l}}_{=0} - \partial_l g_{\phi\phi}) \\
&= -\frac{1}{2} g^{\theta l}\partial_l g_{\phi\phi} \\
&= -\frac{1}{2}\left(g^{\theta\theta}\partial_\theta g_{\phi\phi} + g^{\theta\phi}\partial_\phi g_{\phi\phi}\right) \\
&= -\frac{1}{2}\left(1\cdot\partial_\theta \sin^2\theta + 0\right) \\
&= -\frac{1}{2}\cdot 2\cos\theta\sin\theta \\
&= -\cos\theta\sin\theta;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\phi}_{\phi\phi} &= \frac{1}{2}\, g^{\phi l}(\underbrace{\partial_\phi g_{\phi l}}_{=0} + \underbrace{\partial_\phi g_{\phi l}}_{=0} - \partial_l g_{\phi\phi}) \\
&= -\frac{1}{2} g^{\phi l}\partial_l g_{\phi\phi} \\
&= -\frac{1}{2}\left(g^{\phi\theta}\partial_\theta g_{\phi\phi} + g^{\phi\phi}\partial_\phi g_{\phi\phi}\right) \\
&= 0;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} &= \frac{1}{2}\, g^{\theta l}(\underbrace{\partial_\phi g_{\theta l}}_{=0} + \partial_\theta g_{\phi l} - \partial_l g_{\phi\theta}) \\
&= \frac{1}{2}\left(g^{\theta l}\partial_\theta g_{\phi l} - g^{\theta l}\partial_l g_{\phi\theta}\right) \\
&= \frac{1}{2}\left(g^{\theta\theta}\partial_\theta g_{\phi\theta} + g^{\theta\phi}\partial_\theta g_{\phi\phi} - g^{\theta\theta}\partial_\theta g_{\phi\theta} - g^{\theta\phi}\partial_\phi g_{\phi\theta}\right) \\
&= 0;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} &= \frac{1}{2}\, g^{\phi l}(\underbrace{\partial_\phi g_{\theta l}}_{=0} + \partial_\theta g_{\phi l} - \partial_l \underbrace{g_{\phi\theta}}_{=0}) \\
&= \frac{1}{2}\, g^{\phi l}\partial_\theta g_{\phi l} \\
&= \frac{1}{2}\left(g^{\phi\theta}\partial_\theta g_{\phi\theta} + g^{\phi\phi}\partial_\theta g_{\phi\phi}\right) \\
&= \frac{1}{2}\left(0 + \frac{1}{\sin^2\theta}\,\partial_\theta \sin^2\theta\right) \\
&= \frac{1}{2}\left(\frac{1}{\sin^2\theta}\, 2\cdot\sin\theta\,\cos\theta\right) \\
&= \frac{\cos\theta}{\sin\theta} \\
&= \cot\theta.
\end{aligned}
$$

$\boxed{\text{Victoria!}}$

*Proof of **b)***. We recall the following proposition:

Proposition.

Let V be tangent to a geodesic and X a Killing field on (M, g). Then $V^a X_a = g_{ab} V^a X^b$ is a constant along the geodesic.

Now, note that neither of the metric components is dependent of ϕ; therefore $X = \partial/\partial\phi$ is a Killing field. Thus, given the geodesic vector field

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \dot{\gamma} = \dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi},$$

we deduce that

$$\dot{\gamma}^i X_i = g_{ij}\dot{\gamma}^i X^j = g_{\phi\phi}\dot{\gamma}^\phi X^\phi = \sin^2\theta\,\dot{\phi} = k$$

must be a constant along γ.

Now, since $k = \sin^2\theta\,\dot{\phi}$ is constant, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sin^2\theta\,\dot{\phi} = 0 \qquad \Longrightarrow \qquad \sin^2\theta\,\ddot{\phi} = 0. \tag{1}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}^2 + \frac{k^2}{\sin^2\theta}\right) &= 2\dot{\theta}\ddot{\theta} + 2\sin^2\theta\,\dot{\phi}\ddot{\phi}\\ &= 0 + 0 = 0.\end{aligned}$$

Here the second term vanishes because of the implication (1), whereas the first vanishes because … $\boxed{\text{Victoria!}}$

*Solution to **c)***. We recall that

$$\begin{aligned}R^i{}_{jkl} &= \langle f^i, R(e_k, e_l)e_j\rangle\\ &= \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}.\end{aligned}$$

In the case at hand we want to compute $R_{\theta\phi\theta\phi}$. Thus,

$$\begin{aligned}R_{\theta\phi\theta\phi} &= g_{\theta l}\,R^l{}_{\phi\theta\phi}\\ &= g_{\theta l}\,(\partial_\theta\Gamma^l_{\phi\phi} - \partial_\phi\Gamma^l_{\phi\theta} + \Gamma^m_{\phi\phi}\Gamma^l_{m\theta} - \Gamma^m_{\phi\theta}\Gamma^l_{m\phi}).\end{aligned}$$

Now, at this point we pause to point out that, even though we have a summation over l, $g_{\theta\phi}$ vanishes, so the only term we need to consider is the one multiplying $g_{\theta\theta}$ (also, recall from part a) that the only non-vanishing components of the Levi-Civita connection are $\Gamma^{\theta}_{\phi\phi} = -\cos\theta\sin\theta$ and $\Gamma^{\phi}_{\theta\phi} = \cot\theta$):

$$\begin{aligned}
R_{\theta\phi\theta\phi} &= g_{\theta\theta}\,(\partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{m}_{\phi\phi}\Gamma^{\theta}_{m\theta} - \Gamma^{m}_{\phi\theta}\Gamma^{\theta}_{m\phi}) \\
&= 1\cdot(\partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \underbrace{\Gamma^{\theta}_{\phi\theta}}_{=0} + \Gamma^{\theta}_{\phi\phi}\underbrace{\Gamma^{\theta}_{\theta\theta}}_{=0} + \underbrace{\Gamma^{\phi}_{\phi\phi}\Gamma^{\theta}_{\phi\theta}}_{=0} - \underbrace{\Gamma^{\theta}_{\phi\theta}\Gamma^{\theta}_{\theta\phi}}_{=0} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi}) \\
&= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} \\
&= \partial_\theta(-\cos\theta\,\sin\theta) - \frac{\cos\theta}{\sin\theta}(-\cos\theta\,\sin\theta) \\
&= \sin^2\theta - \cos^2\theta + \cos^2\theta \\
&= \sin^2\theta.
\end{aligned}$$

Next note that

$$R_{\theta\phi\theta\phi} = g_{\theta l}\, R^{l}{}_{\phi\theta\phi} = g_{\theta\theta}\, R^{\theta}{}_{\phi\theta\phi} = 1\cdot R_{\phi\phi} = R_{\phi\phi},$$

so that $R_{\phi\phi} = \sin^2\theta$. Moreover, because $g_{\theta\phi} = g_{\phi\theta} = 0$, the only remaining non-vanishing Riemann curvature tensor is

$$\begin{aligned}
R_{\phi\theta\phi\theta} = g_{\phi l}\, R^{l}{}_{\theta\phi\theta} &= g_{\phi\phi}\, R^{\phi}{}_{\theta\phi\theta} \\
&= g_{\phi\phi}\,(\partial_\phi \underbrace{\Gamma^{\phi}_{\theta\theta}}_{=0} - \partial_\theta \Gamma^{\phi}_{\theta\phi} + \underbrace{\Gamma^{m}_{\theta\theta}}_{=0}\Gamma^{\phi}_{m\phi} - \Gamma^{m}_{\theta\phi}\Gamma^{\phi}_{m\theta}) \\
&= \sin^2\theta\,(-\partial_\theta \Gamma^{\phi}_{\theta\phi} - \Gamma^{m}_{\theta\phi}\Gamma^{\phi}_{m\theta}) \\
&= \sin^2\theta\,(-\partial_\theta \Gamma^{\phi}_{\theta\phi} - \Gamma^{\theta}_{\theta\phi}\underbrace{\Gamma^{\phi}_{\theta\theta}}_{=0} - \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\phi\theta}) \\
&= \sin^2\theta\,(-\partial_\theta(\cot\theta) - \cot^2\theta) \\
&= \sin^2\theta\,(\csc^2\theta - \cot^2\theta) \\
&= \sin^2\theta\cdot 1 \\
&= \sin^2\theta.
\end{aligned}$$

Lastly, note that

$$R_{\phi\theta\phi\theta} = g_{\phi l}\, R^{l}{}_{\theta\phi\theta} = g_{\phi\phi}\, R^{\phi}{}_{\theta\phi\theta} = \sin^2\theta\, R_{\theta\theta},$$

so that $R_{\theta\theta} = 1$.

Collecting these results we have

$$R_{\phi\phi} = \sin^2\theta, \quad R_{\theta\theta} = 1, \quad \text{and} \quad R_{\theta\phi} = R_{\phi\theta} = 0,$$

which implies that $R_{ab} = g_{ab}$, as we wanted to prove. Victoria!

*Proof of **d)**.* The three vector fields are

$$\begin{aligned}
X_1 &= -\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\,\frac{\partial}{\partial\phi} \\
X_2 &= \sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\,\frac{\partial}{\partial\phi} \\
X_3 &= \frac{\partial}{\partial\phi}.
\end{aligned}$$

Recall that if X is a Killing vector field on (M, g), then the covector field $X_a = g_{ab}X^b$ must satisfy ***Killing's equation***

$$\nabla_a X_b + \nabla_b X_a = 0.$$

We will plug in each of the three fields above to check that they do satisfy this equation, in which case we would conclude that they are indeed Killing fields. (Recall from above that the only non-vanishing components of the Levi-Civita connection are $\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta$ and $\Gamma^\phi_{\theta\phi} = \cot\theta$)):

$$\begin{aligned}
\nabla_\theta(X_1)_\phi + \nabla_\phi(X_1)_\theta &= \nabla_\theta(g_{\phi m}(X_1)^m) + \nabla_\phi(g_{\theta m}(X_1)^m) \\
&= \nabla_\theta(g_{\phi\phi}(X_1)^\phi) + \nabla_\phi(g_{\theta\theta}(X_1)^\theta) && (\text{Since } g_{\theta\phi} = g_{\phi\theta} = 0) \\
&= g_{\phi\phi}\,\nabla_\theta(X_1)^\phi + g_{\theta\theta}\,\nabla_\phi(X_1)^\theta && (\text{Since } \nabla g = 0) \\
&= g_{\phi\phi}(\partial_\theta(X_1)^\phi + \Gamma^\phi_{m\theta}(X_1)^m) + g_{\theta\theta}(\partial_\phi(X_1)^\theta + \Gamma^\theta_{m\phi}(X_1)^m) \\
&= g_{\phi\phi}(\partial_\theta(X_1)^\phi + \Gamma^\phi_{\phi\theta}(X_1)^\phi) + g_{\theta\theta}(\partial_\phi(X_1)^\theta + \Gamma^\theta_{\phi\phi}(X_1)^\phi) \\
&= \sin^2\theta\,[\partial_\theta(\cot\theta\sin\phi) + \cot\theta\cot\theta\sin\phi] + \dots \\
&\dots + 1\cdot[\partial_\phi(-\cos\phi) + (-\cos\theta\sin\theta)\cot\theta\sin\phi] \\
&= \sin^2\theta\sin\phi\,\partial_\theta(\cot\theta) + \cot^2\theta\sin\phi\sin^2\theta + \dots \\
&\dots + \partial_\phi(-\cos\phi) - \cos\theta\sin\theta\cot\theta\sin\phi \\
&= \sin^2\theta\sin\phi\,(-\csc^2\theta) + \cot^2\theta\sin\phi\sin^2\theta + \dots \\
&\dots + \sin\phi - \cos\theta\sin\theta\cot\theta\sin\phi \\
&= -\sin^2\theta\sin\phi\,\frac{1}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}\sin\phi\sin^2\theta + \dots \\
&\dots + \sin\phi - \cos\theta\sin\theta\,\frac{\cos\theta}{\sin\theta}\sin\phi \\
&= \sin\phi\left[-1 + \cos^2\theta + 1 - \cos^2\theta\right] \\
&= \sin\phi\cdot 0 \\
&= 0. \qquad \surd
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nabla_\theta(X_2)_\phi + \nabla_\phi(X_2)_\theta &= g_{\phi\phi}(\partial_\theta(X_2)^\phi + \Gamma^\phi_{\phi\theta}(X_2)^\phi) + g_{\theta\theta}(\partial_\phi(X_2)^\theta + \Gamma^\theta_{\phi\phi}(X_2)^\phi) \\
&= \sin^2\theta\,[\partial_\theta(\cot\theta\cos\phi) + \cot\theta\cot\theta\cos\phi] + \dots \\
&\dots + 1\cdot[\partial_\phi(\sin\phi) + (-\cos\theta\sin\theta)\cot\theta\cos\phi] \\
&= \sin^2\theta\cos\phi\,\partial_\theta(\cot\theta) + \sin^2\theta\cot^2\theta\cos\phi + \dots \\
&\dots + \partial_\phi(\sin\phi) - \cos\theta\sin\theta\cot\theta\cos\phi \\
&= \sin^2\theta\cos\phi\,(-\csc^2\theta) + \sin^2\theta\cot^2\theta\cos\phi + \dots \\
&\dots + \cos\phi - \cos\theta\sin\theta\cot\theta\cos\phi \\
&= -\sin^2\theta\cos\phi\,\frac{1}{\sin^2\theta} + \sin^2\theta\frac{\cos^2\theta}{\sin^2\theta}\cos\phi + \dots \\
&\dots + \cos\phi - \cos\theta\sin\theta\frac{\cos\theta}{\sin\theta}\cos\phi \\
&= \cos\phi(-1 + \cos^2\theta + 1 - \cos^2\theta) \\
&= \cos\phi\cdot 0 \\
&= 0. \qquad \surd
\end{aligned}$$

Lastly,

$$\begin{aligned}
\nabla_\theta (X_3)_\phi + \nabla_\phi (X_3)_\theta &= g_{\phi\phi}(\partial_\theta (X_3)^\phi + \Gamma^\phi_{\phi\theta}(X_3)^\phi) + g_{\theta\theta}(\partial_\phi (X_3)^\theta + \Gamma^\theta_{\phi\phi}(X_3)^\phi) \\
&= \sin^2\theta\,[\partial_\theta(1) + \cot\theta \cdot 1] + 1 \cdot [\partial_\phi(0) + (-\cos\theta\sin\theta)\cdot 1] \\
&= \sin^2\theta\cot\theta - \cos\theta\sin\theta \\
&= \sin^2\theta\frac{\cos\theta}{\sin\theta} - \cos\theta\sin\theta \\
&= \cos\theta\sin\theta - \cos\theta\sin\theta \\
&= 0. \qquad \boxed{\text{Victoria!}}
\end{aligned}$$