MATH 742 HW # 1

MARIO L. GUTIERREZ ABED
INSTRUCTOR A. ETKIN

Exercise 1 (Exercise 1-1)**.** *Describe geometrically the sets of points z in the complex plane defined by the following relations:*

a) $|z - z_1| = |z - z_2|$ *where* $z_1, z_2 \in \mathbb{C}$.

b) $1/z = \bar{z}$.

c) $\mathfrak{Re}\, z = 3$.

d) $\mathfrak{Re}\, z > c$, *(resp.,* $\geq c$*) where* $c \in \mathbb{R}$.

e) $\mathfrak{Re}(az + b) > 0$ *where* $a, b \in \mathbb{C}$.

f) $|z| = \mathfrak{Re}\, z + 1$.

g) $\mathfrak{Im}\, z = c$ *with* $c \in \mathbb{R}$.

Solution of a). In simple words, this is nothing but the locus in the complex plane consisting of all points that are an equal distance from both z_1 and z_2. When $z_1 \neq z_2$, this is the line that perpendicularly bisects the line segment from z_1 to z_2. If on the other hand it happens that $z_1 = z_2$, then this is the entire complex plane. $\square$

Solution of b). It is painfully obvious that this is the unit circle. Note that $1/z = \bar{z} \implies z\bar{z} = 1 \implies |z|^2 = 1 \implies |z| = 1$. $\square$

Solution of c). This is just the vertical line in the right half plane that has real value 3. In other words, it is the set $\{x + iy \in \mathbb{C} \mid x = 3\}$. $\square$

Solution of d). This is the open half-plane to the right of the vertical line $x = c$, i.e. $\{x + iy \in \mathbb{C} \mid x > c\}$. In the case when we use "$\geq$," then it is the closed half-plane $\{x + iy \in \mathbb{C} \mid x \geq c\}$. $\square$

Solution of e). Since $a, b, z \in \mathbb{C}$, we have that $az + b = w \in \mathbb{C}$ ($\mathbb{C}$ is itself a $\mathbb{C}$-module, so closure under scalar multiplication and addition is justified). Thus what we have is the open half plane to the right of $\text{Re}(w) = 0$, i.e. $\{w = x + iy \in \mathbb{C} \mid x > 0\}$. $\square$

Solution of f). We have $|z| = \sqrt{x^2 + y^2} = x + 1 \implies y = \sqrt{2x + 1}$. Thus the complex numbers defined by this relation form a parabola opening from the y-axis. $\square$

Solution of g). This is just a line in which $\mathfrak{Re}\, z$ takes on any real value while $\mathfrak{Im}\, z$ remains fixed with a value of c, for c some real constant. $\square$

Exercise 2 (Exercise 1-8)**.** *Suppose U and V are open sets in the complex plane. Prove that if $f\colon U \to V$ and $g\colon V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then*

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar z}\frac{\partial \bar f}{\partial z} \qquad \text{and} \qquad \frac{\partial h}{\partial \bar z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar z} + \frac{\partial g}{\partial \bar z}\frac{\partial \bar f}{\partial \bar z}.$$

This is the complex version of the chain rule.

Proof. By the chain rule we may write the differential of $g \circ f$ as

$$\begin{aligned} d(g\circ f) &= \left(\frac{\partial g}{\partial z}\circ f\right) df + \left(\frac{\partial g}{\partial \bar z}\circ f\right) d\bar f \\ &= \left(\frac{\partial g}{\partial z}\circ f\right)\left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right) + \left(\frac{\partial g}{\partial \bar z}\circ f\right)\left(\frac{\partial \bar f}{\partial z}dz + \frac{\partial \bar f}{\partial \bar z}d\bar z\right) \\ &= \underbrace{\left(\left(\frac{\partial g}{\partial z}\circ f\right)\frac{\partial f}{\partial z} + \left(\frac{\partial g}{\partial \bar z}\circ f\right)\frac{\partial \bar f}{\partial z}\right)}_{=\frac{\partial h}{\partial z}} dz + \underbrace{\left(\left(\frac{\partial g}{\partial z}\circ f\right)\frac{\partial f}{\partial \bar z} + \left(\frac{\partial g}{\partial \bar z}\circ f\right)\frac{\partial \bar f}{\partial \bar z}\right)}_{=\frac{\partial h}{\partial \bar z}} d\bar z. \end{aligned}$$ ☐

Exercise 3 (Exercise 1-9)**.** *Show that in polar coordinates, the Cauchy-Riemann equations take the form*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \qquad (\text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi)$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof. Let $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$. Then,

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= u_x\cos\theta + u_y\sin\theta \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= r(-u_x\sin\theta + u_y\cos\theta). \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} \\ &= v_x\cos\theta + v_y\sin\theta \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} \\ &= r(-v_x\sin\theta + v_y\cos\theta). \end{aligned}$$

Now by using the Cauchy-Riemann equations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x, \end{aligned}$$

we get the desired result

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Now we use these equations to show that

$$\log(re^{i\theta}) = \log r + \log(e^{i\theta}) = \log r + i\theta$$

is holomorphic in the region where $r > 0$ and $-\pi < \theta < \pi$.

Let $u = \log r$ and $v = \theta$. Then,

$$\begin{aligned} u_r &= \frac{1}{r} = \frac{1}{r} v_\theta \\ u_\theta &= 0 = -r v_r. \end{aligned}$$

Hence, since u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on $r > 0$ and $-\pi < \theta < \pi$, we conclude that $\log z$ is holomorphic on this region. □

Exercise 4 (Exercise 1-10)**.** *Show that*

$$4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = \Delta,$$

where Δ *is the* ***Laplacian***

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. This is a straightforward computation. All we need to do is plug in the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

and compute

$$\begin{aligned} 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} &= 4 \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \\ &= \frac{\partial^2}{\partial x^2} \underbrace{-i\frac{\partial^2}{\partial x \partial y} + i\frac{\partial^2}{\partial y \partial x}}_{=0 \text{ (mixed partials are equal)}} + \frac{\partial^2}{\partial y^2} \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta. \end{aligned}$$

□

Exercise 5 (Exercise 1-11)**.** *Use Exercise 10 to prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are* ***harmonic****; that is, their Laplacian is zero.*

Proof. Let $f = u + iv$. Since f is holomorphic on Ω, we know that the Cauchy-Riemann equations are satisfied:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

But then it follows that

$$\Delta = u_{xx} + u_{yy} = \underbrace{v_{yx} - v_{xy}}_{=0 \text{ (mixed partials are equal)}} = 0.$$

This shows that $\mathfrak{Re}\, f = u$ is harmonic. A similar argument shows that $\mathfrak{Im}\, f = v$ is harmonic as well. □

Exercise 6 (Exercise 1-12)**.** *Consider the function defined by*

$$f(x+iy) = \sqrt{|x||y|}, \quad \text{whenever } x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Proof. We have that $\mathfrak{Re}\, f = f$ and $\mathfrak{Im}\, f = 0$. From this it follows that

$$\frac{\partial\, \mathfrak{Im}\, f}{\partial x}\Big|_{(0,0)} = \frac{\partial\, \mathfrak{Im}\, f}{\partial y}\Big|_{(0,0)} = 0.$$

Also, a straight computation shows that

$$\frac{\partial\, \mathfrak{Re}\, f}{\partial x}\Big|_{(0,0)} = \frac{\partial\, \mathfrak{Re}\, f}{\partial y}\Big|_{(0,0)} = 0.$$

On the other hand, a holomorphic function is always differentiable, and this function is not differentiable at the origin. If it were in fact differentiable, the differential would be zero as we computed the partial derivatives at the origin and all were equal to zero. This would imply that the limit

$$\lim_{x+iy\to 0} \frac{|f(x+iy) - f(0)|}{|x+iy|} = 0.$$

But it turns out that

$$\lim_{x\to 0^+} \frac{|f(x+ix)|}{|x+ix|} = \lim_{x\to 0} \frac{\sqrt{|x|^2}}{\sqrt{2}|x|} = \frac{1}{\sqrt{2}} \neq 0.$$

□

Exercise 7 (Exercise 1-13)**.** *Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases one can conclude that f is constant:*

a) $\mathfrak{Re}\, f$ *is constant;*

b) $\mathfrak{Im}\, f$ *is constant;*

c) $|f|$ *is constant.*

Proof of a). Let $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$. Since $\mathfrak{Re}\, f = u(x,y)$ is constant, we have that its partial derivatives are zero:

$$\frac{\partial u}{\partial x} = 0; \qquad \frac{\partial u}{\partial y} = 0.$$

Now since f is assumed to be holomorphic, from the Cauchy-Riemann equations we get

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0.$$

Thus, in Ω,

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0.$$

A function in an open connected set with partial derivatives equal to zero should be constant. Thus it follows that f is constant. □

Proof of b). The argument here is identical to the one in part a). □

Proof of c). We can prove this directly using the Cauchy-Riemann equations and taking derivatives of $|f|$ or of $|f|^2$, but instead let us tackle this with minimal computation. Note that if $|f|$ is constant, then so is $f \cdot \bar{f} = |f|^2$. If $f(z) = 0$ for some z, then f is identically zero and the result follows. Assume therefore that f is nonzero. Then in this case we have that $\bar{f} = f\bar{f}/f$ is holomorphic. But then $\mathfrak{Re}\, f = (f + \bar{f})/2$ is holomorphic as well. This (meaning $\mathfrak{Re}\, f$) is a function with constant imaginary part (it is equal to zero since it is real). Therefore from our results above, it is constant. The same argument implies that $\mathfrak{Im}\, f$ is constant. Hence f is constant, as desired. $\square$