Math 751 Notes

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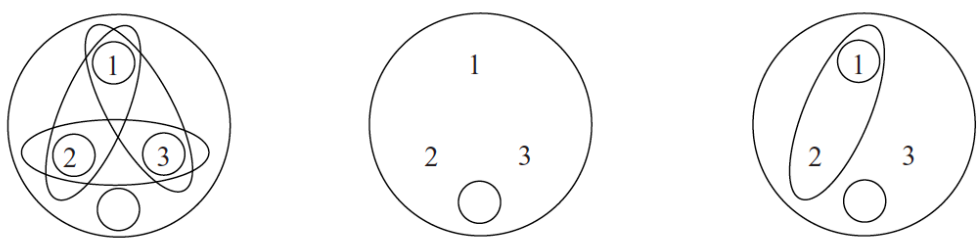
Intro to Topological Spaces

Definition: If X is a set, a topology on X is a collection $\mathcal{T}$ of subsets of X satisfying the following properties:
(i) X and $\emptyset$ are elements of $\mathcal{T}$.
(ii) $\mathcal{T}$ is closed under finite intersections: if $U_1, \ \ldots, \ U_n$ are elements of $\mathcal{T}$, then their intersection $U_1 \bigcap \ldots \bigcap U_n$ is an element of $\mathcal{T}$.
(iii) $\mathcal{T}$ is closed under arbitrary unions: if $(U_\alpha)_{\alpha \in A}$ is any (finite or infinite) family of elements of $\mathcal{T}$, then their union $\bigcup_{\alpha \in A} U_\alpha$ is an element of $\mathcal{T}$.

In other words, a topology on X is a collection of all open sets of X. A pair $(X, \mathcal{T})$ consisting of a set X together with a topology $\mathcal{T}$ on X is called a topological space.

Example (Simple Topologies):



(a) Discrete topology (b) Trivial topology (c) $\{\{1\},\{1,2\},\{1,2,3\},\varnothing\}$

Fig. 2.1: Topologies on $\{1,2,3\}$.

a) Let X be any set whatsoever and let $\mathcal{T}$ be the collection of all subsets of X. Then $\mathcal{T}$ is a topology on X, called the discrete topology on X (see fig. 2.1(a) above), and $(X, \mathcal{T})$ is called a discrete space.

b) Let Y be any set, and let $\mathcal{T} = \{Y, \emptyset\}$ (fig. 2.1(b)). This is called the trivial topology on Y.

c) Let Z be the set $\{1, \ 2, \ 3\}$, and declare the open subsets to be $\{1\}$, $\{1, \ 2\}$, $\{1, \ 2, \ 3\}$, and the empty set. This is the topology illustrated on fig. 2.1 c) above.

We can easily verify that each of the preceding examples is in fact a topology by checking that they satisfy all three conditions stated on our definition.

Definition: Let X be a set and let $\mathcal{T}_f$ be the collection of all subsets $U \subset X$ such that $X \backslash U$ either is finite or is all of of X. Then $\mathcal{T}_f$ is a topology on X, called the finite complement topology.

Remark 1: To show that $\mathcal{T}_f$ is indeed a topology, notice that both X and $\emptyset$ are in $\mathcal{T}_f$, since $X \backslash X$ is finite and $X \backslash \emptyset$ is all of X. Also, if $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$, we have that $\bigcup U_\alpha \subset \mathcal{T}_f$. To see why, compute

$$X \backslash \bigcup U_\alpha = \bigcap (X \backslash U_\alpha).$$

The latter set is finite because each set $X \backslash U_\alpha$ is finite.

Finally, if $U_1, \ldots, U_n$ are nonempty elements of $\mathcal{T}_f$, to show that $\bigcap U_i \subset \mathcal{T}_f$, we compute

$$X \backslash \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \backslash U_i).$$

The latter set is a finite union of finite sets, therefore it is finite.

Remark 2: Let X be a set and let $\mathcal{T}_C$ be the collection of all subsets $U \subset X$ such that $X \backslash U$ either is countable or is all of of X. Then $\mathcal{T}_C$ is also a topology on X, as we may check.

Suppose X is a topological space and A is any subset of X . We define several related subsets as follows:

Definition: The closure of A in X, denoted by $\overline{A}$, is the set $\overline{A} = \bigcap \{B \subseteq X : A \subseteq B,\ B \text{ is closed in } X\}$

Definition: The interior of A, denoted by $\mathrm{Int}(A)$, is $\mathrm{Int}(A) = \bigcup \{C \subseteq X : C \subseteq A,\ C \text{ is open in } X\}$.

Remark: It follows immediately from the properties of open and closed subsets that $\overline{A}$ is closed and $\mathrm{Int}(A)$ is open. To put it succinctly, $\overline{A}$ is “the smallest closed subset containing A” and $\mathrm{Int}(A)$ is “the largest open subset contained in A”.

Definition: The exterior of A, denoted by $\mathrm{Ext}(A)$, is $\mathrm{Ext}(A) = X \backslash \overline{A}$.

Definition: The boundary of A, denoted by ∂A, is $\partial A = X \backslash (\mathrm{Int}(A) \bigcup \mathrm{Ext}(A))$.

Remark: It follows from the above definitions that for any subset $A \subseteq X$, the whole space X is equal to the disjoint union of $\mathrm{Int}(A)$, $\mathrm{Ext}(A)$, and ∂A. Also note that $\mathrm{Int}(A)$ and $\mathrm{Ext}(A)$ are open in X, while $\overline{A}$ and ∂A are closed in X.

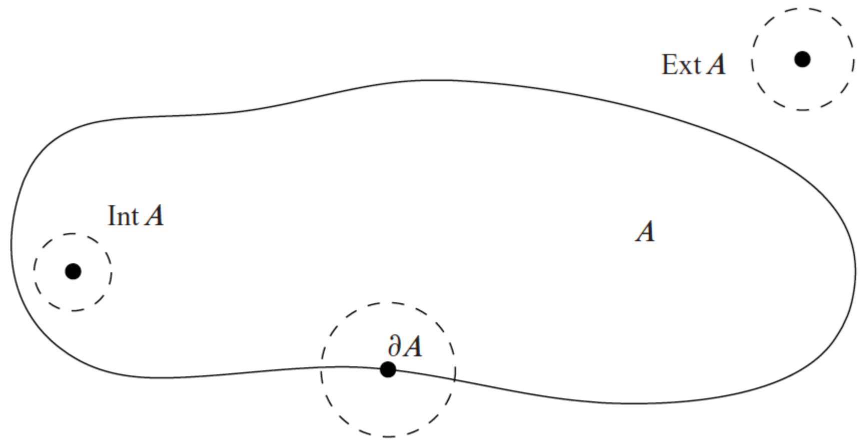


Fig. 2.2: Interior, exterior, and boundary points.

It is also sometimes useful to compare different topologies on the same set:

Definition: Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X, we say that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, and coarser than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. We say that $\mathcal{T}_1$ is comparable to $\mathcal{T}_2$ if either $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Remark: The terminology in this definition is meant to suggest the picture of a subset that is open in a coarser topology being further subdivided into smaller open subsets in a finer topology. It can be shown that the identity map of X is continuous as a map from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$ iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, and furthermore it is a homeomorphism iff $\mathcal{T}_1 = \mathcal{T}_2$.

Here are a few explicit examples of homeomorphisms that we should keep in mind:

Example:
Any open ball in $\mathbb{R}^n$ is homeomorphic to any other open ball: The homeomorphism can easily be constructed as a composition of translations $x \longmapsto x + x_0$ and dilations $x \longmapsto c\,x$.
Similarly, all spheres in $\mathbb{R}^n$ are homeomorphic to each other. These examples illustrate that “size” is not a topological property.

Example:
Let $\mathbb{B}^n \subseteq \mathbb{R}^n$ be the unit ball, and define a map $F: \mathbb{B}^n \longrightarrow \mathbb{R}^n$ by

$$F(x) = \frac{x}{1-|x|}.$$

Direct computation shows that the map $G: \mathbb{R}^n \longrightarrow \mathbb{B}^n$ defined by

$$G(x) = \frac{y}{1+|y|}$$

is an inverse for F. Thus F is bijective, and since F and $F^{-1} = G$ are both continuous, F is a homeomorphism. It follows that $\mathbb{R}^n$ is homeomorphic to $\mathbb{B}^n$, and thus "boundedness" is not a topological property.

Example:

Another illustrative example is the homeomorphism between the surface of a sphere and the surface of a cube:

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, and set

$$C = \{(x,\ y,\ z) : \max\{|x|,\ |y|,\ |z|\} = 1\},$$

which is the cubical surface of side 2 centered at the origin.

Let $\varphi : C \longrightarrow \mathbb{S}^2$ be the map that projects each point of C radially inward to the sphere as shown in the following figure.

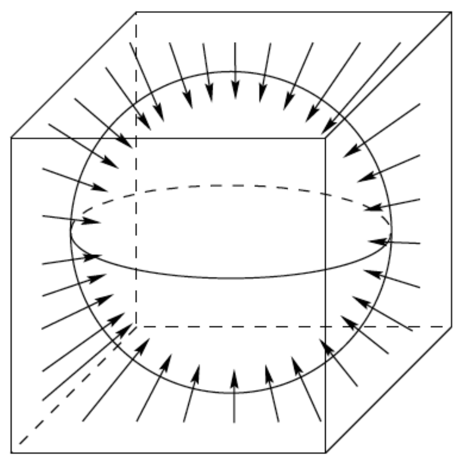


Fig. 2.4: Deforming a cube into a sphere.

More precisely, given a point $p \in C$, the image $\varphi(p)$ is the unit vector in the direction of p. Thus φ is given by the formula

$$\varphi(x,\ y,\ z) = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}},$$

which is continuous on C by the usual arguments of elementary analysis (notice that the denominator is always nonzero on C).

With a little extra effort, it can be shown that φ is a homeomorphism, with

$$\varphi^{-1}(x,\ y,\ z) = \frac{(x, y, z)}{\max\{|x|, |y|, |z|\}}.$$

This demonstrates that "corners" are not topological properties. ✵

Hausdorff Spaces

Definition: A topological space X is said to be a Hausdorff space if given any pair of distinct points $p_1, p_2 \in X$, there exist neighborhoods U_1 of p_1 and U_2 of p_2 with $U_1 \bigcap U_2 = \emptyset$. This property is often summarized by saying that "points can be separated by open subsets."

Example (Hausdorff Spaces):

- Every metric space is Hausdorff: if p_1 and p_2 are distinct, let $r = d(p_1, p_2)$; then the open balls of radius $r/2$ around $p_1$and p_2 are disjoint by the triangle inequality.

- Every discrete space is Hausdorff, because $\{p_1\}$ and $\{p_2\}$ are disjoint open subsets when $p_1 \neq p_2$.

- Every open subset of a Hausdorff space is Hausdorff: if $V \subseteq X$ is open in the Hausdorff space X, and p_1, p_2 are distinct points in V, then in X there are open subsets U_1, U_2 separating $p_1$and p_2, and the sets $U_1 \bigcap V$ and $U_2 \bigcap V$ are open in V, disjoint, and contain p_1 and p_2, respectively.

- Suppose X is a topological space, and for every $p \in X$ there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(\{0\}) = \{p\}$. It can be shown that X is Hausdorff. ✵

Example (Non-Hausdorff Spaces):
The trivial topology on any set containing more than one element is not Hausdorff, nor is the topology on $\{1, 2, 3\}$ described in example 2.1(c) on page 1. Because every metric space is Hausdorff, it follows that these spaces are not metrizable. ✵

Hausdorff spaces have many of the properties that we expect of metric spaces, such as those expressed in the following proposition:

- Proposition:

Let X be a Hausdorff space. Then,
a) Every finite subset of X is closed.
b) If a sequence (p_i) in X converges to a limit $p \in X$, the limit is unique.

Proof:
For part a), consider first a set $\{p_0\}$ containing only one point. Given $p \neq p_0$, the Hausdorff

property says that there exist disjoint neighborhoods U of p and V of p_0. In particular, U is a neighborhood of p contained in $X \setminus \{p_0\}$, so $\{p_0\}$ is closed. It follows that finite subsets are closed, because they are finite unions of one-point sets.
To prove that limits are unique, suppose on the contrary that a sequence (p_i) has two distinct limits p and p'. By the Hausdorff property, there exist disjoint neighborhoods U of p and $U'$of p'. By definition of convergence, there exist $N, N' \in \mathbb{N}$ such that $i \geq N$ implies $p_i \in U$ and $i \geq N'$ implies $p_i \in U'$. But since U and U' are disjoint, this is a contradiction when $i \geq \max\{N, N'\}$.(⇒⇐) ■

Remark: It can be shown that the only Hausdorff topology on a finite set is the discrete topology.

Another important property of Hausdorff spaces is expressed in the following proposition:

• Proposition:
Suppose X is a Hausdorff space and $A \subseteq X$. If $p \in X$ is a limit point of A, then every neighborhood of p contains infinitely many points of A.

Definition: Let X be a topological space. A collection $\mathcal{B}$ of subsets of X is called a basis for the topology of X if the following two conditions hold:
i) Every element of $\mathcal{B}$ is an open subset of X.
ii) Every open subset of X is the union of some collection of elements of $\mathcal{B}$.

• Proposition:
Let X and Y be topological spaces and let $\mathcal{B}$ be a basis for Y. A map $f : X \longrightarrow Y$ is continuous iff for every basis subset $B \in \mathcal{B}$, the subset $f^{-1}(B)$ is open in X.

Remark: Not every collection of sets can be a basis for a topology. The next proposition gives necessary and sufficient conditions for a collection of subsets of a set X to be a basis for some topology on X:

• Proposition:
Let X be a set, and suppose $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X iff it satisfies the following two conditions:
i) $\bigcup_{B \in \mathcal{B}} B = X$
ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \bigcap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \bigcap B_2$.

If so, there is a unique topology on X for which $\mathcal{B}$ is a basis, called the topology generated by $\mathcal{B}$.