MATH 725 HW#4

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Exercise (Exercise 1)**.** *Let V be a Hilbert space. Show that V has a Hilbert basis.*

Proof. We start our proof by showing that every Hilbert space has an orthonormal basis $\mathcal{B}$, and then we show that $\mathcal{B}$ is in fact maximal by showing that $\mathcal{B}^{\perp} = \{0\}$.

Let V be a Hilbert space. To see that $\mathcal{B}$ is an orthonormal basis for V, let $\mathcal{F}$ be the collection of all orthonormal subsets of V ordered by set inclusion. If $\Phi \subset \mathcal{F}$ is linearly ordered then $\cup\Phi$ is an upper bound. But then by Zorn's Lemma,[1] there exists a maximal element $\mathcal{B} \in \mathcal{F}$.

Now assume there exists a unit vector $x \in \mathcal{B}^{\perp} \smallsetminus \{0\}$. Then the set $\mathcal{B} \cup \{x\}$ is an orthonormal set properly containing $\mathcal{B}$, so $\mathcal{B}$ is not maximal. $(\Rightarrow\Leftarrow)$

Thus we have that $\mathcal{B}^{\perp} = \{0\}$, and it follows that every Hilbert space has a Hilbert basis, as desired. $\square$

Exercise (Exercise 2)**.** *Let V be an inner product space and let $A = \{u_i \mid i \in \Lambda\} \subseteq V$, where all the u_i are pointwise orthogonal. Show that A is linearly independent.*

Proof. Let A be defined as above and take scalars $\alpha_i \in \mathbb{F}$. Now suppose that

$$\alpha_1 u_1 + \cdots + \alpha_n u_n = 0.$$

Then, for any $k = 1, \ldots, n$, we have

$$\begin{aligned} 0 &= \langle \alpha_1 u_1 + \cdots + \alpha_n u_n, u_k \rangle \\ &= \alpha_k \langle u_k, u_k \rangle && \text{(Since } \langle u_i, u_j \rangle = 0 \text{ for } i \neq j \text{ due to orthogonality)} \\ \Longrightarrow \alpha_k &= 0 \quad \forall\ k. \end{aligned}$$

Hence, we have that A is linearly independent, as desired. $\square$

Exercise (Exercise 3)**.** *Let $L \in \mathcal{L}(V, W)$ be a bounded linear transformation, where V and W are Banach spaces. Show that its adjoint $L^* \in \mathcal{L}(W, V)$ is also bounded and determine the norm $||L^*||$.*

Proof. Let $v \in V$ and $w \in W$. Then notice that

$$\begin{aligned} |\langle v, L^* w \rangle| &= |\langle Lv, w \rangle| \\ &\leq \|Lv\| \|w\| && \text{(By the Cauchy-Schwarz Inequality)} \\ &\leq \|L\| \|v\| \|w\|, \end{aligned}$$

[1] Here's Zorn's lemma for reference:
Zorn's Lemma: If P is a partially ordered set in which every chain has an upper bound, then P has a maximal element.

which implies

$$\|L^*w\| \le \|L^*\|\|w\| \le \|L\|\|w\|,$$

and hence

$$\|L^*\| \le \|L\|. \tag{$\dagger$}$$

Thus we have shown that L^* is bounded (by the norm of L).

To determine the norm of L^*, notice the following:

$$\begin{aligned}
\|L^*\| = \sup_{w\in W:\|w\|=1} \|L^*w\| &= \sup_{\substack{v\in V:\|v\|=1\\ w\in W:\|w\|=1}} |\langle v, L^*w\rangle| \\
&= \sup_{\substack{v\in V:\|v\|=1\\ w\in W:\|w\|=1}} |\langle Lv, w\rangle| \\
&= \sup_{v\in V:\|v\|=1} \|Lv\| \\
&= \|L\|.
\end{aligned}$$

Thus we have shown that in fact $\|L^*\| = \|L\|$.

We have concluded our proof, but just for reference I am also including a slightly more complex proof that involves the *Hahn-Banach Theorem*:[2]

Suppose $v \in V$, with $Lv \neq 0$, and let

$$w_0 = \frac{Lv}{\|Lv\|} \in W,$$

so that, in particular, $\|w_0\| = 1$. Now let w be a functional such that

$$w(\lambda w_0) = \lambda$$

on the set $S \subset W$ of all elements of the form λw_0. Then we have that $\langle w, w_0\rangle = 1$, where $\|w\|_{\text{on } S} = 1$. Using the Hahn-Banach theorem, we can extend w to a functional on the whole space W such that $\|w\| = 1$ and

$$\langle w, w_0\rangle = 1, \qquad \text{i.e.,} \quad \langle w, Lv\rangle = \|Lv\|.$$

Therefore,

$$\|Lv\| = \langle Lv, w\rangle = |\langle v, L^*w\rangle| \le \|v\|\|L^*w\| \le \|v\|\|L^*\|\|w\| = \|v\|\|L^*\|,$$

which implies

$$\|L\| \le \|L^*\|.$$

Combining this result with the inequality obtained in ($\dagger$), we have that $\|L^*\| = \|L\|$. □

[2]Here's the Hahn-Banach theorem for reference:
(Hahn-Banach Theorem) A linear functional defined on a subspace of a vector space V and which is dominated by (i.e. bounded by) a sublinear function defined on V has a linear extension which is also dominated by the sublinear function.