MATH 3101 HW # 5

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Section 10

Problem 1 (Problem #27). *Let H be a subgroup of a group G and let $g \in G$. Define a one-to-one map of H onto Hg. Prove that your map is bijective.*

Solution. An obvious choice for such map $\phi_g \colon H \to Hg$ is defined by $\phi_g(h) = hg$, where we are fixing g and applying this map to all $h \in H$. Now we just need to show that this map is bijective. To show injection, notice that if $\phi_g(h_1) = \phi_g(h_2)$ for $h_1, h_2 \in H$, then $h_1 g = h_2 g$, hence $h_1 = h_2$ by the cancellation law. Lastly, to show surjection notice that for any element $hg \in Hg$, there exists $h \in H$ such that $\phi_g(h) = hg$. □

Problem 2 (Problem #28). *Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right coset Hg.*

Proof. ($gH \subseteq Hg$) Let $gh \in gH$ where $g \in G$ and $h \in H$. Then $gh = ghg^{-1}g = [(g^{-1})^{-1}hg^{-1}]g$ is in Hg because $(g^{-1})^{-1}hg^{-1}$ is in H by hypothesis. Thus gH is a subset of Hg, as desired.
($Hg \subseteq gH$) Now to prove the other direction, let $hg \in Hg$ where $g \in G$ and $h \in H$. Then $hg = gg^{-1}hg = g(g^{-1}hg)$ is in gH because $g^{-1}hg$ is in H by hypothesis. Thus Hg is a subset of gH and we have concluded our proof. □

Problem 3 (Problem #29). *Let H be a subgroup of a group G. Prove that if the partition of G into left cosets of H is the same as the partition into right cosets of H, then $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. (Note that this is the converse of Exercise 28 above)*

Proof. Let $h_1 \in H$ and $g \in G$. By hypothesis, we have that $Hg = gH$, which implies that $h_1 g = gh_2$ for some $h_2 \in H$. Then, by multiplying by g^{-1} from the left on both sides we get $g^{-1}(h_1 g) = g^{-1}(gh_2) \implies g^{-1}h_1 g = h_2$, showing that $g^{-1}h_1 g \in H$. □

Section 13

Problem 4 (Problem #50). *Let $\phi \colon G \to H$ be a group homomorphism. Show that $\phi[G]$ is abelian if and only if for all $x, y \in G$, we have $xyx^{-1}y^{-1} \in \ker(\phi)$.*

Proof. Let $x_1, y_1 \in \phi[G]$, where $\phi(x) = x_1$ and $\phi(y) = y_1$ for $x, y \in G$. Now we are going to prove both directions at once:

$$\begin{aligned}
\phi[G] \text{ is abelian} &\iff x_1 y_1 = y_1 x_1 \\
&\iff (y_1 x_1)^{-1} x_1 y_1 = x_1^{-1} y_1^{-1} x_1 y_1 = e' \quad \text{(where } e' \text{ is the identity in } \phi[G]) \\
&\iff \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = e' \\
&\iff \phi(x^{-1}y^{-1}xy) = e' \quad \text{(by the homomorphism property of } \phi) \\
&\iff x^{-1}y^{-1}xy \in \ker(\phi). \quad \square
\end{aligned}$$

Section 14

Problem 5 (Problem #21). *A student is asked to show that if H is a normal subgroup of an abelian group G, then G/H is abelian. The student's proof starts as follows:*

"We must show that G/H is abelian. Let a and b be two elements of G/H..."

a) *Why does the instructor reading this proof expect to find nonsense from here on in the student's paper?*
b) *What should the student have written?*
c) *Complete the proof.*

Solution. It seems like the the instructor expects to find nonsense because the student is not defining the elements of the quotient group, which are cosets, in the right format. The right format to define these elements is something like: *"For $a, b \in G$, let aH and bH be two elements of G/H..."*

Now we complete the proof:

Proof. Let G be abelian and H a normal subgroup of G. By the binary operation defined on quotient groups, for $a, b \in G$, we have $(aH)(bH) = (ab)H$. But then since G is abelian, we have $ab = ba$, which in turn indicates that $(ab)H = (ba)H = (bH)(aH)$. Thus $(aH)(bH) = (bH)(aH)$, which shows that G/H is abelian. $\square$

Problem 6 (Problem #22). *A **torsion group** is a group whose elements have finite order. A group is **torsion-free** if the identity is the only element of finite order. A student is asked to prove that if G is a torsion group, then so is G/H for every normal subgroup H of G. The student then writes*

"We must show that each element of G/H is of finite order. Let $x \in G/H$..."

a) *Why does the instructor reading this proof expect to find nonsense from here on in the student's paper?*
b) *What should the student have written?*
c) *Complete the proof.*

Solution. Again, we find the same misuse of notation as on *Problem 21.* The student should've written something like: *"We must show that each element of G/H is of finite order. For $x \in G$, let $xH \in G/H$..."*

Now we complete the proof:

Proof. Because G is a torsion group, we know that for any element $x \in G$, we must have $x^k = e$ for some positive integer k. Now, taking an arbitrary element $x \in G$ we compute

$$(xH)^k = x^k H = eH = H,$$

which shows that xH is of finite order. Since the coset xH is an arbitrary element of G/H, we see that G/H is indeed a torsion group, as we set out to prove. □

Problem 7 (Problem #26)**.** *Prove that the torsion subgroup T of an abelian group G is a normal subgroup of G, and that G/T is torsion-free.*

Proof. Let G be an abelian group and T a torsion subgroup of G (by a previous result, we know that the elements of G of finite order do indeed form a subgroup T of G.) But since G is abelian, we know that every subgroup of G is a normal subgroup, so T is normal in G.

Now suppose that for $x \in G$, we have that xT is of finite order in G/T; in particular, for some $k \in \mathbb{Z}$ we have that $(xT)^k = T$ (where T is the identity on G/T), and $x^k \in T$. Because T is a torsion group, we must have $(x^k)^n = x^{kn} = e$ in G for some $n \in \mathbb{Z}$. Thus x is of finite order in G, which implies that x lives in T, and we have that $xT = T$. Thus the only element of finite order in G/T is the identity T, hence G/T is a torsion-free group, as we set out to prove. □