Analytic Functions
Exam # 1

MARIO L. GUTIERREZ ABED

gutierrez.m101587@gmail.com

Problem 1. *Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.*

Solution. We use the following theorem from Conway's text:

From Conway's

Theorem. *Let G be a connected open set and let $f\colon G \to \mathbb{C}$ be an analytic function. Then the following are equivalent statements:*

- $f \equiv 0$.
- *There is a point $a \in G$ such that $f^{(n)}(a) = 0$ for each $n \geq 0$.*
- $\{z \in G\colon f(z) = 0\}$ *has a limit point in G.*

Now we let $g(z) = f(z) - e^z$ be analytic such that f restricted to $\mathbb{R}$ is e^x. Since $f(x) = e^x$ for $x \in \mathbb{R}$, we have that $\{z \in \mathbb{C} \mid g(z) = 0\}$ has a limit point in $\mathbb{R} \subset \mathbb{C}$. Applying the above theorem while letting $G = \mathbb{C}$, we have that $g(z) = 0$, i.e. $f(z) = e^z$. Hence we have that the only entire function f that satisfies $f(x) = e^x$ for $x \in \mathbb{R}$ is e^z. $\square$

Problem 2. *Show that an entire function f with $\mathfrak{Re}\, f > 0$ must be constant.*

Proof. Let $u = \mathfrak{Re}\, f$ and $v = \mathfrak{Im}\, f$. Then, since $u + iv$ is entire by assumption, so is $g = e^{-u+iv}$. Now note that

$$|g| = |e^{-u+iv}| = \frac{1}{e^u},$$

which is bounded by our assumption that $u > 0$. Then g is an entire bounded function, and hence (by Liouville's Theorem) it is constant. From this, it immediately follows that f is constant as well (g is constant $\implies$ $\log g = -u + iv$ is constant $\implies$ $f = u + iv$ is constant). $\square$

Problem 3. *Find a conformal mapping from a half open unit disk onto the open unit disk.*

Solution. Let $U = \{z \mid \mathfrak{Im}\, z > 0, |z| < 1\}$ be our half-open unit disk; i.e. $U = \mathring{\mathbb{D}}^2_+$, the upper half unit disk. It is clear that the map

$$f(z) = \frac{z - i}{z + i}$$

takes the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z > 0\}$ bijectively to the open unit disk $\mathring{\mathbb{D}}^2 = \{z \in \mathbb{C} : |z| < 1\}$. Since f is holomorphic on $\mathbb{C}_+$, it is clearly conformal. From a straight computation, we have

$$f^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

This is a conformal map that takes the unit disk $\mathring{\mathbb{D}}^2$ to the upper half plane $\mathbb{C}_+$ bijectively. It is not hard to check that $f_1 = f^{-1}|_{\mathring{\mathbb{D}}^2_+}$ maps $\mathring{\mathbb{D}}^2_+$ bijectively to the second quadrant $\mathcal{Q}_2 = \{z \mid \mathfrak{Re}\, z < 0, \mathfrak{Im}\, z > 0\}$.

Now, clearly the map $f_2(z) = -z$, which is the reflection with respect to the imaginary axis, maps $\mathcal{Q}_2$ bijectively onto the first quadrant $\mathcal{Q}_1 = \{z \mid \mathfrak{Re}\, z > 0, \mathfrak{Im}\, z > 0\}$. To fold up $\mathcal{Q}_1$ to the upper half plane $\mathbb{C}_+$, we apply the map $f_3(z) = z^2$. Now define $\Phi = f \circ f_3 \circ f_2 \circ f_1 \colon U \to \mathring{\mathbb{D}}^2$.

$$\underbrace{U = \mathring{\mathbb{D}}^2_+ \xrightarrow{f_1} \mathcal{Q}_2 \xrightarrow{f_2} \mathcal{Q}_1 \xrightarrow{f_3} \mathbb{C}_+ \xrightarrow{f} \mathring{\mathbb{D}}^2}_{\Phi}$$

Since each f_i and f are bijective and conformal, Φ is a conformal map that takes U bijectively onto the open unit disk $\mathring{\mathbb{D}}^2$, as desired. $\square$

Problem 4. *Let Ω be a region and let $f, g \colon \Omega \to \mathbb{C}$ be holomorphic functions satisfying $f(z)g(z) = 0$ for every $z \in \Omega$. Show that either $f \equiv 0$ or $g \equiv 0$.*

Proof. We can find a point $z_0 \in \Omega$ and a sequence $z_n \in \Omega$ which converges to z_0 but never equals z_0. For every n, we have $f(z_n)g(z_n) = 0$, so that either $f(z_n) = 0$ or $g(z_n) = 0$. Thus one of these sets $U = \{n \mid f(z_n) = 0\}$ and/or $V = \{n \mid g(z_n) = 0\}$ must be infinite. Assume, WLOG, that U is infinite. Then there is a subsequence z_{n_k} with $f(z_{n_k}) = 0$. But this implies that f is identically 0 on Ω, as desired. $\square$

Problem 5. *Show that in an arbitrarily small punctured*[1] *disk $\mathring{\mathbb{D}}^2_\varepsilon = \{z : 0 < |z| < \varepsilon\}$ the function $f(z) = e^{1/z}$ takes every nonzero value infinitely often.*

Proof. First we show that under the map $z \mapsto 1/z$, each point of $\mathring{\mathbb{D}}^2_\varepsilon$ is in bijection with the set $\{z : |z| > 1/\varepsilon\}$. Indeed, $z \mapsto 1/z$ is clearly injective, so we only need to show surjectivity. Let $\widetilde{z}$ satisfy $|\widetilde{z}| > 1/\varepsilon$. Then $0 < |1/\widetilde{z}| < \varepsilon$, so $1/\widetilde{z}$ is an element of $\mathring{\mathbb{D}}^2_\varepsilon$ which maps to $\widetilde{z}$ under $z \mapsto 1/z$.

Notice that $e^{1/z} \neq 0$ for any z, so the range of f only takes nonzero values. Given any nonzero w, we want to show that $e^{1/z} = w$ has infinitely many solutions in $\mathring{\mathbb{D}}^2_\varepsilon$. By what we just showed, this is equivalent to showing that $e^{\widetilde{z}} = w$ has infinitely many solutions in the set $\{\widetilde{z} : |\widetilde{z}| > 1/\varepsilon\}$.

Suppose w has polar form $Re^{i\theta}$. Then we can let $\widetilde{z} = \log R + (\theta + 2n\pi)i$, where $n \in \mathbb{Z}$, and we have

$$e^{\widetilde{z}} = e^{\log R + (\theta + 2n\pi)i} = Re^{it} = w,$$

as desired. However, notice that there are infinitely many points of the form $\log R + (\theta + 2n\pi)i$ with only finitely many of them lying in the disk $\{\widetilde{z} : |\widetilde{z}| < 1/\varepsilon\}$. Hence an infinite number of these points lie in the set $\{\widetilde{z} : |\widetilde{z}| > 1/\varepsilon\}$, and we are done. $\square$

[1]On the exam sheet it says "arbitrarily small disk," but I am adding the "punctured'" condition because note that f is not even defined at 0, so the origin cannot be in the domain of f.