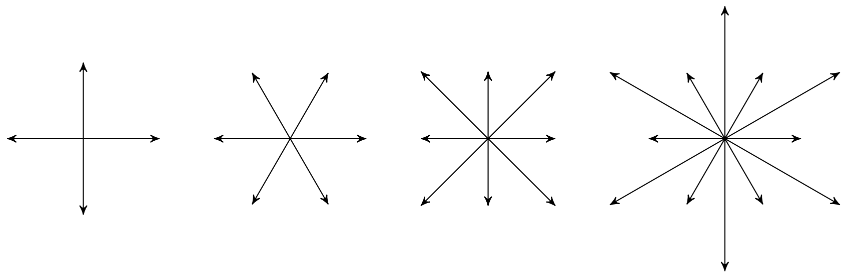
Root systems

Root systems are very important objects, and encode many combinatorial properties that are used throughout mathematics. Since as humans we like to draw pictures to understand things, this workshop mainly focuses on the root systems inside $\mathbb{R}^2$, and the purpose is to prove that the only such root systems are



It turns out that the above root systems correspond to the Dynkin diagrams $A_1 \times A_1$, A_2, B_2 and G_2 from Workshop 1, although we will not prove this here.

Recall that a *euclidean space* E is a real vector space equipped with a positive definite symmetric bilinear form $(-,-)$ (the *inner product*). A subset R of E is called a *root system* if

(R1) R is finite, it spans E, and it does not contain 0.

(R2) If $\alpha \in R$, then the only scalar multiples of α also in R are $\pm\alpha$.

(R3) If $\alpha \in R$, then the reflection $s_\alpha : E \to E$ defined

$$s_\alpha(x) := x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$$

permutes the elements of R.

(R4) If $\alpha, \beta \in R$, then $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$. (Notice that $\langle -,- \rangle$ is only linear in the first slot!)

The standard example of a euclidean space is $\mathbb{R}^n$, with inner product (the so-called "dot product")

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} x_i y_i.$$

The standard example of a root system comes from Lie algebras: if L is a semisimple Lie algebra, consider $E_{\mathbb{R}}$ (defined to be the $\mathbb{R}$-span of all the roots in H^*), equipped with the Killing form, together with the set R of roots.

Angles can be defined for any euclidean space E (motivated by $\mathbb{R}^n$!). Indeed, for $0 \neq u, v \in E$, the angle θ between u and v is defined to be

$$\cos\theta := \frac{(u,v)}{\sqrt{(u,u)}\sqrt{(v,v)}}.$$

Since we have angles, we will often just pretend that our euclidean space is $\mathbb{R}^n$, since this will allow us to draw pictures!

1. Suppose that $R \subset E$ is a root system. Just using the axioms of root systems:
   (a) If $\alpha, \beta \in R$, deduce from the angle formula that

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3, 4\}.$$

Solution: Squaring both sides of the angle formula gives

$$\cos^2\theta = \frac{(u,v)^2}{(u,u)(v,v)}.$$

Thus

$$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 4\frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta,$$

where θ is the angle between α and β. It is thus clear that $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle \geq 0$, and also clear that $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle \leq 4$. But by (R4) it must be an integer, hence the result follows.

(b) If $\alpha,\beta \in R$ with $\beta \neq \pm\alpha$, deduce that $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle \in \{0,1,2,3\}$.

Solution: The only way that $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4$ is if $\cos^2\theta = 1$, in which case θ is an integer multiple of π. But this then implies that α and β are linearly dependent, which contradicts the fact that $\beta \neq \pm\alpha$.

In what follows, suppose that $R \subset E$ is a root system, and $\alpha,\beta \in R$ with $\beta \neq \pm\alpha$. By changing the labels if necessary, we can further assume that $(\beta,\beta) \geq (\alpha,\alpha)$.

2. Deduce that $|\langle\beta,\alpha\rangle| \geq |\langle\alpha,\beta\rangle|$.

Solution: Since $(\beta,\beta) \geq (\alpha,\alpha)$, this is just

$$|\langle\beta,\alpha\rangle| := \frac{2|(\beta,\alpha)|}{(\alpha,\alpha)} \geq \frac{2|(\alpha,\beta)|}{(\beta,\beta)} = |\langle\alpha,\beta\rangle|.$$

3. Use Q1 and Q2 to complete the following table of possibilities:

$\langle\beta,\alpha\rangle$	$\langle\alpha,\beta\rangle$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1		
2			
-2			
3			
-3			

Solution: Since $\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle \in \{0,1,2,3\}$ we obtain

$\langle\beta,\alpha\rangle$	$\langle\alpha,\beta\rangle$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
2	1	$\frac{\pi}{4}$	2
-2	-1	$\frac{3\pi}{4}$	2
3	1	$\frac{\pi}{6}$	3
-3	-1	$\frac{5\pi}{6}$	3

Given $\alpha, \beta \in R$, to classify root systems we need to know when $\alpha+\beta \in R$. The above table will be useful, and gives the following. As notation, let $\measuredangle(\alpha, \beta)$ denote the angle between α and β.

4. Using (R3) and the above table, prove that:
   (a) If $\frac{\pi}{2} < \measuredangle(\alpha, \beta) < \pi$, then $\alpha + \beta \in R$.
   (b) If $0 < \measuredangle(\alpha, \beta) < \frac{\pi}{2}$ and $(\beta, \beta) \geq (\alpha, \alpha)$, then $\alpha - \beta \in R$.

Solution: In either case (by relabelling α and β in part (a) if necessary) we can assume that $(\beta, \beta) \geq (\alpha, \alpha)$, so that we can use the above table freely. By (R3),

$$s_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta \in R.$$

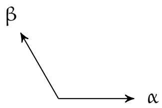
(a) If $\frac{\pi}{2} < \measuredangle(\alpha, \beta) < \pi$ the above table shows that $\langle \alpha, \beta \rangle = -1$, hence $\alpha - (-\beta) = \alpha + \beta \in R$.
(b) Similarly, if $0 < \measuredangle(\alpha, \beta) < \frac{\pi}{2}$, the table shows that $\langle \alpha, \beta \rangle = 1$.

We now classify the root systems in $\mathbb{R}^2$ (with the standard inner product). For this, suppose R is a root system. We first *choose* a root α of smallest possible length. Since R spans $\mathbb{R}^2$ there must be another root, β, such that $\beta \neq \pm\alpha$. By considering $-\beta$ if necessary, we can assume that

$$\frac{\pi}{2} \leq \theta := \measuredangle(\alpha, \beta) < \pi.$$

Moreover, by choosing a different β if necessary, we can further assume that this angle θ is as large as possible. From the table there are four options: $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$.

5. Consider the case $\theta = \frac{2\pi}{3}$. By the last column in the table, the length of β equals the length of α, so we begin



   Further, since θ is as large as possible, there are no roots in the space between β and $-\alpha$. Using (R1) then Q4 (or (R3)), show that we obtain the second root system at the top of page 1.

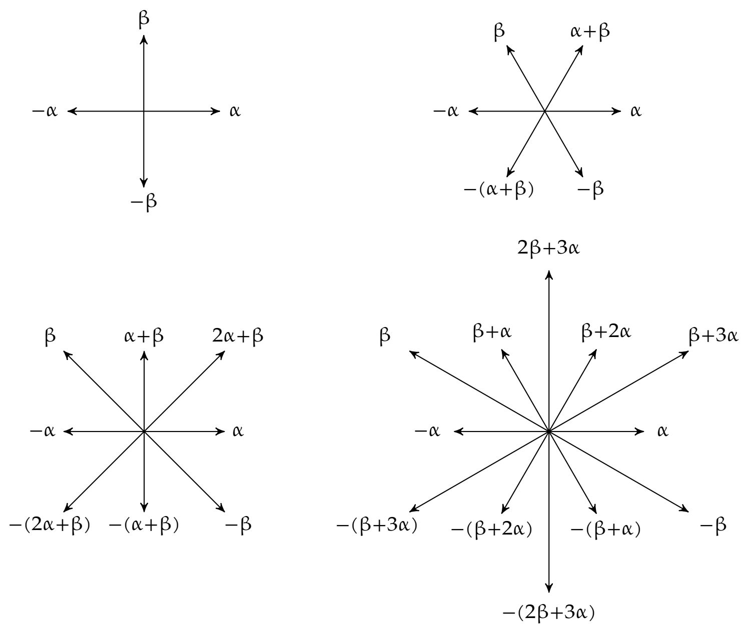
6. Repeating this analysis, show that the cases $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ (in order) correspond to the root systems (in order) at the top of page 1.

Solution: The analysis is always very similar. The idea is to pick two roots α and β with $(\beta, \beta) \geq (\alpha, \alpha)$ and with $\pi > \measuredangle(\alpha, \beta) \geq \frac{\pi}{2}$ as large as possible. Then use the reflections s_α and s_β on α and β repeatedly until we fill in the root system. It is enough to use s_α and s_β because of the identity

$$s_{s_\alpha(\beta)} = s_\alpha \circ s_\beta \circ s_\alpha,$$

which you can easily show using that reflections are orthogonal and idempotent. We then check that the properties (R1) to (R4) for a root system are satisfied and that we cannot add any further roots.

With labels, the roots systems are



Let us work out, say, the case of $\theta = \frac{5\pi}{6}$, which is the last of the root systems. This is the last line in the table, so that $\langle \beta, \alpha \rangle = -3$ and $\langle \alpha, \beta \rangle = -1$. In particular, $(\beta, \beta) = 3(\alpha, \alpha)$. We apply the reflections in turn, but we stop when we arrive at a root we already have:

$$s_\alpha(\beta) = \beta + 3\alpha \qquad s_\beta(\beta + 3\alpha) = 2\beta + 3\alpha \qquad s_\beta(\alpha) = \beta + \alpha \qquad s_\alpha(\beta + \alpha) = \beta + 2\alpha$$

and of course, their negatives. This already fills in the last of the root systems above. It is clear that (R1)-(R3) are satisfied. To see that (R4) is satisfied, notice that since $\langle \alpha, \beta \rangle \in \mathbb{Z}$ and $\langle \beta, \alpha \rangle \in \mathbb{Z}$, it also follows that for all $\gamma \in R$, $\langle \gamma, \alpha \rangle \in \mathbb{Z}$ and $\langle \gamma, \beta \rangle$, since γ is an integer linear combination of α and β. But then since $\langle \gamma, \delta \rangle \langle \delta, \gamma \rangle \in \mathbb{Z}$ for all $\gamma, \delta \in R$, it follows that for all $\gamma \in R$, $\langle \alpha, \gamma \rangle \in \mathbb{Z}$ and $\langle \beta, \gamma \rangle \in \mathbb{Z}$. Therefore again the same is true for anything in the $\mathbb{Z}$-span of α and β, proving that (R4) is satisfied. Finally, to show that we cannot add any more roots, we notice that since $\measuredangle(\alpha, \beta)$ has been chosen to be as large as possible, there are no roots between $-\alpha$ and β and hence, acting with the reflections, there can be no additional roots between any other two roots in R.

Please hand in your solution to either Q5 or one of the parts of Q6 by the start of lecture on Monday 20 November.