

TRRT Midterm Lie Subalgebras of a Direct Sum

MARIO L. GUTIERREZ ABED

s1685113@sms.ed.ac.uk

Direct Sum Lie Algebra

Let L, R be finite-dimensional complex Lie algebras and let $L \oplus R$ denote their direct sum Lie algebra. Recall that elements of $L \oplus R$ are pairs (ℓ, r) with $\ell \in L$ and $r \in R$, and that their bracket is given by

$$[(\ell_1, r_1), (\ell_2, r_2)] = ([\ell_1, \ell_2], [r_1, r_2]). \tag{1}$$

We are now going to prove a structure theorem which characterizes the Lie subalgebras of $L \oplus R$.

Let *K* be a subalgebra of $L \oplus R$ and let

$$K_{L} = \{ \ell \in L \mid (\ell, r) \in K, \exists r \in R \}$$

$$K_{R} = \{ r \in R \mid (\ell, r) \in K, \exists \ell \in L \}$$

$$I_{L} = \{ \ell \in L \mid (\ell, 0) \in K \}$$

$$I_{R} = \{ r \in R \mid (0, r) \in K \}.$$

Note that if $(\ell, r) \in K$, then $\ell \in K_L$ and $r \in K_R$.

Problem 1. Show that K_L and K_R are Lie subalgebras of L and R, respectively.

Proof. First of all, note that K_L and K_R are linear subspaces of L and R, respectively. To see this, take, say, two elements $\ell_1, \ell_2 \in L$ such that there exist $r_1, r_2 \in R$ that satisfy $(\ell_1, r_1) \in K$ and $(\ell_2, r_2) \in K$; in other words, $\ell_1, \ell_2 \in K_L$. But K is subalgebra of $L \oplus R$ (by assumption), so any linear combination of elements in K must also lie in K; in particular,

$$(\ell_1, r_1) + (\ell_2, r_2) = (\ell_1 + \ell_2, r_1 + r_2) \in K \implies \ell_1 + \ell_2 \in K_L$$
 (Note that $r_1 + r_2 \in R$, since R is a linear space).

A similar argument applies to K_R . Hence all that's left is to show that K_L and K_R are closed under Lie brackets, but this follows immediately from (1) ...

Take $\ell_1, \ell_2 \in K_L$ as above; then

$$\left[\left(\ell_{1},r_{1}\right),\left(\ell_{2},r_{2}\right)\right]=\left(\underbrace{\left[\ell_{1},\ell_{2}\right]}_{\in\ L\text{, since L is a Lie algebra}},\underbrace{\left[r_{1},r_{2}\right]}_{\in\ R\text{, since R is a Lie algebra}}\right)\in K.$$

Therefore K_L is closed under the Lie bracket of the ambient Lie algebra L, and it is thus a (Lie) subalgebra of L. This argument yields the desired result for K_R as well.



Problem 2. Show that I_L and I_R are ideals of K_L and K_R , respectively.

Proof. Let us show the right side some love this time and prove that I_R is an ideal of K_R ; the same argument is extended trivially to I_L and K_L . First we show that I_R is closed under linear combinations: Let $r_1, r_2 \in I_R$; then

$$(0,r_1) + (0,r_2) = (0,\underbrace{r_1, +r_2}_{\in R})$$

$$= (0,\widetilde{r}) \qquad (\text{for some } \widetilde{r} \in R).$$

Hence linear combinations of elements in I_R also lie in I_R .

Now let $r' \in I_R$, $r \in K_R$, and $\ell \in K_L$. In order to show that I_R is an ideal of K_R , what we need to show is that $[r', r] \in I_R$. Making use of (1),

$$[(0,r')],(\ell,r)] = (\underbrace{[0,\ell]}_{=0},\underbrace{[r',r]}_{\in R})$$

$$= (0,r'') \qquad (for some r'' \in R).$$

But the last line above shows that, indeed, $r'' = [r', r] \in I_R$, as desired.

Problem 3. By Q2, K_L/I_L and K_R/I_R are Lie algebras, and we will now prove that they are isomorphic. Indeed, we define a map $f: K_L \to K_R/I_R$ as follows: If $\ell \in K_L$, there is some $r \in R$ such that $(\ell, r) \in K$; then we define $f(\ell) = r + I_R$. Now,

- *a)* Show that f is well defined.
- b) Show that f is a homomorphism.
- c) Show that f is surjective.
- *d)* Show that ker $f = I_L$.
- *e)* Show that $K_L/I_L \cong K_R/I_R$.

Proof of a). This one is quite straightforward. In order for f to be well defined, if $\ell_1 = \ell_2$ for some $\ell_1, \ell_2 \in K_L$, then we would need to have $f(\ell_1) = f(\ell_2)$. But if $\ell_1 = \ell_2$, then they have the same image under the injection $\iota \colon K_L \hookrightarrow K$ given by ℓ_1 or $\iota \mapsto (\ell_1$ or $\iota \mapsto (\ell_1$ or $\iota \mapsto (\ell_2)$; i.e., the same $\iota \mapsto (\ell_2)$ is common to both ℓ_1 and ℓ_2 under this injection. Then,

$$f(\ell_1) = r + I_R$$

= $f(\ell_2)$.

¹I did not multiply by a complex scalar here, but it is rather obvious that the result still holds, scalar included or otherwise. I use this slight simplification throughout this paper.



Proof of b). It is rather obvious that f is linear. In order to show that f is a (Lie) homomorphism, we need to demonstrate that f preserves Lie brackets; i.e., that for some $\ell_1, \ell_2 \in K_L$, we have

$$f([\ell_1, \ell_2]) = [f(\ell_1), f(\ell_2)].$$

Consider (ℓ_1, r_1) , (ℓ_2, r_2) , and $([\ell_1, \ell_2], [r_1, r_2])$, all in K, so that

$$f([\ell_1, \ell_2]) = [r_1, r_2] + I_R$$
, $f(\ell_1) = r_1 + I_R$, and $f(\ell_2) = r_2 + I_R$.

Then

$$f([\ell_1, \ell_2]) = [r_1, r_2] + I_R$$

= $[r_1 + I_R, r_2 + I_R]$
= $[f(\ell_1), f(\ell_2)].$

Proof of c). Since we are dealing with finite-dimensional spaces, the following relation is key:

$$\dim K_L = \dim \ker f + \dim \operatorname{im} f.$$

But then

$$\dim \operatorname{im} f = \dim K_L - \dim \ker f$$

$$= \dim K_L - \dim I_L \qquad (\text{By part } d))$$

$$= \dim K_L / I_L.$$

Thus, the rank of f equals the dimension of K_L/I_L . But K_L , when modded by I_L , consists of elements of the form

$$K_L$$
 - elements of $I_L = \{ \ell \in L \mid (\ell, r) \in K, \exists r \in R, r \neq 0 \}.$

Hence, by dimensionality, for every nonzero r with quotient representative $r + I_R \in K_R / I_R$ ($r \notin I_R$), there is a corresponding $\ell \in K_L$ mapped by f. For those $r \in I_R$ (the zero elements), $r + I_R = I_R$ also has a preimage ℓ that lies in $I_L \subset K_L$ (c.f. part d) below). Thus f is indeed surjective. \square

Proof of d). By construction,

$$I_L = \{ \ell \in L \mid (\ell, r) \in K, r = 0 \}.$$

But we defined our map $f: K_L \to K_R/I_R$ such that, if $\ell \in K_L$, there is some $r \in R$ such that $(\ell, r) \in K$ and $f(\ell) = r + I_R$. Thus, if ℓ happens to lie in $I_L \subset K_L$, then r = 0 and $(\ell, 0) \in K$ with $f(\ell) = 0 + I_R = I_R$. This shows that I_L is indeed the nullspace of f.

Proof of e). By the *First Isomorphism Theorem*,

$$K_L/I_L = K_L/\ker f \cong \operatorname{im} f$$

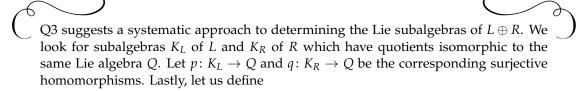
= K_R/I_R (By surjectivity of f).



Problem 4. Prove that

$$\dim K = \dim K_L + \dim I_R = \dim K_R + \dim I_L. \tag{2}$$

Proof. From the result on Q3e), we have that $K_L/I_L \cong K_R/I_R$. Thus dim K_L – dim I_L = dim K_R – dim I_R , which is equivalent to the rightmost equality of (2). Now, to show that K is also of the same dimension, consider the map $\varphi \colon K_R/I_R \to K$ given by $r+I_R \mapsto (\ell,r)$, where this ℓ is the preimage of r by the mapping f from Q3. That φ is well defined and surjective is obvious, and with regards to injectivity note that $(0,0) \in K$ clearly implies that $r \in I_R$; i.e., the kernel of φ is trivial. Hence φ is a linear isomorphism, and (2) holds.

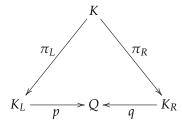




Problem 5. *Show that K is a Lie subalgebra of* $L \oplus R$ *of dimension*

$$\dim K = \dim K_L + \dim K_R - \dim Q. \tag{4}$$

Note that the "obvious" subalgebras (namely, those which are themselves direct sums) are the ones for which Q = 0 and hence $K = K_L \oplus K_R$.



Proof. First we show that K is a linear subspace. Take $\ell_1, \ell_2 \in K_L$ and $r_1, r_2 \in K_R$ such that $p(\ell_i) = q(r_i)$, for i = 1, 2. Then,

$$(\ell_1, r_1) + (\ell_2, r_2) = \underbrace{\left(\underbrace{\ell_1 + \ell_2}_{\in K_L}, \underbrace{r_1 + r_2}_{\in K_L \oplus K_R}\right)}_{\in K_L \oplus K_R}$$



Moreover, since p and q are homomorphisms, we have

$$p(\ell_1 + \ell_2) = p(\ell_1) + p(\ell_2)$$

= $q(r_1) + q(r_2)$
= $q(r_1 + r_2)$.

Thus *K* is indeed closed under linear combinations, as desired.

Next we show that *K* is closed under Lie brackets. Take ℓ_1, ℓ_2, r_1, r_2 as before. Then,

$$[(\ell_1, r_1), (\ell_2, r_2)] = \underbrace{\left(\underbrace{[\ell_1, \ell_2], \underbrace{[r_1, r_2]}_{\in K_L}}_{\in K_L \oplus K_R}\right)}.$$

Moreover, since p and q are (Lie) homomorphisms, they preserve brackets; i.e., we have

$$p([\ell_1.\ell_2]) = [p(\ell_1), p(\ell_2)]$$

$$= [q(r_1), q(r_2)]$$

$$= q([r_1, r_2]).$$

Thus *K* is indeed closed under Lie brackets, and we have proven that *K* is a Lie subalgebra of $L \oplus R$.

Lastly, we show the dimension of K. Since $Q \cong K_R/I_R \cong K_L/I_L$, and we are dealing with finite dimensional linear spaces, these must all have the same dimension. Thus, we recast (4):

$$\dim K_L + \dim K_R - \dim Q = \dim K_L + \underbrace{\dim K_R - \dim K_R / I_R}_{\dim I_R}$$

$$= \dim K_L + \dim I_R$$

$$= \dim K. \qquad (By (2))$$