Geometry of General Relativity
Workshop 3 Hand-In

Mario L. Gutierrez Abed

s1685113@sms.ed.ac.uk

Problem (WS3 Problem 6). *Let T be a $(1,1)$ tensor field, λ a covector field and X, Y vector fields.*

a) *Using the inner Leibniz rule write down an expression for $(\nabla_X T)(\lambda, Y)$. Prove that this does indeed define a $(1,1)$ tensor field $\nabla_X T$.*

b) *Prove that $\nabla_a f^b = -\Gamma^b{}_{ca} f^c$, where $\{f^a\}$ and $\{e_a\}$ are dual bases of covector and vector fields. Hence show that the components of the $(1,2)$ tensor ∇T are*

$$\nabla_c T^a{}_b = e_c(T^a{}_b) + \Gamma^a{}_{dc} T^d{}_b - \Gamma^d{}_{bc} T^a{}_d .$$

Deduce the Kronecker delta tensor is covariantly constant; i.e., $\nabla \delta = 0$.

c) *Repeat part (a) for the Lie derivative and hence write down the components of $\mathcal{L}_X T$ in a coordinate basis. Evaluate $\mathcal{L}_X \delta$.*

*Solution to **a)***. According to the inner Leibniz rule,

$$\nabla_X(T(\lambda, Y)) = (\nabla_X T)(\lambda, Y) + T(\nabla_X \lambda, Y) + T(\lambda, \nabla_X Y), \tag{♠}$$

so that

$$(\nabla_X T)(\lambda, Y) = \nabla_X(T(\lambda, Y)) - T(\nabla_X \lambda, Y) - T(\lambda, \nabla_X Y).$$

Now to demonstrate that $\nabla_X T$ is indeed a $(1,1)$ tensor field, we must show C^∞-bilinearity.[1] Let Ψ be any smooth function. Then,

$$\begin{aligned}
(\nabla_X T)(\Psi\lambda, Y) &= \nabla_X(T(\Psi\lambda, Y)) - T(\nabla_X(\Psi\lambda), Y) - T(\Psi\lambda, \nabla_X Y) \\
&= \nabla_X(\Psi T(\lambda, Y)) - T(\nabla_X(\Psi\lambda), Y) - \Psi T(\lambda, \nabla_X Y) && \text{(Bilinearity of } T) \\
&= X(\Psi) T(\lambda, Y) + \Psi \nabla_X(T(\lambda, Y)) - T(X(\Psi)\lambda + \Psi \nabla_X \lambda, Y) - \Psi T(\lambda, \nabla_X Y) && \text{(Leibniz of } \nabla_X) \\
&= X(\Psi) T(\lambda, Y) + \Psi \nabla_X(T(\lambda, Y)) - X(\Psi) T(\lambda, Y) - \Psi T(\nabla_X \lambda, Y) - \Psi T(\lambda, \nabla_X Y) \\
&= \Psi \nabla_X(T(\lambda, Y)) - \Psi T(\nabla_X \lambda, Y) - \Psi T(\lambda, \nabla_X Y) \\
&= \Psi \left[\nabla_X(T(\lambda, Y)) - T(\nabla_X \lambda, Y) - T(\lambda, \nabla_X Y)\right] \\
&= \Psi (\nabla_X T)(\lambda, Y).
\end{aligned}$$

A similar calculation shows C^∞ linearity on the contravariant slot. $\boxed{\text{Victoria!}}$

*Proof of **b)***. For vector fields X, Y and covector field λ, the Leibniz rule gives us

$$\nabla_X(\lambda \otimes Y) = (\nabla_X \lambda) \otimes Y + \lambda \otimes \nabla_X Y.$$

[1] In fact, we show linearity in one slot only, since by the way $\nabla_X T$ is defined, if linearity holds on either slot, it will obviously hold on the other.

Then applying the contraction $C(\lambda \otimes Y) = \langle \lambda, Y \rangle$ and using the commutativity of ∇_X with contractions, we get

$$\nabla_X \langle \lambda, Y \rangle = \langle \nabla_X \lambda, Y \rangle + \langle \lambda, \nabla_X Y \rangle,$$

or

$$\langle \nabla_X \lambda, Y \rangle = \nabla_X \langle \lambda, Y \rangle - \langle \lambda, \nabla_X Y \rangle. \tag{$\clubsuit$}$$

Now letting $X = e_a, Y = e_b$, and $\lambda = f^c$ on ($\clubsuit$), we get

$$\begin{aligned}
\langle \nabla_a f^c, e_b \rangle &= \nabla_a \langle f^c, e_b \rangle - \langle f^c, \nabla_a e_b \rangle \\
&= \underbrace{\nabla_a \delta^c{}_b}_{=0} - \langle f^c, \Gamma^r_{ba} e_r \rangle \\
&= -\Gamma^r_{ba} \langle f^c, e_r \rangle \\
&= -\Gamma^r_{ba} \delta^c{}_r \\
&= -\Gamma^c_{ba}.
\end{aligned}$$

This result implies that $\nabla_a f^c = -\Gamma^c_{da} f^d$, as desired.[2]

Now to determine the components $(\nabla T)^b{}_{c;a} = \nabla_a T^b{}_c$ of the $(1,2)$ tensor ∇T, we put $X = e_a$, $\lambda = f^b$, and $Y = e_c$ on ($\spadesuit$). Then,

$$\begin{aligned}
\nabla_a (T(f^b, e_c)) = \nabla_a T^b{}_c &= (\nabla_a T)(f^b, e_c) + T(\nabla_a f^b, e_c) + T(f^b, \nabla_a e_c) \\
&= (\nabla_a T)^b{}_c + T(-\Gamma^b_{da} f^d, e_c) + T(f^b, \Gamma^r_{ca} e_r) \\
&= e_a(T^b{}_c) - \Gamma^b_{da} T^d{}_c + \Gamma^r_{ca} T^b{}_r.
\end{aligned}$$

From this result we can easily see that $\nabla \delta = 0$: in components,

$$\begin{aligned}
\nabla_a \delta^b{}_c &= (\nabla_a \delta)(f^b, e_c) + \delta(\nabla_a f^b, e_c) + \delta(f^b, \nabla_a e_c) \\
&= (\nabla_a \delta)^b{}_c + \delta(-\Gamma^b_{da} f^d, e_c) + \delta(f^b, \Gamma^r_{ca} e_r) \\
&= \underbrace{e_a(\delta^b{}_c)}_{=0} - \Gamma^b_{da} \delta^d{}_c + \Gamma^r_{ca} \delta^b{}_r \\
&= -\Gamma^b_{ca} + \Gamma^b_{ca} \\
&= 0.
\end{aligned}$$

$\boxed{\text{Victoria!}}$

*Solution to **c)**.* Analogous to ($\spadesuit$), we have

$$\mathcal{L}_X (T(\lambda, Y)) = (\mathcal{L}_X T)(\lambda, Y) + T(\mathcal{L}_X \lambda, Y) + T(\lambda, \mathcal{L}_X Y),$$

so that

$$(\mathcal{L}_X T)(\lambda, Y) = \mathcal{L}_X (T(\lambda, Y)) - T(\mathcal{L}_X \lambda, Y) - T(\lambda, \mathcal{L}_X Y).$$

In order to show that $\mathcal{L}_X T$ is a $(1,1)$ tensor field, we show C^∞ bilinearity.[3] Let Ξ be any smooth function; then

$$\begin{aligned}
(\mathcal{L}_X T)(\Xi \lambda, Y) &= \mathcal{L}_X (T(\Xi \lambda, Y)) - T(\mathcal{L}_X (\Xi \lambda), Y) - T(\Xi \lambda, \mathcal{L}_X Y) \\
&= \mathcal{L}_X (\Xi\, T(\lambda, Y)) - T(X(\Xi)\, \lambda + \Xi \mathcal{L}_X \lambda, Y) - \Xi T(\lambda, \mathcal{L}_X Y) \\
&= X(\Xi)\, T(\lambda, Y) + \Xi\, \mathcal{L}_X (T(\lambda, Y)) - X(\Xi) T(\lambda, Y) - \Xi\, T(\mathcal{L}_X \lambda, Y) - \Xi T(\lambda, \mathcal{L}_X Y) \\
&= \Xi\, \mathcal{L}_X (T(\lambda, Y)) - \Xi\, T(\mathcal{L}_X \lambda, Y) - \Xi T(\lambda, \mathcal{L}_X Y) \\
&= \Xi\, [\mathcal{L}_X (T(\lambda, Y)) - T(\mathcal{L}_X \lambda, Y) - T(\lambda, \mathcal{L}_X Y)] \\
&= \Xi\, (\mathcal{L}_X T)(\lambda, Y).
\end{aligned}$$

[2] Indeed, a quick check: $\langle -\Gamma^c_{da} f^d, e_b \rangle = -\Gamma^c_{da} \langle f^d, e_b \rangle = -\Gamma^c_{da} \delta^d{}_b = -\Gamma^c_{ba}.$ ✓

[3] As before, showing linearity in one slot only will suffice.

A similar calculation shows C^∞ linearity on the contravariant slot.

Now to determine the components of $\mathcal{L}_X T$ in a coordinate basis, recall that (analogous to equation (2.74) from our course notes)

$$(\mathcal{L}_X T)(\lambda, Y) = X[T(\lambda, Y)] - T(\mathcal{L}_X \lambda, Y) - T(\lambda, [X, Y]). \tag{$\heartsuit$}$$

Thus, in a basis (putting $X = e_a$, $\lambda = f^b$, and $Y = e_c$), we get

$$\begin{aligned}
(\mathcal{L}_a T)(f^b, e_c) &= \mathcal{L}_a(T(f^b, e_c)) - T(\mathcal{L}_a f^b, e_c) - T(f^b, \mathcal{L}_a e_c) \\
&= e_a(T^b{}_c) - T(\mathcal{L}_a f^b, e_c) - T(f^b, \underbrace{[e_a, e_c]}_{=0}) \\
&= e_a(T^b{}_c) - T(\mathcal{L}_a f^b, e_c).
\end{aligned}$$

Question to review on workshop: How can I simplify $T(\mathcal{L}_a f^b, e_c)$ further??

As for $\mathcal{L}_X \delta$, recall that the $(1,1)$ tensor field δ acting on a covector λ and a vector Y yields $\delta(\lambda, Y) = \langle \lambda, Y \rangle = \lambda(Y)$. Using this and (♡) we get

$$\begin{aligned}
(\mathcal{L}_X \delta)(\lambda, Y) &= X(\delta(\lambda, Y)) - \delta(\mathcal{L}_X \lambda, Y) - \delta(\lambda, [X, Y]) \\
&= X(\lambda(Y)) - (\mathcal{L}_X \lambda)(Y) - \lambda([X, Y]) \\
&= X(\lambda(Y)) - X(\lambda(Y)) + \lambda([X, Y]) - \lambda([X, Y]) \\
&= 0,
\end{aligned}$$

where on the third line we used

$$(\mathcal{L}_X \lambda)(Y) = X(\lambda(Y)) - \lambda([X, Y]),$$

which is a result we have derived previously in the course (cf. equation (2.72) from our course notes). $\boxed{\text{Victoria!}}$