MATH 710 HW # 5

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJIAN

Exercise 1 (Exercise 2-1 [DoCarmo]). *Let M be a Riemannian manifold. Let $v \in T_{c(t_0)}M$ and consider the mapping*

$$P = P_{c,t_0,t} \colon T_{c(t_0)}M \to T_{c(t)}M$$

that takes $v \mapsto X_v|_{c(t)}$, where X_v is the parallel vector field along c such that $X_v|_{c(t_0)} = v$, i.e., $X_v|_{c(t)}$ is the vector obtained by parallel transporting the vector v along the curve c. Show that P is a (linear) isometry.

Proof. It is easy to see that P is a linear isometry. We know that for any two vectors $v, u \in T_{c(t_0)}M$, there exist unique parallel vector fields X_v, Y_u such that $X_v|_{c(t_0)} = v$ and $Y_u|_{c(t_0)} = u$. Thus,

$$(\heartsuit) \qquad \begin{aligned} \langle v, u\rangle_{c(t_0)} &= \langle X_v, Y_u\rangle_{c(t_0)} \\ &= \langle X_v, Y_u\rangle_{c(t)} \\ &= \langle P(v), P(u)\rangle_{c(t)}. \end{aligned}$$

Note that $(\heartsuit)$ holds, since X_v, Y_u are both parallel, and thus by definition $\frac{DX_v}{\mathrm{d}t} = \frac{DY_u}{\mathrm{d}t} = 0$, i.e., there is no change whatsoever on these vector fields as t varies and thus the inner product remains constant for all t for which the curve is defined. (Of course, in order for this argument to be valid, I'm assuming that our connection ∇ on $(M, \langle , \rangle)$ is compatible with the metric $\langle , \rangle$, i.e. for any smooth curve γ on M and any pair of parallel vector fields P and P' along γ, we have that $\langle P, P'\rangle = $ constant. But we can indeed make this assumption since we proved the existence of a Levi-Civita connection for any manifold in class.) $\square$

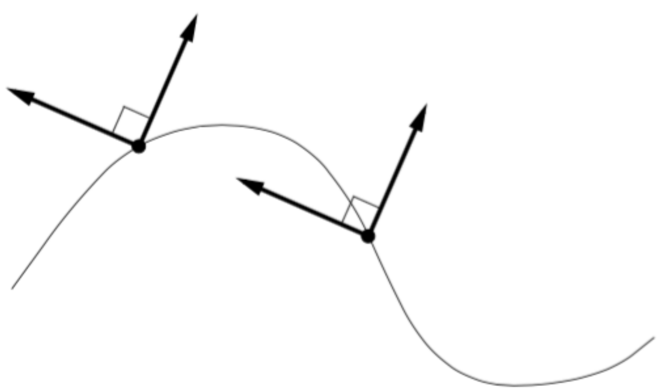

FIGURE 1. Parallel translation is an isometry.

Exercise 2 (Exercise 2-5 [DoCarmo]). *In Euclidean space, the parallel transport of a vector between two points does not depend on the curve joining the two points. Show, by example, that this fact may not be true on an arbitrary Riemannian manifold.*

Solution. The parallel transport map that we used on Exercise 1 above is an isomorphism between tangent spaces at points on a curve. The isomorphisms obtained in this way will in general depend on the choice of the curve: if they do not, then parallel transport along every curve can be used to define parallel vector fields over the entire manifold, which is only possible if the curvature is zero (more on this on Exercise 4-4, DoCarmo's (see HW set #9)). Thus, any Riemannian manifold with nonzero curvature (e.g., $\mathbb{S}^2$) presents an example of a case in which the parallel transport of a vector between two points *does* depend on the curve joining the two points. $\square$