RIEMANNIAN GEOMETRY NOTES

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Riemannian Metrics

Definition. *Let M be a smooth n-manifold. A* ***Riemannian metric*** *on M is a family of (positive definite) inner products*

$$g_p \colon T_pM \times T_pM \longrightarrow \mathbb{R}, \qquad \forall\, p \in M$$

such that, for all smooth vector fields X, Y on M, the map

$$p \mapsto g_p(X(p), Y(p))$$

defines a smooth function $M \to \mathbb{R}$. ★

In other words, a Riemannian metric g is a symmetric covariant 2-tensor field that is positive definite (i.e. $g(X,X) > 0$ for all $X \neq 0$).

Notation. *From now on, we shall use the notation ∂_i to replace $\partial/\partial x^i$.*

In a system of local coordinates on the manifold M given by n real-valued functions $x^1, \ldots, x^n$, the vector fields $\{\partial_1, \ldots, \partial_n\}$ give a basis of tangent vectors at each point of M. Relative to this coordinate system, the components of the metric tensor are

$$g_{ij}(p) := g_p\left((\partial_i)_p \,, (\partial_j)_p \right)$$

at each point p.

Equivalently, the metric tensor can be written in terms of the dual basis $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\}$ of the cotangent bundle as

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Definition. *Endowed with this metric, the smooth manifold (M,g) is a* ***Riemannian manifold****.* ★

Connections

Definition. *An* ***affine connection*** *∇ on a smooth manifold M is a mapping*

$$\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

which is denoted by $(X,Y) \xrightarrow{\nabla} \nabla_X Y$ and which satisfies the following properties

i) $\nabla_X Y$ is linear over $C^\infty(M)$ in X:

$$\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \qquad \text{for } f, g \in C^\infty(M);$$

ii) $\nabla_X Y$ *is linear over* $\mathbb{R}$ *in* Y*:*

$$\nabla_X(\alpha Y_1 + \beta Y_2) = \alpha \nabla_X Y_1 + \beta \nabla_X Y_2 \qquad \textit{for } \alpha, \beta \in \mathbb{R};$$

iii) ∇ *satisfies the following product rule:*

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y \qquad \textit{for } f \in C^\infty(M). \qquad \bigstar$$

Choosing a system of coordinates (x^i) about a point $p \in M$ and writing

$$X = \sum_i X^i \partial_i, \qquad Y = \sum_j Y^j \partial_j,$$

we have

$$\begin{aligned}
\nabla_X Y &= \nabla_{\sum_i X^i \partial_i} \left(\sum_j Y^j \partial_j \right) \\
&= \sum_i X^i \nabla_{\partial_i} \left(\sum_j Y^j \partial_j \right) \\
&= \sum_{i,j} X^i Y^j \nabla_{\partial_i} \partial_j + \sum_{i,j} X^i \partial_i (Y^j) \partial_j \\
&= \sum_k \left(\sum_{i,j} X^i Y^j \Gamma^k_{ij} + X(Y^k) \right) \partial_k,
\end{aligned}$$

where Γ^i_{jk} are the ***Christoffel symbols*** of ∇ with respect to the local coordinate frame, given by

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^n \Gamma^k_{ij} \partial_k.$$

An explicit formula for the Christoffel symbols is given by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

This rather mysterious formula comes up on the proof of the existence and uniqueness of the Levi-Civita connection (see Lee's *Riemannian Manifolds* Pg 68-70).

Definition. *Let* M *be an* n*-dimensional Riemannian manifold with an affine connection* ∇*, and* Y *a vector field defined along a curve* $\gamma(t)$ *in* M*. The* ***covariant derivative*** $DY(t)/\mathrm{d}t$ *of* $Y(t) = Y_{\gamma(t)}$ *is defined by*

$$\frac{DY(t)}{\mathrm{d}t} = \nabla_{\mathrm{d}\gamma/\mathrm{d}t} Y.$$

If Y *is given by* $Y(t) = Y^i(t)(\partial_i)_{\gamma(t)}$ *in local coordinates* (x^i) *and* $\gamma(t)$ *is given by* $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$*, then*

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \sum_{i=1}^n \frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t} \partial_i$$

and

$$\frac{DY(t)}{\mathrm{d}t} = \sum_{i=1}^n \left(\frac{\mathrm{d}Y^i(t)}{\mathrm{d}t} + \sum_{j,k=1}^n \Gamma^i_{jk} \frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t} Y^k(t) \right) \partial_i, \qquad \bigstar$$

Riemannian Connections

Definition. *Let M be a smooth manifold with an affine connection ∇ and a Riemannian metric $\langle , \rangle$. A connection is said to be **compatible** with the metric $\langle , \rangle$, when for any smooth curve γ and any pair of parallel vector fields P and P' along γ, we have $\langle P, P' \rangle =$ constant.* ★

Proposition 1. *Let M be a Riemannian manifold. A connection ∇ on M is compatible with a metric $\langle , \rangle$ if and only if for any vector fields V and W along the smooth curve $\gamma \colon I \to M$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle V, W \rangle = \langle \frac{DV}{\mathrm{d}t}, W \rangle + \langle V, \frac{DW}{\mathrm{d}t} \rangle, \qquad \text{for } t \in I.$$

Corollary 1. *A connection ∇ on a Riemannian manifold M is compatible with the metric if and only if*

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \qquad \text{for } X, Y, Z \in \mathfrak{X}(M).$$

Definition. *An affine connection ∇ on a smooth manifold M is said to be **symmetric** when*

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \text{for all } X, Y \in \mathfrak{X}(M).$$ ★

Theorem 1 (**Levi-Civita**)**.** *Given a Riemannian manifold M, there exists a unique affine connection ∇ on M (known as the **Levi-Civita connection** or **Riemannian connection**) that satisfies the following two conditions:*

i) ∇ *is symmetric.*

ii) ∇ *is compatible with the Riemannian metric.*