Math 351 DNHI 3

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(1) If d is a metric on M, show $|d(x, z) - d(y, z)| \leq d(x, y)$ for any $x, y, z \in M$.

Proof:

Let d be a metric on M. Then $d(x, z) \leq d(x, y) + d(y, z)$ by the triangle inequality. Equivalently,

$$\text{(I)} \quad d(x, z) - d(y, z) \leq d(x, y) .$$

Similarly, $d(y, z) \leq d(y, x) + d(z, x) = d(x, y) + d(x, z)$.

Thus, $d(y, z) - d(x, z) \leq d(x, y)$ or

$$\text{(II)} \quad -(d(x, z) - d(y, z)) \leq d(x, y) .$$

Now (I) and (II) imply that

$$|d(x, z) - d(y, z)| \leq d(x, y) . \qquad \blacksquare$$

(2) As it happens, some of our requirements for a metric are redundant. To see why this is so, let M be a set and suppose that $d : M \times M \longrightarrow \mathbb{R}$ satisfies

i) $d(x, y) = 0$ iff $x = y$

ii) $d(x, y) \leq d(x, z) + d(y, z) \quad \forall\, x, y, z \in M$.

Prove that d is a metric. That is, show that $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ hold for all x, y.

Proof:

First we show that $d(x, y) = d(y, x)$ by using properties i) and ii).

Setting $z = x$ in ii) and observing that $d(x, x) = 0$, we get

$$\text{(I)} \quad d(x, y) \leq d(x, x) + d(y, x) = d(y, x) .$$

Now setting $z = y$ in ii) and observing that $d(y, y) = 0$, we get

$$\text{(II)} \quad d(y, x) \leq d(y, y) + d(x, y) = d(x, y) .$$

Thus, (I) and (II) imply that

$$d(y, x) \leq d(x, y) \leq d(y, x) \ .$$

Hence, $d(x, y) = d(y, x)$.

Now to show that $d(x, y) \geq 0$, we use the fact that any metric with the properties

(i) $\quad d(x, y) = d(y, x) \quad \forall\, x, y \in M$

(ii) $d(x, y) \leq d(x, z) + d(y, z) \quad \forall\, x, y, z \in M$

satisfies $0 \leq |d(x, z) - d(y, z)| \leq d(x, y)$. $\blacksquare$

(3) Let M be a set and suppose that $d: M \times M \longrightarrow [0, \infty)$ satisfies properties (i), (ii), and (iii) for a metric on M and the triangle inequality reversed:
$d(x, y) \geq d(x, z) + d(y, z)$. Prove that M has at most one point.

Proof:
Let $x, y \in M$. Then by properties (i),(ii),(iii), and reverse triangle inequality (rti), we have

$$0 \underset{\text{by (i)}}{=} d(x, x) \underset{\text{by (rti)}}{\geq} d(x, y) + d(y, x) \underset{\text{by (iii)}}{=} d(x, y) + d(x, y) = 2\, d(x, y)$$

Thus, $0 \geq d(x, y)$ and since $d(x, y) \geq 0$ by (i), it follows that $0 = d(x, y)$ and, by (ii), $x = y$. Thus M has at most one point as desired. ■

(4) Let $d: M \times M \longrightarrow [0, \infty)$ be a metric function on the set M. Show that $\rho: M \times M \longrightarrow [0, \infty)$ defined by $\rho(x, y) = \min\{d(x, y), 1\}$ is also a metric function on M.

Proof:
Let d be a metric on M and suppose $\rho: M \times M \longrightarrow [0, \infty)$ is defined by $\rho(x, y) = \min\{d(x, y), 1\}$.
We will prove that ρ is a metric on M by showing that ρ satisfies properties (i) – (iv).
Properties (i) – (iii) are obvious. To prove (iv), note that $\min\{d(x, y), 1\} \leq 1$. So, for any $a \geq 0$,

$$\text{(I)} \quad \min\{d(x, y), 1\} \leq a + 1$$

Now $\min\{d(x, y), 1\} \leq d(x, y) \leq d(x, z) + d(z, y)$.
By (I) we see that

$$\min\{d(x, y), 1\} \leq 1 + d(z, y)$$
$$\min\{d(x, y), 1\} \leq d(x, z) + 1$$
$$\min\{d(x, y), 1\} \leq 1 + 1$$

Hence, $\rho(x, y) \leq \min\{d(x, z), 1\} + \min\{d(y, z), 1\} = \rho(x, z) + \rho(y, z)$. ■

(5) If d_1 and d_2 are both metrics on the same set M, which of the following yield metrics on M:

a) $d_1 + d_2$
Solution:
Let $d = d_1 + d_2$, where d_1 and d_2 are metric functions on M.
Then,
i) For $i = 1$ or 2, $0 \leq d_i(x, y) \leq d_1(x, y) + d_2(x, y) < \infty$ for all pairs $x, y \in M$. ✓

ii) $d_1(x, y) + d_2(x, y) = 0$ iff $d_1(x, y), d_2(x, y) = 0$ iff $x = y$. ✓

iii) $d_1(x, y) + d_2(x, y) = d_1(y, x) + d_2(y, x)$ for all pairs $x, y \in M$. ✓

iv) $d(x,\ y) = d_1(x,\ y) + d_2(x,\ y) \leq d_1(x,\ z) + d_1(y,\ z) + d_2(x,\ z) + d_2(y,\ z)$

$$= (d_1(x,\ z) + d_2(x,\ z)) + (d_1(y,\ z) + d_2(y,\ z))$$

$$= d(x,\ z) + d(y,\ z) \quad ✓$$

Thus, $d = d_1 + d_2$ is another metric on M.

b) $\max\{d_1,\ d_2\}$

<u>Solution:</u>

Now we set $d = \max\{d_1,\ d_2\}$. Clearly d satisfies properties i) – iii) . To see that d satisfies property iv), notice that

$$\begin{aligned} d(x,\ y) &= d_i(x,\ y) \leq d_i(x,\ z) + d_i(y,\ z) \\ &\leq \max_{i \in \{1,2\}} \{d_i(x,\ z)\} + \max_{i \in \{1,2\}} \{d_i(y,\ z)\} \\ &= d(x,\ z) + d(y,\ z) \quad ✓ \end{aligned}$$

Thus, $\max\{d_1,\ d_2\}$ is another metric on M.

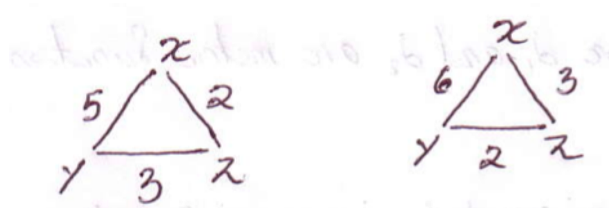
c) $\min\{d_1,\ d_2\}$

<u>Solution:</u>

Setting $d = \min\{d_1,\ d_2\}$, we may find that this time it is not a metric function.

Obviously, d still satisfies properties i) – iii), but what about iv)?

Suppose $M = \{x,\ y,\ z\}$ and d_1 and d_2 are given in the diagram below



Then $d(x,\ y) = 5$, while $d(x,\ z) + d(y,\ z) = 2 + 2 = 4 < 5$. Thus, the function $\min\{d_1,\ d_2\}$ does not always satisfy the triangle inequality and hence it's not a metric.

d) If d is a metric, is d^2 a metric?

<u>Solution:</u>

We will now show that d^2 is not generally a metric function. We might as well do this in more generality.

Let $\alpha > 1$, we show that if $d = |\cdot|$, then d^α is not a metric on $\mathbb{R}$, because it fails to satisfy property iv).

Notice that

$$\left|(0.5)^{1/\alpha} - 0\right| = \left[(0.5)^{1/\alpha}\right]^\alpha = 0.5,$$

while

$$\left|(0.5)^{1/\alpha} - \tfrac{1}{2}(0.5)^{1/\alpha}\right| + \left|\tfrac{1}{2}(0.5)^{1/\alpha} - 0\right| = \left(\tfrac{1}{2}\right)^\alpha 0.5 + \left(\tfrac{1}{2}\right)^\alpha 0.5$$

$$= \frac{1}{2^{\alpha}} < \frac{1}{2} = 0.5$$

Thus, d^2 fails the triangle inequality and thus it is not a metric on M.

(6) Which of the following functions define a metric on $\mathbb{R}$?

a) $d_1(x, y) = |x^7 - y^7|$

A metric . ✓

b) $d_2(x, y) = |x - y|^3$

Not a metric (it is of the form $|x - y|^{\alpha}$, where $\alpha > 1$). ✓

c) $d_3(x, y) = |x - y|^{2/3}$

A metric . ✓

d) $d_4(x, y) = \min\left\{\sqrt{|x - y|}, 1\right\}$

A metric . ✓

e) $d_5(x, y) = \sqrt{|x - y|} + \ln\left(\frac{|x-y|}{1+|x-y|} + 1\right)$

A metric (It is the sum of two metric functions). ✓

(7) Let $0 < \alpha < 1$. Show that if x and y are positive real numbers, then $|x^{\alpha} - y^{\alpha}| \leq |x - y|^{\alpha}$. In particular, $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|}$.

** Hint: Prove that $d(x, y) = |x - y|^{\alpha}$ defines a metric on $\mathbb{R}$ and use exercise 1 **

Proof:

If $0 < \alpha < 1$, then the function $f : [0, \infty) \longrightarrow [0, \infty)$ given by $f(t) = t^{\alpha}$ is zero iff $t = 0$. Notice that $f'(t) > 0 \;\; \forall\, t > 0$ and $f''(t) < 0$. Thus, $f(t + s) \leq f(t) + f(s)$ and for any metric d on M, $f(d)$ is also a metric on M.

We now define $\rho : \mathbb{R}^2 \longrightarrow [0, \infty)$ by $\rho(x, y) = |x - y|^{\alpha}$, then ρ is a metric function on $\mathbb{R}$.

By exercise 1,

$$|\rho(x, 0) - \rho(y, 0)| \leq \rho(x, y),$$

which means that

$$||x - 0|^{\alpha} - |y - 0|^{\alpha}| = ||x|^{\alpha} - |y|^{\alpha}| \leq |x - y|^{\alpha} .$$

Furthermore, if we assume that $x, y > 0$ this yields the desired result. ■

(8) Let $\mathbb{R}^{\infty}$ denote the collection of all real sequences $x = \{x_n\}$. Show that the expression

$$d(x,\ y) = \sum_{n=1}^{\infty} \frac{1}{n!}\ \frac{|x_n - y_n|}{1+|x_n - y_n|}$$

defines a metric on $\mathbb{R}^\infty$. Can you think of other metrics?

Proof:

i) Clearly $0 \le d(x,\ y)$. Notice that $\sum_{n=1}^{\infty} \frac{1}{n!}\ \frac{|x_n - y_n|}{1+|x_n - y_n|} < \sum_{n=1}^{\infty} \frac{1}{n!} = e$ for every $x = \{x_n\}$ and $y = \{y_n\}$. Hence $d(x,\ y) < \infty$. ✓

ii) $d(x,\ y) = 0$ iff $\frac{|x_n - y_n|}{1+|x_n - y_n|} = 0$ and since $\frac{|\cdot|}{1+|\cdot|}$ is a distance function on $\mathbb{R}$, it follows that $x_n = y_n\ \ \forall\, n \in \mathbb{N}$. Thus $x = y$. ✓

iii) Clearly $d(x,\ y) = d(y,\ x)$. ✓

iv) Let $z \in \mathbb{R}^\infty$ with $z = \{z_n\}$. Since the function $\frac{|\cdot|}{1+|\cdot|}$ has the triangle inequality property, we see that

$$\frac{|x_n - y_n|}{1+|x_n - y_n|} \le \frac{|x_n - z_n|}{1+|x_n - z_n|} + \frac{|y_n - z_n|}{1+|y_n - z_n|}\ .$$

Hence,

$$\begin{aligned} d(x,\ y) &= \sum_{n=1}^{\infty} \frac{1}{n!}\ \frac{|x_n - y_n|}{1+|x_n - y_n|} \\ &\le \sum_{n=1}^{\infty} \frac{1}{n!}\ \frac{|x_n - z_n|}{1+|x_n - z_n|} + \sum_{n=1}^{\infty} \frac{1}{n!}\ \frac{|y_n - z_n|}{1+|y_n - z_n|} \\ &= d(x,\ z) + d(y,\ z) \quad ✓ \qquad \blacksquare \end{aligned}$$

(9) Check that

$$d(f,\ g) = \max_{a \le t \le b} |f(t) - g(t)|\ , \qquad \rho(f,\ g) = \int_a^b |f(t) - g(t)|\, dt,$$

$$\text{and} \qquad \sigma(f,\ g) = \int_a^b \min\{|f(t) - g(t)|,\ 1\}\, dt$$

define metrics on $C[a,\ b]$, the vector space of real-valued continuous functions over the closed interval $[a,\ b]$:

Proof:

It is clear that d, ρ, and σ satisfy properties i) and iii). To see that d satisfies ii), observe that if $d(f,\ g) = 0$, then

$$|f(t) - g(t)| \ \le\ \max_{a \le t \le b} |f(t) - g(t)| \ = 0,$$

implying that $f(t) = g(t)\ \ \forall\, t \in [a,\ b]$ and therefore that $f = g$. ✓

To prove property ii) for ρ and σ, observe that $h(t) = |f(t) - g(t)|$ and $k(t) = \min\{|f(t) - g(t)|, 1\} = \frac{h(t) + 1 - |h(t) - 1|}{2}$ are nonnegative continuous functions over $[a, b]$. It will suffice, for now, to say that the area under a nonnegative continuous function is 0 iff that function is identically 0. (Later in the course we will justify this claim more rigorously!). ✓

To establish property iv) for d, note that

$$\max_{a \le t \le b} |f(t) - g(t)| \le \max_{a \le t \le b} \{|f(t) - s(t)| + |s(t) - g(t)|\}$$
$$\le \max_{a \le t \le b} |f(t) - s(t)| + \max_{a \le t \le b} |s(t) - g(t)| . \quad ✓$$

Thus, $d(f, g) \le d(f, s) + d(s, g)$ where $f, g, s \in C[a, b]$.

To establish property iv) for ρ and σ, observe that if $\varphi, \psi \in C[a, b]$ and $\varphi(t) \le \psi(t)$, then $\int_a^b \varphi(t)\, dt \le \int_a^b \psi(t)\, dt$.

For any two functions $f, g \in C[a, b]$, $|f(t) - g(t)|$ and $\min\{|f(t) - g(t)|, 1\}$ are in $C[a, b]$. Furthermore,

$$|f(t) - g(t)| \le |f(t) - s(t)| + |s(t) - g(t)| .$$

Setting

$$\varphi(t) = |f(t) - g(t)| \qquad \text{and} \qquad \psi(t) = |f(t) - s(t)| + |s(t) - g(t)| ,$$

we get that

$$\rho(f, g) = \int_a^b \varphi(t)\, dt \le \int_a^b \psi(t)\, dt$$
$$= \int_a^b |f(t) - s(t)|\, dt + \int_a^b |s(t) - g(t)|\, dt$$
$$= \rho(f, s) + \rho(s, g) . \qquad ✓$$

Thus, ρ has the triangle inequality property.

Now recall that

$$\min\{|f(t) - g(t)|, 1\} \le \min\{|f(t) - s(t)|, 1\} + \min\{|s(t) - g(t)|, 1\} .$$

Thus, to show that σ has property iv), we simply repeat the argument that we used for ρ. ■

(10) We say that a subset A of a metric space M is bounded if there is some $x_0 \in M$ and some constant $C < \infty$ such that $d(a, x_0) \le C \quad \forall\, a \in A$. Show that a finite union of bounded sets is again bounded.

<u>Proof:</u>

Let $A_1, \ldots, A_n$ be bounded sets in (M, d). Then, for each i, there is some $x_i \in M$ and $r_i \in (0, \infty)$ such that $A_i \subset B_{r_i}(x_i)$. Now we let $r = \sum_{i=2}^{n} d(x_i, x_1) + \max_{1 \le j \le n} \{r_j\}$.

Then for any $y \in A = \bigcup_{i=1}^{n} A_i$, $y \in A_j$ for some $j \in \{1, \ldots, n\}$

and

$$d(y, x_1) \leq d(y, x_j) + d(x_j, x_1) < r_j + \sum_{i=2}^{n} d(x_i, x_1) \leq r\,.$$

Thus, $A \subset B_r(x_1)$, which means that A is bounded.
We have thus shown that a finite union of bounded sets is bounded. ■

(11) We define the diameter of a nonempty subset A of M by
$\text{diam}(A) = \sup\{d(a, b) : a, b \in A\}$. Show that A is bounded iff $\text{diam}(A)$ is finite.

Proof:
($\Rightarrow$)
Suppose that A is bounded. That is, let $A \subset B_r(x)$ for some $x \in M$ and $r \in (0, \infty)$. This means that for any $a, b \in A$,

$$d(a, b) \leq d(a, x) + d(b, x) < r + r = 2\,r.$$

This in turn implies that $\sup_{a,b \in A} \{d(a, b)\} \leq 2\,r$, from which follows that $\text{diam}(A)$ is finite.

($\Leftarrow$)
On the other hand, assume that $\text{diam}(A)$ is finite. That is, let $\sup_{a,b \in A} \{d(a, b)\} \leq r$ for some $r \in (0, \infty)$.

Then, for some fixed point $x \in A$, $d(x, b) < r \quad \forall\, b \in A$. That is, $A \subset B_r(x)$, which implies that A is bounded. ■

(12) Show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for any $x \in \mathbb{R}^n$. Also check that $\|x\|_1 \leq n\,\|x\|_\infty$ and $\|x\|_1 \leq \sqrt{n}\,\|x\|_2$.

Proof:
Let x be any vector in $\mathbb{R}^n$ so that $x = (x_1, \ldots, x_n)$. Let's start by showing that $\|x\|_\infty \leq \|x\|_2$:

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| = \sqrt{\left(\max_{1 \leq i \leq n} |x_i|\right)^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|x\|_2\,. \quad \checkmark$$

Now, to see that $\|x\|_2 \leq \|x\|_1$, observe that $x = \sum_{i=1}^{n} x_i\, e_i$, where $e_i = (0, \ldots, 1, \ldots, 0)$ is the standard basis vector with 1 in the i^{th} component and 0 elsewhere.
Then, by triangle inequality,

$$\|x\|_2 = \left\|\sum_{i=1}^{n} x_i\, e_i\right\|_2$$

$$\leq |x_1| \, \| e_1 \|_2 + \ldots + |x_n| \, \| e_n \|_2$$
$$= |x_1| + \ldots + |x_n| = \| x \|_1 . \quad \checkmark$$

Thus we have shown that $\| x \|_\infty \leq \| x \|_2 \leq \| x \|_1$, as desired. ✓

Now to show that $\| x \|_1 \leq n \| x \|_\infty$, observe that

$$\| x \|_1 = |x_1| + \ldots + |x_n|$$
$$\leq \max_{1 \leq i \leq n} |x_i| + \max_{1 \leq i \leq n} |x_i| + \ldots + \max_{1 \leq i \leq n} |x_i|$$
$$= n \max_{1 \leq i \leq n} |x_i| = n \| x \|_\infty . \quad \checkmark$$

Finally, observe that $\| x \|_1 = (1, 1, \ldots, 1) \cdot (|x_1|, |x_2|, \ldots, |x_n|)$.
By Cauchy-Schwarz inequality,

$$\| x \|_1 \leq \| (1, \ldots, 1) \|_2 \; \| (|x_1|, \ldots, |x_n|) \|_2$$
$$= \sqrt{1^2 + \ldots + 1^2} \; \sqrt{|x_1|^2 + \ldots + |x_n|^2}$$
$$= \sqrt{n} \; \| x \|_2 \quad \checkmark$$

Thus we have shown that $\| x \|_1 \leq n \| x \|_\infty$ and $\| x \|_1 \leq \sqrt{n} \; \| x \|_2$, as desired. ✓ ■

(13) Show that $\text{diam}(B_r(x)) \leq 2\,r$, and give an example where strict inequality occurs.

Proof:
If $a, b \in B_r(x)$, then

$$d(a, b) \leq d(a, x) + d(b, x) < r + r = 2\,r.$$

Then,

$$\sup_{a,b \in B_r(x)} d(a, b) \leq 2\,r \implies \text{diam}(B_r(x)) \leq 2\,r .$$

To see that $\text{diam}(B_r(x)) < 2\,r$ can happen, suppose d is discrete. Then $B_1(x) = \{x\}$ and $\text{diam}(B_1(x)) = 0 < 2 \cdot 1$. ■

(14) If $\text{diam}(A) < r$, show that $A \subset B_r(a)$ for some $a \in A$.

Proof:
Suppose $\text{diam}(A) < r$. Let $a \in A$ be any element in A.
Then, for any $b \in A$,

$$d(a, b) \leq \sup_{a,b \in A} d(a, b) = \text{diam}(A) < r .$$

Therefore $B_r(a) \supset A$ as desired. ■

(15) If $A \subset B$, show that $\operatorname{diam}(A) \leq \operatorname{diam}(B)$.

Proof:

Suppose $A \subset B$. Let $S_A = \{d(x, y) : x, y \in A\}$ and $S_B = \{d(x, y) : x, y \in B\}$. Then $S_A \subset S_B \subset [0, \infty)$. Therefore, if α is an upper bound of S_B, then α must also be an upper bound for S_A.
This means that

$$\operatorname{diam}(A) = \sup(S_A) \leq \sup(S_B) = \operatorname{diam}(B) . \qquad \blacksquare$$

(16) Give an example where $\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B)$. If $A \cap B \neq \emptyset$, show that $\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B)$.

Proof:

Let $A = \{1\}$ and $B = \{3\}$. Then $\operatorname{diam}(A) = \operatorname{diam}(B) = 0$, while $\operatorname{diam}(A \bigcup B) = |1 - 3| = 2$. Thus, $\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B)$ as desired. ✓
Now suppose that $A, B \subset M$ for some metric space (M, d). If $A \bigcap B \neq \emptyset$, let $x \in A \bigcap B$. Then, for any $a, b \in A \bigcap B$, we have

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(b, x) \\ &\leq \sup_{a,\, t \in A} d(a, t) + \sup_{b,\, t \in B} d(b, t) \\ &= \operatorname{diam}(A) + \operatorname{diam}(B) . \end{aligned}$$

This actually implies the stronger statement

$$\operatorname{diam}(A \cup B) = \sup_{a, b \in A \bigcup B} d(a, b) \leq \operatorname{diam}(A) + \operatorname{diam}(B) . \qquad \blacksquare$$