Dynkin Diagrams (I)

The aim of this workshop is to introduce Dynkin diagrams and where they come from. These are *very* important towards the end of the course.

> Let V be a real vector space. Recall that a bilinear form is a map
>
> $$B(-,-) \colon V \times V \to \mathbb{R}$$
>
> which is linear in each variable. Further, recall $B(-,-)$ is called
> - **symmetric** if $B(x,y) = B(y,x)$ for all $x, y \in V$.
> - **positive definite** if $B(x,x) > 0$ for all $0 \neq x \in V$.

Bilinear forms appear in many areas of mathematics, and the first part of the workshop will study their basic properties (and is mainly revision from Honours Algebra). The second half of the workshop will study their relationship to Dynkin diagrams.

Throughout, we fix a basis $x_1, \dots, x_n$ of V. For a vector $v = \sum \alpha_i x_i$, we will use $\mathbf{v}$ to denote the column vector

$$\mathbf{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

1. (a) Let $B(-,-)$ be a bilinear form. Consider the matrix $B := (B_{ij})$, where by definition $B_{ij} := B(x_i, x_j)$. Verify that

   $$B(v,w) = \mathbf{v}^T B\, \mathbf{w},$$

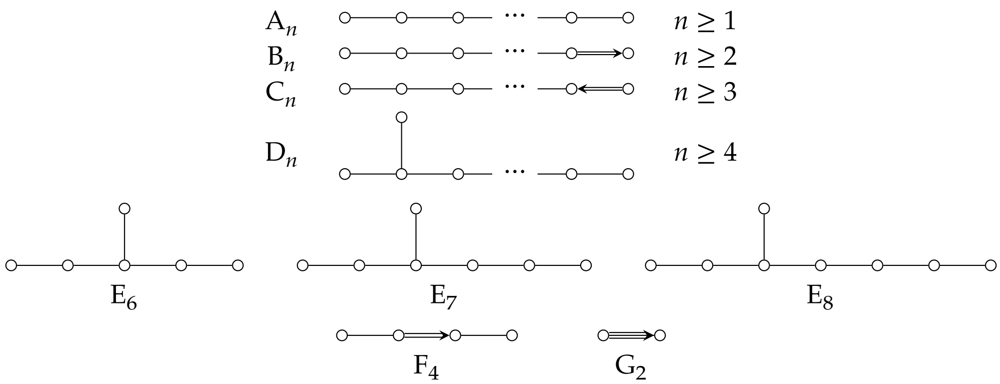
   and that $B(-,-)$ is symmetric if and only if $B := (B_{ij})$ is a symmetric matrix.

   (b) Conversely, given a symmetric matrix $B = (B_{ij})$, verify that

   $$B(v,w) := \mathbf{v}^T B\, \mathbf{w},$$

   defines a symmetric bilinear form.

Now the Dynkin diagrams are simply the following list of graphs, where we allow double and triple edges.



In all cases, the subscript is the number of vertices. You can ignore the arrows for today (they are only important right at the very end of the course), so for now there is no distinction between B_n and C_n.

2. Show that the Dynkin diagrams have the following remarkable property: if we choose a vertex and delete that vertex (and all edges adjacent to it), then we obtain a graph which is either a Dynkin diagram, or a disjoint union of Dynkin diagrams.

To each graph with n nodes, if we further let n_{ij} denote the number of edges between nodes i and j, we can associate a symmetric matrix $\mathrm{B} = (b_{ij})$, where

$$b_{ij} := \begin{cases} 2 & \text{if } i = j \\ -\sqrt{n_{ij}} & \text{if } i \neq j. \end{cases}$$

Thus to each graph, by Q1(b) we can associate a symmetric bilinear form $\mathrm{B}(-,-)$. Note that the square root is only important in the diagrams B_n, C_n, F_4 and G_2, since for all other Dynkin diagrams all the n_{ij} are 0 or 1. (Those Dynkin diagrams for which the n_{ij} are 0 or 1 are said to be "simply-laced" and they are known as the ADE Dynkin diagrams. They appear under many different guises in Mathematics and Physics.)

3. (a) For A_2, show directly that $\mathrm{B}(-,-)$ is positive definite.
   (b) Can you do the same for E_6?

For the remainder of this workshop, we will prove that $\mathrm{B}(-,-)$ is positive definite for all the Dynkin diagrams. This needs some technology, and reduction steps. The powerful and surprising thing is that the converse is also true (with a few assumptions), which is why Dynkin diagrams are so important: they give a visual classification of (certain) positive definite bilinear forms. For this reason, they appear in many different areas of mathematics and physics. We will prove the converse in the next workshop.

To recognise when something is positive definite, the following is very effective:

Sylvester's criterion. Given a real symmetric matrix B, then the associated bilinear form is positive definite if and only if all the leading principle minors are positive. By definition, this just means that the determinant of each of the matrices

$$(b_{11}), \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \dots, \mathrm{B} \tag{1}$$

is positive.

4. Consider the matrix B associated to a Dynkin diagram.
   (a) Give a graph-theoretic interpretation of each of the matrices in (1) above.
   (b) By Q2 and Sylvester's criterion, deduce that $\mathrm{B}(-,-)$ is positive definite for all Dynkin diagrams if and only if $\det \mathrm{B} > 0$ for all Dynkin diagrams.

5. (This will be split amongst the class). Show that the following graphs have positive definite bilinear forms:
   (a) A_n for $n \geq 1$.
   (b) B_n (equivalently C_n since we are ignoring the arrows) for $n \geq 2$.
   (c) D_n for $n \geq 4$.
   (d) F_4 and G_2.
   (e) E_6, E_7 and E_8.

Please hand in your solution to Q4 and one of the parts of Q5 by the start of lecture on Monday 25 September.