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Homework Assignment 6
General Relativity I

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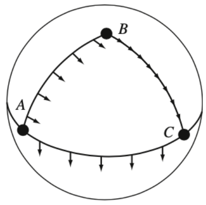
Problem 1 (Exercise 6.10 (Schutz)). *A 'straight line' on a sphere is a great circle, and it is well known that the sum of the interior angles of any triangle on a sphere whose sides are arcs of great circles exceeds* $180°$. *Show that the amount by which a vector is rotated by parallel transport around such a triangle (as in the figure) equals the excess of the sum of the angles over* $180°$.



Solution. Refering to the figure, assume that the sphere is a model of the Earth. Then let the arc $\overline{AC}$ denote a portion of the equator and $\overline{AB}$ and $\overline{BC}$ be two different longitudinal lines which meet at the north pole (point B). By design of the longitude-latitude lattice used to describe coordinates on Earth, it is clear that both $\overline{AB}$ and $\overline{BC}$ form a 90° angle with the equator. Therefore we have $90° + 90° + \measuredangle ABC > 180°$, where $\measuredangle ABC \neq 0$ is the angle formed at the north pole. [1] For a vector X^α being paralelly-transported around the boundary of a closed surface (such as the spherical triangle of this particular example), the total change δX^α it undergoes upon tracing the whole boundary and coming back to its starting point is given by

$$\delta X^\alpha = -\Omega^\alpha{}_\beta X^\beta,$$

where $\Omega^\alpha{}_\beta$ is the *curvature* 2*-form* that satisfies

$$R^\alpha{}_{\beta\mu\nu} = \Omega^\alpha{}_\beta \left(E_\mu, E_\nu\right)$$

for some frame field $\{E_\alpha\}$. A vanishing curvature 2-form (or, equivalently, a vanishing Riemman tensor) indicates the presence of flat space, in which there is no excess of the sum of the triangle's angles over 180°. $\square$

Problem 2 (Exercise 6.30 (Schutz)). *Calculate the Riemann curvature tensor of the cylinder. (Since the cylinder is flat, this should vanish. Use whatever coordinates you like, and make sure you write down the metric properly!)*

Solution. Let's start by looking for the metric of a cylinder in cylindrical coordinates $\{r, \varphi, z\}$, where

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = z,$$

$\{x, y, z\}$ being the typical Cartesian coordinates on $\mathbb{R}^3$. Then

$$\begin{aligned} dl^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= [\mathrm{d}(r\cos\varphi)]^2 + [\mathrm{d}(r\sin\varphi)]^2 + \mathrm{d}z^2 \\ &= [\cos\varphi\,\mathrm{d}r - r\sin\varphi\,\mathrm{d}\varphi]^2 + [\sin\varphi\,\mathrm{d}r + r\cos\varphi\,\mathrm{d}\varphi]^2 + \mathrm{d}z^2 \\ &= \cos^2\varphi\,\mathrm{d}r^2 - 2r\cos\varphi\sin\varphi\,\mathrm{d}r\mathrm{d}\varphi + r^2\sin^2\varphi\,\mathrm{d}\varphi^2 \\ &\quad + \sin^2\varphi\,\mathrm{d}r^2 + 2r\cos\varphi\sin\varphi\,\mathrm{d}r\mathrm{d}\varphi + r^2\cos^2\varphi\,\mathrm{d}\varphi^2 + \mathrm{d}z^2 \\ &= (\cos^2\varphi + \sin^2\varphi)\mathrm{d}r^2 + r^2(\cos^2\varphi + \sin^2\varphi)\mathrm{d}\varphi^2 + \mathrm{d}z^2 \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2. \end{aligned}$$

Thus we have found the line element for a cylinder in cylindrical coordinates

$$\boxed{dl^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2} \tag{1}$$

[1] This is, of course, only an intuitive explanation. A rigorous proof can be found in, e.g., Tai-Pei Cheng's *Relativity, Gravitation and Cosmology: A Basic Introduction*, §5.3.2.

From this expression we get

$$g_{rr} = 1, \qquad g_{\varphi\varphi} = r^2, \qquad g_{zz} = 1,$$

so that calculating the Christoffel symbols from

$$\Gamma^k_{ij} = \frac{1}{2}\, g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{2}$$

would be quite easy, since the only nonvanishing derivatives would be $\partial_r g_{\varphi\varphi}$. However, it is in general a more practical approach to use a Lagrangian formalism, since in this way only the nonvanishing Christoffel symbols will appear in the geodesic equation that extremizes the action, and there is need to waste time calculating expressions that in the end are going to vanish anyway. In this formalism we start from the action

$$I = \frac{1}{2}\int g_{ij}\dot{x}^i \dot{x}^j \,\mathrm{d}\lambda, \tag{3}$$

where the dot, of course, denotes differentiation by the parameter λ. Extremizing this integral leads to the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^i}\right) = 0, \tag{4}$$

where $\mathcal{L}$ is the Lagrangian density

$$\mathcal{L} = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j.$$

Thus, for the $i = r$ index,

$$\begin{aligned} \frac{\partial}{\partial r}\left[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right)\right] - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\frac{\partial}{\partial\dot{r}}\left[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right)\right]\right\} &= 0 \\ r\dot{\varphi}^2 - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\dot{r}\right) &= 0 \\ \ddot{r} - r\dot{\varphi}^2 &= 0. \end{aligned}$$

From this geodesic equation we can easily read off the Christoffel symbol

$$\Gamma^r_{\varphi\varphi} = -r.$$

Similarly, for the $i = \varphi$ index,

$$\begin{aligned} \frac{\partial}{\partial\varphi}\left[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right)\right] - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\frac{\partial}{\partial\dot{\varphi}}\left[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right)\right]\right\} &= 0 \\ 0 - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(r^2\dot{\varphi}\right) &= 0 \\ -r^2\ddot{\varphi} - 2r\,\dot{r}\dot{\varphi} &= 0 \\ \ddot{\varphi} + \frac{2}{r}\,\dot{r}\dot{\varphi} &= 0. \end{aligned}$$

This result showcases why the $1/2$ term in the action (3) is used by convention. Note that this geodesic equation is of the form

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\lambda^2} + \Gamma^\varphi_{r\varphi}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} + \Gamma^\varphi_{\varphi r}\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda} = 0.$$

But by the symmetry of the Christoffel symbols in a torsion-free connection, this is just

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\lambda^2} + 2\Gamma^\varphi_{r\varphi}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = 0.$$

Thus, we have

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}.$$

Now, there's no need to apply this same procedure to the z-index, since ut is clear that the Christoffel symbols vanish: plug in z into (4) and you get $\ddot{z} = 0$.

It turns out to be computationally convenient to write these Christoffel symbols as matrices

$$(\Gamma_k)^i{}_j := \Gamma^i_{kj} \qquad (\text{row} = i,\ \text{column} = j),$$

so that computing the $4^3 = 64$ components [2] of the Riemann curvature tensor

$$R^i{}_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}. \tag{5}$$

[2]We are in three dimensions, of course; in four dimensions it would be $4^4 = 256$ components.

becomes less unwieldy. We write

$$(\mathfrak{R}_{kl})^i{}_j := R^i{}_{jkl} \qquad (\text{row} = i,\ \text{column} = j),$$

which is now just $2^3 = 8$ matrices. From Eq. (5) we see that the $\mathfrak{R}_{kl}$ are defined by the matrix equation

$$\mathfrak{R}_{kl} = \partial_k \Gamma_l - \partial_l \Gamma_k + \Gamma_k \Gamma_l - \Gamma_l \Gamma_k.$$

(Note the antisymmetry, which reduces the number of expressions even further.)

Thus, in our case, we have

$$\Gamma_\varphi = \begin{pmatrix} 0 & -r & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And thus,

$$\begin{aligned}
\mathfrak{R}_{r\varphi} &= \partial_r \Gamma_\varphi - \partial_\varphi \Gamma_r + \Gamma_r \Gamma_\varphi - \Gamma_\varphi \Gamma_r \\
&= \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -r & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -r & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Moreover, $\mathfrak{R}_{\varphi r} = -\mathfrak{R}_{r\varphi}$, so it's also a zero matrix. There are no other nonvanishing components. Thus we conclude that the Riemann tensor vanishes everywhere, as was to be expected since a cylinder is (intrinsically) flat; its only curvature is an extrinsic one, which is nothing but an artifact of its embedding in $\mathbb{R}^3$. □

Problem 3 (Exercise 6.34 (Schutz)). *Establish the following identities for a general metric tensor in a general coordinate system:*

(a) $\Gamma^\mu_{\mu\nu} = 1/2\,(\log|g|)_{,\nu}$

(b) $g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = -1/\sqrt{-g}\,(g^{\alpha\beta}\sqrt{-g})_{,\beta}$

(c) *if $F^{\mu\nu}$ is an antisymmetric tensor, then* $F^{\mu\nu}{}_{;\nu} = 1/\sqrt{-g}\,(\sqrt{-g}F^{\mu\nu})_{,\nu}$

(d) $g^{\alpha\beta}g_{\beta\mu,\nu} = -g^{\alpha\beta}{}_{,\nu}\,g_{\beta\mu}$

(e) $g^{\mu\nu}{}_{,\alpha} = -\Gamma^\mu_{\beta\alpha}\,g^{\beta\nu} - \Gamma^\nu_{\beta\alpha}\,g^{\mu\beta}$

Solution to (a). From Eq. (2), changing $i \leftrightarrow \mu$, $k \leftrightarrow \mu$, and $j \leftrightarrow \nu$, we get

$$\begin{aligned}
\Gamma^\nu_{\mu\nu} &= \frac{1}{2}\,g^{\mu\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \\
&= \frac{1}{2}\,g^{\mu\sigma}\partial_\nu g_{\mu\sigma}.
\end{aligned} \tag{6}$$

Here we used the fact that $g^{\mu\sigma}\partial_\mu g_{\nu\sigma} = g^{\mu\sigma}\partial_\sigma g_{\mu\nu}$, since μ and σ are just dummy summation indices and $g_{\alpha\beta}$ is symmetric. Now, using *Jacobi's formula* for an invertible matrix A,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\det A(x)\right] = \det A(x) \cdot \mathrm{Tr}\left[A^{-1}(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} A(x)\right], \tag{7}$$

note that

$$g^{\mu\sigma}\partial_\nu g_{\mu\sigma} = \mathrm{Tr}\left[g^{-1} \cdot \frac{\partial}{\partial x^\nu} g\right].$$

(Here $\boldsymbol{g}$ is the non-index notation for the metric tensor $g_{\alpha\beta}$, not to be confused with the non-boldface $g \equiv \det \boldsymbol{g}$.) Thus, from (7) we get

$$\frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\mu\sigma} = \frac{1}{2} \frac{1}{g} \partial_\nu g = \frac{1}{2} \partial_\nu (\log |g|), \tag{8}$$

where the last equality follows from the elementary differentiation

$$\partial_y (\log x) = \frac{1}{x} \partial_y x,$$

except we only consider positive values of the determinant, since otherwise the log function is undefined. Now from (6) and (8) we get the desired result, namely,

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} \partial_\nu (\log |g|).$$

□

Solution to (b). We start by expanding the LHS:

$$\begin{aligned} g^{\mu\nu}\Gamma^\alpha_{\mu\nu} &= g^{\mu\nu}\left[\frac{1}{2} g^{\alpha\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\right] \\ &= \frac{1}{2} g^{\mu\nu} g^{\alpha\sigma} \partial_\mu g_{\sigma\nu} + \frac{1}{2} g^{\mu\nu} g^{\alpha\sigma} \partial_\nu g_{\mu\sigma} - \frac{1}{2} g^{\mu\nu} g^{\alpha\sigma} \partial_\sigma g_{\mu\nu} \\ &= g^{\mu\nu} g^{\alpha\sigma} \partial_\mu g_{\sigma\nu} - \frac{1}{2g} g^{\alpha\sigma} \partial_\sigma g. \end{aligned} \tag{9}$$

Now we can simplify the first term on this last equality by observing the following:

$$\partial_\mu(\overbrace{g^{\alpha\sigma} g_{\sigma\nu}}^{=\delta^\alpha{}_\nu}) = 0 \implies g^{\alpha\sigma}\partial_\mu g_{\sigma\nu} = -g_{\sigma\nu}\partial_\mu g^{\alpha\sigma}.$$

Thus, plugging this back into (9),

$$\begin{aligned} g^{\mu\nu}\Gamma^\alpha_{\mu\nu} &= -\underbrace{g^{\mu\nu} g_{\sigma\nu}}_{=\delta^\mu{}_\sigma} \partial_\mu g^{\alpha\sigma} - \frac{1}{2g} g^{\alpha\sigma}\partial_\sigma g \\ &= -\partial_\sigma g^{\alpha\sigma} - \frac{1}{2g} g^{\alpha\sigma}\partial_\sigma g. \end{aligned} \tag{10}$$

Now, expanding the RHS of the expression that we need to prove,

$$\begin{aligned} -\frac{1}{\sqrt{-g}} \partial_\beta \left(g^{\alpha\beta}\sqrt{-g}\right) &= -\frac{1}{\sqrt{-g}}\left[\sqrt{-g}\,\partial_\beta g^{\alpha\beta} + g^{\alpha\beta}\,\partial_\beta \sqrt{-g}\right] \\ &= -\partial_\beta g^{\alpha\beta} - \frac{1}{\sqrt{-g}} \frac{1}{2\sqrt{-g}} g^{\alpha\beta}\,\partial_\beta g \\ &= -\partial_\beta g^{\alpha\beta} - \frac{1}{2g} g^{\alpha\beta}\,\partial_\beta g. \end{aligned}$$

Relabeling, we see that this expression is equal to the the RHS of Eq. (10). Thus we shave shown that

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = -\frac{1}{\sqrt{-g}} \partial_\beta \left(g^{\alpha\beta}\sqrt{-g}\right),$$

as desired. □

Solution to (c). Let $F^{\mu\nu}$ be antisymmetric. Then,

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \Gamma^\mu_{\sigma\nu} F^{\sigma\nu} + \Gamma^\nu_{\sigma\nu} F^{\mu\sigma}. \tag{11}$$

Since $F^{\alpha\beta}$ is antisymmetric and $\Gamma^\lambda_{\alpha\beta}$ is symmetric, the contraction on the second term

$$\Gamma^\mu_{\sigma\nu} F^{\sigma\nu}$$

vanishes. On the other hand, let's expand the other Γ term:

$$\Gamma^\nu_{\sigma\nu} F^{\mu\sigma} = \frac{1}{2} F^{\mu\sigma} g^{\nu\alpha}\left(\partial_\sigma g_{\alpha\nu} + \partial_\nu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\nu}\right)$$

$$
\begin{aligned}
&= \frac{1}{2} F^{\mu\sigma} g^{\nu\alpha} \partial_\sigma g_{\alpha\nu} + \frac{1}{2} F^{\mu\sigma} g^{\nu\alpha} \partial_\nu g_{\sigma\alpha} - \frac{1}{2} F^{\mu\sigma} g^{\nu\alpha} \partial_\alpha g_{\sigma\nu} \\
&= \frac{1}{2} F^{\mu\sigma} g^{\alpha\nu} \partial_\sigma g_{\alpha\nu} \\
&= \frac{1}{2g} F^{\mu\sigma} \partial_\nu g.
\end{aligned} \tag{12}
$$

From the second to the third equality the two rightmost terms vanished by just relabeling, and then on the last equalty we used a previously derived expression.

Now, plugging back into Eq. (11), we get

$$
\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \frac{1}{2g} F^{\mu\sigma} \partial_\nu g.
$$

But this RHS is the same as the RHS of the expression that we need to prove:

$$
\begin{aligned}
\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right) &= \frac{1}{\sqrt{-g}} \left[\sqrt{-g}\, \partial_\nu F^{\mu\nu} + F^{\mu\nu} \partial_\nu \sqrt{-g} \right] \\
&= \partial_\nu F^{\mu\nu} - \frac{1}{\sqrt{-g}} \frac{1}{2\sqrt{-g}} F^{\mu\nu} \partial_\nu g \\
&= \partial_\nu F^{\mu\nu} + \frac{1}{2g} F^{\mu\sigma} \partial_\nu g.
\end{aligned}
$$

Thus we have shown that

$$
\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right). \qquad \square
$$

Solution to (d). We actually already used this result above... Expanding,

$$
\partial_\nu (\underbrace{g^{\alpha\beta} g_{\beta\mu}}_{=\delta^\alpha{}_\mu}) = 0 \implies g^{\alpha\beta} \partial_\nu g_{\beta\mu} = -g_{\beta\mu} \partial_\nu g^{\alpha\beta}. \qquad \square
$$

Solution to (e). This follows from the compatibility of the metric with the Levi-Civita connection ∇. Since $\nabla_\alpha (g^{\mu\nu} g_{\nu\rho}) = 0$, we have

$$
\begin{aligned}
g_{\nu\rho} \nabla_\alpha g^{\mu\nu} &= -g^{\mu\nu} \nabla_\alpha g_{\nu\rho} \\
\implies \underbrace{g^{\lambda\rho} g_{\nu\rho}}_{=\delta^\lambda{}_\nu} \nabla_\alpha g^{\mu\nu} &= -g^{\lambda\rho} g^{\mu\nu} \nabla_\alpha g_{\nu\rho} \\
\implies \nabla_\alpha g^{\mu\lambda} &= -g^{\lambda\rho} g^{\mu\nu} \nabla_\alpha g_{\nu\rho}.
\end{aligned}
$$

Of course, both sides of this expression vanish by compatibility of the metric; thus

$$
\begin{aligned}
\nabla_\alpha g^{\mu\lambda} &= 0 \\
\implies \partial_\alpha g^{\mu\lambda} + \Gamma^\mu_{\beta\alpha} g^{\beta\lambda} + \Gamma^\lambda_{\beta\alpha} g^{\mu\beta} &= 0.
\end{aligned}
$$

Thus, relabeling $\lambda \leftrightarrow \nu$, we get the desired result

$$
g^{\mu\nu}{}_{,\alpha} = -\Gamma^\mu_{\beta\alpha} g^{\beta\nu} - \Gamma^\nu_{\beta\alpha} g^{\mu\beta}. \qquad \square
$$