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Problem Set 3
Numerical Analysis II

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Problem 1. *Following the notions from the lecture notes, carefully prove the convergence of the following projected gradient method with a variable step-length:*

$$u_{n+1} = P_K[u_n - \alpha_n \nabla F(u_n)].$$

Give an example of $\{\alpha_n\}$ satisfying the hypothesis that you need for the convergence.

Solution. Let $K \subset \mathbb{R}^m$ be nonempty, closed, and convex. Let $\nabla F \colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be L-Lipschitz continuous and m-strongly monotone, and let $\bar{u}$ be the unique solution to the stationary problem. Moreover, let $\bar{\alpha} \in \left(0, (2m)/L^2\right)$ satisfy

$$\bar{u} = P_K\left[\bar{u} - \bar{\alpha}\nabla F(\bar{u})\right].$$

Supposing that $\lim_{n\to\infty} \alpha_n \longrightarrow \bar{\alpha}$, since P_K is a nonexpansive mapping, we have

$$\begin{aligned} \|u_{n+1} - \bar{u}\|^2 &\leq \|u_n - \bar{u}\|^2 - 2\langle \alpha_n \nabla F(u_n) - \bar{\alpha}\nabla F(\bar{u}), u_n - \bar{u}\rangle + \|\alpha_n \nabla F(u_n) - \bar{\alpha}\nabla F(\bar{u})\|^2 \\ &\approx \|u_n - \bar{u}\|^2 - 2\bar{\alpha}\langle \nabla F(u_n) - \nabla F(\bar{u}), u_n - \bar{u}\rangle + \bar{\alpha}^2\|\nabla F(u_n) - \nabla F(\bar{u})\|^2. \end{aligned}$$

By the assumptions above we have

$$\begin{aligned} \|\nabla F(u_n) - \nabla F(\bar{u})\| &\leq L\|u_n - \bar{u}\|, \\ \langle \nabla F(u_n) - \nabla F(\bar{u}), u_n - \bar{u}\rangle &\geq m\|u_n - \bar{u}\|, \end{aligned}$$

which imply that

$$\|u_{n+1} - \bar{u}\|^2 \leq \left(1 - 2\alpha m + \alpha^2 L^2\right)\|u_n - \bar{u}\|^2.$$

Letting $\Theta = \sqrt{1 - 2\alpha m + \alpha^2 L^2}$, we end up with

$$\|u_{n+1} - \bar{u}\| \leq \Theta\|u_n - \bar{u}\|,$$

from which we can conclude the convergence $\{u_n\} \longrightarrow \bar{u}$ from the *Banach Contraction Principle.*

As an example of a sequence $\{\alpha_n\}$ satisfying the hypothesis, consider

$$\alpha_N = \frac{2m}{L^2} - \varepsilon_N, \tag{1}$$

where $N \in \mathbb{Z}$ and ε_N satisfies $|\varepsilon_N| < \varepsilon$ for some arbitrarily small $\varepsilon > 0$. Then, from (1) we have

$$\left|\alpha_N - \frac{2m}{L^2}\right| = |\varepsilon_N| < \varepsilon,$$

so our choice of $\{\alpha_n\}$ satisfies the hypothesis, as desired. ♠

Problem 2. *Let n be an even integer, and consider the $n \times n$ matrix A with 3 on the main diagonal, -1 on the super- and sub-diagonal, and $1/2$ in the $(i, n+1-i)$ position for all $i = 1, \dots, n$ except for $i = n/2$ and $n/2+1$. Define a vector $\mathbf{b} = [2.5, 1.5, \dots, 1.5, 1.0, 1.0, 1.5, \dots, 1.5, 2.5]^\top$ where there are $n-4$ repetitions of 1.5 and two repetitions of 1. Write MATLAB codes and solve the system $A\mathbf{x} = \mathbf{b}$ for $n = 100$ (a system of 100 equations with 100 unknowns) using the following methods:*

a) *Steepest Descent with a fixed step length.*

b) *Steepest Descent with a variable step length.*

c) *Conjugate Gradient.*

Solution. The three methods are written in the following snippet:

```
% Steepest Descent with fixed step length
function [x,mag,iter] = sdf(A,b,x0,tol,it_max,alpha)
    r0 = b - A*x0;
    x = x0;
    for i = 1:it_max
        x = x + alpha*r0;
        r0 = b - A*x;
        mag = norm(r0);
        iter = i;
        if mag < tol
            break
        end
    end
end


% Steepest Descent with variable step length
function [x,mag,iter] = sdv(A,b,x0,tol,it_max)
    r0 = b - A*x0;
    x = x0;
    for i = 1:it_max
        alpha = (r0'*r0)/(r0'*A*r0);
        x = x + alpha*r0;
        r0 = b - A*x;
        mag = norm(r0);
        iter = i;
        if mag < tol
            break
        end
    end
end


% Conjugate Gradient method
function [x,mag,iter] = cgm(A,b,x0,tol,it_max)
    r0 = b - A*x0;
    d = r0;
    x = x0;
    for i = 1:it_max
        w = A*d;
        alpha = (r0'*r0)/(d'*w);
        x = x + alpha*d;
        mag2 = norm(r0);
        r0 = r0 - alpha*w;
        mag = norm(r0);
        iter = i;
        if mag < tol
            break
        end
        beta = (mag^2)/(mag2^2);
        d = r0 + beta*d;
    end
end
```

We now implement these methods for the given A and $\mathbf{b}$:

```
% Form matrix A
function A = makeA(n)
    if mod(n,2) ~= 0
        A = eye(1);
    else
        A = 3*eye(n);
        for i = 1:n-1
            A(i+1,i) = - 1;
            A(i,i+1) = - 1;
            A(i,n+1-i) = 1/2;
        end
        A(n,1) = 1/2;
        A(n/2,n+1-(n/2)) = -1;
        A((n/2)+1,n-(n/2)) = -1;
    end
end

% Form vector b
function b = makeb(n)
    b = ones(n,1);
    b(1) = 2.5;
    b(2:(n/2)-1) = 1.5;
    b((n/2)+2:n-1) = 1.5;
    b(n) = 2.5;
end


function [A,b] = makeAb(n)
    A = makeA(n);
    b = makeb(n);
end

% Variables
[A,b] = makeAb(100);
tol = 1e-4;
it_max = 1000;
x = 5+zeros(100,1);

% Steepest descent with fixed step length
[x1,n1,iter1] = sdf(A,b,x,tol,it_max,0.001);

% Steepest descent with variable step length
[x2,n2,iter2] = sdv(A,b,x,tol,it_max);

% Conjugate Gradient method
[x3,n3,iter3] = cgm(A,b,x,tol,it_max);
```

♠

Problem 3. *Find the DFT of the vector* $\mathbf{x} = [0, 1, 0, -1, 0, 1, 0, -1]^\top$.

Proof. Let $\mathbf{y} = [y_0, \dots, y_{n-1}]^\top$, where the k^{th} element is given by the expression

$$y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j e^{-\frac{\imath 2\pi jk}{n}}.$$

Here we use $\imath$ to denote the imaginary number $\sqrt{-1}$. Now, since in this case $n = 8$, we have

$$\omega^j := e^{-(2\pi\imath j)/8} = \cos\frac{\pi}{4}j - \imath\sin\frac{\pi}{4}j.$$

Thus, we collect the ω^i:

$$\omega^0 = 1, \quad \omega^1 = \frac{1-\imath}{\sqrt{2}}, \quad \omega^2 = \imath, \quad \omega^3 = \frac{-1-\imath}{\sqrt{2}}, \quad \omega^4 = -1, \quad \omega^5 = \frac{\imath-1}{\sqrt{2}}, \quad \omega^6 = \imath, \quad \omega^7 = \frac{1+\imath}{\sqrt{2}}.$$

That leads us to the system

$$
\mathbf{y} = \frac{1}{2\sqrt{2}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2}\iota \\ 0 \\ 0 \\ 0 \\ \sqrt{2}\iota \\ 0 \end{bmatrix}. \qquad \spadesuit
$$

Problem 4. *Find the inverse DFT of the vector* $\mathbf{x} = [1, -i, 1, i]^\top$.

Proof. We want to find $\mathbf{x} = F_n^{-1}(\mathbf{y})$, where F_n^{-1} is the inverse DFT. This time the ω^j are given by

$$
\omega^j := e^{-(2\pi\iota j)/4} = \cos\frac{\pi}{2}j - \iota\sin\frac{\pi}{2}j.
$$

Hence we end up with

$$
\mathbf{x} = \frac{1}{2} \begin{bmatrix} (\omega^0)^{-1} & (\omega^0)^{-1} & (\omega^0)^{-1} & (\omega^0)^{-1} \\ (\omega^0)^{-1} & (\omega^1)^{-1} & (\omega^2)^{-1} & (\omega^3)^{-1} \\ (\omega^0)^{-1} & (\omega^2)^{-1} & (\omega^0)^{-1} & (\omega^2)^{-1} \\ (\omega^0)^{-1} & (\omega^3)^{-1} & (\omega^2)^{-1} & (\omega^1)^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ -\iota \\ 1 \\ \iota \end{bmatrix}
$$

$$
= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \iota & -1 & -\iota \\ 1 & -1 & 1 & -1 \\ 1 & -\iota & -1 & \iota \end{bmatrix} \begin{bmatrix} 1 \\ -\iota \\ 1 \\ \iota \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 \\ -\iota \\ 1 \\ \iota \end{bmatrix}. \qquad \spadesuit
$$

Problem 5. *Write down all primitive seventh roots of unity.*

Proof. Since the GCD (greatest common divisor) between 7 and k, where $1 \le k < 7$, is 1 (i.e., 7 is prime), the primitive roots of unity are

$$
e^{\frac{2\pi}{7}\iota}, \quad e^{\frac{4\pi}{7}\iota}, \quad e^{\frac{6\pi}{7}\iota}, \quad e^{\frac{8\pi}{7}\iota}, \quad e^{\frac{10\pi}{7}\iota}, \quad e^{\frac{12\pi}{7}\iota}. \qquad \spadesuit
$$