MATH 710 HW # 6

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Exercise 1 (Exercise 2-2 [DoCarmo])**.** *Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c\colon I \to M$ be an integral curve of X through p, i.e. $c(t_0) = p$ and $dc/dt = X(c(t))$. Prove that the Riemannian connection of M is*

$$(\nabla_X Y)(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} (P^{-1}_{c,t_0,t}(Y(c(t))),$$

where $P_{c,t_0,t}\colon T_{c(t_0)}M \to T_{c(t)}M$ is the parallel transport along c, from t_0 to t (this shows how the connection can be re-obtained from the concept of parallelism).

Proof. Let $(e_i)_{i=1}^n$ be an orthonormal basis for T_pM, with $e_i(t) = P_{c,t_0,t}$, i.e. $\nabla_{c'(t)} e_i(t) = 0$, so that $(e_i(t))_{i=1}^n$ is an orthonormal basis for $T_{c(t)}M$. Indeed,

$$\begin{aligned}
&\nabla_{c'(t)}\langle e_i(t), e_j(t)\rangle = \langle \nabla_{c'(t)} e_i(t), e_j(t)\rangle + \langle e_i(t), \nabla_{c'(t)} e_j(t)\rangle = 0 \\
&\implies \langle e_i(t), e_j(t)\rangle = \langle e_i, e_j\rangle = \delta_i^j.
\end{aligned}$$

Now we can write

$$Y(c(t)) = Y^i(t) e_i(t),$$

and then the following computation follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \left(P^{-1}_{c,t_0,t}(Y(c(t)))\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \left(P^{-1}_{c,t_0,t}(Y^i(t)e_i(t))\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \left(Y^i(t)e_i\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \left(Y^i(t)\right) e_i \\
&= \nabla_{c'(t)}(Y^i(t))e_i(t)\Big|_{t=t_0} \\
&= \nabla_{c'(t)}(Y^i(t)e_i(t))\Big|_{t=t_0} \\
&= (\nabla_X Y)(p).
\end{aligned}$$

□

Exercise 2 (Exercise 2-3 [DoCarmo])**.** *Let $f\colon M^n \to \overline{M}^{n+k}$ be an immersion of a smooth manifold M into a Riemannian manifold $\overline{M}$. Assume that M has the Riemannian metric induced by f (c.f.* Example 2.5, Chapter 1*). Let $p \in M$ and let $U \subset M$ be a neighborhood of p such that $f(U) \subset \overline{M}$ is a submanifold of $\overline{M}$. Further, suppose that X, Y are smooth vector fields on $f(U)$ which extend to smooth vector fields $\overline{X}, \overline{Y}$ on an open set of $\overline{M}$. Define $(\nabla_X Y)(p) =$ the tangential component of $\left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)(p)$, where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Prove that ∇ is the Riemannian connection of M.*

Proof. We have that $(\nabla_X Y)(p)$ is the tangential component of $\left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)(p)$; let us denote that by $\nabla_X Y = \left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)^t$. Using the Riemannian metric $\langle , \rangle$ induced by f, we have the following:

- ∇ is compatible with the metric on M. For all $p \in M$, we get

$$\begin{aligned} X\langle Y, Z\rangle(p) &= \overline{X}\langle \overline{Y}, \overline{Z}\rangle(p) \\ &= \langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\rangle(p) + \langle \overline{Y}, \overline{\nabla}_{\overline{X}}\overline{Z}\rangle(p) \\ &= \langle \overline{\nabla}_{\overline{X}}\overline{Y}, Z\rangle(p) + \langle Y, \overline{\nabla}_{\overline{X}}\overline{Z}\rangle(p) \\ &= \langle \nabla_X Y, Z\rangle(p) + \langle Y, \nabla_X Z\rangle(p). \end{aligned}$$

- ∇ is symmetric. For all $p \in M$, we get

$$\begin{aligned} \left(\nabla_X Y - \nabla_Y X\right)(p) &= \left(\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}\right)^t(p) \\ &= [\overline{X}, \overline{Y}]^t(p) = [X, Y](p). \qquad (\clubsuit) \end{aligned}$$

To get this last equality ($\clubsuit$), note that in local coordinates, we have

$$\begin{aligned} [\overline{X}, \overline{Y}]^t(p) &= \left(\sum_{i,j=1}^{n+k} \left\{ \overline{X}^i \frac{\partial \overline{Y}^j}{\partial x^i} - \overline{Y}^i \frac{\partial \overline{X}^j}{\partial x^i} \right\} \frac{\partial}{\partial x^j} \right)^t (p) \\ &= \left(\sum_{i=1}^{n} \sum_{j=1}^{n+k} \left\{ X^i \frac{\partial \overline{Y}^j}{\partial x^i} - Y^i \frac{\partial \overline{X}^j}{\partial x^i} \right\} \frac{\partial}{\partial x^j} \right)^t (p) \\ &= \left(\sum_{i,j=1}^{n} \left\{ X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right\} \frac{\partial}{\partial x^j} \right) (p) \qquad (\heartsuit) \\ &= [X, Y](p). \end{aligned}$$

This third equality ($\heartsuit$) holds because $\nabla_X Y(p)$ depends only on $X(p)$ and $Y(c(t))$ (where $c(t)$ is an integral curve for X through p).

Thus we have shown that ∇ is the Riemannian connection of M. $\square$

Exercise 3 (Exercise 2-8 [DoCarmo]). *Consider the upper half-plane*

$$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the metric given by $g_{11} = g_{22} = 1/y^2$, and $g_{12} = 0$ (metric of Lobatchevski's non-Euclidean geometry).

a) Show that the Christoffel symbols of the Riemannian connection are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0, \qquad \Gamma^2_{11} = \frac{1}{y}, \qquad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}.$$

b) Let $v_0 = (0, 1)$ be a tangent vector at the point $(0, 1) \in \mathbb{R}^2_+$ (v_0 is the unit vector on the y-axis with origin at $(0, 1)$). Let $v(t)$ be the parallel transport of v_0 along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

Proof of a). We have the following:

$$\begin{aligned}\Gamma_{ij}^{k} &= \frac{1}{2}\, g^{kl}\left(\frac{\partial g_{il}}{\partial x^{j}} + \frac{\partial g_{lj}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{l}}\right)\\ &= \frac{y^2}{2}\left(\frac{\partial g_{ik}}{\partial x^{j}} + \frac{\partial g_{kj}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{k}}\right)\\ &= \frac{y^2}{2}\cdot\frac{-2}{y^3}\left(\frac{\partial x^2}{\partial x^{j}}\,\delta_{ik} + \frac{\partial x^2}{\partial x^{i}}\,\delta_{kj} - \frac{\partial x^2}{\partial x^{k}}\,\delta_{ij}\right).\end{aligned}$$

Thus we have the desired values for the Christoffel symbols

$$\begin{aligned}&\Gamma_{11}^{1} = \Gamma_{12}^{2} = \Gamma_{22}^{1} = 0,\\ &\Gamma_{11}^{2} = \frac{1}{y},\\ &\Gamma_{12}^{1} = \Gamma_{22}^{2} = -\frac{1}{y}.\end{aligned}$$

□

Proof of b). Let $v(t) = (a(t), b(t))$ be the parallel field along the curve $x = t$, $y = 1$ with

$$v(0) = (0,1), \qquad v'(0) = v_0 = (0,1).$$

Then, from the geodesic equations, we have

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} + \Gamma_{12}^{1}\, b = 0,\\ \frac{\mathrm{d}b}{\mathrm{d}t} + \Gamma_{11}^{2}\, a = 0.\end{aligned}$$

Now, taking $a = \cos\theta(t)$ and $b = \sin\theta(t)$ (we can make this assumption since parallel transports preserve inner products), the above equations imply that $\mathrm{d}\theta/\mathrm{d}t = -1$. We have that $v_0 = (0,1)$, so

$$\theta_0 = \frac{\pi}{2} \qquad \text{and thus} \qquad \theta = \frac{\pi}{2} - t,$$

as desired. □