Math Analysis Notes

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Open Sets and Closed Sets

Before we start our discussion of open and closed sets, let's review some facts about sequences and subsequences and about equivalent metrics.

$\boxed{\text{SUBSEQUENCES}}$

Definition: Given a sequence $\{x_n\}_{n=1}^{\infty}$, consider a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\left\{x_{n_k}\right\}_{k=1}^{\infty}$ is called a subsequence of x_n.

Example:

Let $M = \left\{\text{☹}, \text{☺}, \text{😲}\right\}$. Suppose $\{x_n\}_{n=1}^{\infty} \subset M$ is defined by

$$x_n = \begin{cases} \text{☺} & \text{if } \ n = 3\,p \\ \text{☹} & \text{if } \ n = 3\,p + 1 \\ \text{😲} & \text{if } \ n = 3\,p + 2 \end{cases}$$

Let $\{n_k\}_{k=1}^{\infty}$ be given by $n_k = 3\,k$. What is $\left\{x_{n_k}\right\}_{k=1}^{\infty}$?

Solution:
Notice that $n_1 = 3$, $n_2 = 6$, $n_3 = 9$, etc.
So $x_{n_1} = x_3 = \text{☺}$, $x_{n_2} = x_6 = \text{☺}$, and $x_{n_3} = x_9 = \text{☺}$, etc.
Thus, $\left\{x_{n_k}\right\}_{k=1}^{\infty}$ is the constant sequence $\{\text{☺}\}_{k=1}^{\infty}$.

Example: Let $\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

a) Suppose $\{n_k\}_{k=1}^{\infty} = \{2\,k+1\}_{k=1}^{\infty}$. What is $\left\{x_{n_k}\right\}_{k=1}^{\infty}$?
Solution:
$\{x_{n_k}\}_{k=1}^{\infty} = \{x_{2\,k+1}\}_{k=1}^{\infty} = \left\{\frac{1}{2\,k+1}\right\}_{k=1}^{\infty} = \left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right\}$. ✓

b) Suppose $\{n_k\}_{k=1}^{\infty} = \{2^k\}_{k=1}^{\infty}$. What is $\left\{x_{n_k}\right\}_{k=1}^{\infty}$?

Solution:

$\left\{x_{n_k}\right\}_{k=1}^{\infty} = \left\{x_{2^k}\right\}_{k=1}^{\infty} = \left\{\frac{1}{2^k}\right\}_{k=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$. ✓

c) Suppose $\{n_k\}_{k=1}^{\infty} = \{2, 1, 3, 4, 5, 6, \ldots\}$. What is $\left\{x_{n_k}\right\}_{k=1}^{\infty}$? Is it a subsequence of $\{x_n\}_{n=1}^{\infty}$?

Solution:

$\left\{x_{n_k}\right\}_{k=1}^{\infty} = \left\{\frac{1}{2}, 1, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. This sequence does not match the order of $\{x_n\}$ and therefore fails to be a subsequence. Notice that $n_1 = 2 > 1 = n_2$. ✓

d) Suppose $\{n_k\}_{k=1}^{\infty} = \{4, 2, 8, 6, 12, 10, \ldots\}$. What is $\left\{x_{n_k}\right\}_{k=1}^{\infty}$? Is it a subsequence of $\{x_n\}_{n=1}^{\infty}$?

Solution:

$\left\{x_{n_k}\right\}_{k=1}^{\infty} = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{6}, \frac{1}{12}, \frac{1}{10}, \ldots\right\}$. This sequence does not match the order of $\{x_n\}$ and therefore fails to be a subsequence of $\{x_n\}$. In particular $\{n_k\}_{k=1}^{\infty}$ is not an increasing sequence of positive integers. ✓

Note: Subsequences are useful tools that will later help us to describe such important concepts like completeness and compactness. For now however, we will have to be satisfied with the simple analysis of the relationship between subsequences, convergence, and Cauchy sequences.

• Proposition:

If $x_n \overset{d}{\to} x$, then $x_{n_k} \overset{d}{\to} x$ for any subsequence $\left\{x_{n_k}\right\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$.

Proof:

We must show that for any $\varepsilon > 0$, there exists $K > 0$ such that $d(x_{n_k}, x) < \varepsilon$ whenever $k \geq K$.

Since $x_n \overset{d}{\to} x$, we know that $d(x_n, x) < \varepsilon$ whenever $n \geq N$. Setting $K = N$, notice that $n_k \geq K$. Thus, when $k \geq N$, we have $d(x_{n_k}, x) < \varepsilon$. ■

• Proposition:

A Cauchy sequence with a convergent subsequence converges.

Proof:

Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy and suppose that $\left\{x_{n_k}\right\}_{k=1}^{\infty}$ is a convergent subsequence with $x_{n_k} \overset{d}{\to} x$. We must show that $x_n \overset{d}{\to} x$.

For $\varepsilon > 0$, let N_1 be such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ whenever $n, m \geq N_1$. Also let N_2 be such that

$d(x_{n_k}, x) < \frac{\varepsilon}{2}$ whenever $k \geq \mathcal{N}_2$. Now let $\mathcal{N} = \max\{\mathcal{N}_1, \mathcal{N}_2\}$.

If $n, m > \mathcal{N}$, then

$$d(x_n, x_{nm}) \leq d(x_n, x_m) < \frac{\varepsilon}{2} \qquad \text{and} \qquad d(x_{nm}, x) < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_m}) + d(x_{n_m}, x) \\ &\leq d(x_n, x_m) + d(x_{n_m}, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \qquad \blacksquare$$

• Proposition:

Every subsequence of a Cauchy sequence is itself a Cauchy sequence.

Proof:

Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy and suppose that $\{x_{n_k}\}_{k=1}^{\infty}$ is any subsequence. For $\varepsilon > 0$, there is some $\mathcal{N} > 0$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq \mathcal{N}$. Notice that $n_n \geq n$ and $n_m \geq m$. Thus $d(x_{n_n}, x_{n_m}) < \varepsilon$. $\blacksquare$

• Proposition:

If every subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence that converges to x, then $\{x_n\}_{n=1}^{\infty}$ converges to x.

Proof:

Suppose that $\{x_n\}_{n=1}^{\infty}$ does not converge to x, but that every subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence, which converges to x.

If $x_n \overset{\text{not}}{\to} x$ (i.e. if x_n does not converge to x), then infinitely many elements of the sequence $\{x_n\}_{n=1}^{\infty}$ are further from x than some $\varepsilon > 0$. That is, the set $A = \{x_n : d(x_n, x) \geq \varepsilon\}$ is infinite if ε is small enough. Notice that the elements of A form a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. Our assumption dictates that some further subsequence $\left\{x_{n_{k_t}}\right\}_{t=1}^{\infty}$ converges to x. But all the elements of $\left\{x_{n_{k_t}}\right\}_{t=1}^{\infty}$ are elements of A. In other words, $d\left(x_{n_{k_t}}, x\right) \geq \varepsilon \ \ \forall\, t \in \mathbb{N}$, implying that $x_{n_{k_t}} \overset{\text{not}}{\to} x$. $(\Rightarrow\Leftarrow)$ $\blacksquare$

EQUIVALENT METRICS

We have already seen that the metric at hand determines which sequences are Cauchy and which sequences converge. Later we will see that the convergent sequences in (M, d) in turn determine the open and closed sets of (M, d) and therefore the continuous functions on (M, d).

Given another metric function ρ, we have generally no reason to expect the metric spaces (M, d) and (M, ρ) to have the same convergent sequences. In this section we would like to say a few words about metric functions that generate the same convergent sequences.

Definition: Two metrics d and ρ on a set M are said to be equivalent metrics if they generate the same convergent sequences: that is, $d(x_n, x) \to 0$ iff $\rho(x_n, x) \to 0$.

It might be comforting to know that most metric functions on $\mathbb{R}$ (or $[0, \infty)$) hetherto considered are equivalent metrics. The following proposition explains why.

• Proposition:
Let d be a metric on M and suppose that ρ is defined by $\rho(x, y) = f(d(x, y))$, where $f : [0, \infty) \longrightarrow [0, \infty)$ satisfies the following three properties:
i) $f(t) \geq 0$ with equality iff $t = 0$.
ii) $f'(t) > 0$ for $t \in (0, \infty)$
iii) $f''(t) < 0$ for $t \in (0, \infty)$
Then d and ρ are equivalent.

Proof:
Notice that f is continuous and invertible. Since $f'(t) > 0$, we may state that f^{-1} is differentiable and therefore continuous.
Next, we need to observe that if $g : \mathcal{U} \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at 0, then for any sequence $\{t_n\}_{n=1}^{\infty} \subset \mathcal{U}$ with $t_n \to 0$, we have $g(t_n) \to g(0)$ (this will be discussed later on when we consider continuous functions and identify their properties).
Now suppose that $\{x_n\}_{n=1}^{\infty} \subset M$ with $x_n \overset{d}{\to} x$. Then $d(x_n, x) \to 0$. Since f is continuous, we see that $f(d(x_n, x)) \to f(0) = 0$ by setting $t_n = d(x_n, x)$.
Thus,

$$d(x_n, x) \to 0 \Longrightarrow f(d(x_n, x)) = \rho(x_n, x) \to 0.$$

On the other hand, if $\rho(x_n, x) \to 0$, then $d(x_n, x) = f^{-1}(\rho(x_n, x)) \to 0$ because f^{-1} is also continuous at 0. We have thus established that $d(x_n, x) \to 0$ iff $\rho(x_n, x) \to 0$. Hence d and ρ are equivalent. ■

Example:
The following are all equivalent metrics on $\mathbb{R}$:
• $d(x, y) = |x - y|$ • $\rho(x, y) = \sqrt{|x - y|}$ • $\partial(x, y) = \frac{|x-y|}{1+|x-y|}$

• $\zeta(x, y) = \ln(|x - y| + 1)$ • $\varphi(x, y) = \frac{\sqrt{\ln(|x-y|+1)}}{1+\sqrt{\ln(|x-y|+1)}}$

Note: Equivalent metrics preserve convergent sequences. Must they also have the same cauchy sequences? The answer is NO, as we can verify in the following example.

Example:

Let $M = (0, \infty)$. Then $d(x, y) = |x - y|$ and $\rho(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ are equivalent metrics on M that do not generate the same Cauchy sequences.

To see that d and ρ are equivalent, observe that the function $f(t) = \frac{1}{t}$ is continuous on $(0, \infty)$.

Notice also that $f^{-1}(t) = f(t)$. That is, f is its own inverse.

Now, if $x_n \overset{d}{\to} x$, then $f(x_n) \to f(x)$, or $|f(x_n) - f(x)| \to 0$. Hence $x_n \overset{d}{\to} x$ implies that $\left|\frac{1}{x_n} - \frac{1}{x}\right| \to 0$ or $x_n \overset{\rho}{\to} x$.

On the other hand, $x_n \overset{\rho}{\to} x$ implies that $f^{-1}(x_n) \to f^{-1}(x)$.

That is,

$$x_n \overset{\rho}{\to} x \Longrightarrow |f^{-1}(x_n) - f^{-1}(x)| = \left|\frac{1}{x_n} - \frac{1}{x}\right| \to 0 .$$

This in turn implies that $\left|f\left(\frac{1}{x_n}\right) - f\left(\frac{1}{x}\right)\right| = |x_n - x| \to 0$.

Thus we have shown that d and ρ are equivalent. ✓

To see that d and ρ fail to generate the same Cauchy sequences, notice that $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence when it is considered under the metric $d(x, y) = |x - y|$. Under the metric ρ however, $\rho\left(\frac{1}{n}, \frac{1}{m}\right) = |n - m| \geq 1$ if $m \neq n$.

Thus $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is not Cauchy under ρ. ✓

Example:

Let $M = \mathbb{R}^n$.

Then $d_1(x, y) = \|x - y\|_1$, $d_2(x, y) = \|x - y\|_2$, and $d_\infty(x, y) = \|x - y\|_\infty$ are all equivalent metrics on $\mathbb{R}^n$ because

$$\|x - y\|_\infty \leq \|x - y\|_2 \leq \|x - y\|_1$$

while

$$\|x - y\|_1 \leq n\,\|x - y\|_\infty \quad \text{and} \quad \|x - y\|_1 \leq \sqrt{n}\,\|x - y\|_2.$$

We should verify that d_1, d_2, and d_3 all generate the same Cauchy sequences on $\mathbb{R}^n$.

The last example is very important as it allows us to make the following definition:

Definition: Given two metric spaces (M, d) and (N, ρ), we can define a metric on the product

$M \times N$ in a variety of ways. Our only requirement is that a sequence of pairs (a_n, x_n) in $M \times N$ should converge precisely when both coordinate sequences $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ converge in (M, d) and (N, ρ), respectively.

Each of the following define metrics on $M \times N$ that enjoy this property. Moreover, they are equivalent :

- $d_1((a, x), (b, y)) = d(a, b) + \rho(x, y)$
- $d_2((a, x), (b, y)) = \left(d(a, b)^2 + \rho(x, y)^2\right)^{1/2}$
- $d_\infty((a, x), (b, y)) = \max\{d(a, b), \rho(x, y)\}$

Henceforth, any implicit reference to "the" metric on $M \times N$, sometimes called the product metric, will mean one of d_1, d_2, or d_∞. Any one of them will serve equally well.

OPEN SETS

Definition: A set $\mathcal{U}$ in a metric space (M, d) is called an open set if $\mathcal{U}$ contains a neighborhood of each of its points. In other words, $\mathcal{U}$ is an open set if, given $x \in \mathcal{U}$, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathcal{U}$.

Example:

a) In any metric space, the whole space M is an open set. The empty set $\emptyset$ is also open (by default).

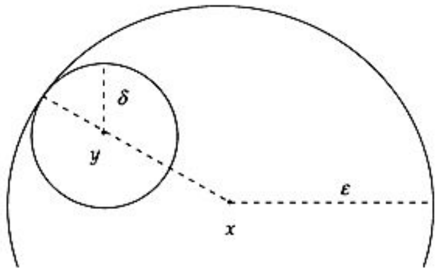
b) In $\mathbb{R}$, any open interval is an open set. Indeed, given $x \in (a, b)$, let $\varepsilon = \min\{x - a, b - x\}$. Then $\varepsilon > 0$ and $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. The cases (a, ∞) and $(-\infty, b)$ are similar.
While we're at it, notice that the interval $[0, 1)$, for example, is not open in $\mathbb{R}$ because it does not contain an entire neighborhood of 0.

c) In a discrete space, $B_1(x) = \{x\}$ is an open set for any x. It follows that every subset of a discrete space is open.

Before we get too carried away, we should follow the lead suggested by the last example and check that every open ball is in fact an open set.

• Proposition:
For any $x \in M$ and any $\varepsilon > 0$, the open ball $B_\varepsilon(x)$ is an open set.

Proof:
Consult the drawing below to understand the motivation behind the argument of the proof.



Let $y \in B_\varepsilon(x)$. Then $d(x, y) < \varepsilon$ and hence $\delta = \varepsilon - d(x, y) > 0$.
We will show that $B_\delta(y) \subset B_\varepsilon(x)$.
Indeed, if $d(y, z) < \delta$, then, by the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + \varepsilon - d(x, y) = \varepsilon. \qquad \blacksquare$$

Let's collect our thoughts. First, every open ball is open. Next, it follows from the definition of open sets that an open set must actually be a union of open balls. In fact, if $\mathcal{U}$ is open, then
$\mathcal{U} = \bigcup\{B_\varepsilon(x) : B_\varepsilon(x) \subset \mathcal{U}\}$.
Moreover, any arbitrary union of open balls is again an open set. Here's what all of this means:

• Theorem:
An arbitrary union of open sets is again open. That is, if $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is any collection of open sets, then $V = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ is open.

Proof:
If $x \in V$, then $x \in \mathcal{U}_\alpha$ for some $\alpha \in A$. But then, since $\mathcal{U}_\alpha$ is open, it follows that $B_\varepsilon(x) \subset \mathcal{U}_\alpha \subset V$ for some $\varepsilon > 0$. $\blacksquare$

Intersections aren't nearly as generous:

• Theorem:
A finite intersection of open sets is open. That is, if each of $\mathcal{U}_1, \ldots, \mathcal{U}_n$ is open, then so is $V = \mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n$.

Proof:
If $x \in V$, then $x \in \mathcal{U}_i$ for all $i = 1, \ldots, n$. Thus, for each i there is an $\varepsilon_i > 0$ such that $B_{\varepsilon_i} \subset \mathcal{U}_i$. But then, setting $\varepsilon = \min\{\varepsilon_1, ..., \varepsilon_n\} > 0$, we have $B_\varepsilon(x) \subset \bigcap_{i=1}^{n} \mathcal{U}_i = V$. $\blacksquare$

Example:

The word "finite" is crucial in the above theorem because $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, and $\{0\}$ is not open in $\mathbb{R}$.

Now, since the real line $\mathbb{R}$ is of special interest to us, let's characterize the open subsets of $\mathbb{R}$. This will come in handy later. But it should be stressed that while this characterization holds for $\mathbb{R}$, it does not have a satisfactory analogue even in $\mathbb{R}^2$. (As we will see in a later chapter, not every open set in the plane can be written as a union of disjoint open disks.)

• Theorem:

If $\mathcal{U}$ is an open subset of $\mathbb{R}$, then $\mathcal{U}$ may be written as a countable union of disjoint open intervals. That is, $\mathcal{U} = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (a_n, b_n)$ (these may be unbounded) and $I_n \bigcap I_m = \emptyset$ for $n \neq m$.

Proof:

We know that $\mathcal{U}$ can be written as a union of open intervals (because each $x \in \mathcal{U}$ is in some open interval I with $I \subset \mathcal{U}$). What we need to show is that $\mathcal{U}$ is a union of disjoint open intervals (such a union must be countable (to see why, check exercise 2.15 on Carother's)) .

We first claim that each $x \in \mathcal{U}$ is contained in a maximal open interval $I_x \subset \mathcal{U}$ in the sense that if $x \in I \subset \mathcal{U}$, where I is an open interval, then we must have $I \subset I_x$. Indeed, given $x \in \mathcal{U}$, let

$$a_x = \inf \{a : (a, x] \subset \mathcal{U}\} \quad \text{and} \quad b_x = \sup \{b : [x, b) \subset \mathcal{U}\} .$$

Then, $I_x = (a_x, b_x)$ satisfies $x \in I_x \subset \mathcal{U}$, and I_x is clearly maximal. (Check this!)

Next, notice that for any $x, y \in \mathcal{U}$ we have either $I_x \bigcap I_y = \emptyset$ or $I_x = I_y$. Why? Because if $I_x \bigcap I_y \neq \emptyset$, then $I_x \bigcup I_y$ is an open interval containing both I_x and I_y. By maximality we would then have $I_x = I_y$.

It follows then that $\mathcal{U}$ is the union of disjoint (maximal) intervals: $\mathcal{U} = \bigcup_{x \in \mathcal{U}} I_x$. ■

Now any time we make up a new definition in a metric space setting, it is usually very helpful to find an equivalent version stated exclusively in terms of sequences. To motivate this in the particular case of open sets, let's recall:

$$x_n \to x \iff \{x_n\} \text{ is eventually in } B_\varepsilon(x) \text{ for any } \varepsilon > 0$$

and hence

$$x_n \to x \iff \{x_n\} \text{ is eventually in } \mathcal{U}, \text{ for any open set } \mathcal{U} \text{ containing } x.$$

This last statements essentially characterizes open sets:

• Theorem:
A set $\mathcal{U}$ in (M, d) is open iff, whenever a sequence $\{x_n\}_{n=1}^{\infty}$ in M converges to a point $x \in \mathcal{U}$, we have $x_n \in \mathcal{U}$ for all but finitely many n.

Proof:
The forward implication is clear from the remarks preceding the theorem. Let's see why the new condition implies that $\mathcal{U}$ is open:
If $\mathcal{U}$ is not open , then there is an $x \in \mathcal{U}$ such that $B_\varepsilon(x) \bigcap \mathcal{U}^c \neq \emptyset$ for all $\varepsilon > 0$. In particular , for each n there is some $x_n \in B_{1/n}(x) \bigcap \mathcal{U}^c$. But then $\{x_n\}_{n=1}^{\infty} \subset \mathcal{U}^c$ and $x_n \to x$. Thus, the new condition also fails.(⇒⇐) ■

In slightly different language, the above theorem is saying that the only way to reach a member of an open set is by traveling well inside the set; there are no inhabitants on the "frontier". In essence, you cannot visit a single resident without seeing a whole neighborhood!

CLOSED SETS

Definition: A set F in a metric space (M, d) is said to be a closed set if its complement $F^c = M \setminus F$ is open.

We can draw several immediate (although not terribly enlightening) conclusions:

Example:
a) Ø and M are always closed. (This means that it is possible for a set to be both open and closed!)

b) An arbitrary intersection of closed sets is closed. A finite union of closed sets is closed.

c) Any finite set is closed. Indeed, it is enough to show that $\{x\}$ is always closed. Why? Given any $y \in M \setminus \{x\}$ (that is, any $y \neq x$), note that $\varepsilon = d(x, y) > 0$, and hence $B_\varepsilon(y) \subset M \setminus \{x\}$.

d) In $\mathbb{R}$, each of the intervals $[a, b]$, $[a, \infty)$, and $(-\infty, b]$ is closed. Also, $\mathbb{N}$ and Δ (the Cantor set, which we'll study very soon) are closed sets.

e) In a discrete space, every subset is closed (and as we previously saw, they are also open. That means that in a discrete space, every subset is clopen!).

f) Sets are not "doors"! For instance, $(0, 1]$ is neither open nor closed in $\mathbb{R}$!

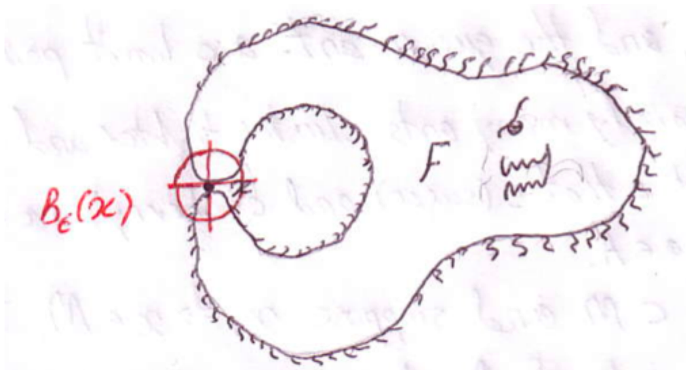
As yet, our definition is not terribly useful. It would be nice if we had an intrinsic characterization of closed sets (something that did not depend on a knowledge of open sets), something in terms of sequences, for example. For this let's first make an observation: F is closed iff F^c is open.
So F is closed iff

$$x \in F^c \Longrightarrow B_\varepsilon(x) \subset F^c \quad \text{for some } \varepsilon > 0 .$$

But this is the same as saying: F is closed iff

$$B_\varepsilon(x) \bigcap F \neq \emptyset \quad \text{for every } \varepsilon > 0 \Longrightarrow x \in F .$$

This is our first characterization of closed sets. Intuitively speaking, closed sets are like blobs from a 60's horror movie. A closed set will devour all points that reside in arbitrary proximity to the set.



In order to better understand the nature of these "predatory" sets, let's bring in a few definitions.

Definition: Let A be a subset of M. A point $x \in M$ is called a limit point of A if every neighborhood of x contains a point of A that is different from x itself, that is, if $(B_\varepsilon(x) \setminus \{x\}) \bigcap A \neq \emptyset$ for any $\varepsilon > 0$.

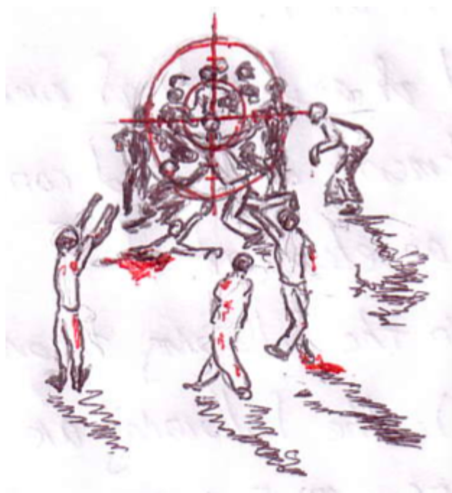
Example:
a) Suppose you place a sugar cube, a cherry, and a queen-ant around an ant nest. Several moments later you take a snapshot of the event that ensues:

If we let A be the set of all ants in the picture, then the sugar cube, the cherry, and the queen-ant are all limit points of A (assuming that infinitely many ants cluster tighter and tighter around each object). Notice that s (sugar) and c (cherry) are not members of A, whereas $Q \in A$.

b) Let $A = \{x_n\}_{n=1}^{\infty} \subset M$ and suppose $x_n \overset{d}{\to} x \in M$. Then x is the only limit point of A.
Note however, that a non-convergent sequence can have a huge number of limit points. For instance, let $\mathbb{Q}$ be arranged into a sequence $\{r_n\}_{n=1}^{\infty}$. Then every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$. ✵

Definition: A set F is a closed set iff it contains all of its limits points.

Example:
Let $\mathcal{Z}$ be the set of all zombies. Is $\mathcal{Z}$ a closed set?
Solution:
To answer this question, we must decide whether $\mathcal{Z}$ contains all of its limit points. To put this problem in "life or death" terms, suppose x is a limit point of $\mathcal{Z}$. Then, would you shoot x with your sniper rifle?



If the concentric circles of the sniper scope never present the target in isolation, that is, if the heads of other zombies are always present within each target's circle (no matter how small it is), then the target is a limit point of zombies. It will inevitably be "in contact" with the living dead and therefore become a zombie (if it is not already a zombie). Therefore it appears that $\mathcal{Z}$ is a closed set (and that shooting the target might be the most humane course of action). ✵

Notice that the characterization of closed sets in terms of limit points can readily be translated into a sequential description. To see why, suppose x is a limit point of some set F. Then, by definition, $B_\varepsilon(x) \cap F \neq \emptyset$ for every $\varepsilon > 0$. But this means that for each $1/n$, we can find some $x_n \in B_{1/n}(x) \cap F$.

Thus, the sequence $\{x_n\}_{n=1}^{\infty} \subset F$ converges to x. In other words, any limit point of F is necessarily a point of convergence of some sequence of elements of F.
This means that F is closed iff every sequence $\{x_n\}_{n=1}^{\infty}$ that consists of elements of F and converges in

$M \supset F$, must actually converge to an element of F.

We summarize our results in the following theorem:

• Theorem:
Given a set F in (M, d), the following are equivalent:
(i) F is closed; that is, $F^c = M \setminus F$ is open.
(ii) If $B_\varepsilon(x) \bigcap F \neq \emptyset$ for every $\varepsilon > 0$, then $x \in F$.
(iii) If a sequence $\{x_n\}_{n=1}^{\infty} \subset F$ converges to some point $x \in M$, then $x \in F$.

Proof:
(i)$\Longleftrightarrow$(ii): This is clear from our observations above and the definition of an open set.

(ii)$\Longrightarrow$(iii): Suppose that $\{x_n\}_{n=1}^{\infty} \subset F$ and $x_n \xrightarrow{d} x \in M$. Then $B_\varepsilon(x)$ contains infinitely many x_n for any $\varepsilon > 0$, and hence $B_\varepsilon(x) \bigcap F \neq \emptyset$ for any $\varepsilon > 0$. Thus, $x \in F$ by (ii).

(iii)$\Longrightarrow$(ii): If $B_\varepsilon(x) \bigcap F \neq \emptyset$ for all $\varepsilon > 0$, then for each n there is an $x_n \in B_{1/n}(x) \bigcap F$. The sequence $\{x_n\}_{n=1}^{\infty}$ satisfies $\{x_n\}_{n=1}^{\infty} \subset F$ and $x_n \to x$. Hence, $x \in F$ by (iii). ■

Note: Most authors take (iii) as the definition of a closed set. In other words, condition (iii) says that a closed set must contain all of its limit points. That is, "closed" means closed under the operation of taking of limits.

Now, as we've seen, some sets are neither open nor closed. However, it is possible to describe the "open part" of a set and the "closure" of a set. Here's what we'll do:

Definition: Given a set A in (M, d), we define the interior of A, written $\text{int}(A)$ or A^o, to be the largest open set contained in A.
That is,

$$\begin{aligned} \text{int}(A) = A^\circ &= \bigcup \{U : U \text{ is open and } U \subset A\} \\ &= \bigcup \{B_\varepsilon(x) : B_\varepsilon(x) \subset A \text{ for some } x \in A,\ \varepsilon > 0\} \qquad \text{(Why?)} \\ &= \{x \in A : B_\varepsilon(x) \subset A \text{ for some } \varepsilon > 0\}. \end{aligned}$$

Note that A^o is clearly an open subset of A.
We next define the closure of A, written $\text{cl}(A)$ or $\overline{A}$, to be the smallest closed set containing A. That is,

$$\text{cl}(A) = \overline{A} = \bigcap \{F : F \text{ is closed and } A \subset F\} .$$

Please take note of the "dual" nature of our two new definitions. Now it is clear that $\overline{A}$ is a closed set containing A (and necessarily the smallest one). But it's not so clear which points are in $\overline{A}$ or, more precisely, which points are in $\overline{A} \backslash A$.
We could use a description of $\overline{A}$ that is a little easier to "test" on a given set A. It follows from our last theorem that $x \in \overline{A}$ iff $B_\varepsilon(x) \cap \overline{A} \neq \emptyset$ for every $\varepsilon > 0$. The description that we are looking for simply removes this last reference to $\overline{A}$.

• Proposition:
$x \in \overline{A}$ iff $B_\varepsilon(x) \cap A \neq \emptyset$ for every $\varepsilon > 0$.

Proof:
($\Leftarrow$)
If $B_\varepsilon(x) \cap A \neq \emptyset$ for every $\varepsilon > 0$, then $B_\varepsilon(x) \cap \overline{A} \neq \emptyset$ for every $\varepsilon > 0$ (since $A \subset \overline{A}$), and hence $x \in \overline{A}$ (since closed sets contain their limit points) .
($\Rightarrow$)
Now let $x \in \overline{A}$ and let $\varepsilon > 0$. If $B_\varepsilon(x) \cap A = \emptyset$, then A is a subset of $(B_\varepsilon(x))^c$, a closed set. Thus, $\overline{A} \subset B_\varepsilon((x))^c$. (since $A \subset \overline{A}$) But this is a contradiction, because $x \in \overline{A}$ while $x \notin (B_\varepsilon(x))^c$.($\Rightarrow\Leftarrow$) ■

• Corollary:
$x \in \overline{A}$ iff there is a sequence $\{x_n\}_{n=1}^\infty \subset A$ with $x_n \to x$.
That is, $\overline{A}$ is the set of all limits of convergent sequences in A (including limits of constant sequences).

Example:
In $(\mathbb{R}, |\cdot|)$:
a) $\mathrm{int}((0,\ 1]) = (0,\ 1)$ and $\mathrm{cl}((0,\ 1]) = [0,\ 1]$
b) $\mathrm{int}\left(\left\{\frac{1}{n}\right\}_{n=1}^\infty\right) = \emptyset$ and $\mathrm{cl}\left(\left\{\frac{1}{n}\right\}_{n=1}^\infty\right) = \left\{\frac{1}{n}\right\}_{n=1}^\infty \cup \{0\}$
c) $\mathrm{int}(\mathbb{Q}) = \emptyset$ and $\mathrm{cl}(\mathbb{Q}) = \mathbb{R}$