MATH 742 HW # 3

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Exercise 1 (Exercise 2.1 [Stein])**.** *Prove that*

$$\int_0^\infty \sin x^2 \,\mathrm{d}x = \int_0^\infty \cos x^2 \,\mathrm{d}x = \frac{\sqrt{2\pi}}{4}.$$

These are called the ***Fresnel integrals****. Here,* $\int_0^\infty$ *is interpreted as* $\lim_{R\to\infty}\int_0^R$*. [Hint: Integrate the function* e^{-x^2} *over the path in* Figure 1*. Recall that* $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$*.]*

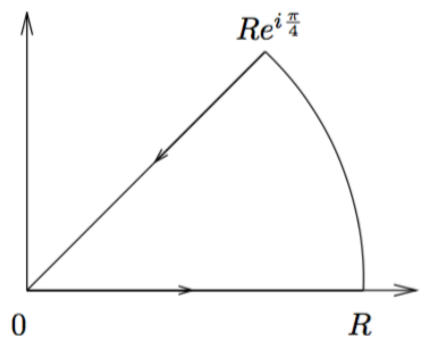

FIGURE 1. The contour in Exercise 1.

Solution. Let $f(z) = e^{iz^2}$. We integrate $f(z)$ around a circular sector of radius R running from $\theta = 0$ to $\pi/4$. Along the x-axis the integral is $\int_0^R e^{ix^2}\,\mathrm{d}x$. Along the curved part we have $z = Re^{i\theta}$ and the integral is

$$\int_0^{\pi/4} e^{iR^2e^{2i\theta}}\, iRe^{i\theta}\,\mathrm{d}\theta = iR\int_0^{\pi/4} e^{-R^2\sin(2\theta)}\, e^{i(\theta+iR^2\cos(2\theta))}\,\mathrm{d}\theta.$$

Finally, along the segment at angle $\pi/4$ we have $z = re^{i\pi/4}$ and the integral is $\int_R^0 e^{-r^2}e^{i\pi/4}\,\mathrm{d}r$. The total integral is zero since f is analytic everywhere.

As $R\to\infty$, the integral over the third piece approaches

$$-e^{i\pi/4}\int_0^\infty e^{-x^2}\,\mathrm{d}x = -e^{i\pi/4}\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i.$$

To estimate the integral over the curved piece, we used the fact that $\sin(2\phi) \geq (4\phi)/\pi$ for $0 \leq \phi \leq \pi/4$; this follows from the concavity of $\sin(2\phi)$. Using this, we have

$$\begin{aligned}\left| iR \int_0^{\pi/4} e^{-R^2\sin(2\theta)}\, e^{i(\theta + iR^2\cos(2\theta))}\,\mathrm{d}\theta \right| &\leq R \int_0^{\pi/4} \left| e^{-R^2\sin(2\theta)}\, e^{i(\theta+iR^2\cos(2\theta))} \right| \mathrm{d}\theta \\ &= R \int_0^{\pi/4} e^{-R^2\sin(2\theta)}\,\mathrm{d}\theta \\ &\leq R \int_0^{\pi/4} e^{-4R^2\theta/\pi}\,\mathrm{d}\theta \\ &= -\frac{\pi}{4R} e^{-4R^2\theta/\pi}\Big|_0^{\pi/4} \\ &= \frac{\pi(1-e^{-R^2})}{4R}.\end{aligned}$$

As $R \to \infty$, this approaches zero and we are left with

$$\int_0^\infty e^{-ix^2}\,\mathrm{d}x - \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4} i = 0.$$

Taking real and imaginary parts, we have

$$\int_0^\infty \cos x^2\,\mathrm{d}x = \int_0^\infty \sin x^2\,\mathrm{d}x = \frac{\sqrt{2\pi}}{4}. \qquad \square$$

Exercise 2 (Exercise 2.2 [Stein]). *Show that*

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

[Hint: The integral equals $\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}\,\mathrm{d}x$*. Use the indented semicircle.]*

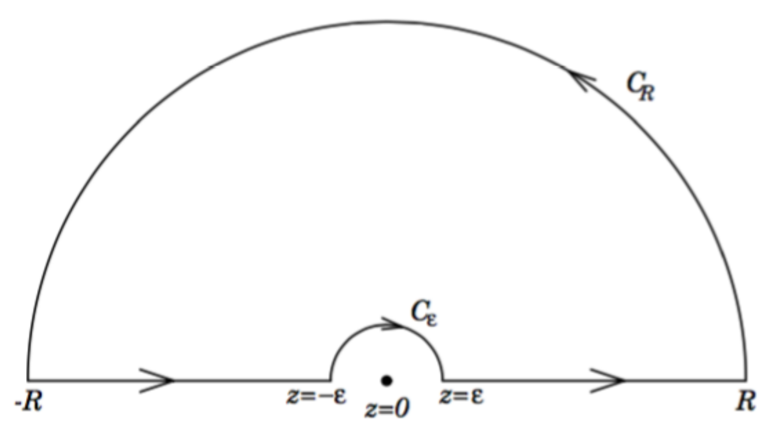

FIGURE 2. Indented contour.

Solution. We integrate $f(z) = e^{iz}/z$ around an indented semicircular contour bounded by circles of radius ε and R in the upper half plane. The integrals along the two portions of the real axis add up to

$$\int_{-R}^{\varepsilon} \frac{\cos x + i \sin x}{x}\, \mathrm{d}x + \int_{\varepsilon}^{R} \frac{\cos x + i \sin x}{x}\, \mathrm{d}x = 2i \int_{\varepsilon}^{R} \frac{\sin x}{x}\, \mathrm{d}x$$

because cosine is even and sine is odd. The integral around the arc C_R of radius R tends to zero as $R \to \infty$ by the Jordan lemma.

Quick digression

Since Jordan's lemma isn't mentioned in our book (at least on this chapter), I am going to include here a proof for this specific case for the sake of completeness: On this arc, $z = Re^{i\theta}$, so the integral is

$$\int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}}\, iRe^{i\theta}\, \mathrm{d}\theta = i \int_0^{\pi} e^{-R\sin\theta}\, e^{iR\cos\theta}\, \mathrm{d}\theta.$$

The absolute value of this integral is at most

$$\int_0^{\pi} e^{-R\sin\theta}\, \mathrm{d}\theta = 2\int_0^{\pi} e^{-R\sin\theta}\, \mathrm{d}\theta.$$

by symmetry. Now $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$ by the concavity of the sine function, so this is at most

$$2\int_0^{\pi} e^{-2R\theta/\pi}\, \mathrm{d}\theta = -\frac{\pi e^{-2R\theta/\pi}}{R}\bigg|_0^{\pi/2} = \frac{\pi(1-e^{-R})}{R},$$

which tends to 0 as $R \to \infty$.

Finally, the integral over the inner semicircle C_ε tends to $-\pi i$; we can see that this holds from the fact that $e^{iz}/z = 1/z + O(1)$ as $z \to 0$ and, since the length of the semicircle is tending to zero, the integral over it approaches the integral of $1/z$ over it, which is

$$\int_{\pi}^{0} \frac{1}{\varepsilon e^{i\theta}}\, i\varepsilon\, e^{i\theta}\, \mathrm{d}\theta = -\int_0^{\pi} \mathrm{d}\theta = -\pi i.$$

Putting the pieces together and letting $R \to \infty$ and $\varepsilon \to 0$, we have

$$2i\int_0^{\infty} \frac{\sin x}{x}\, \mathrm{d}x - \pi i = 0 \implies \int_0^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}. \qquad \square$$

Exercise 3 (Exercise 2.3 [Stein]). *Evaluate the integrals*

$$\int_0^{\infty} e^{-ax}\cos bx\, \mathrm{d}x \qquad \textit{and} \qquad \int_0^{\infty} e^{-ax}\sin bx\, \mathrm{d}x \qquad \textit{with } a > 0$$

by integrating e^{-Az}, *where* $A = \sqrt{a^2+b^2}$, *over an appropriate sector with angle* ω, *with* $\cos\omega = a/A$.

Solution. As indicated, we integrate $f(z) = e^{-Az}$ around a circular sector of radius R with angle θ satisfying $0 \leq \theta \leq \omega$, where $\omega = \arccos(a/A)$ is strictly between 0 and $\pi/2$. (Here we assume $b \neq 0$, since otherwise the integrals are trivially equal to $1/a$ and 0, respectively). The integral along the x-axis is

$$\int_0^{R} e^{-Ax}\, \mathrm{d}x \to \int_0^{\infty} e^{-Ax}\, \mathrm{d}x = \frac{1}{A} \quad \text{as } R \to \infty.$$

To estimate the integral over the curved part we use the fact that $\cos\theta \geq 1 - 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$, which follows from the concavity of the cosine in the first quadrant. Then we have

$$\begin{aligned}\left|\int_0^\omega e^{-ARe^{i\theta}}\, Re^{i\theta}\,\mathrm{d}\theta\right| &\leq \int_0^\omega \left|e^{-ARe^{i\theta}}\, Re^{i\theta}\right|\mathrm{d}\theta \\ &= R\int_0^\omega e^{-AR\cos\theta}\,\mathrm{d}\theta \\ &\leq R\int_0^\omega e^{-AR}\, e^{2AR\theta/\pi}\,\mathrm{d}\theta \\ &= Re^{-AR}\frac{\pi}{2AR}\, e^{2AR\theta/\pi}\Big|_0^\omega \\ &= \frac{\pi}{2A}\left(e^{-AR(1-\frac{2\omega}{\pi})} - e^{-AR}\right).\end{aligned}$$

Since $1 - 2\omega/\pi$ is a positive constant, this tends to 0 as $R \to \infty$.

Finally, on the segment with $\theta = \omega$ we have that $z = re^{i\omega} = r(a+bi)/A$, so the integral is

$$\int_R^0 e^{-Ar(a+ib)/A}\,\frac{a+ib}{A}\,\mathrm{d}r = \frac{a+ib}{A}\int_R^0 e^{-ar}e^{-ibr}\,\mathrm{d}r.$$

Putting the pieces together and letting $R \to \infty$, we have

$$\frac{a+ib}{A}\int_\infty^0 e^{-ax}e^{-ibx}\,\mathrm{d}x + \frac{1}{A} = 0 \implies \int_0^\infty e^{-ax}e^{ibx}\,\mathrm{d}x = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}.$$

Comparing the real and imaginary parts, we have

$$\int_0^\infty e^{-ax}\cos bx\,\mathrm{d}x = \frac{a}{a^2+b^2} \qquad \text{and} \qquad \int_0^\infty e^{-ax}\sin bx\,\mathrm{d}x = \frac{b}{a^2+b^2}.$$ □

Exercise 4 (Exercise 2.6 [Stein]). *Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then*

$$\int_T f(z)\,\mathrm{d}z = 0.$$

Proof. Let γ_ε be a circle of radius ε centered at w, where ε is sufficiently small such that γ_ε lies within the interior of T. Since f is holomorphic in the region R between T and γ_ε, we have

$$\int_{\partial R} f(z)\,\mathrm{d}z = \int_T f(z)\,\mathrm{d}z - \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = 0.$$

Thus, $\int_T f(z)\,\mathrm{d}z = \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z$. But f is bounded near w and the length of γ_ε goes to 0 as $\varepsilon \to 0$, so $\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z \to 0$ and therefore $\int_T f(z)\,\mathrm{d}z = 0$, as desired.
(Note: If we're not allowed to use Cauchy's theorem for a region bounded by two curves, we can use a "keyhole contour" instead; the result is the same.) □

Exercise 5 (Exercise 2.7 [Stein]). *Suppose $f\colon \mathbb{D}^2 \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}^2} |f(z) - f(w)|$ of the image of f satisfies*

$$2|f'(0)| \leq d.$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$. [Hint: $2f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)-f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta$ whenever $0 < r < 1$.]

Proof. By the Cauchy derivative formula, we have

$$f'(0) = \frac{1}{2\pi i}\oint_{C_r} \frac{f(\zeta)}{\zeta^2}\,\mathrm{d}\zeta,$$

where C_r is the circle of radius r centered at 0, with $0 < r < 1$. Substituting $-\zeta$ for ζ and adding the two equations yields

$$2f'(0) = \frac{1}{2\pi i}\oint_{C_r} \frac{f(\zeta)-f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta.$$

So we get

$$|2f'(0)| \leq \frac{1}{2\pi}\oint_{C_r} \frac{|f(\zeta)-f(-\zeta)|}{r^2}\,\mathrm{d}\zeta \leq \frac{d_{\zeta,r}}{r} \leq \frac{d}{r},$$

where

$$d_{\zeta,r} = \sup_{|\zeta|=r} |f(\zeta) - f(-\zeta)|.$$

Letting $r \to 1$ yields the desired result. □

Exercise 6 (Exercise 2.10 [Stein])**.** *Weierstrass's theorem states that a continuous function on $[0,1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z?*

Solution. The answer is positive. This is essentially the Stone-Weierstrass Theorem for complex valued functions defined on the closed unit disc $\mathbb{D}^2 \subset \mathbb{C}$. In order to apply the Stone-Weierstrass Theorem, we need to consider polynomials in z and $\bar{z}$. Let $h(z)$ be a continuous function. Then we can write $h(z) = f(z) + ig(z)$, where f and g are real valued functions. These can be approximated as real valued functions with polynomials $p_f(x,y)$ and $p_g(x,y)$ in x, y by the real version of the Stone-Weierstrass Theorem. But now, any polynomial in x, y can be transformed into a polynomial in the variables z and $\bar{z}$, so that we can make

$$\|h(z) - (p_f(z,\bar{z}) + ip_g(z,\bar{z}))\|_{\text{sup}}$$

arbitrarily small, as desired. □

Exercise 7 (Exercise 2.13 [Stein])**.** *Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial. [Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.]

Proof. First we need to digress for a second to prove the following lemma:

Quick digression

Lemma. *Let $S \subset \mathbb{C}$ be a subset of the plane with no accumulation points. Then S is at most countable.*

Proof. For each $x \in S$, since x is not an accumulation point of S, there exists a $r_x > 0$ such that $\mathbb{B}_{r_x} \cap S = \{x\}$. Then $\{\mathbb{B}_{r_x/2}(x) \mid x \in S\}$ is a disjoint family of open sets; since each contains a distinct rational point, it is at most countable. But this set is bijective with S, so S is at most countable. □

Now suppose to the contrary that f is not a polynomial. Then none of its derivatives can be identically zero, because if $f^{(n)}$ were identically zero, then $f^{(k)}$ would be zero for $k \geq n$ and f would be a polynomial of degree less than or equal to $n-1$. Since the derivatives of f are entire functions that are not everywhere zero, the set of zeros of $f^{(n)}$ has no accumulation points, so it is at most countable by the above lemma. The set of zeros of *any* derivative of f must then be countable since it is a countable union of countable sets. But by hypothesis, every point $z \in \mathbb{C}$ is a zero of some derivative of f, since if $f(z) = \sum c_n (z - z_0)^n$ and $c_k = 0$, then we have

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} f(z)\bigg|_{z_0} = 0.$$

Since $\mathbb{C}$ is uncountable, this is a contradiction, so f must be a polynomial. $(\Rightarrow\Leftarrow)$ □

Exercise 8 (Exercise 2.15 [Stein]). *Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}^2$ that is holomorphic in $\mathbb{D}^2$. Prove that if*

$$|f(z)| = 1 \quad \textit{whenever } |z| = 1,$$

then f is constant. [Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\bar{z})}$ whenever $|z| > 1$, and argue as in the Schwarz reflection principle.]

Proof. Let us define

$$F(z) = \begin{cases} f(z) & \text{if } |z| \leq 1, \\ 1/f(1/z) & \text{otherwise.} \end{cases}$$

Then F is obviously continuous for $|z| < 1$ and $|z| > 1$. For the case when $|z| = 1$, we clearly have continuity from the inside, and if $w \to z$ with $|w| > 1$, then

$$\frac{1}{\bar{w}} \to \frac{1}{\bar{z}} = z \quad \text{and} \quad \frac{1}{\bar{f}\left(\frac{1}{\bar{w}}\right)} \to \frac{1}{\bar{f}\left(\frac{1}{\bar{z}}\right)} = f(z) = F(z).$$

Hence F is continuous everywhere. It is also known by assumption to be holomorphic for $|z| < 1$. Now for $|z| > 1$ we can compute $\partial f / \partial \bar{z} = 0$; alternatively, if Γ is any contour lying in the region $|z| > 1$, let $\widetilde{\Gamma}$ be the image of Γ under the map $w = 1/z$. Then $\widetilde{\Gamma}$ is a contour lying in the region $|w| < 1$ and excluding the origin from its interior (since the point at infinity does not lie within Γ), so

$$\oint_\Gamma F(z)\,\mathrm{d}z = \oint_{\widetilde{\Gamma}} \frac{-1}{w^2} \frac{1}{f(\bar{w})}\,\mathrm{d}w = 0 \qquad \text{since } \frac{1}{w^2 f(\bar{w})} \text{ is analytic on and inside } \widetilde{\Gamma}.$$

To show that F is analytic at points on the unit circle we follow the same procedure as with the *Schwarz reflection principle*, by subdividing a triangle which crosses the circle into triangles which either have a vertex on the circle or an edge lying "along" the circle (i.e., a chord of the circle). In the former case we may move the vertex by ε to conclude that the integral around the triangle is zero. In the case where a side of the triangle is a chord of the circle, we subdivide into smaller triangles (take the midpoint of the circular arc spanned by the chord) until the chord lies within ε of the circle and apply the same argument. The result is that F is entire. But F is also bounded since $f(\overline{\mathbb{D}}^2)$ is a compact set which excludes 0 and hence excludes a neighborhood of zero; so $1/f$ is bounded on $\mathbb{D}^2$. Since F is a bounded entire function, it is constant, and so f is constant as well. $\square$

Problem 1 (Problem 2.1 [Stein])**.** *Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed: Let f be a function defined in the unit disc $\mathbb{D}^2$, with boundary circle $\mathbb{S}^1$. A point w on $\mathbb{S}^1$ is said to be* ***regular*** *for f if there is an open neighborhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D}^2 \cap U$. A function f defined on $\mathbb{D}^2$ cannot be continued analytically past the unit circle if no point of $\mathbb{S}^1$ is regular for f.*

a) *Let*

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \qquad \text{for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc. [Hint: Suppose $\theta = 2\pi p/2^k$, where p and k are positive integers. Let $z = re^{i\theta}$; then $|f(re^{i\theta})| \to \infty$ as $r \to 1$.]

b) *Fix $0 < \alpha < \infty$. Show that the analytic function f defined by*

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \qquad \text{for } |z| < 1$$

extends continuously to the unit circle, but cannot be analytically continued past the unit circle. [Hint: There is a nowhere differentiable function lurking in the background. See Chapter 4 in Book I.]

Proof of a). Since all of the coefficients are either 0 or 1, it is clear that the radius of convergence is 1. Now let $\theta = (2\pi p)/2^k$, with p and k positive integers. We then have, for $r < 1$,

$$\begin{aligned} f(re^{i\theta}) &= \sum_{n=0}^{k-1} r^{2^n} e^{i\frac{2\pi i p\, 2^n}{2^k}} + \sum_{n=k}^{\infty} r^{2^n} e^{i\frac{2\pi i p\, 2^n}{2^k}} \\ &= \sum_{n=0}^{k-1} r^{2^n} e^{i\frac{2\pi i p\, 2^n}{2^k}} + \sum_{n=k}^{\infty} r^{2^n}. \end{aligned}$$

Note that the first sum is bounded for all $r < 1$ and all values of p and k. Letting r approach 1, we see that the $f(re^{i\theta})$ diverges. Therefore the function $f(z)$ has singularities at all points of the form $e^{(2\pi i p)/2^k}$, that is, at all 2^{nd} roots of unity. Such points are dense on the unit circle. Now, given any point w on the boundary $\mathbb{S}^1$ of $\mathbb{D}^2$, let U be an open set in $\mathbb{C}$ containing w. Since the set of 2^{nd} roots of unity is dense in $\mathbb{S}^1$, there exists a point of the form $e^{(2\pi i p)/2^k}$ contained in U. If

w were regular, then there would exist a holomorphic function g which satisfied $f = g$ on $U \cap \mathbb{D}^2$. We would then necessarily have

$$\lim_{r\to 1^-} g(re^{(2\pi i p)/2^k}) = \lim_{r\to 1^-} f(re^{(2\pi i p)/2^k}) = \infty,$$

which contradicts the holomorphicity of g on U. $(\Rightarrow\Leftarrow)$

Therefore, no point of $\mathbb{S}^1$ is regular, and f cannot be continued analytically past $\mathbb{D}^2$. $\square$

Proof of b). For each α, let $f_\alpha(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$. Again, the function $f_\alpha(z)$ converges (uniformly) on the open unit disc $\mathring{\mathbb{D}}^2$. Now, fixing θ we have

$$\left| \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i\theta 2^n} \right| \leq \sum_{n=0}^{\infty} 2^{-n\alpha} = \frac{1}{1-2^{-\alpha}} < \infty \quad \text{since we assume } \alpha > 0.$$

Now set $G_\theta := \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i\theta 2^n}$, and define a function $F_\alpha(z)$ on the closed unit disc $\mathbb{D}^2$ by

$$F_\alpha(re^{i\theta}) = \begin{cases} f_\alpha(re^{i\theta}) & \text{if } r < 1, \\ G_\theta & \text{if } r = 1. \end{cases}$$

By using Abel's Theorem,[1] we have

$$\begin{aligned} \lim_{r\to 1^-} F_\alpha(re^{i\theta}) &= \lim_{r\to 1^-} f_\alpha(re^{i\theta}) \\ &= \lim_{r\to 1^-} \sum_{n=0}^{\infty} 2^{-n\alpha} r^{2^n} e^{i\theta 2^n} \\ &\overset{\text{Abel}}{=\!=} G_\theta = F_\alpha(e^{i\theta}). \end{aligned}$$

Since $F_\alpha(e^{i\theta})$ is continuous as a function of θ, we obtain that $F_\alpha(z)$ is a continuous function on $\mathbb{D}^2$, which is holomorphic on $\mathring{\mathbb{D}}^2$ (note that it doesn't make sense to talk about holomorphicity on the boundary of $\mathbb{D}^2$, since holomorphicity is defined on open sets).

Assume now that $F_\alpha(z)$ can be analytically continued past the closed unit disc. This means that there exists an open region Ω satisfying $\Omega \cap \mathbb{D}^2 \neq \emptyset$ with $\Omega \not\subset \mathbb{D}^2$, and a holomorphic function $\Phi(z)$ on Ω such that $\Phi(z) = F_\alpha(z)$ on $\Omega \cap \mathbb{D}^2$.

Assume first that $0 < \alpha < 1$. Since Ω is open, connected, and not entirely contained in $\mathbb{D}^2$, the intersection $\Omega \cap \partial\mathbb{D}^2$ is nonempty, and must contain a segment of the unit circle $\mathbb{S}^1$. Since $\Phi(z)$ is holomorphic on Ω, its restriction to $\Omega \cap \partial\mathbb{D}^2$ will be differentiable. However, the restriction of $\Phi(z)$ to $\Omega \cap \partial\mathbb{D}^2$ is equal to $F_\alpha(z)$, and then we know from Chapter 4 Book I (as suggested in the hint), that the function of θ

$$\sum_{n=0}^{\infty} 2^{-n\alpha} e^{i\theta 2^n}$$

is not differentiable. Thus we obtain a contradiction. $(\Rightarrow\Leftarrow)$

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[1] Here's Abel's Theorem, for reference:

Theorem (Abel's Theorem). (IF A SERIES CONVERGES, THEN IT IS ABEL SUMMABLE WITH THE SAME LIMIT) *Suppose $\sum_{n=1}^{\infty} a_n$ converges. Then*

$$\lim_{\substack{r\to 1 \\ r<1}} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n.$$

Now assume that $\alpha > 1$, where α is not an integer, and assume further that $F_\alpha(z)$ extends past $\mathbb{D}^2$. We define an operator D on holomorphic functions by

$$D(g(z)) := z\,\frac{\mathrm{d}g(z)}{\mathrm{d}z}.$$

Note that on the open unit disc $\mathring{\mathbb{D}}^2$, we have

$$D(f_\alpha(z)) = f_{\alpha-1}(z).$$

Now let Ω and $\Phi(z)$ be as before. Note that the condition $\Omega \cap \mathbb{D}^2 \neq \emptyset$ actually implies $\Omega \cap \mathring{\mathbb{D}}^2 \neq \emptyset$. Moreover, the set $\Omega \cap \mathbb{D}^2 \neq \emptyset$ is open. Choose a positive integer m such that $0 < \alpha - m < 1$. Then on the open set $\Omega \cap \mathbb{D}^2 \neq \emptyset$ we have

$$\Phi(z) = F_\alpha(z) = f_\alpha(z).$$

which implies

$$\underbrace{D \circ \cdots \circ D}_{m \text{ times}}(\Phi(z)) = D^m(\Phi(z)) = D^m(f_\alpha(z)) = f_{\alpha-m}(z).$$

Since $\Phi(z)$ is a holomorphic function on Ω, so is $D^m(\Phi(z))$. But now we obtain a contradiction as above: if such a function $\Phi(z)$ existed, then $D^m(\Phi(z))$ would have to agree with $F_{\alpha-m}(z)$ on $\Omega \cap \partial\mathbb{D}^2$, which is not differentiable.

Finally, if $\alpha = m$ is an integer, then any extension $\Phi(z)$ of f_m would have to satisfy

$$D^m(\Phi(z)) = D^m(f_m(z)) = f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

on $\Omega \cap \mathring{\mathbb{D}}^2$, where $f(z)$ is the function of part a). Such a function $\Phi(z)$ would not even extend to $\Omega \cap \partial\mathbb{D}^2$, and we obtain a contradiction. $(\Rightarrow\Leftarrow)$ $\square$