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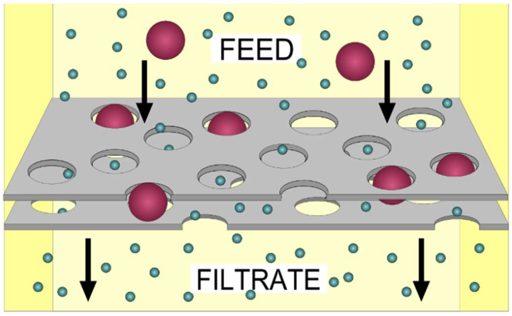
1 Elementary Filter-Clogging Models

Various manufacturing processes involve filtration, passing a liquid through a filter to remove large particles. At Eastman Kodak we filtered photographic emulsions to remove oversized silver halide grains and coupler droplets.



Because filters trap the oversized particles, they accumulate material over time and get clogged. There are three elementary models of how this clogging works.

They are all based on modeling the filter as a sequence of parallel tubes through which the inflowing fluid –the *feed*– is pushed by a fixed pressure difference Δp. If there are N tubes per square centimeter of filter face, then the volumetric flow rate Q, in $\mathrm{cm}^3/(\mathrm{hour}\cdot\mathrm{cm}^2)$, is given by

$$Q = \frac{N\pi r^4}{8\mu L}\Delta p. \tag{1}$$

Here, r, μ, and L are the radius of the tubes, the viscosity of the fluid, and the length of the tubes, respectively. The three models are as follows:

- In the *pore-blocking* model, the volumetric flow diminishes with time because pores get blocked. The number of pores N diminishes in such a way that the probability per unit liquid filtered of any pores becoming plugged is constant.
- In the *cake* model, the volumetric flow diminishes with time because filtered material cakes onto the front of the filter, thus lengthening the pores: L increases linearly with the total amount of liquid that has been filtered.
- In the *arteriosclerosis* model (hysterical name ☺), the volumetric flow rate diminishes over time because the radius of the pores diminishes. That radius diminishes because each bit of fluid that flows through the filter deposits some amount of material on the inside of the pore; the pore volume diminishes at a rate proportional to the volumetric flux.

1.1 First Question

Write a differential equation for $Q(t)$ for each of the three models. Solve those three equations and graph the solutions, using a set of parameter that makes the various solutions comparable. For example, be sure to use the same viscosity in all three cases. Comment on the models.

Solution. We tackle each of the three models separately.

- [Pore-blocking model] Let σ be the constant that denotes the probability per unit liquid filtered of any pores becoming clogged. Then we have the following relation:

$$N(t+\Delta t) = N(t) - \sigma N(t) Q\Delta t, \tag{2}$$

  which yields

$$\begin{aligned} \frac{N(t+\Delta t) - N(t)}{\Delta t} &= -\sigma N(t) Q \\ \frac{\mathrm{d}N}{\mathrm{d}t} &= -\sigma N(t) Q. \end{aligned} \tag{3}$$

This is a straightforward separable equation for N, so this time I won't be killing a mosquito with a shotgun! No numerical solution this time ☺... Instead,

$$
\begin{aligned}
\int \frac{1}{N^2}\mathrm{d}N &= -\int \frac{\sigma\pi r^4\Delta p}{8\mu L}\mathrm{d}t \\
N &= \left(\frac{\sigma\pi r^4\Delta p}{8\mu L}t + N_0\right)^{-1},
\end{aligned} \tag{4}
$$

where $N_0 = N(0)$ is the number of pores initially clogged at $t = 0$. Hence, the volumetric flow rate Eq. (1) for this particular pore-blocking model is given by

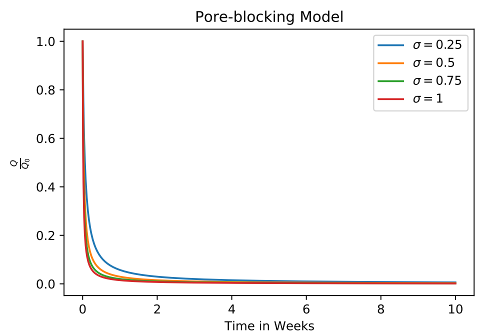
$$
Q = \frac{\pi r^4\Delta p}{8\mu L}\left(\frac{\sigma\pi r^4\Delta p}{8\mu L}t + N_0\right)^{-1}. \tag{5}
$$

For simplicity, we shall now set all parameters to unity (except the probability parameter σ, since it may be interesting to look at the dependence of our results on such parameter). This simplifies Eq. (5) into

$$
Q = \frac{\pi}{8}\left(\frac{\sigma\pi}{8}t + 1\right)^{-1}. \tag{6}
$$

[Note that at $t = 0$, $Q_0 = Q(0) = \pi/8$.] Moreover, since the Figure on the following question gives time in weeks, (and eventually we want to compare our models to this idealized simulated data),[1] we will multiply $t \times 24 \times 7$ and graph instead

$$
\frac{Q}{Q_0} = \frac{1}{21\sigma\pi t + 1}. \tag{7}
$$



- [Cake model] In this model, L increases linearly with the amount of liquid that has been filtered. This means that L takes the form

$$
L(t) = \alpha V(t) + L_0, \tag{8}
$$

where $L_0 = L(0)$ is the initial pore length at $t = 0$, and $\alpha \in (0, \infty)$ is some positive constant (note that a quick dimensional analysis reveals that α must have units cm^2). Thus, since $Q = \mathrm{d}V/\mathrm{d}t$,

$$
\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \alpha Q \\
\int L\mathrm{d}L &= \frac{\alpha N\pi r^4}{8\mu}\Delta p\int \mathrm{d}t \\
L &= \sqrt{\frac{\alpha N\pi r^4\Delta p}{4\mu}t + L_0^2}.
\end{aligned} \tag{9}
$$

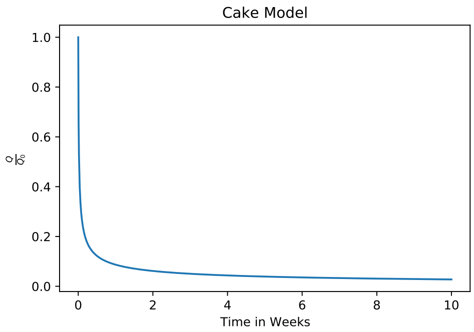
Plugging this result back into Eq. (1) yields

$$
Q = \frac{N\pi r^4\Delta p}{8\mu\sqrt{\frac{\alpha N\pi r^4\Delta p}{4\mu}t + L_0^2}}. \tag{10}
$$

Dividing by Q_0 and setting all the parameters to unity as before (including N of course, since this time is constant), we have (in terms of weeks)

$$
\frac{Q}{Q_0} = \frac{1}{\sqrt{42\pi t + 1}}. \tag{11}
$$

[1] Of course, our graphs won't fit exactly, because of our choice of setting all parameters to unity, but at least we would like to get an overall *shape* that matches that of the Figure.



- [Arteriosclerosis model] In this model the pores' radius (and consequently, their volume) diminishes, with the volume decreasing at a rate proportional to the volumetric flux. Since the pore's volume is given by $V = \pi r^2 L$, and since volumetric flux is the flow rate divided by the area A of the filter, we have, for some positive constant $\nu \in (0, \infty)$,

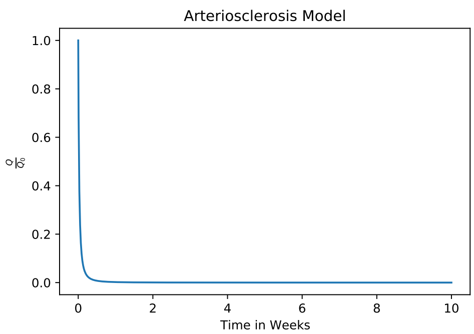
$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= -\nu \frac{Q}{A} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\pi r^2 L\right) &= -\nu \frac{N\pi r^4 \Delta p}{8\mu L A} \\
2\pi r L \,\mathrm{d}r &= -\nu \frac{N\pi r^4 \Delta p}{8\mu L A}\mathrm{d}t \\
\int \frac{\mathrm{d}r}{r^3} &= -\nu \frac{N\Delta p}{16\mu L^2 A}\int \mathrm{d}t \\
r &= \frac{1}{\sqrt{\frac{\nu N \Delta p}{8\mu L^2 A}t + \frac{1}{r_0^2}}},
\end{aligned} \tag{12}$$

where $r_0 = r(0)$ is the inital radius of the pores at time $t = 0$. Plugging back into Eq. (1) yields

$$Q = \frac{N\pi\Delta p}{8\mu L\left(\frac{\nu N\Delta p}{8\mu L^2 A}t + \frac{1}{r_0^2}\right)^2} \tag{13}$$

Setting all the parameters to unity and diving by Q_0, as before, we have (in terms of weeks)

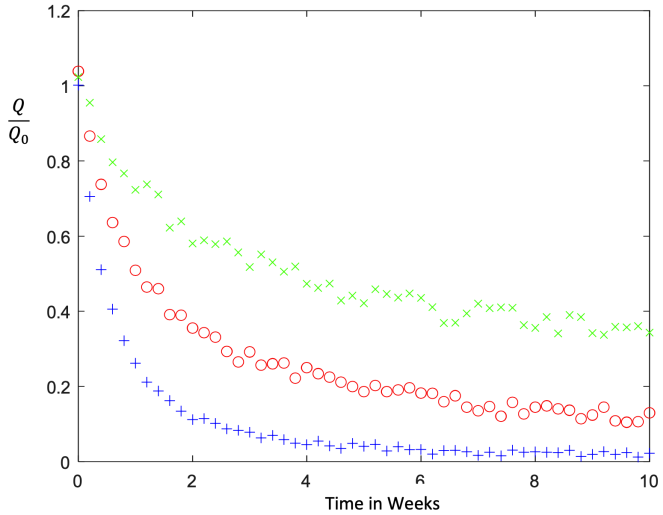
$$\frac{Q}{Q_0} = \frac{1}{(21t+1)^2}. \tag{14}$$



□

1.2 Second Question

In the Figure we have very idealized simulated data of normalized volumetric flow rates over time for the three different models. Which is which?



Solution. Admittedly, my models would have been much closer to these results had I used more realistic values for the paramaters, but alas, I set them all to 1! At any rate, we can still discern that the arteriosclerosis model is the one given by the $+$'s on the Figure (also, this can be seen by noticing in this model the flow rate decays as t^{-2}, so it is no surprise that it diminishes faster than the other models). The cake and pore-blocking models are much harder to tell apart from my graphs…I could play around with the parameters, but we know that the pore-blocking model diminishes as t^{-1}, while the cake model decays as $t^{-1/2}$, so the latter must take longer to decrease. Hence, the cake model is represented on the Figure by the $\times$'s and the pore-blocking model by the $\circ$'s. $\square$