MATH 742 HW # 2

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Exercise 1 (Exercise 3.4 [Conway]). *Discuss the mapping properties of z^n and $z^{1/n}$ for $n \geq 2$. (Hint: use polar coordinates.)*

Solution. Letting $z = re^{i\theta}$, we have $z^n = r^n e^{in\theta}$ and $z^{1/n} = r^{1/n} e^{\frac{i\theta}{n}}$. In the case of $w = z^n$, we have that when $|z| > 1$ the ray that extends from the origin to w stretches by the power of n while w wraps around the circle of radius r at a faster rate (faster by a factor of n). When $|z| < 1$, w still wraps around the circle faster, but now the ray is shrunk. Lastly, when $|z| = 1$, w once again wraps around the circle faster, although this time the ray remains unchanged (we are on the unit circle). For the case when $v = z^{1/n}$, v wraps around the circle slower now (by a factor of $1/n$). When $|z| > 1$, the ray this time won't extend arbitrarily large; its largest value is when $n = 2$ and it shrinks as n increases until v reaches the unit circle as $n \to \infty$ (for $|z| = 1$, v is also on the unit circle). Lastly, for $|z| < 1$, v has its smallest value when $n = 2$, and it increases as $n \to \infty$ (although this upper bound (i.e. 1) is not in this set $|z| < 1$). $\square$

Exercise 2 (Exercise 3.5 [Conway]). *Find the fixed points of a dilation, translation, and inversion on $\mathbb{C}_\infty$.*

Solution. Recall that a Möbius transformation S is of the form

$$S(z) = \frac{az + b}{cz + d}, \qquad \text{where } ad - bc \neq 0.$$

- To get a dilation $S(z) = az$, we have that $b, c = 0$, $d = 1$, and $a > 0$. To find a fixed point, we have to find all z such that $S(z) = z$; in this case $az = 0$. Obviously $z = 0$ is a fixed point. Also $z = \infty$ is a fixed point, since $a \cdot \infty = \infty$ (for $a > 0$).
- To get a translation $S(z) = z + b$, we have that $c = 0$, $a, d = 1$, and $b \in \mathbb{C}$. To find a fixed point, we have to find all z such that $z + b = z$, which is true only when $z = \infty$ (or for all z in the trivial case where $b = 0$).
- To get an inversion $S(z) = 1/z$, we have that $a, d = 0$ and $b, c = 1$. To find a fixed point, we have to find all z such that $1/z = z$, which is equivalent to $z^2 = 1$. Thus, $z = 1$ and $z = -1$ are the fixed points. $\square$

Exercise 3 (Exercise 3.6 [Conway]). *Evaluate the following cross ratios:*

a) $(7 + i, 1, 0, \infty)$.

b) $(2, i - 1, 1, 1 + i)$.

c) $(0, 1, i, -1)$.

d) $(i - 1, \infty, 1 + i, 0)$.

Solution. Recall that

$$S(z) = \frac{\frac{z-z_3}{z-z_4}}{\frac{z_2-z_3}{z_2-z_4}} \qquad \text{if } z_2, z_3, z_4 \in \mathbb{C}, \tag{1}$$

$$S(z) = \frac{z-z_3}{z-z_4} \qquad \text{if } z_2 = \infty, \tag{2}$$

$$S(z) = \frac{z_2-z_4}{z-z_4} \qquad \text{if } z_3 = \infty, \tag{3}$$

$$S(z) = \frac{z-z_3}{z_2-z_3} \qquad \text{if } z_4 = \infty. \tag{4}$$

If $z \in \mathbb{C}_\infty$, then (z, z_2, z_3, z_4) (known as the ***cross ratio*** of z, z_2, z_3, and z_4) is the image of z under the unique Möbius transformation which takes

$$\begin{aligned} z_2 &\mapsto 1 \\ z_3 &\mapsto 0 \\ z_4 &\mapsto \infty. \end{aligned}$$

Now, to solve *a)*, note that by (4), we have

$$(7+i, 1, 0, \infty) = \frac{7+i-0}{1-0} = 7+i.$$

To solve *b)*, note that by (1), we have

$$(2, i-1, 1, 1+i) = \frac{\frac{2-1}{2-1-i}}{\frac{1-i-1}{1-i-1-i}} = \frac{\frac{1}{1-i}}{\frac{-i}{-2i}} = \frac{2}{1-i} = 2\frac{1+i}{(1-i)(1+i)} = 2\frac{1+i}{2} = 1+i.$$

To solve *c)*, note that by (1), we have

$$(0, 1, i, -1) = \frac{\frac{0-i}{0+i}}{\frac{1-i}{1+1}} = \frac{-i}{\frac{1-i}{2}} = -i\frac{2}{1-i}\,\frac{1+i}{1+i} = -i\frac{2}{2}(1+i) = -i - i^2 = 1-i.$$

To solve *d)*, note that by (2), we have

$$(i-1, \infty, 1+i, 0) = \frac{i-1-1-i}{i-1-0} = \frac{-2}{i-1}\,\frac{i+1}{i+1} = \frac{-2}{-2}(i+1) = i+1.$$

□

Exercise 4 (Exercise 3.7 [Conway]). *If $Tz = (az+b)/(cz+d)$, find z_2, z_3, z_4 (in terms of a, b, c, d) such that $Tz = (z, z_2, z_3, z_4)$.*

Solution. Just evaluate the inverse of T:

$$T^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Now everything follows:

$$\begin{aligned} T^{-1}(1) = z_2 &= \frac{d-b}{a-c} \\ T^{-1}(0) = z_3 &= -\frac{b}{a} \\ T^{-1}(\infty) = z_4 &= -\frac{d}{c}. \end{aligned}$$

□

Exercise 5 (Exercise 3.8 [Conway])**.** *If $Tz = (az+b)/(cz+d)$, show that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ if and only if we can choose a, b, c, d to be real numbers.*

Proof. $(\Rightarrow)$ We first assume that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Then let $z_0 \in \mathbb{R}_\infty$ such that $Tz_0 = 0$. Observe that this implies $az_0 = -b$, so $-b/a \in \mathbb{R}_\infty$. Set $r_1 \equiv b/a \in \mathbb{R}_\infty$. Likewise, if $z_\infty \in \mathbb{R}_\infty$ such that $Tz_\infty = \infty$, then $cz_\infty + d = 0 \implies z_\infty = -d/c$; thus we may set $r_2 \equiv d/c \in \mathbb{R}_\infty$. Now let $z_1 \in \mathbb{R}_\infty$ such that $Tz_1 = 1$. Then,

$$\begin{aligned}
\frac{az_1+b}{cz_1+d} &= 1 \\
\implies az_1 + b &= cz_1 + d \\
\implies z_1\left(1 - \frac{c}{a}\right) &= \frac{d}{a} - \frac{b}{a} \\
\implies \frac{z_1}{c} - \frac{z_1}{a} - \frac{r_2}{a} + \frac{r_1}{c} &= 0 \\
\implies \frac{z_1 + r_1}{c} &= \frac{z_1 + r_2}{a} \\
\implies \frac{z_1 + r_1}{z_1 + r_2} &= \frac{c}{a} \in \mathbb{R}_\infty.
\end{aligned}$$

Now letting $r_3 = c/a$, we have

$$\frac{d}{a} = \frac{d}{c} \cdot \frac{c}{a} = r_2 r_3 \in \mathbb{R}_\infty.$$

Hence,

$$Tz = \frac{az+b}{cz+d} = \frac{z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}} = \frac{z + r_1}{r_3 z + r_2 r_3},$$

and we have thus found real coefficients for T.

$(\Leftarrow)$ Now we prove the converse by contradiction, assuming that $T(\mathbb{R}_\infty) \neq \mathbb{R}_\infty$. Recognizing that $\mathbb{R}_\infty$ is a circle in $\mathbb{C}_\infty$, and knowing the fact that Möbius transformations map circles onto circles, we may conclude that $T(\mathbb{R}_\infty)$ is some other circle in $\mathbb{C}_\infty$. In particular, this means that the intersection $T(\mathbb{R}_\infty) \cap (\mathbb{C}_\infty \setminus \mathbb{R}_\infty)$ is nonempty, which is to say that there must be some value $w \in \mathbb{R}_\infty$ for which $Tw \notin \mathbb{R}_\infty$. Now, if there is some representation of T in which a, b, c, d are real, then

$$Tw = \frac{aw+b}{cw+d}$$

is clearly an element of $\mathbb{R}_\infty$, contradicting the observation that Tw is not real. Hence we have shown that a, b, c, d are all real if and only if $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. $\square$

Exercise 6 (Exercise 3.9 [Conway])**.** *If $Tz = (az+b)/(cz+d)$, find necessary and sufficient conditions such that $T(\mathbb{S}^1) = \mathbb{S}^1$.*

Solution. We want to find necessary and sufficient conditions so that $|z| = |z|^2 = z\bar{z} = 1$ implies $T(z)\overline{T(z)} = 1$. Note the following

$$\begin{aligned}
T(z)\overline{T(z)} = 1 &\iff \frac{az+b}{cz+d}\,\overline{\frac{az+b}{cz+d}} = 1 \\
&\iff \frac{(az+b)(\bar{a}\bar{z}+\bar{b})}{(cz+d)(\bar{c}\bar{z}+\bar{d})} = 1 \\
&\iff az\bar{a}\bar{z} + b\bar{a}\bar{z} + az\bar{b} + b\bar{b} = cz\bar{c}\bar{z} + d\bar{c}\bar{z} + \bar{d}cz + d\bar{d} \\
&\iff z\bar{z}(a\bar{a} - c\bar{c}) + z(a\bar{b} - c\bar{d}) + \bar{z}(b\bar{a} - d\bar{c}) + b\bar{b} - d\bar{d} = 0. \qquad (5)
\end{aligned}$$

Now note that equation (5) is equivalent to saying that $z\bar{z} - 1 = 0$ if

$$\begin{aligned} a\bar{a} - c\bar{c} = |a|^2 - |c|^2 &= 1 \\ b\bar{b} - d\bar{d} = |b|^2 - |d|^2 &= -1 \\ a\bar{b} - c\bar{d} &= 0 \\ b\bar{a} - d\bar{c} &= 0. \end{aligned}$$

Hence we have the necessary and sufficient conditions

$$\begin{aligned} |a|^2 + |b|^2 &= |c|^2 + |d|^2 \\ a\bar{b} - c\bar{d} = b\bar{a} - d\bar{c} &= 0. \end{aligned} \tag{6}$$

Let $c = \delta\bar{b}$, so that $a\bar{b} - c\bar{d} = 0$ yields

$$a\bar{b} - \delta\bar{b}\bar{d} = 0 \iff a = \delta\bar{d} \iff d = \frac{\bar{a}}{\bar{\delta}}.$$

Plugging this into (6), we get

$$|a|^2 + |b|^2 = |\delta|^2|b|^2 + \frac{|a|^2}{|\delta|^2} \implies |\delta| = |\delta|^2 = \delta\bar{\delta} = 1 \implies \delta = \frac{1}{\bar{\delta}}.$$

Thus the Möbius transformation is of the form

$$T(z) = \frac{az + b}{\delta(\bar{b}z + \bar{a})} \quad \text{or} \quad T(z) = \bar{\delta}\,\frac{az + b}{(\bar{b}z + \bar{a})}, \qquad \text{where } |\delta| = 1.$$

Now taking $\delta = e^{-i\theta}$, we get

$$T(z) = e^{i\theta}\,\frac{az + b}{(\bar{b}z + \bar{a})} \qquad \text{for some } \theta.$$

This mapping transforms $|z| = 1$ into $|T(z)| = 1$; in other words, it satisfies $T(\mathbb{S}^1) = \mathbb{S}^1$, as desired. □

Exercise 7 (Exercise 3.10 [Conway]). *Consider the interior of the unit disk* $\mathring{\mathbb{D}}^2 = \{z : |z| < 1\}$. *Find all Möbius transformations* T *such that* $T(\mathring{\mathbb{D}}^2) = \mathring{\mathbb{D}}^2$.

Solution. Let $w \in \mathring{\mathbb{D}}^2$ be such that $T(w) = 0$. Recall that the symmetric point w^* of a point w is one that satisfies

$$w^* - a = \frac{R^2}{\bar{w} - \bar{a}}.$$

Hence in this case with $a = 0$ and $R = 1$, the symmetric point of w with respect to the unit circle is

$$w^* = \frac{1}{\bar{w}}.$$

Therefore, we have that $T(w^*) = \infty$, and thus T is the form

$$T(z) = \lambda\,\frac{z - w}{\bar{w}z - 1}, \qquad \text{where } \lambda \text{ is a constant.} \tag{$\heartsuit$}$$

(It is easy to see that (♡) satisfies $T(w) = 0$ and $T(w^*) = T(1/\bar{w}) = \infty$.) Finally, we are going to choose the constant λ in such a way that $|T(z_0)| = 1$, where $z_0 = e^{i\theta}$. We have

$$T(z_0) = \lambda \, \frac{e^{i\theta} - w}{\bar{w}e^{i\theta} - 1},$$

and therefore,

$$\begin{aligned} 1 = |T(z_0)| &= |\lambda| \, \frac{|e^{i\theta} - w|}{|\bar{w}e^{i\theta} - 1|} \\ &= |\lambda| \, \frac{(e^{i\theta} - w)(e^{-i\theta} - \bar{w})}{|e^{i\theta}| \cdot |\bar{w} - e^{-i\theta}|} \\ &= |\lambda| \, \frac{(e^{i\theta} - w)(e^{-i\theta} - \bar{w})}{|e^{i\theta}| \cdot |\bar{w} - e^{-i\theta}|} \\ &= |\lambda| \, \frac{(e^{i\theta} - w)(e^{-i\theta} - \bar{w})}{1 \cdot (\bar{w} - e^{-i\theta})(w - e^{i\theta})} \\ &= -|\lambda|. \end{aligned}$$

This indicates that $|\lambda| = 1$, which in turn implies that $\lambda = e^{i\theta}$ for some real θ. Hence we conclude that all of the Möbius transformations satisfying $T(\mathring{\mathbb{D}}^2) = \mathring{\mathbb{D}}^2$ are of the form

$$T(z) = e^{i\theta} \, \frac{z - w}{\bar{w}z - 1}, \qquad \text{for some real } \theta. \qquad \square$$

Exercise 8 (Exercise 3.13 [Conway])**.** *Give a discussion of the mapping $f(z) = \frac{1}{2}(z + 1/z)$.*

Solution. The function

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

can be defined for all $z \in \mathbb{C} \setminus \{0\}$ and therefore also in the interior of the punctured disk $\mathring{\mathbb{D}}^2 \setminus \{0\} = \{z : 0 < |z| < 1\}$. To see that in this domain the function is injective, let z_1, z_2 be two numbers in the domain of f such that $f(z_1) = f(z_2)$; then

$$0 = f(z_1) - f(z_2) = \frac{z_1^2 + 1}{z_1} - \frac{z_2^2 + 1}{z_2} = \frac{z_1^2 z_2 + z_2 - z_1 z_2^2 - z_1}{z_1 z_2} = \frac{(z_1 z_2 - 1)(z_1 - z_2)}{z_1 z_2}.$$

Since we assumed that $0 < |z_i| < 1$, for $i = 1, 2$, the factor $z_1 z_2 - 1$ is always nonzero and we conclude that $z_1 = z_2$, proving thus the injectivity of f.

Now to determine the range of the function, let us write $z = re^{i\theta}$ in polar coordinates and let $f(z) = w = a + ib$, for $a, b \in \mathbb{R}$. Then,

$$f(z) = f(re^{i\theta}) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = \frac{1}{2}\left[\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\right].$$

Note that for the real and imaginary parts of w the following equations must hold

$$(\spadesuit) \qquad \begin{aligned} a &= \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta \\ b &= \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$$

Then, if $f(z) = w$ has imaginary part $b = 0$, we have that $\sin\theta = 0$ and $|\cos\theta| = 1$. Therefore points of the form $a + ib$, with $a \in [-1, 1]$ and $b = 0$, cannot be in the range of f. For all other points the equations ($\spadesuit$) can be solved for r and θ uniquely (after restricting the argument to $[-\pi, \pi)$). Therefore we conclude that the range of the function is $\mathbb{C} \smallsetminus \{z \in \mathbb{C} \mid \mathfrak{Re} z \in [-1.1] \text{ and } \mathfrak{Im} z = 0\}$.

Lastly, to give a geometric notion of the function, note that given any value of $r \in (0,1)$, the graph of $f(re^{i\theta})$ as a function of θ looks like an ellipse. In fact, from the formulas ($\spadesuit$) we see that

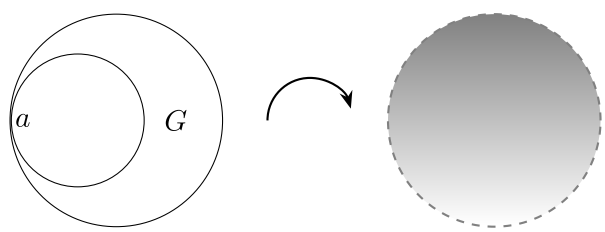
$$\left(\frac{a}{\frac{1}{2}\left(r+\frac{1}{r}\right)}\right)^2 + \left(\frac{b}{\frac{1}{2}\left(r-\frac{1}{r}\right)}\right)^2 = 1.$$

If we fix the argument θ and let r vary in $(0,1)$, it follows from equations ($\spadesuit$) that the graph of $f(re^{i\theta})$ is a hyperbola and it degenerates to rays if z is purely real or imaginary. In the case when $\theta \in \{(2k+1)\pi \mid k \in \mathbb{Z}\}$, the graph of f in dependence on r is on the imaginary axis and for $\theta \in \{2k\pi \mid k \in \mathbb{Z}\}$ the graph of $f(re^{i\theta})$ is either $(-\infty, -1)$ or $(1, \infty)$. If $\cos\theta, \sin\theta \neq 0$, then

$$\left(\frac{a}{\cos\theta}\right)^2 - \left(\frac{b}{\sin\theta}\right)^2 = 1. \qquad \square$$

Exercise 9 (Exercise 3.14 [Conway])**.** *Suppose that one circle is contained inside another and that they are tangent at the point a. Let G be the region between the two circles and map G conformally onto the open unit disk $\mathring{\mathbb{D}}^2$. (Hint: first try $(z-a)^{-1}$.)*

Solution. The situation is presented in the following picture:



Now, using the provided hint, define the Möbius transformation $T(z) = (z-a)^{-1}$ which sends the region G to a region between two lines. Afterwards, applying a rotation followed by a translation, it is possible to send this region to any other region bounded by any two parallel lines we want. Hence, choose $T'(z) = cz + d$, where $|c| = 1$, such that

$$(T' \circ T)(G) = \left\{x + iy \mid 0 < y < \frac{\pi}{2}\right\}.$$

Applying the exponential function to this region yields the right half plane

$$(\exp \circ T' \circ T)(G) = \{x + iy \mid x > 0\}.$$

Finally, the Möbius transformation

$$R(z) = \frac{z-1}{z+1}$$

maps the right half plane onto the unit disk (we can see all the details of this mapping on page 53, Conway's). Hence the function ψ defined by $\psi(z) = (R \circ \exp \circ T' \circ T)(z)$ maps G onto $\mathring{\mathbb{D}}^2$ and is the desired conformal mapping (ψ is a composition of conformal mappings, hence it is also conformal). Doing some simplifications we obtain

$$\psi(z) = \frac{e^{\frac{\alpha}{z-a}+\beta} - 1}{e^{\frac{\alpha}{z-a}+\beta} + 1}$$

where the constants α, β will depend on the circle's location. $\square$

Exercise 10 (Exercise 3.18 [Conway]). *Let* $-\infty < a < b < \infty$ *and put* $Mz = (z - ia)/(z - ib)$. *Define the lines* $L_1 = \{z \mid \mathfrak{Im}\, z = b\}$, $L_2 = \{z \mid \mathfrak{Im}\, z = a\}$, *and* $L_3 = \{z \mid \mathfrak{Re}\, z = 0\}$. *Determine which of the regions* A, B, C, D, E, F *in* Figure 1, *are mapped by* M *onto the regions* U, V, W, X, Y, Z *in* Figure 2.

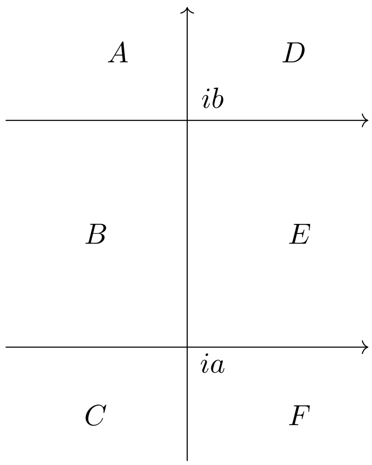


Figure 1

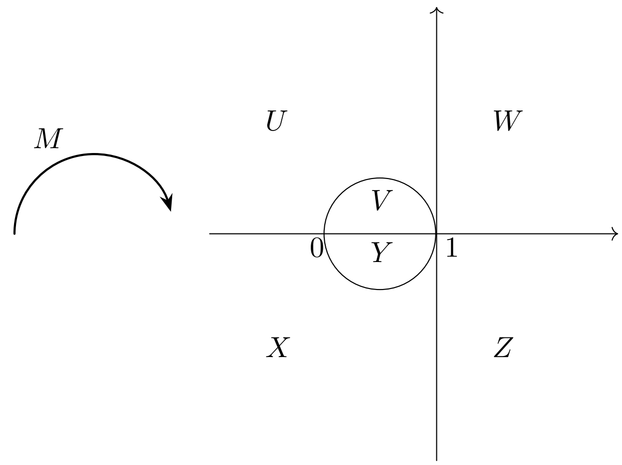


Figure 2

Solution. Note that $M(ia) = (ia - ia)/(ia - ib) = 0$. Therefore the regions B, C, E. and F which touch the line ia are mapped in some form to the regions U, X, V, and Y which touch 0. Similarly we have $M(ib) = \infty$ and therefore the regions B, E, A, and D which touch the line ib are mapped somehow to the regions U, W, Z, and X which touch ∞. Thus we must have that C or F goes to either V or Y. Let us find out what's really going on: Let x, y be small positive real numbers such that the point $z = x + iy + ia \in E$. Thus, the imaginary part of Mz is a positive number multiplied by $x(b - a)$ and therefore also positive. Thus we conclude that M maps E to U and B to X. Because B and C meet at the line ia, we conclude that X and $M(C)$ do meet there as well. Hence, M maps C to Y and F to V. By a similar argument, we obtain that M maps A to Z and D to W. $\square$