MATH 750 HW # 1

MARIO L. GUTIERREZ ABED
PROF. H. PARLIER

Ex 1-7: A linear transformation $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **norm preserving** if $|T(x)| = |x|$, and similarly it is **inner product preserving** if $\langle Tx, Ty\rangle = \langle x, y\rangle$.

a) Prove that T is norm preserving iff T is inner product preserving.
b) Prove that such a linear transformation T is 1-1 and that its inverse T^{-1} is of the same sort.

Proof. For part a), let us assume first that T is norm preserving, so that $|T(x)| = |x|$ holds. Then we have

$$\begin{aligned}
\langle Tx, Ty\rangle &= \frac{1}{4}(|Tx+Ty|^2 - |Tx-Ty|^2) && \text{(By the polarization identity)}\\
&= \frac{1}{4}(|T(x+y)|^2 - |T(x-y)|^2) && \text{(By linearity of } T\text{)}\\
&= \frac{1}{4}(|x+y|^2 - |x-y|^2) && \text{(Since } T \text{ is norm preserving)}\\
&= \langle x, y\rangle && \text{(By the polarization identity)}
\end{aligned}$$

Hence we have shown that T is inner product preserving.

Now we assume that T is inner product preserving, so that $\langle Tx, Ty\rangle = \langle x, y\rangle$ holds. By *Theorem 1.2* part *(4)* on our text, we have that $|x| = \sqrt{\langle x, x\rangle}$, which is true because

$$\begin{aligned}
\langle x, x\rangle &= x_1^2 + x_2^2 + \ldots x_n^2\\
\Longrightarrow \sqrt{\langle x, x\rangle} &= \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}\\
&= |x|.
\end{aligned}$$

Using this result then, we have

$$|Tx| = \sqrt{\langle Tx, Tx\rangle} = \sqrt{\langle x, x\rangle} = |x|,$$

thus showing that T is also norm preserving, and we are done.

Now for part b), take two elements $Tx, Ty \in \mathbb{R}^n$ with $Tx = Ty$, so that $Tx - Ty = 0$. Then

$$\begin{aligned} 0 &= \langle 0, 0 \rangle \\ &= \langle Tx - Ty, Tx - Ty \rangle \\ &= \langle x - y, x - y \rangle && \text{(Since } T \text{ is inner product preserving).} \end{aligned}$$

This result implies that $x = y$, and thus we have that T is injective, as desired.

Lastly, since T is an injective linear operator, it is invertible. Hence $T^{-1} \in \mathcal{L}(\mathbb{R}^n)$ exists and, since T is norm preserving and inner product preserving, for every $x, y \in \mathbb{R}^n$ we have

$$\|T^{-1}x\| = \|T(T^{-1}x)\| = \|x\|,$$

and

$$\langle T^{-1}x, T^{-1}y \rangle = \langle T(T^{-1}x), T(T^{-1}y) \rangle = \langle x, y \rangle.$$

Therefore T^{-1} is also norm preserving and inner product preserving, and this concludes our proof. □

Ex 1-10: If $T\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $|T(\mathbf{h})| \leq M|h|$ for $\mathbf{h} \in \mathbb{R}^m$.
(*Hint:* Estimate $|T(\mathbf{h})|$ in terms of $|\mathbf{h}|$ and the entries in the matrix of T.)

Proof. Let A be the matrix associated with the linear map T. That is,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} := \begin{pmatrix} \mathbf{v}^1 \\ \vdots \\ \mathbf{v}^n \end{pmatrix},$$

so that

$$T(\mathbf{h}) = A\mathbf{h} = \begin{pmatrix} \langle \mathbf{v}^1, \mathbf{h} \rangle \\ \vdots \\ \langle \mathbf{v}^n, \mathbf{h} \rangle \end{pmatrix}.$$

Then we have

$$\begin{aligned}|T(\mathbf{h})|^2 &= \sum_{j=1}^{n} \langle \mathbf{v}^j, \mathbf{h}\rangle^2 \\ &\leq \sum_{j=1}^{n} (|\mathbf{v}^j| \cdot |\mathbf{h}|)^2 && \text{(By \textit{Theorem 1-1} part (2))} \\ &= \left(\sum_{j=1}^{n} |\mathbf{v}^j|^2\right) \cdot |\mathbf{h}|^2.\end{aligned}$$

Thus, letting $M = \sqrt{\sum_{j=1}^{n} |\mathbf{v}^j|^2}$, we have that $|T(\mathbf{h})| \leq M|h|$, as desired. □

Ex 1-18: If $A \subset [0,1]$ is the union of open intervals (a_i, b_i) such that each rational number in $(0,1)$ is contained in some (a_i, b_i), show that $\partial A = [0,1] \smallsetminus A$.

Proof. Let $K = [0,1]$. It is clear that A is open since by hypothesis we have $A = \cup_i (a_i, b_i)$, which is a union of open sets (which is open). Hence we must have that $A^c = K \smallsetminus A$ is closed in K, which implies that $\overline{K \smallsetminus A} = K \smallsetminus A$.

Now since

$$\partial A = \bar{A} \cap \overline{K \smallsetminus A} = \bar{A} \cap (K \smallsetminus A),$$

it suffices to show that $K \smallsetminus A \subseteq \bar{A}$, which holds if and only if $\bar{A} = K$.

Now take any $x \in K$ and any open neighborhood U of x in K. Since $\mathbb{Q}$ is dense, there exists a rational $r \in U$. Since there is some i such that $r \in (a_i, b_i)$, we know that $U \cap (a_i, b_i) \neq \emptyset$, which means that $x \in \bar{A}$. Hence A is dense in K (i.e. $K = \bar{A}$), which implies that $K \smallsetminus A \subseteq \bar{A}$, and thus $\partial A = [0,1] \smallsetminus A$, as desired. □

Ex 1-21: a) If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.
b) If A is closed, B is compact, and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \geq d$ for all $y \in A$ and $x \in B$. (*Hint:* For each $b \in B$, find an open set U containing b such that this relation holds for $x \in U \cap B$.)
c) Give a counterexample in $\mathbb{R}^2$ if A and B are closed but neither is compact.

Proof. a) By hypothesis, A^c is open, since A is closed. Since $x \in A^c$, there exists an open ball $\mathcal{B}_d(x)$ with radius $d > 0$ such that $x \in \mathcal{B}_d(x) \subset A^c$. Then we have that $|y - x| \geq d$ for all $y \in A$, as desired.

b) What we need to show is that the distance from A to B, which we denote by $d(A,B)$, is greater than 0. Since A is closed and $A \cap B = \emptyset$, for each point $x \in B$, there exists $\delta_x > 0$ so that $d(x,A) > 3\delta_x$. Since the union of open balls $\bigcup_{x\in B} \mathcal{B}_{2\delta_x}(x)$ covers B, and B is compact, we may find a subcover, which we denote by $\bigcup_{j=1}^{N} \mathcal{B}_{2\delta_j}(x_j)$. If we let $\delta = \min\{\delta_1, \ldots, \delta_N\}$, then we must have $d(A,B) \geq \delta > 0$. Indeed, if $x \in B$ and $y \in A$, then for some j we have $|x_j - x| \leq 2\delta_j$, and by construction $|y - x_j| \geq 3\delta_j$. Therefore

$$|y - x| \geq |y - x_j| - |x_j - x| \geq 3\delta_j - 2\delta_j \geq \delta,$$

as desired.

c) Let A be the x-axis and B be the graph of the exponential function. *Figure 1* below shows this counterexample:

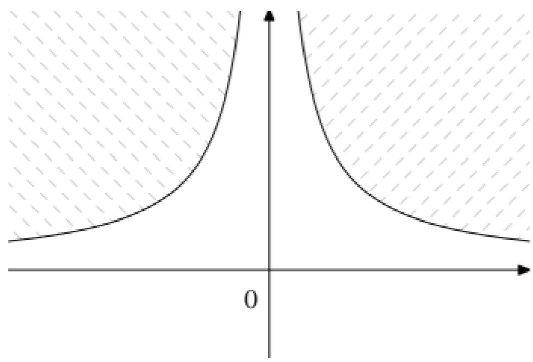

FIGURE 1. Counterexample in $\mathbb{R}^2$ if A and B are closed but neither is compact. ☐

Ex 1-22: If U is open and $C \subset U$ is compact, show that there is a compact set D such that $C \subset \text{int}(D)$ and $D \subset U$.

Proof. Because $C \subset U$, and U is open, then for each $x \in C$ there is an open rectangle U_x with $x \in U_x \subset U$. Now consider the collection of all of these rectangles, one for each point in C. For each x, we will minimize U_x to get a smaller open rectangle V_x in the following way:

The i^{th} side of U_x should be an interval $(x_i - \delta_i, x_i + \varepsilon_i)$ containing x_i. Now we trim off half the distance from x_i to the boundary on each side, so that the i^{th} side of V_x is $(x_i - \delta_i/2, x_i + \varepsilon_i/2)$. Then the closed rectangle $\overline{V_x}$ is a proper subset of $U_x \subset U$.

Now let $\mathcal{O}$ the open cover for C consisting of all the V_x. Since there is one V_x for each $x \in C$, this certainly covers C. But C is compact, so we only need finitely many of these rectangles, say $V_{x_1}, \ldots, V_{x_k}$.

Now let

$$D = \overline{V_{x_1}} \bigcup \cdots \bigcup \overline{V_{x_1}}.$$

Since each $\overline{V_{x_j}}$ is compact, and there are finitely many of them in the union forming D, we have that D must be compact. Moreover, $C \subset (V_{x_1} \bigcup \cdots \bigcup V_{x_k}) = \text{int}(D)$ and $D \subset U$, as we set out to prove. $\square$

Ex 1-28: If $A \subset \mathbb{R}^n$ is not closed, show that there is a continuous function $f\colon A \longrightarrow \mathbb{R}$ which is not bounded. (*Hint:* If $x \in \mathbb{R}^n \smallsetminus A$ but $x \notin \text{int}(\mathbb{R}^n \smallsetminus A)$, let $f(y) = 1/|y-x|$.)

Proof. Since A is not closed, we have $A \cap \partial A = \emptyset$. Now we choose $x \in \partial A$, so that $x \notin A$, and let $f(y) = 1/|y-x|$ for all $y \in A$. Clearly f is not bounded, since the closer we pick $y \in A$ to a point $x \in \partial A$, the more this function will blow up to infinity. Thus what's left for us to show is that f is indeed a continuous function:

Pick an arbitrary $p \in A$, and then for any $\varepsilon > 0$ we choose

$$\delta = \min\left\{\frac{|p-x|}{2}, \frac{\varepsilon|p-x|^2}{2}\right\},$$

so that for any y with $0 < |y-p| < \delta$, we have $|y-x| \geq |p-x|/2$.

Then,

$$\begin{aligned} |f(y) - f(p)| = \left|\frac{1}{|y-x|} - \frac{1}{|p-x|}\right| &= \frac{||p-x| - |y-x||}{|y-x||p-x|} \\ &\leq \frac{|p-y|}{|y-x||p-x|} \qquad (\clubsuit) \\ &< \frac{2\delta}{|p-x|^2} \leq \varepsilon. \end{aligned}$$

This result proves the continuity of f that we desire, but before concluding our proof we need to show why it is true that the inequality ($\clubsuit$) holds. That is, we need to show that for any x, y it is always the case that $|\|x\| - \|y\|| \leq \|x - y\|$:

$$\begin{aligned}\|x-y\|^2 &= \sum_{j=1}^{n}(x_j - y_j)^2 \\ &= \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^{n} x_j y_j \\ &\geq \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \\ &= (\|x\| - \|y\|)^2.\end{aligned}$$

Taking square roots yields the desired result, and our proof is done. □

Ex 1-29: If A is compact, prove that every continuous function $f\colon A \to \mathbb{R}$ takes on a maximum and a minimum value.

Proof. Since A is compact and f is continuous, *Theorem 1-9* guarantees that $f(A) \subset \mathbb{R}$ is compact, and hence it is closed and bounded. Let m and M be the greatest lower bound and least upper bound, respectively, of $f(A)$. Then m and M are boundary points of $f(A)$, both of which are in $f(A)$ since this is a closed set. Hence m and M are the minimum and maximum values, respectively, attained by any continuous function f with compact domain A. □