Math 260 HW # 8

Mario L. Gutierrez Abed

(Problem 1) Suppose $T \in \mathcal{L}(V)$ is invertible and λ is a nonzero scalar. Prove that λ is an eigenvalue of T iff $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

Proof:

($\Rightarrow$)

Suppose that λ is an eigenvalue of T. Then $\exists\, x \in V$ such that $T\,x = \lambda\,x,\ \ x \neq 0$. But then since T is invertible we have that $T^{-1}\,T\,x = T^{-1}\,\lambda\,x \Longrightarrow x = T^{-1}\,\lambda\,x$. Dividing this last equation by λ we get $\frac{1}{\lambda}\,x = T^{-1}\,x$. This in turn implies that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

($\Leftarrow$)

To prove in the other direction we simply go backwards. That is, suppose that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}, then it is true that $\frac{1}{\lambda}\,x = T^{-1}\,x$. Multiplying this last equation by λ we get $x = T^{-1}\,\lambda\,x$. Then since T is invertible we have that $T\,x = T\,T^{-1}\,\lambda\,x \Longrightarrow T\,x = \lambda\,x$. This in turn implies that λ is an eigenvalue of T. ■

Section 5.1

(4) For the following linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

e) $V = P_2(\mathbb{R})$ and $T(f(x)) = x\,f'(x) + f(2)\,x + f(3)$

Solution:

Let γ be the standard basis for $P_2(\mathbb{R})$, i.e. $\gamma = \{1,\ x,\ x^2\}$.

Now we want to compute $[T]_\gamma$:

$\multimap\ T(1) = x(0) + 1\,x + 1 = x + 1$

$\multimap\ T(x) = x(1) + 2\,x + 3 = 3\,x + 3$

$\multimap\ T(x^2) = x(2\,x) + 4\,x + 9 = 2\,x^2 + 4\,x + 9$

Hence $[T]_\gamma = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$.

Now we want to find the eigenvalues of T. We let $[T]_\gamma = A$, then we have

$$\text{char}(A) = \det(A - t\,I) = \det\left(\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}\right) = 0$$

$$= \det\begin{pmatrix} 1-t & 3 & 9 \\ 1 & 3-t & 4 \\ 0 & 0 & 2-t \end{pmatrix} = (2-t)[(1-t)\,(3-t)-3] = (2-t)\,(3-t-3\,t+t^2-3) = 0$$

$$= (2-t)\,(t^2-4\,t) = t(2-t)\,(t-4) = 0$$

Hence $\lambda = 0,\ 2,\ 4$ are the eigenvalues of T. ✓

Now we want to find the eigenbasis β that makes $[T]_\beta$ a diagonal matrix with the three computed eigenvalues as the diagonal entries. In order to find this basis we must find the eigenvectors associated with these eigenvalues :

$\bullet\!\!\rightarrow$ For $\lambda = 0$:

$$(A-(0)\,I)\,x = 0 \implies \left(\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From here we have $x_3 = 0$, $x_1 = -3\,x_2$. Letting $x_2 = \alpha \in \mathbb{R}$, we our solution set is $\begin{pmatrix} -3\,\alpha \\ \alpha \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$. Hence $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ $(\{-3+x\})$ is an eigenvector corresponding to $\lambda = 0$. ✓

$\bullet\!\!\rightarrow$ For $\lambda = 2$:

$$(A-(2)\,I)\,x = 0 \implies \left(\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} -1 & 3 & 9 \\ 0 & 4 & 13 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From here we have $4\,x_2 = -13\,x_3 \implies x_2 = -\frac{13}{4}\,x_3$ and

$x_1 - 3\left(-\frac{13}{4}\,x_3\right) - 9\,x_3 = 0 \implies x_1 = -\frac{3}{4}\,x_3$

Letting $x_3 = \zeta \in \mathbb{R}$, we our solution set is

$\begin{pmatrix} -\frac{3}{4}\zeta \\ -\frac{13}{4}\zeta \\ \zeta \end{pmatrix} = \zeta \begin{pmatrix} -\frac{3}{4} \\ -\frac{13}{4} \\ 1 \end{pmatrix} = \frac{1}{4}\zeta \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$. Hence $\begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$ $(\{-3 - 13x + 4x^2\})$ is an eigenvector corresponding to $\lambda = 2$. ✓

⟼ For $\lambda = 4$:

$(A - (4) I) x = 0 \implies \left(\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\implies \begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

From here we have $x_3 = 0$, $x_1 = x_2$. Letting $x_1, x_2 = \xi \in \mathbb{R}$, we our solution set is

$\begin{pmatrix} \xi \\ \xi \\ 0 \end{pmatrix} = \xi \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $(\{1 + x\})$ is an eigenvector corresponding to $\lambda = 4$. ✓

The eigenvectors we just computed are linearly independent by a previous lemma. Hence we have the eigenbasis

$$\beta = \{-3 + x,\ -3 - 13x + 4x^2,\ 1 + x\}.$$

Now we want to compute $[T]_\beta$:

** Recall from above that $T(f(x)) = x f'(x) + f(2)x + f(3)$ **

⟼ $T(-3 + x) = x(1) + (-1)x + 0 = 0$
$= 0(-3 + x) + 0(-3 - 13x + 4x^2) + 0(1 + x)$

⟼ $T(-3 - 13x + 4x^2) = x(8x - 13) + (16 - 26 - 3)x + 36 - 39 - 3 = 8x^2 - 26x - 6$
$= 0(-3 + x) + 2(-3 - 13x + 4x^2) + 0(1 + x)$

⟼ $T(1 + x) = x(1) + (3)x + 4 = 4x + 4$
$= 0(-3 + x) + 0(-3 - 13x + 4x^2) + 4(1 + x)$

Hence, FINALLY we have $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ ✡

Section 5.2

(8) Suppose that $A \in M_{n\times n}(\mathbb{F})$ has two distinct eigenvalues $\lambda_1$and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

Proof:
Suppose that $A \in M_{n\times n}(\mathbb{F})$, then by a previous theorem we know that A has at most n eigenvalues. We are given that A actually has two distinct eigenvalues λ_1 and λ_2. We are also given that $\dim(E_{\lambda_1}) = n - 1$ and this in turn implies that $\dim\left(E_{\lambda_2}\right) = 1$. We know that $\dim(E_{\lambda_1})$ is also the multiplicity of λ_1, i.e. $\text{mult}(\lambda_1) = \dim(E_{\lambda_1}) = n - 1$. This means that λ_1 appears $n - 1$ times on the diagonal and this in turn implies that λ_2 appears only once, therefore it must have multiplicity 1. Thus we have that multiplicity $(\lambda_i) = \dim(E_{\lambda_i})$ for $i = 1, 2$. We also know that $\text{char}(A)$ splits over $\mathbb{F}$, since both eigenvalues are unique and that allows us to write $\text{char}(A)$ as a multiplication of linear terms. Hence we have proven that A is diagonalizable. $\blacksquare$