# MATH 725 HW#5

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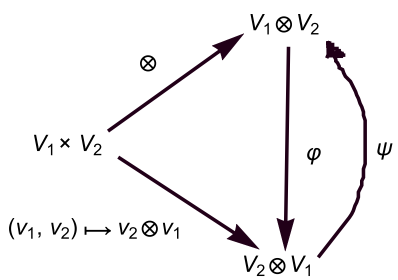
**Exercise** (Exercise 1)**.** *Show that* $V_1 \otimes V_2 \cong V_2 \otimes V_1$*.*

*Proof.* We want to create a linear map $V_1 \otimes V_2 \to V_2 \otimes V_1$ sending $v_1 \otimes v_2$ to $v_2 \otimes v_1$. To do this, we back up and start off with a map out of $V_1 \times V_2$ to the desired target space $V_2 \otimes V_1$. Let us define this map $V_1 \times V_2 \to V_2 \otimes V_1$ by $(v_1, v_2) \mapsto v_2 \otimes v_1$. This is a bilinear map since $v_2 \otimes v_1$ is bilinear in $v_2$ and $v_1$. Therefore, by the universal mapping property of the tensor product, there exists a unique linear map $\varphi\colon V_1 \otimes V_2 \to V_2 \otimes V_1$ such that $\varphi(v_1 \otimes v_2) = v_2 \otimes v_1$ on elementary tensors, that is, the diagram on the figure below commutes:




Following the above argument with the roles of $V_1$ and $V_2$ interchanged, there is a unique linear map $\psi\colon V_2 \otimes V_1 \to V_1 \otimes V_2$ where $\psi(v_2 \otimes v_1) = v_1 \otimes v_2$ on elementary tensors. We will now show that $\varphi$ and $\psi$ are inverses of each other.

To show that $\varphi(\psi(x)) = x$ for all $x \in V_2 \otimes V_1$, it suffices to check this when $x$ is an elementary tensor, since both sides are linear in $x$ and $V_2 \otimes V_1$ is spanned by its elementary tensors:

$$\varphi(\psi(v_2 \otimes v_1)) = \varphi(v_1 \otimes v_2) = v_2 \otimes v_1.$$

Therefore $\varphi(\psi(x)) = x$ for all $x \in V_2 \otimes V_1$. The proof that $\psi(\varphi(y)) = y$ for all $y \in V_1 \otimes V_2$ is similar. Thus we have shown that $\varphi$ and $\psi$ are inverses of each other, which in turn indicates that $V_1 \otimes V_2 \cong V_2 \otimes V_1$, as desired. □

**Exercise** (Exercise 2)**.** *Let* $S_1$ *and* $S_2$ *be subspaces of* $V_1$*. Then prove that*

$$(S_1 \otimes V_2) \cap (S_2 \otimes V_2) = (S_1 \cap S_2) \otimes V_2. \tag{†}$$

*Proof.* Let $\mathcal{B} = \{v_i\}$ be a basis of $V_2$ (we know of the existence of such basis by a previous theorem we proved in class). Let $z \in (S_1 \otimes V_2) \cap (S_2 \otimes V_2)$ and then write

$$z = \sum_{i=1}^{m} x_i \otimes v_{j_i} = \sum_{i=1}^{m} y_i \otimes v_{k_i},$$

where $x_i \in S_1$ and $y_i \in S_2$. After reindexing, we have

$$z = \sum_{i=1}^{r} x_i \otimes v_{j_i} = \sum_{i=1}^{r} y_i \otimes v_{j_i}$$

so that

$$\sum_{i=1}^{r} (x_i - y_i) \otimes v_{j_i} = 0.$$

Since the $\{v_{j_i}\}$ are linearly independent, $x_i = y_i$ for each $i$ and therefore $z \in (S_1 \cap S_2) \otimes V_2$. The other inclusion is obvious, since if $x_i \in S_1 \cap S_2$, then $x_i \in S_1$ and $x_i \in S_2$, so that (†) immediately follows. $\square$