Math 260 Extra Credit

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Let V be a finite dimensional VS over a field $\mathbb{F}$. Let $\beta = \{\beta_1, \ldots, \beta_n\}$ be an ordered basis for V. Then, $\dim(V) = n$. Consider the singleton subsets of $\beta : \{\beta_i\}$, for $i = 1, \ldots, n$. Let $W_i = \operatorname{span}(\{\beta_i\})$. Then, each of the W_i's are 1-dimensional subspaces of V.

Complete the following :

(1) Show that $\sum_{i=1}^{n} W_i = V$.

Solution:
We have that $W_i = \operatorname{span}\{\beta_i\}$ for $i = 1, .., n$. We also know that span $\{\beta_i\}$ is the set of all linear combination representations of the singleton subsets $\{\beta_i\}$, for $i = 1, \ldots, n$.
We have n singleton subsets that span n one-dimensional subspaces W_i. Hence by adding all the linear combination representations of the n singletons $\{\beta_i\}$ we get a linear combination representation of the spanning set $\{\beta_1, \ldots, \beta_n\}$, which is a basis for V. Thus we can see that $\sum_{i=1}^{n} W_i = V$. ✡

(2) Prove that $W_i \cap \left(\sum_{j \neq i} W_j\right) = \{0\}$.

Proof:
We are given that each of the W_i's are one-dimensional subspaces of V. We also know that the sum of subspaces of a VS is also a subspace in that ambient VS, hence $\sum_{j \neq i} W_j$ is also a subspace of V. By an earlier theorem we proved in class, we know that the intersection of these subspaces is also a subspace of V. Hence the zero vector must lie in every W_i and W_j $(j \neq i)$ and in the intersection of these subspaces as well. Now suppose $\exists\ v \in W_i \cap \left(\sum_{j \neq i} W_j\right)$ such that $v \neq 0$. Then, to satisfy one of the properties of VS's there must be an additive inverse $-v$ such that $v + (-v) = 0$, where $v \in W_i$ and $-v \in \sum_{j \neq i} W_j$. But then by the unique representation of the zero vector as the sum of a vector in W_i and a vector in $\sum_{j \neq i} W_j$, we must have that $v = 0$. $(\Rightarrow\Leftarrow)$
Hence we have proven that $W_i \cap \left(\sum_{j \neq i} W_j\right) = \{0\}$. ■

(3) Use the above results to show that $V = W_1 \oplus ... \oplus W_n$. This is known as a direct sum decomposition of V.

Solution:

We have proven that $\sum_{i=1}^{n} W_i = V$ and also that $W_i \cap \left(\sum_{j \neq i} W_j\right) = \{0\}$. Since each W_i is a one-dimensional subspace and so is the summation of all W_j with $j \neq i$, by Proposition 1.9 (Axler's), we are guaranteed that $V = W_1 \oplus ... \oplus W_n$.