Math 351 Assignment 4

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(1) Let G be an open subset of $(M,\ d)$ and F be a closed subset of $(M,\ d)$. Prove or disprove: $G \backslash F$ is an open subset of $(M,\ d)$.

Proof:
It is painfully obvious that this is a true statement, since by definition $G \backslash F = \{x \in G : x \notin F\}$ and we have that for any $\varepsilon > 0$, $B_\varepsilon(x) \subset G \ \ \forall\, x \in G$. Thus $B_\varepsilon(x) \subset G \backslash F \ \ \forall\, x \in G \backslash F$, which implies that $G \backslash F$ is an open subset of $(M,\ d)$. ■

(2) Let $G = \{(x,\ y) \in \mathbb{R}^2 : x \neq y\}$. Prove that G is an open subset of $\mathbb{R}^2$.

Proof:
Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x,\ y) = x - y$. This function is a polynomial in two variables (and even a linear map) and is therefore continuous. Notice that the set G can be expressed as the intersection

$$\{(x,\ y) \in \mathbb{R}^2 : f(x,\ y) < 0\} \bigcup \{(x,\ y) \in \mathbb{R}^2 : f(x,\ y) > 0\} = f^{-1}(-\infty,\ 0) \bigcup f^{-1}(0,\ \infty).$$

This is a union of open sets which must be open. ■

(3) Suppose that E is a nowhere dense subset of $(M,\ d)$. Prove or disprove: E^c is everywhere dense in M.

Proof:
Suppose that E is a nowhere dense set in M, i.e. $(\overline{E})^o = \emptyset$. Observe that
$E^c = M \backslash E \Longrightarrow \overline{E^c} = \overline{M \backslash E} = \overline{M} = M$. Hence, E^c is everywhere dense in M. ■

(4) Let $\diamond$ be a set constructed out of the interval $[0,\ 1]$ as follows:

• Step 1: Partition the interval into 5 parts of equal length and remove every other (open) part. Thus, you obtain the set $I_1 = \left[0,\ \frac{1}{5}\right] \cup \left[\frac{2}{5},\ \frac{3}{5}\right] \cup \left[\frac{4}{5},\ 1\right]$.

• Step 2: Partition each of the interval segments of I_1 further into 5 parts of equal length and remove every other (open) part. Thus, you obtain the set

$I_2 = \left[0, \frac{1}{25}\right] \cup \left[\frac{2}{25}, \frac{3}{25}\right] \cup \left[\frac{4}{25}, \frac{1}{5}\right] \cup \left[\frac{2}{5}, \frac{11}{25}\right] \cup \left[\frac{12}{25}, \frac{13}{25}\right] \cup \left[\frac{14}{25}, \frac{3}{5}\right] \cup$
$\bigcup \left[\frac{4}{5}, \frac{21}{25}\right] \cup \left[\frac{22}{25}, \frac{23}{25}\right] \cup \left[\frac{24}{25}, 1\right]$

• Step n: Partition each of the interval segments of I_{n-1} into 5 parts of equal length and remove every other (open) part to obtain I_n.

Set $\diamond = \bigcap_{n=1}^{\infty} I_n$.

a) Is $\diamond$ closed as a subset of $\mathbb{R}$?

Solution:
It is a closed subset of $\mathbb{R}$ because $\diamond$ is an infinite intersection of closed sets, which is always closed. ✓

b) Is $\diamond$ countable or uncountable?

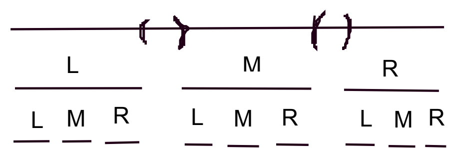
Solution:
It follows from the nested interval theorem that $\diamond$ is nonempty. What's more, $\diamond$ is uncountable, since we can construct a map $f : \diamond \longrightarrow [0, 1]$ that is onto, as I show on part g). ✓

c) Is $\diamond$ dense, nowhere dense, or neither?

Solution:
It is nowhere dense, since it contains no intervals. As I show on part e) it has "length" zero. ✓

d) Is $\diamond$ perfect?

Solution:
It is perfect, since it is closed (as explained on part a)) and all of its points are limit points. To see why the latter is true, look at the following figure



We can see that as we get to the "dust" of this construction, any point in $\diamond$ is a string of choices, *LRMLRMRLLRLMMLLR*…, which describes a unique "path" from the top level all the way down to $\diamond$. Said another way, each such "path" determines a unique sequence of nested subintervals, one from each level, whose intersection is a single point of $\diamond$. Conversely, each point $x \in \diamond$ is a limit point that lies at the end of exactly one such path. ✓

e) What is the "length" (i.e measure) of $\diamond$?

Solution:
In order to determine the length of $\diamond$, we will compute the length of its complement $[0,\ 1]\backslash\diamond$.
We can see that at the n^{th} stage we construct I_n from I_{n-1} by removing $2\cdot 3^{n-1}$ disjoint, open intervals from I_{n-1}, each of length 5^{-n}. Hence by induction, the total length of the $2\cdot 3^{n-1}$ disjoint intervals is $\frac{2\cdot 3^{n-1}}{5^n}$.
Thus, the total length of $[0,\ 1]\backslash\diamond$ must be

$$\sum_{n=1}^{\infty} \frac{2\cdot 3^{n-1}}{5^n} = \frac{2}{5} \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^{n-1} = \frac{2}{5} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^{n} = \frac{2}{5}\ \frac{1}{1-\frac{3}{5}} = 1$$

Thus, since length$([0,\ 1]\backslash\diamond) = 1$, $\diamond$ has zero measure. ✓

f) The elements of $\diamond$ can be most easily described in terms of some base-p decimal expansion. What p should we choose? In terms of the decimal expansion base-p, how would you decide whether x is an element of $\diamond$?

Solution:
As I showed while computing the length of $\diamond$ in part e), the base of choice for $\diamond$ is 5, since we are splitting the intervals in fifths.

To decide whether an element x is in ◇, we must check that it satisfies $x = \sum_{n=1}^{\infty} \frac{a_n}{5^n}$, where each a_n is $0, 2,$ or 4 (this way we can detect which elements are in ◇ while avoiding any ambiguity at the endpoints). ✓

g) Construct a Cantor-like function for ◇.

<u>Solution:</u>
As we saw on part f), we can express any $x \in$ ◇ as $x = \sum_{n=1}^{\infty} \frac{a_n}{5^n}$, where each a_n is $0, 2,$ or 4. We are going to construct a function that takes these elements that live in ◇ and maps them to the interval $[0, 1]$, each of them represented in base 3.
Observe that

$$f\left(\sum_{n=1}^{\infty} \frac{2\, b_n}{5^n}\right) = \sum_{n=1}^{\infty} \frac{b_n}{3^n} \qquad (\text{where } b_n = 0,\ 1,\ 2)$$

is a surjective map (even though it's not injective). This indicates that ◇ is uncountable, as we stated on part b). ✓