MATH 725 HW#3

MARIO L. GUTIERREZ ABED
PROF. J. LOUSTAU

Exercise (Exercise 1)**.** *Show that the derivative operator D is not bounded on $C^1[0,1]$.*

Proof. For a linear transformation $T\colon V \to W$ (where V and W are Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively) to be ***bounded***, there must exist an $M \geq 0$ such that

$$\|Tu\|_W \leq M\|u\|_V \quad \text{for each } u \in V.$$

If T is a bounded linear operator, then its norm $\|T\|$ is the smallest M that satisfies the above inequality. That is,

$$\|T\| = \sup_{u \neq 0} \frac{\|Tu\|_W}{\|u\|_V}.$$

Now let $V = W = L^2[0,1]$ and consider the derivative operator D with domain $C^1[0,1] \subset L^2[0,1]$ defined by

$$Du = u' \quad \forall\, u \in C^1[0,1].$$

Now consider the sequence of functions defined by $\varphi_n(x) = x^n$ for $n \in \mathbb{N}$, and notice that φ_n is continuously differentiable for each $n = 1, 2, \dots$, i.e. $\varphi_n \in C^1[0,1]$ $\forall n \in \mathbb{N}$. Taking L^2 norms (which functions in $C^1[0,1]$ inherit from $L^2[0,1]$) and squaring them, we have

$$\|\varphi_n\|_{L^2}^2 = \int_0^1 |x^n|^2\, dx = \int_0^1 x^{2n}\, dx = \frac{1}{2n+1},$$

and

$$\|D(\varphi_n)\|_{L^2}^2 = \int_0^1 |D(x^n)|^2\, dx = \int_0^1 n^2 x^{2n-2}\, dx = \frac{n^2}{2n-1}.$$

Now notice that

$$\frac{\|D(\varphi_n)\|_{L^2}^2}{\|\varphi_n\|_{L^2}^2} = n\sqrt{\frac{2n+1}{2n-1}} \to \infty \quad \text{as } n \to \infty.$$

This result shows that D is an unbounded linear operator. □

Exercise (Exercise 2)**.** *Prove that the sequence $f_n(x) = x^n$ does not converge uniformly on $[0,1]$.*

Proof. Notice that, for $0 \leq x < 1$, our sequence $f_n(x) = x^n$ converges to 0 for all $n > N$ for some large N, while $f_n(x) = 1$ for $x = 1$. In other words f_n converges pointwise to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Notice that this convergence is indeed pointwise because given some $\varepsilon > 0$, and given $x \in [0,1]$, we have to find an $N = N(x,\varepsilon)$ (that is, an N that depends on both ε and x), that satisfies

$$(\dagger) \qquad |f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \geq N.$$

We now want to show that this convergence is not uniform, which is the same as saying that there is no $N = N(\varepsilon)$ (that is, an N that depends only on ε) for which (†) holds for all $x \in [0,1]$. Assume, to the contrary, that there is such an N. Then we may pick $\varepsilon = 1/4$ and notice that for all $x \in [0,1)$, (†) implies

$$x^n < \frac{1}{4} \qquad \forall\, n \geq N$$

But then taking the limit as $x \to 1^-$, we would have that $1^n = 1 < 1/4$, which is not possible. □

Exercise (Exercise 3)**.** *Let V^{**} denote the* ***double algebraic dual space*** *which consists of all linear functionals $v^{**}\colon V^* \to \mathbb{F}$. In other words, an element $v^{**} \in V^{**}$ is a linear map that assigns a scalar to each linear functional on V. Now let $v \in V$ and consider the map $v^{**}\colon V^* \to \mathbb{F}$ defined by*

$$v^{**}(f) = f(v),$$

which sends the linear functional f to the scalar $f(v)$. This map v^{**} is called the ***evaluation at*** *v. Then*

- *a) Show that v^{**} is indeed an element of V^{**}.*
- *b) Show that the map $v \mapsto v^{**}$ is linear and injective.*

Proof of a). Let $f, g \in V^*$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned} v^{**}(\alpha f + \beta g) &= (\alpha f + \beta g)(v) \\ &= \alpha f(v) + \beta g(v) && \text{(By linearity of } f \text{ and } g\text{)} \\ &= \alpha v^{**}(f) + \beta v^{**}(g). \end{aligned}$$

Thus we have proven linearity of v^{**}, and we have that $v^{**} \in V^{**}$. □

Proof of b). Let $u, v \in V$ and $\alpha, \beta \in \mathbb{F}$, and define the map $\varphi\colon V \to V^{**}$ by $\varphi(v) = v^{**}$ (this is the ***canonical map*** from V to V^{**}).

Notice that, for all $f \in V^*$, we have

$$\begin{aligned} (\alpha u + \beta v)^{**}(f) &= f(\alpha u + \beta v) \\ &= \alpha f(u) + \beta f(v) && \text{(By linearity of } f\text{)} \\ &= \alpha u^{**}(f) + \beta v^{**}(f) \\ &= (\alpha u^{**} + \beta v^{**})(f) && \text{(By the linearity of } v^{**} \text{ showed in part a))} \end{aligned}$$

Hence, the map φ is indeed linear.

Lastly, to show injectivity notice that

$$\begin{aligned} \varphi(v) = 0 &\Longrightarrow v^{**} = 0 \\ &\Longrightarrow v^{**}(f) = 0 \qquad \text{for all } f \in V^* \\ &\Longrightarrow f(v) = 0 \qquad \text{for all } f \in V^* \\ &\Longrightarrow v = 0 \qquad (\clubsuit) \\ &\Longrightarrow \ker(\varphi) = \{0\} \\ &\Longrightarrow \varphi \text{ is injective.} \end{aligned}$$

Notice that ($\clubsuit$) holds by a previous theorem that says that a vector $v \in V$ is zero if and only if $f(v) = 0$ for all $f \in V^*$. It turns out that in the finite-dimensional case, since $\dim(V^{**}) = \dim(V^*) = \dim(V)$, it follows that φ is also surjective, hence an isomorphism. $\square$