MATH 710 HW # 3

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Problem 1 (Problem 9-1)**.** *Suppose M is a smooth manifold, $X \in \mathfrak{X}(M)$, and γ is a maximal integral curve of X.*

a) *We say γ is* ***periodic*** *if there is a number $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Show that exactly one of the following holds:*
   - *γ is constant.*
   - *γ is injective.*
   - *γ is periodic and nonconstant.*

b) *Show that if γ is periodic and nonconstant, then there exists a unique positive number T (called the* ***period of*** *γ) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.*

c) *Show that the image of γ is an immersed submanifold of M; diffeomorphic to $\mathbb{R}$, $\mathbb{S}^1$, or $\mathbb{R}^0$.*

Proof of a). If γ is constant, then it obviously cannot be nonconstant periodic nor can it be injective. Thus let us we assume that γ is nonconstant. If γ is not injective then $\gamma(t_0) = \gamma(t_1)$ for some $t_0 \neq t_1$ (say, WLOG, $t_0 < t_1$), and γ is defined on at least $(t_0 - \varepsilon, t_1 + \varepsilon)$ for some $\varepsilon > 0$. Let $T = t_1 - t_0$. Then γ and $t \mapsto \gamma(t_1 + t)$ are both integral curves of X starting at $\gamma(t_0)$, and thus γ must be defined on at least $(t_0 - \varepsilon, t_1 + T + \varepsilon)$. By induction, $t_0 + kT$ is in the domain of γ for all $k \in \mathbb{Z}$, so γ is defined on $\mathbb{R}$ and has period T. □

Proof of b). Let $\mathcal{A}$ be the set of all $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$, which is nonempty since γ is assumed to be periodic. If we can show that $\mathcal{A}$ is closed, then $T_0 = \inf \mathcal{A} \in \mathcal{A}$ is the period of γ. But if $T \notin \mathcal{A}$, then $\gamma(t+T) \neq \gamma(t)$ for some $t \in \mathbb{R}$, so

$$\gamma^{-1}(M \setminus \{\gamma(t)\}) - t' = \{t - t' : \gamma(t) \neq \gamma(t')\}$$

is a neighborhood of T contained in $\mathbb{R} \setminus \mathcal{A}$. □

Proof of c). If γ is constant, it is obvious that the image of γ is diffeomorphic to $\mathbb{R}^0$. Otherwise, Proposition 9.21[1] shows that γ is a smooth immersion. If γ is injective, then Proposition 5.18[2] shows

[1] Here's Proposition 9.21, for reference:

Proposition. *Let V be a smooth vector field on a smooth manifold M, and let $\theta\colon \mathfrak{D} \to M$ be the flow generated by V. If $p \in M$ is a singular point of V, then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$. If p is a regular point, then $\theta^{(p)}\colon \mathfrak{D}^{(p)} \to M$ is a smooth immersion.*

[2] Here's Proposition 5.18, for reference

Proposition (**Images of Immersions as Submanifolds**)**.** *Suppose M is a smooth manifold (with or without boundary), N is a smooth manifold, and $F\colon N \to M$ is an injective smooth immersion. Let $S = F(N)$. Then S has a unique topology and smooth structure such that it is a smooth submanifold of M and such that $F\colon N \to S$ is a diffeomorphism onto its image.*

that the image of γ is diffeomorphic to $\mathbb{R}$, and if γ is periodic then it descends to a smooth injective immersion from $\mathbb{S}^1$ to M (using a suitable smooth covering of $\mathbb{S}^1$ by $\mathbb{R}$), which is a diffeomorphism onto its image. $\square$

Problem 2 (Problem 9-3)**.** *Compute the flow of each of the following vector fields on* $\mathbb{R}^2$*:*

a) $V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

b) $W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$.

c) $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

d) $Y = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

Solution. Let $\gamma(t) = (x(t), y(t))$ a curve. We have the differential equations

$$\begin{array}{r|l} x'(t) = y(t) & y'(t) = 1 \\ x'(t) = x(t) & y'(t) = 2y(t) \\ x'(t) = x(t) & y'(t) = -y(t) \\ x'(t) = y(t) & y'(t) = x(t) \end{array}$$

Therefore the flows are, respectively,

$$\begin{aligned} \theta_t(x,y) &= \left(x + ty + \frac{1}{2}t^2, y + t\right), \\ \theta_t(x,y) &= (xe^t, ye^{2t}) \\ \theta_t(x,y) &= (xe^t, ye^{-t}) \\ \theta_t(x,y) &= \frac{1}{2}\left(e^t(x+y) + e^{-t}(x-y), e^t(x+y) - e^{-t}(x-y)\right). \end{aligned}$$

$\square$

Problem 3 (Problem 9-4)**.** *For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ by $\theta(t,z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth nonvanishing vector field on $\mathbb{S}^{2n-1}$. [Remark: in the case $n = 2$, the integral curves of X are the curves γ_z of Problem 3-6, so this provides a simpler proof that each γ_z is smooth.]*

Proof. For all $z \in \mathbb{C}^n$ let $\gamma_z(t) = \theta(t,z)$, so that the infinitesimal generator $\theta^{(z)\prime}(0) = \gamma_z'(0) = ie^{it}z|_{t=0} = iz\,\partial/\partial z$, which is clearly smooth and nonvanishing (it would only vanish at $z = z_0 = 0$ but $z_0 \notin \mathbb{S}^{2n-1}$). $\square$

Problem 4 (Problem 9-5)**.** *Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. Show that there exists a smooth map $F\colon M \to M$ that is homotopic to the identity and has no fixed points.*

Proof. Let V be a nowhere vanishing smooth vector field on M. Since M is compact, we know that there is a global flow $\theta\colon \mathbb{R} \times M \to M$ of V (Corollary 9.17). Then by Theorem 9.22,[3] each p has a neighborhood U_p in which V has the coordinate representation $\partial/\partial s^1$. Choosing a sufficiently small neighborhood V_p of p, there is some $T_p > 0$ such that $\theta_t(x) = x + (t, 0, \dots, 0)$ in local coordinates for all $0 \leq t \leq T_p$ and $x \in V_p$. Since $\{V_p : p \in M\}$ is an open cover of M, there is a finite subcover $\{V_{p_1}, \dots, V_{p_n}\}$. Let $T = \min\{T_{p_1}, \dots, T_{p_n}\}$. Then θ_T has no fixed points, and the map $H\colon M \times I \to M$ given by $(x, t) \mapsto \theta(tT, x)$ is a homotopy from the identity to θ_T. $\square$

Problem 5 (Problem 9-6 **(The Escape Lemma)**). *Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma\colon J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, show that for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in any compact subset of M.*

Proof. Let θ be the flow of V. Let $\{b_n\}$ be an increasing sequence contained in $[t_0, b)$ and converging to b. Since $\gamma([t_0, b))$ is contained in a compact set $E \subseteq M$, there is a subsequence $\{\gamma(b_{n_k})\}$ of $\{\gamma(b_n)\}$ that converges to a point $p \in E$. Choose some $\varepsilon > 0$ and a neighborhood U of p such that θ is defined on $(-2\varepsilon, 2\varepsilon) \times U$, and choose an integer m such that $b_m \in (b - \varepsilon, b)$ and $\gamma(b_m) \in U$. Define $\gamma_1\colon [t_0, b + \varepsilon) \to M$ by

$$\gamma_1(t) = \begin{cases} \gamma(t) & \text{if } t_0 \leq t < b, \\ \theta^{(\gamma(b_m))}(t - b_m) & \text{if } b \leq t < b + \varepsilon. \end{cases}$$

For all $t \in (b_m, b)$ we have

$$\theta^{(\gamma(b_m))}(t - b_m) = \theta_{t - b_m}(\gamma(b_m)) = \gamma(t),$$

so γ_1 is smooth because γ and $t \mapsto \theta^{(\gamma(b_m))}(t - b_m)$ agree where they overlap. But γ_1 extends γ at b, which contradicts the maximality of γ. $(\Rightarrow\Leftarrow)$ $\square$

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[3] Here's Theorem 9.22, for reference:

Theorem (**Canonical Form Near a Regular Point**). *Let V be a smooth vector field on a smooth manifold M; and let $p \in M$ be a regular point of V. There exist smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\partial/\partial s^1$. If $S \subseteq M$ is any embedded hypersurface with $p \in S$ and $V_p \notin T_pS$, then the coordinates can also be chosen so that s^1 is a local defining function for S.*