TOPOLOGICAL QUANTUM FIELD THEORIES

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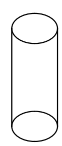
Category Theory

Definition. *A* **category** $\mathcal{C}$ *consists of*

- *a class of* ***objects****, denoted* $\mathrm{Ob}(\mathcal{C})$*.*
- *given two objects* $x, y \in \mathcal{C}$*, a set* $\mathrm{Hom}(x, y)$ *of* ***morphisms****. Generalizing from the categories where* $\mathrm{Hom}(x, y)$ *is a set of functions, we denote* $f \in \mathrm{Hom}(x, y)$ *by* $f\colon x \to y$*. Morphisms satisfy the following properties:*
  - *given morphisms* $f\colon x \to y$ *and* $g\colon y \to z$*, we can compose them and obtain* $g \circ f\colon x \to z$*. When there is no possibility of confusion* $g \circ f$ *is abbreviated* gf*.*

$$\begin{tikzcd} & y \arrow[dl, "g"'] & \\ z & & x \arrow[ul, "f"'] \arrow[ll, "gf"] \end{tikzcd}$$

  - *for any* $x \in \mathcal{C}$*, there is an* ***identity*** *morphism* $1_x\colon x \to x$ *such that, for any* $f\colon x \to y$*, we have* $f1_x = f = 1_y f$*. For example,*



$$\begin{tikzcd} S \arrow[d, "{[0,1]\times S}"] \\ S \end{tikzcd}$$

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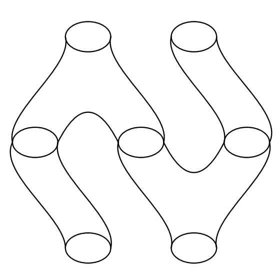
Examples of categories are:

- Set, where objects are sets and morphisms are functions.
- nCob,where objects are $(n-1)$-dimensional oriented compact manifolds, and morphisms are n-dimensional cobordisms.

$$\begin{tikzcd} S \amalg S \arrow[d, "M^* \amalg 1_S"] \\ S \amalg S \amalg S \arrow[d, "1_S \amalg M"] \\ S \amalg S \end{tikzcd}$$



- Vect, where objects are (finite-dimensional, complex) vector spaces, and morphisms are linear operators.
- Hilb, where objects are (finite-dimensional, complex) Hilbert spaces, and morphisms are linear operators.

Quantum mechanics uses Hilb rather than Vect because (among other things)

- given state vectors (i.e. unit vectors) in a Hilbert space, say ϕ and ψ, then $\langle \phi \mid \psi \rangle$ is the **amplitude** and $|\langle \phi \mid \psi \rangle|^2$ is the **probability** that a system prepared in state ψ will be found in state ϕ. There is no such structure in Vect.
- given an operator $T \colon \mathcal{H} \to \mathcal{H}'$, the condition $\langle T^*\phi \mid \psi \rangle = \langle \phi \mid T\psi \rangle$ defines an **adjoint** operator $T^* \colon \mathcal{H}' \to \mathcal{H}$. In Vect, the best we can get is the dual $T^* \colon \mathcal{H}'^* \to \mathcal{H}^*$.
- **observables** in quantum mechanics are represented by self-adjoint operators $A \colon \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is the space of states of the system and $A = A^*$. Such an operator[1] has associated an orthonormal basis $\{\psi_i\}$ of $\mathcal{H}$ such that $A\psi_i = a_i\psi_i$ with $a_i \in \mathbb{R}$. The interpretation is that ψ_i is a state in which A will always be measured to be a_i.

The fact that in Hilb we have a canonical antiisomorphism $\mathcal{H} \to \mathcal{H}^*$ induced by $\langle \cdot \mid \cdot \rangle$ is very different from Vect or Set, but a lot like nCob, where the "dual" of a space is the same space with the opposite orientation, and the "adjoint" of an n-cobordism is its time-reversal. Time reversal is of utmost importance in physics.

Definition. *Given categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** $F \colon \mathcal{C} \to \mathcal{D}$ consists of:*

a) A map sending any object $x \in \mathcal{C}$ to an object $F(x) \in \mathcal{D}$.

b) For any pair of objects x and y, a map sending morphisms $f \colon x \to y$ to morphisms $F(f) \colon F(x) \to F(y)$, such that these laws hold:

- *for any object $x \in \mathcal{C}$, we have $F(1_x) = 1_{F(x)}$.*
- *for any pair of morphisms $f \colon x \to y$ and $g \colon y \to z$, we have $F(gf) = F(g)F(f)$.*

In short: F sends objects to objects, morphisms to morphisms, and preserves sources, targets, identities, and composition. ★

Definition. *We say that a category $\mathcal{C}$ has* **adjoints** *or* **duals for morphisms** *or is a* $*$**-category** *if there is a contravariant functor $* \colon \mathcal{C} \to \mathcal{C}$ which takes objects to themselves and such that $*^2 = 1$ (the identity functor). For any object x or morphism f, the dual is denoted $*(x) = x^*$ or $*(f) = f^*$.* ★

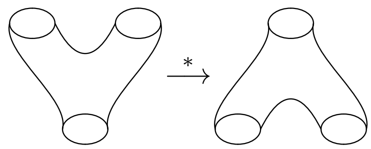
Spelling out the definition, $*$ has to satisfy the following properties:

- $x^* = x$ for any $x \in \mathcal{C}$,
- for any $f \colon x \to y$ there is a morphism $f^* \colon y \to x$ (this is what "contravariant" means),
- for any $x \in \mathcal{C}$, $(1_x)^* = 1_{x^*} = 1_x$,
- for any morphisms $f \colon x \to y$ and $g \colon y \to z$, we have $(gf)^* = f^*g^*$, and
- $(f^*)^* = f$ for any morphism f.

[1] More generally, any **normal** operator, i.e. any operator such that $NN^* = N^*N$, has an orthonormal basis of eigenvectors with complex eigenvalues.

Examples of $*$-categories are:

- nCob, where M^* is obtained by exchanging the roles of input and output. If the cobordism is imbedded, this can be represented as reflection along the "time" direction.



- Hilb, where the adjoint T^* of a linear operator $T\colon \mathcal{H} \to \mathcal{H}'$ is defined by $\langle T^*\phi \mid \psi\rangle_{\mathcal{H}} = \langle \phi \mid T\psi\rangle_{\mathcal{H}'}$.
- any **groupoid** (a category where every morphism is invertible), as then the inverse has the properties required of $*$.

Definition. *We say that $F\colon \mathcal{C} \to \mathcal{D}$ is a $*$-functor if given $f\colon x \to y$, we have $F(f^*) = F(f)^*\colon F(x) \to F(y)$.* ★

Definition. *A category $\mathcal{C}$ is* **monoidal** *if it is equipped with an operation $\otimes$ with the following properties:*

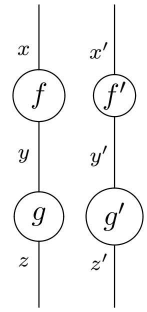
- *for any $x, y \in \mathcal{C}$, there is an object $x \otimes y \in \mathcal{C}$;*
- *for any $f\colon x \to x'$ and $g\colon y \to y'$, there is a morphism $f \otimes g\colon x \otimes y \to x' \otimes y'$.*
- *for any objects $x, y, z \in \mathcal{C}$ there is an isomorphism $a_{xyz}\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ called the* **associator** *and satisfying the* **pentagon identity***:*

$$\begin{tikzcd}
 & (w\otimes x)\otimes(y\otimes z) \arrow[dr, "a_{wx(yz)}"] & \\
\big((w\otimes x)\otimes y\big)\otimes z \arrow[ur, "a_{(wx)yz}"] \arrow[d, "a_{wxy}\otimes 1_z"'] & & w\otimes\big(x\otimes(y\otimes z)\big) \\
\big(w\otimes(x\otimes y)\big)\otimes z \arrow[rr, "a_{w(xy)z}"] & & w\otimes\big((x\otimes y)\otimes z\big) \arrow[u, "1_w\otimes a_{xyz}"']
\end{tikzcd}$$

- *there is an object 1 such that, for any object $x \in C$, there are isomorphisms $l_x\colon 1 \otimes x \to x$ and $r_x\colon x \otimes 1 \to x$ called* **units** *satisfying the* **other identity***:*

$$\begin{tikzcd}
 & x \otimes 1 \arrow[dr, "r_x"] & \\
(1 \otimes x) \otimes 1 \arrow[ur, "l_x\otimes 1_1"] \arrow[d, "a_{1x1}"'] & & x \\
1 \otimes (x \otimes 1) \arrow[rr, "1_1\otimes r_x"] & & 1 \otimes x \arrow[u, "l_x"']
\end{tikzcd}$$

- *finally, given $f\colon x \to y$, $g\colon y \to z$, $f'\colon x' \to y'$ and $g'\colon y' \to z'$, we require that $(g \otimes g')(f \otimes f') = (gf) \otimes (g'f')$, which just says that the following diagram is unambiguous:*



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MacLane's theorem guarantees that if the above two diagrams commute, then any diagram that can be constructed from the associator and the units commutes.

Examples of monoidal categories are

- Grp: objects are groups, morphisms are group homomorphisms and $\otimes$ is the direct product of groups.
- nCob: the $\otimes$, both for objects and for morphisms, is the disjoint union of manifolds.
- Vect or Hilb: the $\otimes$ is the tensor product. This is how, in quantum mechanics, two things are put together.
- Elect: it has just one object, morphisms are electrical circuit elements, composition is serial combination of components, and $\otimes$ is parallel or shunted combination of components.

Note: An algebra equipped with a nondegenerate trace is called a **Frobenius algebra**. We've just seen that given any TQFT, the Hilbert space $Z(S^1)$ is a commutative Frobenius algebra with multiplication given by

$$Z(m)\colon Z(S^1) \otimes Z(S^1) \to Z(S^1),$$

unit given by

$$Z(i)\colon \mathbb{C} \to Z(S^1),$$

and trace given by

$$Z(i^*)\colon Z(S^1) \to \mathbb{C}.$$

But the cool part is the converse: for any Hilbert space with the structure of a commutative Frobenius algebra, there exists a unique $2d$ TQFT. Actually, uniqueness isn't hard. The only real work is to figure out a formula for $Z(m^*)$ in terms of the 3 maps just listed.

Faithfulness, Fullness, Equivalences

Definition. *A functor $F\colon C \to D$ is called **faithful** if for each pair of objects $X, Y \in C$, the map $F_{X,Y}\colon \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_D(F(X),F(Y))$ is injective. F is called **full** if the maps $F_{X,Y}\colon \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_D(F(X),F(Y))$ are all surjective. If C is a subcategory of D, then the inclusion functor is always faithful. If it is full, C is called a **full subcategory**.* ★

Definition. *A functor is called* ***essentially surjective*** *if every object in D is isomorphic to an image under F of an object of C. A functor is called an* ***equivalence*** *if it is faithful, full, and essentially surjective.* ★