MATH 709 NOTES
SUBMERSIONS, IMMERSIONS, & EMBEDDINGS

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Embedded Submanifolds

Definition. *Suppose M is a smooth manifold (with or without boundary). An* ***embedded submanifold*** *of M is a subset $S \subseteq M$ that is a manifold (without boundary) in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding. (Note that embedded submanifolds are also called* ***regular submanifolds*** *by some authors).* ★

Definition. *If S is an embedded submanifold of M, the difference $\dim M - \dim S$ is called the* ***codimension of*** *S* ***in*** *M, and the containing manifold M is called the* ***ambient manifold*** *for S. An* ***embedded hypersurface*** *is an embedded submanifold of codimension 1. (The empty set is an embedded submanifold of any dimension).* ★

Remark: The easiest embedded submanifolds to understand are those of codimension 0. Recall that for any smooth manifold M we defined an open submanifold of M to be any open subset with the subspace topology and with the smooth charts obtained by restricting those of M.

Proposition 1 (**Open Submanifolds**). *Suppose M is a smooth manifold. The embedded submanifolds of codimension 0 in M are exactly the open submanifolds.*

The next few propositions demonstrate several other ways to produce embedded submanifolds:

Proposition 2 (**Images of Embeddings as Submanifolds**). *Suppose M is a smooth manifold (with or without boundary), N is a smooth manifold, and $F\colon N \to M$ is a smooth embedding. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.*

Proof. If we give S the subspace topology that it inherits from M, then the assumption that F is an embedding means that F can be considered as a homeomorphism from N onto S, and thus S is a topological manifold. We give S a smooth structure by taking the smooth charts to be those of the form $\big(F(U), \varphi \circ F^{-1}\big)$, where (U, φ) is any smooth chart for N; smooth compatibility of these charts follows immediately from the smooth compatibility of the corresponding charts for N. With this smooth structure on S, the map F is a diffeomorphism onto its image (essentially by definition), and this is obviously the only smooth structure with this property. The inclusion map $S \hookrightarrow M$ is equal to the composition of a diffeomorphism followed by a smooth embedding:

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M,$$

and therefore it is a smooth embedding. □

Remark: Since every embedded submanifold is the image of a smooth embedding (namely its own inclusion map), the previous proposition shows that embedded submanifolds are exactly the images of smooth embeddings.

Proposition 3 (**Slices of Product Manifolds**). *Suppose M and N are smooth manifolds. For each $p \in N$, the subset $M \times \{p\}$ (called a **slice** of the product manifold) is an embedded submanifold of $M \times N$ diffeomorphic to M.*

Proof. The set $M \times \{p\}$ is the image of the smooth embedding $x \mapsto (x, p)$. □

Proposition 4 (**Graphs as Submanifolds**). *Suppose M is a smooth m-manifold (without boundary), N is a smooth n-manifold (with or without boundary), $U \subseteq M$ is open, and $f\colon U \to N$ is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote the graph of f:*

$$\Gamma(f) = \{(x, y) \in M \times N \mid x \in U, y = f(x)\}.$$

Then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$ (see Figure 1 *below).*

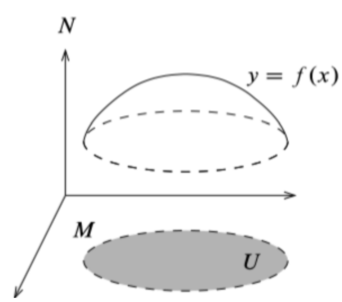

FIGURE 1. A graph is an embedded submanifold.

Proof. Define a map $\gamma_f\colon U \to M \times N$ by

(♣) $$\gamma_f(x) = (x, f(x))\,.$$

It is a smooth map whose image is $\Gamma(f)$. Because the projection $\pi_M\colon M \times N \to M$ satisfies $\pi_M \circ \gamma_f(x) = x$ for $x \in U$, the composition $d(\pi_M)_{(x,f(x))} \circ d(\gamma_f)_x$ is the identity on T_xM for each $x \in U$. Thus, $d(\gamma_f)_x$ is injective, so f is a smooth immersion. It a homeomorphism onto its image because $\pi_M|_{\Gamma(f)}$ is a continuous inverse for it. Thus, $\Gamma(f)$ is an embedded submanifold diffeomorphic to U. □

Recall the following proposition from topology:

Proposition 5 (**Sufficient Conditions for Properness**). *Suppose X and Y are topological spaces, and $F\colon X \to Y$ is a continuous map.*

- *a) If X is compact and Y is Hausdorff, then F is proper.*
- *b) If F is a closed map with compact fibers, then F is proper.*
- *c) If F is a topological embedding with closed image, then F is proper.*

d) *If Y is Hausdorff and F has a continuous left inverse (i.e., a continuous map $G\colon Y \to X$ such that $G \circ F = \mathrm{Id}_X$), then F is proper.*

e) *If F is proper and $A \subseteq X$ is a subset that is saturated with respect to F, then $F|_A\colon A \to F(A)$ is proper.*

Definition. *An embedded submanifold $S \subseteq M$ is said to be **properly embedded** if the inclusion $S \hookrightarrow M$ is a proper map.* ★

Proposition 6. *Suppose M is a smooth manifold (with or without boundary) and $S \subseteq M$ is an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M.*

Corollary 1. *Every compact embedded submanifold is properly embedded.*

Proof. Compact subsets of Hausdorff spaces are closed. □

Graphs of globally defined functions are common examples of properly embedded submanifolds:

Proposition 7 (Global Graphs Are Properly Embedded). *Suppose M is a smooth manifold, N is a smooth manifold (with or without boundary), and $f\colon M \to N$ is a smooth map. With the smooth manifold structure of* Proposition 4*, $\Gamma(f)$ is properly embedded in $M \times N$.*

Proof. Proof. In this case, the projection $\pi_M\colon M \times N \to M$ is a smooth left inverse for the embedding $\gamma_f\colon M \to M \times N$ defined by equation (♣) above. Thus γ_f is proper by Proposition 5. □

As Theorem 1 below will show, embedded submanifolds are modeled locally on the standard embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, identifying $\mathbb{R}^k$ with the subspace

$$\{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \mid x^{k+1} = \cdots = x^n = 0\} \subseteq \mathbb{R}^n.$$

Somewhat more generally:

Definition. *If U is an open subset of $\mathbb{R}^n$ and $k \in \{0, \dots, n\}$, a k-**dimensional slice of** U (or simply a k-**slice**) is any subset of the form*

$$S = \{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U \mid x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$

for some constants $c^{k+1}, \dots, c^n$. (When $k = n$, this just means $S = U$.) ★

Definition. *Let M be a smooth n-manifold, and let (U, φ) be a smooth chart on M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that S is a k-**slice of** U.* ★

Remark: Although in general we allow our slices to be defined by arbitrary constants $c^{k+1}, \dots, c^n$, it is sometimes useful to have slice coordinates for which the constants are all zero, which can easily be achieved by subtracting a constant from each coordinate function.

Definition. *Given a subset $S \subseteq M$ and a nonnegative integer k, we say that S satisfies the **local k-slice condition** if each point of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ is a single k-slice in U. Any such chart is called a **slice chart for** S **in** M, and the corresponding coordinates $(x^1, \dots, x^n)$ are called **slice coordinates**.* ★

Theorem 1 (**Local Slice Criterion for Embedded Submanifolds**). *Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.*

Proof. ($\Rightarrow$) First suppose that $S \subseteq M$ is an embedded k-dimensional submanifold. Since the inclusion map $S \hookrightarrow M$ is an immersion, the rank theorem shows that for any $p \in S$ there are smooth charts (U, φ) for S (in its given smooth manifold structure) and (V, ψ) for M, both centered at p, in which the inclusion map $\iota\big|_U : U \to V$ has the coordinate representation

$$(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

Choose $\varepsilon > 0$ small enough that both U and V contain coordinate balls of radius ε centered at p, and denote these coordinate balls by $U_0 \subseteq U$ and $V_0 \subseteq V$. It follows that $U_0 = \iota(U_0)$ is exactly a single slice in V_0. Because S has the subspace topology, the fact that U_0 is open in S means that there is an open subset $W \subseteq M$ such that $U_0 = W \cap S$. Setting $V_1 = V_0 \cap W$, we obtain a smooth chart $\left(V_1, \psi\big|_{V_1}\right)$ for M containing p such that $V_1 \cap S = U_0$, which is a single slice of V_1.

($\Leftarrow$) Conversely, suppose S satisfies the local k-slice condition. With the subspace topology, S is Hausdorff and second countable, because both properties are inherited by subspaces. To see that S is locally Euclidean, we construct an atlas. The basic idea of the construction is that if $(x^1, \dots, x^n)$ are slice coordinates for S in M, we can use $(x^1, \dots, x^k)$ as local coordinates for S.

For this proof, let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ denote the projection onto the first k coordinates. Let (U, φ) be any slice chart for S in M (see Figure 2), and define

$$V = U \cap S, \qquad \widehat{V} = \pi \circ \varphi(V), \qquad \psi = \pi \circ \varphi\big|_V : V \to \widehat{V}.$$

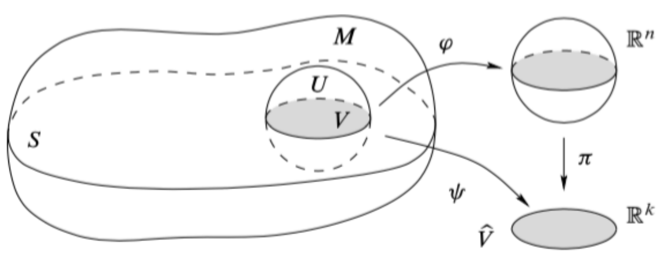

FIGURE 2. A chart for a subset satisfying the k-slice condition.

By definition of slice charts, $\varphi(V)$ is the intersection of $\varphi(U)$ with a certain k-slice $A \subseteq \mathbb{R}^n$ defined by setting $x^{k+1} = c^{k+1}, \dots, x^n = c^n$, and therefore $\varphi(V)$ is open in A. Since $\pi|_A$ is a diffeomorphism from A to $\mathbb{R}^k$, it follows that $\widehat{V}$ is open in $\mathbb{R}^k$. Moreover, ψ is a homeomorphism because it has a continuous inverse given by $\varphi^{-1} \circ j\big|_{\widehat{V}}$, where $j : \mathbb{R}^k \to \mathbb{R}^n$ is the map

$$j(x^1, \dots, x^k) = (x^1, \dots, x^k, c^{k+1}, \dots, c^n)$$

Thus S is a topological k-manifold, and the inclusion map $\iota\colon S \hookrightarrow M$ is a topological embedding.

To put a smooth structure on S, we need to verify that the charts constructed above are smoothly compatible. Suppose (U, φ) and (U', φ') are two slice charts for S in M, and let (V, ψ), (V', ψ') be the corresponding charts for S. The transition map is given by $\psi' \circ \psi^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ j$, which is a composition of four smooth maps (see Figure 3)

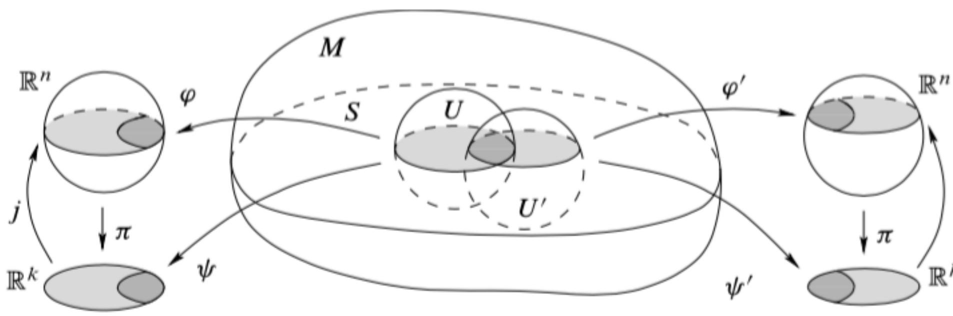

FIGURE 3. Smooth compatibility of slice charts.

Thus the atlas we have constructed is in fact a smooth atlas, and it defines a smooth structure on S. In terms of a slice chart (U, φ) for M and the corresponding chart (V, ψ) for S, the inclusion map $S \hookrightarrow M$ has a coordinate representation of the form

$$(x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, c^{k+1}, \ldots, c^n),$$

which is a smooth immersion. Since the inclusion is a smooth immersion and a topological embedding, S is an embedded submanifold. $\square$

Remark: Notice that the local slice condition for $S \subseteq M$ is a condition on the subset S only; it does not presuppose any particular topology or smooth structure on S. As we will see later on on Theorem 6, the smooth manifold structure constructed in the preceding theorem is the unique one in which S can be considered as a submanifold, so a subset satisfying the local slice condition is an embedded submanifold in only one way.

Example 1 (**Spheres as Submanifolds**)**.** *For any $n \geq 0$, $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$, because it is locally the graph of a smooth function: the intersection of $\mathbb{S}^n$ with the open subset $\{x\colon x^i > 0\}$ is the graph of the smooth function*

$$x^i = f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1}),$$

where $f\colon \mathbb{B}^n \to \mathbb{R}$ is given by $f(u) = \sqrt{1 - |u|^2}$. Similarly, the intersection of $\mathbb{S}^n$ with $\{x \mid x^i < 0\}$ is the graph of $-f$. Since every point in $\mathbb{S}^n$ is in one of these sets, $\mathbb{S}^n$ satisfies the local n-slice condition and is thus an embedded submanifold of $\mathbb{R}^{n+1}$. The smooth structure thus induced on $\mathbb{S}^n$ is the same as the one we have previously defined in class: in fact, the coordinates for $\mathbb{S}^n$ determined by these slice charts are exactly the graph coordinates previously defined.

If M is a smooth manifold with nonempty boundary and $S \subseteq M$ is an embedded submanifold, then S might intersect ∂M in very complicated ways, so we will not attempt to prove any general results about the existence of slice charts for S in M in that case. However, in the special case in which the submanifold is the boundary of M itself, the boundary charts for M play the role of slice charts for ∂M in M, and we do have the following result:

Theorem 2. *If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.*

Level Sets

Definition. *If $\Phi\colon M \to N$ is any map and c is any point of N, we call the set $\Phi^{-1}(c)$ a* ***level set of*** Φ *(see* Figure 4*). (In the special case when $N = \mathbb{R}^k$ and $c = 0$, the level set $\Phi^{-1}(0)$ is usually called the* ***zero set of*** Φ*).* ★

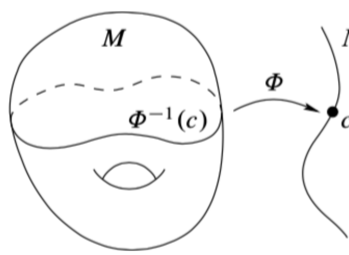

Figure 4. A level set.

Example 2. *It is easy to find level sets of smooth functions that are not smooth submanifolds. For instance, consider the three smooth functions $\Theta, \Phi, \Psi\colon \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$\Theta(x,y) = x^2 - y, \qquad \Phi(x,y) = x^2 - y^2, \qquad \Psi(x,y) = x^2 - y^3.$$

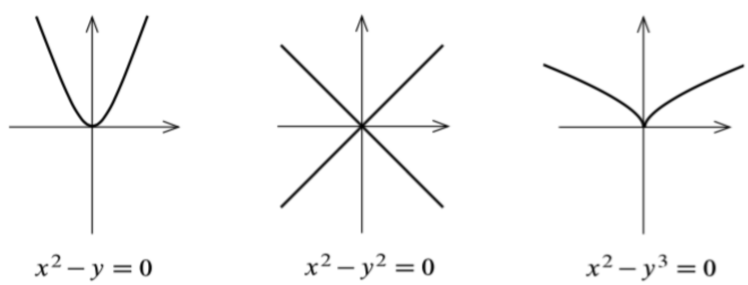

Figure 5. Level sets may or may not be embedded submanifolds.

Although the zero set of Θ (a parabola) is an embedded submanifold of $\mathbb{R}^2$ (because it is the graph of the smooth function $f(x) = x^2$), it can be shown that neither the zero set of Φ nor that of Ψ is an embedded submanifold (see Problem 5-11*, HW set # 6). In fact, without further assumptions on the smooth function, the situation is about as bad as could be imagined: as it has been shown by a previous theorem, every closed subset of M can be expressed as the zero set of some smooth real-valued function.*

Theorem 3 (Constant-Rank Level Set Theorem). *Let M and N be smooth manifolds, and let $\Phi\colon M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.*

Corollary 2 (**Submersion Level Set Theorem**). *If M and N be smooth manifolds and $\Phi\colon M \to N$ is a smooth submersion, then each level set of Φ is a properly embedded submanifold whose codimension is equal to the dimension of N.*

Proof. Every smooth submersion has constant rank equal to the dimension of its codomain. □

Remark: This result should be compared to the corresponding result in linear algebra: if $L\colon \mathbb{R}^m \to \mathbb{R}^r$ is a surjective linear map, then the kernel of L is a linear subspace of codimension r by the rank-nullity law. The vector equation $Lx = 0$ is equivalent to r linearly independent scalar equations, each of which can be thought of as cutting down one of the degrees of freedom in $\mathbb{R}^m$, leaving a subspace of codimension r. In the context of smooth manifolds, the analogue of a surjective linear map is a smooth submersion, each of whose (local) component functions cuts down the dimension by one.

Corollary 2 can be strengthened considerably, because we need only check the submersion condition on the level set we are interested in.

Definition. *If $\Phi\colon M \to N$ is a smooth map, a point $p \in M$ is said to be a* ***regular point of*** *Φ if $\mathrm{d}\Phi_p\colon T_pM \to T_{\Phi(p)}N$ is surjective; it is a* ***critical point of*** *Φ otherwise. (This means, in particular, that every point of M is critical if $\dim M < \dim N$, and every point is regular if and only if F is a submersion.)* ★

Definition. *A point $c \in N$ is said to be a* ***regular value of*** *Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point; c is called a* ***critical value*** *otherwise. (In particular, if $\Phi^{-1}(c) = \emptyset$, then c is a regular value.* ★

Definition. *A level set $\Phi^{-1}(c)$ is called a* ***regular level set*** *if c is a regular value of Φ; in other words, a regular level set is a level set consisting entirely of regular points of Φ (points p such that $\mathrm{d}\Phi_p$ is surjective).* ★

Corollary 3 (**Regular Level Set Theorem**). *Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.*

Remark: It is worth noting that the previous corollary also applies to empty level sets, which are both regular level sets and properly embedded submanifolds.

Example 3 (**Spheres**). *Now we can give a much easier proof that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$. The sphere is a regular level set of the smooth function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = |x|^2$, since $\mathrm{d}f_x(v) = 2\sum_i x^i v^i$ which is surjective except at the origin.*

Not all embedded submanifolds can be expressed as level sets of smooth submersions. However, the next proposition shows that every embedded submanifold is at least locally of this form:

Proposition 8. *Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$.*

Proof. (⇒) First suppose S is an embedded k-submanifold. If $(x^1, \ldots, x^m)$ are slice coordinates for S on an open subset $U \subseteq M$, then the map $\Phi\colon U \to \mathbb{R}^{m-k}$ given in coordinates by $\Phi(x) = (x^{k+1}, \ldots, x^m)$ is easily seen to be a smooth submersion, one of whose level sets is $S \cap U$ (see Figure 6).

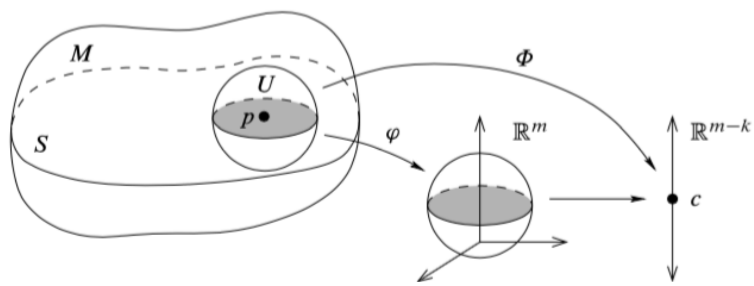

FIGURE 6. An embedded submanifold is locally a level set.

(⇐) Conversely, suppose that around every point $p \in S$ there is a neighborhood U and a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$ such that $S \cap U$ is a level set of Φ. Then by the submersion level set theorem (Corollary 2), we have that $S \cap U$ is an embedded submanifold of U, so it satisfies the local slice condition; it follows that S is itself an embedded submanifold of M. □

Definition. *If $S \subseteq M$ is an embedded submanifold, a smooth map $\Phi\colon M \to N$ such that S is a regular level set of Φ is called a **defining map for** S. In the special case when $N = \mathbb{R}^{m-k}$ (so that Φ is a real-valued or vector-valued function), it is usually called a **defining function**.* ★

For instance, from Example 3, we have that $f(x) = |x|^2$ is a defining function for the sphere. More generally:

Definition. *If U is an open subset of M and $\Phi\colon U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called a **local defining map** (or **local defining function**) **for** S.* ★

Proposition 8 says that every embedded submanifold admits a local defining function in a neighborhood of each of its points. In specific examples, finding a (local or global) defining function for a submanifold is usually just a matter of using geometric information about how the submanifold is defined together with some computational ingenuity. Here is an example:

Example 4 (**Surfaces of Revolution**). *Let H be the half-plane $\{(r, z) \mid r > 0\}$, and suppose $C \subseteq H$ is an embedded 1-dimensional submanifold. The **surface of revolution** determined by C is the subset $S_C \subseteq \mathbb{R}^3$ given by*

$$S_C = \left\{(x, y, z) : \left(\sqrt{x^2 + y^2}, z\right) \in C\right\}.$$

*The set C is called its **generating curve** (see* Figure 7*).*

If $\varphi\colon U \to \mathbb{R}$ is any local defining function for C in H, we get a local defining function Φ for S_C by

$$\Phi(x, y, z) = \varphi\left(\sqrt{x^2 + y^2}, z\right),$$

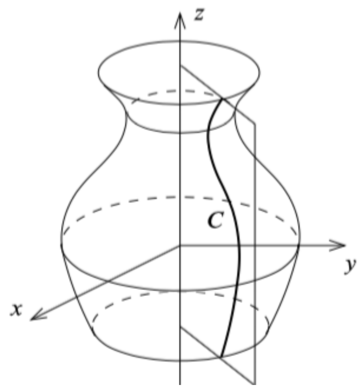

FIGURE 7. A surface of revolution.

defined on the open subset

$$\widetilde{U} = \left\{(x,y,z) : \left(\sqrt{x^2+y^2}, z\right) \in U\right\} \subseteq \mathbb{R}^3.$$

A computation shows that the Jacobian matrix of Φ is

$$D\Phi(x,y,z) = \left(\frac{x}{r}\frac{\partial\varphi}{\partial r}(r,z)\ \frac{y}{r}\frac{\partial\varphi}{\partial r}(r,z)\ \frac{\partial\varphi}{\partial z}(r,z)\right),$$

where we have written $r = \sqrt{x^2+y^2}$. At any point $(x,y,z) \in S_C$, at least one of the components of $D\Phi(x,y,z)$ is nonzero, so S_C is a regular level set of Φ and is thus an embedded 2-dimensional submanifold of $\mathbb{R}^3$.

For a specific example, the doughnut-shaped torus of revolution is the surface of revolution obtained from the circle $(r-2)^2 + z^2 = 1$. It is a regular level set of the function $\Phi(x,y,z) = \left(\sqrt{x^2+y^2}-2\right)^2 + z^2$, which is smooth on $\mathbb{R}^3$ minus the z-axis.

Immersed Submanifolds

Definition. *Suppose M is a smooth manifold (with or without boundary). An* ***immersed submanifold of*** *M is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace topology) with respect to which it is a topological manifold (without boundary), and a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth immersion. (As in the case of embedded submanifolds, we define the* ***codimension of S in M*** *to be $\dim M - \dim S$).* ★

Remark: Note that every embedded submanifold is also an immersed submanifold. Because immersed submanifolds are the more general of the two types of submanifolds, we adopt the convention that the term ***smooth submanifold*** without further qualification means an immersed one, which includes an embedded submanifold as a special case. Similarly, the term ***smooth hypersurface*** without qualification means an immersed submanifold of codimension 1.

Immersed submanifolds often arise in the following way:

Proposition 9 (Images of Immersions as Submanifolds). *Suppose M is a smooth manifold (with or without boundary), N is a smooth manifold, and $F\colon N \to M$ is an injective smooth* *immersion. Let $S = F(N)$. Then S has a unique topology and smooth structure such that it is a smooth submanifold of M and such that $F\colon N \to S$ is a diffeomorphism onto its image.*

The following observation is sometimes useful when thinking about the topology of an immersed submanifold:

Proposition 10. *Suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. Then every subset of S that is open in the subspace topology is also open in its given submanifold topology, and the converse is true if and only if S is embedded.*

Proposition 11 (Sufficient Conditions for Immersed Submanifolds to be Embedded). *Suppose M is a smooth manifold (with or without boundary), and $S \subseteq M$ is an immersed submanifold. If any of the following holds, then S is embedded.*

a) S has codimension 0 in M.

b) The inclusion map $S \hookrightarrow M$ is proper.

c) S is compact.

Although many immersed submanifolds are not embedded, the next proposition shows that the local structure of an immersed submanifold is the same as that of an embedded one:

Proposition 12 (Immersed Submanifolds Are Locally Embedded). *If M is a smooth manifold (with or without boundary), and $S \subseteq M$ is an immersed submanifold, then for each $p \in S$ there exists a neighborhood U of p in S that is an embedded submanifold of M.*

Proof. By a previous theorem, we have that each $p \in S$ has a neighborhood U in S such that the inclusion $\iota\big|_U\colon U \hookrightarrow M$ is an embedding. $\square$

Remark: It is important to be clear about what this proposition does and does not say: given an immersed submanifold $S \subseteq M$ and a point $p \in S$, it is possible to find a neighborhood U of p (in S) such that U is embedded; but it may not be possible to find a neighborhood V of p (in M) such that $V \cap S$ is embedded (see Figure 8).

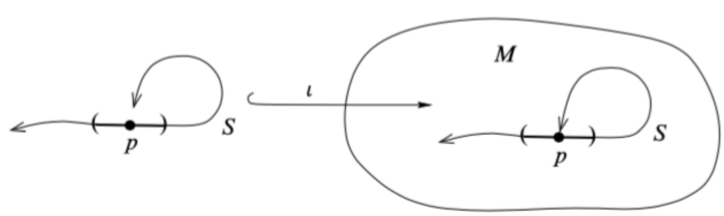

FIGURE 8. An immersed submanifold is locally embedded.

Definition. *Suppose $S \subseteq M$ is an immersed k-dimensional submanifold. A* ***local parametrization of S*** *is a continuous map $\Psi\colon U \to M$ whose domain is an open subset $U \subseteq \mathbb{R}^k$, whose image is an open subset of S, and which, considered as a map into S, is a homeomorphism onto its image. It is called a* ***smooth local parametrization*** *if it is a diffeomorphism onto its image (with respect to S's smooth manifold structure). If the image of Ψ is all of S, it is called a* ***global parametrization****.* ★

Proposition 13. *Suppose M is a smooth manifold (with or without boundary), $S \subseteq M$ is an immersed k-submanifold, $\iota\colon S \hookrightarrow M$ is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map $\Psi\colon U \to M$ is a smooth local parametrization of S if and only if there is a smooth coordinate chart (V, φ) for S such that $\Psi = \iota \circ \varphi^{-1}$. Therefore, every point of S is in the image of some local parametrization.*

Restricting Maps to Submanifolds

Given a smooth map $F\colon M \to N$, it is important to know whether F is still smooth when its domain or codomain is restricted to a submanifold. In the case of restricting the domain, the answer is easy:

Theorem 4 (Restricting the Domain of a Smooth Map). *If M and N are smooth manifolds (with or without boundary), $F\colon M \to N$ is a smooth map, and $S \subseteq M$ is an immersed or embedded submanifold (see Figure 9), then $F\big|_S\colon S \to N$ is smooth.*

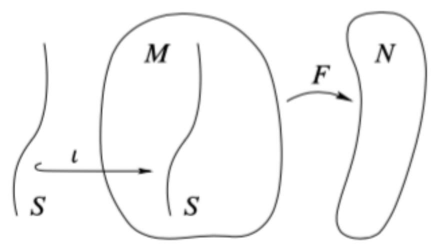

Figure 9. Restricting the domain.

Proof. The inclusion map $\iota\colon S \hookrightarrow M$ is smooth by definition of an immersed submanifold. Since $F\big|_S = F \circ \iota$, the result follows. □

When the codomain is restricted, however, the resulting map may not be smooth, as the following example shows:

Example 5. *Let $S \subseteq \mathbb{R}^2$ be the figure-eight submanifold (lemniscate), with the topology and smooth structure induced by the immersion $\beta\colon (-\pi, \pi) \to \mathbb{R}^2$ defined by $\beta(t) = (\sin 2t, \sin t)$. Define a smooth map $G\colon \mathbb{R} \to \mathbb{R}^2$ by $G(t) = (\sin 2t, \sin t)$. (We are using the same formula that we used to define β, but now the domain is extended to the whole real line instead of being just a subinterval.) It is easy to check that the image of G lies in S. However, as a map from $\mathbb{R}$ to S, G is not even continuous, because $\beta^{-1} \circ G$ is not continuous at $t = \pi$.*

The next theorem gives sufficient conditions for a map to be smooth when its codomain is restricted to an immersed submanifold. It shows that the failure of continuity is the only thing that can go wrong:

Theorem 5 (Restricting the Codomain of a Smooth Map). *Suppose M is a smooth manifold (without boundary), $S \subseteq M$ is an immersed submanifold, and $F\colon N \to M$ is a smooth map whose image is contained in S (see Figure 10). If F is continuous as a map from N to S, then $F\colon N \to S$ is smooth.*

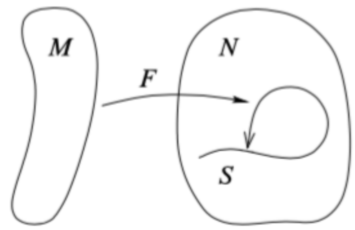

FIGURE 10. Restricting the codomain.

Remark: This theorem is stated only for the case in which the ambient manifold M is a manifold without boundary, because it is only in that case that we have constructed slice charts for embedded submanifolds of M. But the conclusion of the theorem is still true when M has nonempty boundary (see Problem 9-13).

In the special case in which the submanifold S is embedded, the continuity hypothesis is always satisfied:

Corollary 4 (**Embedded Case**). *Let M be a smooth manifold and $S \subseteq M$ be an embedded submanifold. Then every smooth map $F\colon N \to M$ whose image is contained in S is also smooth as a map from N to S.*

Proof. Since $S \subseteq M$ has the subspace topology, a continuous map $F\colon N \to M$ whose image is contained in S is automatically continuous into S, by the characteristic property of the subspace topology. □

Although the conclusion of the preceding corollary fails for some immersed submanifolds such as the lemniscate, it turns out that there are certain immersed but nonembedded submanifolds for which it holds. To distinguish them, we introduce the following definition:

Definition. *If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold, then S is said to be* ***weakly embedded in*** *M if every smooth map $F\colon N \to M$ whose image lies in S is smooth as a map from N to S. (Be aware that weakly embedded submanifolds are called* ***initial submanifolds*** *by some authors.)* ★

Remark: Corollary 4 shows that every embedded submanifold is weakly embedded. It follows from Example 5 that the lemniscate is not weakly embedded. However, the dense curve on the torus is weakly embedded (see Problem 5-13).

Using the preceding results about restricting maps to submanifolds, we can now prove the promised uniqueness theorem for the smooth manifold structure on an embedded submanifold:

Theorem 6. *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. The subspace topology on S and the smooth structure described earlier in* Theorem 1 *are the only topology and smooth structure with respect to which S is an embedded or immersed submanifold.*

Proof. See page 114, Lee's *Smooth Manifolds.* □

Remark: Thanks to this uniqueness result, we now know that a subset $S \subseteq M$ is an embedded submanifold if and only if it satisfies the local slice condition, and if so, its topology and smooth structure are uniquely determined. Because the local slice condition is a local condition, if every point $p \in S$ has a neighborhood $U \subseteq M$ such that $U \cap S$ is an embedded k-submanifold of U, then S is an embedded k-submanifold of M .

The preceding theorem is false in general if S is merely immersed; but we do have the following uniqueness theorem for the smooth structure of an immersed submanifold once the topology is known:

Theorem 7. *Suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. For the given topology on S, there is only one smooth structure making S into an immersed submanifold.*

It is certainly possible for a given subset of M to have more than one topology making it into an immersed submanifold (see Problem 5–15). However, for weakly embedded submanifolds we have a stronger uniqueness result:

Theorem 8. *If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, then S has only one topology and smooth structure with respect to which it is an immersed submanifold.*

Lemma 1 (**Extension Lemma for Functions on Submanifolds**). *Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$.*

a) If S is embedded, then there exist a neighborhood U of S in M and a smooth function $\widetilde{f} \in C^\infty(U)$ such that $\widetilde{f}\big|_S = f$.

b) If S is properly embedded, then the neighborhood U in part a) can be taken to be all of M.

The Tangent Space to a Submanifold

Let M be a smooth manifold (with or without boundary), and let $S \subseteq M$ be an immersed or embedded submanifold. Since the inclusion map $\iota\colon S \hookrightarrow M$ is a smooth immersion, at each point $p \in S$ we have an injective linear map $\mathrm{d}\iota_p\colon T_pS \to T_pM$. In terms of derivations, this injection works in the following way: for any vector $\nu \in T_pS$, the image vector $\widetilde{\nu} = \mathrm{d}\iota_p(\nu) \in T_pM$ acts on smooth functions on M by

$$\widetilde{\nu} f = \mathrm{d}\iota_p(\nu) f = \nu(f \circ \iota) = \nu(f\big|_S).$$

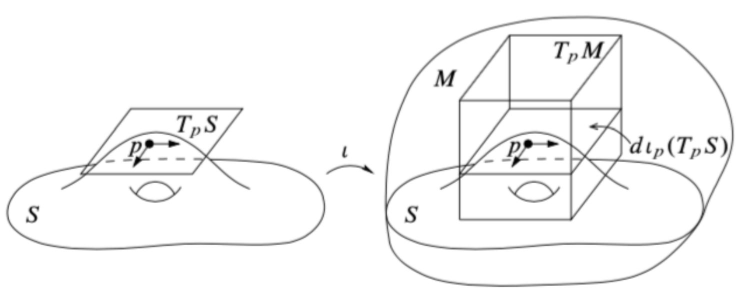

Figure 11. The tangent space to an embedded submanifold.

We adopt the convention of identifying T_pS with its image under this map, thereby thinking of T_pS as a certain linear subspace of T_pM (see Figure 11). This identification makes sense regardless of whether S is embedded or immersed.

There are several alternative ways of characterizing T_pS as a subspace of T_pM. The first one is the most general; it is just a straightforward generalization of a previous proposition:

Proposition 14. *Suppose M is a smooth manifold (with or without boundary), $S \subseteq M$ is an embedded or immersed submanifold, and $p \in S$. A vector $\nu \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma\colon J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = \nu$.*

The next proposition gives a useful way to characterize T_pS in the embedded case (Problem 5–20 shows that this does not work in the nonembedded case):

Proposition 15. *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded submanifold, and $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by*

$$T_pS = \{\nu \in T_pM \mid \nu f = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f\big|_S = 0\}.$$

If an embedded submanifold is characterized by a defining map, the defining map gives a concise characterization of its tangent space at each point, as the next proposition shows:

Proposition 16. *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi\colon U \to N$ is any local defining map for S, then $T_pS = \ker \mathrm{d}\Phi_p\colon T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap U$.*

When the defining function Φ takes its values in $\mathbb{R}^k$, it is useful to restate the proposition in terms of component functions of Φ.

Corollary 5. *Suppose $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi^1, \dots, \Phi^k)\colon M \to \mathbb{R}^k$. A vector $\nu \in T_pM$ is tangent to S if and only if $\nu\Phi^1 = \cdots = \nu\Phi^k = 0$.*