Numerical Relativity

The Initial Data Problem

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Overview

Can't even trust black holes nowadays smh...

Introduction

Equations...Equations ...Equations anyone?

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= \chi[\alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) \\
&- 8\pi\alpha(S_{ij} - \frac{1}{2}\underbrace{\gamma_{ij}}_{\text{no TF}}(S-\rho)) - D_i D_j \alpha \\
&+ \mathcal{L}_{\vec{\beta}} K_{ij}]^{\mathrm{TF}} + \chi^{-1}\bar{A}_{ij}\,[\frac{2}{3}\chi(\alpha K - \partial_i \beta^i) \\
&+ \beta^i \partial_i \chi] \\
&= [\alpha\chi R_{ij} + \alpha\chi\chi^{-1} K\left(\bar{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right) \\
&- 2\alpha\chi\chi^{-1}\left(\bar{A}_{ik} + \frac{1}{3}\bar{\gamma}_{ik}K\right)\left(\bar{A}^k{}_j + \delta^k{}_j K\right)\ldots \\
&\cdots - 8\pi\alpha\chi S_{ij} - \chi D_i D_j \alpha]^{\mathrm{TF}} + \ldots
\end{aligned}
$$

Einstein Field Equations

1915 Relation between Geometry and Matter [Ein15]

$$\underbrace{G_{ab}}_{\text{Geometry}} = 8\pi \underbrace{T_{ab}}_{\text{Matter-Energy}}$$

1952 Can be posed as a Cauchy problem (with ***constraints***!) [Fou52]

- Not trivial. In GR space and time are on equal footing!

1969 Extension from local to "global" (MGHD) [CG69]

- Foliation is possible in *globally hyperbolic* spacetimes [strongest causality condition; physically relevant spacetimes must satisfy it]

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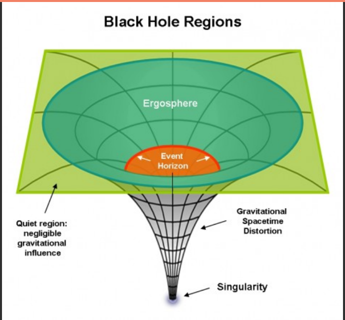
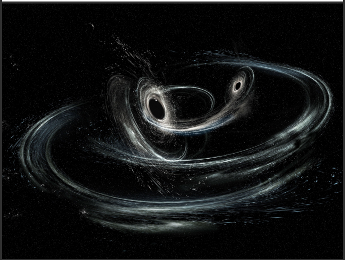
Why numerical solutions? (Symmetry/Asymmetry)



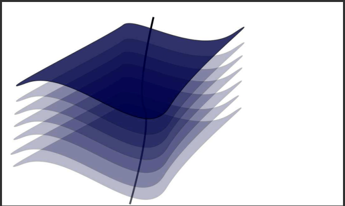


- Symmetries $\implies$ exact solutions to EFE's;
  - Schwarzschild, Kerr, Reissner-Nordström, Kerr-Newman, ...
- No symmetries $\implies$ solutions to EFE's unlikely;
  - Indispensable for study of black hole mergers (GW's!)

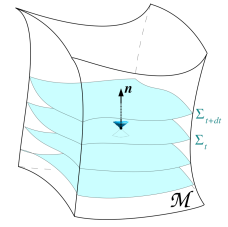
ADM Formalism

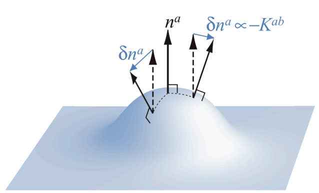
3+1 Approach to Numerical Relativity

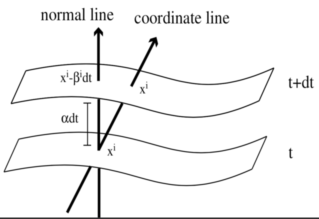
- Formulate EFE's as a Cauchy problem with constraints.
    - Not trivial! In GR space and time are on equal footing ☹
- Foliation is possible in ***globally hyperbolic*** spacetimes.
    - Strongest causality condition (physically relevant spacetimes must satisfy it)



Some Key Ingredients







Some Key Ingredients

- Intrinsic metric & curvature

$$\boxed{\begin{aligned} \gamma_{ab} &= g_{ab} + n_a n_b \\ K_{ab} &= -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d \\ &= -\frac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ab} \end{aligned}}$$

- Projections

$$\boxed{\begin{aligned} \gamma_a{}^e \gamma_b{}^f \gamma_c{}^g \gamma_d{}^h\, {}^{(4)}R_{efgh} &= R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{cb} \\ \gamma_a{}^e \gamma_b{}^f \gamma_c{}^g n^h\, {}^{(4)}R_{efgh} &= D_b K_{ac} - D_a K_{bc} \\ \gamma_a{}^q \gamma_b{}^r n^c n^d\, {}^{(4)}R_{qcrd} &= \mathcal{L}_{\vec{n}} K_{ab} + \frac{1}{\alpha} D_a D_b \alpha + K_b{}^c K_{ac} \end{aligned}}$$

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ADM Equations

- **Evolution Equations:**

$$\boxed{\begin{aligned}\partial_t\gamma_{ij} &= 2D_{(i}\beta_{j)} - 2\alpha K_{ij}\\ \partial_t K_{ij} &= \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right)\\ &\quad - D_iD_j\alpha + \beta^k D_k K_{ij} + 2K_{k(j}D_{i)}\beta^k\end{aligned}}$$

- **Constraint Equations:**

$$\boxed{\begin{aligned}R + K^2 - K_{ij}K^{ij} &= 16\pi\rho\\ D_j\left(K^{ij} - \gamma^{ij}K\right) &= 8\pi S^i\end{aligned}}$$

Problems with ADM

- Straightforward and relatively easy...but practically useless
  - Numerical simulations violently unstable
  - Equations in this form are “weakly hyperbolic”
- “More hyperbolic” form needed
  - use conformal rescaling
  - messier, but effective

BSSN Formalism

Advantages of BSSN

- Messier, but effective! ☺
  - Stable numerical simulations
  - Equations in this form are "more hyperbolic"
- (One of) ***the*** standard(s) in NR

Setting up the stage

Theorem

Two ***strongly causal*** *Lorentzian metrics $g_{ab}^{(1)}$ and $g_{ab}^{(2)}$ for some manifold $\mathcal{M}$ determine the same future and past sets at all points (events) if and only if the two metrics are* ***globally conformal****, i.e., if $g_{ab}^{(1)} = \Psi g_{ab}^{(2)}$, for some smooth function $\Psi \in C^{\infty}(\mathcal{M})$. In this case, both spacetimes $(\mathcal{M}, g_{ab}^{(1)})$ and $(\mathcal{M}, g_{ab}^{(2)})$ belong to the same conformal class and share the same causal structure.*

BSSN Approach

- **BSSN Variables:**

$$\boxed{\{\chi, \bar{\gamma}_{ij}, \bar{A}_{ij}, K, \bar{\Gamma}^i\}}$$

- **Conformal Rescalings:**

$$\boxed{\begin{aligned} \bar{\gamma}_{ij} &= \chi \gamma_{ij} \\ \bar{\gamma} &= 1 \\ \chi &= \gamma^{-1/3} \\ \bar{A}_{ij} &= \chi K_{ij} - \frac{1}{3}\bar{\gamma}_{ij} K \\ \bar{\Gamma}^i &= \bar{\gamma}^{jk}\,\bar{\Gamma}^i_{jk} \\ &= -\partial_j \bar{\gamma}^{ij} \end{aligned}}$$

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BSSN Evolution Eqs

- **Evolution Equations:**

$$\partial_t \chi = \frac{2}{3}\chi(\alpha K - \partial_i \beta^i) + \beta^i \bar{D}_i \chi$$

$$\partial_t \bar{\gamma}_{ij} = -2\alpha \bar{A}_{ij} + \beta^k \partial_k \bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j \beta^k + \bar{\gamma}_{kj}\partial_i \beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k \beta^k$$

$$\partial_t K = \alpha \left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2 \right) + 4\pi\alpha(\rho + S) - D^2\alpha + \beta^i \bar{D}_i K$$

$$\begin{aligned} \partial_t \bar{A}_{ij} = {} & \left[\chi(\alpha R_{ij} - 8\pi\alpha S_{ij} - D_i D_j \alpha)\right]^{\mathrm{TF}} - \alpha(2\bar{A}_{ik}\bar{A}^k{}_j + \bar{A}_{ij}K) \\ & + \beta^k \partial_k \bar{A}_{ij} + \bar{A}_{ik}\partial_j \beta^k + \bar{A}_{kj}\partial_i \beta^k - \frac{2}{3}\bar{A}_{ij}\partial_k \beta^k \end{aligned}$$

$$\begin{aligned} \partial_t \bar{\Gamma}^i = {} & -2\alpha \left(\frac{3}{2\chi}\bar{A}^{ij}\bar{D}_j \chi + \frac{2}{3}\bar{D}^i K + 8\pi \bar{S}^i - \bar{\Gamma}^i_{jk}\bar{A}^{jk} \right) - 2\bar{A}^{ij}\bar{D}_j \alpha \\ & + \beta^j \partial_j \bar{\Gamma}^i + \bar{\gamma}^{jk}\partial_j \partial_k \beta^i - \bar{\Gamma}^j \partial_j \beta^i + \frac{2}{3}\bar{\Gamma}^i \partial_j \beta^j + \frac{1}{3}\bar{\gamma}^{ij}\partial_j \partial_k \beta^k \end{aligned}$$

BSSN Approach (cont)

- **Constraint Equations:**

$$
\boxed{
\begin{aligned}
16\pi\bar{\rho} &= \bar{R} + 2\,\bar{D}^2(\log\chi) - \frac{1}{2}\bar{D}_k(\log\chi)\bar{D}^k(\log\chi) \\
&\quad + \frac{4}{3\chi}K^2 - \frac{1}{\chi}\bar{A}_{ij}\bar{A}^{ij} \\
8\pi\bar{S}^i &= \bar{D}_j\bar{A}^{ij} - \frac{3}{2\chi}\bar{A}^{ij}\,\bar{D}_j\chi - \frac{2}{3}\bar{D}^i K
\end{aligned}
}
$$

Initial Data

Initial Data Problem

Solving the constraints

$$\boxed{\begin{aligned} R + K^2 - K_{ij}K^{ij} &= 16\pi\rho \\ D_j\left(K^{ij} - \gamma^{ij}K\right) &= 8\pi S^i \end{aligned}}$$

is not so trivial.

- there's a total 12 DoF on the system ($\{\gamma_{ij}, K_{ij}\}$);
- the constraints are just 4 equations (thus removing 4 DoF);
- no a priori preference for which eight of the total data to use as ***free*** parameters and which four as ***constrained*** quantities.

York-Lichnerowicz Conformal Transformations

Re-scale the 3-metric γ_{ij} and the (traceless part of) the extrinsic curvature A_{ij}:

$$\boxed{\begin{aligned} \bar{\gamma}_{ij} &= \psi^{-4}\gamma_{ij} \\ \widetilde{A}_{ij} &= \psi^{2}A_{ij} \end{aligned}}$$

where

$$\psi = \chi^{-1/4} = \gamma^{1/12}$$
$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K.$$

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These conformal transformations lead to the ***constraints***

$$\boxed{\begin{aligned}\bar{D}^2\psi + \frac{1}{8}\left(\psi^{-7}\widetilde{A}_{ij}\widetilde{A}^{ij} - \psi\bar{R}\right) - \frac{1}{12}\psi^5K^2 &= -2\pi\psi^{-3}\widetilde{\rho}\\ \bar{D}_j\widetilde{A}^{ij} - \frac{2}{3}\psi^6\bar{D}^iK &= 8\pi\widetilde{S}^i\end{aligned}}$$

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Conformal Transverse-Traceless (CTT) method

$$\boxed{\widetilde{A}^{ij} = \widetilde{A}^{ij}_{\mathrm{L}} + \widetilde{A}^{ij}_{\mathrm{TT}}}$$

with the ***transverse-traceless*** (TT) part of $\widetilde{A}^{ij}$ satisfying

$$\bar{D}_j \widetilde{A}^{ij}_{\mathrm{TT}} = 0 \qquad \text{(Transverse)}$$
$$\bar{\gamma}_{ij} \widetilde{A}^{ij}_{\mathrm{TT}} = 0. \qquad \text{(Traceless)}$$

The ***longitudinal*** (L) part of $\widetilde{A}^{ij}$ is expressed in terms of the ***conformal Killing operator*** $(\bar{\mathbb{L}}X)^{ij}$:

$$\widetilde{A}^{ij}_{\mathrm{L}} \equiv (\bar{\mathbb{L}}X)^{ij} := 2\bar{D}^{(i}X^{j)} - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k X^k.$$

The vector field X is determined from

$$\bar{D}_j(\bar{\mathbb{L}}X)^{ij} = \bar{D}_j\widetilde{A}^{ij}.$$

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The vector field $\boldsymbol{X}$ is determined from the ***conformal vector Laplacian***

$$\bar{\Delta}_{\bar{\mathbb{L}}} X^i \equiv \bar{D}_j(\bar{\mathbb{L}}\boldsymbol{X})^{ij} = \bar{D}_j \widetilde{A}^{ij}.$$

Conformal Transverse-Traceless (CTT) method

Expanding $\bar{\Delta}_{\mathbb{L}} X^i$:

$$\begin{aligned}
\bar{\Delta}_{\mathbb{L}} X^i = \bar{D}_j (\mathbb{L}\boldsymbol{X})^{ij} &= \bar{D}_j \left(\bar{D}^i X^j + \bar{D}^j X^i - \frac{2}{3} \bar{\gamma}^{ij} \bar{D}_k X^k \right) \\
&= \underbrace{\bar{D}_j \bar{D}^i X^j}_{= \bar{R}^i{}_j X^j + \bar{D}^i \bar{D}_j X^j} + \bar{D}^2 X^i - \frac{2}{3} \bar{D}^i \bar{D}_j X^j \\
&= \bar{D}^2 X^i + \bar{R}^i{}_j X^j + \frac{1}{3} \bar{D}^i \bar{D}_j X^j. \qquad \text{(by Ricci identity)}
\end{aligned}$$

Replacing $\bar{\Delta}_{\mathbb{L}} X^i \leftrightarrow \bar{D}_j \widetilde{A}^{ij}$ in the constraint

$$\bar{D}_j \widetilde{A}^{ij} - \frac{2}{3} \psi^6 \bar{D}^i K = 8\pi \widetilde{S}^i,$$

we rewrite the ***momentum constraints*** as

$$\boxed{\bar{D}^2 X^i + \bar{R}^i{}_j X^j + \frac{1}{3} \bar{D}^i \bar{D}_j X^j - \frac{2}{3} \psi^6 \bar{D}^i K = 8\pi \widetilde{S}^i}$$

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Conformal Transverse-Traceless (CTT) method

To sum up, in CTT,

$$\boxed{\begin{aligned} \bar{D}^2\psi + \frac{1}{8}\left(\psi^{-7}\widetilde{A}_{ij}\widetilde{A}^{ij} - \psi\bar{R}\right) - \frac{1}{12}\psi^5K^2 &= -2\pi\psi^{-3}\widetilde{\rho} \\ \bar{D}_j\widetilde{A}^{ij} - \frac{2}{3}\psi^6\bar{D}^iK &= 8\pi\widetilde{S}^i \end{aligned}}$$

becomes

$$\boxed{\begin{aligned} \bar{D}^2\psi + \frac{1}{8}\left(\psi^{-7}\widetilde{A}_{ij}\widetilde{A}^{ij} - \psi\bar{R}\right) - \frac{1}{12}\psi^5K^2 &= -2\pi\psi^{-3}\widetilde{\rho} \\ \bar{D}^2X^i + \bar{R}^i{}_jX^j + \frac{1}{3}\bar{D}^i\bar{D}_jX^j - \frac{2}{3}\psi^6\bar{D}^iK &= 8\pi\widetilde{S}^i \end{aligned}}$$

Conformal Transverse-Traceless (CTT) method

- **CTT approach (12 DoF)**
  - `Constrained Data (4 DoF):` ψ, X^i
  - `Free Data (8 DoF):` $\widetilde{A}^{ij}_{\mathrm{TT}}, \bar{\gamma}_{ij}, K,$ (matter terms (ρ, S^i), if present)

The physical solution is then constructed from

$$
\begin{aligned}
\gamma_{ij} &= \psi^4 \bar{\gamma}_{ij} \\
K_{ij} &= \psi^{-2} \widetilde{A}_{ij} + \frac{1}{3}\gamma_{ij} K.
\end{aligned}
$$

Conformal Thin-Sandwich (CTS) method

- Consider $\bar{\gamma}_{ij}$ on Σ and its vicinity; i.e., $\bar{\gamma}_{ij}$ and $\bar{v}_{ij} \equiv \partial_t \bar{\gamma}_{ij}$
- We'll use (from BSSN)

$$\bar{v}_{ij} \equiv \partial_t \bar{\gamma}_{ij} = 2\bar{D}_{(i}\beta_{j)} - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k\beta^k - 2\alpha\bar{A}_{ij},$$

with

$$\bar{A}_{ij} = \psi^{-4}A_{ij} = \psi^{-6}\widetilde{A}_{ij}.$$

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with

$$\bar{A}_{ij} = \psi^{-4}A_{ij} = \psi^{-6}\widetilde{A}_{ij}.$$

- Thus,

$$\boxed{\widetilde{A}_{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{\mathbb{L}}\boldsymbol{\beta})_{ij} - \bar{v}_{ij}\right)}$$

with the ***densitized lapse***

$$\bar{\alpha} \equiv \psi^{-6}\alpha.$$

Conformal Thin-Sandwich (CTS) method

Expand the divergence $\bar{D}_j \widetilde{A}^{ij}$,

$$\begin{aligned}
\bar{D}_j \widetilde{A}^{ij} &= \bar{D}_j \left[\frac{1}{2\bar{\alpha}} \left((\bar{\mathbb{L}}\boldsymbol{\beta})^{ij} - \bar{v}^{ij} \right) \right] \\
&= \frac{1}{2\bar{\alpha}} \bar{\Delta}_{\mathbb{L}} \beta^i - \frac{1}{2\bar{\alpha}^2} (\bar{\mathbb{L}}\boldsymbol{\beta})^{ij} \bar{D}_j \bar{\alpha} - \frac{1}{2\bar{\alpha}} \bar{D}_j \bar{v}^{ij} + \frac{1}{2\bar{\alpha}^2} \bar{v}^{ij} \bar{D}_j \bar{\alpha} \\
&= \frac{1}{2\bar{\alpha}} \left(\bar{\Delta}_{\mathbb{L}} \beta^i - (\bar{\mathbb{L}}\boldsymbol{\beta})^{ij} \bar{D}_j (\log \bar{\alpha}) - \bar{D}_j \bar{v}^{ij} + \bar{v}^{ij} \bar{D}_j (\log \bar{\alpha}) \right).
\end{aligned}$$

Then plugging in the momentum constraints,

$$\bar{D}_j \widetilde{A}^{ij} - \frac{2}{3} \psi^6 \bar{D}^i K = 8\pi \widetilde{S}^i$$

$$\frac{1}{2\bar{\alpha}} [\bar{\Delta}_{\mathbb{L}} \beta^i - (\bar{\mathbb{L}}\boldsymbol{\beta})^{ij} \bar{D}_j (\log \bar{\alpha}) \underbrace{- \bar{D}_j \bar{v}^{ij} + \bar{v}^{ij} \bar{D}_j (\log \bar{\alpha})}_{= -\bar{\alpha} \bar{D}_j (\bar{\alpha}^{-1} \bar{v}^{ij})}] - \frac{2}{3} \psi^6 \bar{D}^i K = 8\pi \widetilde{S}^i$$

$$\bar{\Delta}_{\mathbb{L}} \beta^i - (\bar{\mathbb{L}}\boldsymbol{\beta})^{ij} \bar{D}_j (\log \bar{\alpha}) - \bar{\alpha} \bar{D}_j (\bar{\alpha}^{-1} \bar{v}^{ij}) - \frac{4}{3} \bar{\alpha} \psi^6 \bar{D}^i K = 16\pi \bar{\alpha} \widetilde{S}^i.$$

Conformal Thin-Sandwich (CTS) method

Expand the divergence $\bar{D}_j\widetilde{A}^{ij}$,

$$\begin{aligned}
\bar{D}_j\widetilde{A}^{ij} &= \bar{D}_j\left[\frac{1}{2\bar{\alpha}}\left((\bar{\mathbb{L}}\boldsymbol{\beta})^{ij} - \bar{v}^{ij}\right)\right] \\
&= \frac{1}{2\bar{\alpha}}\bar{\Delta}_{\mathbb{L}}\beta^i - \frac{1}{2\bar{\alpha}^2}(\bar{\mathbb{L}}\boldsymbol{\beta})^{ij}\bar{D}_j\bar{\alpha} - \frac{1}{2\bar{\alpha}}\bar{D}_j\bar{v}^{ij} + \frac{1}{2\bar{\alpha}^2}\bar{v}^{ij}\bar{D}_j\bar{\alpha} \\
&= \frac{1}{2\bar{\alpha}}\left(\bar{\Delta}_{\mathbb{L}}\beta^i - (\bar{\mathbb{L}}\boldsymbol{\beta})^{ij}\bar{D}_j(\log\bar{\alpha}) - \bar{D}_j\bar{v}^{ij} + \bar{v}^{ij}\bar{D}_j(\log\bar{\alpha})\right).
\end{aligned}$$

Then plugging in the momentum constraints,

$$\bar{D}_j\widetilde{A}^{ij} - \frac{2}{3}\psi^6\bar{D}^iK = 8\pi\widetilde{S}^i$$

$$\frac{1}{2\bar{\alpha}}[\bar{\Delta}_{\mathbb{L}}\beta^i - (\bar{\mathbb{L}}\boldsymbol{\beta})^{ij}\bar{D}_j(\log\bar{\alpha})\underbrace{-\bar{D}_j\bar{v}^{ij} + \bar{v}^{ij}\bar{D}_j(\log\bar{\alpha})]}_{=-\bar{\alpha}\bar{D}_j(\bar{\alpha}^{-1}\bar{v}^{ij})} - \frac{2}{3}\psi^6\bar{D}^iK = 8\pi\widetilde{S}^i$$

$$\bar{\Delta}_{\mathbb{L}}\beta^i - (\bar{\mathbb{L}}\boldsymbol{\beta})^{ij}\bar{D}_j(\log\bar{\alpha}) - \bar{\alpha}\bar{D}_j(\bar{\alpha}^{-1}\bar{v}^{ij}) - \frac{4}{3}\bar{\alpha}\psi^6\bar{D}^iK = 16\pi\bar{\alpha}\widetilde{S}^i.$$

Conformal Thin-Sandwich (CTS) method

Hence, in CTS,

$$\boxed{\begin{aligned} \bar{D}^2\psi + \frac{1}{8}\left(\psi^{-7}\widetilde{A}_{ij}\widetilde{A}^{ij} - \psi\bar{R}\right) - \frac{1}{12}\psi^5 K^2 &= -2\pi\psi^{-3}\widetilde{\rho} \\ \bar{D}_j\widetilde{A}^{ij} - \frac{2}{3}\psi^6\bar{D}^i K &= 8\pi\widetilde{S}^i \end{aligned}}$$

becomes

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Conformal Thin-Sandwich (CTS) method

- **CTS approach (16 DoF)**
  - `Constrained Data (4 DoF)`: ψ, β^i
  - `Free Data (12 DoF)`: $\bar{\gamma}_{ij}, \bar{v}_{ij}, K, \bar{\alpha}$, (matter terms ($\rho, S^i$), if present)

The physical solution is then constructed from

$$
\begin{aligned}
\alpha &= \psi^6 \bar{\alpha} \\
\gamma_{ij} &= \psi^4 \bar{\gamma}_{ij} \\
K_{ij} &= \psi^{-2} \widetilde{A}_{ij} + \frac{1}{3} \gamma_{ij} K.
\end{aligned}
$$

Extended Conformal Thin-Sandwich (XCTS) method

- Replaces CTS's free datum $\bar{\alpha}$ with $\partial_t K$
  - `Constrained Data:` ψ, β^i, α (or, equivalently, $\bar{\alpha}$);
  - `Free Data:` $\bar{\gamma}_{ij}, \bar{v}_{ij}, K, \partial_t K$, (matter terms (ρ, S^i), if present).
- Additional constraint equation

$$\boxed{\begin{aligned}\bar{D}^2(\alpha\psi) = \alpha\psi\left(\frac{7}{8}\psi^{-8}\widetilde{A}_{ij}\widetilde{A}^{ij} + \frac{5}{12}\psi^4 K^2 + \frac{1}{8}\bar{R} + 2\pi\psi^{-4}(2\widetilde{S} + \widetilde{\rho})\right) \\ + \psi^5\left(\beta^i \bar{D}_i K - \partial_t K\right)\end{aligned}}$$

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The slippery slope of having $\bar{\gamma}_{ij}$ as free data

- Conformally-related metric $\bar{\gamma}_{ij}$ as free data
  - flat? ($\bar{\gamma}_{ij} = f_{ij}$) $\longrightarrow$ spurious GW bursts at start of sim
  - Kerr-Schild-like? ($\bar{\gamma}_{ij} = f_{ij} + 2H\ell_i\ell_j$)
  - more control over $\bar{\gamma}_{ij}$ in CTS than in CTT, as $\partial_t\bar{\gamma}_{ij}$ is also free
  - even more control in XCTS ($\partial_t\bar{\gamma}_{ij}$ *and* $\partial_t K$ are free)
    - setting $\partial_t\bar{\gamma}_{ij} = \partial_t K = 0$ is great for equilibrium/quasi-equilibrium settings
- Waveless Approximation ([SUF04])
  - replaces $\bar{\gamma}_{ij} \leftrightarrow \partial_t \grave{A}_{ij}$ as free data
  - impose helical symmetry of $\partial_t\bar{\gamma}_{ij}$ & $\partial_t \grave{A}_{ij}$ in the near zone, and set them $= 0$ in the far zone. (Very accurate for binaries!)

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THANK YOU!