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Final Problem Set
Numerical Methods for PDEs

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Problem 1. *a) Consider the piecewise linear function $f(x)$ on $[0,1]$ given by*

$$f(x) = \begin{cases} 8x, & \text{if } 0 \le x \le 0.25 \\ -2x + 2.5, & \text{if } 0.25 \le x \le 0.5 \\ -4x + 3.5, & \text{if } 0.5 \le x \le 0.75 \\ 6x - 4, & \text{if } 0.75 \le x \le 1. \end{cases}$$

Write $f(x)$ as a linear combination of the standard "hat" basis functions on the given partition of $[0,1]$. Include the expressions of the hat basis functions. **Hint**: *Note that since $f(1) \neq 0$, you will also need an additional basis function ("half of a hat", based at $x = 1$) to the standard "hat" functions to represent the function.*

b) Let $0 = x_0 < x_1 < x_2 < x_3 = 1$, where $x_1 = 1/2$ and $x_2 = 3/4$, be a (non-uniform) partition of the interval $[0,1]$ into three subintervals, and let V_h be the space of continuous, piecewise linear functions on this partition that vanish at end-points $x = 0$ and $x = 1$.

i. Find the stiffness matrix K whose entries are given by $K_{ij} = \int_0^1 \phi_j'(x)\phi_i'(x)\,dx$ for $i,j = 1,2$.

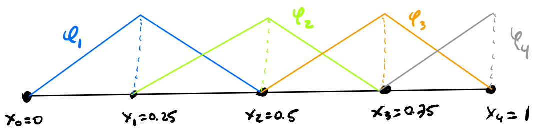
ii. Find the load vector $\boldsymbol{F}$, with $f(x) = 1$, whose entries are given by $F_i = \int_0^1 f(x)\phi_i(x)\,dx$ for $i = 1,2$.

iii. Solve the linear system $K\boldsymbol{U} = \boldsymbol{F}$ where $\boldsymbol{U}$ is a vector of nodal values of the finite element solution u_h at the interior nodes. Plot the finite element solution u_h.

Do the computations in parts (a) and (b) by hand.

Solution to a). The hat functions $\{\varphi_j\}$ at the nodes $\{x_i\}$ are defined by

$$\varphi_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$



Setting $h = 0.25$ and writing $I_i = [x_{i-1}, x_i]$, we have, for $i \in \{1,2,3,4\}$,

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & \text{if } x \in I_i; \\ \frac{x_{i+1} - x}{h} & \text{if } x \in I_{i+1}; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then,

$$\begin{aligned} f(x) &\approx \sum_{i=1}^{4} f(x_i)\varphi_i(x) \\ &= f(0.25)\varphi_1(x) + f(0.5)\varphi_2(x) + f(0.75)\varphi_3(x) + f(1)\varphi_4(x) \\ &= [8(0.25)]\,\varphi_1 + [-2(0.5) + 2.5]\,\varphi_2 + [-4(0.75) + 3.5]\,\varphi_3 + [6(1) - 4]\,\varphi_4 \\ &= 2\varphi_1 + 1.5\varphi_2 + 0.5\varphi_3 + 2\varphi_4, \end{aligned}$$

where

$$\varphi_1(x) = \begin{cases} \frac{x}{0.25} & \text{if } x \in [0, 0.25], \\ \frac{0.5 - x}{0.25} & \text{if } x \in [0.25, 0.5], \\ 0 & \text{otherwise;} \end{cases}$$

$$\varphi_2(x) = \begin{cases} \frac{x-0.25}{0.25} & \text{if } x \in [0.25, 0.5], \\ \frac{0.75-x}{0.25} & \text{if } x \in [0.5, 0.75], \\ 0 & \text{otherwise}; \end{cases}$$

$$\varphi_3(x) = \begin{cases} \frac{x-0.5}{0.25} & \text{if } x \in [0.5, 0.75], \\ \frac{1-x}{0.25} & \text{if } x \in [0.75, 1], \\ 0 & \text{otherwise}; \end{cases}$$

$$\varphi_4(x) = \begin{cases} \frac{x-0.75}{0.25} & \text{if } x \in [0.75, 1], \\ 0 & \text{otherwise}. \end{cases}$$

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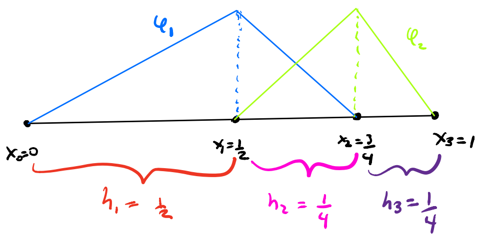
Solution to b). Using the same notation as before for the subintervals $I_i = [x_{i-1}, x_i]$, but now using nonuniform spacing $h_i = x_i - x_{i-1}$, we get from Eq. (1)

$$\varphi_i'(x) = \begin{cases} \frac{1}{h_i} & \text{if } x \in I_i; \\ -\frac{1}{h_{i+1}} & \text{if } x \in I_{i+1}; \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$



Even though numerical quadrature is not a necessity for the integrals in this problem, we shall use Simpson's quadrature; recall that on an interval $I = [x_{i-1}, x_i]$, Simpson's method takes the form

$$\int_I f \approx \frac{h_i}{6} \left[f(x_{i-1}) + 4f(m_i) + f(x_i) \right],$$

where x_m is the midpoint $m_i = \frac{1}{2}(x_i + x_{i-1})$. Furthermore, in the following derivation we will write the entries of the stiffness matrix in the more general case where there is a nontrivial function $k(x)$; then we will set this function to unity at the end of the derivation. Whence, without further ado, the diagonal entries of $\boldsymbol{K}$ are then given by

$$\begin{aligned} K_{ii} &= \int_{[0,1]} k \left(\varphi_i'\right)^2 \mathrm{d}x \\ &= \int_{x_{i-1}}^{x_i} k \left(\varphi_i'\right)^2 \mathrm{d}x + \int_{x_i}^{x_{i+1}} k \left(\varphi_i'\right)^2 \mathrm{d}x \\ &= \frac{h_i}{6} \left[k(x_{i-1}) \cdot \left(\frac{1}{h_i}\right)^2 + 4 \cdot k(m_i) \cdot \left(\frac{1}{h_i}\right)^2 + k(x_i) \cdot \left(\frac{1}{h_i}\right)^2 \right] \\ &+ \frac{h_{i+1}}{6} \left[k(x_i) \cdot \left(-\frac{1}{h_{i+1}}\right)^2 + 4 \cdot k(m_{i+1}) \cdot \left(-\frac{1}{h_{i+1}}\right)^2 + k(x_{i+1}) \cdot \left(-\frac{1}{h_{i+1}}\right)^2 \right] \\ &= \frac{1}{6h_i} \left[k(x_{i-1}) + 4k(m_i) + k(x_i) \right] + \frac{1}{6h_{i+1}} \left[k(x_i) + 4k(m_{i+1}) + k(x_{i+1}) \right]. \end{aligned}$$

But, since $k(x) \equiv 1$, we end up with

$$\boxed{K_{ii} = \frac{1}{h_i} + \frac{1}{h_{i+1}}.} \tag{3}$$

Similarly, for the subdiagonal entries,

$$
\begin{aligned}
K_{i+1,i} &= \int_{[0,1]} k\varphi_i'\varphi_{i+1}'\,\mathrm{d}x \\
&= \int_{x_i}^{x_{i+1}} k\varphi_i'\varphi_{i+1}'\,\mathrm{d}x \\
&= \frac{h_{i+1}}{6}\left[k(x_i)\cdot\left(-\frac{1}{h_{i+1}}\right)\left(\frac{1}{h_{i+1}}\right) + 4\cdot k(m_{i+1})\cdot\left(-\frac{1}{h_{i+1}}\right)\left(\frac{1}{h_{i+1}}\right) + k(x_{i+1})\cdot\left(-\frac{1}{h_{i+1}}\right)\left(\frac{1}{h_{i+1}}\right)\right] \\
&= -\frac{1}{6h_{i+1}}\left[k(x_i) + 4k(m_{i+1}) + k(x_{i+1})\right].
\end{aligned}
$$

Thus, using $k(x) \equiv 1$, we have

$$\boxed{K_{i+1,i} = -\tfrac{1}{h_{i+1}}.} \tag{4}$$

By symmetry, $K_{i+1,i} = K_{i,i+1}$, so we don't need another calculation. Now, for the load vector $\boldsymbol{F}$:

$$
\begin{aligned}
F_i &= \int_{[0,1]} f\varphi_i\,\mathrm{d}x \\
&= \int_{x_{i-1}}^{x_i} f\varphi_i\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} f\varphi_i\,\mathrm{d}x \\
&= \frac{h_i}{6}\left[f(x_{i-1})\,\varphi_i(x_{i-1}) + 4f(m_i)\,\varphi_i(m_i) + f(x_i)\varphi_i(x_i)\right] \\
&\quad + \frac{h_{i+1}}{6}\left[f(x_i)\,\varphi_i(x_i) + 4f(m_{i+1})\,\varphi_i(m_{i+1}) + f(x_{i+1})\varphi_i(x_{i+1})\right] \\
&= \frac{h_i}{6}\left[f(x_{i-1})\cdot 0 + 4f(m_i)\cdot\left(\frac{1}{2}\right) + f(x_i)\cdot 1\right] \\
&\quad + \frac{h_{i+1}}{6}\left[f(x_i)\cdot 1 + 4f(m_{i+1})\cdot\left(\frac{1}{2}\right) + f(x_{i+1})\cdot 0\right] \\
&= \frac{h_i}{6}\left[2f(m_i) + f(x_i)\right] + \frac{h_{i+1}}{6}\left[f(x_i) + 2f(m_{i+1})\right].
\end{aligned}
$$

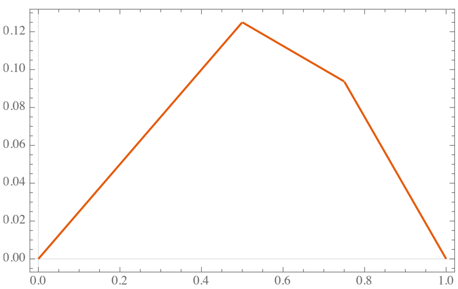
Then, since $f(x) \equiv 1$, we end up with

$$\boxed{F_i = \tfrac{h_i + h_{i+1}}{2}.} \tag{5}$$

Hence we have the system

$$
\begin{aligned}
\underbrace{\begin{bmatrix} \frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{4}} & -\frac{1}{\frac{1}{4}} \\ -\frac{1}{\frac{1}{4}} & \frac{1}{\frac{1}{4}} + \frac{1}{\frac{1}{4}} \end{bmatrix}}_{\boldsymbol{K}}
\underbrace{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}}_{\boldsymbol{U}}
&= \underbrace{\begin{bmatrix} \frac{\frac{1}{2}+\frac{1}{4}}{2} \\ \frac{\frac{1}{4}+\frac{1}{4}}{2} \end{bmatrix}}_{\boldsymbol{F}} \\
\begin{bmatrix} 6 & -4 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} &= \begin{bmatrix} \frac{3}{8} \\ \frac{1}{4} \end{bmatrix} \\
\implies \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{8} \\ \frac{3}{32} \end{bmatrix}.
\end{aligned}
$$

Finally, here is the plot of $u_h(x) \approx U_1\varphi_1(x) + U_2\varphi_2(x)$:



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Problem 2. *Consider the elliptic boundary value problem*

$$\begin{aligned} -\nabla \cdot (c(x,y)\nabla u) + a(x,y)u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where $\Omega = [-1,1]^2$.

a) *Show the derivation of the weak formulation of the problem using test function (and solution) space* $V = H_0^1(\Omega)$.

b) *Modify the MATLAB script* `SolvePDE.m` *and use it solve the problem for* $c = 1$, $a = 4 + xy$, *and* $f = 5e^y \cos\left(\frac{3}{2}x\right)$. *Use* `Hmax` *value anywhere between* 0.025 *and* 0.05 *when generating the finite element mesh. Plot the approximate solution and the finite element mesh.*

c) *Modify the MATLAB script* `EstimateError.m` *to compute the in* L^∞ *norm errors and numerical order of convergence for the problem with* $c = 1$, $a = x + y$, *and* $f = (10\pi^2 + x + y)\sin(\pi x)\sin(3\pi y)$. *Exact solution of this problem is given by* $u = \sin(\pi x)\sin(3\pi y)$. *Use* `Hmax` *values of* 0.2, 0.1, 0.05, 0.025, 0.0125 *and list the corresponding errors as well as approximate orders of convergence in a table.*

Solution to a). To simplify the notation we drop the explicit dependence on (x,y):

$$-\nabla \cdot (c\nabla u) + au = f. \tag{6}$$

We will make use of test functions from the space

$$H_0^1(\Omega) = \left\{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega),\ v|_{\partial\Omega} = 0 \right\}. \tag{7}$$

Multipliying Eq. (6) by $v \in H_0^1(\Omega)$ and integrating, we get

$$-\int_\Omega \nabla \cdot (c\nabla u)v + \int_\Omega auv = \int_\Omega fv. \tag{8}$$

We notice that an application of the product rule yields

$$c\nabla uv\Big|_{\partial\Omega} = \int_\Omega \nabla \cdot (c\nabla u\, v) = \int_\Omega \nabla \cdot (c\nabla u)v + \int_\Omega c\nabla u \nabla v. \tag{9}$$

But the term $c\nabla uv|_{\partial\Omega}$ vanishes due to the condition $v|_{\partial\Omega} = 0$. Thus, substituting back into Eq. (8), we get the weak form of the BVP

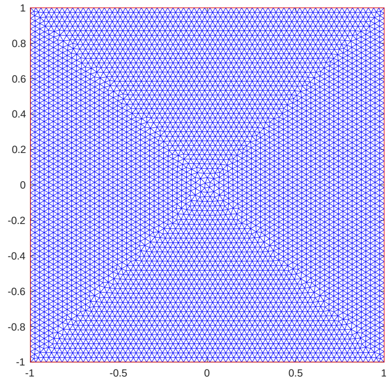
$$\boxed{\int_\Omega (c\nabla u\nabla v + auv) = \int_\Omega fv.} \tag{10}$$

This form has the advantage of having first-order gradients, as opposed to the original form which had second-order derivatives. ♠

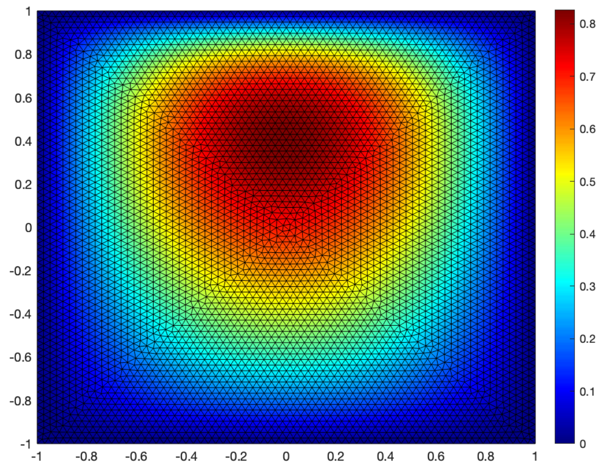
Solution to b). The only lines that needed to be modified were

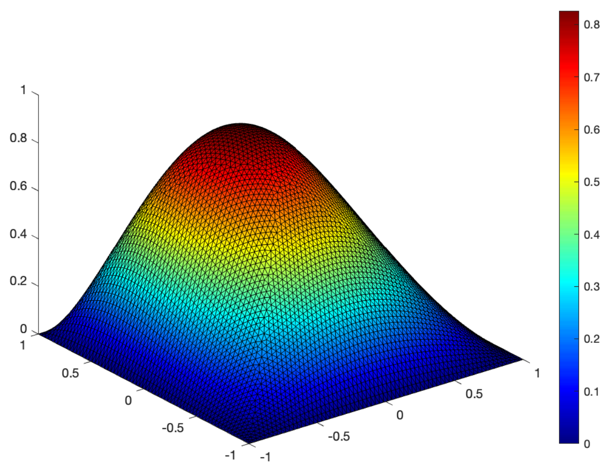
```
c = 1;

%function a (a = 4 + xy)
a = @(location,state) 4 + (location.x .* location.y);

%function f (f = 5 e^y cos((3/2)x)
f = @(location,state) 5 .* exp(location.y) .* cos( (3/2) .* location.x );
```

Using `Hmax` = 0.03 and the above modifications, I got the resulting mesh



as well as the contour and surface plots





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Solution to c). The modification is trivial and similar to the one in part b). The following screenshot shows the output of the code, which includes the desired errors (and order) table:

```
>> EstimateError2_mod
Solve -div (grad u) = f in square [-1,1]^2, u = 0 on the boundary
Computing the approximate solution with hmax = 0.2 ...
Error: 0.1802446, Number of triangles: 228
Computing the approximate solution with hmax = 0.1 ...
Error: 0.043729493, Number of triangles: 904
Computing the approximate solution with hmax = 0.05 ...
Error: 0.011729217, Number of triangles: 3652
Computing the approximate solution with hmax = 0.025 ...
Error: 0.0026786648, Number of triangles: 14672
Computing the approximate solution with hmax = 0.0125 ...
Error: 0.00073830994, Number of triangles: 58548
approximate order of convergence
h = 0.2, order: n/a
h = 0.1, order 2.043
h = 0.05, order 1.898
h = 0.025, order 2.131
h = 0.0125, order 1.859
```

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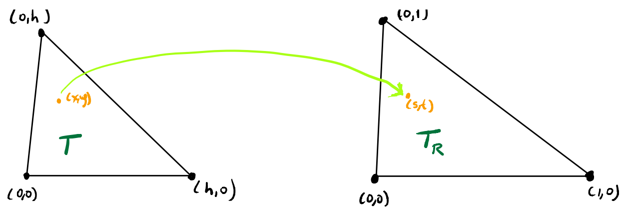
Problem 3. *Consider the triangle T with vertices $(0,0)$, $(h,0)$, and $(0,h)$.*

a) Find the linear basis functions $\psi_1(x,y)$, $\psi_1(x,y)$, and $\psi_1(x,y)$ on T.

b) Show that the element stiffness matrix with entries $K_{ij} = \int_T \nabla\psi_i \cdot \nabla\psi_j$ is given by

$$K = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}.$$



Solution to a). We translate from triangle T to the reference triangle T_R, as in the figure. Then, we have

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} h-0 & 0-0 \\ 0-0 & h-0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}}_{:=J} \begin{bmatrix} s \\ t \end{bmatrix}. \end{aligned} \tag{11}$$

Given the Jacobian J that we just defined on the last equality, let us write its inverse transpose, since we shall need it:

$$J^{-\top} = \begin{bmatrix} 1/h & 0 \\ 0 & 1/h \end{bmatrix}. \tag{12}$$

The linear basis functions $\gamma_i(s,t)$ from the triangle T_R are

$$\gamma_1 = 1 - s - t, \qquad \gamma_2 = s, \qquad \gamma_3 = t. \tag{13}$$

Moreover, from Eq. (11) we have

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1/h & 0 \\ 0 & 1/h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{14}$$

Hence we recover the linear basis functions $\psi_i(x,y)$ on the triangle T:

$$\psi_1 = 1 - \frac{x}{h} - \frac{y}{h}, \qquad \psi_2 = \frac{x}{h}, \qquad \psi_3 = \frac{y}{h}. \qquad \spadesuit \tag{15}$$

Solution to b). From Eq. (15) we get the gradients

$$\nabla\psi_1 = \begin{bmatrix} -1/h \\ -1/h \end{bmatrix}, \qquad \nabla\psi_2 = \begin{bmatrix} 1/h \\ 0 \end{bmatrix}, \qquad \nabla\psi_3 = \begin{bmatrix} 0 \\ 1/h \end{bmatrix}. \tag{16}$$

We can now compute that elements of the stiffness matrix. Starting with K_{11}:

$$\begin{aligned}
K_{11} &= \int_T \nabla\psi_1 \cdot \nabla\psi_1 \\
&= \int_T \begin{bmatrix} -1/h \\ -1/h \end{bmatrix} \cdot \begin{bmatrix} -1/h \\ -1/h \end{bmatrix} \\
&= \int_T \frac{2}{h^2} \\
&= \frac{2}{h^2} \cdot \frac{h^2}{2} \\
&= 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
K_{12} &= \int_T \nabla\psi_1 \cdot \nabla\psi_2 \\
&= \int_T \begin{bmatrix} -1/h \\ -1/h \end{bmatrix} \cdot \begin{bmatrix} 1/h \\ 0 \end{bmatrix} \\
&= -\int_T \frac{1}{h^2} \\
&= -\frac{1}{h^2} \cdot \frac{h^2}{2} \\
&= -\frac{1}{2} \\
&= K_{21} = K_{31} = K_{13}.
\end{aligned}$$

$$\begin{aligned}
K_{22} &= \int_T \nabla\psi_2 \cdot \nabla\psi_2 \\
&= \int_T \begin{bmatrix} 1/h \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/h \\ 0 \end{bmatrix} \\
&= \int_T \frac{1}{h^2} \\
&= \frac{1}{h^2} \cdot \frac{h^2}{2} \\
&= \frac{1}{2} \\
&= K_{33}.
\end{aligned}$$

$$\begin{aligned}
K_{23} &= \int_T \nabla\psi_2 \cdot \nabla\psi_3 \\
&= \int_T \begin{bmatrix} 1/h \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/h \end{bmatrix} \\
&= \int_T 0 \\
&= 0 \\
&= K_{32}.
\end{aligned}$$

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