Complex Variables Notes

Mario L. Gutierrez Abed

» Things to keep in mind:

- $\operatorname{Re}(z) = x = \dfrac{z+\bar{z}}{2}$
- $\operatorname{Im}(z) = y = \dfrac{z-\bar{z}}{2i}$
- $\cos(x) = \dfrac{e^{ix}+e^{-ix}}{2}$
- $\sin(x) = \dfrac{e^{ix}-e^{-ix}}{2i}$
- $\cosh(x) = \dfrac{e^{x}+e^{-x}}{2}$
- $\sinh(x) = \dfrac{e^{x}-e^{-x}}{2}$
- $\cosh(x) = \cos(i\,x)$
- $i\sinh(x) = \sin(i\,x)$
- $|z|^2 = x^2 + y^2 = z\,\bar{z}$
- We know that $i = e^{\frac{\pi}{2}i}$. Hence if we had to evaluate i^i, that would be $\left(e^{\frac{\pi}{2}i}\right)^i = e^{-\pi/2}$.
- $(z^2+a^2)^2 = (z+a\,i)^2\,(z-a\,i)^2$ **this is from the Fundamental Theorem of Algebra**

» Triangle Inequality:

For any $z_1,\ z_2 \in \mathbb{C}$, we have

$$\big|\,|z_1| - |z_2|\,\big| \le |z_1 + z_2| \le |z_1| + |z_2|\,.$$

» Ratio Test for Convergence of Power Series:

A power series converges for all values of z where :

$$\operatorname*{limit}_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |z - z_0| < 1$$

By letting $|z - z_0| = R$, where R is the radius of convergence of the power series we have

$$R = \operatorname*{limit}_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

» Taylor's series:

Let $f(z)$ be analytic in $|z| \leq R$. Then a power series for $f(z)$ about the point $z = z_0$, is given by

$$f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j, \text{ where } b_j = \frac{f^{(j)}(z_0)}{j!}.$$

» Cauchy's Integral Formula:

Let $f(z)$ be analytic interior to and on a simple closed contour C. Then at any interior point $z = a$ we have

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.$$

Extension of this theorem

Let $f(z)$ be analytic interior to and on a simple closed contour C, then $f^{(k)}(z)$, $k = 1, 2, \ldots$ exists in the domain D interior to C and

$$f^{(k)}(a) = \frac{k!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{k+1}}\, dz.$$

» Isolated Singularities:

Suppose $f(z)$ (or any single-valued branch of $f(z)$, if $f(z)$ is multivalued) is analytic in the region $0 < |z - z_0| < R$ (i.e., in a neighborhood of $z = z_0$), but not at the point z_0. Then the point $z = z_0$ is called an isolated singular point of $f(z)$.

An isolated singular point (or isolated singularity) at z_0 of $f(z)$ is said to be a pole if $f(z)$ has the following representation:

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

where m is a positive integer, $m \geq 1$, $\phi(z)$ is analytic in a neighborhood of z_0, and $\phi(z_0) \neq 0$. We generally say that $f(z)$ has an m^{th}-order pole if $m \geq 2$ and it has a simple pole if $m = 1$.

If $f(z)$ is analytic in the region $0 < |z - z_0| < R$ (but not at z_0), and if $f(z)$ can be made analytic at $z = z_0$ by assigning an appropriate value for $f(z_0)$, then $z = z_0$ is called a removable singularity.

An isolated singular point that is neither removable nor a pole is called an essential singular point.

» Calculating the Residue of a Function that has a Pole:

If $f(z)$ has an essential singular point at $z = z_0$, then expansion in terms of a Laurent series is the only general method to evaluate the residue.

If, however, $f(z)$ has a pole in the neighborhood of z_0, then let $f(z)$ have the representation of a function with a pole at z_0, i.e. $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, where m is a positive integer and $\phi(z)$ is analytic in the neighborhood of $z = z_0$. If $\phi(z_0) \neq 0$, then f is said to have a pole of order m (in the case that $\phi(z_0) = 0$, f has a pole of order $m-1$). Then the residue of $f(z)$ at z_0 is given by

$$\text{Res}(f(z); z_0) = \frac{1}{(m-1)!} \left(\frac{d^{m-1}}{dz^{m-1}} \phi(z) \right) \Bigg|_{z=z_0}$$

or

$$\text{Res}(f(z); z_0) = \frac{1}{(m-1)!} \left(\frac{d^{m-1}}{dz^{m-1}} ((z-z_0)^m f(z)) \right) \Bigg|_{z=z_0}$$

In the case that we have a simple pole ($m = 1$), the formula above simplifies to the following:

$$\text{Res}(f(z); z_0) = \lim_{z \to z_0} ((z-z_0) f(z)).$$

Alternatively, given that $f(z)$ has a simple pole ($m = 1$) and it has the form $f(z) = \frac{N(z)}{D(z)}$, where $N(z)$ and $D(z)$ are both analytic and $D(z)$ has a simple zero at $z = z_0$ and $N(z) \neq 0$, it is often convenient to use the following formula instead of the limit:

$$\text{Res}(f(z); z_0) = \lim_{z \to z_0}((z-z_0) f(z)) = \frac{N(z_0)}{D'(z_0)}$$

Either method yields the same result, though often times it's simpler to compute $\frac{N(z_0)}{D'(z_0)}$ than it is to find $\lim_{z \to z_0}((z-z_0) f(z))$.

» Fourier Transform:

The Fourier transform of a real valued function $f(x)$ is another function $\hat{F}(k)$ (where k is a real variable) given by:

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-i k x}\, dx.$$

» Inverse Fourier Transform:

The inverse Fourier transform of a function $\hat{F}(k)$ is given by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(k)\, e^{i k x}\, dk.$$

• Theorem (Conformal Maps):

Assume that $f(z)$ is analytic and nonconstant in a domain D of the complex z plane. For any point $z \in D$ for which $f'(z) \neq 0$, this mapping is conformal, that is, it preserves the angle between two differentiable arcs.