MATH 709 HW # 8

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Problem 1 (Problem 6-1). *Use Proposition 6.5*[1] *to give a simple proof (that does not use Sard's theorem) of the corollary that says that "if M and N are smooth manifolds (with or without boundary), and $F\colon M \to N$ is a smooth map, then if $\dim M < \dim N$, $F(M)$ has measure zero in N." (Hint: given a smooth map $F\colon M \to N$, define a suitable map from $M \times \mathbb{R}^k$ to N, where $k = \dim N - \dim M$).*

Proof. Let $n = \dim N$. For $k = \dim N - \dim M$, define $\widetilde{F}\colon M \times \mathbb{R}^k \to N$ by $(x, y) \mapsto F(x)$ for all $x \in M$ and $y \in \mathbb{R}^k$. This map is smooth because it can be written as the composition $\pi \circ (F \times \mathrm{Id}_{\mathbb{R}^k})$, where $\pi\colon N \times \mathbb{R}^k \to N$ is the canonical projection. It suffices then to show that the image of $\widetilde{F}$ has measure zero in N.

Let $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ and (V, ψ) be smooth charts for $M \times \mathbb{R}^k$ and N, respectively, such that $U_1 \subseteq M$ and $U_2 \subseteq \mathbb{R}^k$. To simplify notation a bit, let $U = U_1 \times U_2$ and $\varphi = \varphi_1 \times \varphi_2$. Then we have

$$\begin{aligned}\psi\left(\widetilde{F}(U) \cap V\right) &= \psi\left(\widetilde{F}\left(U \cap \widetilde{F}^{-1}(V)\right)\right)\\ &= \left(\psi \circ \widetilde{F} \circ \varphi^{-1}\right)\left(\varphi\left(U \cap \widetilde{F}^{-1}(V)\right)\right)\\ &\subseteq \left(\psi \circ \widetilde{F} \circ \varphi^{-1}\right)\left(\varphi(U) \cap \left(\varphi_1\left(\widetilde{F}^{-1}(V)\right) \times \{0\}\right)\right),\end{aligned}$$

which has measure zero in $\mathbb{R}^n$ by Proposition 6.5. Therefore $\widetilde{F}(U)$ has measure zero. But $\widetilde{F}(M \times \mathbb{R}^k)$ can be written as a union of countably many sets of this form, so $\widetilde{F}(M \times \mathbb{R}^k) = F(M)$ has measure zero in N, as desired. □

Problem 2 (Problem 6-9). *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map $F(x, y) = (e^y \cos x, e^y \sin x, e^{-y})$. For which positive numbers r is F transverse to the sphere $S_r(0) \subseteq \mathbb{R}^3$? For which positive numbers r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?*

Solution. Note that we have $\|F(x, y)\| = r$ only when

$$r^2 = e^{2y}\cos^2 x + e^{2y}\sin^2 x + e^{-2y} = e^{2y} + e^{-2y};$$

that is, when

$$y = \frac{1}{2}\log\left(\frac{1}{2}\left(r^2 \pm \sqrt{r^4 - 4}\right)\right).$$

Hence $\|F(x, y)\| \geq \sqrt{2}\ \forall (x, y) \in \mathbb{R}^2$, and therefore F is trivially transverse to S_r for $r \in [0, \sqrt{2})$ ($F^{-1}(S_r(0))$ is empty in this case). Now, for every $(x, y) \in \mathbb{R}^2$, we must have $F(x, y) \perp T_{F(x,y)} S_r$

[1] The proposition states the following:
If $A \subseteq \mathbb{R}^n$ has measure zero and $F\colon A \to \mathbb{R}^n$ is a smooth map, then $F(A)$ has measure zero.

if $F(x,y) \in S_r$. Then we compute the inner product of $F(x,y)$ with the columns of $DF(x,y)$:

$$(e^y \cos x, e^y \sin x, e^{-y}) \cdot \begin{pmatrix} -e^y \sin x \\ e^y \cos x \\ 0 \end{pmatrix} = 0 \qquad \text{and} \qquad (e^y \cos x, e^y \sin x, e^{-y}) \cdot \begin{pmatrix} e^y \cos x \\ e^y \sin x \\ -e^{-y} \end{pmatrix} = e^{2y} - e^{-2y}.$$

The second inner product is zero only when $y = 0$, which is the case only when $\|F(x,y)\| = \sqrt{2}$. Hence, since $T_{F(x,y)}S_r$ and $DF_{(x,y)}(T_{(x,y)}\mathbb{R}^2)$ span $T_{F(x,y)}\mathbb{R}^3$ only when $r \neq \sqrt{2}$, we conclude that F is transverse to S_r for $r \in [0, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

Now, by part a) of *Theorem 6.30*[2] from the text, we are guaranteed that $F^{-1}(S_r(0))$ is an embedded submanifold of $\mathbb{R}^2$ for $r \in [0, \sqrt{2}) \cup (\sqrt{2}, \infty)$; so we only need to check when $r = \sqrt{2}$. But note that in this case $F^{-1}(S_{\sqrt{2}}(0))$ is just a line in $\mathbb{R}^2$, so it is an embedded submanifold as well. $\square$

[2] Here's the statement, for reference:
Suppose N and M are smooth manifolds and $S \subseteq M$ is an embedded submanifold. If $F\colon N \to M$ is a smooth map that is transverse to S, then $F^{-1}(S)$ is an embedded submanifold of N whose codimension is equal to the codimension of S in M.