Math 353 HW 12

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Section 4.5

(6) Suppose we are given the differential equation $\frac{d^2u}{dx^2} - \omega^2\, u = -f(x)$, with $u(x = \pm\infty) = 0, \;\; \omega > 0$. Then,

a) Take the Fourier transform of this equation to find $\hat{U}(k) = \frac{\hat{F}(k)}{k^2+\omega^2}$, where $\hat{U}(k)$ and $\hat{F}(k)$ are the Fourier transform of $u(x)$ and $f(x)$, respectively.

Solution:

$$-f(x) = \frac{d^2u}{dx^2} - \omega^2\, u \Longrightarrow -\hat{F}(k) = (i\, k)^2\, \hat{U}(k) - \omega^2\, \hat{U}(k)$$

$$\Longrightarrow -\hat{F}(k) = \hat{U}(k)\,(-k^2 - \omega^2) \Longrightarrow \hat{U}(k) = \frac{\hat{F}(k)}{k^2+\omega^2}.$$

b) Use the convolution product to deduce that

$$u(x) = \frac{1}{2\,\omega} \int_{\infty}^{\infty} e^{-\omega\,|x-\zeta|}\, f(\zeta)\, d\zeta$$

and thereby obtain the solution to the differential equation.

Solution:

From part a) we have that $\hat{U}(k) = \frac{\hat{F}(k)}{k^2+\omega^2}$, where $\hat{G}(k) = \frac{1}{k^2+\omega^2}$ is the Fourier transform of the function $g(x) = \frac{1}{2\,\omega}\, e^{-w\,|x|}$ (we calculated this transform on HW#11).

From this information we see that $\hat{U}(k) = \hat{F}(k)\, \hat{G}(k)$.

This means that

$$u(x) = (f * g)(x) = \int_{\infty}^{\infty} f(\zeta)\, g(\zeta)\, d\zeta = \frac{1}{2\,\omega} \int_{\infty}^{\infty} e^{-\omega\,|x-\zeta|}\, f(\zeta)\, d\zeta\,.$$