Math 351 Assignment 1

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Let $A,\ B \subset \mathbb{R}$ be nonempty.

(1) Define $A+B=\{x+y : x \in A\ ,\ y \in B\}$. Compute $\sup(A+B)$ in terms of $\sup A$ and $\sup B$. Repeat exercise for $\inf(A+B)$. Justify your answer.

<u>Solution:</u>
We know that $\mathbb{R}$ has the least upper bound property, according to the existence theorem. In addition, the sets A and B are both defined to be nonempty proper subsets of $\mathbb{R}$. Thus, we are guaranteed that there is some element $\alpha \in \mathbb{R}$ that satisfies $x \leq \alpha \quad \forall\ x \in A$, i.e. $\alpha = \sup A$. Similarly, there must be some $\beta \in \mathbb{R}$ such that $y \leq \beta \quad \forall\ y \in B$, that is $\beta = \sup B$. It follows then, that $\sup A + \sup B = \alpha + \beta$ is an upper bound for $\sup(A+B)$. That is, for any $x \in A,\ \ y \in B$, we have $x + y \leq \alpha + \beta$.
Now, if we choose any $\varepsilon > 0$, we have

$$\alpha + \beta - \varepsilon = \left(\alpha - \tfrac{\varepsilon}{2}\right) + \left(\beta - \tfrac{\varepsilon}{2}\right)$$

Thus, since $\alpha - \frac{\varepsilon}{2}$ is not an upper bound for A and $\beta - \frac{\varepsilon}{2}$ is not an upper bound for B, it must be true that $\alpha + \beta - \varepsilon$ is not an upper bound for $A+B$. Thus, we have proven that $\alpha + \beta$ is in fact the least upper bound for $A+B$.
That is, $\sup A + \sup B = \alpha + \beta = \sup(A+B)$. ✓

Now let $\inf A = \gamma,\ \ \inf B = \lambda$ for $\gamma,\ \lambda \in \mathbb{R}$.
Then, for any $\varepsilon > 0$ we have

$$\gamma + \lambda + \varepsilon = \left(\gamma + \tfrac{\varepsilon}{2}\right) + \left(\lambda + \tfrac{\varepsilon}{2}\right)$$

Thus, since $\gamma + \frac{\varepsilon}{2}$ is not a lower bound for A and $\lambda + \frac{\varepsilon}{2}$ is not a lower bound for B, then $\gamma + \lambda + \varepsilon$ cannot be a lower bound for $A+B$. Thus, we have shown that $\gamma + \lambda$ is in fact the greatest lower bound for $A+B$. That is, $\inf A + \inf B = \gamma + \lambda = \inf(A+B)$. ✓ ✡

(2) Let $c > 0$. Define $c\,A = \{c\,x : x \in A\}$. Compute $\sup c\,A$ in terms of $\sup A$. What happens if $c < 0$? Repeat exercise for $\inf c\,A$.

<u>Solution:</u>
We have that c is a positive scalar. As shown on part (1), A is bounded above and has a least upper

bound, call it α. It follows that $x \leq \alpha \quad \forall\, x \in A$. This in turn implies $c\,x \leq c\,\alpha$ for every positive c, meaning that $c\,\alpha$ is an upper bound of $c\,A$.
Now, for any $\varepsilon > 0$ we have

$$c\,\alpha - \varepsilon = c\left(\alpha - \frac{\varepsilon}{c}\right)$$

Thus, since $\alpha - \frac{\varepsilon}{c}$ is not an upper bound of A, it follows that $c\,\alpha - \varepsilon$ cannot be an upper bound of $c\,A$. This in turn implies that $c\,\alpha$ is in fact the least upper bound of $c\,A$. In other words, $\sup c\,A = c\,\alpha = c \sup A$. ✓

In the case that $c < 0$, we have a different result. That is, $x \leq \alpha \implies c\,x \geq c\,\alpha \;\; \forall\, x \in A$ if c is negative. As a consequence, $c\,\alpha$ turns out to be a lower bound of $c\,A$ when c is negative. ✓

Now, assuming that c is positive, we want to find $\inf c\,A$. Let β be the greatest lower bound of A. Then we have $x \geq \beta \quad \forall\, x \in A$, which implies $c\,x \geq c\,\beta$ for every positive c. Thus, $c\,\beta$ is a lower bound of $c\,A$.
For any $\varepsilon > 0$ we have

$$c\,\beta + \varepsilon = c\left(\beta + \frac{\varepsilon}{c}\right)$$

Thus, since $\beta + \frac{\varepsilon}{c}$ is not a lower bound of A, it follows that $c\,\beta + \varepsilon$ cannot be a lower bound of $c\,A$. This in turn implies that $c\,\beta$ is in fact the greatest lower bound of $c\,A$. In other words, $\inf c\,A = c\,\beta = c \inf A$. ✓

Once again, if we let $c < 0$, we get a different result. That is, $x \geq \beta \implies c\,x \leq c\,\beta \quad \forall\, x \in A$ if c is negative. As a consequence, $c\,\beta$ is an upper bound of $c\,A$ when c is negative. ✓

(3) Define $A\,B = \{x\,y : x \in A,\;\; y \in B\}$. Assuming that the elements of A and the elements of B are nonnegative, compute $\sup AB$ in terms of $\sup A$ and $\sup B$. Is your answer still true if we drop the assumption that A and B are nonnegative?

Solution:
As shown on part (1), both A and B are bounded above and each has a least upper bound, call them α and β, respectively, i.e. $\sup A = \alpha$, $\sup B = \beta$. Thus, since $x \leq \alpha$ and $y \leq \beta \quad \forall\, x \in A,\;\; y \in B$, by the properties of fields it must be true that $x\,y \leq \alpha\,\beta$ (since x and y are elements of A and B, respectively, both of which reside in $\mathbb{R}$. Therefore x and y are both field elements of $\mathbb{R}$). Using this information we have that $\alpha\,\beta$ is an upper bound of $A\,B$.
Now, for any $\varepsilon > 0$ we have

$$\begin{aligned} \alpha\,\beta - \varepsilon &= \alpha\left(\beta - \frac{\varepsilon}{\alpha}\right) \\ &= \beta\left(\alpha - \frac{\varepsilon}{\beta}\right) \end{aligned}$$

Thus, since $\alpha - \frac{\varepsilon}{\beta}$ is not an upper bound of A and $\beta - \frac{\varepsilon}{\alpha}$ is not an upper bound of B, it follows that $\alpha\,\beta - \varepsilon$ cannot be an upper bound of AB. This in turn implies that $\alpha\,\beta$ is in fact the least upper bound of AB. In other words, $\sup AB = \alpha\,\beta = \sup A \sup B$. ✓

Without assuming that A and B are nonnegative however, our previous result is no longer valid. Assume for instance, that the elements of A and B are nonpositive with least upper bounds ϕ and λ, respectively, i.e. $\sup A = \phi$ and $\sup B = \lambda$. Note that $\phi,\ \lambda \le 0$, otherwise our assumption that the elements of A and B are strictly nonpositive wouldn't be valid. It follows that $x \le \phi$ and $y \le \lambda$ $\forall\ x \in A,\ y \in B$. But then, according to the properties of fields, we have that each $x\,y \in AB$ must be nonnegative if both x and y are nonpositive. This indicates that $\phi\,\lambda \le x\,y \quad \forall\ x\,y \in AB$.
This in turn implies that $\phi\,\lambda$ is a lower bound of AB, which is obviously a different result from the one obtained when assuming that the elements of A and B are nonnegative. ✓

(4) Suppose $f : A \longrightarrow \mathbb{R}$ and $g : A \longrightarrow \mathbb{R}$ are real valued functions. Define

$$f(A) \oplus g(A) = \{f(x) + g(x) : x \in A\} \quad \text{and} \quad f(A) + g(A) = \{f(x) + g(y) : x,\ y \in A\}.$$

What is the relationship between $\sup(f(A) \oplus g(A))$ and $\sup(f(A) + g(A))$? Repeat exercise for $\inf(f(A) \oplus g(A))$.

<u>Solution:</u>
Since A is a proper subset of $\mathbb{R}$, it is finite and bounded above. So we have that the collection of all images of the elements of A under f, denoted $f(A)$, is also finite and bounded above. This also applies to the set of all images of the elements of A under g, denoted $g(A)$.
Since the sets of images under f and g are bounded above, for all $s \in A$ we must have $f(s) \le \alpha$ and $g(s) \le \beta$, for some $\alpha,\ \beta \in \mathbb{R}$. That is, $\sup(f(A)) = \alpha$ and $\sup(g(A)) = \beta$. Thus, $\alpha + \beta$ is an upper bound for $f(A) + g(A)$ (note that $f(A) \oplus g(A)$ is a subset of $f(A) + g(A)$).
Hence, for any $\varepsilon > 0$ we have

$$\alpha + \beta - \varepsilon = \left(\alpha - \frac{\varepsilon}{2}\right) + \left(\beta - \frac{\varepsilon}{2}\right)$$

Thus, since $\alpha - \frac{\varepsilon}{2}$ is not an upper bound for $f(A)$, and $\beta - \frac{\varepsilon}{2}$ is not an upper bound for $g(A)$, it must be true that $\alpha + \beta - \varepsilon$ is not an upper bound for $f(A) + g(A)$. Thus, we have shown that $\alpha + \beta$ is in fact the least upper bound for $f(A) + g(A)$.
Hence, since $f(A) \oplus g(A) \subset f(A) + g(A)$, it follows that

$$\sup(f(A) \oplus g(A)) \le \alpha + \beta = \sup(f(A) + g(A)) \qquad ✓$$

Now, let $\inf(f(A)) = \eta$ and $\inf(g(A)) = \psi$, for $\eta,\ \psi \in \mathbb{R}$. Thus, $\eta + \psi$ is a lower bound for $f(A) + g(A)$.
Hence, for any $\varepsilon > 0$ we have

$$\eta + \psi + \varepsilon = \left(\eta + \frac{\varepsilon}{2}\right) + \left(\psi + \frac{\varepsilon}{2}\right)$$

Thus, since $\eta + \frac{\varepsilon}{2}$ is not a lower bound for $f(A)$, and $\psi + \frac{\varepsilon}{2}$ is not a lower bound for $g(A)$, it must be true that $\eta + \psi + \epsilon$ is not a lower bound for $f(A) + g(A)$. Thus, we have shown that $\eta + \psi$ is in fact the greatest lower bound for $f(A) + g(A)$.

Hence, since $f(A) \oplus g(A) \subset f(A) + g(A)$, it follows that

$$\inf(f(A) \oplus g(A)) \geq \eta + \psi = \inf(f(A) + g(A)) \qquad \checkmark$$