# Math 351 DNHI 6

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(1) Let $f:[0,\ 1]\longrightarrow\mathbb{R}$ be defined as follows in each case. Determine the points at which $f$ is continuous:

a) $f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$

Solution:

$x = 0$ is the only point at which $f$ is continuous.

If $y \in (-\varepsilon,\ \varepsilon)$, we have $|f(y) - 0| \leq |y| < \varepsilon$. Thus, $\lim\limits_{y\to 0} f(y) = f(0)$.

If $x \neq 0$, there is a sequence of rationals $r_n \to x$ and irrationals $p_n \to x$. Clearly, $f(r_n) = r_n \to x$, while $f(p_n) = 0 \overset{\text{not}}{\to} x$. Therefore the sequence $f(r_1),\ f(p_1),\ f(r_2),\ f(p_2),\ \ldots$ does not converge even though the sequence $r_1,\ p_1,\ r_2,\ p_2,\ \ldots$ converges to $x$. Hence $f$ is not continuous at $x$. ✓

b) $f(x) = \begin{cases} 1-x & \text{if } x \notin \mathbb{Q} \\ x & \text{if } x \in \mathbb{Q} \end{cases}$

Solution:

$f$ is continuous at a point $x$ iff $\lim\limits_{n\to\infty} f(r_n) = \lim\limits_{n\to\infty} f(p_n)$, where $\{r_n\}$ is any sequence of rationals satisfying $r_n \to x$ and $\{p_n\}$ is any sequence of irrationals satisfying $p_n \to x$.

Notice that $\lim\limits_{n\to\infty} f(r_n) = \lim\limits_{n\to\infty} r_n = x$, while $\lim\limits_{n\to\infty} f(p_n) = \lim\limits_{n\to\infty} (1 - p_n) = 1 - x$. Hence $f$ is continuous at $x$ iff $x = 1 - x$ or when $x = \frac{1}{2}$. ✓

c) $f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \ \text{(in lowest terms)} \end{cases}$

Solution:

Clearly $f$ is not continuous at any rational $x > 0$.

If $\{p_n\} \subset [0,\ 1]\backslash\mathbb{Q}$ is any sequence satisfying $p_n \to x$, then $f(p_n) = 0 \overset{\text{not}}{\to} x$.

If $x = 0$ or $x \in [0,\ 1]\backslash\mathbb{Q}$, then $f$ is continuous at $x$:

For any $\varepsilon > 0$ there is an integer $N$ such that $\frac{1}{N} < \varepsilon$. Since $x \neq \frac{m}{n}$ for any $m, n \in \mathbb{N}$, there is some $\delta_k > 0$ such that the interval $(x - \delta_k, x + \delta_k)$ has no points of the form $\frac{m}{k+1}$. Let $\delta = \min\{\delta_1, \ldots, \delta_{N-1}\}$. Then the interval $(x - \delta, x + \delta)$ contains no points of the form $\frac{m}{n}$ for $n = 2, 3, \ldots, N$. Hence, if $y \in (x - \delta, x + \delta)$, we have $|f(y) - f(x)| = |f(y) - 0| \leq \frac{1}{n}$ for $n \geq N + 1$ so $|f(y) - 0| < \varepsilon$ which proves that $f$ is continuous at $x$. ✓

(2) Given a subset $A$ of some "universal" set $S$, we define $X_A : S \longrightarrow \mathbb{R}$, the characteristic function of $A$, by $X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$.

Prove or disprove the following formulas and state what corrections are necessary :

a) $X_{A \cup B} = X_A + X_B$

Solution:

This formula is not correct. For instance, let $x \in A \cup B$, with $x \in A$ and $x \in B$. then we have:

$$1 = X_{A \cup B}(x) \neq X_A(x) + X_B(x) = 1 + 1 = 2 .$$

The correct formula would actually be $X_{A \cup B} = X_A + X_B - X_{A \cap B}$ . ✓

b) $X_{A \cap B} = X_A \cdot X_B$

Solution:

This formula is correct.

We can see that if $x \in A \cap B$, then

$$1 = X_{A \cap B}(x) = X_A(x) \cdot X_B(x) = 1 \cdot 1 = 1 .$$

Otherwise, if $x \notin A \cap B$, then we may assume WLOG that $x \notin A$. Then,

$$0 = X_{A \cap B}(x) = X_A(x) \cdot X_B(x) = 0 \cdot 1 = 0 .$$ ✓

c) $X_{A \setminus B} = X_A - X_B$.

Solution:

This formula is not correct. Notice that if $x \notin A$ and $x \in B$, then

$$0 = X_{A \setminus B}(x) \neq X_A(x) - X_B(x) = 0 - 1 = -1 .$$

The correct formula would actually be $X_{A \setminus B} = X_{A \cup B} - X_B$ . ✓

(3) If $f : A \longrightarrow B$ and $C \subset B$, what is $X_C \circ f$ (as a characteristic function)?

Solution:

$X_C(f(x)) = 1$ iff $f(x) \in C$. Thus $X_C \circ f = X_{f(A) \cap C}$ .

(4) Show that $X_\Delta : \mathbb{R} \longrightarrow \mathbb{R}$, the characteristic function of the Cantor set, is discontinuous at each point of $\Delta$ .

Proof:
Notice that $X_\Delta^{-1}(B_{1/3}(1)) = \Delta$. Since $\Delta$ is nowhere dense, we see that $\text{int}(\Delta) = \emptyset$, which means that $\Delta$ contains no open intervals. Thus, $X_\Delta$ is not continuous at any $x \in \Delta$ (otherwise we would have $X_\Delta(B_\delta(x)) \subset B_{1/3}(1)$ for some $\delta > 0$). Similarly, since $X_\Delta^{-1}(B_{1/3}(0)) = \mathbb{R} \setminus \Delta$ and $\mathbb{R} \setminus \Delta$ is open, we see that $X_\Delta$ is continuous at each $x \in \mathbb{R} \setminus \Delta$. ■

(5) If $A \subset \mathbb{R}$ , show that $X_A$ is continuous at each point of int $(A)$. Are there any other points of continuity?

Proof:
Suppose $A \subset \mathbb{R}$ and $x \in \text{int}(A)$. Then, for some $\delta > 0$, we have

$$B_\delta(x) = (x - \delta,\ x + \delta) \subset \text{int}(A) \subset A \quad \text{and} \quad \{1\} = X_A(B_\delta(x)) \subset B_\varepsilon(1) = B_\varepsilon(X_A(x))$$

for any $\varepsilon > 0$. Hence $X_A$ is continuous at each $x \in \text{int}(A)$.

Similarly, if $x \in \text{int}(A^c)$, then for some $\delta > 0$ we have that $B_\delta(x) \subset \text{int}(A^c) \subset A^c$. Hence $\{0\} = X_A(B_\delta(x)) \subset B_\varepsilon(0) = B_\varepsilon(X_A(x))$ for any $\varepsilon > 0$, implying that $X_A$ is continuous at each $x \in \text{int}(A^c)$.

Lastly, if $x$ is on the boundary of $A$, that is, if $x \notin \text{int}(A)$ and $x \notin \text{int}(A^c)$, then $x$ is a limit point of both $A$ and $A^c$. Therefore $X_A$ is discontinuous at $x$. ■

(6) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Show that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$ and that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$. If $f(x) = 0$ whenever $x$ is rational, show that $f(x) = 0$ for every real $x$.

Proof:
Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Then for $x \in \{x : f(x) > 0\}$, we have $f(x) = r > 0$. To show that $\{x : f(x) > 0\}$ is open, consider the open ball

$$B_{r/4}(f(x)) = \left(f(x) - \frac{r}{4},\ f(x) + \frac{r}{4}\right) = \left(\frac{3r}{4},\ \frac{5r}{4}\right).$$

Because $f$ is continuous, the inverse image of any open set is open. In particular, $f^{-1}(B_{r/4}(f(x)))$ is open and contains $x$. Therefore $B_\delta(x) \subset f^{-1}(B_{r/4}(f(x)))$ for some $\delta > 0$. Notice, however that this implies that

$$f(B_\delta(x)) \subset B_{r/4}(f(x)) = \left(\frac{3r}{4}, \frac{5r}{4}\right).$$

Hence, if $y \in B_\delta(x)$, then $f(y) > \frac{3r}{4} > 0$ and we see that $B_\delta(x) \subset \{x : f(x) > 0\}$, which proves that this set is open.

Notice also that the set $\{x : f(x) < 0\}$ is open since it is identical to the set $\{x : -f(x) > 0\}$. Thus the set $\{x : f(x) \neq 0\} = \{x : f(x) > 0\} \bigcup \{x : f(x) < 0\}$ is open as well, implying that $\{x : f(x) = 0\} = \{x : f(x) \neq 0\}^c$ is closed. ■

(7)
a) If $f : M \longrightarrow \mathbb{R}$ is continuous and $a \in \mathbb{R}$, show that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open subsets of $M$.

Solution:
Suppose that $f : M \longrightarrow \mathbb{R}$ is continuous. Fix $a \in \mathbb{R}$. Then $g : M \longrightarrow \mathbb{R}$ given by $g(x) = f(x) - a$ is also continuous. Observe that the set $\{x : g(x) > 0\} = \{x : f(x) > a\}$ and $\{x : g(x) < 0\} = \{x : f(x) < a\}$. Now repeat the argument from problem (6) above to prove that $\{x : g(x) > 0\}$ and $\{x : g(x) < 0\}$ are open.
✓

b) Conversely, if the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every $a \in \mathbb{R}$, show that $f$ is continuous.

Solution:
Suppose the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every $a \in \mathbb{R}$. Fix $y \in M$ and $\varepsilon > 0$ and consider

$$\begin{aligned} f^{-1}(f(y) - \varepsilon, f(y) + \varepsilon) &= \{x : f(y) - \varepsilon < f(x) < f(y) + \varepsilon\} \\ &= \{x : f(x) > f(y) - \varepsilon\} \bigcap \{x : f(x) < f(y) + \varepsilon\}. \end{aligned}$$

Observe that $f^{-1}(B_\varepsilon(f(y))) = f^{-1}(f(y) - \varepsilon, f(y) + \varepsilon)$ is the intersection of two open sets and must therefore be open.
Let $\delta > 0$ satisfy $B_\delta(y) \subset f^{-1}(B_\varepsilon(f(y)))$ [Clearly, $y \in \{x : f(y) - \varepsilon < f(x) < f(y) + \varepsilon\}$].
This proves that $f$ is continuous at $y$. Since $y$ is arbitrary, it follows that $f$ is continuous. ✓

c) Show that $f$ is continuous even if we assume only that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every rational $a$.

Solution:
For $y \in M$ and $\varepsilon > 0$, fix rational numbers $\alpha$ and $\beta$ satisfying

$$f(y) - \varepsilon < \alpha < f(y) < \beta < f(y) + \varepsilon.$$

Then $f(y) \in (\alpha, \beta) \subset (f(y) - \varepsilon, f(y) + \varepsilon)$.
Hence
$$f^{-1}(\alpha, \beta) = \{x : \alpha < f(x) < \beta\} = \{x : f(x) > \alpha\} \bigcap \{x : f(x) < \beta\}$$
is open.
Furthermore,
$$y \in f^{-1}(\alpha, \beta) \subset f^{-1}(f(y) - \varepsilon, f(y) + \varepsilon) = f^{-1}(B_\varepsilon(f(y)))$$
so $B_\delta(y) \subset f^{-1}(\alpha, \beta) \subset f^{-1}(B_\varepsilon(f(y)))$.

This proves that $f$ is continuous at $y$. Again, since $y$ is arbitrary, we have shown that $f$ is continuous. ✓

(8) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Then,

a) If $f(0) > 0$, show that $f(x) > 0$ for all $x$ in some interval $(-a, a)$.

Solution:
Let $f(0) = r > 0$. Then $B_{r/2}(f(0)) = \left(\frac{r}{2}, \frac{3r}{2}\right)$. Since $f$ is continuous at 0, there is some $a > 0$ that satisfies
$$f(B_a(0)) = f(-a, a) \subset B_{r/2}(f(0)) = \left(\frac{r}{2}, \frac{3r}{2}\right).$$
Hence $f(x) > \frac{r}{2} > 0$ for all $x \in (-a, a)$. ✓

b) If $f(x) \geq 0$ for every rational $x$, show that $f(x) \geq 0$ for all real $x$.

Solution:
If there were an $x$ for which $f(x) < 0$, by a slight modification of the argument in part a), there would be some interval $(x - a, x + a)$ on which $f$ is negative. However, since the interval $(x - a, x + a)$ contains rationals and $f$ is nonnegative on every rational, such interval cannot exist.
Thus, we have that $f(x) \geq 0 \ \ \forall x$. ✓

(9) Let $A = (0, 1] \cup \{2\}$ be considered as a subset of $\mathbb{R}$. Show that every function $f : A \longrightarrow \mathbb{R}$ is continuous, relative to $A$, at 2.

Proof:
Let $A = (0, 1] \cup \{2\}$ have the usual metric function of $\mathbb{R}$. Then the open ball
$B^A_{1/2}(2) = \left(2 - \frac{1}{2}, 2 + \frac{1}{2}\right) \bigcap A = \{2\}$ is an open subset of $A$.
If $f : A \longrightarrow \mathbb{R}$ is any function and $\varepsilon > 0$, then $f\left(B^A_{1/2}(2)\right) = \{f(2)\} \subset B_\varepsilon(f(2))$.
This shows that any function $f : A \longrightarrow \mathbb{R}$ is continuous at 2. ∎

(10) Let $A$ and $B$ be subsets of $M$, and let $f : M \longrightarrow \mathbb{R}$. Prove or disprove the following statements. If either statement is not true in general, what modifications are necessary to make it so?

a) If $f$ is continuous at each point of $A$ and $f$ is continuous at each point of $B$, then $f$ is continuous at each point of $A \bigcup B$.

Solution:

If $f$ is continuous at each point of $A$ (relative to $M$) and at each point of $B$ (relative to $M$), then $f$ is continuous at each point of $A \bigcup B$ (relative to $M$). This is so because for each $x \in A \bigcup B$, we may assume WLOG that $x \in A$. Therefore by hypothesis, $x$ is a point of continuity of $f$ relative to $M$.
✓

b) If $f|_A$ is continuous, relative to $A$ and $f|_B$ is continuous, relative to $B$, then $f|_{A \bigcup B}$ is continuous, relative to $A \bigcup B$.

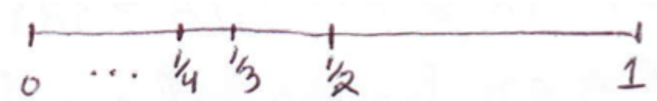
Solution:

Suppose $f|_B$ (i.e. $f : M \longrightarrow \mathbb{R}$ restricted to $B$) is continuous relative to $B$ and $f|_A$ is continuous relative to $A$. Then it is not necessarily true that $f|_{A \bigcup B}$ is continuous relative to $A \bigcup B$. To see this, consider $X_{\mathbb{Q}} : \mathbb{R} \longrightarrow \mathbb{R}$ and set $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. Then $X_{\mathbb{Q}}\big|_A = 1$, $X_{\mathbb{Q}}\big|_B = 0$ are constant functions. Therefore $X_{\mathbb{Q}}\big|_A$ is continuous relative to $A$ and $X_{\mathbb{Q}}\big|_B$ is continuous relative to $B$. However, $X_{\mathbb{Q}}\big|_{A \bigcup B} = X_{\mathbb{Q}}$ is not continuous everywhere on $A \bigcup B$.
Notice that $f|_{A \bigcup B}$ is continuous whenever $f|_A$ and $f|_B$ are continuous with the added hypothesis that $A$ and $B$ are open subsets of $M$. ✓

(11) Let $I = (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$ with its usual metric. Prove that there is a continuous function $g$ mapping $I$ onto $\mathbb{Q} \cap [0, 1]$.

Proof:

First partition the interval $[0, 1]$ into interval subsegments using the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.




For each integer $n \in \mathbb{N}$ set $I_n = \left(\frac{1}{n+1}, \frac{1}{n}\right) \bigcap I$, where $I = (\mathbb{R} \setminus \mathbb{Q}) \bigcap [0, 1]$. Then the $I_n$ are pairwise disjoint, nonempty open subsets of (relative to) $I$.
Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers $\mathbb{Q} \bigcap [0, 1]$ and define $g : (\mathbb{R} \setminus \mathbb{Q}) \bigcap [0, 1] \longrightarrow \mathbb{Q} \bigcap [0, 1]$ by $g(x) = q_n$ if $x \in I_n$.

Clearly, $g$ is onto. To see that $g$ is continuous, observe that $g^{-1}(\{q_n\}) = I_n$. That is, the inverse image under $g$ of any singleton is an open set. From this it immediately follows that for any $V \subset \mathbb{Q} \bigcap [0, 1]$, $g^{-1}(V)$ is an open subset of $I$.
Thus the inverse image of any open set is open and therefore $g$ is continuous. ■

(12) Let $f, g : (M, d) \longrightarrow (N, \rho)$ be continuous, and let $D$ be a dense subset of $M$. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If $f$ is onto, show that $f(D)$ is dense in $N$.

Proof:
Define $k : M \longrightarrow \mathbb{R}$ by $k(x) = \rho(f(x), g(x))$. We will prove that $k(x) = 0$ for all $x \in M$, showing that $f(x) = g(x)$.
Notice that

$$\begin{aligned} |k(x) - k(y)| &= |\rho(f(x), g(x)) - \rho(f(y), g(y))| \\ &\leq |\rho(f(x), g(x)) - \rho(f(y), g(y))| + |\rho(f(x), g(x)) - \rho(f(y), g(y))| \\ &\leq \rho(g(x), g(y)) + \rho(f(x), f(y)) \end{aligned}$$

Since $f, g : (M, d) \longrightarrow (N, \rho)$ are continuous, the calculation above implies that $k : M \longrightarrow \mathbb{R}$ is continuous.
Since $f(x) = g(x)$ for any $x \in D$, we have $k(x) = 0 \ \ \forall x \in D$. If, for some $y \in M$, $k(y) \neq 0$, we may assume WLOG that $k(y) > 0$. By a slight modification of problem (8)a), there is a neighborhood $B_\delta^d(y) \subset M$ such that $k(z) > 0$ for any $z \in B_\delta^d(y)$. But $D \bigcap B_\delta^d(y) \neq \emptyset$, since $D$ is dense in $M$.
Let $\omega \in D \bigcap B_\delta^d(y)$. Then $k(\omega) = 0$, contradicting the assertion that $k$ is positive on $B_\delta^d(y)$. We conclude that $k(x) = 0 \ \ \forall x \in M$, from which the desired result follows.

Now suppose that $f : (M, d) \longrightarrow (N, \rho)$ is onto. Then any $z \in N$ is of the form $f(x)$ for some $x \in M$. Since $f$ is continuous at $x$, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $f(B_\delta^d(x)) \subset B_\varepsilon^\rho(f(x))$. Since $D \bigcap B_\delta^d(x)$ is not empty, $f(D) \bigcap B_\varepsilon^\rho(f(x))$ is also not empty. In particular, either $f(x) \in f(D)$ or $f(x)$ is a limit point of $f(D)$. This establishes that $f(D)$ is dense in $N$. ■

(13) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to satisfy a Lipschitz condition if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K\,|x - y|$ for all $x, y \in \mathbb{R}$. More economically, we may say that $f$ is Lipschitz (or Lipschitz with constant $K$ if a particular constant seems to matter). Show that $\sin(x)$ is Lipschitz with constant $K = 1$. Prove that a Lipschitz function is continuous.

Proof:
Let $f(x) = \sin(x)$, which is everywhere differentiable. Thus, for any fixed $x, y \in \mathbb{R}$, $f$ is continuous on $[x, y]$ and differentiable on $(x, y)$. Therefore, by the mean-value theorem, there is some

$c \in (x, y)$ such that $\frac{f(x)-f(y)}{x-y} = f'(c)$. This means that $\sin(x) - \sin(y) = \cos(c)(x-y)$, so $|\sin(x) - \sin(y)| = |\cos(c)|\,|x-y| \leq |x-y|$. In particular, $f(x) = \sin(x)$ is Lipschitz with constant $K = 1$.

Observe that every Lipschitz function of order $K$ is continuous :

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x-y| < \frac{\varepsilon}{K}. \qquad \blacksquare$$

(14) In a more general setting, a function $f : (M, d) \longrightarrow (N, \rho)$ is called Lipschitz if there is a constant $K < \infty$ such that $\rho(f(x), f(y)) \leq K\, d(x, y)$ for all $x, y \in M$. Prove that a Lipschitz mapping is continuous.

Proof:

If $f : (M, d) \longrightarrow (N, \rho)$ is Lipschitz, then $\rho(f(x), f(y)) \leq K\, d(x, y)$ means that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \frac{\varepsilon}{K}$. In particular, $f$ is continuous. $\blacksquare$

(15) Show that the map $L(f) = \int_a^b f(t)\, dt$ is Lipschitz with constant $K = b-a$ for $f \in C[a, b]$.

Proof:

Let $f, g \in C[a, b]$. define $L : C[a, b] \longrightarrow \mathbb{R}$ by $L(f) = \int_a^b f(t)\, dt$.

Then,

$$\begin{aligned} |L(f) - L(g)| &= \left| \int_a^b (f(t) - g(t))\, dt \right| \\ &\leq \int_a^b |(f(t) - g(t))|\, dt \\ &\leq \int_a^b \| f - g \|_\infty\, dt \\ &= |b-a|\, \| f - g \|_\infty . \end{aligned}$$

Setting $K = |b-a|$, we see that $L$ is Lipschitz with constant $K$. $\blacksquare$

(16) Define $g : \ell_2 \longrightarrow \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}$. Is $g$ continuous?

Solution:

Define $g : \ell_2 \longrightarrow \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}$.

Then,

$$|g(x) - g(y)| = \left| \sum_{n=1}^{\infty} \frac{x_n - y_n}{n} \right| \leq \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{n}$$

$$\leq \left( \sum_{n=1}^{\infty} \left| x_n - y_n \right|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2}$$
$$= K \,||\, x - y \,||_2.$$

Therefore $g$ is Lipschitz. ∎