Math 351 Assignment 2

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(1) Calculate $\frac{3}{7}$ in base 3.

Solution:

We know that we can express any $x \in [0,\ 1]$ as a sum $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$, or in short form $x = 0.\, a_1\, a_2 \ldots a_n$, where $\{a_n\}$ is a sequence of integers with $0 \leq a_n \leq p-1$, for some base $p \geq 2$.
Thus, in this case we have

$$x = \frac{3}{7} = \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Then, our task is to find the sequence of integers $\{a_n\}$ that satisfies this equation. This is a rather trivial operation albeit it requires a little bit of work.
First, notice that since $\frac{3}{7}$ equals the the entire infinite sum, each partial sum must be less than the desired result of $\frac{3}{7}$. Moreover, we must also keep in mind the restriction $0 \leq a_n \leq p-1$, and choose the largest a_n that satisfies these conditions.
That is,

$$\begin{aligned}
&\frac{a_1}{3} < \frac{3}{7} \Longrightarrow a_1 = 1 \\
&\frac{a_1}{3} + \frac{a_2}{3^2} < \frac{3}{7} \Longrightarrow a_2 = 0 \\
&\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} < \frac{3}{7} \Longrightarrow a_3 = 2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

By continuing in this fashion, we get the entire sequence of digits that we're looking for. However, even though we may very well drag on with this rather tedious computation, there's a very simple yet equally efficient method to obtain the desired result.

We are going to perform "kindergarten" division, only we'll do it on base 3 instead of the usual base 10. We will execute this operation until we get finite result or a repeating pattern. The algorithm is displayed below:

$$\begin{array}{ll}
3 & |\underline{7\qquad\qquad\qquad\qquad} \\
\underline{\times 3} & 0.102120
\end{array}$$

$$\begin{array}{l}
\underline{9\ -\ 7} \\
\quad 2 \\
\underline{\times 3} \\
\quad \underline{6 - 0} \\
\qquad 6 \\
\quad \underline{\times 3} \\
\qquad \underline{18 - 14} \\
\qquad\qquad 4 \\
\qquad \underline{\times 3} \\
\qquad\quad \underline{12\ -\ 7} \\
\qquad\qquad\quad 5 \\
\qquad\qquad \underline{\times 3} \\
\qquad\qquad\quad \underline{15 - 14} \\
\qquad\qquad\qquad 1 \\
\qquad\qquad\quad \underline{\times 3} \\
\qquad\qquad\qquad \underline{3\ -\ 0} \\
\qquad\qquad\qquad\quad 3 \\
\qquad\qquad\qquad \underline{\times 3} \\
\qquad\qquad\qquad\quad 9.....
\end{array}$$

Since we got back to 9 again, we see that the pattern will repeat from this point on. Hence we have found that $\frac{3}{7} = 0.\overline{102\,120}$ (base 3) ✓

(2) Let $A = \{a, b, c\}$. Define $F : A \longrightarrow \mathcal{P}(A)$ by

$$F(x) = \begin{cases} \{a, b\} & \text{if } x = a \\ \{a, c\} & \text{if } x = b \\ \{b\} & \text{if } x = c \end{cases}$$

Compute $S_F = \{x \in A : x \notin F(x)\}$.

<u>Solution:</u>
We compute $F(a) = \{a, b\}$, and we see that $a \in F(a)$, so this does not contribute any element to S_F. Now we compute $F(b) = \{a, c\}$, where we observe that $b \notin F(b)$, so b is in S_F. Finally we have $F(c) = \{b\}$, where $c \notin F(c)$, so c is also in S_F.
Hence, $S_F = \{b, c\}$. ✓

(3) Let A be a proper infinite subset of some set X. If x, y are two distinct elements of X that are not

in A, we may set $B = \{x,\ y\} \bigcup A$. What is the cardinality of B in terms of the cardinality of A? Justify your answer.

<u>Solution:</u>

We have that $A \subset X$, where $x,\ y \in X$ and $x,\ y \notin A$. If A was a finite set, then clearly $\text{card}(B) = \text{card}(A) + \text{card}(\{x,\ y\}) = \text{card}(A) + 2$. However, in this case A is infinite, thus $|B| = |A| + 2 = |A|$. We are now going to show that this is actually the case.

We are not given any information on whether A is countable. However, we know from a previous theorem that every infinite set has a countable subset.

Therefore we proceed by choosing a countably infinite subset of A, call it A_c such that $A_c = \{a_n : n \in \mathbb{N}\}$, and we define a function $f : B \longrightarrow A$ that satisfies

$$f(b) = \begin{cases} a_1 & \text{if } b = x \\ a_2 & \text{if } b = y \\ a_{n+2} & \text{if } b = a_n \\ b & \text{if } b \in A \backslash A_c \end{cases}$$

Clearly, f is both injective and surjective and thus a bijection between A and B. Hence, we have proven that $A \sim B$, or equivalently, $\text{card}(B) = \text{card}(A)$. ✓

(4) Find a transfinite number that represents the cardinality of the open interval $(0,\ 1)$ in terms of $\aleph_0$. Justify your answer.

<u>Solution:</u>

We know that $\aleph_0$ is the transfinite number that represents the cardinality of $\mathbb{N}$ or any equivalent set, (i.e. any countable set). We also know that the cardinality of the power set of any set that has cardinality n is 2^n, therefore $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$.

We are now going to show that there is a 1-1 correspondence between $\mathcal{P}(\mathbb{N})$ and $(0,\ 1)$, and therefore $|(0,\ 1)| = 2^{\aleph_0}$.

Using the binary base (i.e. 0's and 1's only), let $\zeta : \mathcal{P}(\mathbb{N}) \longrightarrow (0,\ 1)$ be defined by

$$\zeta(S) = \{0.\, x_1\, x_2\, x_3 \ldots \mid S \in \mathcal{P}(\mathbb{N}) \text{ and } x_i = 1 \text{ if } n_i \in S \text{ for } n \in \mathbb{N}, \text{ otherwise } x_i = 0\}$$

For instance, let's take $S = \{1,\ 2,\ 4,\ 5,\ 7\}$. Then $\zeta(S) = 0.1101101$. We can see right away that, even though we're getting somewhere, there's still two major problems with this function.

One problem is the ambiguity that decimal approximations create, meaning that each of the images under ζ can be written as either a finite or an infinite expansion. For instance, using the same S as above as an example, we see that the image under ζ can also be expressed as

$\zeta(S) = 0.1101100111111\ldots$. This seemingly unharmful issue will prevent ζ from being injective, and that's something we want to avoid! However, we can get around this ambiguity by just imposing the restriction that ζ only maps to infinite approximations. There's certainly no reason why we can't impose such restriction.

The other problem arises when ζ maps $S = \mathbb{N}$ (which is certainly a set in $\mathcal{P}(\mathbb{N})$), in which case we get $\zeta(S) = 0.111111111\ldots\ldots$, which is the same as $\zeta(S) = 1$, and obviously 1 is not even included in the interval $(0, 1)$. However, we can easily save our souls from this brutal punishment by simply removing $\mathbb{N}$ from our set of preimages. As we know from a previous theorem, removing one element (or finitely many elements for that matter) will not change the cardinality of an infinite set. In other words, it is true that $\text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(\mathcal{P}(\mathbb{N})\backslash\mathbb{N})$.

Now that we have touched on the issues that we had with our original map, we can now modify ζ to be a bijective function as follows

$$\zeta(S) = \{0.\, x_1\, x_2\, x_3 \ldots \mid S \in \mathcal{P}(\mathbb{N})\backslash\mathbb{N} \text{ and } x_i = 1 \text{ if } n_i \in S \text{ for } n \in \mathbb{N}, \\ \text{otherwise } x_i = 0 \text{ and } 0.\, x_1\, x_2\, x_3 \ldots \text{ is an infinite expansion}\}.$$

Since we were able to find a bijection between $\mathcal{P}(\mathbb{N})$ and the interval $(0, 1)$, we have found that $|(0, 1)| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$. ✓