MATH 750 HW # 2

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Ex 2-1: Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then it is continuous at a.

Proof. Let f be differentiable at $a \in \mathbb{R}^n$. Then there exists a linear map $\lambda\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0,$$

or equivalently,

$$f(a+h) - f(a) = \lambda(h) + R(h) \tag{$\dagger$}$$

where the remainder $R(h)$ satisfies

$$\lim_{h\to 0} \frac{\|R(h)\|}{\|h\|} = 0. \tag{$\dagger\dagger$}$$

Now let $h \to 0$ in ($\dagger$) and notice that the error term $R(h)$ approaches 0 by ($\dagger\dagger$). In addition, the linear term $\lambda(h)$ also tends to 0 because if $h = \sum_{i=1}^n h_i e_i$, where where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$, then by linearity we have $\lambda(h) = \sum_{i=1}^n h_i \lambda(e_i)$ and each term on the right tends to 0 as $h_i \to 0$ for all $1 \leq i \leq n$ (that is, as $h \to 0$).

Hence we are left with

$$\lim_{h\to 0} [f(a+h) - f(a)] = 0.$$

That is, $\lim_{h\to 0} f(a+h) = f(a+0) = f(a)$. Thus f is continuous at a, as desired. $\square$

Ex 2-6: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$. Show that f is not differentiable at $(0,0)$.

Proof. First notice that

$$\lim_{h\to 0} \frac{\|f(h,0)\|}{\|h\|} = 0 = \lim_{k\to 0} \frac{\|f(0,k)\|}{\|k\|}.$$

Hence, if f is differentiable at $(0,0)$ we must have that $Df(0,0)(x,y) = 0$, since the derivative is unique if it exists. However, if we let $h = k > 0$, and take a sequence $\{(h,h)\} \to (0,0)$, then we have

$$\begin{aligned}
\lim_{(h,h)\to(0,0)} \frac{\|f[(0,0)+(h,h)] - f(0,0) - 0\|}{\|(h,h)\|} &= \lim_{(h,h)\to(0,0)} \frac{\|f(h,h) - f(0,0) - 0\|}{\|(h,h)\|} \\
&= \lim_{(h,h)\to(0,0)} \frac{\|f(h,h)\|}{\|(h,h)\|} \\
&= \lim_{(h,h)\to(0,0)} \frac{\sqrt{h^2}}{\sqrt{h^2+h^2}} \\
&= \frac{1}{\sqrt{2}} \neq 0.
\end{aligned}$$

Therefore, f is not differentiable. □

Ex 2-7: Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function such that $\|f(x)\| \leq \|x\|^2$. Show that f is differentiable at 0.

Proof. Evaluating at $x = 0$, notice that $\|f(0)\| \leq \|0\|^2 = 0$ implies that $f(0) = 0$. Thus,

$$\lim_{x\to 0} \frac{\|f(0+x) - f(0)\|}{\|x\|} = \lim_{x\to 0} \frac{\|f(x)\|}{\|x\|} \leq \lim_{x\to 0} \frac{\|x\|^2}{\|x\|} = \lim_{x\to 0} \|(x)\| = 0.$$

Hence, $Df(0) = 0$, and thus f is differentiable at 0, as desired. □

Ex 2-14: Let E_i (with $i = 1, \ldots, k$) be Euclidean spaces of various dimensions, and consider the map $f\colon E_1 \times \cdots \times E_k \to \mathbb{R}^p$. Then

a) If f is multilinear and $i \neq j$, show that for $h = (h_1, \ldots, h_k)$, with $h_\ell \in E_\ell$, we have

$$(\heartsuit) \qquad \lim_{h\to 0} \frac{\|f(a_1, \ldots, h_i, \ldots, h_j, \ldots, a_k)\|}{\|h\|} = 0$$

(*Hint:* If $g(x,y) = f(a_1, \ldots, x, \ldots, y, \ldots, a_k)$, then g is bilinear).

b) Prove that

$$Df(a_1, \ldots, a_k)(x_1, \ldots, x_k) = \sum_{i=1}^{k} f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k).$$

Proof. a) Before proceeding to prove the case of multilinearity, we are going to show that this result holds in the less general case of bilinear functions. That is, we are going to show that for a bilinear function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, it is true that

$$(\clubsuit) \qquad \lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} = 0.$$

Let $(e_1, \dots, e_n)$ and $(e_1, \dots, e_m)$ be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we have

$$x = \sum_{i=1}^{n} x_i e_i \qquad \text{and} \qquad y = \sum_{j=1}^{m} y_j e_j.$$

Therefore, considering the bilinearity of f, we have

$$\begin{aligned} f(x,y) = f\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{m} y_j e_j\right) &= \sum_{i=1}^{n} f\left(x_i e_i, \sum_{j=1}^{m} y_j e_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i e_i, y_j e_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j f\left(e_i, e_j\right). \end{aligned}$$

Now we let $M = \sum_{i,j} \|f(e_i, e_j)\|$, so that we have

$$\begin{aligned} \|f(x,y)\| = \|\sum_{i,j} x_i y_j f\left(e_i, e_j\right)\| &\le \sum_{i,j} |x_i y_j| \, \|f\left(e_i, e_j\right)\| \\ &\le M \max_i\{|x_i|\} \max_j\{|y_i|\} \\ &= M\|x\|_\infty \|y\|_\infty. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} &\le \lim_{(h,k)\to 0} \frac{M\|h\|\,\|k\|}{\|(h,k)\|} \\ &= \lim_{(h,k)\to 0} \frac{M\|h\|\,\|k\|}{\sqrt{\|h\|^2 + \|k\|^2}} \\ &\le \lim_{(h,k)\to 0} \frac{M(\|h\|^2 + \|k\|^2)}{\sqrt{\|h\|^2 + \|k\|^2}} \qquad (\bigstar) \\ &= \lim_{(h,k)\to 0} M\sqrt{\|h\|^2 + \|k\|^2} \\ &= 0. \end{aligned}$$

From this result the desired equality on ($\clubsuit$) follows. Also, note that the inequality ($\bigstar$) holds because

$$\|h\|\,\|k\| \leq \begin{cases} \|h\|^2 & \text{if } \|k\| \leq \|h\|, \\ \|k\|^2 & \text{if } \|h\| \leq \|k\|. \end{cases}$$

Hence, $\|h\|\,\|k\| \leq \|h\|^2 + \|k\|^2$, and ($\bigstar$) follows.

Now that we have proved the bilinear case, the multilinear case stated on ($\heartsuit$) follows immediately. Following the hint given on the text, we define a function $g\colon E_i \times E_j \to \mathbb{R}^p$ such that $g(x_i, y_j) = f(a_1, \ldots, x_i, \ldots, y_j, \ldots, a_k)$. Then g is bilinear and so

$$\lim_{(h_i,h_j)\to 0} \frac{\|g(h_i, h_j)\|}{\|h\|} = 0,$$

by the results obtained above for bilinear functions. Since this result holds for all $1 \leq i.j \leq k$, it must be the case that ($\heartsuit$) is true, and we conclude our proof.

Now for part b), the multilinear case is also a simple extension of the bilinear case. Letting $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be bilinear as before, we want to show that

$$Df(a,b)(x,y) = f(a,y) + f(x,b).$$

Notice that

$$\lim_{(h,k)\to 0} \frac{\|f(a+h, b+k) - f(a,b) - f(a,k) - f(h,b)\|}{\|(h,k)\|}$$

$$\begin{aligned}
&= \lim_{(h,k)\to 0} \frac{\|f(a,b) + f(a,k) + f(h,b) + f(h,k) - f(a,b) - f(a,k) - f(h,b)\|}{\|(h,k)\|} \\
&= \lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} \\
&= 0 \qquad \text{(By part a) above)}.
\end{aligned}$$

Hence we have that $Df(a,b)(x,y) = f(a,y) + f(x,b)$, as desired. The multilinear case follows immediately from this result. $\square$

Ex 2-16: Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and has a differentiable inverse $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$. Show that $(f^{-1})'(a) = [f'(f^{-1}(a))]^{-1}$.
(*Hint:* Note that $f \circ f^{-1}(x) = x$).

Proof. We have that $f \circ f^{-1}(x) = x$, as indicated by the hint. On the one hand $D(f \circ f^{-1})(a)(x) = x$, since $f \circ f^{-1}$ is linear. On the other hand,

$$D(f \circ f^{-1})(a)(x) = \left[Df(f^{-1}(a)) \circ Df^{-1}(a)\right](x).$$

Therefore, $Df^{-1}(a) = \left[Df(f^{-1}(a))\right]^{-1}$, as desired. □

Ex 2-17: Find the partial derivatives of the following functions:

a) $f(x, y, z) = x^y$.

Solution.

- $D_1 f(x, y, z) = yx^{y-1}$
- $D_2 f(x, y, z) = x^y \log x$
- $D_3 f(x, y, z) = 0$.

b) $f(x, y, z) = z$.

Solution.

- $D_1 f(x, y, z) = D_2 f(x, y, z) = 0$
- $D_3 f(x, y, z) = 1$.

c) $f(x, y) = \sin(x \sin y)$.

Solution.

- $D_1 f(x, y) = (\sin y) \cos(x \sin y)$
- $D_2 f(x, y) = x \cos y \cos(x \sin y)$.

d) $f(x, y, z) = \sin(x \sin(y \sin z))$.

Solution.

- $D_1 f(x, y, z) = \sin(y \sin z) \cos(x \sin(y \sin z))$
- $D_2 f(x, y, z) = \cos(x \sin(y \sin z)) x \cos(y \sin z) \sin z$.
- $D_3 f(x, y, z) = \cos(x \sin(y \sin z)) x \cos(y \sin z) y \cos z$.

e) $f(x, y, z) = x^{y^z}$.

Solution.

- $D_1 f(x, y, z) = y^z x^{y^z - 1}$
- $D_2 f(x, y, z) = x^{y^z} z y^{z-1} \log x$
- $D_3 f(x, y, z) = y^z \log y \left(x^{y^z} \log x\right)$.

f) $f(x, y, z) = x^{y+z}$.

Solution.

- $D_1 f(x, y, z) = (y + z)x^{y+z-1}$
- $D_2 f(x, y, z) = D_3 f(x, y, z) = x^{y+z} \log x$

g) $f(x, y, z) = (x + y)^z$.

Solution.

- $D_1 f(x, y, z) = D_2 f(x, y, z) = z(x + y)^{z-1}$.
- $D_3 f(x, y, z) = (x + y)^z \log(x + y)$.

h) $f(x, y) = \sin(xy)$.

Solution.

- $D_1 f(x, y) = y \cos(xy)$
- $D_2 f(x, y) = x \cos(xy)$.

i) $f(x, y) = [\sin(xy)]^{\cos 3}$.

Solution.

- $D_1 f(x, y) = \cos 3[\sin(xy)]^{\cos 3-1} y \cos(xy)$
- $D_1 f(x, y) = \cos 3[\sin(xy)]^{\cos 3-1} x \cos(xy)$

Ex 2-20: Find the partial derivatives of f in terms of the derivatives of g and h:

a) $f(x, y) = g(x)h(y)$.

Solution.

- $D_1 f(x, y) = g'(x)h(y)$
- $D_2 f(x, y) = g(x)h'(y)$

b) $f(x, y) = g(x)^{h(y)}$.

Solution.

- $D_1 f(x, y) = h(y)g(x)^{h(y)-1} g'(x)$
- $D_2 f(x, y) = h'(y)g(x)^{h(y)} \log g(x)$

c) $f(x, y) = g(x)$.

Solution.

- $D_1 f(x,y) = g'(x)$
- $D_2 f(x,y) = 0$

d) $f(x,y) = g(y)$.

Solution.

- $D_1 f(x,y) = 0$
- $D_2 f(x,y) = g'(y)$

e) $f(x,y) = g(x+y)$.

Solution.

- $D_1 f(x,y) = D_2 f(x,y) = g'(x+y)$

Ex 2-32: a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable at 0 but f' is not continuous at 0.

b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2) \sin \frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq 0, \\ 0 & \text{if } (x,y) = 0. \end{cases}$$

Show that f is differentiable at $(0,0)$ but $D_i f$ is not continuous at $(0,0)$.

Proof. For part a), we have

$$\lim_{x\to 0} \frac{\|f(x+0) - f(0)\|}{\|x\|} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x\to 0} x \sin \frac{1}{x} = 0.$$

Hence, $f'(0) = 0$. Moreover, for any $x \neq 0$, we have

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

It is clear that $\lim_{x\to 0} f'(x)$ does not exist because the $-\cos 1/x$ term keeps oscillating and does not approach a fixed limit. Therefore, f' is not continuous at 0.

Now for part b), since

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\sin\frac{1}{\sqrt{x^2+y^2}} = 0,$$

we know that $f'(0,0) = (0,0)$. Now take any $(x,y) \neq (0,0)$. Then

$$D_1 f(x,y) = 2x \sin\frac{1}{\sqrt{x^2+y^2}} - 2x\cos\frac{1}{\sqrt{x^2+y^2}}.$$

As in part a), we have $\lim_{x\to 0} D_1 f(x,0)$ does not exist because of the infinitely oscillating terms. The proof follows similarly for $D_2 f$. $\square$

Ex 2-37: a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f is not 1-1. [*Hint:* If, for example, $D_1 f(x,y) \neq 0$ for all (x,y) in some open set A, consider $g\colon A \to \mathbb{R}^2$ defined by $g(x,y) = (f(x,y), y)$].
b) Generalize this result to the case of a continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$.

Proof. For part a), let f be a continuously differentiable function, so that both $D_1 f$ and $D_2 f$ are continuous. In addition let us assume that f is 1-1. Now suppose that there is $(x_0, y_0) \in \mathbb{R}^2$ such that $D_1 f(x_0, y_0) \neq 0$ (the case for $D_2 f(x_0, y_0)$ is analogous). The continuity of $D_1 f$ implies that there is an open set $A \subset \mathbb{R}^2$ containing (x_0, y_0) such that $D_1 f(x,y) \neq 0$ for all $(x,y) \in A$.

Now define a function $g\colon A \to \mathbb{R}^2$ with

$$g(x,y) = (f(x,y), y).$$

Then for all $(x,y) \in A$, we have

$$g'(x,y) = \begin{pmatrix} D_1 f(x,y) & D_2 f(x,y) \\ 0 & 1 \end{pmatrix},$$

so that $\det(g'(x,y)) = D_1 f(x,y) \neq 0$. Furthermore, g is continuously differentiable on A and is also 1-1. But then since A is open and f (and hence g) is a continuously differentiable 1-1 map on A with $\det(g'(x,y)) \neq 0$, we must have that $g(A)$ is open and that the inverse function $g^{-1}\colon g(A) \to A$ is differentiable (we know this from a previous proposition). The inverse function is clearly of the form $(h(x,y), y)$ and so

$$(f(h(x,y),y),y) = (x,y) \qquad \text{for all} \qquad (x,y) \in V = \{(f(x,y),y) \mid (x,y) \in A\}.$$

Now V is open but each horizontal line intersects A at most once since f is 1-1. This is a contradiction since A is nonempty and open. $(\Rightarrow\Leftarrow)$

This proves that f cannot be 1-1.

For part b) we can write our function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ as $f = (f^1, \ldots, f^m)$, where $f^i\colon \mathbb{R}^n \to \mathbb{R}$ for every $i = 1, \ldots, m$. As in part a), there is a mapping, say WLOG f^1, a point $a \in \mathbb{R}^n$, and an open set A containing a such that $D_1 f^1(x) \neq 0$ for all $x \in A$. Define $g\colon A \to \mathbb{R}^m$ as

$$g\left(x^1, x^\gamma\right) = \left(f(x), x^\gamma\right),$$

where $x^\gamma = (x^2, \ldots, x^n)$. Then as in part a), it follows that f cannot be 1-1. $\square$