MATH 709 HW # 6

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJAN

Problem 1 (Problem 5-4). *Show that the image of the curve $\beta\colon (-\pi,\pi) \to \mathbb{R}^2$ defined by $\beta(t) = (\sin 2t, \sin t)$ (the image of this curve is the lemniscate that we discussed in the previous chapter) is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.]*

Proof. In order for $\beta\left((-\pi,\pi)\right) \subset \mathbb{R}^2$ to be an embedded submanifold $\mathbb{R}^2$, we would need to have that $\beta\left((-\pi,\pi)\right)$ is a manifold (without boundary) in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $\beta\left((-\pi,\pi)\right) \hookrightarrow \mathbb{R}^2$ is a smooth embedding. But the image of β with the subset topology is not a topological manifold at all. This can be seen by considering the point $(0,0)$, around which there is no coordinate map in the subspace topology into $\mathbb{R}$. Another way to see this is to think of what happens when the origin is removed: we get four connected components in the lemniscate but only two in $(-\pi,\pi)$, so any map between the two cannot be an homeomorphism. (Note however that the lemniscate, being the image of an injective smooth immersion, is an immersed submanifold when given an appropriate topology and smooth structure.) $\square$

Problem 2 (Problem 5-10). *For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by*

$$M_a = \{(x,y) \mid y^2 = x(x-1)(x-a)\}.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

Proof. Let $F_a\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $F_a(x,y) = y^2 - x(x-1)(x-a) = y^2 - x^3 + (a+1)x^2 - ax$, so that

$$DF_a(x,y) = \begin{bmatrix} -(3x^2 - (2a+2)x + a) & 2y \end{bmatrix}.$$

We now compute the rank of this Jacobian for various values of a and $(x,y) \in M_a$:

$$\begin{aligned}
x \notin \{0,1,a\} &\implies y \neq 0 && \implies \text{the rank is 1},\\
x = 0 &\implies -3x^2 + 2(a+1)x - a = -a && \implies \text{the rank is 1 iff } a \neq 0,\\
x = 1 &\implies -3x^2 + 2(a+1)x - a = a - 1 && \implies \text{the rank is 1 iff } a \neq 1,\\
x = a &\implies -3x^2 + 2(a+1)x - a = a(a-1) && \implies \text{the rank is 1 iff } a(a-1) = 0.
\end{aligned}$$

So we conclude that if $a \notin \{0,1\}$, then M_a is an embedded submanifold of $\mathbb{R}^2$. Given an appropriate topology and smooth structure however, M_a is an immersed submanifold for all $a \in \mathbb{R}$. $\square$