Analytic Functions
Exam # 2

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Problem 1. *Find the value of*

$$\oint_C \frac{1}{1+z^2}\,\mathrm{d}z$$

a) *when C is the circumference* $\|z-i\|=1$,

b) *when C is the circumference* $\|z\|=2$.

Solution of a). Note that by partial fractions decomposition, we have

$$\frac{1}{1+z^2}=\frac{1}{(z+i)(z-i)}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right). \tag{†}$$

In this particular case for $\|z-i\|=1$, we have $z=i$ outside the curve and $\eta(C;-i)=1$, so from (†) we get

$$\oint_{\|z-i\|=1}\frac{1}{1+z^2}\,\mathrm{d}z=2\pi i\left(-\frac{1}{2i}\right)=-\pi. \qquad \square$$

Solution of b). In this particular case for $\|z\|=2$, we have both $z=i$ and $z=-i$ inside the curve and $\eta(C;-i)=\eta(C;i)=1$. Thus from (†) we get

$$\oint_{\|z\|=2}\frac{1}{1+z^2}\,\mathrm{d}z=2\pi i\left(\frac{1}{2i}-\frac{1}{2i}\right)=0. \qquad \square$$

Quick aside

Now we state two results that we are going to use on the following two problems:

Theorem 1 (Rouché's Theorem). *Suppose that f and g are analytic inside and on a regular closed curve γ and that $\|f(z)\|>\|g(z)\|$ for all $z\in\gamma$. Then*

$$Z(f+g)=Z(f) \quad \textit{inside } \gamma,$$

where $Z(f)$ = the number of zeroes of f inside γ.

Theorem 2 (Maximum Modulus Theorem). *If f is analytic in a region Ω and a is a point in Ω with $\|f(a)\|\geq\|f(z)\|$ for all $z\in\Omega$, then f must be a constant function.*

Problem 2. *Find the number of zeros of $f(z) = \frac{1}{3}e^z - z$ in $\|z\| \leq 1$.*

Solution. Let $g(z) = -z$. Both $f(z)$ and $g(z)$ are entire functions, and on $\|z\| = 1$, we have

$$\|f(z) - g(z)\| = \left\| \frac{1}{3}\, e^z \right\| = \frac{1}{3}\, e^{\mathfrak{Re}\, z} \leq \frac{1}{3}\, e^{\|z\|} = \frac{1}{3}\, e < 1 = \|g(z)\|.$$

Therefore, by Rouché's theorem f and g have the same number of zeroes in $\|z\| \leq 1$, which is 1. □

Problem 3. *Let Ω be a region and suppose that $f \colon \Omega \to \mathbb{C}$ is holomorphic and $\|f\|$ obtains its minimum value at an interior point $a \in \Omega$, i.e. $\|f(a)\| \leq \|f(z)\|$ for all $z \in \Omega$. Show that either $f(a) = 0$ or f is a constant function.*

Proof. Clearly the result holds trivially if $f(a) = 0$. Thus, suppose that $f(a) \neq 0$. We want to show that $\|f(a)\| \leq \|f(z)\|$ for all $z \in \Omega$ implies that f is constant. Since $f(a) \neq 0$, we have that $\|f(a)\| > 0$, which implies that $f(z) \neq 0$ for all $z \in \Omega$. Hence, we can consider the reciprocal inequality of $\|f(a)\| \leq \|f(z)\|$; that is, $\|1/f(z)\| \leq \|1/f(a)\|$. Now since $f(a)$ is nonzero, the quantity on the right-hand side is finite and, since $f(z)$ is also nonzero, we have that $1/f(z)$ is analytic in Ω. Hence by the *Maximum Modulus Theorem*, we have that $1/f(z)$ must be constant. Therefore $f(z)$ is constant as well. □

Problem 4. *Give an example of a closed rectifiable curve γ in $\mathbb{C}$ such that, for any integer k, there is a point $a \notin \gamma$ with winding number $\eta(\gamma; a) = k$.*

Solution. Let $k \in \mathbb{Z}$ and consider the curve $\gamma \colon [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{kit}$; that is γ wraps around the circle $\mathbb{S}^1$ k times. Then for any point a in the bounded region determined by γ (that is, for any $a \in \mathring{\mathbb{D}}^2$), we have $\eta(\gamma; a) = k$. □

Problem 5. *Suppose f is holomorphic in an open set containing the closed unit disk, except for a simple pole at z_0 on the unit circle. Show that if $\sum_{n=0}^{\infty} a_n z^n$ denotes the power series expansion of f in the open unit disk, then $\lim_{n\to\infty} a_n / a_{n+1} = z_0$.*

Proof. Let us start by replacing z with z/z_0, assuming WLOG that the pole is at $z_0 = 1$. Now let Ω be an open set containing $\overline{\mathbb{D}}^2$ such that f is holomorphic on Ω except for a pole at 1. Then we have that

$$g(z) = f(z) - \sum_{k=1}^{N} \frac{a_{-k}}{(z-1)^k}$$

is holomorphic on Ω for some N and $a_{-1}, \ldots, a_{-N}$, where N is the order of the pole at 1. Next, we note that Ω must contain some disk of radius $1 + \delta$ with $\delta > 0$ (the set $\{z : \|z\| \leq 2\} \smallsetminus \Omega$ is compact, so its image under the map $z \mapsto \|z\|$ is also compact and hence attains a lower bound, which must be strictly greater than 1 since the unit circle is contained in Ω). Now since g converges on the disk $\|z\| < 1 + \delta$, we can expand it in a power series $\sum_{n=0}^{\infty} b_n z^n$ on this disk, and we must have $b_n \to 0$. (This follows from the fact that $\limsup b_{n+1}/b_n < 1$ when the radius of convergence is greater than 1.) Now for $\|z\| < 1$, we have

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{k=1}^{N} \frac{a_{-k}}{(z-1)^k} + \sum_{n=0}^{\infty} b_n z^n.$$

Now we use the following fact

$$\frac{1}{(z-1)^k} = \frac{(-1)^k}{(k-1)!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \left(\frac{1}{z-1} \right) = \frac{(-1)^k}{(k-1)!} \sum_{\ell=0}^{\infty} \frac{(\ell+k-1)!}{\ell!} z^\ell \quad \text{for } \|z\| < 1,$$

so that we can write

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{\ell=0}^{\infty} \left(\sum_{k=1}^{N} \frac{(-1)^k a_{-k}}{(k-1)!} \frac{(\ell+k-1)!}{\ell!} \right) z^\ell + \sum_{n=0}^{\infty} b_n z^n \implies a_n = P(n) + b_n,$$

where $P(n)$ is a polynomial in n of degree at most $N-1$. Here the rearrangements of the series are justified by the fact that all these series converge uniformly on compact subsets of $\mathbb{D}^2$. Then since $b_n \to 0$, we have

$$\frac{a_n}{a_{n+1}} \to \lim_{n\to\infty} \frac{P(n)}{P(n+1)} = 1.$$

Note that every polynomial P has the property that $P(n)/P(n+1) \to 1$ since, if the leading coefficient is $c_k n^k$, then we have

$$\frac{P(n)}{P(n+1)} \approx \frac{cn^k}{c(n+1)^k} = \left(1 - \frac{1}{n+1}\right)^k \to 1. \qquad \square$$