Geometry of General Relativity 2018
Workshop sheet 6

Gravitational waves

The purpose of this workshop is to show that the Einstein equations predict the existence of gravitational waves.

Consider a spacetime $(\mathbb{R}^4, g)$ with cartesian coordinates $x^\mu, \mu = 0, 1, 2, 3$, and a metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric and the components $h_{\mu\nu}$ are small compared to 1. Thus our spacetime is a small perturbation of Minkowski spacetime. In all subsequent calculations neglect any terms which are of quadratic order in $h_{\mu\nu}$. All indices are raised and lowered with respect to $\eta_{\mu\nu}$ (except for $g^{\mu\nu}$).

1. (a) Show the inverse metric is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and the Christoffel symbols are

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\nu\rho})$$

   Hence show that the linearised Ricci tensor is

$$R_{\mu\nu} = -\frac{1}{2}\partial^2 h_{\mu\nu} + \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial_\mu \partial_\nu h$$

   where $\partial^2 = \partial^\rho \partial_\rho$ and $h = h^\mu{}_\mu$.

   (b) Define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ and show that the linearised Einstein equation is

$$-\frac{1}{2}\partial^2 \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\mu} \bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = 8\pi G T_{\mu\nu}$$

2. The diffeomorphism invariance of GR implies that the linearised Einstein equations are invariant under the transformations $h \to h + \mathcal{L}_\xi \eta$ where ξ is a vector field.

   (a) Verify the linearised Einstein equation is invariant under $h \to h + \mathcal{L}_\xi \eta$.

   (b) Argue this can be used to fix $\partial^\rho \bar{h}_{\rho\mu} = 0$; what transformations preserve this condition? Deduce that the linearised Einstein equations reduces to

$$\partial^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

3. The linearised *vacuum* Einstein equation reduces to the wave equation $\partial^2 \bar{h}_{\mu\nu} = 0$ where $\partial^\rho \bar{h}_{\rho\mu} = 0$. Consider a plane wave solution of the form

$$\bar{h}_{\mu\nu} = \mathrm{Re}(\epsilon_{\mu\nu} e^{ik_\mu x^\mu})$$

   where the wave vector k^μ and polarisation $\epsilon_{\mu\nu}$ are constant.

   (a) Show that wave vector k^μ is null and the polarisation is transverse $\epsilon_{\mu\nu}k^\nu = 0$.

   (b) Argue that gravitational waves only two have independent 'physical' polarisations. You will need to consider the residual transformations $h \to h + \mathcal{L}_\xi \eta$.

Solution.

1. (a) We can directly verify

$$g^{\mu\rho}g_{\rho\nu} = (\eta^{\mu\rho}-h^{\mu\rho})(\eta_{\rho\nu}+h_{\rho\nu}) = \eta^{\mu\rho}\eta_{\rho\nu}+\eta^{\mu\rho}h_{\rho\nu}-h^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu+h^\mu{}_\nu-h^\mu{}_\nu = \delta^\mu_\nu$$

where we have neglected terms quadratic in h and raised and lowered indices with the Minkwoski metric. This shows that in this approximation $g^{\mu\nu}$ is the inverse metric.

Now consider the Christoffel symbols. Clearly we have $\partial_\mu g_{\nu\rho} = \partial_\mu h_{\nu\rho}$ since $\eta_{\nu\rho}$ are constants in these coordinates. Therefore, using the general expression for the Christoffel symbols and the inverse metric, it is clear that to first order in h they are given by the above expression.

Now consider the Riemann tensor. Since Γ is first order in h the terms quadratic in Γ can be neglected. Thus,

$$\begin{aligned} R_{\mu\nu} &= R^\rho{}_{\mu\rho\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\rho} \\ &= \frac{1}{2}\eta^{\rho\sigma}(\partial_\rho\partial_\mu h_{\sigma\nu} + \partial_\rho\partial_\nu h_{\sigma\mu} - \partial_\rho\partial_\sigma h_{\mu\nu}) - (\nu \leftrightarrow \rho) \\ &= -\frac{1}{2}\partial^2 h_{\mu\nu} + \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial_\mu\partial_\nu h \end{aligned}$$

where $(\nu \leftrightarrow \rho)$ stands for the same terms with ν and ρ swapped.

(b) We need the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ and thus we need the Ricci scalar. Since $R_{\mu\nu}$ is first order in h we find that

$$R = g^{\mu\nu}R_{\mu\nu} = \eta^{\mu\nu}\left(-\frac{1}{2}\partial^2 h_{\mu\nu} + \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial_\mu\partial_\nu h\right) = -\partial^2 h + \partial^\rho\partial^\sigma h_{\rho\sigma}$$

where we have relabelled an index in the last equality. Thus to first order

$$G_{\mu\nu} = -\frac{1}{2}\partial^2 h_{\mu\nu} + \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial_\mu\partial_\nu h + \frac{1}{2}\eta_{\mu\nu}\partial^2 h - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma}$$

Now convert to $\bar{h}$ defined above. In particular, we have

$$\begin{aligned} \partial^\rho\partial_{(\mu}h_{\nu)\rho} &= \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} + \frac{1}{4}\partial^\rho\partial_\mu(h\eta_{\nu\rho}) + \frac{1}{4}\partial^\rho\partial_\nu(h\eta_{\mu\rho}) = \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} + \frac{1}{2}\partial_\mu\partial_\nu h \\ \partial^\rho\partial^\sigma h_{\rho\sigma} &= \partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} + \frac{1}{2}\partial^2 h \end{aligned}$$

which then gives

$$G_{\mu\nu} = -\frac{1}{2}\partial^2\bar{h}_{\mu\nu} + \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma}$$

Using the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ then gives the answer.

2. (a) First note that in Minkowski spacetime $\mathcal{L}_\xi\eta_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. Hence the diffeomorphism invariance can be written as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

which implies

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\rho \xi_\rho$$

Thus,

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu \bar{h}_{\mu\nu} + \partial^2 \xi_\nu + \partial_\nu \partial^\mu \xi_\mu - \partial_\nu \partial^\rho \xi_\rho = \partial^\mu \bar{h}_{\mu\nu} + \partial^2 \xi_\nu$$

Now plug into the Einstein tensor to check that $G_{\mu\nu} \to G_{\mu\nu}$; you will need to commute partial derivatives to check all terms depending on ξ actually cancel.

(b) From the previous part

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu \bar{h}_{\mu\nu} + \partial^2 \xi_\nu$$

Therefore if we choose ξ to obey the wave equation $\partial^2 \xi_\nu = -\partial^\mu \bar{h}_{\mu\nu}$ then we can fix $\partial^\mu \bar{h}_{\mu\nu} \to 0$. It also follows that any transformation which preserves $\partial^\mu \bar{h}_{\mu\nu} = 0$ must obey $\partial^2 \xi = 0$.

It then follows $\partial^\rho \partial_\mu \bar{h}_{\nu\rho} = 0$ and hence the Einstein equations reduce as claimed.

3. We will surpress Re in all equations.

(a) Differentiating $\partial_\rho \bar{h}_{\mu\nu} = \partial_\rho(\epsilon_{\mu\nu} e^{ik_\sigma x^\sigma}) = \epsilon_{\mu\nu} i k_\sigma \delta^\sigma_\rho e^{ik_\sigma x^\sigma} = ik_\rho \bar{h}_{\mu\nu}$. Repeating we get $\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -k^\rho k_\rho \bar{h}_{\mu\nu}$. Hence the wave equation is obeyed iff $k^\rho k_\rho = 0$, i.e. if k is null.

The condition $0 = \partial^\mu \bar{h}_{\mu\nu} = ik^\mu \epsilon_{\mu\nu} e^{ix^\rho k_\rho}$ gives $k^\mu \epsilon_{\mu\nu} = 0$.

(b) As shown above residual transformations are solutions to $\partial^2 \xi_\mu = 0$. We may solve this wave equation in the same way: $\xi_\mu = v_\mu e^{ik_\rho x^\rho}$ where v_μ is constant which obeys wave equation since k is null. The residual symmetry therefore acts as

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + ik_\mu v_\nu + ik_\nu v_\mu - i\eta_{\mu\nu} k^\rho v_\rho$$

The $\epsilon_{\mu\nu}$ is a symmetric constant tensor subject to four algebraic conditions $k^\mu \epsilon_{\mu\nu} = 0$ and the residual symmetry may be used to impose four other algebraic conditions (since it is parameterised by a vector v_μ). Hence the number of independent polarisation components is $10 - 4 - 4 = 2$.

A standard choice is to use this to set $\epsilon_{0\mu} = 0$ and $\epsilon^\mu{}_\mu = 0$. If we suppose the wave travels in the z-direction so $k^\mu = (\omega, 0, 0, \omega)$ it can then be shown that the general ϵ which is transverse and satisfies these standard choices is

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where ϵ_+ and $\epsilon_\times$ are constants which correspond to the two independent polarisations.