MATH 709 NOTES
INTRODUCTION TO SMOOTH MANIFOLDS

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Some Preliminaries

Definition. *A topological space X is*

- ***connected*** *if there do not exist two disjoint, nonempty, open subsets of X whose union is X.*
- ***path-connected*** *if every pair of points in X can be joined by a path in X.*
- ***locally path-connected*** *if X has a basis of path-connected open subsets.* ★

Proposition 1. *If M is a topological manifold, then*

a) *M is locally path-connected.*

b) *M is connected if and only if it is path-connected (In general, connected $\not\Rightarrow$ path-connected, but in manifolds this is always true.)*

c) *The connected components of M are exactly the path-connected components.*

d) *M has countably many components, each of which is open in M and is a connected topological manifold.*

Definition. *A topological space X is said to be* ***locally compact*** *if every point has a neighborhood contained in a compact subset of X.* ★

Definition. *A subset of a topological space X is said to be* ***precompact*** *in X if its closure in X is compact.* ★

Proposition 2. *For a Hausdorff space X the following are equivalent:*

a) *X is locally compact.*

b) *Each point of X has a precompact neighborhood.*

c) *X has a basis of precompact open subsets.*

Lemma 1. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proposition 3. *Every topological manifold is locally compact.*

Proof. From Lemma 1 we have that every point in a topological manifold is contained in a precompact coordinate ball. □

Let X be a topological space. Then we have the following definitions:

Definition. *A collection $\{S_i\}$ of subsets of X is said to be **locally finite** if each point of X has a neighborhood that intersects at most finitely many of the sets in $\{S_i\}$.* ★

Definition. *Given a cover $\mathcal{U}$ of X, another cover $\mathcal{V}$ is called a **refinement** of $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$.* ★

Definition. *We say that X is **paracompact** if every open cover of X admits an open, locally finite refinement.* ★

Theorem 1. *Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{U}$ of M, and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.*

Theorem 2. *Let M be a topological manifold. Then $\pi_1(M, *)$ has countably many elements.*

Proof. See proof on page 10, Lee's *Smooth Manifolds.* □

Smooth Manifolds

Definition. *A **coordinate chart** (or just a **chart**) on a topological manifold M is a pair (U, φ), where U is an open subset of M and $\varphi\colon U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (see* Figure 1 *below.)* ★

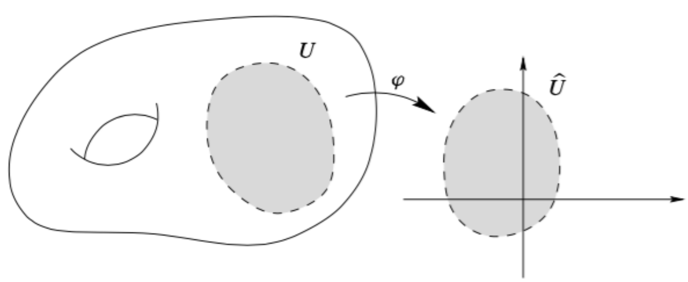

Figure 1. A coordinate chart.

Definition. *Let M be a topological n-manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is called the **transition map** from φ to ψ (see* Figure 2 *below). It is a composition of homeomorphisms, and is therefore itself a homeomorphism.* ★

Definition. *Two charts (U, φ) and (V, ψ) are said to be **smoothly compatible** if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders.* ★

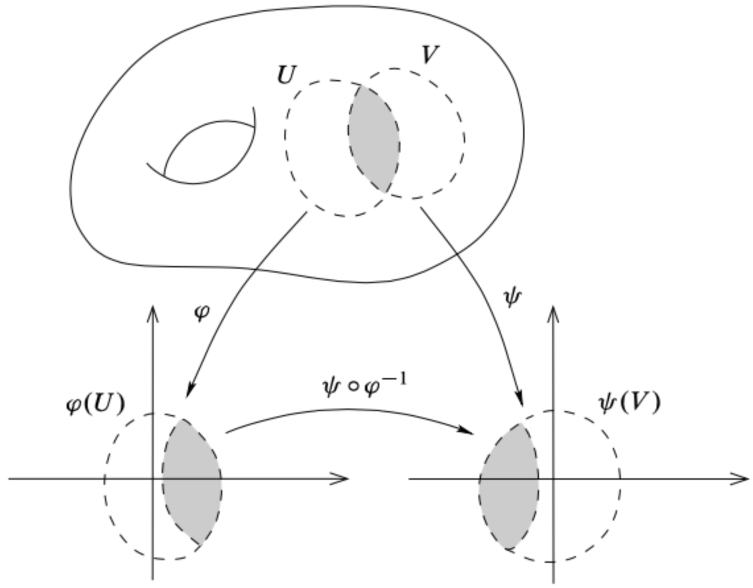

FIGURE 2. A transition map.

Definition. *We define an **atlas** for a manifold M to be a collection of charts whose domains cover M. An atlas $\mathscr{A}$ is called a **smooth atlas** if any two charts in $\mathscr{A}$ are smoothly compatible with each other.* ★

Remark 1: Note that to show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U, φ) and (V, ψ) are charts in $\mathscr{A}$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$ is one of the transition maps we have already shown to be smooth.

Remark 2: Alternatively, given two particular charts (U, φ) and (V, ψ) it is often easiest to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to the following proposition:

Proposition 4. *Suppose $U \subseteq \mathbb{R}^n$ is an open subset, and $F \colon U \to \mathbb{R}^n$ is a smooth function whose Jacobian determinant is nonzero at every point in U.*

a) F is an open map.

b) If F is injective, then $F \colon U \to F(U)$ is a diffeomorphism.

Our plan is to define a "smooth structure" on M by giving a smooth atlas, and to define a function $f \colon M \to \mathbb{R}$ to be smooth if and only if $f \circ \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, φ) in the atlas. There is one minor technical problem with this approach: in general, there will be many possible atlases that give the "same" smooth structure, in that they all determine the same collection of smooth functions on M. For example, consider the following pair of atlases on $\mathbb{R}^n$:

$$\mathcal{A}_1 = \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\},$$
$$\mathcal{A}_2 = \{(B_1(x), \mathrm{Id}_{B_1(x)}) \mid x \in \mathbb{R}^n\}.$$

Although these are different smooth atlases, clearly a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is smooth with respect to either atlas if and only if it is smooth in the sense of ordinary calculus. We could choose to define a smooth structure as an equivalence class of smooth atlases under an appropriate equivalence relation. However, it is more straightforward to make the following definitions:

Definition. *A smooth atlas $\mathscr{A}$ on M is* ***maximal*** *(or* ***complete****) if it is not properly contained in any larger smooth atlas. This just means that any chart that is smoothly compatible with every chart in $\mathscr{A}$ is already in $\mathscr{A}$.* ★

Definition. *If M is a topological manifold, a* ***smooth structure*** *on M is a maximal smooth atlas. A* ***smooth manifold*** *is a pair $(M, \mathscr{A})$, where M is a topological manifold and $\mathscr{A}$ is a smooth structure on M.* ★

It is generally not very convenient to define a smooth structure by explicitly describing a maximal smooth atlas, because such an atlas contains very many charts. Fortunately, we need only specify some smooth atlas, as the next proposition shows:

Proposition 5. *Let M be a topological manifold.*

a) *Every smooth atlas $\mathscr{A}$ for M is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathscr{A}$.*
b) *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

Remark: For example, if a topological manifold M can be covered by a single chart, the smooth compatibility condition is trivially satisfied, so any such chart automatically determines a smooth structure on M. An example would be $\mathscr{A}_1 = \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\}$.

Definition. *A set $B \subseteq M$ is called a* ***regular coordinate ball*** *if there is a smooth coordinate ball $B' \supseteq \overline{B}$ and a smooth coordinate map $\varphi\colon B' \to \mathbb{R}^n$ such that for some positive real numbers $r < r'$, we have*

$$\varphi(B) = B_r(0), \quad \varphi(\overline{B}) = \overline{B}_r(0), \quad \text{and} \quad \varphi(B') = B_{r'}(0).$$

Proposition 6. *Every smooth manifold has a countable basis of regular coordinate balls.*

Usually we construct a smooth manifold structure in two stages: we start with a topological space and check that it is a topological manifold, and then we specify a smooth structure. However it is often more convenient to combine these two steps into a single construction, especially if we start with a set that is not already equipped with a topology. The following lemma provides a shortcut; it shows how, given a set with suitable "charts" that overlap smoothly, we can use the charts to define both a topology and a smooth structure on the set:

Lemma 2 (**Smooth Manifold Chart Lemma**)**.** *Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:*

a) For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

b) For each α and β, the sets $\varphi_\alpha(U_\alpha \bigcap U_\beta)$ and $\varphi_\beta(U_\alpha \bigcap U_\beta)$ are open in $\mathbb{R}^n$.

c) Whenever $U_\alpha \bigcap U_\beta \neq \emptyset$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\alpha \bigcap U_\beta)$ is smooth.

d) Countably many of the sets U_α cover M.

e) Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof. See proof on Page 22, Lee's *Smooth Manifolds*. □