Math 260 HW # 5

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Section 2.3

(11) Let V be a VS, and let $T : V \longrightarrow V$ be linear. Prove that $T^2 = T_0$ iff $R(T) \subseteq N(T)$.

Proof:
($\Rightarrow$)
Assume that T is linear and $T^2 = T_0$, then we need to prove that $R(T) \subseteq N(T)$.
Let's pick an element $w \in R(T)$ so that $w = T(x)$ for some $x \in V$. But then since $T^2 = T_0$, we have $T^2(x) = T\,(T(x)) = T\,w = T_0(x) = 0$. Hence $w \in N(T)$, proving that any element contained in $R(T)$ is also contained in $N(T)$.
($\Leftarrow$)
We assume that $R(T) \subseteq N(T)$, then we need to prove that $T^2 = T_0$.
We pick an element $w \in R(T)$ so that $w = T(x)$ for some $x \in V$. But then since $R(T) \subseteq N(T)$, we know that $w \in N(T)$. Hence $T^2(x) = T(T(x)) = T\,w = 0 = T_0(x)$. ■

Section 2.4

(2) For the following linear transformation T, determine whether T is invertible and justify your answer.

b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(a_1,\, a_2) = (3\,a_1 - a_2,\, a_2,\, 4\,a_1)$

Solution:
T cannot possibly be invertible because in order to be invertible both the domain and codomain must have the same dimension and we have $\dim(\mathbb{R}^2) = 2 \neq 3 = \dim(\mathbb{R}^3)$. ✵

(17) Let V and W be finite dimensional VS's and $T : V \longrightarrow W$ be an isomorphism. Let V_0 be a subspace of V. Then

a) Prove that $T(V_0)$ is a subspace of W.

Proof:
⟜ Since V_0 is a subspace of V, we know that $0_V \in V_0$. Since T is an isomorphism it is also injective,

thus we have $T(0_V) = 0_W \in T(V_0) \subseteq W$. ✓

⟷ Pick two elements $T\,v_1,\ T\,v_2 \in T(V_0)$, with $v_1,\ v_2 \in V_0$. Since T is an isomorphism it is also linear, thus we have $T\,v_1 + T\,v_2 = T(v_1 + v_2) \in T(V_0) \subseteq W$. ✓

⟷ Pick a scalar c and an element $T\,v_1 \in T(V_0)$, with $v_1 \in V_0,\ \ c \in \mathbb{F}$.

Since T is linear we have $c\,T\,v_1 = T(c\,v_1) \in T(V_0) \subseteq W$ ✓

Hence since $T(V_0)$ is closed under addition and scalar multiplication, and also contains the zero vector in W, we conclude that $T(V_0)$ is a subspace of W. ■

b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Proof:

We are given that T is an isomorphism that maps elements from V to W. Since V_0 is a subspace of V and we just proved that $T(V_0)$ is a subspace of W, we may consider a linear map T' that maps elements from V_0 to $T\,(V_0)$, i.e. $T' \in \mathcal{L}(V_0,\ T(V_0))$. This T' is also an isomorphism, since T is just an extension of this linear map applied to the ambient vector spaces V and W. Hence, since T' is also an isomorphism, the dimension of the domain and the codomain have to be the same, i.e. $\dim(V_0) = \dim(T(V_0))$. ■

(Extra Problem) Prove that $\mathbb{C} \cong \mathbb{R}^2$ as vector spaces over $\mathbb{R}$ and use this to show that $\dim(\mathbb{C}) = 2$. Furthermore, show that $\mathbb{C}^n \cong \mathbb{R}^{2\,n}$, and $\dim(\mathbb{C}^n) = 2\,n$.

(Hint: Construct a map from $\mathbb{C}$ to $\mathbb{R}^2$ and prove that this map is linear and an isomorphism. Modify this isomorphism into a map between $\mathbb{C}^n$ and $\mathbb{R}^{2\,n}$ and prove that this map is also linear and an isomorphism.)

Note This isomorphism is the reason why we can graph complex numbers using the standard $x\,y$ plane.

Proof:

We pick an arbitrary element $a_1 + i\,b_1 \in \mathbb{C}$ and we map it to an image $(a_1,\ b_1) \in \mathbb{R}^2$. We call this map T and we must first show that it is a linear transformation, i.e. $T \in \mathcal{L}(\mathbb{C},\ \mathbb{R}^2)$, and furthermore we must prove that this map is also bijective. By proving this we are showing that T is an isomorphism and thus $\mathbb{C} \cong \mathbb{R}^2$.

⟷ Check for linearity:

Given a scalar $c \in \mathbb{R}$ and two elements $v_1,\ v_2 \in \mathbb{C}$ such that $v_1 = a_1 + i\,b_1$ and $v_2 = a_2 + i\,b_2$, we must show that $T(c\,v_1 + \ v_2) = c\,T\,v_1 + T\,v_2$. Thus we have

$$\begin{aligned} T(c\,v_1 + \ v_2) &= T(c(a_1 + i\,b_1) + a_2 + i\,b_2) = T(c\,a_1 + i\,c\,b_1 + a_2 + i\,b_2) \\ &= T(c\,a_1 + a_2 + i\,(c\,b_1 + \ b_2) = (c\,a_1 + a_2,\ c\,b_1 + \ b_2) \\ &= (c\,a_1,\ c\,b_1) + (a_2,\ b_2) = c(\,a_1,\ b_1) + (a_2,\ b_2) \end{aligned}$$

$$= c\,T(a_1 + i\,b_1) + T(a_2 + i\,b_2) = c\,T\,v_1 + T\,v_2 \quad \checkmark$$

Thus we have proven that T is linear.

⟼ Check for injectivity:
We know that injectivity implies that $\mathcal{N}(T) = \{0\}$. We have that $T(a_1 + i\,b_1) = (a_1,\ b_1)$.
Thus,

$$T(a_1 + i\,b_1) = (0,\ 0) \Longrightarrow a_1,\ b_1 = 0$$
$$\Longrightarrow T(0 + i\,0) = (0,\ 0) \Longrightarrow T(0) = (0,\ 0)$$
$$\Longrightarrow \text{nullity}(T) = 0 \Longrightarrow \mathcal{N}(T) = \{0\}$$

Hence T is injective. ✓

⟼ Check for surjectivity:
Surjectivity implies that $R(T) = \mathbb{R}^2$. Clearly by construction the map T is surjective, since T maps all $a_i + i\,b_i \in \mathbb{C}$ to $(a_i,\ b_i) \in \mathbb{R}^2\ \forall\ i$. In other words, we can express $(a_i,\ b_i) = a_i(1,\ 0) + b_i(0,\ 1)\ \forall\ i$. Hence $\{(1,\ 0),\ (0,\ 1)\}$ is a spanning set for $R(T)$ which is also linearly independent, hence it's a basis (standard basis for $\mathbb{R}^2$). Thus, the range of this linear map T is simply the entire codomain $\mathbb{R}^2$.
Hence T is surjective. ✓

We have proven that T is linear and furthermore that it's bijective. Hence T is an isomorphism and thus $\mathbb{C} \cong \mathbb{R}^2$. ✓

By the Rank-Nullity theorem we know that $\dim(\mathbb{C}) = \text{rank}\ T + \text{nullity}\ T$. Hence in this case we have $\dim(\mathbb{C}) = \dim(\mathbb{R}^2) + 0 = 2 + 0 = 2$. ✓

Now we only need to show that $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and that $\dim(\mathbb{C}^n) = 2\,n$.
We have an $n-$tuple in $\mathbb{C}^n$, and we want to map it to $\mathbb{R}^{2n}$. Let's call this map T'. So we have
$T'(a_1 + i\,b_1,\ a_2 + i\,b_2,\ \ldots,\ a_n + i\,b_n) = (a_1,\ b_1,\ a_2,\ b_2,\ \ldots,\ a_n,\ b_n)$.
It is immediately obvious that T' is also an isomorphism. We prove that it's linear by a similar argument as we used for T. The nullspace is also trivial (thus T' is injective) since $T'(a_1 + i\,b_1,\ a_2 + i\,b_2,\ \ldots,\ a_n + i\,b_n) = (0,\ 0,\ \ldots,\ 0) \Longrightarrow a_i,\ b_i = 0\ \forall\ i \Longrightarrow \mathcal{N}(T') = \{0\}$. Also by construction T' is surjective since the range is simply the entire codomain $\mathbb{R}^{2n}$,i.e. $R(T') = \mathbb{R}^{2n}$. Thus, since T' is an isomorphism we have that $\mathbb{C}^n \cong \mathbb{R}^{2n}$. ✓

Lastly, by the Rank-Nullity theorem we have

$$\dim(\mathbb{C}^n) = \text{nullity}(T') + \text{rank}(T') = 0 + \dim(\mathbb{R}^{2n}) = 0 + 2\,n = 2\,n. \quad \checkmark \qquad \blacksquare$$