MATH 752
OPTIONAL PROBLEM SET # 1

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Problem 1. *Let $\mathcal{C}$ be a category with congruence $\sim$. Let $\mathcal{C}' = \mathcal{C}/\sim$ denote the quotient category. Show that there is a functor from $\mathcal{C}$ to $\mathcal{C}'$ which takes each morphism of $\mathcal{C}$ to its equivalence class.*

Proof. Let us define $F\colon \mathcal{C} \to \mathcal{C}'$ as $F(A) = A$ for every $A \in \mathrm{obj}(\mathcal{C})$. For $A, B \in \mathrm{obj}(\mathcal{C})$, and for a morphism $f\colon A \to B$, denote the equivalence class of f by $[f]$. Define $F(f)$ to be $[f]$. Then $F(f) \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. By the definition of the quotient category, $[g] \circ [f] = [g \circ f]$ if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Thus $F(g) \circ F(f) = F(g \circ f)$. If 1_A is the identity morphism $A \to A$ in the category $\mathcal{C}$, then $F(1_A) = [1_A]$ is the identity morphism $A \to A$ in the category $\mathcal{C}'$. Thus F is a functor. $\square$

Problem 2. *For an abelian group G, let*

$$tG = \{x \in G \mid x \text{ has finite order}\}$$

denote its ***torsion subgroup****.*

a) Show that t defines a functor from **Ab** *to* **Ab** *if one defines $t(f) = f|_{tG}$ for every homomorphism f.*

b) Show that if f is injective, then $t(f)$ is injective.

c) Give an example of a surjective homomorphism f for which $t(f)$ is not surjective.

Proof of a). To show that t defines a functor, we need to demonstrate the following:

- Let G be an abelian group. Since all subgroups of abelian groups are abelian, it follows that tG is abelian.
- Let G and G' be abelian groups and assume $f\colon G \to G'$ is a homomorphism. Let $x \in tG$. Then by definition $x^n = e$ for some $n \in \mathbb{N}$, where e denotes the identity element of G. Since f is a homomorphism, it follows that $f(x)^n = f(x^n) = f(e) = e'$, where e' denotes the identity element of G'. Thus $f(tG) \subset tG'$, and it follows that the restriction homomorphism $t(f)\colon tG \to tG'$ is well defined.
- Let G_1, G_2, and G_3 be abelian groups and let $f\colon G_1 \to G_2$ and $g\colon G_2 \to G_3$ be homomorphisms. Then for every $x \in G_1$, we have
$$\begin{aligned}(t(g) \circ t(f))(x) &= t(g)(t(f)(x)) = t(g)(f(x)) = g(f(x)) \\ &= (g \circ f)(x) = t(g \circ f)(x).\end{aligned}$$
Thus, we have that $t(g) \circ t(f) = t(g \circ f)$.
- Finally, let G be an abelian group, and let $1_G\colon G \to G$ be the identity homomorphism. Then the restriction of 1_G to tG is the identity homomorphism $tG \to tG$, i.e., $t(1_G) = 1_{tG}$. $\square$

Proof of b). Restrictions of injective maps are injective. Thus $t(f)\colon tG \to tG'$ is injective if $f\colon G \to G'$ is injective. □

Solution of c). Consider $\mathbb{S}^1$, which is an abelian group when equipped with complex multiplication. The set $\mathbb{R}$ of all real numbers equipped with addition is also an abelian group. Define

$$f\colon \mathbb{R} \to \mathbb{S}^1 \qquad \text{such that} \qquad x \mapsto e^{ix}.$$

Then f is a surjective group homomorphism. Now, $t\mathbb{R} = \{0\}$ and $\{-1, 1\} \subset t\mathbb{S}^1$. It follows that $t(f)\colon t\mathbb{R} \to t\mathbb{S}^1$ cannot be surjective. □

Problem 3. *Show that there is no functor from* **Group** *to* **Ab** *sending each group G to its center.*[1] *(Consider $S_2 \to S_3 \to S_2$, where S_2 and S_3 denote symmetric groups.)*

Proof. Let $\bar{n} = \{1, \dots, n\}$, for $n \geq 2$. Let S_n denote the group of permutations of elements of $\bar{n}$. Let $\sigma \in S_n$. Then σ is a bijection $\bar{n} \to \bar{n}$. The number of inversions of σ means the number of pairs of elements $x, y \in \bar{n}$ such that $x < y$ but $\sigma(x) > \sigma(y)$. The sign of a permutation σ is

$$\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)},$$

where $N(\sigma)$ denotes the number of inversions in σ. Identify the group S_2 with the multiplication group $\{-1, 1\}$. Then

$$\operatorname{sgn}\colon S_n \to S_2 \qquad \text{such that} \qquad \sigma \mapsto \operatorname{sgn}(\sigma)$$

is a group homomorphism. Denote the center of a group G by $Z(G)$. Then $Z(S_2) = S_2$ while $Z(S_3)$ is trivial. Let $f\colon S_2 \to S_3$ be the group homomorphism taking the nontrivial element of S_2 to the element of S_3 that permutes the numbers 1 and 2. Then $\operatorname{sgn} \circ f\colon S_2 \to S_2$ is the identity homomorphism. Assume there is a functor F from **Group** to **Ab** sending each group G to its center. Then $\operatorname{sign}\colon S_3 \to S_2$ induces the (trivial) homomorphism $F(\operatorname{sgn})\colon \{e\} \to S_2$. Since F is a functor, it follows that

$$1_{S_2} = F(1_{S_2}) = F(\operatorname{sgn} \circ f) = F(\operatorname{sign}) \circ F(f).$$

But this is impossible since $F(\operatorname{sgn})F(f)$ is the trivial homomorphism. $(\Rightarrow\Leftarrow)$

Hence such a functor F cannot exist. □

Problem 4. *Let X be a topological space. Let $x_0, x_1 \in X$ and let $f_i\colon X \to X$ for $i = 0, 1$ denote the constant map at x_i. Prove that $f_0 \simeq f_1$ if and only if there is a continuous $F\colon I \to X$ with $F(0) = x_0$ and $F(1) = x_1$.*

Proof. $(\Rightarrow)$ Assume first that there is a homotopy $H\colon f_0 \simeq f_1$, i.e., a continuous map $H\colon X \times I \to X$ with $H(x, 0) = f_0(x) = x_0$ and $H(x, 1) = f_1(x) = x_1$ for all $x \in X$. Let

$$F\colon I \to X \qquad \text{such that} \qquad t \mapsto H(x_0, t).$$

Then $F(0) = H(x_0, 0) = x_0$ and $F(1) = H(x_0, 1) = x_1$. Since the map

$$h_0\colon I \to I \times \{x_0\} \qquad \text{such that} \qquad t \mapsto (t, x_0)$$

is continuous and $F = H \circ h_0$, it follows that F is continuous.

[1] Recall that the center of a group G is the set of elements that commute with every element of G.

($\Leftarrow$) Conversely, assume that there is a continuous $F\colon I \to X$ with $F(0) = x_0$ and $F(1) = x_1$. Let $\pi_I\colon X \times I \to I$ be the projection and let $H = F \circ \pi_I\colon X \times I \to X$. Then H is continuous, and $H(x, 0) = F(0) = x_0$ and $H(x, 1) = F(1) = x_1$ for all $x \in X$. Thus $f_0 \simeq f_1$, as desired. $\square$

Problem 5. *Let X and Y be subspaces of a Euclidean space. Assume X is convex and X and Y have the same homotopy type. Show that Y does not need to be convex.*

Proof. Let $X = \{(x, 0) \mid x \in \mathbb{R}\}$ and let $Y = \{(x, x^2) \mid x \in \mathbb{R}\}$. Then X and Y are both subsets of $\mathbb{R}^2$. Clearly, X is convex but Y is not convex. Let

$$f\colon X \to Y \qquad \text{such that} \qquad (x, 0) \mapsto (x, x^2),$$

and let

$$g\colon Y \to X \qquad \text{such that} \qquad (x, x^2) \mapsto (x, 0).$$

Then f is a bijection and g is its inverse function. Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ denote the projection $(x, y) \mapsto x$, and let $\pi|_Y$ denote the restriction of π to Y. Take the inclusion $\iota\colon \mathbb{R} \to \mathbb{R}^2$ that send $x \mapsto (x, 0)$, so that $g = \iota \circ \pi|_Y$. Since both ι and $\pi|_Y$ are continuous, it follows that g is continuous. Similarly, f is continuous. Thus the sets X and Y are homeomorphic, which implies that they have the same homotopy type. $\square$