M746 NOTES
DIFFERENTIATION

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The Averaging Problem

The following question is referred to as the **averaging problem**:

Suppose f is integrable on $\mathbb{R}^d$. Is it true that

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x), \quad \text{for a.e. } x\ ?$$

Note of course that in the special case when f is continuous at x, the limit does converge to $f(x)$. Indeed, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

Since

$$f(x) - \frac{1}{m(B)} \int_B f(y)\,dy = \frac{1}{m(B)} \int_B (f(x) - f(y))\,dy,$$

we find that whenever B is a ball of radius $< \delta/2$ that contains x, then

$$\left| f(x) - \frac{1}{m(B)} \int_B f(y)\,dy \right| \leq \frac{1}{m(B)} \int_B |f(x) - f(y)|\,dy < \varepsilon.$$

as desired.

The averaging problem has an affirmative answer, but to establish that fact, which is qualitative in nature, we need to make some quantitative estimates bearing on the overall behavior of the averages of f. This will be done in terms of the maximal averages of $|f|$, to which we now turn.

Hardy-Littlewood Maximal Function

If f is integrable on $\mathbb{R}^d$, we define its **maximal function** f^* by

$$f^*(x) = \sup_{B\colon x\in B} \frac{1}{m(B)} \int_B |f(y)|\, dy, \qquad x \in \mathbb{R}^d,$$

where the supremum is taken over all balls containing the point x. In other words, we replace the limit in the statement of the averaging problem by a supremum, and f by its absolute value.

The main properties of f^* we shall need are summarized in the following theorem:

Theorem 1. *Suppose f is integrable on $\mathbb{R}^d$. Then,*

i) *f^* is measurable.*

ii) *$f^*(x) < \infty$ for a.e. x.*

iii) *f^* satisfies*

$$m(\{x \in \mathbb{R}^d \mid f^*(x) > \alpha\}) \leq \frac{3^d}{\alpha}\|f\|_{L^1(\mathbb{R}^d)} \quad \forall\, \alpha > 0. \tag{$\clubsuit$}$$

Proof of i). (i) The assertion that f^* is measurable is pretty simple. Indeed, the set $E_\alpha = \{x \in \mathbb{R}^d\colon f^*(x) > \alpha\}$ is open, because if $\bar{x} \in E_\alpha$, there exists a ball B such that $\bar{x} \in B$ and

$$\frac{1}{m(B)} \int_B |f(y)|\, dy > \alpha.$$

Now any point x close enough to $\bar{x}$ will also belong to B; hence $x \in E_\alpha$ as well. $\square$

Proof of ii). This condition follows directly from (iii) once we observe that

$$\{x \mid f^*(x) = \infty\} \subset \{x \mid f^*(x) > \alpha\} \qquad \forall\ \alpha.$$

Taking the limit as α tends to infinity, the third property yields $m(\{x \mid f^*(x) = \infty\}) = 0$. $\square$

Proof of iii). The proof of inequality ($\clubsuit$) relies on an elementary version of a Vitali covering argument, which is stated in the following lemma:

Lemma 1. *Suppose $B = \{B_1, B_2, \ldots, B_N\}$ is a finite collection of open balls in $\mathbb{R}^d$. Then there exists a disjoint sub-collection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of B that satisfies*

$$(\heartsuit) \qquad m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^d \sum_{j=1}^{k} m(B_{i_j}).$$

Loosely speaking, the lemma tells us that we may always find a disjoint subcollection of balls that covers a fraction of the region covered by the original collection of balls.

Now the proof of *iii*) is within reach. If we let $E_\alpha = \{x \colon f^*(x) > \alpha\}$, then for each $x \in E_\alpha$ there exists a ball B_x that contains x, and such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)| \, dy > \alpha.$$

Therefore, for each ball B_x we have

$$(\dagger) \qquad m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)| \, dy.$$

Fix a compact subset K of E_α. Since K is covered by $\bigcup_{x \in E_\alpha} B_x$, we may select a finite subcover of K, say $K \subset \bigcup_{l=1}^{N} B_l$. The covering lemma discussed above guarantees the existence of a sub-collection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of disjoint balls that satisfies inequality ($\heartsuit$). Now since these balls $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ are disjoint and satisfy ($\heartsuit$) as well as ($\dagger$), we find that

$$\begin{aligned} m(K) \leq m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^d \sum_{j=1}^{k} m(B_{i_j}) &\leq \frac{3^d}{\alpha} \sum_{j=1}^{k} \int_{B_{i_j}} |f(y)| \, dy \\ &= \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f(y)| \, dy \\ &\leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, dy. \end{aligned}$$

Since this inequality is true for all compact subsets K of E_α, the proof of the weak type inequality for the maximal operator is complete. $\square$

Remark 1: Let us clarify the nature of the main conclusion *iii*). As we shall observe later when we prove the *Lebesgue Differentiation Theorem*, one has that $f^*(x) \geq |f(x)|$ for a.e. x; the effect of *iii*) is that, broadly speaking, f^* is not much larger than $|f|$. From this point of view, we would have liked to conclude that f^* is integrable, as a result of the assumed integrability of f. However, this is not the case, and *iii*) is the best substitute available (see exercises 4 and 5, Stein's Chapter 3).

Remark 2: An inequality of the type (♣) is called a **weak-type** inequality because it is weaker than the corresponding inequality for the L^1-norms. Indeed, this can be seen from the *Tchebychev Inequality*, which states that for an arbitrary integrable function g,

$$m(\{x\colon |g(x)| \geq \alpha\}) \leq \frac{1}{\alpha}\|g\|_{L^1(\mathbb{R}^d)} \quad \forall\, \alpha > 0.$$

We should add that the value of 3^d in the inequality (♣) is unimportant for us. What matters is that this constant be independent of α and f.

The estimate obtained for the maximal function now leads to a solution of the averaging problem:

Theorem 2 (**The Lebesgue Differentiation Theorem**). *If f is integrable on $\mathbb{R}^d$, then*

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)}\int_B f(y)\,dy = f(x), \quad \textit{for a.e. } x. \tag{1}$$

Proof. It suffices to show that for each $\alpha > 0$ the set

$$E_\alpha = \left\{ x : \limsup_{\substack{m(B)\to 0\\ B:x\in B}} \left| \frac{1}{m(B)}\int_B f(y)\,dy - f(x) \right| > 2\alpha \right\}$$

has measure zero, because this assertion then guarantees that the set $E = \bigcup_{n=1}^{\infty} E_{1/n}$ has measure zero, and the limit in (1) holds at all points of E^c. We fix α, and invoke a previous theorem, which states that for each $\varepsilon > 0$ we may select a continuous function g of compact support with $\|f-g\|_{L^1(\mathbb{R}^d)} < \varepsilon$. Then since g is continuous, (1) holds not just for a.e. x, but in fact for all x [1]. That is,

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)}\int_B g(y)\,dy = g(x), \quad \text{for all } x.$$

Since we may write the difference $\frac{1}{m(B)}\int_B f(y)\,dy - f(x)$ as

$$\frac{1}{m(B)}\int_B (f(y)-g(y))\,dy + \frac{1}{m(B)}\int_B g(y)\,dy - g(x) + g(x) - f(x)$$

we find that

$$\limsup_{\substack{m(B)\to 0\\ B:x\in B}} \left| \frac{1}{m(B)}\int_B f(y)\,dy - f(x) \right| \leq (f-g)^*(x) + |g(x)-f(x)|,$$

[1] This is shown on *Page 1* of these notes in the discussion of the *averaging problem*.

where the symbol $*$ indicates the maximal function. Consequently, if

$$F_\alpha = \{x : (f-g)^*(x) \geq \alpha\} \quad \text{and} \quad G_\alpha = \{x : |f(x) - g(x)| \geq \alpha\},$$

then $E_\alpha \subset (F_\alpha \bigcup G_\alpha)$, because if u_1 and u_2 are positive, then $u_1 + u_2 > 2\alpha$ only if $u_i > \alpha$ for at least one u_i. On the one hand, *Tchebychev's Inequality* yields

$$m(G_\alpha) \leq \frac{1}{\alpha}\|f-g\|_{L^1(\mathbb{R}^d)},$$

and on the other hand, the weak type estimate for the maximal function gives

$$m(F_\alpha) \leq \frac{3^d}{\alpha}\|f-g\|_{L^1(\mathbb{R}^d)}.$$

The function g was selected so that $\|f-g\|_{L^1(\mathbb{R}^d)} < \varepsilon$. Hence we get

$$m(E_\alpha) \leq \frac{3^d}{\alpha}\varepsilon + \frac{1}{\alpha}\varepsilon.$$

Since ε is arbitrary, we must have $m(E_\alpha) = 0$, and the proof of the theorem is complete. $\square$

Remark: Note that as an immediate consequence of the theorem applied to $|f|$, we see that $f^*(x) \geq |f(x)|$ for a.e. x.

We have worked so far under the assumption that f is integrable. This "global" assumption is slightly out of place in the context of a "local" notion like differentiability. Indeed, the limit in Lebesgues theorem is taken over balls that shrink to the point x, so the behavior of f far from x is irrelevant. Thus, we expect the result to remain valid if we simply assume integrability of f on every ball, as we shall see next.

Definition. *A measurable function f on $\mathbb{R}^d$ is **locally integrable**, if for every ball B the function $f(x)\chi_B(x)$ is integrable. We shall denote the space of all locally integrable functions by $L^1_{loc}(\mathbb{R}^d)$.* ★

Remark: Loosely speaking, the behavior at infinity does not affect the local integrability of a function. For example, the functions $e^{|x|}$ and $|x|^{-1/2}$ are both locally integrable, but not integrable on $\mathbb{R}^d$.

Now that we have the notion of local integrability, we can see that the *Lebesgue Differentiation Theorem* holds under weaker assumptions:

Theorem 3. *If $f \in L^1_{loc}(\mathbb{R}^d)$, then*

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x), \quad \text{for a.e. } x.$$

Definition. *If E is a measurable set and $x \in \mathbb{R}^d$, we say that x is a point of* ***Lebesgue density*** *of E if*

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{m(B \cap E)}{m(B)} = 1$$

Loosely speaking, this condition says that small balls around x are almost entirely covered by E. More precisely, for every $\alpha < 1$ close to 1, and every ball of sufficiently small radius containing x, we have

$$m(B \cap E) \geq \alpha m(B).$$

Thus E covers at least a proportion α of B. ★

Corollary 1. *Suppose E is a measurable subset of $\mathbb{R}^d$. Then:*

- *Almost every $x \in E$ is a point of density of E.*
- *Almost every $x \notin E$ is not a point of density of E.*

Definition. *If f is locally integrable on $\mathbb{R}^d$, the* ***Lebesgue set*** *of f consists of all points $\bar{x} \in \mathbb{R}^d$ for which $f(\bar{x})$ is finite and*

$$\lim_{\substack{m(B)\to 0\\ \bar{x}\in B}} \frac{1}{m(B)} \int_B |f(y) - f(\bar{x})|\,dy = 0.$$

★

Remark: At this stage, two simple observations about this definition are in order. First, $\bar{x}$ belongs to the Lebesgue set of f whenever f is continuous at $\bar{x}$. Second, if $\bar{x}$ is in the Lebesgue set of f, then

$$\lim_{\substack{m(B)\to 0\\ \bar{x}\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(\bar{x}).$$

Corollary. *If f is locally integrable on $\mathbb{R}^d$, then almost every point belongs to the Lebesgue set of f.*

Functions of Bounded Variation

Definition. *A function F is said to be of* ***bounded variation*** *on an interval $[a,b]$ if the variations of F over all partitions of such interval are bounded, that is, there exists $M < \infty$ so that*

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| \le M$$

for all partitions $a = t_0 < t_1 < \cdots < t_N = b$. ★

Remark 1: Notice that if F is real-valued, monotonic, and bounded, then F is of bounded variation. Indeed, if for example F is nondecreasing and bounded by M, we see that

$$\begin{aligned}\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| &= \sum_{j=1}^{N} F(t_j) - F(t_{j-1}) \\ &= F(b) - F(a) \le 2M.\end{aligned}$$

Remark 2: Notice also that if F is differentiable at every point, and F' is bounded, then F is of bounded variation. Indeed, if $|F'| \le M$, then the mean value theorem implies

$$|F(x) - F(y)| \le M|x-y|, \qquad \forall\, x, y \in [a,b],$$

which in turn implies that $\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| \le M(b-a)$.

Example: Look at the function

$$F(x) = \begin{cases} x^a \sin(x^{-b}) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0. \end{cases}$$

We have that F is of bounded of variation if and only if $a > b$. ▲

Definition. *The* ***total variation*** *of a function F on $[a,x]$ (where $a \le x \le b$) is defined by*

$$T_F(a,x) = \sup \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|,$$

where the sup is taken over all partitions of $[a,x]$. ★

Definition. *The* ***positive variation*** *of a function F on $[a, x]$ (where $a \leq x \leq b$) is defined by*

$$P_F(a, x) = \sup \sum_{(+)} (F(t_j) - F(t_{j-1})),$$

where the sum is taken over all j such that $F(t_j) \geq F(t_{j-1})$, and the sup is taken over all partitions of $[a, x]$. ★

Definition. *The* ***negative variation*** *of a function F on $[a, x]$ (where $a \leq x \leq b$) is defined by*

$$N_F(a, x) = \sup \sum_{(-)} (F(t_{j-1}) - F(t_j)),$$

where the sum is taken over all j such that $F(t_{j-1}) \geq F(t_j)$, and the sup is taken over all partitions of $[a, x]$. ★

Lemma 2. *Suppose F is real-valued and of bounded variation on $[a, b]$. Then for all $a \leq x \leq b$, we have*

$$F(x) - F(a) = P_F(a, x) - N_F(a, x), \qquad \text{and} \qquad T_F(a, x) = P_F(a, x) + N_F(a, x).$$

Theorem 4. *A real-valued function F on $[a, b]$ is of bounded variation iff F is the difference of two increasing bounded functions.*

Proof. ($\Leftarrow$) Clearly, if $F = F_1 - F_2$, where each F_j is bounded and increasing, then F is of bounded variation.

($\Rightarrow$) Conversely, suppose F is of bounded variation. Then, we let $F_1(x) = P_F(a, x) + F(a)$ and $F_2(x) = N_F(a, x)$. Clearly, both F_1 and F_2 are increasing and bounded, and by the above lemma $F(x) = F_1(x) - F_2(x)$. □

The next result lies at the heart of the theory of differentiation:

Theorem 5. *If F is of bounded variation on $[a, b]$, then F is differentiable almost everywhere. In other words, the quotient*

$$\lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$

exists for almost every $x \in [a, b]$.

The proof of this theorem relies on the lemma discussed below as well as on the concept of the so called *Dini* numbers.

Lemma 3 (**Riesz's Lemma**). *Suppose G is real-valued and continuous on $\mathbb{R}$. Let*

$$E = \{x \in \mathbb{R} : G(x+h) > G(x) \text{ for some } h = h_x > 0\}$$

If E is non-empty, then it must be open, and hence can be written as a countable disjoint union of open intervals $E = \bigcup(a_k, b_k)$. If (a_k, b_k) is a finite interval of this union, then we have that

$$G(b_k) - G(a_k) = 0.$$

Proof. Since G is continuous, it is clear that E is open whenever it is non-empty and can therefore be written as a disjoint union of countably many open intervals. If (a_k, b_k) denotes a finite interval in this decomposition, then $a_k \notin E$; therefore we cannot have $G(b_k) > G(a_k)$.

We now suppose that $G(b_k) < G(a_k)$. By continuity, there exists $a_k < c < b_k$ so that

$$G(c) = \frac{G(a_k) + G(b_k)}{2},$$

and in fact we may choose c farthest to the right in the interval (a_k, b_k). Since $c \in E$, there must exist a $d > c$ in E so that $G(d) > G(c)$. Since $b_k \notin E$, we must have $G(x) \leq G(b_k)$ for all $x \geq b_k$; therefore $d < b_k$. Since $G(d) > G(c)$, there exists (by continuity) $c' > d$ with $c' < b_k$ and $G(c') = G(c)$, which contradicts the fact that c was chosen farthest to the right in (a_k, b_k). $(\Rightarrow\Leftarrow)$

This shows that we must have $G(a_k) = G(b_k)$, and the lemma is proved. □

Remark: This result is usually called the **rising sun lemma** for the following reason. If one thinks of the sun rising from the east (at the right) with the rays of light parallel to the x-axis, then the points $(x, G(x))$ on the graph of G, with $x \in E$, are precisely the points which are in the shade; these points appear in bold in Figure 1 below:

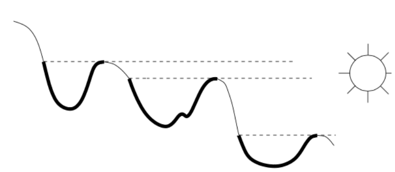

FIGURE 1. Visual representation of the *Rising Sun Lemma*.

Corollary 2. *If F is increasing and continuous, then F' exists almost everywhere. Moreover F' is measurable, non-negative, and*

$$\int_a^b F'(x)\,dx \le F(b) - F(a).$$

In particular, if F is bounded on $\mathbb{R}$, then F' is integrable on $\mathbb{R}$.

Remark: Note that if we had equality, then the above corollary would give us the *Fundamental Theorem of Calculus*. However, we cannot go any farther than the inequality above if we allow all continuous increasing functions, as it is illustrated by the following important example, the so so-called *Cantor-Lebesgue function*.

The Cantor-Lebesgue function

The following simple construction yields a continuous function $F : [0,1] \longrightarrow [0,1]$ that is increasing with $F(0) = 0$ and $F(1) = 1$, but $F'(x) = 0$ almost everywhere! Hence F is of bounded variation, but

$$\int_a^b F'(x)\,dx \neq F(b) - F(a).$$

Consider the standard triadic Cantor set $\mathcal{C} \subset [0,1]$, where $\mathcal{C} = \bigcap_{k=0}^{\infty} C_k$ and each C_k is a disjoint union of 2^k closed intervals. For example, $C_1 = [0, 1/3] \bigcup [2/3, 1]$.

Now let $F_1(x)$ be the continuous increasing function on $[0,1]$ (and linear on C_1) that satisfies

$$F_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/2 & \text{if } 1/3 \le x \le 2/3, \\ 1 & \text{if } x = 1. \end{cases}$$

Similarly, let $F_2(x)$ (see figure 2 on the next page) be the continuous increasing function on $[0,1]$ (and linear on C_2) that satisfies

$$F_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/4 & \text{if } 1/9 \le x \le 2/9, \\ 1/2 & \text{if } 1/3 \le x \le 2/3, \\ 3/4 & \text{if } 7/9 \le x \le 8/9, \\ 1 & \text{if } x = 1. \end{cases}$$

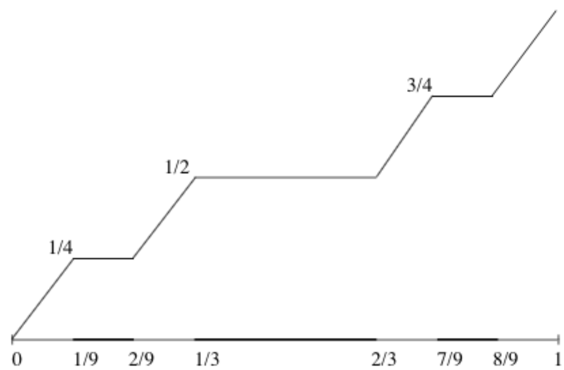

FIGURE 2. Here's a visualization of the construction of F_2.

This process yields a sequence of continuous increasing functions $\{F_n\}_{n=1}^{\infty}$ such that clearly

$$|F_{n+1}(x) - F_n(x)| \leq \frac{1}{2^{n+1}} .$$

Hence $\{F_n\}_{n=1}^{\infty}$ converges uniformly to a continuous limit F, which is called the **Cantor-Lebesgue function** (see figure 3 below).

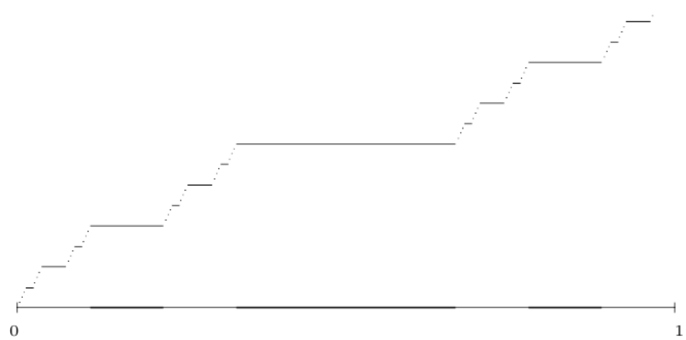

FIGURE 3. Here's a visualization of the *Cantor-Lebesgue* function.

By construction F is increasing, $F(0) = 0$, $F(1) = 1$, and we see that F is constant on each interval of the complement of the Cantor set. Since $m(\mathcal{C}) = 0$, we find that $F'(x) = 0$ almost everywhere, as desired.

The considerations in this section, as well as this last example, show that the assumption of bounded variation guarantees the existence of a derivative almost everywhere, but not the validity of the formula

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In order to achieve this equality we need to consider *absolutely continuous* functions, which we turn to next.

Absolutely Continuous Functions

Definition. *A function f defined on some interval $[a,b]$ is said to be **absolutely continuous** if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon \quad \textit{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta,$$

where the intervals (a_k, b_k) (for $k = 1, \ldots, N$) are disjoint intervals. ★

Remark 1: From the definition, it is clear that absolutely continuous functions are continuous, and in fact uniformly continuous.

Remark 2: If F is absolutely continuous on a bounded interval, then it is also of bounded variation on the same interval. Moreover, its total variation is continuous (in fact absolutely continuous). As a consequence, the decomposition of such a function F into two monotonic functions given previously in *Theorem 3.3* shows that each of these functions is continuous.

Remark 3: If $F(x) = \int_a^x f(y)\,dy$ where f is integrable, then F is absolutely continuous.

In fact, this last remark shows that absolute continuity is a necessary condition to impose on F if we hope to prove the desired equality $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Theorem 6. *If F is absolutely continuous on $[a,b]$, then $F'(x)$ exists almost everywhere. Moreover, if $F'(x) = 0$ for a.e. x, then F is constant.*

The following lemma and corollary are used in the proof of *Theorem 6.* First we need to state the definition of a *Vitali covering*:

Definition. *A collection $\mathcal{B}$ of balls $\{B\}$ is said to be a* ***Vitali covering*** *of a set E if for every $x \in E$ and any $\eta > 0$ there is a ball $B \in \mathcal{B}$, such that $x \in B$ and $m(B) < \eta$. Thus every point is covered by balls of arbitrarily small measure.* ★

Lemma 4. *Suppose E is a set of finite measure and $\mathcal{B}$ is a Vitali covering of E. For any $\delta > 0$, we can find finitely many balls $B_1, \ldots, B_N$ in $\mathcal{B}$ that are disjoint and so that*

$$\sum_{i=1}^{N} m(B_i) \geq m(E) - \delta.$$

Corollary 3. *We can arrange the choice of the balls from the Vitali covering given in the above lemma so that*

$$m\left(E - \bigcup_{i=1}^{N} B_i\right) < 2\delta.$$

The culmination of all our efforts is contained in the next theorem. In particular, it resolves the problem of establishing the reciprocity between differentiation and integration, known as the *Fundamental Theorem of Calculus*:

Theorem 7 (**Fundamental Theorem of Calculus**)**.** *Suppose F is absolutely continuous on $[a, b]$. Then F' exists almost everywhere and is integrable. Moreover,*

$$F(x) - F(a) = \int_a^x F'(y)\, dy, \qquad \forall\, a \leq x \leq b.$$

By selecting $x = b$ we get $F(b) - F(a) = \int_a^b F'(y)\, dy$.

Conversely, if f is integrable on $[a, b]$, then there exists an absolutely continuous function F such that $F'(x) = f(x)$ almost everywhere, and in fact, we may take $F(x) = \int_a^x f(y)\, dy$.

Proof. Since we know that a real-valued absolutely continuous function is the difference of two continuous increasing functions, *Corollary 3.7* shows that F' is integrable on $[a, b]$. Now let $G(x) = \int_a^x F'(y)\, dy$. Then G is absolutely continuous; hence so is the difference $G(x) - F(x)$. By the *Lebesgue Differentiation Theorem*, we know that $G'(x) = F'(x)$ for a.e. x; hence the difference $F - G$ has derivative 0 almost everywhere. By *Theorem 3.8*, we conclude that $F - G$ is constant, and evaluating this expression at $x = a$ gives the desired result.

The converse is a consequence of the observation we made earlier, namely that $\int_a^x f(y)\,dy$ is absolutely continuous, and the *Lebesgue Differentiation Theorem*, which gives $F(x) = f(x)$ almost everywhere. $\square$