MATH 751 NOTES
THE QUOTIENT TOPOLOGY

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The Quotient Topology

Definition. *Let X and Y be topological spaces, and let $p\colon X \to Y$ be a surjective map. The map p is said to be a* ***quotient map*** *provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.* ★

Remark: Note that this condition is stronger than continuity; some mathematicians call it "strong continuity." An equivalent condition is to require that a subset A of Y be closed if and only if $p^{-1}(A)$ is closed in X. Equivalence of the two conditions follow from the equation

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$

Another way of describing a quotient map is as follows:

We say that a subset C of X is ***saturated*** (with respect to the surjective map $p\colon X \to Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. In other words, C is saturated if it equals the complete inverse image of a subset of Y. Thus to say that p is a quotient map is equivalent to saying that p is continuous and p maps saturated open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).

Two special kinds of quotient maps are the open maps and the closed maps. It follows immediately from the definition that if $p\colon X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map. The converse however does not always hold; that is, there are quotient maps that are neither open nor closed.

Example 1: Let X be the subspace $[0,1] \cup [2,3]$ of $\mathbb{R}$ and let Y be the subspace $[0,2]$ of $\mathbb{R}$. The map $p\colon X \to Y$ defined by

$$p(x) = \begin{cases} x & \text{for } x \in [0,1], \\ x-1 & \text{for } x \in [2,3]. \end{cases}$$

is readily seen to be surjective, continuous, and closed. Therefore it is a quotient map. It is not, however, an open map; the image of the open set $[0,1]$ of X is not open in Y.

Note that if A is the subspace $[0,1) \cup [2,3]$ of X, then the map $q\colon A \to Y$ obtained by restricting p is continuous and surjective, but it is not a quotient map. The reason is that the set $[2,3]$ is open in A and is saturated with respect to q, but its image is not open in Y.

Example 2: Let $\pi_1\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first coordinate; then π_1 is continuous and surjective. Furthermore, π_1 is an open map. For if $U \times V$ is a nonempty basis element for $\mathbb{R} \times \mathbb{R}$, then $\pi_1(U \times V) = U$ is open in $\mathbb{R}$; it follows that π_1 carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$. However, π_1 is not a closed map. The subset

$$C = \{x \times y \mid xy = 1\}$$

of $\mathbb{R} \times \mathbb{R}$ is closed, but $\pi_1(C) = \mathbb{R} \setminus \{0\}$, which is not closed in $\mathbb{R}$.

Note that if A is the subspace of $\mathbb{R} \times \mathbb{R}$ that is the union of C and the origin $\{0\}$, then the map $q\colon A \to \mathbb{R}$ obtained by restricting π_1 is continuous and surjective, but it is not a quotient map. The reason is that the one-point set $\{0\}$ is open in A and is saturated with respect to q, but its image is not open in $\mathbb{R}$.

We now show how the notion of a quotient map can be used to construct a topology on a set:

Definition. *If X is a space and A is a set and if $p\colon X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it called the* ***quotient topology*** *induced by p.* ★

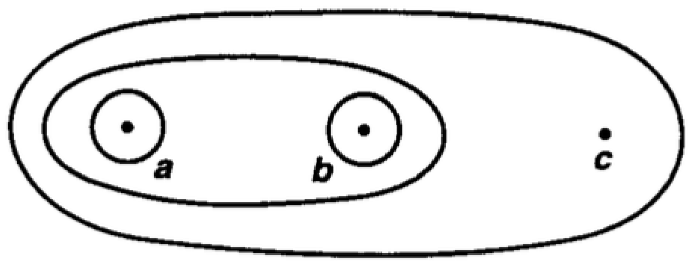
Remark: The topology $\mathcal{T}$ mentioned in the above definition is of course defined by letting it consist of those subsets U of A such that $p^{-1}(U)$ is open in X. It is easy to check that $\mathcal{T}$ is a topology:

- The sets $\emptyset$ and A are open because
$$p^{-1}(\emptyset) = \emptyset \qquad \text{and} \qquad p^{-1}(A) = X.$$
- The other two conditions follow from the equations
$$p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha)$$
and
$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i).$$

<u>*Example 3:*</u> Let p be the map of the real line $\mathbb{R}$ onto the three-point set $A = \{a, b, c\}$ defined by

$$p(x) = \begin{cases} a & \text{if } x > 0, \\ b & \text{if } x < 0, \\ c & \text{if } x = 0. \end{cases}$$

You can check that the quotient topology on A induced by p is the one indicated in the figure below:



There is a special situation in which the quotient topology occurs particularly frequently. It is the following:

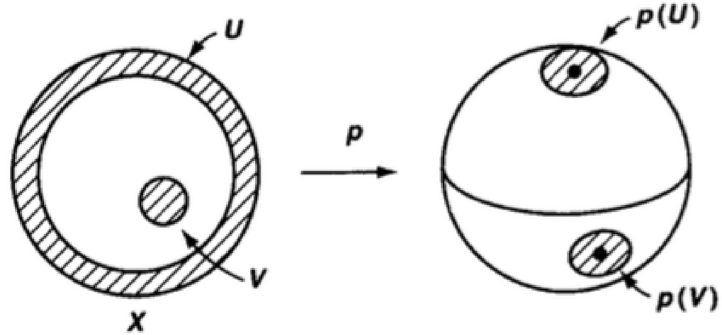
Definition. *Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p\colon X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called the* ***quotient space*** *of X.* ★

<u>*Example 4:*</u> Let X be the closed unit ball

$$\mathbb{B}^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

in $\mathbb{R}^2$, and let X^* be the partition of X consisting of all the one-point sets $\{(x, y)\}$ for which $x^2 + y^2 < 1$, along with the set $\mathbb{S}^1 = \{(x, y) \mid x^2 + y^2 = 1\}$. Typical saturated open sets in X are pictured by the shaded regions in the figure below. One can show that X^* is homeomorphic with the subspace of $\mathbb{R}^3$ called the ***unit 2-sphere***, defined by

$$\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$


U
p
p(U)
V
X
p(V)