Math 260 HW # 3

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Section 1.4

(3) For the following list of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.

d) $\{(2,\,-1,\,0),\,(1,\,2,\,-3),\,(1,\,-3,\,2)\}$

<u>Solution:</u>

$$\begin{aligned} a(1,\,2,\,-3)\,+\,b(1,\,-3,\,2) &= (2,\,-1,\,0) \\ a+b &= 2 \\ 2a-3b &= -1 \\ -3a+2b &= 0 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -5 & -5 \\ 0 & 5 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 5 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

We can see in the resulting matrix that the system is inconsistent. Hence the system has no solution and we can conclude that $(2,\,-1,\,0)$ cannot be written as a linear combination of the other two vectors.

(5) In each part, determine whether the given vector is in the span of S.

e) $-x^3+2x^2+3x+3\,,\quad S=\{x^3+x^2+x+1,\ x^2+x+1,\ x+1\}$

<u>Solution:</u>

$$a(x^3+x^2+x+1)+b(x^2+x+1)+c(x+1) = -x^3+2x^2+3x+3$$

$$\begin{aligned} a x^3 &= -x^3 \\ a x^2 + b x^2 &= 2 x^2 \\ a x + b x + c x &= 3 x \\ a + b + c &= 3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a &= -1 \\ a + b &= 2 \\ a + b + c &= 3 \\ a + b + c &= 3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

As we can see in the above matrix we have a consistent system with the following set of solutions: $\{a, b, c\} = \{-1, 3, 1\}$. Hence we can conclude that $-x^3 + 2x^2 + 3x + 3$ is in the span of S.

h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

Solution:

$$a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} 1a + 0b + 1c &= 1 \\ 0a + 1b + 1c &= 0 \\ -1a + 0b + 0c &= 0 \\ 0a + 1b + 0c &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

We can see in the resulting matrix above that the system is inconsistent. Therefore we can conclude that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not in the span of S.

(13) Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

Proof:

We want to prove that span(S_1) $\subseteq$ span(S_2). We choose an arbitrary vector $v \in S_1$ and we express it as $v = a_1 v_1 + \ldots + a_n v_n$, where each v_i is contained in S_1. But since
$S_1 \subseteq S_2$, then we know that each v_i is also in S_2. Hence span(S_1) $\subseteq$ span(S_2). ✓
Since we know that span(S_1) is a subspace contained within span(S_2), if span(S_1) $= V$, then necessarily span(S_2) $= V$ as well. ■

Section 1.5

(9) Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent iff u or v is a multiple of the other.

<u>Proof:</u>
($\Rightarrow$)
Assume $\{u, v\}$ is linearly dependent, then we need to show that u or v is a multiple of the other.
Since $\{u, v\}$ is linearly dependent the zero vector can be expressed as $a\,u + b\,v = \hat{0}$ for $a, b \in \mathbb{F}$ and a or $b \neq 0$.
But then this means that $b\,v = -a\,u$.
Hence we have shown that if $\{u, v\}$ is linearly dependent then u or v is a multiple of the other.
($\Leftarrow$)
Suppose u or v is a multiple of the other, then we need to show that $\{u, v\}$ is linearly dependent.
WLOG, we let $u = k\,v$ with $k \in \mathbb{F}$, $k \neq 0$. Then $u - k\,v = 0$. But then this means that zero can be expressed as a non-trivial linear combination of the vectors u and v , with at least one nonzero coefficient. Thus we have determined that $\{u, v\}$ is linearly dependent. ■

Section 1.6

(2) Determine if the following set is a basis for $\mathbb{R}^3$:

b) $\{(2, -4, 1), (0, 3, -1), (6, 0, -1)\}$

<u>Solution:</u>
In order to determine whether the given set is a basis for $\mathbb{R}^3$ we need to show that the set is linearly independent and that it spans $\mathbb{R}^3$:

• Linear independence:
$a(2, -4, 1) + b(0, 3, -1) + c(6, 0, -1) = (0, 0, 0)$
We need to show that the only solution to this linear system is the trivial one, i.e
$a = b = c = 0$.

$$2\,a + 0\,b + 6\,c = 0$$

$$\begin{aligned} -4a + 3b + 0c &= 0 \\ a - b - c &= 0 \end{aligned}$$

$$\begin{pmatrix} 2 & 0 & 6 \\ -4 & 3 & 0 \\ 1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 12 \\ 0 & -1 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

We can see from the resulting matrix that c is a free variable. That is, the system has an infinite number of solutions and therefore the zero vector in $\mathbb{R}^3$ can be expressed in infinitely many ways and not just with the trivial solution $a = b = c = 0$. Therefore we may conclude that our given set of vectors is not a basis for $\mathbb{R}^3$.

(3) Determine if the following set is a basis for $P_2(\mathbb{R})$:

b) $\{1 + 2x + x^2,\ 3 + x^2,\ x + x^2\}$

Solution:
We need to show that the given set is linearly independent and that it spans $P_2(\mathbb{R})$.
• Linear independence:

$$a(1 + 2x + x^2) + b(3 + x^2) + c(x + x^2) = 0$$
$$(a + b + c)x^2 + (2a + c)x + (a + 3b)x^0 = 0$$
$$\begin{aligned} a + b + c &= 0 \\ 2a + 0b + c &= 0 \\ a + 3b + 0c &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As shown in the resulting matrix the zero vector has a unique representation, which is the trivial one $a = b = c = 0$. ✓

• Span:
We don't need to show that the given set spans $P_2(\mathbb{R})$ because we know that $\dim(P_2(\mathbb{R})) = 3 =$ cardinality of the given set. ✓

Thus since the given set $\{1 + 2x + x^2,\ 3 + x^2,\ x + x^2\}$ is linearly independent and spans $P_2(\mathbb{R})$, it is a basis for $P_2(\mathbb{R})$.

(13) The set of solutions to the system of linear equations

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Solution:
First let us find the subspace $\{x_1, x_2, x_3\}$:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Hence we have $x_2 = x_3 = x_1$. The solution set is
$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = x_3\} = \{a(1, 1, 1) : a \in \mathbb{R}\}$. Thus S is a line in $\mathbb{R}^3$.
Consider the set $B = \{(1, 1, 1)\}$. We will show that this is a basis for S. Since B is a nonzero singleton set, B is linearly independent. Now we only need to show that $\text{span}(B) = S$. By theorem 1.5 (Friedberg's), we know that $\text{span}(B) \subseteq S$, thus we only need to show that $S \subseteq \text{span}(B)$:
Let $\hat{x} \in S$. Then $\hat{x} = t(1, 1, 1)$ for some $t \in \mathbb{R}$. This is clearly a linear combination of the vector in B. Thus $\hat{x} \in \text{span}(B) \Longrightarrow S \subseteq \text{span}(B)$. Then $\text{span}(B) = S$. So B is a basis for S.

(Extra Problem) Suppose $\{v_1, \ldots, v_n\}$ is linearly independent in V and $w \in V$. Prove that if $\{v_1 + w, \ldots, v_n + w\}$ is linearly dependent, then $w \in \text{span}\{v_1, \ldots, v_n\}$.

Proof:
Suppose $\{v_1 + w, \ldots, v_n + w\}$ is linearly dependent. Then we know that the zero vector in V can be expressed as

$$a_1(v_1 + w) + a_2(v_2 + w) + \ldots + a_n(v_n + w) = \hat{0} \quad \forall\, a_i \in \mathbb{F} \text{ and } \exists\ a_k \neq 0$$

Then we have

$$a_k(v_k + w) = -a_1(v_1 + w) - \ldots - a_n(v_n + w)$$
$$\Longrightarrow (v_k + w) = -\frac{a_1}{a_k}(v_1 + w) - \ldots - \frac{a_n}{a_k}(v_n + w)$$
$$\Longrightarrow v_k = -\frac{a_1}{a_k} v_1 - \frac{a_1}{a_k} w - \ldots - \frac{a_n}{a_k} v_n - \frac{a_n}{a_k} w - w$$
$$\Longrightarrow \frac{a_1}{a_k} v_1 + v_k + .. + \frac{a_n}{a_k} v_n = -\frac{a_1}{a_k} w - \ldots - \frac{a_n}{a_k} w - w$$

$$\Longrightarrow \frac{a_1}{a_k} v_1 + v_k + .. + \frac{a_n}{a_k} v_n = w\left(- \frac{a_1}{a_k} - ... - \frac{a_n}{a_k} - 1\right)$$

$$\Longrightarrow \frac{\frac{a_1}{a_k} v_1 + v_k + .. + \frac{a_n}{a_k} v_n}{\left(- \frac{a_1}{a_k} - ... - \frac{a_n}{a_k} - 1\right)} = w$$

$$\Longrightarrow \frac{\frac{a_1}{a_k}}{\left(- \frac{a_1}{a_k} - ... - \frac{a_n}{a_k} - 1\right)} v_1 + \frac{1}{\left(- \frac{a_1}{a_k} - ... - \frac{a_n}{a_k} - 1\right)} v_k + .. + \frac{\frac{a_n}{a_k}}{\left(- \frac{a_1}{a_k} - ... - \frac{a_n}{a_k} - 1\right)} v_n = w$$

Hence w can be written as a linear combination of vectors from $\{v_1, ..., v_n\}$.
However the proof is not yet complete. We still have to show that the denominator on the left hand side is not zero. We do this by contradiction. Suppose $a_1 + ... + a_n = 0$. Then we have
$w = 0 = -a_1 v_1 - ... - a_n v_n$.
But since $\{v_1, ..., v_n\}$ is assumed to be linearly independent, the only representation the zero vector has is the trivial one. Thus, $-a_i = 0 \ \forall \ i$, which contradicts the fact that at least one of the a_i's must be nonzero from above. Thus $a_1 + ... + a_n \neq 0$ and $w \in \text{span}\{v_1, ..., v_n\}$. ■