Math 260 HW # 9

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Section 6.1

(8) Provide reasons why the following is not an inner product on the given vector space:

a) $\langle (a,\ b),\ (c,\ d) \rangle = a\,c - b\,d$ on $\mathbb{R}^2$.

Solution:
We can easily see that the positive definitiveness axiom ($\langle x,\ x \rangle = \|x\|^2 \geq 0$) fails. We have that $\langle (a,\ b),\ (a,\ b) \rangle = a\,a - b\,b = a^2 - b^2$, which can be negative if $b^2 > a^2$.
Therefore we conclude that the above is not an inner product on $\mathbb{R}^2$.

(17) Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Proof:
Let T be linear operator on an IPS V. Since we are given that $\|T(x)\| = \|x\|\ \ \forall\, x$, that means that $\|T(0)\| = \|0\| = 0$. Accordingly, it is obvious that if we plug in any nonzero value for the preimage x, we get a nonzero value for the image x, since we know that $\|x\| = 0$ iff $x = 0$. Hence the null space of T is trivial, which means that T has to be injective. ■

Section 6.2

(2) Apply the Gram-Schmidt process to the given subset S of the IPS V to obtain an orthogonal basis for span S. Then normalize the vectors in this basis to obtain an orthonormal basis β for span S.

c) $V = P_2(\mathbb{R})$ with the inner product $\langle f(x),\ g(x) \rangle = \int_0^1 f(t)\, g(t)\, dt$, and $S = \{1,\ x,\ x^2\}$.

Solution:
We apply the Gram-Schimdt process to the standard basis S.
Define $\{w_1,\ w_2,\ w_3\} = \{1,\ x,\ x^2\}$, then we want to find an orthogonal basis $\{v_1,\ v_2,\ v_3\}$. Thus we have :

$\bullet\!\!\rightarrow\ v_1 = w_1 = 1$ ✓

$\longmapsto v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$

where $\langle w_2, v_1 \rangle = \langle x, 1 \rangle = \int_0^1 t \, dt = \left. \frac{t^2}{2} \right|_0^1 = \frac{1}{2}$.

Hence

$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = x - \frac{\frac{1}{2}}{1} 1 = x - \frac{1}{2}$ ✓

$\longmapsto v_3 = w_3 - \sum_{j=1}^{2} \frac{\langle w_3, v_j \rangle}{\|v_j\|^2} v_j = w_3 - \left(\frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 \right)$

where $\langle w_3, v_1 \rangle = \langle x^2, 1 \rangle = \int_0^1 t^2 \, dt = \left. \frac{t^3}{3} \right|_0^1 = \frac{1}{3}$

,

$\langle w_3, v_2 \rangle = \left\langle x^2, x - \frac{1}{2} \right\rangle = \int_0^1 \left(t^3 - \frac{1}{2} t^2 \right) dt = \left. \left(\frac{t^4}{4} - \frac{t^3}{6} \right) \right|_0^1 = \frac{1}{12}$,

and $\| v_2 \|^2 = \langle v_2, v_2 \rangle = \int_0^1 \left(t - \frac{1}{2} \right)^2 dt = \left. \frac{1}{3} \left(t - \frac{1}{2} \right)^3 \right|_0^1 = \frac{1}{12}$.

Hence $v_3 = w_3 - \sum_{j=1}^{2} \frac{\langle w_3, v_j \rangle}{\|v_j\|^2} v_j = w_3 - \left(\frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 \right)$

$\Longrightarrow v_3 = x^2 - \left(\frac{1}{3} (1) + \frac{\frac{1}{12}}{\frac{1}{12}} \left(x - \frac{1}{2} \right) \right) = x^2 - \left(x - \frac{1}{2} \right) - \frac{1}{3} = x^2 - x + \frac{1}{6}.$ ✓

Thus we have that $\left\{ 1, x - \frac{1}{2}, x^2 - x + \frac{1}{6} \right\}$ is an orthogonal basis.

Now we want to normalize this basis :

$\longmapsto \| 1 \| = \sqrt{1} = 1$. Thus $\frac{1}{\|1\|} = 1$. ✓

$\longmapsto \| x - \frac{1}{2} \|^2 = \frac{1}{12}$ (as computed above). This implies that $\| x - \frac{1}{2} \| = \frac{1}{\sqrt{12}}$.

Thus, $\frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = \sqrt{12} \left(x - \frac{1}{2} \right) = 2 \sqrt{3} \left(x - \frac{1}{2} \right) = 2 \sqrt{3} \, x - \sqrt{3} = \sqrt{3} \, (2 x - 1)$ ✓

$\longmapsto \| x^2 - x + \frac{1}{6} \|^2 = \left\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6} \right\rangle = \int_0^1 \left(t^2 - t + \frac{1}{6} \right)^2 dt$

$= \int_0^1 \left(\left(t - \frac{1}{2} \right)^2 - \frac{1}{12} \right)^2 dt = \int_0^1 \left(\left(t - \frac{1}{2} \right)^2 - \frac{1}{12} \right)^2 dt$

We can let $u = t - \frac{1}{2}$; $du = dt$. Then we have

$\int_{\frac{-1}{2}}^{\frac{1}{2}} \left(u^2 - \frac{1}{12} \right)^2 du = \int_{\frac{-1}{2}}^{\frac{1}{2}} \left(u^4 - \frac{1}{6} u^2 + \frac{1}{144} \right)^2 du = \frac{1}{180}$

Hence, $\int_0^1 \left(t^2 - t + \frac{1}{6}\right)^2 dt = \frac{1}{180}$.

That implies that $|| x^2 - x + \frac{1}{6} || = \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}}$.

Thus, $\frac{x^2-x+\frac{1}{6}}{||x^2-x+\frac{1}{6}||} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right) = \sqrt{5}\,(6x^2 - 6x + 1)$ ✓

Thus, at last we have our orthonormal basis $\beta = \left\{1,\ \sqrt{3}\,(2x - 1),\ \sqrt{5}\,(6x^2 - 6x + 1)\right\}$.