Math 260 Extra Credit

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Definition: Let S be a subset of the dual space V^* of a finite dimensional vector space V. Define $S^o = \{v \in V \mid f(v) = 0 \ \ \forall\ f \in S\}$. S^o is called the annihilator of S in V.

Prove any two of the following:

(1) Prove that S^o is a subspace of V.

Proof:
If we take any $x,\ y \in S^o$ and any scalar $\alpha \in \mathbb{F}$, we have

• $0 = f(x+y) = f(x) + f(y)$ (by the linearity of the linear functionals f)
$\quad = 0 + 0 = 0$ ✓ (closed under addition)

• $0 = f(\alpha \cdot x) = \alpha\, f(x)$ (also by linearity)
$\quad = \alpha \cdot 0 = 0$ ✓ (closed under scalar multiplication)

Since S^o is closed under addition and scalar multiplication, and therefore contains the zero vector of V, we conclude that S^o is a subspace of V. Moreover, this subspace turns out to be the null space of all linear maps (functionals) f in S. ■

(3) Prove that $S^o = (\text{span } S)^o$ for any $S \subset V^*$. In other words, prove that the annihilator of a subset and the subspace spanned by that same subset are equal.

Proof:
Let S be any subset in V^*, $S = \{v_1,\ \ldots,\ v_n\} \subset V^*$, with scalars $\alpha_i \in \mathbb{F}$ and let f be a linear functional in S. We have that span $S = \alpha_1\, v_1 + \ldots + \alpha_n\, v_n$. Then the annihilator of the subspace spanned by S is given by $(\text{span } S)^o = (\alpha_1\, v_1 + \ldots + \alpha_n\, v_n)^o$. Now if we apply f to this subspace spanned by S we have

$f(\alpha_1\, v_1 + \ldots + \alpha_n\, v_n) = f(\alpha_1\, v_1) + \ldots + f(\,\alpha_n\, v_n)$ (by linearity of the functional f)
$\quad = \alpha_1\, f(v_1) + \ldots + \alpha_n\, f(\,v_n)$ (also by linearity)
$\quad = \alpha_1(0) + \ldots + \alpha_n(0) = 0$ ✓

Obviously S^o is by definition the set all elements in V that are mapped to zero via any linear functional f in V. Thus we have proven that $S^o = (\text{span } S)^o$ for any $S \subset V^*$. ■