# MATH 722 HOMEWORK

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**Problem 1.** *i) Prove that every ring $A \neq 0$ has at least one maximal ideal.*
*ii) Prove that if $A$ is a ring and $\mathfrak{m}$ is a maximal ideal of $A$, then $A/\mathfrak{m}$ is a field.*

*Proof of i).* Let $\Sigma$ be the set of all ideals not equal to $\langle 1\rangle$ in $A$. Now order $\Sigma$ by inclusion (note that $\Sigma$ is not empty, since $0 \in \Sigma$). We must show that every chain in $\Sigma$ has an upper bound in $\Sigma$. Let $(\mathfrak{a}_\alpha)$ be a chain of ideals in $\Sigma$, so that for each pair of indices $\alpha, \beta$ we have either

$$\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta \qquad \text{or} \qquad \mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha.$$

Let $\mathfrak{a} = \bigcup_\alpha \mathfrak{a}_\alpha$. We claim that $\mathfrak{a}$ is an ideal. Indeed, $\mathfrak{a}$ is clearly closed under multiplication by $A$, so we show closure under addition. Let $x, y \in \mathfrak{a}$. Then $x \in \mathfrak{a}_\alpha$ and $y \in \mathfrak{a}_\beta$ for some $\alpha, \beta$. Then one of these ideals contains the other since they are elements of a chain, and we therefore have $x, y$ contained in the same ideal and thus $x+y \in \mathfrak{a}$. Note that $1 \notin \mathfrak{a}$ since $1 \notin \mathfrak{a}_\alpha$ for any $\alpha$. Hence $\mathfrak{a} \in \Sigma$ and $\mathfrak{a}$ is an upper bound of the chain. Thus, by Zorn's Lemma, $\Sigma$ contains a maximal element, which is a maximal ideal in $A$. $\square$

*Proof of ii).* Let $\mathfrak{m}$ be maximal. Then since, by a previous proposition[1], there is a 1-1 correspondence between ideals of $A/\mathfrak{m}$ and ideals containing $\mathfrak{m}$, the maximality of $\mathfrak{m}$ says that $A/\mathfrak{m}$ has no nontrivial ideals. But since we know that a ring with no nontrivial ideals must be a field, $A/\mathfrak{m}$ is guaranteed to be a field. $\square$

**Problem 2.** *Let $A$ be a ring in which every element $x$ satisfies $x^n = x$ for some $n > 1$ (depending on $x$). Show that every prime ideal in $A$ is maximal.*

*Proof.* Let $\mathfrak{p}$ be an arbitrary prime ideal of $A$. We need to show that the only ideal of $A$ properly containing $\mathfrak{p}$ is $\langle 1\rangle$. Let $\mathfrak{a}$ be an ideal such that $\mathfrak{p} \subsetneq \mathfrak{a}$. Then there exists an element $x \in a \smallsetminus \mathfrak{p}$. But by assumption, $x^n = x$ for some $n > 1$. That is, $x - x^n = x(1 - x^{n-1}) = 0 \in \mathfrak{p}$, which implies that $(1 - x^{n-1}) \in \mathfrak{p} \subsetneq \mathfrak{a}$ since $\mathfrak{p}$ is prime and $x \notin \mathfrak{p}$. But then we have $1 = (1 - x^{n-1}) + x^{n-1} \in \mathfrak{a}$. Hence $\mathfrak{a} = \langle 1\rangle$ is the unit ideal and thus $\mathfrak{p}$ must be maximal. Since $\mathfrak{p}$ was arbitrary, we conclude that every prime ideal in $A$ is maximal, as desired. $\square$

**Problem 3.** *If $A$ is a ring and $\mathfrak{p}$ is a prime ideal of $A$, show that $S = A \smallsetminus \mathfrak{p}$ is multiplicatively closed. Also, show that $S^{-1}A$ is a local ring.*

*Proof.* It is obvious that $S = A \smallsetminus \mathfrak{p}$ is multiplicatively closed. To see why, note that if $s, t \in A \smallsetminus \mathfrak{p}$, then $st$ must also be in $A \smallsetminus \mathfrak{p}$ since, if otherwise $st \in \mathfrak{p}$, we must have that either $s \in \mathfrak{p}$ or $t \in \mathfrak{p}$ by primality of $\mathfrak{p}$. But this contradicts our assumption that $s, t \in A \smallsetminus \mathfrak{p}$. $(\Rightarrow\Leftarrow)$

[1] The proposition states that there is a 1-1 correspondence between the ideals $\mathfrak{b}$ containing the ideal $\mathfrak{a}$ and the ideals $\bar{\mathfrak{b}}$ of $A/\mathfrak{a}$, given by $\mathfrak{b} = \phi^{-1}(\bar{\mathfrak{b}})$.

Now, to show that $S^{-1}$ is a local ring, note that the elements $a/s$ (with $a \in \mathfrak{p}$ and $s \in S$) form an ideal $\mathfrak{m}$ in $S^{-1}A$. If an element $b/t \notin \mathfrak{m}$, then $b \notin \mathfrak{p}$, hence $b \in S$ and therefore $b/t$ is a unit in $S^{-1}A$. It then follows that if $\mathfrak{a}$ is an ideal in $S^{-1}A$ and $\mathfrak{a} \not\subseteq \mathfrak{m}$, then $\mathfrak{a}$ contains a unit and is therefore the whole ring. Hence $\mathfrak{m}$ is the only maximal ideal in $S^{-1}A$, i.e., $S^{-1}A$ is a local ring. $\square$