Math 353 HW 4

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Section 2.2

(1) Find the location of the branch points and discuss possible branch cuts for the following functions:

a) $w = \dfrac{1}{(z-1)^{1/2}}$

Solution:

By letting $z' = z - 1$ we have $w = \dfrac{1}{(z')^{1/2}}$. Now let us we write $(z')^{1/2}$ in polar form as

$\left(R\, e^{i(\theta_p + 2\pi n)}\right)^{1/2} = R^{1/2}\, e^{i\left(\frac{\theta_p}{2} + \pi n\right)}$. Then letting $n = 0$, we have $\dfrac{1}{(z')^{1/2}} = R^{-1/2}\, e^{-i\,\frac{\theta_p}{2}}$.

Now we let θ_p go from 0 to 2π:

$\theta_p = 0 \Longrightarrow \dfrac{1}{(z')^{1/2}} = R^{-1/2}$

$\theta_p = 2\pi \Longrightarrow \dfrac{1}{(z')^{1/2}} = -R^{-1/2}$.

As we can see our function w does not return to its initial value after traversing a small circle of radius R at the point $0 = z' = z - 1$, i.e. when $z = 1$. Therefore $z' = 0$ (or $z = 1$) is a branch point of our function. Also $z' = \infty$ is another branch point of this function. We can see this by substituting the value $z' = \dfrac{1}{t}$ back into our function and analyzing what happens at the point $t = 0$. Since $w = (z')^{-1/2} = t^{1/2}$, by a similar argument as above we can show that $t = 0$ (or $z' = \infty = z$) is also a branch point for w. For convenience and simplicity we can choose a line parallel to the positive real axis starting at $z' = 0$ (or $z = 1$) as a branch cut.

b) $w = (z + 1 - 2\,i)^{1/4}$

Solution:

We let $z' = z + 1 - 2\,i$. Then we have $w = (z')^{1/4}$. Now by writing $(z')^{1/4}$in polar form and letting $n = 0$, we have $(z')^{1/4} = R^{1/4}\, e^{i\,\frac{\theta_p}{4}}$.

Now we let θ_p go from 0 to 2π:

$\theta_p = 0 \Longrightarrow (z')^{1/4} = R^{1/4}$

$\theta_p = 2\pi \Longrightarrow (z')^{1/4} = i\, R^{1/4}$.

As we can see our function w does not return to its initial value after traversing a small circle of radius R at the point $0 = z' = z + 1 - 2\,i$, i.e. when $z = 2\,i - 1$. Therefore $z' = 0$ (or $z = 2\,i - 1$) is a branch point of our function. Also $z' = \infty$ is another branch point of this function. We can see this by substituting the value $z' = \frac{1}{t}$ back into our function and analyzing what happens at the point $t = 0$. Since $w = (z')^{1/4} = t^{-1/4}$, by a similar argument as above we can show that $t = 0$ (or $z' = \infty = z$) is also a branch point for w. We can choose the branch cut to be a line passing through $z = 2\,i - 1$ and parallel to the positive real axis.

c) $w = 2 \log z^2 = 4 \log z$

Solution:

$e^w = e^4\, z \Longrightarrow z = \frac{1}{e^4}\, e^w$. Now we let $w = u + i\, v$ and write z in polar form, then $R\, e^{i\theta_p} = \frac{1}{e^4}\, e^u\, e^{i\, v} \Longrightarrow$
$R = e^{u-4}$, $v = \theta_p + 2\,\pi\, n$, for n an integer.

Thus, letting $n = 0$ and substituting back into our function w we have

$$\log e^w = \log\left(e^4\, e^{u-4}\, e^{i\theta_p}\right)$$
$$\Longrightarrow w = \log\left(e^u\, e^{i\theta_p}\right) = \log e^u + \log e^{i\theta_p}$$
$$= u + i\,\theta_p$$

Now if we allow θ_p to go from 0 to 2π we have

$\theta_p = 0 \Longrightarrow w = u$

$\theta_p = 2\,\pi \Longrightarrow w = u + i\, 2\,\pi$

We can see that our function w does not return to its initial value after traversing a small circle of radius R at the point $z = 0$. In fact $w = 4 \log z$ has an infinite number of values (a differente value for each time we traverse a small circle around $z = 0$). Hence $z = 0$ is a branch point and so is $z = \infty$, since we can write $z = \frac{1}{t}$ and evaluate $4 \log \frac{1}{t} = -4 \log t$ at $t = 0$ and we obtain a similar result. As a branch cut we can choose the positive real axis.

d) $z^{\sqrt{2}}$

Solution:

We write this in polar form $\left(R\, e^{i\theta_p}\right)^{\sqrt{2}} = R^{\sqrt{2}}\, e^{i\,\sqrt{2}\,\theta_p}$.

Now if we let θ_p go from 0 to 2π we have :

$\theta_p = 0 \Longrightarrow z^{\sqrt{2}} = R^{\sqrt{2}}$

$\theta_p = 2\,\pi \Longrightarrow z^{\sqrt{2}} = R^{\sqrt{2}}\, e^{i 2\,\sqrt{2}\,\pi}$.

As we can see our function w does not return to its initial value after traversing a small circle of radius $R^{\sqrt{2}}$ at the point $0 = z^{\sqrt{2}}$, i.e. when $z = 0$. Therefore $z = 0$ is a branch point of our function. Also $z = \infty$ is another branch point of this function. We can see this by substituting the value $z = \frac{1}{t}$ back into our function and analyzing what happens at the point $t = 0$. Since $w = z^{\sqrt{2}} = t^{-\sqrt{2}}$, by a similar argument as above we can show that $t = 0$ (or $z = \infty$) is also a branch point for w. We can take the positive real axis to be the branch cut .

(2) Determine all possible values and give the principal value of the following numbers (put in the form $x + i\, y$):

a) $i^{1/2}$

Solution:
First we want to write our function in polar form:

$$x = 0;\ y = 1;\ \ \theta_p = \frac{\pi}{2};\ R = 1.$$

Hence

$$i^{1/2} = \left(e^{i\left(\frac{\pi}{2}+2\,\pi\,n\right)}\right)^{1/2} = e^{i\,\frac{\pi}{4}}\,e^{i\pi\,n}\,.$$

- For $n = 0$: $i^{1/2} = e^{i\,\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \boxed{\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}}$ (principal value)

- For $n = 1$: $i^{1/2} = e^{i\,\frac{\pi}{4}}\,e^{i\pi} = -e^{i\,\frac{\pi}{4}} = -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \boxed{-\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}}$

b) $\frac{1}{(1+i)^{1/2}}$

Solution:
First we want to write our function in polar form:

$$x = 1\ ;\quad y = 1\quad;\ \theta_p = \frac{\pi}{4}\quad;\ R = \sqrt{2}\,.$$

Hence

$$\frac{1}{(1+i)^{1/2}} = \left(\sqrt{2}\,e^{i\left(\frac{\pi}{4}+2\,\pi\,n\right)}\right)^{-1/2} = \left(\sqrt[4]{2}\,e^{i\,\frac{\pi}{8}}\,e^{i\pi\,n}\right)^{-1}.$$

- For $n = 0$: $\frac{1}{(1+i)^{1/2}} = \left(\sqrt[4]{2}\,e^{i\,\frac{\pi}{8}}\right)^{-1}$

  $= \boxed{\frac{1}{\sqrt[4]{2}\,\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}}}$ (principal value)

▸ For $n=1$: $\dfrac{1}{(1+i)^{1/2}} = \left(-\sqrt[4]{2}\; e^{i\,\frac{\pi}{8}}\right)^{-1}$

$$= -\frac{1}{\sqrt[4]{2}\;\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}}$$

c) $\log\left(1+\sqrt{3}\; i\right)$

Solution:

First we want to write our function in polar form:

$$x = 1\;;\quad y = \sqrt{3}\;;\quad \theta_p = \frac{\pi}{3};\quad R = 2.$$

Hence

$$\log\left(1+\sqrt{3}\; i\right) = \log\!\left(2\, e^{i\left(\frac{\pi}{3}+2\,\pi \mathrm{n}\right)}\right)$$

$$= \log 2 + \log e^{i\,\frac{\pi}{3}} + \log e^{i 2\pi\, n}$$

$$= \log 2 + i\left(\frac{\pi}{3} + 2\,\pi\, n\right).$$

▸ For $n=0$: $\log\left(1\;+\;\sqrt{3}\; i\right) = \log 2 + i\,\frac{\pi}{3}$ (principal value)

▸ For $n=1$: $\log\left(1+\sqrt{3}\; i\right) = \log 2 + i\left(\frac{\pi}{3} + 2\,\pi\right) = \log 2 + i\,\frac{7\,\pi}{3}$

.....With the log function we obtain an infinite number of solutions (one for each choice of n).

Hence the general solution is $\log 2 + i\left(\frac{\pi}{3} + 2\,\pi\, n\right)$, for $n = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots.$

d) $\log i^3 = 3\log i$

Solution:

First we want to write our function in polar form:

$$x = 0;\quad y = 1;\quad \theta_p = \frac{\pi}{2};\quad R = 1.$$

Hence

$$3\log i = 3\log\!\left(e^{i\left(\frac{\pi}{2}+2\,\pi\, n\right)}\right)$$

$$= 3\log e^{i\,\frac{\pi}{2}} + 3\log e^{i 2\pi\, n} = 3\, i\left(\frac{\pi}{2} + 2\,\pi\, n\right)$$

▸ For $n=0$: $3\log i = i\,\frac{3\,\pi}{2}$ (principal value)

As in the previous case, since we are dealing with a logarithmic function, we have an infinite number of solutions. Hence our general solution is of the form $i\left(\frac{3\,\pi}{2} + 6\,\pi\, n\right)$, for $n = 0,\ 1,\ 2,\ 3,\ \ldots$

(3) Solve for z:

a) $z^5 = 1$

Solution:
$z = 1^{1/5}$. Now we put our function in polar form

$$x = 1; \quad y = 0\,; \quad \theta = 0; \quad R = 1.$$

Hence

$$z = e^{i\,\frac{2\pi n}{5}}, \text{ for } n = \text{any five consecutive integers.}$$

- For $n = 0 : z = 1$
- For $n = 1$: $z = e^{i\,\frac{2\pi}{5}}$
- For $n = 2$: $z = e^{i\,\frac{4\pi}{5}}$
- For $n = 3$: $z = e^{i\,\frac{6\pi}{5}}$
- For $n = 4$: $z = e^{i\,\frac{8\pi}{5}}$.

Hence $z = \left\{1, e^{i\,\frac{2\pi}{5}}, e^{i\,\frac{4\pi}{5}}, e^{i\,\frac{6\pi}{5}}, e^{i\,\frac{8\pi}{5}}\right\}$.

b) $3 + 2\,e^{z-i} = 1$

Solution:

$$2\,e^{z-i} = -2 \Longrightarrow e^{z-i} = -1$$

$$\Longrightarrow \frac{e^z}{e^i} = -1 \Longrightarrow e^z = -e^i$$

$$\Longrightarrow z = \log(-1) + i\,.$$

Now we write $\log(-1)$ in polar form:

$$x = -1; \quad y = 0; \quad \theta_p = \pi \quad ; \; R = 1.$$

Then

$$\log(-1) = \log e^{i(\pi + 2\pi n)} = i(\pi + 2\pi n)\,.$$

Hence

$$z = i(\pi + 2\pi n) + i = i\,(\pi + 2n\pi + 1)$$

As it's the case with logarithmic functions, z has an infinite number of solutions. In this case we have $z = i(\pi + 2n\pi + 1)$, for $n = 0, 1, 2, 3, \ldots$

(5) Derive the following formulae:

a) $\coth^{-1} z = \frac{1}{2} \log \frac{z+1}{z-1}$

Solution:

$$w = \coth^{-1} z \Longrightarrow \coth w = z \Longrightarrow \frac{e^{w} + e^{-w}}{e^{w} - e^{-w}} = z\ .$$

$$\frac{\frac{e^{2\,w}+1}{e^{w}}}{\frac{e^{2\,w}-1}{e^{w}}} = \frac{e^{2\,w}+1}{e^{2\,w}-1} \Longrightarrow e^{2\,w} + 1 = z\,e^{2\,w} - z$$

$$\Longrightarrow e^{2\,w}(1-z) = -z-1$$

$$\Longrightarrow e^{2\,w} = \frac{-z-1}{1-z} \cdot \frac{-1}{-1}$$

$$\Longrightarrow e^{2\,w} = \frac{z+1}{z-1} \Longrightarrow 2\,w = \log \frac{z+1}{z-1}$$

$$\Longrightarrow w = \frac{1}{2} \log \frac{z+1}{z-1}$$

b) $\operatorname{sech}^{-1} z = \log \frac{1 + (1-z^2)^{1/2}}{z}$

Solution:

$$w = \operatorname{sech}^{-1} z \Longrightarrow \operatorname{sech} w = z \Longrightarrow \frac{2}{e^{w} + e^{-w}} = z.$$

$$\frac{2\,e^{w}}{e^{2\,w}+1} = z \Longrightarrow 2\,e^{w} = z\,e^{2\,w} + z$$

$$\Longrightarrow z\,e^{2\,w} - 2\,e^{w} + z = 0$$

$$\Longrightarrow e^{2\,w} - \frac{2}{z}\,e^{w} + 1 = 0$$

$$\Longrightarrow e^{w} = \frac{\frac{2}{z} + \sqrt{\frac{4}{z^2} - 4}}{2} = \frac{\frac{2}{z} + \sqrt{4\left(\frac{1}{z^2} - 1\right)}}{2}$$

$$= \frac{2\left(\frac{1}{z} + \sqrt{\left(\frac{1-z^2}{z^2}\right)}\right)}{2} = \frac{1}{z} + \frac{1}{z}\sqrt{(1-z^2)}$$

$$= \frac{1 + \sqrt{1-z^2}}{z}$$

$$\Longrightarrow \log e^{w} = \log \frac{1+\sqrt{1-z^2}}{z}$$

$$\Longrightarrow w = \operatorname{sech}^{-1} z = \log \frac{1 + \sqrt{1-z^2}}{z}$$

(6) Deduce the following derivative formulae:

a) $\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$

Solution:

$$w = \tan^{-1} z \Longrightarrow \tan w = z \Longrightarrow \frac{\frac{e^{i w}- e^{-i w}}{2 i}}{\frac{e^{i w}+ e^{-i w}}{2}} = z\,.$$

Then we have

$$\frac{-i(e^{i w}- e^{-i w})}{e^{i w}+ e^{-i w}} = z \Longrightarrow \frac{i(e^{-i w}- e^{i w})}{e^{i w}+ e^{-i w}} = z$$
$$\Longrightarrow \frac{i- ie^{2 i w}}{e^{2 i w}+ 1} = z \Longrightarrow i - ie^{2 i w} = z\, e^{2 i w} + z$$
$$\Longrightarrow z\, e^{2 i w} + ie^{2 i w} = i - z \Longrightarrow e^{2 i w}(z + i) = i - z$$
$$\Longrightarrow e^{2 i w} = \frac{i- z}{i+z} \Longrightarrow e^{w} = \left(\frac{i- z}{i+z}\right)^{1/2 i}$$
$$\Longrightarrow w = \tan^{-1} z = \log\left(\frac{i- z}{i+z}\right)^{1/2 i} = \frac{1}{2 i} \log \frac{i- z}{i+z}\,.$$

So now we differentiate :

$$\frac{d}{dz}\left(\frac{1}{2 i} \log \frac{i- z}{i+z}\right) = \frac{1}{2 i}\, \frac{d}{dz}\left[\log(i - z) - \log(i + z)\right]$$
$$= \frac{1}{2 i}\left(-\frac{1}{i-z} - \frac{1}{i+z}\right) = \frac{i}{2}\left(\frac{1}{i-z} + \frac{1}{i+z}\right)$$
$$= \frac{i}{2}\, \frac{1\,(i+z) + 1\,(i-z)}{(i^2- z^2)} = \frac{i^2}{(i^2- z^2)}$$
$$= -\frac{1}{(-1 - z^2)} = \frac{1}{1+ z^2}$$

c) $\frac{d}{dz}\sinh^{-1} z = \frac{1}{(1+z^2)^{1/2}}$

Solution:

$$w = \sinh^{-1} z \Longrightarrow \sinh w = z \Longrightarrow \frac{e^{w}- e^{-w}}{2} = z\,.$$

Then we have

$e^{w} - e^{-w} = 2\, z \Longrightarrow e^{2 w} - 1 = 2\, z\, e^{w} \;\Longrightarrow e^{2 w} - 2\, z\, e^{w} - 1 = 0$

$$\Longrightarrow e^{w} = \frac{2\,z + \sqrt{4\,z^2 + 4}}{2} = \frac{2\,z + 2\,\sqrt{(z^2 + 1)}}{2}$$

$$= z + \sqrt{(z^2\ +\ 1)}\ \Longrightarrow w = \log\left(z + \sqrt{(z^2\ +\ 1)}\ \right).$$

Now we can differentiate:

$$\frac{d}{dz}\left[\log\left(z + \sqrt{(z^2\ +\ 1)}\right)\right] = \frac{1}{z+\sqrt{(z^2+1)}}\left[1\ +\ \frac{1}{2}\,(z^2+1)^{-1/2}\,(2\,z)\right]$$

$$= \frac{1+\dfrac{z}{\sqrt{z^2+1}}}{z+\sqrt{(z^2+1)}} = \frac{\dfrac{\sqrt{z^2+1}+z}{\sqrt{z^2+1}}}{z+\sqrt{(z^2+1)}} = \boxed{\frac{1}{\sqrt{z^2+1}}}$$