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1 Newton's Law of Cooling

Newton's Law of Cooling is a model of the cooling of a warm object. Its basic premise is that the rate at which an object cools is proportional to the difference between the object's temperature and the temperature of the object's surroundings. So, according to this model, if I am in a room at 72 degrees Fahrenheit and my coffee, sitting on the table in front of me, is at 184 degrees, it is losing heat at twice the rate that your coffee—you're sitting next to me— which is at 128 degrees, is losing heat. Mathematically, the model is expressed in a differential equation. If $x(t)$ is the temperature of the object a time t after we start the clock, then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(c - x), \tag{1}$$

where c is the temperature of the object's surroundings –the air, in the case of the coffee– and k is a proportionality constant.

1.1 First Question

Does the right-hand side of the ODE (1) make sense? Does its dependence on x make sense? What about its dependence on c?

Answer. The RHS of the ODE (1) does seem to pose a sensible model for the rate of change of a body's temperature in time, relating the heat transfer to/from the body itself to its surroundings. The RHS of the equation states that this heat transfer is manifested via a linear relation (the difference) between the body's temperature and that of its surroundings. If the body is hotter than its surroundings (i.e., $x > c$), then $c - x$ is negative, which in turn indicates that the rate of change of the body's temperature $\mathrm{d}x/\mathrm{d}t$ (LHS of the equation) is also negative and so we have a heat loss from the body, transferred to its surroundings. When the temperature of the surroundings is higher than that of the object (i.e., $c > x$), then $c - x$ is positive, which indicates a postive rate of change of the body's temperature; in this case the heat is being transferred from the surroundings to the object. Of course, when $x = c$ the RHS is zero, which indicates that there is no heat transfer whatsoever; we have thermal equilibrium. As for the proportionality constant k, it is here to warn us that this heat transfer does not exclusively depend on the object's temperature and that of its surrroundings, but rather there may well be other factors to be considered in our model such as the area and nature of the surface of the body under consideration, as well as perhaps the turbulance (or lack thereof) of the surroundings (e.g., windy air). □

1.2 Second Question

Solve the equation; find its general solution.

Solution. Let us define

$$T(t) \equiv k(c - x(t)). \tag{2}$$

Then,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(k(c - x(t))) = -k\frac{\mathrm{d}x}{\mathrm{d}t}, \tag{3}$$

where on the last equality we used the fact that c is considered to be constant in our model, so its derivative vanishes. Now from (1) and (2) we see that (3) can be rewritten as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -kT, \tag{4}$$

whose solution is the familiar exponential,

$$T(t) = Ce^{-kt}, \tag{5}$$

for some constant C that depends on our (unspecified) initial temperature values. Now we plug (5) into (2),

$$Ce^{-kt} = k(c - x) \quad \Longrightarrow \quad \frac{C}{k}e^{-kt} = c - x \quad \Longrightarrow \quad Ce^{-kt} = c - x, \tag{6}$$

where we have absorbed the constant k into C in the coefficient of the exponential (if we were being pedantic we would re-label this "new C" with some other symbol, but it is redundant). Solving now for x, we get the general solution to Newton's Law of Cooling (1):

$$\boxed{x(t) = c - Ce^{-kt}} \tag{7}$$

□

1.3 Third Question

Discuss the dependence of the solution on c and k, and discuss whether it seems sensible to you. Do the roles of these parameters in the solution seem consonant with their roles in the ODE?

Answer. An interesting observation that arises from comparing (1) and its solution (7) is that, if in the latter the proportionality constant k is very large, then the object's temperature approaches the value of that of its surroundings, i.e., $\lim_{k\to\infty} x = c$. If we try the same, i.e., let $k \to \infty$ in (1), we note that the rate of change of the object's temperature goes wildly to ∞ as well. Had we let $k \to -\infty$ instead, both (1) and (7) go awry. This unphysical, unrealistic scenario warns us that the proportionality constant must be, relatively speaking, a small number. Nevertheless, one interpretation that I can think of for the former scenario (i.e., $k \to +\infty$) is that the *very rapid rate of change* of the body's temperature in (1) indicates an *instanteneous* heat transfer to/from the body to its surrounding, as suggested by the fact that $\lim_{k\to\infty} x = c$ in (7). This last observation does seem to suggest a level of consistency in the roles of c and k in both the ODE and its solution. ☐

1.4 Fourth Question

There should be a parameter in your general solution that, until now, we have not mentioned. What is it? How does the solution depend on it? Does this make sense?

Answer. The parameter C is the one that did not show up until we derived the solution (7). It gives us information about the initial temperature of the body: at time $t = 0$, (7) yields $x(0) = c - C$. It does make sense that such term exists since otherwise we would have $x(0) = c$, which is not very feasible. In other words, we should not expect that the body's temperature is equal to that of its surroundings from the onset; reaching thermal equilibrium more often than not requires some time. ☐

1.5 Fifth Question

Does the model, generally, seem sensible to you? What are some implicit assumptions of the model that have not yet been mentioned? Do they make sense?

Answer. The model is quite adequate, but not entirely realistic. It assumes, for instance, that the medium's temperature c is constant, while a more realistic model would allow c to be time-dependent. This is not a deal-breaker in most daily situations though, as often the time it takes an object (such as a cup of coffee) to reach thermal equilibirum with its surroundings is much shorter than the time it would take the medium (e.g., air) to change its ambient temperature. As I alluded to before on the answer to the First Question, there may be other factors in play such as the area and nature of the surface of the body under consideration, as well as perhaps the turbulance (or lack thereof) of the surroundings (e.g., windy air, viscous fluid). However, with some care these factors may very well be absorbed into the proportianlity constant k. ☐

1.6 Sixth Question

We sometimes stir coffee or blow over the coffee's surface to cool it. And these methods work. What does the efficacy of these methods tell you about the model?

Answer. These methods present hands-on proof of the validity of Newton's Law of Cooling. By either blowing over the surface of our coffee or stirring the latter we are simply speeding up the heat transfer from coffee to air. It is clear from both (1) and (7) that this speed-up of heat transfer (large $\mathrm{d}x/\mathrm{d}t$ in (1)) is largely attributed to the value of the proportionality constant k. ☐

2 Crystals' Volume Growth

In some contexts the volumes of crystals grow at rates proportional to their surface areas. Suppose that a spherical crystal of radius $x(t)$ grows such that its volume increases at a rate proportional to its surface area. This model is expressed in the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi x^3\right) = 4\pi k x^2. \tag{8}$$

Here, again, k is a proportionality constant.

2.1 First Question

Solve this equation, find its general solution; that is, find a general expression for $x(t)$. (Again, this general solution will involve a parameter that has not been provided.)

Solution. The ODE (8) states that the rate of change of the crystal's volume is proportional to its surface area, i.e.,

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \propto A(t), \tag{9}$$

where V and A are the volume and surface area, respectively, of the crystal; the proportionality is given by the parameter k. The solution is straightforward:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi x(t)^3\right) &= 4\pi k x(t)^2 \\ \frac{4}{3}\pi \cdot 3x^2 \frac{\mathrm{d}x}{\mathrm{d}t} &= 4\pi k x^2 \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= k. \end{aligned} \tag{10}$$

Solving then for x yields

$$\boxed{x(t) = kt + C} \tag{11}$$

□

2.2 Second Question

Does the role of k in the solution make sense? Does its role in the ODE make sense? Do those two roles correspond?

Solution. The role of the parameter k in both the ODE and in the solution is well defined; it represents the rate of change of the crystal's radius (we know this from (10)). We can see that this interpretation of k makes sense on the solution (11):

radius at time t = rate of change of radius $\times$ time unit (+ some constant that sets the initial radius at $t = 0$).

In the original ODE (8) it also makes sense that the crystal's surface area changes accordingly with the radius (as it either shrinks or expands), and that it turns produces a change in the volume (LHS). Thus we have determined that this role of k as the rate of change of the crystal's radius is consistent in both the ODE and its solution. □

2.3 Third Question

Does the model make sense? Does the rate at which the crystal's volume changes increase or decrease with time? Does this make sense?

Solution. The model does in general make sense ... but there are some caveats. Whether the crystal's volume increases or decreases with time is entirely up to the rate of change of the radius (i.e., k). If $k > 0$ we have an ever-expanding volume, while if $k < 0$ the crystal's volume shrinks until it (theoretically) becomes a singularity. The case $k = 0$ of course represents a static, time-independent volume. This is where the model breaks down; there must be some other factors taken into account (e.g., surface tension) that impede the crystal's volume from ever expanding/shrinking. Now, to answer the second question, the *rate* at which the crystal's volume changes increase or decrease with time depend on whether k is constant. In other words, if $\mathrm{d}k/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2 \neq 0$, then the rate at which the crystal's volume changes will increase or decrease accordingly. □

2.4 Fourth Question

How would the ODE change if the crystal were cubic?

Solution. With a cubic crystal we have

$$\frac{\mathrm{d}V_{\cube}(t)}{\mathrm{d}t} \propto A_{\cube}(t), \tag{12}$$

where $V_{\cube}(t) = x(t)^3$ and $A_{\cube}(t) = 6x(t)^2$ are the volume and surface area, respectively, of a cube. Using k as the proportionality constant again, and expanding (12), we get

$$\begin{aligned} \frac{\mathrm{d}V_{\cube}(t)}{\mathrm{d}t} &= kA_{\cube}(t) \\ 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} &= 6kx^2 \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= 2k. \end{aligned} \tag{13}$$

What this result indicates is that, if the crystal's surface is cubic, the proportionality constant k is now half the rate of change of the crystal's radius. Comparing this with our previous spherical example, we see that the cubic shape of a crystal yields a (half) slower growth/shrink. □

2.5 Fifth Question

Consider a model in which the spherical crystal's volume increases at a rate proportional (use k again for the proportionality constant) to its radius. Write down an ODE for this model and find its general solution.

Solution. In this model we have the proportionality law

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \propto x(t). \tag{14}$$

Using k again and expanding this relation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi x(t)^3\right) &= kx(t)\\ x\mathrm{d}x &= \frac{k}{4\pi}\mathrm{d}t\\ \int x\mathrm{d}x &= \frac{k}{4\pi}\int \mathrm{d}t\\ x &= \pm\sqrt{2\left(\frac{kt}{4\pi}+C\right)}.\end{aligned}$$

Since the radius cannot (on physical grounds) be a negative number, we choose the positive answer (and absorb 2 into the constant C) to get the general solution to (14):

$$\boxed{x(t) = \sqrt{\frac{kt}{2\pi}+C}} \tag{15}$$

□

2.6 Sixth Question

Consider a model in which the spherical crystal's volume increases at a rate proportional (use k again for the proportionality constant) to its volume. Write down an ODE for this model and find its general solution.

Solution. In this case we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \propto V(t), \tag{16}$$

which indicates that its solution will be in the form of an exponential. Indeed, using k again and expanding this relation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi x(t)^3\right) &= k\frac{4}{3}\pi x(t)^3\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{k}{3}x\\ \frac{3}{k}\int \frac{1}{x}\mathrm{d}x &= \int \mathrm{d}t\\ \frac{3}{k}\log x &= t+\log C\\ x^{3/k} &= e^{t+\log C}\\ x &= C^{k/3}e^{kt/3}.\end{aligned}$$

Redefining C as $C^{k/3}$, we get the general solution to (16):

$$\boxed{x(t) = Ce^{kt/3}} \tag{17}$$

□

2.7 Seventh Question

Consider a model in which the spherical crystal's volume increases at a rate inversely proportional (use k again for the proportionality constant) to its surface area. Write down an ODE for this model and find its general solution.

Solution. We now have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \propto \frac{1}{A(t)}. \tag{18}$$

Using k again and expanding,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi x(t)^3\right) &= \frac{k}{4\pi x(t)^2} \\
\frac{4}{3}\pi \cdot 3x^2 \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{k}{4\pi x^2} \\
16\pi^2 \int x^4 \mathrm{d}x &= k \int \mathrm{d}t \\
\frac{16\pi^2}{5} x^5 &= kt + C.
\end{aligned} \tag{19}$$

Redefining C as $5/(16\pi^2)C$ and solving for x we get

$$\boxed{x(t) = \sqrt[5]{\frac{5kt}{16\pi^2} + C}} \tag{20}$$

□

2.8 Eight Question

Consider a model in which the spherical crystal's volume increases at a rate proportional (use k again for the proportionality constant) to the product of its surface area and e^{-x}. Write down an ODE for this model and find its general solution.

Solution. In this last model we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \propto A(t)e^{-x(t)}. \tag{21}$$

We proceed as usual, with k as the proportionality constant:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi x(t)^3\right) &= 4\pi k x(t)^2 e^{-x(t)} \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= ke^{-x} \\
\int e^x \mathrm{d}x &= k \int \mathrm{d}t \\
e^x &= kt + C,
\end{aligned}$$

which leads to the general solution

$$\boxed{x(t) = \log(kt + C)} \tag{22}$$

where C once again is a constant that sets the value of the radius at initial time $t = 0$ (without C in this case we would have a negatively infinite initial "radius"!). □

2.9 Ninth Question

Discuss, briefly, all of these models and whether they make sense. Discuss, for each model, its general solution, how that general solution reflects the ODE, and how fast the volume of the crystal grows in the long run in the model.

Solution. To make the answer a bit more readable, I will list my comments of each model as a bullet-point list:

- We could say that the model (14) makes just about as much sense as our original ODE (8). We expect radius growth to be directly correlated to volume growth, so the ODE does make sense. As for the solution (15), we can see that the result is quite similar to the original ODE's result (11), albeit the radius does grow at a slower rate in (15). Consequently, of course, the volume in (14) also grows slower than in (8).

- On the other hand, the solution (20) to the model (18) also seems to have a similar behavior to (15) (although the radius increases at an even slower pace this time), but the model itself (18) does not make a whole lot of sense. Making the volume inversely proportional to the surface area is completely non-sensical, as the growth of these two quantities must go hand-in-hand. This particular example serves as a warning: just because either the model or its solution yields a result that somewhat makes sense, we cannot assume that we are in the right. We need to consider both the model and its solution carefully.
- There is absolutely nothing wrong with the model (16), although as we can see in the solution (17) care must be taken when setting the parameter k, since otherwise the (exponential) growth of the radius (and consequently, the volume) will grow very (exponentially!) fast.
- Lastly, we have the model (21) which also makes as much sense as our original ODE (8). The only difference is that this time we have a damping term e^{-x}, which slows down tremendously the growth of the volume for small radius x (this damping effect becomes irrelevant, however, for large x). This slow growth of the volume, of course, indicates a slow growth of the radius (thus the damping effect does stick around for a while!). The latter can also be easily seen in the solution (22), since a logarithmic growth is indeed quite slow! $\square$