Math 353 HW 10

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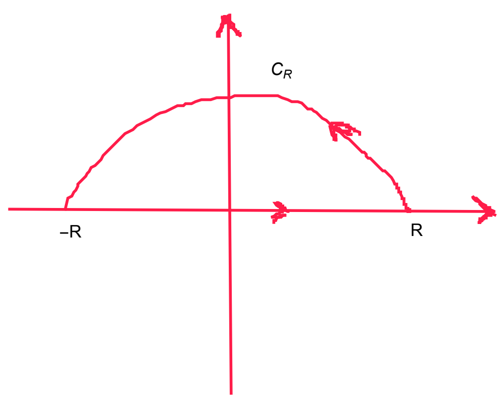
Section 4.2

(1) Evaluate the following real integrals:

b) $\int_0^{\infty} \frac{1}{(x^2+a^2)^2}\, dx \quad ; \; a^2 > 0$

Solution:
We can solve complicated real integrals using complex variables by integrating inside a closed contour:



Following the figure above, we have

$$\oint_C \frac{1}{(z^2+a^2)^2}\, dz = \int_{-R}^{R} \frac{1}{(z^2+a^2)^2}\, dz + \int_{C_R} \frac{1}{(z^2+a^2)^2}\, dz.$$

Now, taking the limit as the radius R approaches infinity we have

$$\lim_{R\to\infty} \oint_C \frac{1}{(z^2+a^2)^2}\, dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(z^2+a^2)^2}\, dz + \lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+a^2)^2}\, dz$$

$$\Longrightarrow 2\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\, dx + \lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+a^2)^2}\, dz$$

We have that $\lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+a^2)^2}\, dz = 0$, since by a theorem we discussed in class we know that if $f(z)$ is a rational function and the degree of the denominator is at least two higher than the degree

of the numerator, then $\lim\limits_{R\to\infty} \int_{C_R} f(z)\, dz = 0$. In this case we have a denominator of degree 4 and a numerator of degree 0, so the theorem can be applied.
Consequently, we have

$$2\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\, dx.$$

Now let's find the singularities:

$$(z^2+a^2)^2 = 0 \Longrightarrow z^2+a^2 = 0 \Longrightarrow z = \pm i\, a.$$

Of these two singularities however only $z = i\, a$ is enclosed in the region, so we just need to find the residue at this point, which is a pole of order 2:

$$\begin{aligned}
\operatorname{Res}(f(z);\, i\, a) &= \frac{1}{1!} \frac{d}{dz}\left[(z - i\, a)^2 \frac{1}{(z^2+a^2)^2}\right]\Bigg|_{z=i\,a} \\
&= \frac{d}{dz}\left[(z - i\, a)^2 \frac{1}{(z+a\, i)^2\,(z-a\, i)^2}\right]\Bigg|_{z=i\,a} \\
&= \frac{d}{dz}\left[\frac{1}{(z+a\, i)^2}\right]\Bigg|_{z=i\,a} = \frac{-2}{(z+a\, i)^3}\Big|_{z=i\,a} \\
&= \frac{-2}{(2\, a\, i)^3} = \frac{-2}{8\, a^3\,(-i)} = \frac{1}{4\, a^3\, i}.
\end{aligned}$$

Hence

$$2\pi i \frac{1}{4\, a^3\, i} = \frac{\pi}{2\, a^3} = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\, dx$$

$$\Longrightarrow \int_{0}^{\infty} \frac{1}{(x^2+a^2)^2}\, dx = \boxed{\frac{\pi}{4\, a^3}} \quad \text{(since the function is even).}$$

c) $\displaystyle\int_0^{\infty} \frac{1}{(x^2+a^2)\,(x^2+b^2)}\, dx$; $\quad a^2,\ b^2 > 0$

<u>Solution:</u>
Following the same method used on part *b*), we have

$$\lim_{R\to\infty} \oint_C \frac{1}{(z^2+a^2)\,(z^2+b^2)}\, dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(z^2+a^2)\,(z^2+b^2)}\, dz + \lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+a^2)\,(z^2+b^2)}\, dz.$$

The denominator of our function once again has degree 4 whereas the numerator has degree 0, so we know that

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+a^2)\,(z^2+b^2)}\, dz = 0.$$

Then we're left with

$$2\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)\,(x^2+b^2)}\, dx.$$

This time $f(z)$ has singularities at $z = \pm a\, i,\ \pm b\, i$. However, only $z = a\, i,\ b\, i$ lie in the region (we're still using the upper half plane as in part *b*)) so we only need to find the residue at these points, which are both simple poles:

• $\text{Res}(f(z); a\,i) = \dfrac{1}{\frac{d}{dz}[(z^2+a^2)\,(z^2+b^2)]\big|_{z=a\,i}} = \dfrac{1}{\frac{d}{dz}[(z^4+b^2\,z^2+a^2\,z^2+a^2\,b^2]\big|_{z=a\,i}}$

$$= \frac{1}{(4\,z^3+2\,b^2\,z+2\,a^2\,z)|_{z=a\,i}} = \frac{1}{4\,(a\,i)^3+2\,b^2\,(a\,i)+2\,a^2\,(a\,i)}$$

$$= \frac{1}{-4\,a^3\,i+2\,b^2\,a\,i+2\,a^3\,i} = \frac{1}{i\,(-4\,a^3\ +2\,b^2\,a\ +2\,a^3)}$$

$$= \frac{1}{i\,(-2\,a^3\ +2\,b^2\,a)}.$$

• $\text{Res}(f(z); b\,i) = \dfrac{1}{\frac{d}{dz}[(z^4+b^2\,z^2+a^2\,z^2+a^2\,b^2]\big|_{z=b\,i}}$

$$= \frac{1}{(4\,z^3+2\,b^2\,z+2\,a^2\,z)|_{z=b\,i}} = \frac{1}{4\,(b\,i)^3+2\,b^2\,(b\,i)+2\,a^2\,(b\,i)}$$

$$= \frac{1}{-4\,b^3\,i+2\,b^3\,i+2\,a^2\,b\,i} = \frac{1}{i\,(-2\,b^3\ +2\,a^2\,b)}$$

Then we have

$$2\,\pi i\cdot\left(\frac{1}{i\,(-2\,a^3\ +2\,b^2\,a)} + \frac{1}{i\,(-2\,b^3\ +2\,a^2\,b)}\right)$$

$$= \pi\cdot\left(\frac{1}{-a^3\ +b^2\,a} + \frac{1}{-b^3\ +a^2\,b}\right) = \pi\cdot\left(\frac{1}{a\,(\,b^2-a^2)} + \frac{1}{b\,(a^2-b^2)}\cdot\frac{-1}{-1}\right)$$

$$= \pi\cdot\left(\frac{1}{a\,(\,b^2-a^2)} - \frac{1}{b\,(b^2-a^2)}\right) = \pi\cdot\left(\frac{b-a}{a\,b\,(\,b^2-a^2)}\right) = \pi\cdot\left(\frac{b-a}{a\,b\,(\,b+a)\,(b-a)}\right)$$

$$= \frac{\pi}{a\,b(b+a)} = \int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)\,(x^2+b^2)}\ dx\,.$$

Hence, since the function is even our solution is half of this, i.e.

$$\int_0^{\infty}\frac{1}{(x^2+a^2)\,(x^2+b^2)}\ dx = \frac{\pi}{2\,a\,b(b+a)}.$$

d) $\displaystyle\int_0^{\infty}\frac{1}{x^6+1}\ dx$

Solution:

Again, following the same method used on part *b*), we have

$$\lim_{R\to\infty}\oint_C\frac{1}{z^6+1}\ dz = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{z^6+1}\ dz + \lim_{R\to\infty}\int_{C_R}\frac{1}{z^6+1}\ dz.$$

The denominator of our function this time has degree 6 whereas the numerator has degree 0, so we know that $\displaystyle\lim_{R\to\infty}\int_{C_R}\frac{1}{z^6+1}\ dz = 0.$

Then we're left with

$$2\,\pi\,i\cdot(\text{sum of residues}) = \int_{-\infty}^{\infty}\frac{1}{x^6+1}\ dx.$$

First let's find the singularities:

$z^6 + 1 = 0 \implies z^6 = -1 \implies z^6 = e^{i(\pi + 2\pi n)}$ for $n = 0, 1, 2, 3, \dots$

$\implies z = e^{i\left(\frac{\pi}{6} + \frac{\pi}{3} n\right)}$.

$\bullet\!\!\rightarrow$ For $n = 0 : z = e^{i\frac{\pi}{6}}$

$\bullet\!\!\rightarrow$ For $n = 1 : z = e^{i\frac{\pi}{2}} = i$

$\bullet\!\!\rightarrow$ For $n = 2 : z = e^{i\frac{5\pi}{6}}$

$\bullet\!\!\rightarrow$ For $n = 3 : z = e^{i\frac{7\pi}{6}}$

$\bullet\!\!\rightarrow$ For $n = 4 : z = e^{i\frac{3\pi}{2}} = -i$

$\bullet\!\!\rightarrow$ For $n = 5 : z = e^{i\frac{11\pi}{6}}$

For $n > 5$, the values of the roots start repeating, hence we only have six unique roots, which means that we have six singularities. However we only need to consider the singularities that are enclosed in the region (we are once again considering the upper half plane just like in parts *b*) and *c*)). These singularities are $z = e^{i\frac{\pi}{6}}, i, e^{i\frac{5\pi}{6}}$, which are all simple poles.

Now we need to calculate the residues at these poles:

$$\bullet\ \mathrm{Res}\left(f(z); e^{i\frac{\pi}{6}}\right) = \frac{1}{\frac{d}{dz}(z^6+1)\big|_{z=e^{i\pi/6}}} = \frac{1}{6z^5\big|_{z=e^{i\pi/6}}} = \frac{1}{6e^{i\frac{5\pi}{6}}} = \frac{1}{6}e^{-i\frac{5\pi}{6}}$$

$$\bullet\ \mathrm{Res}(f(z); i) = \frac{1}{6z^5\big|_{z=i}} = \frac{1}{6i^5} = \frac{1}{6i} = -\frac{i}{6}$$

$$\bullet\ \mathrm{Res}\left(f(z); e^{i\frac{5\pi}{6}}\right) = \frac{1}{6z^5\big|_{z=e^{i5\pi/6}}} = \frac{1}{6e^{i\frac{25\pi}{6}}} = \frac{1}{6e^{i\left(\frac{24\pi}{6}+\frac{\pi}{6}\right)}} = \frac{1}{6e^{i\left(4\pi+\frac{\pi}{6}\right)}} = \frac{1}{6(1)e^{i\frac{\pi}{6}}} = \frac{1}{6}e^{-i\frac{\pi}{6}}$$

Hence we have

$$2\pi i\cdot\left(\frac{1}{6}e^{-i\frac{5\pi}{6}} - \frac{i}{6} + \frac{1}{6}e^{-i\frac{\pi}{6}}\right) = \frac{1}{3}\pi i\cdot\left(e^{-i\frac{5\pi}{6}} - i + e^{-i\frac{\pi}{6}}\right)$$

$$= \frac{1}{3}\pi i\cdot\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right) - i + \cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{3}\pi i\cdot\left[-\frac{\sqrt{3}}{2} - \frac{i}{2} - i + \frac{\sqrt{3}}{2} - \frac{i}{2}\right] = \frac{1}{3}\pi i\cdot(-2i) = \frac{2\pi}{3} = \int_{-\infty}^{\infty}\frac{1}{x^6+1}\,dx.$$

Hence, since the function is even, we have

$$\int_0^{\infty}\frac{1}{x^6+1}\,dx = \frac{\pi}{3}.$$

(2) Evaluate the following real integrals by residue integration :

a) $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\, dx \quad a^2 > 0$

Solution:

This integral is of the form $\int_{-\infty}^{\infty} f(x) \sin(k\, x)\, dx$, where $k = 1$ in this case. In order to solve this integral we use a technique similar to the one we used on problem (1), i.e. we evaluate an integral on a closed contour going on the x-axis from $-R$ to R, and then over the curve C_R (we choose the upper half plane in this case, since $k > 0$). The difference this time is that we have a sine term in the integrand so we are going to introduce an extra term, namely $e^{i\,k\,z}$, with $k = 1$ in this case.

Thus we have

$$\underset{R\to\infty}{\text{limit}} \oint_C \frac{z\, e^{i\, z}}{z^2+a^2}\, dz = \underset{R\to\infty}{\text{limit}} \int_{-R}^{R} \frac{x\, e^{i\, x}}{x^2+a^2}\, dx + \underset{R\to\infty}{\text{limit}} \int_{C_R} \frac{z\, e^{i\, z}}{z^2+a^2}\, dz.$$

The term $\underset{R\to\infty}{\text{limit}} \int_{C_R} \frac{z\, e^{i\, z}}{z^2+a^2}\, dz$ goes to zero by Jordan's lemma, since the degree of the denominator is greater than the numerator's.

Consequently, we're left with

$$\underset{R\to\infty}{\text{limit}} \oint_C \frac{z\, e^{i\, z}}{z^2+a^2}\, dz = \underset{R\to\infty}{\text{limit}} \int_{-R}^{R} \frac{x\, e^{i\, x}}{x^2+a^2}\, dx$$

or

$$2\, \pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{x\, e^{i\, x}}{x^2+a^2}\, dx\ .$$

Now we can see that $\frac{z e^{i\, z}}{z^2+a^2}$ has two simple poles at $z = \pm a\, i$. However, only $z = a\, i$ is enclosed in the region (upper half plane). So we only need to find the residue at this pole:

$$\text{Res}(f(z);\, z = a\, i) = \underset{z\to a\, i}{\text{limit}}\, (z - a\, i)\, \frac{z e^{i\, z}}{(z+a\, i)\,(z-a\, i)}$$

$$= \underset{z\to a\, i}{\text{limit}} \frac{z e^{i\, z}}{(z+a\, i)} = \frac{a\, i\, e^{-a}}{(2\, a\, i)} = \frac{e^{-a}}{2}.$$

Hence we have

$$2\, \pi i \cdot \frac{e^{-a}}{2} = \pi\, i\, e^{-a} = \int_{-\infty}^{\infty} \frac{x\, e^{i\, x}}{x^2+a^2}\, dx = \int_{-\infty}^{\infty} \frac{x \cos x}{x^2+a^2}\, dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\, dx$$

$$\Longrightarrow i\pi e^{-a} = i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\, dx \Longrightarrow \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\, dx = \pi e^{-a}.$$

h) $\int_0^{2\pi} \frac{1}{(5-3\sin(\theta))^2}\, d\theta$

Solution:

We want to change variables from θ to z and turn this into a contour integral on the circle $|z| = 1$. To that end we make the following substitutions:

$$z = e^{i\theta} \; ; dz = ie^{i\theta}\, d\theta \Longrightarrow d\theta = \frac{dz}{iz}.$$

Thus we have

$$\int_0^{2\pi} \frac{1}{\left(5-3\,\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2}\, d\theta = \oint_{|z|} \frac{1}{\left(5-3\,\frac{z-\frac{1}{z}}{2i}\right)^2}\,\frac{dz}{iz}$$

$$= \oint_{|z|} \frac{1}{\left(5-3\left(\frac{z^2-1}{2iz}\right)\right)^2}\,\frac{dz}{iz} = \oint_{|z|} \frac{1}{\left(\frac{10iz-3z^2+3}{2iz}\right)^2}\,\frac{dz}{iz}$$

$$= \oint_{|z|} \frac{(2iz)^2}{(10iz-3z^2+3)^2}\,\frac{dz}{iz} = \oint_{|z|} \frac{-4z^2}{(10iz-3z^2+3)^2}\,\frac{dz}{iz}$$

$$= 4i\oint_{|z|} \frac{z}{(-3z^2+10iz+3)^2}\, dz\,.$$

Our integrand then has a pole of order 2 when $-3z^2+10iz+3=0$. Therefore, using the quadratic formula we have

$$z = \frac{-10i \pm \sqrt{(10i)^2-4(-3)3}}{2(-3)} = \frac{-10i \pm \sqrt{-100+36}}{-6} = \frac{-10i \pm \sqrt{-64}}{-6} = \frac{-10i \pm 8i}{-6}.$$

So $z = 3i,\ \frac{1}{3}i$. However, only $\frac{1}{3}i$ lies inside the enclosed region $|z| = 1$. Now we need to find the residue at this point:

$$\mathrm{Res}\left(f(z);\, z = \tfrac{1}{3}i\right) = \frac{1}{(2-1)!}\,\frac{d}{dz}\left(\left(z-\tfrac{1}{3}i\right)^2 \frac{z}{(-3z^2+10iz+3)^2}\right)\Bigg|_{z=\frac{1}{3}i}$$

$$= \frac{d}{dz}\left(\left(z-\tfrac{1}{3}i\right)^2 \frac{z}{\left(-3(z-3i)\left(z-\frac{1}{3}i\right)\right)^2}\right)\Bigg|_{z=\frac{1}{3}i} = \frac{d}{dz}\left(\frac{z}{9(z-3i)^2}\right)\Bigg|_{z=\frac{1}{3}i}$$

$$= \frac{1}{9}\,\frac{(z-3i)^2-2z(z-3i)}{(z-3i)^4}\Bigg|_{z=\frac{1}{3}i} = \frac{1}{9}\,\frac{(z-3i)-2z}{(z-3i)^3}\Bigg|_{z=\frac{1}{3}i}$$

$$= \frac{1}{9}\,\frac{\left(\frac{1}{3}i-3i\right)-\frac{2}{3}i}{\left(\frac{1}{3}i-3i\right)^3} = \frac{1}{9}\,\frac{\left(-\frac{10}{3}i\right)}{\left(-\frac{8}{3}i\right)^3}$$

$$= \frac{1}{9}\left(-\frac{10}{3}\right)\left(\frac{(-3)^3}{8^3}\right)i^2 = -\frac{5}{256}.$$

So we have that

$$4i\oint_{|z|} \frac{z}{(-3z^2+10iz+3)^2}\, dz = 4i(2\pi i)\left(-\frac{5}{256}\right) = \frac{5\pi}{32}.$$

Consequently,

$$\int_0^{2\pi} \frac{1}{(5-3\sin(\theta))^2}\, d\theta = 4i\oint_{|z|} \frac{z}{(-3z^2+10iz+3)^2}\, dz = \frac{5\pi}{32}.$$

(Problem C) Evaluate $\int_0^\infty \frac{\cos m x}{(x^2+1)^2}\, dx \quad ; \; m > 0$

<u>Solution:</u>

We use the same trick as in problem (2). We are given that $m > 0$ so we evaluate our closed contour on the upper half plane.

We have that

$$\lim_{R\to\infty} \oint_C \frac{e^{i m z}}{(z^2+1)^2}\, dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i m x}}{(x^2+1)^2}\, dx + \lim_{R\to\infty} \int_{C_R} \frac{e^{i m z}}{(z^2+1)^2}\, dz.$$

The term $\lim_{R\to\infty} \int_{C_R} \frac{e^{i m z}}{(z^2+1)^2}\, dz$ goes to zero by Jordan's lemma, since we have that $f(z) = \frac{1}{(z^2+1)^2}$ and the degree of the denominator is greater than that of the numerator.

Thus we're left with

$$\lim_{R\to\infty} \oint_C \frac{e^{i m z}}{(z^2+1)^2}\, dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i m x}}{(x^2+1)^2}\, dx$$

or

$$2\pi i \cdot (\text{sum of residues}) = \int_{-\infty}^{\infty} \frac{x e^{i m x}}{(x^2+1)^2}\, dx.$$

We can see that $\frac{e^{i m z}}{(z^2+1)^2}$ has two poles of order 2 at $z = \pm i$. However, only $z = i$ is enclosed in the region (upper half plane). So we only need to find the residue at this pole:

$$\text{Res}(f(z); z = i) = \frac{1}{(2-1)!} \frac{d}{dz}\left[(z-i)^2 \frac{e^{i m z}}{(z+i)^2 (z-i)^2}\right]\Bigg|_{z=i}$$

$$= \frac{d}{dz} \frac{e^{i m z}}{(z+i)^2}\Bigg|_{z=i} = \frac{i m e^{i m z}(z+i)^2 - e^{i m z}\, 2(z+i)}{(z+i)^4}\Bigg|_{z=i}$$

$$= \frac{i m e^{-m}(2i)^2 - e^{-m}\, 2(2i)}{(2i)^4} = \frac{-4 i m e^{-m} - 4 i e^{-m}}{16}$$

$$= -\frac{i e^{-m}(m+1)}{4}.$$

Then we have

$$2\pi i \cdot \left(-\frac{i e^{-m}(m+1)}{4}\right) = \frac{\pi e^{-m}(m+1)}{2} = \int_{-\infty}^{\infty} \frac{x e^{i m x}}{(x^2+1)^2}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x \cos m x}{(x^2+1)^2}\, dx + i \int_{-\infty}^{\infty} \frac{x \sin m x}{(x^2+1)^2}\, dx$$

$$\Longrightarrow \frac{\pi e^{-m}(m+1)}{2} = \int_{-\infty}^{\infty} \frac{x \cos m x}{(x^2+1)^2}\, dx$$

$$\Longrightarrow \int_0^\infty \frac{x \cos m x}{(x^2+1)^2}\, dx = \frac{\pi e^{-m}(m+1)}{4}.$$