TRRT
Workshop 1 Hand-In

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Problem 1 (Solution to Exercise 4)**.** *a) We may attach a diagram D_n to any one of these symmetric matrices (b_{ij}) by using the rule suggested on the workshop sheet:*

$$b_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -\sqrt{n_{ij}} & \text{if } i \neq j, \end{cases}$$

where n_{ij} denotes the number of edges between nodes i and j on our corresponding diagram. Note that removing a node from a diagram (which leaves us with yet another Dynkin diagram, or a disjoint union of these) corresponds to removing an entire row and column from the corresponding matrix (which analogously leaves us with another symmetric matrix (b_{ij})). For instance, let us take

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

which corresponds to a Dynkin diagram (A_3)

Then by removing any combination of j^{th} row + j^{th} column, we end up with a smaller matrix that corresponds to a different Dynkin diagram (or a disjoint union of Dynkin diagrams). Let's illustrate this in two instances:

i) Eliminate third row + third column. Then our new matrix is

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

which corresponds to A_2:

ii) Eliminate second row + second column. Then our new matrix is

$$\begin{pmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

which corresponds to $A_1 \amalg A_1$:

and so on . . . □

b) *Sylvester's criterion states that, given a real symmetric matrix B, the associated bilinear form is positive definite if and only if all the leading principle minors are positive; that is, if the determinant of each of the matrices*

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{1n} & \dots & b_{nn} \end{pmatrix}, \begin{pmatrix} b_{11} & \dots & b_{1(n-1)} \\ \vdots & \ddots & \vdots \\ b_{1(n-1)} & \dots & b_{(n-1)(n-1)} \end{pmatrix}, \dots, (b_{11})$$

is positive. But we have previously established that each one of these symmetric matrices corresponds to some Dynkin diagram (or a disjoint union thereof); hence, by Sylvester's criterion, it follows that the associated bilinear form $B(-,-)$ is positive definite for all Dynkin diagrams if and only if the determinant of the symmetric matrix associated to each diagrams is positive. □

Problem 2 (Solution to Exercise 5)**.** *The matrix*

$$B_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \vdots \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & 0 & 0 & -1 & 2 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -\sqrt{2} \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 2 \end{pmatrix},$$

is a finite-type[1] *Cartan matrix associated to a finite Dynkin diagram. We now show that such matrices are necessarily positive definite. First assume arbitrarily that B_n is of finite or affine type. Then there exists $v > 0$ such that $B_n v \geq 0$. For $\lambda > 0$, we have $(B_n + \lambda I)v > 0$; thus $B_n + \lambda I$ is of finite type and hence nondegenerate. Therefore $\det(B_n + \lambda I) \neq 0$ for all $\lambda > 0$ and the eigenvalues of B_n are nonnegative. Since a symmetric matrix with all positive eigenvalues is positive definite, we have the desired result.*

Alternatively, we could have used the recurrence relation

$$f_i = d_i f_{i-1} - c_i a_{i-1} f_{i-2},$$

where the determinants

$$f_i = \begin{vmatrix} d_1 & a_1 & & & & & \\ c_2 & d_2 & a_2 & & & & \\ & c_3 & d_3 & a_3 & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & c_{i-1} & d_{i-1} & a_{i-1} \\ & & & & & c_i & d_i \end{vmatrix}$$

[1]By a ***finite type*** matrix A, me mean $\det A \neq 0$; there exists $u > 0$ such that $Au > 0$ and $Av \geq 0$ implies that $v > 0$ or $v = 0$. Similarly, an ***affine type*** matrix $\widetilde{A}$ is one such that $\mathrm{corank}(\widetilde{A}) = 1$; there exists $u > 0$ such that $\widetilde{A}u = 0$ and $\widetilde{A}v \geq 0$ implies that $\widetilde{A}v = 0$.

are minors of some tridiagonal matrix

$$A = \begin{pmatrix} d_1 & a_1 & & & & & \\ c_2 & d_2 & a_2 & & & & \\ & c_3 & d_3 & a_3 & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & c_{i-1} & d_{i-1} & a_{i-1} \\ & & & & & c_n & d_n \end{pmatrix}.$$

Applying this method to our tridiagonal matrix B_n and using induction, it is easy to see that all the minors are postive, and therefore by Sylvester's Criterion the associated bilinear form to B_n is positive definite. □