Linear Algebra Notes

Mario L. Gutierrez Abed

Linear Transformations and Matrices

LINEAR MAPS, NULLSPACE, AND RANGE

Properties of linear maps:

Given a tranformation T from a VS V to a VS W, the following properties hold :

1) If T is linear, $T(\hat{0}_v) = \hat{0}_w$. (it maps the zero vector in V to the zero vector of W)

2) T is linear iff $T(c\,u + v) = c\,T\,u + T\,v$.

Examples:

a) The zero map $T_o : V \longrightarrow W$, such that $T_o\, v = \hat{0}_w \;\; \forall\, v \in V$.

Let $u, v \in V$ and $c \in \mathbb{F}$. Then

$T_o(c\,u + v) = \hat{0}_w$

$c\,T_o\,u + T_o\,v = c\,\hat{0}_w + \hat{0}_w = \hat{0}_w$

Thus the zero map is linear.

b) The identity map $\text{Id} : V \longrightarrow V$, such that $Id(v) = v \;\; \forall\, v \in V$.

Let $u, v \in V$ and $c \in \mathbb{F}$. Then

$Id(c\,u + v) = c\,u + v = c\,Id(u) + Id(v)$.

Thus Id is linear.

c) The differential operator $\frac{d}{dx} : P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$.

$\frac{d}{dx}(c\,f + g) = c\,\frac{df}{dx} + \frac{dg}{dx}$

So $\frac{d}{dx}$ is linear.

d) The integral map $\int_a^b : P_n(\mathbb{R}) \longrightarrow \mathbb{R}$.

$\int_a^b (c\,f + g)\, dx = c\int_a^b f\, dx + \int_a^b g\, dx$

Thus the integral map is linear.

Definition: Let V and W be VS's. Then the set of all possible linear maps form V to W is a VS over $\mathbb{F}$ and it's denoted $\mathcal{L}(V, W)$.

Definition: Let V and W be VS's over the same field and let $T \in \mathcal{L}(V, W)$. Then the nullspace of T (denoted $\mathcal{N}(T)$ or null(T)), is the set of all vectors in V whose image is $\hat{0}_w$, i.e. $\mathcal{N}(T) = \{v \in V : T v = \hat{0}\}$.

Definition: The range of T (denoted $R(T)$ or range(T) or $\mathrm{Im}(T)$) is the set of all image vectors in W that have a preimage in V, i.e. $R(T) = \{w \in W : T v = w \ \ \forall v \in V\}$.

• Theorem:

For $T \in \mathcal{L}(V, W)$,

a) $\mathcal{N}(T) \subseteq V$ is a subspace .

b) $R(T) \subseteq W$ is a subspace .

Proof:

a) We show that $\mathcal{N}(T)$ is a subspace of V.

i) $T(\hat{0}_v) = \hat{0}_w$. Hence $\hat{0}_v \in \mathcal{N}(T)$.

ii) Let $u, v \in \mathcal{N}(T)$. We must show that $u + v \in \mathcal{N}(T)$.

$T(u+v) = T u + T v = \hat{0}_w + \hat{0}_w = \hat{0}_w$

$\Longrightarrow u + v \in \mathcal{N}(T)$

iii) Let $u \in \mathcal{N}(T)$ and $c \in \mathbb{F}$. We must show that $c u \in \mathcal{N}(T)$.

$T(c u) = c(T u) = c(\hat{0}_w) = \hat{0}_w$.

$\Longrightarrow c u \in \mathcal{N}(T)$.

Thus $\mathcal{N}(T)$ is a subspace of V. ✓

b) Now we show that $R(T)$ is a subspace of W.

i) There exists a preimage of $\hat{0}_w$ such that $T(\hat{0}_v) = \hat{0}_w$

Hence $\hat{0}_w \in R(T)$.

ii) Let $u, v \in R(T)$. We must show that $u+v \in R(T)$.

$\exists\ x, y \in V$ such that $T x = u$ and $T y = v$.

$\Longrightarrow u + v = T x + T y = T(x + y)$.

$\Longrightarrow u + v \in R(T)$.

iii) Let $u \in R(T)$ and $c \in \mathbb{F}$.

$\Longrightarrow \exists\ x \in V$ such that $T x = u$.

$\Longrightarrow c u = c T x = T(c x)$

$\Longrightarrow c u \in R(T)$.

Thus $R(T)$ is a subspace of W. ✓ ■

• Theorem:

Let V, W be VS's and let $T \in \mathcal{L}(V, W)$. Supose $\dim(V) = n < \infty$, and let $B = \{v_1, \ldots, v_n\}$ be a basis for V. Then,

$R(T) = \text{span}(T(B)) = \text{span}(T v_1, \ldots, T v_n)$. $(\Longrightarrow \dim(R(T)) \leq \dim(V))$.

Proof:

We wish to show that $R(T) = \text{span}(T(B))$.

$(\supseteq)$

Let $w \in \text{span}(T(B))$.

Then, $w = a_1 T v_1 + \ldots + a_n T v_n$ for $a_i \in \mathbb{F}$.

$\Longrightarrow w = T(a_1 v_1 + \ldots + a_n v_n) = T v$.

$\Longrightarrow w \in R(T)$ ✓

$(\subseteq)$

Let $s \in R(T)$.

Then, $\exists\ x \in V$ such that $T x = s$.

Then $x = b_1 v_1 + \ldots + b_n v_n$ with $b_i \in \mathbb{F}$, since B is a basis for V.

$\Longrightarrow s = T x = T(b_1 v_1 + \ldots + b_n v_n) = b_1 T v_1 + \ldots + b_n T v_n$

$\Longrightarrow s \in \text{span}(T(B))$ ✓

Thus $R(T) = \text{span}(T(B))$. ■

Definition: Let V, W be VS's and $T \in \mathcal{L}(V, W)$. Then,

i) The nullity of T, $\text{nullity}(T) = \dim(\mathcal{N}(T))$.

ii) The rank of T, $\text{rank}(T) = \dim(R(T))$.

iii) T is said to be injective (or one-to-one), if every image has a unique pre-image. In other words, $T u = T v \Longrightarrow u = v \quad \forall\ u, v \in V$.

iv) T is said to be surjective (or onto) if every vector in the codomain has a preimage. In other words, $R(T) = W$.

• Theorem:

Let V, W be VS's and $T \in \mathcal{L}(V, W)$. Then T is injective iff $\mathcal{N}(T) = \{\hat{0}_v\}$.

Proof:

$(\Rightarrow)$

Suppose T is injective, we wish to show that the nullspace of T is trivial, i.e $\mathcal{N}(T) = \{\hat{0}_v\}$.

Let's suppose $\exists\ u \in \mathcal{N}(T)$ such that $u \neq \hat{0}_v$.

But by the definition of injectivity,

$$T u = \hat{0}_w \Longrightarrow T u = T(\hat{0}_v) \Longrightarrow u = \hat{0}_v.\ (\Rightarrow\Leftarrow) \quad ✓$$

($\Leftarrow$)
Suppose $\mathcal{N}(T) = \{\hat{0}_v\}$. We wish to show that T is injective. Let $u, v \in V$ such that $T u = T v$, then we have to show that $u = v$.
By linearity we have

$$T\ u = T\ v \Longrightarrow T\ u - T\ v = \hat{0}_w \Longrightarrow T\ (u - v) = \hat{0}_w \Longrightarrow u - v = \hat{0}_v \Longrightarrow u = v.$$

Hence T has to be injective. ✓ ■

• Dimension Theorem/Rank-Nullity Theorem:
Let V, W be VS's and dim $(V) < \infty$, and let $T \in \mathcal{L}(V, W)$.
Then $\dim(V) = \text{nullity}\,(T) + \text{rank}\,(T) = \dim(\mathcal{N}\,(T)) + \dim\,(R\,(T))$.

Proof:
Let $\{v_1, \ldots, v_m\}$ be a basis for $\mathcal{N}(T)$. Then nullity$(T) = m$. Considering $\{v_1, \ldots, v_m\}$ as a subset of V, it is still a linearly independent set in V (even though it doesn't necessarily span V). But then we know by a previous theorem that we can extend $\{v_1, \ldots, v_m\}$ to a basis for V, $\{v_1, \ldots, v_m, w_1, \ldots, w_k\}$. Then $\dim(V) = m + k$. We know that nullity$(T) = m$, so we need to show that rank$(T) = k$.

Now by an earlier theorem we know that

$$R(T) = \text{span}(T\ v_1, \ldots, T\ v_m, T\ w_1, \ldots, T\ w_k).$$

But since it's given that $\{v_1, \ldots, v_m\}$ is a basis for $\mathcal{N}(T)$, we know that $\{T\ v_1, \ldots, T\ v_m\}$ are all zero. Since every set that contains the zero vector is by definition linearly dependent, then by the LDL we can remove vectors from our set and it won't affect the span, hence we remove the vectors $\{T\ v_1, \ldots, T\ v_m\}$ and we're left with $R(T) = \text{span}(T\ w_1, \ldots, T\ w_k)$. This implies that rank$(T) \leq k$.

Now to show equality we need to show that $\{T\ w_1, \ldots, T\ w_k\}$ is linearly independent.
We let $a_i \in \mathbb{F}$ and then we need to show that $a_1\ T\ w_1 + \ldots + a_k\ T\ w_k = \hat{0}_w$, with $a_i = 0 \ \ \forall\ i$.
By linearity we have $T(a_1\ w_1 + \ldots + a_k\ w_k) = \hat{0}_w$. Then $a_1\ w_1 + \ldots + a_k\ w_k \in \mathcal{N}(T)$. Since $\{v_1, \ldots, v_m\}$ is a basis for $\mathcal{N}(T)$, there are unique scalars $b_1, \ldots, b_m$ such that $w = b_1\ v_1 + \ldots + b_m\ v_m$.
This implies that

$$\begin{aligned} & a_1\ w_1 + \ldots + a_k\ w_k = b_1\ v_1 + \ldots + b_m\ v_m \\ \Longrightarrow\ & b_1\ v_1 + \ldots + b_m\ v_m - a_1\ w_1 - \ldots - a_k\ w_k = \hat{0} \\ \Longrightarrow\ & b_i = -a_i = 0 \Longrightarrow a_i = 0. \end{aligned}$$

Thus $\{T\ w_1, \ldots T\ w_k\}$ is a linearly independent spanning set for rank(T). Thus $|\{T\ w_1, \ldots T\ w_k\}| = k = \text{rank}(T)$. ■

• Corollary:
Suppose V, W are both finite dimensional VS's and $\dim(V) > \dim(W)$. Then no linear map from V to W will be injective.

Proof:

Injectivity implies that nullity$(T) = 0$, i.e. the nullspace is trivial.

Let T be a linear map from V to W, i.e. $T \in \mathcal{L}(V,\ W)$. Then we know that

$\dim(V) = \text{nullity}(T) + \text{rank}(T)$.

$\implies$ nullity $(T) = \dim(V) - \text{rank}(T)$

$\implies$ nullity $(T) \leq \dim(V) - \dim(W)$

$\implies$ nullity$(T) > 0$ (since $\dim(V) > \dim(W)$)

Since nullity (T) is not trivial, T is not injective. ■

• Corollary:

Suppose V, W are finite dimensional VS's and $\dim(V) < \dim(W)$. Then no linear map from V to W will be surjective.

Proof:

Surjectivity implies that $R(T) = W$.

We have that

$\dim(V) = \text{nullity}(T) + \text{rank}(T)$

$\implies$ rank $(T) = \dim(V) - \text{nullity}(T)$

$\implies$ rank $(T) \leq \dim(V)$

$\implies$ rank$(T) < \dim(W)$ (since $\dim(V) < \dim(W)$)

$\implies R(T) \subset W$.

$\implies T$ is not surjective. ■

• Theorem:

Suppose V is a VS and dim $(V) < \infty$, and let $T \in \mathcal{L}(V)$. Then the following are equivalent:

i) T is injective .

ii) T is surjective .

iii) rank$(T) = \dim(V)$.

Proof:

Suppose that $T \in \mathcal{L}(V)$ is injective. Then

$N(T) = \{\hat{0}\} \implies$ nullity$(T) = 0$.

So we have dim $(V) = \text{nullity}(T) + \text{rank}(T) = \text{rank}(T)$.

Treating T now as a codomain,

dim $(V) = \text{rank}(T) \implies \dim(V) = \dim(R(T)) \implies V = R(T)$

Hence T is surjective. ■

Read the following theorem in the book (Friedberg pg 72-73)

• Theorem:

Let V,W be finite dimensional VS's with the same dimension, and let $\{v_1,\ \ldots,\ v_n\}$ be a basis for V

and $\{w_1, \ldots, w_n\}$ a basis for W. Then $\exists$ a unique $T : V \longrightarrow W$ such that $T\, v_i = w_i$ for some $w_i \in W$. (In other words if we pick some random vectors w_i in the codomain, then there exists a unique linear transformation T that will map the vectors in the basis in V to these w_i).

• Corollary:
Let V, W be VS's and $\dim(V) < \infty$. Let $\{v_1, \ldots, v_n\}$ be a basis for V. If U, $T \in \mathcal{L}(V, W)$ are such that $U(v_i) = T(v_i)$ $\forall\, i$, then $U = T$.

Definition: Let V be a VS and let $T : V \longrightarrow V$ be linear. A subspace W of V is said to be T − invariant if $T(x) \in W$ for every $x \in W$, i.e. $T(W) \subseteq W$. If W is T-invariant, we define the restriction of T on W to be the function $T_W : W \longrightarrow W$ defined by $T_W(x) = T(x)$ $\forall$ $x \in W$.

THE MATRIX REPRESENTATION OF A LINEAR MAP

Definition: Let $B = \{v_1, \ldots, v_n\}$ be an ordered basis for V, and let $x \in V$. Then $\exists$ unique scalars $a_i \in \mathbb{F}$ such that $x = a_1 v_1 + \ldots + a_n v_n$. Then the coordinate vector of x with respect to B, denoted by $[x]_B$ is given by

$$[x]_B = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}$$

We can define a map $[\ ]_B : V \longrightarrow \mathbb{F}^n$ ($x \mapsto [x]_B$). This map is linear.
That is, $\forall\, x, v \in V$ and $\forall\, c \in \mathbb{F}$, the following properties hold:
i) $[x + v]_B = [x]_B + [v]_B$
ii) $[c\, x]_B = c[x]_B$

• Theorem:
Let $T \in \mathcal{L}(V, W)$, and let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V and $\gamma = \{w_1, \ldots, w_m\}$ a basis for W.
Then
$\mathcal{M}(T\, x) = \mathcal{M}(T)\, \mathcal{M}(x)$, i.e. $[T\, x]_\gamma = [T\, x]_\beta^\gamma [x]_\beta$ $\forall\, x \in V$.

Proof:
Let $x \in V$. Then, $T\, x = a_1 w_1 + \ldots + a_m w_m$.

We know that x can be written as a linear combination $x = a_1\, v_1 + ... + a_n\, v_n$ with $a_i \in \mathbb{F}$.
Then $T\, x = T(a_1\, v_1 + ... + a_n\, v_n) = a_1\, T\, v_1 + ... + a_n\, T\, v_n$.
Now, we look at $T\, v_i$:
$T\, v_i = \sum_{j=1}^{m} c_{i,j}\, w_j = c_{i,1}\, w_1 + ... + c_{i,m}\, w_m$.
Then,

$$\begin{aligned} T\, x &= a_1\, T\, v_1 + ... + a_n\, T\, v_n \\ &= a_1\left(\sum_{j=1}^{m} c_{1,j}\, w_j\right) + a_2\left(\sum_{j=1}^{m} c_{2,j}\, w_j\right) + a_3\left(\sum_{j=1}^{m} c_{3,j}\, w_j\right) + ... + a_n\left(\sum_{j=1}^{m} c_{n,j}\, w_j\right) \\ &= \sum_{j=1}^{m} \left(a_1\, c_{1,j} + a_2\, c_{2,j} + ... + a_n\, c_{n,j}\right) w_j \end{aligned}$$

Hence we have

$$[T\, x]_\gamma = \begin{pmatrix} a_1\, c_{1,1} + a_2\, c_{2,1} + ... + a_n\, c_{n,1} \\ a_1\, c_{1,2} + a_2\, c_{2,2} + ... + a_n\, c_{n,2} \\ . \\ . \\ a_1\, c_{1,m} + a_2\, c_{2,m} + ... + a_n\, c_{n,m} \end{pmatrix}$$

Thus
$\mathcal{M}(T)\, \mathcal{M}(x) = [T\, x]_\beta^\gamma\, [x]_\beta$

$$\Longrightarrow \begin{pmatrix} c_{11} & c_{12} & . & . & c_{1\,n} \\ c_{21} & c_{22} & . & . & c_{2\,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{\mathrm{m}1} & c_{\mathrm{m}2} & . & . & c_{\mathrm{mn}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} = \begin{pmatrix} a_1\, c_{1,1} + a_2\, c_{2,1} + ... + a_n\, c_{n,1} \\ a_1\, c_{1,2} + a_2\, c_{2,2} + ... + a_n\, c_{n,2} \\ . \\ . \\ a_1\, c_{1,m} + a_2\, c_{2,m} + ... + a_n\, c_{n,m} \end{pmatrix} = [T\, x]_\gamma$$

■

Note: Suppose V, W are finite dimensional VS's and let $\beta = \{v_1, ..., v_n\}$ be an ordered basis for V and $\gamma = \{w_1, ..., w_m\}$ be an ordered basis for W. Let $T \in \mathcal{L}(V, W)$.
Then,

$T\, v_1 = a_1^1\, w_1 + ... + a^1{}_m\, w_m$
$T\, v_2 = a^2{}_1\, w_1 + ... + a^2{}_m\, w_m$
... ...
... ...
$T\, v_n = a^n{}_1\, w_1 + ... + a^n{}_m\, w_m$

where $a_j \in \mathbb{F}$.

$$\text{Then, } [T\,v_1]_\gamma = \begin{pmatrix} a_1^1 \\ a_2^1 \\ . \\ . \\ a_m^1 \end{pmatrix}, \quad [T\,v_2]_\gamma = \begin{pmatrix} a_1^2 \\ a_2^2 \\ . \\ . \\ a_m^2 \end{pmatrix}, \ldots\ldots\ldots, [T\,v_n]_\gamma = \begin{pmatrix} a_1^n \\ a_2^n \\ . \\ . \\ a_m^n \end{pmatrix}.$$

$$\begin{pmatrix} a_1^1 \\ a_2^1 \\ . \\ . \\ a_m^1 \end{pmatrix}\begin{pmatrix} a_1^2 \\ a_2^2 \\ . \\ . \\ a_m^2 \end{pmatrix}\begin{pmatrix} a_1^3 \\ a_2^3 \\ . \\ . \\ a_m^3 \end{pmatrix} \cdots \begin{pmatrix} a_1^n \\ a_2^n \\ . \\ . \\ a_m^n \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 & a_1^3 & & a_1^n \\ a_2^1 & a_2^2 & a_2^3 & & a_2^n \\ . & . & . & \cdots & . \\ . & . & . & & . \\ a_m^1 & a_m^2 & a_m^3 & & a_m^n \end{pmatrix} = [T]_\beta^\gamma \quad \checkmark$$

Definition: Let $A \in M_{m\times n}(\mathbb{F})$, and let $A_{ij} = a_i^j$, for $1 \le i \le m$ and $1 \le j \le n$.

Then the matrix representation of the linear map T with respect to the bases β and γ is denoted $A = [T]_\beta^\gamma$.

Thus we can define a map $\mathcal{M}: \mathcal{L}(V, W) \longrightarrow M_{m\times n}(\mathbb{F})$ such that $\mathcal{M}(T) = [T]_\beta^\gamma$ and $\mathcal{M}$ is a linear map.

That is, let $S, T \in \mathcal{L}(V, W)$. Then

i) $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T) \Longrightarrow [S+T]_\beta^\gamma = [S]_\beta^\gamma + [T]_\beta^\gamma$ (Additivity) ✓

ii) $\mathcal{M}(c\,T) = c\,\mathcal{M}(T) \Longrightarrow [c\,T]_\beta^\gamma = c\,[T]_\beta^\gamma$ (Homogeneity) ✓

COMPOSITION OF LINEAR MAPS AND MATRIX MULTIPLICATION

First we need a concept of "multiplication" in $\mathcal{L}(V, W)$.

We use composition, i.e. if $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, Z)$, then $S \circ T = ST = S$ "times" T.

Then we wish to show that $S \circ T : V \longrightarrow Z$ is linear.

• Theorem:

Let V, W, Z be VS's and let $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, Z)$. Then assuming V, W, Z are properly defined (so that $S \circ T$ makes sense), we have $S \circ T \in \mathcal{L}(V, Z)$.

Proof:

First, assume $S \circ T : V \longrightarrow Z$ is defined. Then let $v, w \in V$.

i) We want to show that $S \circ T(v+w) = (S \circ T)\,v + (S \circ T)\,w$:

$$(S \circ T)(v+w) = S(T(v+w)) = S(T\,v + T\,w)$$

$$= S(T\,v) + S(T\,w)$$
$$= (S \circ T)\,v + (S \circ T)\,w \quad \text{(Additivity)} \quad ✓$$

ii) Now we want to show that $(S \circ T)\,(c\,w) = c\,(S \circ T)\,(w)$:

$$(S \circ T)\,(c\,w) \;=\; S(T\,(c\,w)) \;=\; S(c\,T\,(w))$$
$$= c\,(S(T\,w)) = c(S \circ T)\,(w) \quad \text{(Homogeneity)} ✓ \qquad ■$$

Note: Since $S \circ T$ (which is usually denoted by ST) is in $\mathcal{L}\,(V, Z)$, it has a matrix representation $(\mathcal{M} : \mathcal{L}(V, Z) \longrightarrow M_{p\times n}(\mathbb{F})\,)$.

What is $\mathcal{M}(S\,T)$? (i.e. what is $[S\,T]_\beta^\gamma$?)

To define matrix multiplication , we want the following to be true :

$\mathcal{M}(S\,T) = \mathcal{M}(S)\,\mathcal{M}(T)$

Let V^n, W^m, Z^p be finite dimensional VS's and $\beta = \{v_1, \ldots, v_n\}$ be a basis for V, $\gamma = \{w_1, \ldots, w_m\}$ a basis for W, and $\alpha = \{z_1, \ldots, z_p\}$ a basis for Z.
Let $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, Z)$. Then $S\,T \in \mathcal{L}(V, Z)$.
Then for $1 \le j \le n$,

$$\begin{aligned}(S\,T)\left(v_j\right) &= S\left(T\,v_j\right) = S\left(\textstyle\sum_{k=1}^m b_{k,j}\,w_k\right) \\ &= \textstyle\sum_{k=1}^m b_{k,j}\,S(w_k) \\ &= \textstyle\sum_{k=1}^m b_{k,j}\left(\sum_{i=1}^p a_{i,k}\,z_i\right) \\ &= \textstyle\sum_{i=1}^p \left(\sum_{k=1}^m a_{i,k}\,b_{k,j}\right) z_i\ .\end{aligned}$$

$$\Longrightarrow \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1\,m} \\ a_{21} & a_{22} & . & . & a_{2\,m} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{\mathrm{p}1} & a_{\mathrm{p}2} & . & . & a_{\mathrm{pm}} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & . & . & b_{1\,n} \\ b_{21} & b_{22} & . & . & b_{2\,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{\mathrm{m}1} & b_{\mathrm{m}2} & . & . & b_{\mathrm{mn}} \end{pmatrix} = \mathcal{M}(S)\,\mathcal{M}(T) = \mathcal{M}(ST)\,.$$

Notation: Let $A \in M_{m\times n}(\mathbb{F})$. Then we let $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ such that $L_A(x) = A\,x$.

Then we have the following properties:
Let $A \in M_{m\times n}(\mathbb{F})$ and β, γ be standard bases for $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively.
Then,

a) $[L_A]_\beta^\gamma = A$.

b) $L_A = L_B$ iff $A = B$.

c) $L_{A+B} = L_A + L_B$ and $L_{c\,A} = c\,L_A$ for $c \in \mathbb{F}$.

d) If $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$, then $\exists$ a unique matrix $C \in M_{m\times n}(\mathbb{F})$ such that $T = L_C$.

e) $L_{AE} = L_A L_E$, $E \in M_{n\times p}(\mathbb{F})$.

f) $L_{I_n} = I_{\mathbb{F}^n}$ (this is the identity map from $\mathbb{F}^n$ to $\mathbb{F}^n$)

$\Longrightarrow L_{I_n}(x) = (I)(x) = x.$

• Theorem:

Matrix multiplication is associative.

Proof:

$L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B) L_C = L_{AB} L_C = L_{(AB)C}$. ■

INVERTIBILITY AND ISOMORPHISMS

Definition: Let $T \in \mathcal{L}(V, W)$. A linear map $S : W \longrightarrow V$ is said to be an inverse of T if $TS = I_W$ and $ST = I_V$. If such an S exists then T is invertible.

• Lemma:

The inverse of a linear map is unique.

Proof:

Let $T \in \mathcal{L}(V, W)$ and let $S, S' \in \mathcal{L}(V, W)$ be two distinct inverses of T, i.e. $S \neq S'$.

Then

$$S = SI = S(T S') = (ST) S' = IS' = S' \quad (\Rightarrow\Leftarrow) \qquad ■$$

• Theorem:

Let $T \in \mathcal{L}(V, W)$. Then T is invertible iff T is bijective.

Definition: Let $A \in M_{n\times n}(\mathbb{F})$. Then A is invertible iff $\exists$ $B \in M_{n\times n}(\mathbb{F})$ such that $AB = BA = I_n$.

• Lemma:

Let T be an invertible map $T \in \mathcal{L}(V, W)$. Then V is finite dimensional iff W is finite dimensional. Furthermore, $\dim(V) = \dim(W)$.

Proof:

(⇒)
Suppose V is finite dimensional. Let $\beta = \{v_1, \ ..., v_n\}$ be a basis for V. Then $R(T) = \text{span}\,(T\,(\beta))$.
Since T is invertible, it is also bijective, hence $R(T) = W$. So, $\text{span}\,(T\,(\beta)) = W$. Since W is spanned by a finite set $(T(\beta))$, W is finite dimensional. ✓
(⇐)
Exact same argument in the opposite direction using T^{-1} instead. ✓

Now we only need to prove that $\dim(V) = \dim(W)$:
Suppose V, W are finite dimensional. Since T is invertible, T is also bijective, and hence injective.
Then
$\mathcal{N}(T) = \{0\} \Longrightarrow \text{nullity}(T) = 0$
By the rank-nullity theorem we have
$\dim(V) = \text{nullity}(T) + \text{rank}(T) = 0 + \text{rank}(T) = \text{rank}(T)$.
Hence by surjectivity, $\dim(V) = \dim(W)$. ✓ ■

• Theorem:
Let V, W be finite-dimensional VS's with bases β and γ respectively and let $T \in \mathcal{L}(V, W)$.
Then T is invertible iff $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\beta^\gamma = \left([T]_\beta^\gamma\right)^{-1}$.

Proof:
(⇒)
Suppose T is invertible. Then ∃ a unique $T^{-1} : W \longrightarrow V$ with $\dim\,(W) = \dim\,(V)$.
Consider $T\,T^{-1} = I_W \Longrightarrow [T\,T^{-1}]_\gamma = [I_W]_\gamma$.
Then,

$$\begin{aligned} &\mathcal{M}(T\,T^{-1}) = \mathcal{M}(I) \\ \Longrightarrow &\mathcal{M}(T)\,\mathcal{M}(T^{-1}) = I_n \\ \Longrightarrow &[T]_\beta^\gamma\,[T^{-1}]_\gamma^\beta = I_n\,. \end{aligned}$$

Similarly,

$$T^{-1}\,T = I_V \Longrightarrow [T^{-1}]_\gamma^\beta\ [T]_\beta^\gamma = I_n.$$

Thus $[T]_\beta^\gamma$ is invertible. Moreover, $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$. ✓
(⇐)
Suppose that $[T]_\beta^\gamma$ is invertible. Let $A \in M_{n\times n}(\mathbb{F})$ such that $A = [T]_\beta^\gamma$.
Then ∃ $B \in M_{n\times n}(\mathbb{F})$ such that $AB = BA = I_n$. Then (by theorem 2.6 in Friedberg's), ∃ a unique $U \in \mathcal{L}(W, V)$ such that

$$U(w_j) = \sum_{i=1}^n B_{i\,j}\,v_i$$

with $1 \le j \le n$ and $w_j \in \gamma$, $v_i \in \beta$, $B_{i\,j}$ entries of B.
By construction, $[U]_\gamma^\beta = B$.

Now we wish to show that $UT = I = TU$.
Then $[UT]_\beta = [U]_\gamma^\beta\,[T]_\beta^\gamma = BA = I_n = [I]_\beta$. This implies that $UT = I_V$.
Similarly, $TU = I_W \Longrightarrow U$ is the inverse of $T \Longrightarrow T$ is invertible. ✓ ■

Definition: Let V, W be VS's. V is said to be isomorphic to W if $\exists\ T \in \mathcal{L}(V, W)$ that is invertible. Such a map T is called an isomorphism.

Notation $V \cong W$ means V is isomorphic to W.

• Theorem:
Let V, W be finite dimensional VS's . Then $V \cong W$ iff $\dim(V) = \dim(W)$.

Proof:
($\Rightarrow$)
Suppose $V \cong W$. Then $\exists\ T \in \mathcal{L}(V, W)$ that is invertible. But then we know from a previous theorem that $\dim(V) = \dim(W)$. ✓
($\Leftarrow$)
Suppose $\dim(V) = \dim(W)$. Then let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V and $\gamma = \{w_1, \ldots, w_n\}$ a basis for W. Then by a previous theorem $\exists\ T : V \longrightarrow W$ such that $T\,v_i = w_i\ \ \forall\ i$.
Then $R(T) = \text{span}\,(T\,(\beta)) = \text{span}\,(\gamma) = W$.
Since $R(T) = W$, T must be surjective. Now we only need to show that T is injective. To do this we use the Rank-Nullity theorem:

$$\dim(V) = \text{nullity}(T) + \text{rank}\,(T)\ .$$

Since we are assuming that $\dim(V) = \dim(W)$, we have that

$$\dim(W) = \text{nullity}(T) + \text{rank}\,(T)$$

Since $\text{rank}\,(T) = \dim\,(W)$ they cancel and we're left with

$$\text{nullity}(T) = 0 \Longrightarrow T \text{ is injective.}$$

Thus we have that T is bijective, which implies that T is invertible. ✓ ■

• Corollary:
Let V be a VS over $\mathbb{F}$. Then $V \cong \mathbb{F}^n$ iff $\dim\,(V) = n$.

• Theorem:
Let V, W be finite dimensional VS's over $\mathbb{F}$ of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then the map $\mathcal{M} : \mathcal{L}(V, W) \longrightarrow M_{m\times n}(\mathbb{F})$ defined by $\mathcal{M}(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$ is an isomorphism. ($\mathcal{L}(V, W) \cong M_{n\times n}(\mathbb{F})$ is called a canonical isomorphism)

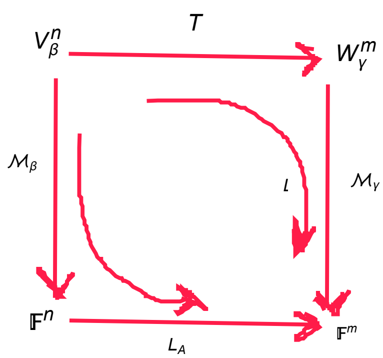
Proof:
We've previously shown that $\mathcal{M}$ is linear. So we simply need to show that $\mathcal{M}$ is both injective and

surjective.
This is accomplished if we show that for every $m \times n$ matrix A, $\exists$ a unique linear transformation $T : V \longrightarrow W$, such that $\mathcal{M}(T) = A$.
Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ and let A be a given $m \times n$ matrix. Then by theorem 2.6 (Friedberg's pg 72)(**there seems to be a contradiction here because this theorem is stated to work for two VS's with the same dimension but here we have $\dim(V) = n$ and $\dim(W) = m$. I'm still gonna follow the text and use theorem 2.6 to prove this one but it doesn't really make sense to me**), $\exists$ a unique linear transformation $T : V \longrightarrow W$, such that

$$T(v_j) = \sum_{i=1}^{m} A_{ij}\, w_i \text{ for } 1 \le j \le n.$$

But this means that $[T]_\beta^\gamma = A$, or $\mathcal{M}(T) = A$. Thus $\mathcal{M}$ is an isomorphism. ■

Note: Let V^n and W^m be finite dimensional VS's. Then we have



$V^n \cong F^n$ and $W^m \cong F^m$.
Let $A = [T]_\beta^\gamma$. Then $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ such that $x \longmapsto A_x$.

CHANGE – OF – COORDINATES MATRIX

Let V^n be a finite dimensional VS over $\mathbb{F}$. Let β, β' be two distinct ordered bases for V. What's the relationship between β and β'?
To answer this we first let $x \in V$. Then $x = a_1\, \beta_1 + \ldots + a_n\, \beta_n$ and also $x = b_1\, \beta_1' + \ldots + b_n\, \beta_n'$. Then we have that

$$[x]_\beta = \begin{pmatrix} a_1 \\ . \\ . \\ a_n \end{pmatrix} \text{ and } [x]_{\beta'} = \begin{pmatrix} b_1 \\ . \\ . \\ b_n \end{pmatrix} \text{ so that } [x]_\beta \neq [x]_{\beta'}.$$

We look at the following map

$Id : V_{\beta'} \longrightarrow V_\beta$ such that $x \longmapsto x$.

Then notice that $[x]_\beta = [Id(x)]_\beta = [Id]_{\beta'}^{\beta}[x]_{\beta'}$. (Here we have found the relationship between $[x]_\beta$ and $[x]_{\beta'}$).

Now we let $Q = [Id]_{\beta'}^{\beta}$, so that $[x]_\beta = Q\,[x]_{\beta'}$. Q is called the change-of-coordinates matrix.

• Theorem:

Q is invertible.

• Corollary:

$Q^{-1} = [I]_{\beta}^{\beta'}$

• Theorem:

Let $T \in \mathcal{L}(V^n)$ and let $\beta,\ \beta'$ be ordered bases for V. Let Q be the change of coordinates matrix that changes β' coordinates into β coordinates, i.e. $Q = [Id]_{\beta'}^{\beta}$.

Then $[T]_{\beta'} = Q^{-1}[T]_\beta\,Q$.

Proof:

Saying that $[T]_{\beta'} = Q^{-1}[T]_\beta\,Q$ is the same as $Q\,[T]_{\beta'} = [T]_\beta\,Q$.

Hence consider

$$Q\,[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [I\,T]_{\beta'}^{\beta} = [T\,I]_{\beta'}^{\beta} = [T]_\beta[I]_{\beta'}^{\beta} = [T]_\beta\,Q. \qquad \blacksquare$$

Definition: Let $A,\ B \in M_{n\times n}(\mathbb{F})$. Then B is said to be similar to A if $\exists$ an invertible $P \in M_{n\times n}(\mathbb{F})$ such that $B = P^{-1}\,A\,P$.