TRRT Midterm
Lie Subalgebras of a Direct Sum

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Direct Sum Lie Algebra

Let L, R be finite-dimensional complex Lie algebras and let $L \oplus R$ denote their direct sum Lie algebra. Recall that elements of $L \oplus R$ are pairs (ℓ, r) with $\ell \in L$ and $r \in R$, and that their bracket is given by

$$[(\ell_1, r_1), (\ell_2, r_2)] = ([\ell_1, \ell_2], [r_1, r_2]) . \tag{1}$$

We are now going to prove a structure theorem which characterizes the Lie subalgebras of $L \oplus R$.

Let K be a subalgebra of $L \oplus R$ and let

$$\begin{aligned} K_L &= \{\ell \in L \mid (\ell, r) \in K, \exists\, r \in R\} \\ K_R &= \{r \in R \mid (\ell, r) \in K, \exists\, \ell \in L\} \\ I_L &= \{\ell \in L \mid (\ell, 0) \in K\} \\ I_R &= \{r \in R \mid (0, r) \in K\}. \end{aligned}$$

Note that if $(\ell, r) \in K$, then $\ell \in K_L$ and $r \in K_R$.

Problem 1. *Show that K_L and K_R are Lie subalgebras of L and R, respectively.*

Proof. First of all, note that K_L and K_R are linear subspaces of L and R, respectively. To see this, take, say, two elements $\ell_1, \ell_2 \in L$ such that there exist $r_1, r_2 \in R$ that satisfy $(\ell_1, r_1) \in K$ and $(\ell_2, r_2) \in K$; in other words, $\ell_1, \ell_2 \in K_L$. But K is subalgebra of $L \oplus R$ (by assumption), so any linear combination of elements in K must also lie in K; in particular,

$$(\ell_1, r_1) + (\ell_2, r_2) = (\ell_1 + \ell_2, r_1 + r_2) \in K \implies \ell_1 + \ell_2 \in K_L \qquad \text{(Note that } r_1 + r_2 \in R\text{, since } R \text{ is a linear space).}$$

A similar argument applies to K_R. Hence all that's left is to show that K_L and K_R are closed under Lie brackets, but this follows immediately from (1) ...

Take $\ell_1, \ell_2 \in K_L$ as above; then

$$[(\ell_1, r_1), (\ell_2, r_2)] = \left(\underbrace{[\ell_1, \ell_2]}_{\in L,\text{ since } L \text{ is a Lie algebra}}, \underbrace{[r_1, r_2]}_{\in R,\text{ since } R \text{ is a Lie algebra}} \right) \in K.$$

Therefore K_L is closed under the Lie bracket of the ambient Lie algebra L, and it is thus a (Lie) subalgebra of L. This argument yields the desired result for K_R as well. □

Problem 2. *Show that I_L and I_R are ideals of K_L and K_R, respectively.*

Proof. Let us show the right side some love this time and prove that I_R is an ideal of K_R; the same argument is extended trivially to I_L and K_L. First we show that I_R is closed under linear combinations: Let $r_1, r_2 \in I_R$; then

$$\begin{aligned}(0, r_1) + (0, r_2) &= (0, \underbrace{r_1, +r_2}_{\in R}) \\ &= (0, \widetilde{r}) \qquad\qquad\qquad\qquad (\text{for some } \widetilde{r} \in R).\end{aligned}$$

Hence linear combinations of elements in I_R also lie in I_R.[1]

Now let $r' \in I_R$, $r \in K_R$, and $\ell \in K_L$. In order to show that I_R is an ideal of K_R, what we need to show is that $[r', r] \in I_R$. Making use of (1),

$$\begin{aligned}[(0, r')], (\ell, r)] &= (\underbrace{[0, \ell]}_{=0}, \underbrace{[r', r]}_{\in R}) \\ &= \big(0, r''\big) \qquad\qquad\qquad\qquad (\text{for some } r'' \in R).\end{aligned}$$

But the last line above shows that, indeed, $r'' = [r', r] \in I_R$, as desired. □

Problem 3. *By Q2, K_L/I_L and K_R/I_R are Lie algebras, and we will now prove that they are isomorphic. Indeed, we define a map $f\colon K_L \to K_R/I_R$ as follows: If $\ell \in K_L$, there is some $r \in R$ such that $(\ell, r) \in K$; then we define $f(\ell) = r + I_R$. Now,*

a) Show that f is well defined.

b) Show that f is a homomorphism.

c) Show that f is surjective.

d) Show that $\ker f = I_L$.

e) Show that $K_L/I_L \cong K_R/I_R$.

Proof of a). This one is quite straightforward. In order for f to be well defined, if $\ell_1 = \ell_2$ for some $\ell_1, \ell_2 \in K_L$, then we would need to have $f(\ell_1) = f(\ell_2)$. But if $\ell_1 = \ell_2$, then they have the same image under the injection $\iota\colon K_L \hookrightarrow K$ given by $\ell_{1 \text{ or } 2} \mapsto (\ell_{1 \text{ or } 2}, r)$; i.e., the same $r \in R$ is common to both ℓ_1 and ℓ_2 under this injection. Then,

$$\begin{aligned}f(\ell_1) &= r + I_R \\ &= f(\ell_2).\end{aligned}$$

□

[1] I did not multiply by a complex scalar here, but it is rather obvious that the result still holds, scalar included or otherwise. I use this slight simplification throughout this paper.

Proof of b). It is rather obvious that f is linear. In order to show that f is a (Lie) homomorphism, we need to demonstrate that f preserves Lie brackets; i.e., that for some $\ell_1, \ell_2 \in K_L$, we have

$$f([\ell_1, \ell_2]) = [f(\ell_1), f(\ell_2)] .$$

Consider (ℓ_1, r_1), (ℓ_2, r_2), and $([\ell_1, \ell_2], [r_1, r_2])$, all in K, so that

$$f([\ell_1, \ell_2]) = [r_1, r_2] + I_R, \qquad f(\ell_1) = r_1 + I_R, \qquad \text{and} \qquad f(\ell_2) = r_2 + I_R.$$

Then

$$\begin{aligned} f([\ell_1, \ell_2]) &= [r_1, r_2] + I_R \\ &= [r_1 + I_R, r_2 + I_R] \\ &= [f(\ell_1), f(\ell_2)]. \end{aligned}$$ □

Proof of c). Since we are dealing with finite-dimensional spaces, the following relation is key:

$$\dim K_L = \dim \ker f + \dim \mathrm{im} f.$$

But then

$$\begin{aligned} \dim \mathrm{im} f &= \dim K_L - \dim \ker f \\ &= \dim K_L - \dim I_L && \text{(By part d))} \\ &= \dim K_L / I_L. \end{aligned}$$

Thus, the rank of f equals the dimension of K_L/I_L. But K_L, when modded by I_L, consists of elements of the form

$$K_L - \text{ elements of } I_L = \{\ell \in L \mid (\ell, r) \in K, \exists\, r \in R, r \neq 0\}.$$

Hence, by dimensionality, for every nonzero r with quotient representative $r + I_R \in K_R/I_R$ ($r \notin I_R$), there is a corresponding $\ell \in K_L$ mapped by f. For those $r \in I_R$ (the zero elements), $r + I_R = I_R$ also has a preimage ℓ that lies in $I_L \subset K_L$ (c.f. part d) below). Thus f is indeed surjective. □

Proof of d). By construction,

$$I_L = \{\ell \in L \mid (\ell, r) \in K, r = 0\}.$$

But we defined our map $f \colon K_L \to K_R/I_R$ such that, if $\ell \in K_L$, there is some $r \in R$ such that $(\ell, r) \in K$ and $f(\ell) = r + I_R$. Thus, if ℓ happens to lie in $I_L \subset K_L$, then $r = 0$ and $(\ell, 0) \in K$ with $f(\ell) = 0 + I_R = I_R$. This shows that I_L is indeed the nullspace of f. □

Proof of e). By the *First Isomorphism Theorem*,

$$\begin{aligned} K_L/I_L = K_L/\ker f &\cong \mathrm{im} f \\ &= K_R/I_R && \text{(By surjectivity of } f\text{).} \end{aligned}$$ □

Problem 4. *Prove that*

$$\dim K = \dim K_L + \dim I_R = \dim K_R + \dim I_L. \tag{2}$$

Proof. From the result on Q3e), we have that $K_L/I_L \cong K_R/I_R$. Thus $\dim K_L - \dim I_L = \dim K_R - \dim I_R$, which is equivalent to the rightmost equality of (2). Now, to show that K is also of the same dimension, consider the map $\varphi\colon K_R/I_R \to K$ given by $r + I_R \mapsto (\ell, r)$, where this ℓ is the preimage of r by the mapping f from Q3. That φ is well defined and surjective is obvious, and with regards to injectivity note that $(0,0) \in K$ clearly implies that $r \in I_R$; i.e., the kernel of φ is trivial. Hence φ is a linear isomorphism, and (2) holds. $\square$

Q3 suggests a systematic approach to determining the Lie subalgebras of $L \oplus R$. We look for subalgebras K_L of L and K_R of R which have quotients isomorphic to the same Lie algebra Q. Let $p\colon K_L \to Q$ and $q\colon K_R \to Q$ be the corresponding surjective homomorphisms. Lastly, let us define

$$K = \{(\ell, r) \in K_L \oplus K_R \mid p(\ell) = q(r)\}. \tag{3}$$

Problem 5. *Show that K is a Lie subalgebra of $L \oplus R$ of dimension*

$$\dim K = \dim K_L + \dim K_R - \dim Q. \tag{4}$$

Note that the "obvious" subalgebras (namely, those which are themselves direct sums) are the ones for which $Q = 0$ and hence $K = K_L \oplus K_R$.

$$\begin{tikzcd} & K \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ K_L \arrow[r, "p"'] & Q & K_R \arrow[l, "q"] \end{tikzcd}$$

Proof. First we show that K is a linear subspace. Take $\ell_1, \ell_2 \in K_L$ and $r_1, r_2 \in K_R$ such that $p(\ell_i) = q(r_i)$, for $i = 1, 2$. Then,

$$(\ell_1, r_1) + (\ell_2, r_2) = \underbrace{\Bigg(\underbrace{\ell_1 + \ell_2}_{\in K_L}, \underbrace{r_1 + r_2}_{\in K_R}\Bigg)}_{\in K_L \oplus K_R}.$$

Moreover, since p and q are homomorphisms, we have

$$\begin{aligned} p(\ell_1 + \ell_2) &= p(\ell_1) + p(\ell_2) \\ &= q(r_1) + q(r_2) \\ &= q(r_1 + r_2). \end{aligned}$$

Thus K is indeed closed under linear combinations, as desired.

Next we show that K is closed under Lie brackets. Take ℓ_1, ℓ_2, r_1, r_2 as before. Then,

$$[(\ell_1, r_1), (\ell_2, r_2)] = \underbrace{\left(\underbrace{[\ell_1, \ell_2]}_{\in K_L}, \underbrace{[r_1, r_2]}_{\in K_R} \right)}_{\in K_L \oplus K_R}.$$

Moreover, since p and q are (Lie) homomorphisms, they preserve brackets; i.e., we have

$$\begin{aligned} p([\ell_1 . \ell_2]) &= [p(\ell_1), p(\ell_2)] \\ &= [q(r_1), q(r_2)] \\ &= q([r_1, r_2]). \end{aligned}$$

Thus K is indeed closed under Lie brackets, and we have proven that K is a Lie subalgebra of $L \oplus R$.

Lastly, we show the dimension of K. Since $Q \cong K_R / I_R \cong K_L / I_L$, and we are dealing with finite dimensional linear spaces, these must all have the same dimension. Thus, we recast (4):

$$\begin{aligned} \dim K_L + \dim K_R - \dim Q &= \dim K_L + \underbrace{\dim K_R - \dim K_R / I_R}_{\dim I_R} \\ &= \dim K_L + \dim I_R \\ &= \dim K. && \text{(By (2))} \qquad \square \end{aligned}$$