Math 3101 HW # 4

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Section 9

In exercises 1 through 6, find all orbits of the given permutation.

(#1) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$

Solution:
The orbits are $\{1, 5, 2\}$, $\{4, 6\}$, and $\{3\}$.

(#2) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$

Solution:
The orbits are $\{1, 5, 8, 7\}$, $\{2, 6, 3\}$, and $\{4\}$.

(#3) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$

Solution:
The orbits are $\{1, 2, 3, 5, 4\}$, $\{7, 8\}$, and $\{6\}$.

(#4) $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$, where $\sigma(n) = n + 1$

Solution:
There's only one orbit $\{\ldots, -1, 0, 1, \ldots\} = \mathbb{Z}$.

(#5) $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$, where $\sigma(n) = n + 2$

Solution:
There are two orbits, $\{2n : n \in \mathbb{Z}\}$ and $\{2n+1 : n \in \mathbb{Z}\}$.

(#6) $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$, where $\sigma(n) = n - 3$

Solution:
There are three orbits, $\{3n : n \in \mathbb{Z}\}$, $\{3n+1 : n \in \mathbb{Z}\}$ and $\{3n+2 : n \in \mathbb{Z}\}$.

In exercises 7 through 9, compute the indicated product of cycles that are permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

(#7) $(1, 4, 5)(7, 8)(2, 5, 7)$

Solution:

$$(1, 4, 5)(7, 8)(2, 5, 7) = (1, 4, 5)\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 3 & 4 & 7 & 6 & 2 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 3 & 5 & 1 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 3 & 4 & 8 & 6 & 2 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 8 & 6 & 2 & 7 \end{pmatrix}$$

(#8) $(1, 3, 2, 7)(4, 8, 6)$

Solution:

$$(1, 3, 2, 7)(4, 8, 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 2 & 4 & 5 & 6 & 1 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 8 & 5 & 4 & 7 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 2 & 8 & 5 & 4 & 1 & 6 \end{pmatrix}$$

(#9) $(1, 2)(4, 7, 8)(2, 1)(7, 2, 8, 1, 5)$

Solution:

$$(1, 2)(4, 7, 8)(2, 1)(7, 2, 8, 1, 5)$$

$$= (1, 2)(4, 7, 8)\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 3 & 4 & 7 & 6 & 2 & 1 \end{pmatrix}$$

$$= (1,\ 2)\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 7 & 5 & 6 & 8 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 3 & 4 & 7 & 6 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 7 & 8 & 6 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 7 & 8 & 6 & 2 & 1 \end{pmatrix}$$

(#36) Let G be a group and let a be a fixed element of G. Show that the map $\lambda_a : G \longrightarrow G$, given by $\lambda_a(g) = a\,g$ for $g \in G$, is a permutation of the set G.

Proof:
Let G be a group and fix $a \in G$. Then the map λ_a is given by $\lambda_a(g) = \{a\,g : g \in G\}$. We need to show that this map is bijective:

Showing that the map is injective is trivial; if we pick two images $\lambda_a(g_1) = a\,g_1$ and $\lambda_a(g_2) = a\,g_2$ such that $a\,g_1 = a\,g_2$, we have that $g_1 = g_2$ by the cancellation law, where $g_1,\ g_2 \in G$. Hence λ_a is injective. This map is obviously surjective as well, since by definition for each image $a\,g \in G$ we have a preimage $g \in G$.

Since λ_a is bijection from the group G onto itself, we have that λ_a is a permutation on G. ■

(#37) Referring to exercise 36, show that $H = \{\lambda_a : a \in G\}$ is a subgroup of S_G, the group of all permutations of G.

Proof:
To show that H is a subgroup of S_G, we need to show that the identity element and inverse element of S_G are in H, and we also need to show that H is closed under the binary operation defined on G (permutation multiplication):

- To show closure, let $\lambda_a(g),\ \lambda_b(g) \in H$, where $a,\ b,\ g \in G$. Then,

$$\lambda_a \circ \lambda_b(g) = \lambda_a(\lambda_b(g)) = \lambda_a(b\,g) = a\,b\,g = \lambda_{a\,b}(g) \in H$$

Hence H is closed under permutation multiplication. ✓

- Since G is a group, for any $a \in G\ \exists\ a^{-1} \in G$. Thus the map $\lambda_{a\,a^{-1}} = \lambda_e$ represents our identity on H, since $\lambda_e(g) = e\,g = g$. ✓

- For $a,\ a^{-1},\ g \in G$ and $\lambda_a \in H$, we have

$$\lambda_a \circ \lambda_{a^{-1}}(g) = \lambda_a(\lambda_{a^{-1}}(g)) = \lambda_a(a^{-1}\,g) = a\,a^{-1}\,g = e\,g = \lambda_e(g).$$

Hence $\lambda_{a^{-1}}$ is the inverse element of H. ✓

Since H is closed under the binary operation defined on S_G, and it contains the identity and inverse elements of S_G, we have that H is a subgroup of S_G. ■