MATH 709 HW # 7

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Problem 1 (Problem 5-6)**.** *Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:*

$$UM = \{(x, v) \in T\mathbb{R}^n \mid x \in M, v \in T_x M, |v| = 1\}.$$

It is called the ***unit tangent bundle of*** *M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Let $(x, v) \in UM$. Since M is an embedded submanifold of $\mathbb{R}^n$, we can choose a smooth chart (U, φ) for $\mathbb{R}^n$ containing x such that

$$\varphi(M \cap U) = \{(x^1, \dots, x^n) \in \varphi(U) \mid x^{m+1} = \cdots = x^n = 0\}.$$

Similarly, $\mathbb{S}^{m-1}$ is an embedded submanifold of $\mathbb{R}^m$, so we can choose a smooth chart (V, ψ) for $\mathbb{R}^m$ containing v such that

$$\psi(\mathbb{S}^{m-1} \cap V) = \{(x^1, \dots, x^m) \in \psi(V) \mid x^m = 0\}.$$

Now we write $\varphi = (x^1, \dots, x^n)$ and $\widetilde{U} = \varphi^{-1}(U)$. By definition, the map $\widetilde{\varphi} \colon \widetilde{U} \to \mathbb{R}^{2n}$ given by

$$v^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto (x^1(p), \dots, x^n(p), v^1, \dots, v^n)$$

is a coordinate map for $T\mathbb{R}^n$. By shrinking $\widetilde{U}$, we can assume that $V \subseteq \pi(\widetilde{\varphi}(\widetilde{U}))$, where $\pi \colon \mathbb{R}^{2n} \to \mathbb{R}^m$ is the projection onto the coordinates $n+1, \dots, n+m$. Now define

$$\theta(x^1, \dots, x^n, v^1, \dots, v^n) = (x^1, \dots, x^n, \psi(v^1, \dots, v^m), v^{m+1}, \dots, v^n),$$

so that θ is a diffeomorphism onto its image, and $\theta \circ \widetilde{\varphi} \colon \widetilde{U} \to \mathbb{R}^{2n}$ is still a coordinate map for $T\mathbb{R}^n$. Furthermore,

$$(\theta \circ \widetilde{\varphi})(UM \cap \widetilde{U}) = \{(x^1, \dots, x^n, v^1, \dots, v^n) \in (\theta \circ \widetilde{\varphi})(\widetilde{U}) \mid x^{m+1} = \cdots = x^n = v^m = \cdots = v^n = 0\},$$

so UM satisfies the local $(2m-1)$-slice condition. By the theorem of the local slice criterion for embedded submanifolds, we have that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n$, as desired. □

Problem 2 (Problem 5-19)**.** *Suppose $S \subseteq M$ is an embedded submanifold and $\gamma \colon J \to M$ is a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.*

Proof. By a previous result we have that if M and N are smooth manifolds and $S \subseteq M$ is an embedded submanifold, then every smooth map $F \colon N \to M$ whose image is contained in S is also smooth as a map from N to S. Hence in this case the aforementioned result shows that γ is smooth as a map into S. Let us denote this map by $\gamma_0 \colon J \to S$, and let $\iota \colon S \hookrightarrow M$ be the inclusion map.

Then $\gamma = \iota \circ \gamma_0$, and thus $\gamma'(t) = d\iota_{\gamma(t)}(\gamma_0'(t))$ by a previous proposition[1]. Therefore $\gamma'(t) \in T_{\gamma(t)}S \subseteq T_{\gamma(t)}M$, as desired. Note that this need not hold, however, if S is merely an immersed submanifold. For example, consider a curve γ that crosses the point of self-intersection in the figure-eight curve β (lemniscate) that we discussed in class. In this case we have that γ is not continuous as a map into the image of β. $\square$

[1] Here's the proposition, for reference:

Proposition (**The Velocity of a Composite Curve**). *Let $F\colon M \to N$ be a smooth map, and let $\gamma\colon J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma\colon J \to N$ is given by*

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$