Math 353 HW 3

Mario L. Gutierrez Abed

Section 2.1

(1) Which of the following satisfy the Cauchy-Riemann equations? If they satisfy the C-R equations, give the analytic function of z.

b) $f(x, y) = y^3 - 3x^2 y + i(x^3 - 3xy^2 + 2)$

Solution:

$u(x, y) = y^3 - 3x^2 y \quad ; \quad v(x, y) = x^3 - 3xy^2 + 2$

$u_x = -6xy \quad ; \quad u_y = 3y^2 - 3x^2$

$v_x = 3x^2 - 3y^2 \quad ; \quad v_y = -6xy$

Then we have that $u_x = v_y$ and $u_y = -v_x$.

Hence the C-R equations are satisfied. ✓

Now to find $f(z)$ we use the fact that $x = \text{Re}(z) = \frac{z + \bar{z}}{2}$ and $y = \text{Im}(z) = \frac{z - \bar{z}}{2i}$.

Then we have

$$
\begin{aligned}
f(z) &= \left(\frac{z-\bar{z}}{2i}\right)^3 - 3\left(\frac{z+\bar{z}}{2}\right)^2 \frac{z-\bar{z}}{2i} + i\left[\left(\frac{z+\bar{z}}{2}\right)^3 - 3\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)^2 + 2\right] \\
&= \frac{1}{-i8}(z^3 - 3z^2\bar{z} + 3z\bar{z}^2 - \bar{z}^3) - \frac{3}{8i}(z^2 + 2z\bar{z} + \bar{z}^2)(z - \bar{z}) + \\
&\quad i\left[\frac{1}{8}(z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3) + \frac{3}{8}(z + \bar{z})(z^2 - 2z\bar{z} + \bar{z}^2) + 2\right] \\
&= \frac{i}{8}(z^3 - 3z^2\bar{z} + 3z\bar{z}^2 - \bar{z}^3) + \frac{3i}{8}(z^2 + 2z\bar{z} + \bar{z}^2)(z - \bar{z}) + \\
&\quad \frac{1}{8}i(z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3) + \frac{3}{8}i(z + \bar{z})(z^2 - 2z\bar{z} + \bar{z}^2) + 2i \\
&= \frac{i}{8}[z^3 - 3z^2\bar{z} + 3z\bar{z}^2 - \bar{z}^3 + (3z^2 + 6z\bar{z} + 3\bar{z}^2)(z - \bar{z}) + \\
&\quad z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3 + (3z + 3\bar{z})(z^2 - 2z\bar{z} + \bar{z}^2) + 16] \\
&= \frac{i}{8}[z^3 - 3z^2\bar{z} + 3z\bar{z}^2 - \bar{z}^3 + 3z^3 + 6z^2\bar{z} + 3z\bar{z}^2 - 3z^2\bar{z} - 6z\bar{z}^2 - 3\bar{z}^3 + \\
&\quad z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3 + 3z^3 - 6z^2\bar{z} + 3z\bar{z}^2 + 3z^2\bar{z} - 6z\bar{z}^2 + 3\bar{z}^3 + 16] \\
&= \frac{i}{8}(8z^3 + 16) = i(z^3 + 2)
\end{aligned}
$$

As a quick check we substitute $z = x + iy$ into our answer and compare the result with $f(x, y)$:

$$
\begin{aligned}
i\left[(x + iy)^3 + 2\right] &= i(x^3 + 3x^2 iy - 3xy^2 - iy^3) + 2i \\
&= ix^3 - 3x^2 y - i3xy^2 + y^3 + 2i \\
&= y^3 - 3x^2 y + i(x^3 - 3xy^2 + 2) = f(x, y) \quad ✓
\end{aligned}
$$

Thus $f(z) = i(z^3 + 2)$.

c) $f(x, y) = e^y(\cos x + i \sin y)$

Solution:
$u(x, y) = e^y \cos x$; $v(x, y) = e^y \sin y$
$u_x = -e^y \sin x$ $\quad$ $v_y = e^y(\cos y + \sin y)$
$u_y = e^y \cos x$ $\quad$ $v_x = 0$
We can see that $u_y = -v_x$ only when $\cos x = 0$, i.e. when $x = \frac{\pi}{2} + \pi n$ for $n \in \mathbb{Z}$. However that implies that $u_x = -e^y$ for odd multiples of n and $u_x = e^y$ for even multiples of n, and we can see that $u_x \neq v_y$ no matter which value we pick for y. Hence the C-R conditions do not hold and we can conclude that $f(x, y)$ is not analytic.

(2) In the following we are given the real part of an analytic function of z. Find the imaginary part and the function of z.

a) $\text{Re}(z) = 3x^2 y - y^3$

Solution:
$u(x, y) = 3x^2 y - y^3$; $u_x = 6xy$; $u_y = 3x^2 - 3y^2$

We are given that $f(z)$ is analytic so the C-R conditions must be met, i.e. $u_x = v_y$ and $u_y = -v_x$. Hence $v_y = 6xy$. In order to determine $v(x, y)$ now we integrate v_y with respect to y:

$$\int v_y \, dy = \int 6xy \, dy \Longrightarrow v(x, y) = 3xy^2 + v(x)$$

Then we differentiate $v(x, y)$ with respect to x to get
$v_x = 3y^2 + v'(x)$.

But then we know that $v_x = -u_y$. Thus
$3y^2 + v'(x) = 3y^2 - 3x^2 \Longrightarrow v'(x) = -3x^2 \Longrightarrow v(x) = -x^3 + C$.
Hence, choosing $C = 0$ we have $v(x, y) = \text{Im}(z) = 3xy^2 - x^3$ ✓

Now that we have $f(x, y) = 3x^2 y - y^3 + i(3xy^2 - x^3)$, we need to write $f(x, y)$ in terms of z to get the function $f(z)$.

Once again we use $x = \mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$ and $y = \mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$. Then

$$\begin{aligned}
f(z) &= 3\left(\frac{z+\bar{z}}{2}\right)^2\left(\frac{z-\bar{z}}{2i}\right)-\left(\frac{z-\bar{z}}{2i}\right)^3+i\left[3\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)^2-\left(\frac{z+\bar{z}}{2}\right)^3\right] \\
&= \tfrac{3}{8i}\left(z^2+2z\bar{z}+\bar{z}^2\right)(z-\bar{z})+\tfrac{1}{8i}\left(z^3-3z^2\bar{z}+3z\bar{z}^2-\bar{z}^3\right)- \\
&\quad \tfrac{3i}{8}(z+\bar{z})\left(z^2-2z\bar{z}+\bar{z}^2\right)-\tfrac{i}{8}\left(z^3+3z^2\bar{z}+3z\bar{z}^2+\bar{z}^3\right) \\
&= -\tfrac{3i}{8}\left(z^2+2z\bar{z}+\bar{z}^2\right)(z-\bar{z})-\tfrac{i}{8}\left(z^3-3z^2\bar{z}+3z\bar{z}^2-\bar{z}^3\right)- \\
&\quad \tfrac{3i}{8}(z+\bar{z})\left(z^2-2z\bar{z}+\bar{z}^2\right)-\tfrac{i}{8}\left(z^3+3z^2\bar{z}+3z\bar{z}^2+\bar{z}^3\right) \\
&= -\tfrac{i}{8}\Big[3\left(z^2+2z\bar{z}+\bar{z}^2\right)(z-\bar{z})+z^3-3z^2\bar{z}+ \\
&\qquad 3z\bar{z}^2-\bar{z}^3+3(z+\bar{z})\left(z^2-2z\bar{z}+\bar{z}^2\right)+z^3+3z^2\bar{z}+3z\bar{z}^2+\bar{z}^3\Big] \\
&= -\tfrac{i}{8}\Big[\left(3z^2+6z\bar{z}+3\bar{z}^2\right)(z-\bar{z})+z^3-3z^2\bar{z}+ \\
&\qquad 3z\bar{z}^2-\bar{z}^3+(3z+3\bar{z})\left(z^2-2z\bar{z}+\bar{z}^2\right)+z^3+3z^2\bar{z}+3z\bar{z}^2+\bar{z}^3\Big] \\
&= -\tfrac{i}{8}\Big[3z^3+6z^2\bar{z}+3z\bar{z}^2-3z^2\bar{z}-6z\bar{z}^2-3\bar{z}^3+z^3-3z^2\bar{z}+3z\bar{z}^2- \\
&\qquad \bar{z}^3+3z^3-6z^2\bar{z}+3z\bar{z}^2+3z^2\bar{z}-6z\bar{z}^2+3\bar{z}^3+z^3+3z^2\bar{z}+3z\bar{z}^2+\bar{z}^3\Big] \\
&= -\frac{i}{8}\left(8z^3\right) = -i\,z^3
\end{aligned}$$

As a quick check we substitute $z = x + i\,y$ into our answer and compare the result with $f(x, y)$:

$$\begin{aligned}
-i(x+i\,y)^3 &= -i\left(x^3+3x^2\,i\,y-3xy^2-i\,y^3\right) \\
&= -i\,x^3+3x^2y+i\,3xy^2-y^3 \\
&= 3x^2y-y^3+i\left(3xy^2-x^3\right) = f(x, y) \quad \checkmark
\end{aligned}$$

Hence $f(z) = -i\,z^3$.

c) $\mathrm{Re}(z) = \dfrac{y}{x^2+y^2}$

<u>Solution:</u>

$$u(x, y) = \frac{y}{x^2+y^2}\ ;\ u_x = -\frac{2xy}{(x^2+y^2)^2}\ ;\ u_y = \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Since $f(z)$ is analytic we know that the C-R equations hold.
Hence

$$v_y = -\frac{2xy}{(x^2+y^2)^2} \implies \int v_y\,dy = -\int \frac{2xy}{(x^2+y^2)^2}\,dy$$

Now using substitution we define $w = x^2 + y^2$, then $dw = 2\,y\,dy$.

Then the integral becomes

$$-\int \frac{x}{w^2}\, dw = -x\int w^{-2}\, dw = \frac{x}{w} + v(x) = \frac{x}{x^2+y^2} + v(x)$$

Thus $v(x, y) = \dfrac{x}{x^2+y^2} + v(x)$.

Now we differentiate $v(x, y)$ with respect to x to get

$$v_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + v'(x) = \frac{y^2-x^2}{(x^2+y^2)^2} + v'(x)$$

Since $v_x = -u_y$, we must have

$$\frac{y^2-x^2}{(x^2+y^2)^2} + v'(x) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

From here we can see that $v'(x) = 0$, therefore $v(x) = 0$.

Hence $v(x, y) = \text{Im}(z) = \dfrac{x}{x^2+y^2}$. ✓

Now that we have $f(x, y) = \dfrac{y}{x^2+y^2} + i\,\dfrac{x}{x^2+y^2}$ we need to write $f(x, y)$ in terms of z to get the function $f(z)$:

$$f(z) = \frac{\frac{z-\bar{z}}{2i}}{\left(\frac{z+\bar{z}}{2}\right)^2+\left(\frac{z-\bar{z}}{2i}\right)^2} + i\,\frac{\frac{z+\bar{z}}{2}}{\left(\frac{z+\bar{z}}{2}\right)^2+\left(\frac{z-\bar{z}}{2i}\right)^2}$$

$$= \frac{\frac{z-\bar{z}}{2i}}{\frac{1}{4}(z+\bar{z})^2-\frac{1}{4}(z-\bar{z})^2} + i\,\frac{\frac{z+\bar{z}}{2}}{\frac{1}{4}(z+\bar{z})^2-\frac{1}{4}(z-\bar{z})^2}$$

$$= \frac{\frac{2(z-\bar{z})}{i}}{(z+\bar{z})^2-(z-\bar{z})^2} + i\,\frac{2(z+\bar{z})}{(z+\bar{z})^2-(z-\bar{z})^2}$$

$$= -2i\,\frac{(z-\bar{z})}{(z+\bar{z})^2-(z-\bar{z})^2} + 2i\,\frac{z+\bar{z}}{(z+\bar{z})^2-(z-\bar{z})^2}$$

$$= 2i\left[\frac{z+\bar{z}}{(z+\bar{z})^2-(z-\bar{z})^2} - \frac{z-\bar{z}}{(z+\bar{z})^2-(z-\bar{z})^2}\right] = 2i\left[\frac{2\bar{z}}{(z+\bar{z})^2-(z-\bar{z})^2}\right]$$

$$= 2i\left[\frac{2\bar{z}}{z^2+2z\bar{z}+\bar{z}^2-\left(z^2-2z\bar{z}+\bar{z}^2\right)}\right] = 2i\left[\frac{2\bar{z}}{4z\bar{z}}\right] = \frac{i}{z}$$

As a quick check we substitute $z = x + i\,y$ into our answer and compare the result with $f(x, y)$:

$$\frac{i}{z} = \frac{i}{x+i\,y}\cdot\frac{x-i\,y}{x-i\,y} = \frac{i\,x+y}{x^2+y^2} = \frac{y}{x^2+y^2} + i\,\frac{x}{x^2+y^2} = f(x, y) \quad ✓$$

Hence $f(z) = \dfrac{i}{z}$.

d) $\text{Re}(z) = \cos x \cosh y$

Solution:

$u(x,\ y) = \cos x \cosh y$; $u_x = -\sin x \cosh y$; $u_y = \cos x \sinh y$

Since $f(z)$ is analytic we know that the C-R equations hold. Hence

$$v_y = -\sin x \cosh y \implies \int v_y\, dy = -\int \sin x \cosh y\, dy$$

Hence $v(x,\ y) = -\sin x \sinh y + v(x)$.

Now we differentiate $v(x,\ y)$ with respect to x :

$$v_x = -\cos x \sinh y + v'(x) .$$

Since $v_x = -u_y$ we must have

$$-\cos x \sinh y + v'(x) = -\cos x \sinh y.$$

This implies that $v'(x) = 0$ and so $v(x) = 0$.

Thus $v(x,\ y) = \text{Im}(z) = -\sin x \sinh y$. ✓

We have that $f(x,\ y) = \cos x \cosh y - i \sin x \sinh y$. Now let's rewrite $f(x,\ y)$ in terms of z to get $f(z)$:

$$\begin{aligned} f(x,\ y) &= \cos x \cosh y - i \sin x \sinh y \\ &= \cos x \cos(i\ y) - i \sin x \frac{\sin(i\ y)}{i} \\ &= \cos x \cos(i\ y) - \sin x \sin(i\ y) \\ &= \cos(x + i\ y) = \cos z \end{aligned}$$

Hence $f(z) = \cos z$.

(3) Determine whether the following functions are analytic. Discuss whether they have any singular points or if they are entire.

a) $\tan z$

Solution:

We know that $\tan z = \dfrac{\sin z}{\cos z}$. Therefore, if we can prove that both $\sin z$ and $\cos z$ are analytic, then $\tan z$ is also analytic except where the function is not defined (i.e. where $\cos z = 0$).

We can show that $\sin z$ is analytic as follows:

$$\begin{aligned} \sin z &= \sin(x + i\ y) = \sin x \cos(i\ y) + \sin(i\ y) \cos x \\ &= \sin x \cosh(y) + i \sinh y \cos x \end{aligned}$$

Thus, $\text{Re}(z) = u(x,\ y) = \sin x \cosh y$ and $\text{Im}(z) = v(x,\ y) = \sinh y \cos x$.

Then we have

$u_x = \cos x \cosh y \qquad v_y = \cos x \cosh y$

$u_y = \sin x \sinh y \qquad v_x = -\sin x \sinh y$

Since the C-R conditions hold and the partial derivatives are all continuous, we have proven that $\sin z$ is holomorphic. ✓

Now we also need to prove that $\cos z$ is analytic:
$\cos z = \cos(x + i\, y) = \cos x \cos(i\, y) - \sin x \sin(i\, y)$
$\quad = \cos x \cosh y - i \sinh y \sin x$

Thus, $\operatorname{Re}(z) = u(x, y) = \cos x \cosh y$ and $\operatorname{Im}(z) = v(x, y) = -\sin x \sinh y$.

Then we have
$u_x = -\sin x \cosh y \qquad v_y = -\sin x \cosh y$
$u_y = \cos x \sinh y \qquad v_x = -\cos x \sinh y$

Since the C-R conditions hold and the partial derivatives are all continuous, we have proven that $\cos z$ is also holomorphic. ✓

Hence $\tan z$ is analytic for all $z \setminus \cos z = 0$ (i.e. when $z = \frac{\pi}{2} + \pi\, n$ for $n = 0, 1, 2, 3, \ldots$). These values of z are singular points.

c) $e^{1/(z-1)}$

Solution:
$\frac{d}{dz}\left(e^{1/(z-1)}\right) = -\frac{1}{(z-1)^2}\, e^{1/(z-1)}$
Since the derivative is continous at any point except when $z = 1$, we conclude that our function is analytic $\forall\; z \in \mathbb{C} : z \neq 1$, i.e. $z = 1$ is a singular point.

d) $e^{\bar{z}}$

Solution:

$$e^{x-i\, y} = \frac{e^x}{e^{i\, y}} = \frac{e^x}{\cos y + i \sin y} \cdot \frac{\cos y - i \sin y}{\cos y - i \sin y}$$
$$= \frac{e^x(\cos y - i \sin y)}{\cos^2 y + \sin^2 y} = e^x(\cos y - i \sin y)$$

Thus, $\operatorname{Re}(z) = u(x, y) = e^x \cos y$ and $\operatorname{Im}(z) = v(x, y) = -e^x \sin y$.

Then we have

$u_x = e^x \cos y \qquad v_y = -e^x \cos y$

$u_y = -e^x \sin y \qquad v_x = -e^x \sin y$

We can see that u_x and v_y are equal only when $\cos y = 0$, that is when $y = \frac{\pi}{2} + \pi n$. However in order for $u_y = -v_x$ to be satisfied, $\sin y$ must be equal to zero. But as we see, $y = \frac{\pi}{2} + \pi n$, hence $\sin y$ cannot equal zero and hence the C-R conditions do not hold. Thus we conclude that the function $e^{\bar{z}}$ is non-analytic everywhere.

e) $\dfrac{z}{z^4 + 1}$

Solution:

Since this is a rational function we already know that this function is analytic for all $z \backslash z^4 = -1$, i.e. $\forall\ z \backslash z = e^{\frac{\pi}{4} i + 2\pi n}$, for $n =$ any four consecutive integers. In other words, this function only has four singular points and it's holomorphic elsewhere.

(5) Let $f(z)$ be analytic in some domain. Show that $f(z)$ is necessarily a constant if either the function $\overline{f(z)}$ is analytic or $f(z)$ assumes only pure imaginary values in the domain.

Proof:

‣ Case 1: $\overline{f(z)}$ is analytic.

Assume that $f(z)$ is not constant. We are given that both $f(z)$ and $\overline{f(z)}$ are analytic. Then we can express $f(z)$ as $u(x, y) + i\, v(x, y)$ and $\overline{f(z)}$ can be expressed as $u(x, y) - i\, v(x, y)$, where $u(x, y)$ and $v(x, y)$ are not constant functions.

Since both functions are analytic the C-R conditions must hold on both cases (i.e. $u_x = v_y$ and $u_y = -v_x$). But in order for these conditions to hold on both cases it must be true that $v(x, y) = -v(x, y)$, which is impossible. $(\Rightarrow\Leftarrow)$

Hence the only way that both functions are analytic is if $f(z)$ is a constant function, since in that case all the derivatives are zero and the C-R conditions always hold. ✓

‣ Case 2: $f(z)$ assumes only pure imaginary values in the domain.

If $f(z)$ assumes only imaginary values then $\text{Re}(z) = u(x, y) = 0$, therefore $f(z) = i\, v(x, y)$. We are given that $f(z)$ is analytic which means that the C-R conditions must hold. Since $u(x, y) = 0$, then $u_x = u_y = 0$ and so v_x and v_y must also be zero, i.e. the C-R conditions hold. ✓

Thus we have proved that $f(z)$ has to be a constant function if either the function $\overline{f(z)}$ is analytic or $f(z)$ assumes only pure imaginary values in the domain. ■