Math 353 HW 6

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Section 2.6

(1) Evaluate the integrals $\oint_C f(z)\, dz$, where C is the unit circle centered at the origin and $f(z)$ is given by the following :

a) $\dfrac{\sin z}{z}$

Solution:

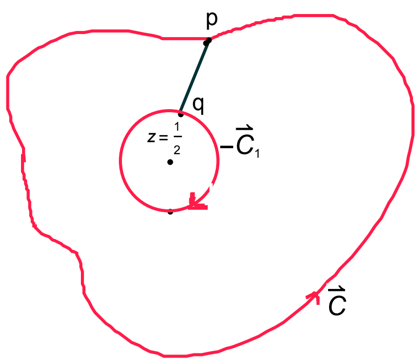
$\sin 0 = \frac{1}{2\pi i} \oint_C \frac{\sin z}{z}\, dz \Longrightarrow \oint_C \frac{\sin z}{z}\, dz = 0.$

b) $\dfrac{1}{(2z-1)^2}$

Solution:

$$\frac{1}{(2z-1)^2} = \frac{1}{\left(2\left(z-\frac{1}{2}\right)\right)^2} = \frac{1}{4}\frac{1}{\left(z-\frac{1}{2}\right)^2}.$$

Now we have to solve $\frac{1}{4}\oint_C \frac{1}{\left(z-\frac{1}{2}\right)^2}\, dz$. In order to solve this we are going to use a cross-cut and evaluate this integral around a circle around the point $z = \frac{1}{2}$.



We know from previous work that $\oint_C + \oint_{pq} + \oint_{-C_1} + \oint_{qp} = 0 \Longrightarrow \oint_C = \oint_{C_1}$. Thus letting $w = z - \frac{1}{2}$, $dw = dz$, we have

$$\frac{1}{4}\oint_{C_1} \frac{1}{\left(z-\frac{1}{2}\right)^2}\, dz = \frac{1}{4}\oint_C \frac{1}{w^2}\, dw = 0 \quad \left(\text{since } \oint_C \frac{1}{w^n} = 0 \ \ \forall\, n \neq 1\right)$$

Hence $\oint_C \frac{1}{(2z-1)^2}\, dz = 0$.

c) $\frac{1}{(2z-1)^3}$

<u>Solution:</u>

$$\frac{1}{(2z-1)^3} = \frac{1}{\left(2\left(z-\frac{1}{2}\right)\right)^3} = \frac{1}{8}\frac{1}{\left(z-\frac{1}{2}\right)^3}.$$

By a similar argument as in b), we have $\frac{1}{8}\oint_{C_1} \frac{1}{\left(z-\frac{1}{2}\right)^3}\, dz = 0$.

Hence $\oint_C \frac{1}{(2z-1)^3}\, dz = 0$.

d) $\frac{e^z}{z}$

<u>Solution:</u>

$$e^0 = \frac{1}{2\pi i}\oint_C \frac{e^z}{z-0} \Longrightarrow \oint_C \frac{e^z}{z} = 2\pi i.$$

e) $e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right)$

Solution:

We have

$$\oint_C e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right) dz = \oint_C \frac{e^{z^2}}{z^2}\, dz - \oint_C \frac{e^{z^2}}{z^3}\, dz.$$

Then we solve the first integral...

$$\frac{d}{dz} e^{z^2}\Big|_{z=0} = \frac{1!}{2\pi i} \oint_C \frac{e^{z^2}}{z^2}\, dz \Longrightarrow \oint_C \frac{e^{z^2}}{z^2}\, dz = 0\ .$$

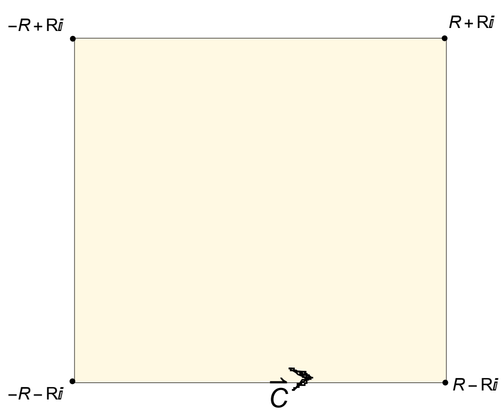
And the other...

$$\frac{d^2}{dz^2} e^{z^2}\Big|_{z=0} = \frac{2!}{2\pi i} \oint_C \frac{e^{z^2}}{z^3}\, dz \Longrightarrow \oint_C \frac{e^{z^2}}{z^3}\, dz = 2\pi i\ .$$

Thus we have

$$\oint_C e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right) dz = \oint_C \frac{e^{z^2}}{z^2}\, dz - \oint_C \frac{e^{z^2}}{z^3}\, dz = 0 - 2\pi i = -2\pi i.$$

(2) Evaluate the integrals $\oint_C f(z)\, dz$ over a contour C, where C is the boundary of a square with diagonal opposite corners at $z = -(1+i)\,R$ and $(1+i)\,R$, where $R > a > 0$, and where $f(z)$ is given by the following :



a) $\frac{e^z}{z - \frac{\pi i}{4}\, a}$

Solution:

$$e^{\frac{\pi i}{4}a} = \frac{1}{2\pi i}\oint_C \frac{e^z}{z-\frac{\pi i}{4}a}\,dz \Longrightarrow \oint_C \frac{e^z}{z-\frac{\pi i}{4}a}\,dz = 2\pi i\, e^{\frac{\pi i}{4}a} = 2\pi i\left(\frac{\sqrt{2}}{2}+i\,\frac{\sqrt{2}}{2}\right)^a.$$

b) $\dfrac{e^z}{\left(z-\frac{\pi i}{4}a\right)^2}$

Solution:

$$\left.\frac{d}{dz}e^z\right|_{z=\frac{\pi i}{4}a} = \frac{1!}{2\pi i}\oint_C \frac{e^z}{\left(z-\frac{\pi i}{4}a\right)^2}\,dz$$

$$\Longrightarrow \oint_C \frac{e^z}{\left(z-\frac{\pi i}{4}a\right)^2}\,dz = 2\pi i\, e^{\frac{\pi i}{4}a} = 2\pi i\left(\frac{\sqrt{2}}{2}+i\,\frac{\sqrt{2}}{2}\right)^a.$$

c) $\dfrac{z^2}{2z+a}$

Solution:

We have that $\dfrac{z^2}{2z+a} = \dfrac{z^2}{2\left(z+\frac{a}{2}\right)}$.

Thus,

$$\frac{1}{2}\left(-\frac{a}{2}\right)^2 = \frac{1}{2\pi i}\,\frac{1}{2}\oint_C \frac{z^2}{\left(z+\frac{a}{2}\right)}\,dz \Longrightarrow \oint_C \frac{z^2}{2\left(z+\frac{a}{2}\right)}\,dz = \pi i\,\frac{a^2}{4}$$

$$\Longrightarrow \oint_C \frac{z^2}{2z+a}\,dz = i\,\frac{\pi a^2}{4}.$$

d) $\dfrac{\sin z}{z^2}$

Solution:

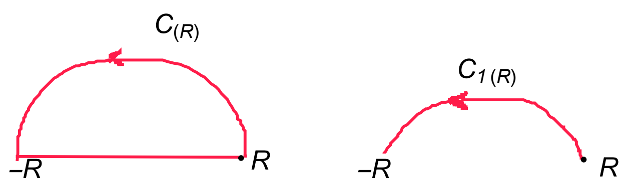
$$\left.\frac{d}{dz}\sin z\right|_{z=0} = \frac{1!}{2\pi i}\oint_C \frac{\sin z}{z^2}\,dz \Longrightarrow 1 = \frac{1}{2\pi i}\oint_C \frac{\sin z}{z^2}\,dz$$

$$\Longrightarrow \oint_C \frac{\sin z}{z^2}\,dz = 2\pi i.$$

(3) Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x+i)^2}\,dx$ by considering $\displaystyle\oint_{C_{(R)}} \frac{1}{(z+i)^2}\,dz$, where $C_{(R)}$ is the closed

semicircle in the upper half plane with corners at $z = -R$ and $z = R$, plus the x axis. Hint: Show that $\lim_{R\to\infty} \int_{C_{1\,(R)}} \frac{1}{(z+i)^2}\, dz = 0$, where $C_{1\,(R)}$ is the open semicircle in the upper half plane (not including the x axis).

<u>Solution:</u>



We look at the integral over the open semicircle $\int_{C_{1\,(R)}} \frac{1}{(z+i)^2}\, dz$ and we use triangle inequality...

$$\begin{aligned} ||z| - |i|| \le |z+i| &\Longrightarrow ||R e^{i t}| - |i|| \le |z+i| \\ &\Longrightarrow ||R| - |i|| \le |z+i| \\ &\Longrightarrow |R-1| \le |z+i| \Longrightarrow (|R-1|)^2 \le (|z+i|)^2 \\ &\Longrightarrow \frac{1}{(|R-1|)^2} \ge \frac{1}{(|z+i|)^2} \Longrightarrow \frac{1}{(R-1)^2} \ge \frac{1}{(z+i)^2}. \end{aligned}$$

Hence $M = \frac{1}{(R-1)^2}$ is our upper bound.

We also know that the arc length is $L = \pi R$ and so

$$\left| \int_{C_{1\,(R)}} \frac{1}{(z+i)^2}\, dz \right| \le ML = \frac{\pi R}{(R-1)^2}.$$

Now,

$$\lim_{R\to\infty} \int_{C_{1\,(R)}} \frac{1}{(z+i)^2}\, dz \le \left| \lim_{R\to\infty} \int_{C_{1\,(R)}} \frac{1}{(z+i)^2}\, dz \right| \le \lim_{R\to\infty} \frac{\pi R}{(R-1)^2} = 0 .$$

Hence, since $\lim_{R\to\infty} \int_{C_{1\,(R)}} \frac{1}{(z+i)^2}\, dz = 0$, we have that $\int_{-\infty}^{\infty} \frac{1}{(x+i)^2}\, dx = 0$.

(9) From Morera's Theorem, what can be said about the following function?

d) $\frac{e^z}{z}$

Solution:

This function is not continuous in any simply connected domain that encloses the origin (there is a singularity at $z = 0$). Furthermore we have that $e^0 = \frac{1}{2\pi i} \oint_C \frac{e^z}{z}\, dz \Longrightarrow \oint_C \frac{e^z}{z}\, dz = 2\pi i \neq 0$. Hence $\frac{e^z}{z}$ is not analytic on such domain.