MATH 746 TAKE HOME EXAM

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Chapter 2

Problem 3) Suppose f is integrable on $(-\pi, \pi]$ and extended to $\mathbb{R}$ by making it periodic of period 2π. Show that

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_I f(x)\,dx,$$

where I is any interval in $\mathbb{R}$ of length 2π.
(Hint: I is contained in two consecutive intervals of the form $\big(k\pi, (k+2)\,\pi\big)$.)

Proof. By assumption we have that f is periodic, so that $f(x) = f(x + 2\pi n)$ for any $n \in \mathbb{Z}$. Now, since as the hint suggests, an arbitrary interval $I = (a, b]$ (where $b - a = 2\pi$) is contained in two consecutive intervals of the form $\big(k\pi, (k+2)\,\pi\big)$ for some $k \in \mathbb{Z}$, we must have

$$I = (a, b] \subset (k\pi, (k+4)\pi].$$

Now let us define an element $c = (k+2)\pi \in (a, b]$, and observe that we can break up the integral over $(a, b]$ as follows:

$$\begin{aligned} \int_{(a,b]} f(x)\,dx &= \int_{(a,c]} f(x)\,dx + \int_{(c,b]} f(x)\,dx \\ &= \int_{(a,c]} f(x)\,dx + \int_{(k\pi,a]} f(x)\,dx \qquad (\spadesuit) \end{aligned}$$

Note that $(\spadesuit)$ is a valid equality because

$$f(x)|_{(c,b]} = f(x)|_{((k+2)\pi,b]} = f(x-2\pi)|_{((k+2)\pi-2\pi,b-2\pi]} = f(x-2\pi)|_{(k\pi,a]} = f(x)|_{(k\pi,a]}$$

by the periodicity of f.

Putting all this together, we have that

$$\int_{(a,b]} f(x)\,dx = \int_{(k\pi,c]} f(x)\,dx = \int_{(k\pi,(k+2)\pi]} f(x)\,dx.$$

Now we can break up this integral as follows

$$\int_{(k\pi,(k+2)\pi]} f(x)\,dx = \int_{(k\pi,(k+1)\pi]} f(x)\,dx + \int_{((k+1)\pi,(k+2)\pi]} f(x)\,dx,$$

and then by 2π-periodicity again, it follows that

$$\int_{(a,b]} f(x)\,dx = \int_{(k\pi,(k+2)\pi]} f(x)\,dx = \int_{((k+1)\pi,(k+3)\pi]} f(x)\,dx,$$

thus we can see that this equality of integrals holds for any integer k.

By taking the integral over the interval $(k\pi,(k+2)\pi]$ and setting $k=-1$, we have the desired equality

$$\int_{(a,b]} f(x)\,dx = \int_I f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx. \qquad \square$$

Problem 4) Suppose f is integrable on $[0,b]$ and

$$g(x) = \int_x^b \frac{f(t)}{t}\,dt \qquad \text{for } 0 < x \leq b.$$

Prove that g is integrable on $[0,b]$ and

$$\int_0^b g(x)\,dx = \int_0^b f(t)\,dt.$$

Proof. Let f be integrable on $[0,b]$. We may assume WLOG that f is non-negative, since otherwise we can analyze f^+ and f^- separately. Let $E = \{x \mid 0 < x \leq t \leq b\}$, which is clearly a measurable set, and let

$$h(x,t) = \frac{f(t)}{t}\chi_E.$$

Notice that h is non-negative and is clearly measurable since it is a quotient of measurable functions times another measurable function. Hence, the integral $\int_x^b f(t)/t\,\chi_E\,dt$ is a measurable function of x, which is equal to $g(x)$ for $0 < x \leq b$ and equals 0 elsewhere. Thus g is measurable (in general, we have that $g\colon (0,b] \to \mathbb{R}$ is measurable iff $g \cdot \chi_{(0,b]}\colon \mathbb{R} \to \mathbb{R}$ is also measurable) and, by an application of Fubini's theorem, we have

$$\begin{aligned}
\int_0^b g(x)\,dx &= \int_0^b \left(\int_x^t \frac{f(t)}{t}\,dt \right) dx \\
&= \int_{\mathbb{R}\times\mathbb{R}} h(x,t) \\
&= \int_0^b \left(\int_0^t h(x,t)\,dx \right) dt \\
&= \int_0^b \left(\int_0^t \frac{f(t)}{t}\,dx \right) dt \\
&= \int_0^b t\frac{f(t)}{t}\,dt \\
&= \int_0^b f(t)\,dt.
\end{aligned}$$

Thus we have proven that $\int_0^b g(x)\,dx = \int_0^b f(t)\,dt$, which in turn implies that g is integrable on $[0,b]$, and this concludes our proof. □

Problem 7) Let $\Gamma \subset \mathbb{R}^d \times \mathbb{R}$, $\Gamma = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} \mid y = f(x)\}$, and assume f is measurable on $\mathbb{R}^d$. Show that Γ is a measurable subset of $\mathbb{R}^{d+1}$, and $m(\Gamma) = 0$.

Proof. First, we partition $\mathbb{R}^d$ into almost disjoint closed unit cubes $\{Q_k\}_{k=1}^{\infty}$. We start by looking at the restriction of Γ to each of these closed cubes

$$\Gamma_k = \{(x,y) \in Q_k \times \mathbb{R} \mid y = f(x)\},$$

so that

$$(\clubsuit) \qquad \Gamma = \bigcup_{k=1}^{\infty} \Gamma_k.$$

From here we define the d-dimensional sets

$$F_{k,n}^i = \left\{ x \in Q_k \mid \frac{i}{2^n} \le f(x) < \frac{i+1}{2^n} \right\}.$$

Now let

$$E_{k,n}^i = F_{k,n}^i \times \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right),$$

and finally:

$$E_{k,n} = \bigcup_{i=-\infty}^{\infty} E_{k,n}^i.$$

The fact that these sets are measurable follows by the measurability of f. Notice from the above definitions that

$$(\star) \qquad \Gamma_k \subset E_{k,n} \quad \forall n \in \mathbb{N},$$

and also

$$(\star\star) \qquad E_{k,n+1} \subset E_{k,n} \quad \forall n \in \mathbb{N}.$$

Now observe that

$$\begin{aligned}
m(E_{k,n}) &\le \sum_{i=-\infty}^{\infty} m(E_{k,n}^i) && \text{(By subadditivity)} \\
&\le \sum_{i=-\infty}^{\infty} m(F_{k,n}^i) \cdot m\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right) && \text{(By construction)} \\
&= \frac{1}{2^n} \sum_{i=-\infty}^{\infty} m(F_{k,n}^i) \\
&\le \frac{1}{2^n} \cdot m(Q_k) && \text{(By construction)} \\
&= \frac{1}{2^n}. && \text{(since } Q_k \text{ is a unit closed cube)}
\end{aligned}$$

Now this result, combined with $(\star)$ and $(\star\star)$, will yield the desired outcome. Observe, by $(\star\star)$, that since the sets $E_{k,n}$ are collapsing, with finite measure, we must have that the Γ_k are measurable since:

$$\begin{aligned}
m_*(\Gamma_k) &\le \lim_{n\to\infty} m(E_{k,n}) && \text{(by } (\star\star) \text{ and by monotonicity on } (\star)) \\
&= \lim_{n\to\infty} \frac{1}{2^n} \\
&= 0.
\end{aligned}$$

We know from a previous result in class that any set of outer measure 0 is measurable. Thus the Γ_k are indeed measurable, and moreover, it follows from this fact and from $(\clubsuit)$ that Γ is also measurable since any countable union of measurable sets is measurable.

Finally, observe that

$$\begin{aligned} m(\Gamma) &\leq \sum_{k=1}^{\infty} m(\Gamma_k) && \text{(By monotonicity on } (\clubsuit)) \\ &= 0. \end{aligned}$$

By a previous result, we know that any subset of a set that has outer measure 0 is also measurable with measure 0. Hence Γ is measurable with $m(\Gamma) = 0$, as we set out to prove. □

Problem 9) **(Tchebychev Inequality)** Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) \geq \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f.$$

Proof. First of all, notice that since f is integrable, it is measurable, and so E_α is also measurable by construction (this simple argument assures us that $m(E_\alpha)$ is well defined.) Now proving the inequality is easy as π (☺); all we need to do is rewrite E_α as

$$E_\alpha = \left\{ x : \frac{f(x)}{\alpha} \geq 1 \right\},$$

and then observe that

$$\begin{aligned} m(E_\alpha) &= \int_{E_\alpha} \\ &\leq \int_{E_\alpha} \frac{f(x)}{\alpha} && \text{(Since } f(x)/\alpha \geq 1) \\ &\leq \frac{1}{\alpha} \int f. \end{aligned}$$

□

Problem 19) Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x\colon |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)| \, dx = \int_0^\infty m(E_\alpha) \, d\alpha.$$

Proof. Notice that since f is integrable it is measurable, and for each $\alpha > 0$, E_α is a measurable set. Moreover, we know that for each $\alpha > 0$, $m(E_\alpha) = \int_{E_\alpha} dx = \int_{\mathbb{R}^d} \chi_{E_\alpha} \, dx$. Now, putting all this together and applying Fubini's (Tonelli's) Theorem, we have

$$\begin{aligned}
\int_0^\infty m(E_\alpha)d\alpha &= \int_0^\infty \left(\int_{E_\alpha} dx\right) d\alpha \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_{E_\alpha}\, dx\right) d\alpha \\
&= \int_{\mathbb{R}^d} \left(\int_0^\infty \chi_{\{|f(x)|>\alpha\}}\, d\alpha\right) dx \qquad \text{(By Tonelli's Theorem)} \\
&= \int_{\mathbb{R}^d} m((0, |f(x)|])\, dx \\
&= \int_{\mathbb{R}^d} \big|f(x)\big|\; dx.
\end{aligned}$$
□

Chapter 3

Problem 10) Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.

Solution. Let $\{r_n\}_{n=1}^\infty$ be an enumeration of the rationals. Let us then define functions f_n such that

$$f_n(x) = \begin{cases} 0, & \text{if } x < r_n \\ \frac{1}{2^n}, & \text{if } x \geq r_n, \end{cases}$$

and then let

$$f(x) = \sum_{n=1}^\infty f_n(x).$$

Our claim is that f is an increasing function whose set of discontinuities is precisely $\mathbb{Q}$. Notice that the function is indeed strictly increasing since for $x, y \in \mathbb{R}$, if $x < y$, there is some rational r_n so that $x < r_n < y$ and $f(y) \geq f(x) + 1/2^n > f(x)$ (Note that the existence of r_n is guaranteed by the fact that the rationals are dense in $\mathbb{R}$.) Our next step then is to show that f is continuous at the irrational points. Let's choose an arbitrary point $i \in \mathbb{I}$ and put $\alpha_n = 1/2^n$. Then, for $\varepsilon > 0$, there exists some large $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^\infty \alpha_n < \varepsilon.$$

Consider the finite list of rationals $r_1, \ldots, r_N$. Since our point $i \in \mathbb{I}$ is not on this list, we can find a $\delta > 0$ so that none of the points in the list is in the interval $(i-\delta, i+\delta)$. Suppose that $i < x < i + \delta$. Then $f(x) - f(i)$ is the sum of the α_n's for which the corresponding point r_n is in the interval $(i, x]$. According to our construction, none of the points r_n for $n \leq N$ is in this interval, so

$$f(x) - f(i) = \sum_{r_n \in (i,x]} \alpha_n \leq \sum_{n=N+1}^{\infty} \alpha_n < \varepsilon.$$

Similarly, if $i - \delta < x < i$, then $f(i) - f(x)$ is the sum of the α_n's such that r_n is in $(x, i]$. By the same reasoning as above $f(i) - f(x) < \varepsilon$. Hence we have shown that f is continuous in the irrationals.

Our last step is then to show that f is discontinuous at the rationals. To see this, let r_k be a rational point. Define a function g by

$$g(x) = \sum_{\substack{1 \leq n < \infty \\ n \neq k}} f_n(x).$$

By our previous work above, we have that g is continuous at r_k and we have $f(x) = f_k(x) + g(x)$. But then notice that

$$f(r_k+) = f_k(r_k+) + g(r_k+) = 1/2^k + g(r_k),$$

and similarly,

$$f(r_k-) = 0 + g(r_k).$$

Hence,

$$f(r_k+) - f(r_k-) = (1/2^k + g(r_k)) - g(r_k) = 1/2^k,$$

which means that f has a jump discontinuity at r_k. This shows that f is discontinuous on $\mathbb{Q}$ and thus we have constructed a continuous function whose set of discontinuities is exactly the rationals, as desired. $\square$

Problem 13) Show directly from the definition that the Cantor-Lebesgue function is not absolutely continuous.

Proof. In order for a function f defined on some interval $[a, b]$ to be absolutely continuous, we need to have that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{k=1}^{N} \left|f(b_k) - f(a_k)\right| < \varepsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta,$$

where the intervals (a_k, b_k) (for $k = 1, \ldots, N$) are disjoint intervals. For simplicity, let us take the interval $[0, 1]$, where the Lebesgue-Cantor function f satisfies the conditions

$f(0) = 0$ and $f(1) = 1$. If we were to use *Theorem 3.11* from our text, we could easily conclude that f is not absolutely continuous on $[0, 1]$ since it doesn't satisfy

$$f(1) - f(0) = \int_0^1 f'(x)\,dx. \tag{$\circledast$}$$

Notice that left-hand side of ($\circledast$) is equal to 1 whereas the integral on the right-hand side equals 0 because the derivative f' equals 0 on the complement of the Cantor set, which is almost everywhere. Then *Theorem 3.11* tells us that, since ($\circledast$) does not make sense, f is not absolutely continuous.

This application of *Theorem 3.11* would make our lives very easy, but well, life ain't easy ☹. We are being asked to show that f is not absolutely continuous ***directly from the definition***, so that's exactly what we're going to attempt now:

We know that f is the limit of the sequence of continuous increasing functions $\{f_k\}_{k=1}^{\infty}$, where $\left|f_{k+1}(x) - f_k(x)\right| \le 1/2^{k+1}$. Now let us pick $0 < \varepsilon < 1$. Then, for every $\delta > 0$, we can find a collection of intervals (a_k, b_k) that cover the Cantor points in $[0, 1]$ such that $\sum_k (b_k - a_k) < \delta$, since the Cantor set has measure zero. However, notice that since f only changes on the Cantor set, we have that $\sum_k \left|f(b_k) - f(a_k)\right| = f(1) - f(0) = 1 > \varepsilon$, hence absolute continuity is not achieved. □

Problem 15) Suppose F is of bounded variation and continuous. Prove that $F = F_1 - F_2$, where both F_1 and F_2 are monotonic and continuous.

Proof. We know from a previous theorem discussed in class that every function of bounded variation is a difference of increasing bounded functions, so let us write $F = G_1 - G_2$ where G_1 and G_2 are increasing and of bounded variation. As shown in lemmas 3.12 and 3.13 on our text, an increasing bounded function is a continuous increasing function plus a jump function. Hence $G_1 = F_1 + J_1$, where F_1 is continuous and increasing, and J_1 is a jump function; similarly, $G_2 = F_2 + J_2$. Then $F = (F_1 - F_2) + (J_1 - J_2)$. But $J_1 - J_2$ is a jump function, and we know that jump functions are continuous only if they're constant. Since F is continuous, this implies that $J_1 - J_2$ is constant. Let us say, WLOG, that $J_1 - J_2 = 0$ (otherwise we could redefine $F_1^{'} = F_1 + (J_1 - J_2)$ and $F_1^{'}$ would also be continuous and increasing.) Hence $F = F_1 - F_2$, where F_1 and F_2 are monotonic and continuous. □

Problem 19) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then

a) f maps sets of measure zero to sets of measure zero.

b) f maps measurable sets to measurable sets.

Solution. a) Let f be absolutely continuous and suppose $E \subset \mathbb{R}$ has measure zero. Let $\varepsilon > 0$, and then by absolute continuity we must have a $\delta > 0$ such that $\sum \left|f(b_j) - f(a_j)\right| < \varepsilon$ whenever $\sum \left|b_j - a_j\right| < \delta$, for disjoint intervals (a_j, b_j). Since $m(E) = 0$, there is an open set $\mathcal{O} \supset E$ with $m(\mathcal{O}) < \delta$. Every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals, so

$$\mathcal{O} = \bigcup_{j=1}^{\infty} (a_j, b_j) \quad \text{with} \quad \sum_{j=1}^{\infty} (b_j - a_j) < \delta.$$

Now for each j let $m_j, M_j \in [a_j, b_j]$ be values of x such that

$$f(m_j) = \min_{x \in [a_j, b_j]} f(x) \quad \text{and} \quad f(M_j) = \max_{x \in [a_j, b_j]} f(x).$$

Both m_j and M_j must exist because f is continuous and $[a_j, b_j]$ is compact. Then

$$f(\mathcal{O}) \subset \bigcup_{j=1}^{\infty} [f(m_j), f(M_j)].$$

Hence $f(E)$ is a subset of a set of measure less that ε. This is true for all ε, so $f(E)$ has measure zero.

b) Let $E = F \bigcup G$, where $E \subset \mathbb{R}$ is measurable, F is F_σ, and G has measure zero. Since closed subsets of $\mathbb{R}$ are σ-compact, F is σ-compact. But then $f(F)$ is also σ-compact since f is continuous. Then $f(E) = f(F) \bigcup f(G)$ is a union of an F_σ set and a set of measure zero. Hence $f(E)$ is measurable. □

Problem 24) Suppose F is an increasing function on $[a, b]$. Then,

a) Prove that we can write

$$F = F_A + F_C + F_J,$$

where each of the functions F_A, F_C, and F_J is increasing and:

(i) F_A is absolutely continuous.

(ii) F_C is continuous, but $F_C'(x) = 0$ for a.e. x.

(iii) F_J is a jump function.

b) Moreover, each component F_A, F_C, F_J is uniquely determined up to an additive constant.

Note: The above is the **Lebesgue decomposition** of F. There is a corresponding decomposition for any F of bounded variation.

Proof. Intuitively, this decomposition makes perfect sense. We can take any increasing function and break it up into a jump function -which is basically a summation of all jump discontinuities, if any-, a singular function -which remains constant almost everywhere-, and the rest is composed by an absolutely continuous function. Now let us try to prove this intuitive notion.
Since F is a monotone function defined on a closed interval $[a, b]$, it is clearly bounded on $[a, b]$ since it is bounded by $f(a)$ on one side and by $f(b)$ on the other end. Hence, since F is nondecreasing and bounded, according to part *(ii)* of *Lemma 3.13* from our text, we can represent F as a sum

$$F(x) = C + F_J, \tag{$\dagger$}$$

where C is a nondecreasing continuous function and F_J is a jump function.
We now let

$$F_A(x) = \int_a^x C'(t)\,dt, \quad \text{and} \quad F_C(x) = C(x) - F_A(x). \tag{$\dagger\dagger$}$$

Now F_A is absolutely continuous (this is a result that I believe we will prove next class?), while F_C is a continuous function that is in fact a singular function, since it satisfies

$$F_C'(x) = C'(x) - \frac{d}{dx}\int_a^x C'(t)\,dt = 0 \quad \text{almost everywhere.}$$

Now combining equations (†) and (††), we have that an increasing function F defined on a closed interval $[a, b]$ can be written as

$$F = F_A + F_C + F_J,$$

as we set out to prove. □