Algebraic Topology
HW Set # 3

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Problem 1. *Prove that every map $f\colon \mathbb{S}^2 \to \mathbb{S}^1$ is homotopic to the trivial map. [Hint: Use the covering space $E\colon \mathbb{R} \to \mathbb{S}^1$. If you can show that every map $f\colon \mathbb{S}^2 \to \mathbb{S}^1$ lifts to a map $\widetilde{f}\colon \mathbb{S}^2 \to \mathbb{R}$, then you can conclude that f is nullhomotopic because $\mathbb{R}$ is contractible.]*

Proof. Following the hint, we use the covering space $E\colon \mathbb{R} \to \mathbb{S}^1$ and we want to show that every map $f\colon \mathbb{S}^2 \to \mathbb{S}^1$ lifts to a map $\widetilde{f}\colon \mathbb{S}^2 \to \mathbb{R}$. But recall the following proposition:

Proposition

Suppose given a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and a map $f\colon (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

Applying the proposition while letting $Y = \mathbb{S}^2$, $\widetilde{X} = \mathbb{R}$, and $X = \mathbb{S}^1$ (and picking any arbitrary basepoints y_0, $\widetilde{x}_0$, and x, respectively, in each space), we indeed have the existence of a lift $\widetilde{f}\colon \mathbb{S}^2 \to \mathbb{R}$ since $f_*(\pi_1(\mathbb{S}^2, y_0)) = 1 \subset p_*(\pi_1(\mathbb{R}, \widetilde{x}_0)) = 1$. Hence we have that $f = E \circ \widetilde{f}$, where $\widetilde{f}$ is homotopic to a constant map since $\mathbb{R}$ is contractible. From this it follows that $f = E \circ \widetilde{f}$ is homotopic to the trivial (i.e. constant) map, as desired. □

Problem 2. *Let G be topological group and let H be a discrete subgroup of G. Prove that there is a neighborhood U of the identity e such that the sets hU, for $h \in H$, are pairwise disjoint. [Hint: First choose a neighborhood $V \subset G$ of e such that $V \cap H = \{e\}$. Now use the map $f\colon G \times G \to G$ defined by $f(x, y) = xy^{-1}$ to prove that there is an open set U containing e such that $\{xy^{-1} \mid x, y \in U\} \subset V$.]*

Proof. Let U be a neighborhood of e and choose a neighborhood $V \subset G$ of e such that $V \cap H = \{e\}$. Now we claim that for all $h \in H$ such that $h \neq e$, the sets U and hU are disjoint. Assume, to the contrary, that this is not the case. Then there would exist $x, y \in U$ and $h \in H$ such that $x = hy$, with $h \neq e$. Now we use our construction of U to conclude that $h = xy^{-1} \in V$. Thus $h \in V \cap H = \{e\}$, a contradiction of the assumption that $h \neq e$. This proves our claim. □

Problem 3. *Let G be a simply connected topological group and let H be a discrete normal subgroup. Prove that $\pi_1(G/H, e) = H$. (For example: $G = \mathbb{R}$, $H = \mathbb{Z}$, then $G/H = \mathbb{S}^1$.) [Hint: Use the previous problem to show that $G \to G/H$ is a covering space.]*

Proof. We prove this in two steps. First we show that the projection $p\colon G \to G/H$ is the universal cover of G/H. Then we conclude that the group of deck transformations of p is isomorphic to H. Since $\pi_1(G/H)$ is isomorphic to the group of deck transformations of p, this will prove the theorem.

To show that p is a covering map, we use the result from the previous problem, letting U and V be stated as before. Fix $g \in G$ and let $W = p(Ug)$. Note that W contains $p(g)$ and is an open set in G/H, since $p^{-1}(W) = \amalg_{h\in H} hUg$ is open (this is the definition of the quotient topology). Moreover, the restrictions $p|_{hUg}\colon hUg \to W$ are homeomorphisms for each $h \in H$, again by the definition of the quotient topology. This is exactly what needs to happen for p to be a covering map.

Having proved that p is a covering map, it is not hard to show now that the group of deck transformations of p is isomorphic to H. For each $h \in H$, let $L_h\colon G \to G$ be the left translation map $L_h(g) = hg$. Note that $L_{h_1} \circ L_{h_2} = L_{h_1 h_2}$, so the set $\{L_h \mid h \in H\}$ forms a group isomorphic to H under composition. Every L_h is a deck transformation of p. On the other hand, suppose φ is a deck transformation of p. Then $\varphi(e) \in H$, so that $L_{\varphi(e)^{-1}} \circ \varphi$ is a deck transformation of p that fixes $e \in G$. The only deck transformation that fixes a point is the identity, so $L_{\varphi(e)^{-1}} \circ \varphi = \mathrm{Id}$, and hence $\varphi = L_{\varphi(e)}$. This proves that the group of deck transformations of p is exactly $\{L_h \mid h \in H\} \cong H$. □

Problem 4. *Let M_1 and M_2 be n-dimensional connected manifolds, where $n > 2$. Let $M_1\#M_2$ be their connected sum. Show that $\pi_1(M_1\#M_2) = \pi_1(M_1) * \pi_1(M_2)$.*

The proof relies on some auxiliary results, which I am listing as propositions as an aside in order to avoid breaking the flow of the main argument. The propositions that appear on our text (*Massey's*) are merely stated; the other results that do not appear on our text are stated and proved.

Propositions

Proposition 1. *Let $M_1\#M_2$ be a connected sum of n-manifolds M_1 and M_2. There are open subsets $U_1, U_2 \subseteq M_1\#M_2$ and points $p_i \in M_i$ such that $U_i \cong M_i \smallsetminus \{p_i\}$, $U_1 \cap U_2 \cong \mathbb{R}^n \smallsetminus \{0\}$, and $U_1 \cup U_2 = M_1\#M_2$.*

Proof. For $i = 1, 2$, let $\mathbb{B}_i \subseteq M_i$ be the regular coordinate ball around $p_i \in M_i$ and let $C_i \supseteq \bar{\mathbb{B}}_i$ be the larger coordinate balls around p_i. Let $j_i\colon M_i \smallsetminus \mathbb{B}_i \to M_1\#M_2$ be the injections. Take $U_1 = j_1(M_1 \smallsetminus \mathbb{B}_1) \cup j_2(C_2 \smallsetminus \mathbb{B}_2)$ and $U_2 = j_1(C_1 \smallsetminus \mathbb{B}_1) \cup j_2(M_2 \smallsetminus \mathbb{B}_2)$. It is clear that $U_i \cong M_i \smallsetminus \{p_i\}$ and $U_1 \cup U_2 = M_1\#M_2$. Also, note that

$$\begin{aligned} U_1 \cap U_2 &\cong j_1(C_1 \smallsetminus \mathbb{B}_1) \cup j_2(C_2 \smallsetminus \mathbb{B}_2) \\ &\cong \mathbb{S}^{n-1} \times (0,1) \\ &\cong \mathbb{R}^n \smallsetminus \{0\}. \end{aligned}$$ □

Proposition 2 (Simply Connected Intersection)**.** *Assume the hypotheses of the Seifert-Van Kampen theorem, letting as usual X be covered by open sets U and V, and suppose in addition that $U \cap V$ is simply connected. Then $\pi_1(X,p) \cong \pi_1(U,p) * \pi_1(V,p)$.*

Proposition 3 (One Simply Connected Set)**.** *Assume the hypotheses of the Seifert-Van Kampen theorem, letting as usual X be covered by open sets U and V, and suppose in addition that U is simply connected. Then the inclusion $V \hookrightarrow X$ induces an isomorphism*

$$\pi_1(X,p) \cong \pi_1(V,p)/\overline{j_* \, \pi_1(U \cap V, p)},$$

where $\overline{j_* \, \pi_1(U \cap V, p)}$ is the normal closure of $j_* \, \pi_1(U \cap V, p)$.

Proposition 4. *For any $n \geq 1$, $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{R}^n \smallsetminus \{0\}$ and of $\bar{\mathbb{B}}^n \smallsetminus \{0\}$.*

Proof. Define a homotopy $H \colon (\mathbb{R}^n \smallsetminus \{0\}) \times I \to \mathbb{R}^n \smallsetminus \{0\}$ by

$$H(x,t) = (1-t)x + t\frac{x}{|x|}.$$

This is just the straight-line homotopy from the identity map to the retraction onto the sphere (see Figure 1). The same formula works for $\bar{\mathbb{B}}^n \smallsetminus \{0\}$. □

Corollary 1. *For $n \geq 3$, both $\mathbb{R}^n \smallsetminus \{0\}$ and $\bar{\mathbb{B}}^n \smallsetminus \{0\}$ are simply connected.*

Proposition 5. *Suppose M is a connected manifold of dimension at least 3, and $p \in M$. Then the inclusion $M \smallsetminus \{p\} \to M$ induces an isomorphism $\pi_1(M \smallsetminus \{p\}) \cong \pi_1(M)$.*

Proof. Let $\mathbb{B}$ be a coordinate ball around p and let $U = \mathbb{B}$ and $V = M \smallsetminus \{p\}$ in Proposition 3. Choose some base point q in $\mathbb{B} \smallsetminus \{p\}$. Then the inclusion $M \smallsetminus \{p\} \hookrightarrow M$ induces an isomorphism

$$\pi_1(M,q) \cong \pi_1(M \smallsetminus \{p\}, q)/\overline{j_* \, \pi_1(\mathbb{B} \smallsetminus \{p\}, q)},$$

where $j \colon \mathbb{B} \smallsetminus \{p\} \hookrightarrow M \smallsetminus \{p\}$ is the inclusion. But $\pi_1(\mathbb{B} \smallsetminus \{p\}, q)$ is trivial by Corollary 1, so $\pi_1(M,q) \cong \pi_1(M \smallsetminus \{p\}, q)$. □

Proof of Problem 4. By Proposition 1, there are open sets $U_1, U_2 \subseteq M_1 \# M_2$ and points $p_i \in M_i$ such that U_i is homeomorphic to $M_i \smallsetminus \{p_i\}$, $U_1 \cap U_2$ is homeomorphic to $\mathbb{R}^n \smallsetminus \{0\}$, and $U_1 \cup U_2 = M_1 \# M_2$. Choose a base point $q \in U_1 \cap U_2$. Now, since $\mathbb{R}^n \smallsetminus \{0\}$ is simply connected when $n > 2$, by using Proposition 2 and Proposition 5, we have

$$\begin{aligned} \pi_1(M_1 \# M_2) &\cong \pi_1(U_1, q) * \pi_1(U_2, q) \\ &\cong \pi_1(M_1 \smallsetminus \{p_1\}) * \pi_1(M_2 \smallsetminus \{p_2\}) \\ &\cong \pi_1(M_1) * \pi_1(M_2). \end{aligned}$$ □

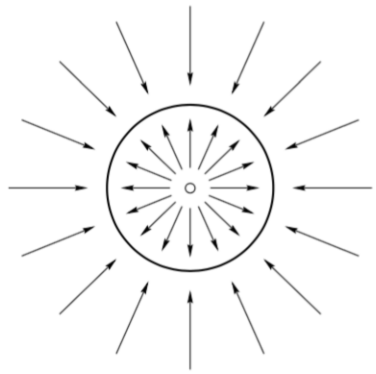

Figure 1: *Strong deformation retraction of* $\mathbb{R}^2 \smallsetminus \{0\}$ *onto* $\mathbb{S}^1$.

Problem 5. *Let F be a finitely generated free group. Prove that there is an n-manifold M, for $n > 2$, with $\pi_1(M) = F$.*

Proof. First, note that we can realize a free group on one generator by taking the product of $\mathbb{S}^1$ with $\mathbb{S}^{n-1}$. Let M' denote the resulting smooth orientable compact n-manifold. To get this result, we must use the fact that for any spaces X and Y, we have $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ (as we proved in class), coupled with the fact that $\pi_1(\mathbb{S}^1) = \langle \alpha \rangle$ and $\mathbb{S}^{n-1}$ is simply-connected for $n > 2$. Then, by the result in Problem 4 above, we can take the connected sum of k copies of M', $M = \#_{i=1}^k M'$, to realize the free group on k generators $\pi_1(M) = \pi_1(\#_{i=1}^k M') = \ast_{i=1}^k M' = F$. $\square$

Problem 6. *Let G be any finitely presented group. Show that there is a 4-manifold that has G as its fundamental group.*

Proof. Let G have presentation $\langle g_1, \dots, g_i \mid r_1, \dots, r_k \rangle$ and let M be the orientable 4-manifold such that $G = \pi_1(M)$ (we know of the existence of M by Problem 5 above). Let $[\alpha] \in G$ and let $G' = \langle g_1, \dots, g_i \mid r_1, \dots, r_k, \alpha \rangle$. We are going to show the existence of a 4-manifold M' that satisfies $\pi_1(M') = G'$.

Let α be represented by C, a smooth simple closed curve in M (such a C is guaranteed to exist for $\dim M = n > 2$). We are going to consider a tubular neighborhood N of C, which is homeomorphic to $\mathbb{S}^1 \times \bar{\mathbb{B}}^3$. Notice that the boundary of N is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$. In addition, $\mathbb{S}^1 \times \mathbb{S}^2$ happens to be the boundary of the orientable 4-manifold with boundary $\bar{\mathbb{B}}^2 \times \mathbb{S}^2$. So, if we let $\widetilde{M}$ be the complement of the interior of N, we can perform surgery on C by identifying the boundaries of $\widetilde{M}$ and $\bar{\mathbb{B}}^2 \times \mathbb{S}^2$. This will yield a new space

$$M' = \left(\widetilde{M} \amalg \bar{\mathbb{B}}^2 \times \mathbb{S}^2\right) / \sim .$$

We can reference Milnor here and claim that the quotient of two smooth compact orientable 4-manifolds with boundary yields a smooth compact orientable 4-manifold. Thus it remains only to show that M' satisfies $\pi_1(M') = G'$. The first step will be to show that $\pi_1(\widetilde{M}) \cong \pi_1(M)$. Then we will be able to complete our proof by showing that

$$\pi_1(M') \cong \frac{\pi_1(M)}{\langle \alpha \rangle}.$$

Now, since $M = \widetilde{M} \cup N$ and $\widetilde{M} \cap N \neq \emptyset$, we can apply the Van Kampen theorem letting $U = \widetilde{M}$ and $V = N$. This will show that $\pi_1(M)$ is in terms of $\pi_1(\widetilde{M})$. First, note that $\widetilde{M} \cap N \cong \mathbb{S}^1 \times \mathbb{S}^2$, while $N \cong \mathbb{S}^1 \times \bar{\mathbb{B}}^3$, so clearly the homomorphism $\pi_1(\widetilde{M} \cap N) \to \pi_1(N)$ induced by the inclusion map is actually an isomorphism (since $\mathbb{S}^2$ and $\bar{\mathbb{B}}^3$ are simply-connected, this follows directly from the fact that $\pi_1(\widetilde{M} \cap N) = \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^2)$ and $\pi_1(N) = \pi_1(\mathbb{S}^1) \times \pi_1(\bar{\mathbb{B}}^3)$). Thus it must follow that the homomorphism $\pi_1(\widetilde{M}) \to \pi_1(M)$ is an isomorphism.

Now we finally compute $\pi_1(M')$. Using the Van Kampen theorem, this time we let $U = \widetilde{M}$ and $V = \bar{\mathbb{B}}^2 \times \mathbb{S}^2$ (we can do this because note that $M' = \widetilde{M} \cup (\bar{\mathbb{B}}^2 \times \mathbb{S}^2)$ and $\widetilde{M} \cap (\bar{\mathbb{B}}^2 \times \mathbb{S}^2) = \widetilde{M} \cap N \neq \emptyset$). Now, since $V = \bar{\mathbb{B}}^2 \times \mathbb{S}^2$ is simply-connected, we may apply Proposition 3 (from the list of propositions given in Problem 4), so that we have $\pi_1(M') \cong \pi_1(\widetilde{M})/T$, where T is the smallest normal subgroup containing the image of $\pi_1(\widetilde{M} \cap N) \to \pi_1(\widetilde{M})$. But $\pi_1(\widetilde{M} \cap N)$ is generated by one loop that generates $\pi_1(N)$ itself. Moreover, this loop is homotopic to our loop C. So the image of $\pi_1(\widetilde{M} \cap N)$ in $\pi_1(\widetilde{M})$ corresponds to the image of $\pi_1(C)$ in $\pi_1(M)$ under the isomorphism $\pi_1(\widetilde{M}) \xrightarrow{\cong} \pi_1(M)$. Thus T is equivalent to $\overline{\langle \alpha \rangle}$, and so $\pi_1(M') \cong \pi_1(M)/\overline{\langle \alpha \rangle}$, as desired. $\square$