Math 260 HW # 4

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Section 2.1

(28) Given that $T : V \longrightarrow V$ is linear, prove that the subspaces $\{\hat{0}\}$, V, $R(T)$, and $N(T)$ are all T-invariant.

Proof:

We need to prove that the subspace $\{\hat{0}\} \subseteq V$ is T-invariant, i.e. $T(x) \in \{\hat{0}\}$ for every $x \in \{\hat{0}\}$. But the zero subspace contains only the zero vector, i.e. $x = \hat{0} \;\; \forall \;\; x \in \{\hat{0}\}$. Then we have $T(\hat{0}) = \hat{0} \in \{\hat{0}\}$. Thus we conclude that $\{\hat{0}\}$ is T-invariant. ✓

Clearly V is T-invariant since T is the linear operator $T : V \longrightarrow V$. In other words, T is the linear map that takes preimages in V and map them to images that are also in V. Hence $T(x) \in V \;\; \forall \;\; x \in V$, and we conclude that V is T-invariant. ✓

By the same reasoning as above it is also obvious that the subspace $R(T)$ is T-invariant, since $R(T)$ is the subspace that contains all images of the linear map T, which is a linear operator. ✓

Yet again by the same reasoning it is clear that $N(T)$ is T-invariant, since $N(T)$ is the subspace containg all the preimages of V that are being mapped to the zero vector (image) in V by the linear operator. ✓ ■

Section 2.2

(3) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Compute $[T]_\beta^\gamma$.
If $\alpha = \{(1, 2), (2, 3)\}$, compute $[T]_\alpha^\gamma$.

Solution:

• We have $\beta = \{(1, 0), (0, 1)\}$. Then

$\hookrightarrow T(1, 0) = (1, 1, 2) = a(1, 1, 0) + b(0, 1, 1) + c(2, 2, 3)$

$$a + 0\,b + 2\,c = 1$$
$$a + b + 2\,c = 1$$
$$0\,a + b + 3\,c = 2$$

$$\begin{pmatrix} 1 & 0 & 2 & | & 1 \\ 1 & 1 & 2 & | & 1 \\ 0 & 1 & 3 & | & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 3 & | & 2 \\ 0 & 1 & 0 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 3 & | & 2 \end{pmatrix}$$

So $c = \frac{2}{3}$, $b = 0$, and $a = -\frac{1}{3}$. Thus
$T(1, 0) = -\frac{1}{3}(1, 1, 0) + 0\,(0, 1, 1) + \frac{2}{3}(2, 2, 3)$

$\bullet\!\!\to T(0, 1) = (-1, 0, 1) = r\,(1, 1, 0) + s\,(0, 1, 1) + t\,(2, 2, 3)$

$$r + 0\,s + 2\,t = -1$$
$$r + s + 2\,t = 0$$
$$0\,r + s + 3\,t = 1$$

$$\begin{pmatrix} 1 & 0 & 2 & | & -1 \\ 1 & 1 & 2 & | & 0 \\ 0 & 1 & 3 & | & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 1 & 0 & | & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & | & -1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & -3 & | & 0 \end{pmatrix}$$

So $t = 0$, $s = 1$, and $r = -1$. Thus
$T(0, 1) = -1\,(1, 1, 0) + 1\,(0, 1, 1) + 0\,(2, 2, 3)$

Hence

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix} \qquad \checkmark$$

• Now we use the basis α in $\mathbb{R}^2$.

$\bullet\!\!\to T(1, 2) = (-1, 1, 4) = a(1, 1, 0) + b(0, 1, 1) + c\,(2, 2, 3)$

$$a + 0\,b + 2\,c = -1$$
$$a + b + 2\,c = 1$$
$$0\,a + b + 3\,c = 4$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 2 \end{array}\right)$$

So $c = \frac{2}{3}$, $b = 2$, and $a = -\frac{7}{3}$. Thus
$T(1,\ 2) = -\frac{7}{3}\,(1,\ 1,\ 0) + 2\,(0,\ 1,\ 1) + \frac{2}{3}\,(2,\ 2,\ 3)$

$\bullet\!\!\longrightarrow T(2,\ 3) = (-1,\ 2,\ 7) = r\,(1,\ 1,\ 0) + s\,(0,\ 1,\ 1) + t\,(2,\ 2,\ 3)$

$$\begin{aligned} r + 0\,s + 2\,t &= -1 \\ r + s + 2\,t &= 2 \\ 0\,r + s + 3\,t &= 7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 4 \end{array}\right)$$

So $t = \frac{4}{3}$, $s = 3$, and $r = -\frac{11}{3}$. Thus
$T(2,\ 3) = -\frac{11}{3}\,(1,\ 1,\ 0) + 3\,(0,\ 1,\ 1) + \frac{4}{3}\,(2,\ 2,\ 3)$

Hence

$$[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} \quad \checkmark$$

(Extra Problem) Prove that if T is a linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that $\mathcal{N}(T) = \{(x_1,\ x_2,\ x_3,\ x_4) \in \mathbb{R}^4 : x_1 = 5\,x_2 \text{ and } x_3 = 7\,x_4\}$, then T is surjective.

<u>Proof:</u>
Let $T \in \mathcal{L}(\mathbb{R}^4,\ \mathbb{R}^2)$ such that $\mathcal{N}(T)$ is defined as above. We wish to show that T is surjective, i.e. $\text{rank}(T) = \dim(\mathbb{R}^2) = 2$.
Let $x = (x_1,\ x_2,\ x_3,\ x_4) \in \mathcal{N}(T)$ be an arbitrary vector. Then, $x_1 = 5\,x_2$ and $x_3 = 7\,x_4$, implying $x = (5\,x_2,\ x_2,\ 7\,x_4,\ x_4)$. Note that each of the x_i's are scalars individually since they are the entries of the 4-tuple. Using a bit of arithmetic, we have

$$\begin{aligned} x = (5\,x_2,\ x_2,\ 7\,x_4,\ x_4) &= (5\,x_2,\ x_2,\ 0,\ 0) + (\ 0,\ 0,\ 7\,x_4,\ x_4) \\ &= x_2(5,\ 1,\ 0,\ 0) + x_4(0,\ 0,\ 7,\ 1) \end{aligned}$$

Since x_2 and x_4 are arbitrary scalars, we can relabel them as $a = x_2$ and $b = x_4$. Then,

$x = a(5,\ 1,\ 0,\ 0) + b(0,\ 0,\ 7,\ 1)$, with $a,\ b \in \mathbb{R}$. So, an arbitrary vector $x \in \mathcal{N}(T)$ can be expressed as a linear combination of the vectors $(5,\ 1,\ 0,\ 0)$ and $(0,\ 0,\ 7,\ 1)$.

Thus, $\beta = \{(5,\ 1,\ 0,\ 0),\ (0,\ 0,\ 7,\ 1)\}$ is a spanning set for $\mathcal{N}(T)$. We can easily check that $(5,\ 1,\ 0,\ 0)$ and $(0,\ 0,\ 7,\ 1)$ are in $\mathcal{N}(T)$ by checking that they satisfy the requirements of $\mathcal{N}(T)$ as stated above. To show that β is a basis, we must show linear independence.
By problem 9 of section 1.5, a set of two vectors is linearly dependent iff one is a multiple of the other. Then, to show linear independence, we show that $(5,\ 1,\ 0,\ 0)$ is not a scalar multiple of $(0,\ 0,\ 7,\ 1)$. This is clearly true, so β is a basis for $\mathcal{N}(T)$. Then, nullity $T = |\beta| = 2$. Since the domain is finite-dimensional. the Rank−Nullity theorem applies. Thus,

$$\dim V = \text{nullity}(T) + \text{rank}(T) \Longrightarrow \text{rank}(T) = \dim(V) - \text{nullity}(T) = 4 - 2 = 2.$$

Thus, $\text{rank}(T) = 2$ and T is surjective. ■