CHEAT SHEET

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1. Relativity & Cosmology

- **Riemann curvature tensor:** $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.
In components,
$$R^i{}_{jkl} = \langle f^i, R(e_k, e_l)e_j \rangle$$
$$= \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}.$$

Proposition 1. *Let ∇ be torsionless. Then,*

a) $R^a{}_{[bcd]} = 0$

(Note that $R^a{}_{[bcd]} = \frac{1}{3}(R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc})$ due to antisymmetry in the last two indices)

b) $R^a{}_{bcd} = \frac{2}{3}\left(R^a{}_{(bc)d} - R^a{}_{(bd)c}\right)$

c) $\nabla_{[c} R^a{}_{|b|de]} = 0.$ **(Bianchi Identity)**

Due to the fact that $R^a{}_{b(cd)} = 0$, the Bianchi Identity reduces to
$$\nabla_c R^a{}_{bde} + \nabla_d R^a{}_{bec} + \nabla_e R^a{}_{bcd} = 0.$$

Other symmetries, this time of $R_{abcd} = g_{ae} R^e{}_{bcd}$:

d) $R_{a[bcd]} = 0$

e) $R_{abcd} = -R_{abdc}$

f) $R_{abcd} = R_{cdab}$.

- **Ricci Identity:** Let ∇ be torsionless. Then for a vector field Z^a,
$$\nabla_c \nabla_d Z^a - \nabla_d \nabla_c Z^a = R^a{}_{bcd}\, Z^b,$$
while for a covector field $Z_a = g_{ab} Z^b$,
$$\nabla_c \nabla_d Z_a - \nabla_d \nabla_c Z_a = -R^b{}_{acd}\, Z_b.$$

- **Ricci tensor:** $R(X,Y) = \langle f^a, R(e_a, Y)X \rangle$. In components, $R_{ab} = R^d{}_{adb}$.
- **Einstein Curvature Tensor:** The ***Einstein curvature tensor*** is given by
$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab},$$
where $R = g^{ab} R_{ab}$ is the ***Ricci scalar***. The ***contracted Bianchi identity***[1] is then
$$\nabla^a G_{ab} = 0.$$

[1] see proof of this identity on Pg. 50 from the GGR course notes.

- **Geodesic Deviation Equation:** Let ∇ be torsionless, X the tangent vector to a smooth 1-parameter family of geodesics, and Z the deviation vector (so that $[X, Z] = 0$).
  Then,
  $$\nabla_X \nabla_X Z = R(X, Z)X.$$
  In coordinates,
  $$X^b \nabla_b (X^c \nabla_c Z^a) = R^a{}_{bcd} X^b X^c Z^d.$$

- **Levi-Civita Connection:** For a Levi-Civita connection (i.e. a connection ∇ that is metric-compatible and torsionless), we have, for vector fields X, Y, and Z,
  $$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\{&X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &+ g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)\}. \end{aligned}$$
  In a coordinate basis $\{e_i = \partial/\partial x^i\}$ (recalling that for a coordinate basis $[e_i, e_j] = 0$),
  $$\begin{aligned} g(\nabla_k e_j, e_l) &= \frac{1}{2}\left(e_k(g_{jl}) + e_j(g_{lk}) - e_l(g_{kj})\right) \\ \implies g_{ml}\,\Gamma^m_{jk} &= \frac{1}{2}\left(\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{kj}\right) \\ \implies \Gamma^i_{jk} &= \frac{1}{2} g^{il}\left(\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{kj}\right) \qquad \text{(Multiplying by } g^{il}\text{)}. \end{aligned}$$

- **Derivatives:** Take, for example, a $\binom{1}{2}$ tensor field $T^\alpha{}_{\beta\gamma}$. Then, the its ***covariant derivative*** is given by
  $$\nabla_\delta T^\alpha{}_{\beta\gamma} = \partial_\delta T^\alpha{}_{\beta\gamma} + \Gamma^\alpha_{\epsilon\delta} T^\epsilon{}_{\beta\gamma} - \Gamma^\epsilon_{\gamma\delta} T^\alpha{}_{\beta\epsilon} - \Gamma^\epsilon_{\beta\delta} T^\alpha{}_{\gamma\epsilon}.$$
  Meanwhile, the ***Lie derivative*** of $T^\alpha{}_{\beta\gamma}$ in the direction of a vector field X^α is given in coordinates by
  $$(\mathcal{L}_X T)^\alpha{}_{\beta\gamma} = X^\epsilon \partial_\epsilon T^\alpha{}_{\beta\gamma} - \partial_\epsilon X^\alpha T^\epsilon{}_{\beta\gamma} + \partial_\beta X^\epsilon T^\alpha{}_{\epsilon\gamma} + \partial_\gamma X^\epsilon T^\alpha{}_{\beta\epsilon}. \tag{1}$$
  Remark: Note that because the Lie derivative is a tensor, it can be computed in normal coordinates with the consequence that the partial derivatives in (1) can be replaced by covariant derivatives with respect to any metric without changing the definition.

- **Killing Fields:** In a coordinate basis a Killing field X must satisfy
  $$(\mathcal{L}_X g)_{ij} = X^k \partial_k g_{ij} + g_{kj} \partial_i X^k + g_{ik} \partial_j X^k = 0.$$
  Theorem. *Let X be a Killing vector field on (M, g). Then, the covector field $X_a = g_{ab} X^b$ satisfies* ***Killing's equation****,*
  $$\nabla_a X_b + \nabla_b X_a = 0,$$
  where ∇ is the Levi-Civita connection.

  (Very easy proof using normal coordinates –see pg 51, GGR Notes.)

  Proposition. *Let X be a Killing vector field on (M, g). Then*
  $$\nabla_a \nabla_b X_c = R^d{}_{abc} X_d.$$

  Remark: An important consequence of the above proposition is that a Killing field X^a is completely determined by the values of X^a and the anti-symmetric tensor $L_{ab} = \nabla_a X_b$ at any point $p \in M$. This implies that on a manifold of dimension n there can be at most $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ linearly independent Killing fields.

- **Schwarzschild metric:** In Schwarzschild coordinates,

(2) $$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = -\left(1-\frac{r_S}{r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{r_S}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2),$$

where $r_S = 2GM/c^2$ is the ***Schwarzschild radius***. The surface $r_S = r$ (where there's a coordinate singularity) is called the ***event horizon***.

In geometric units ($G = c = 1$), (2) becomes

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

In order to remove the coordinate singularity at $r = 2M$, we can rewrite the Schwarzschild metric in *Kruskal-Szekeres* coordinates:

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 = \frac{32M^3}{r}e^{-r/2M}\left(-\mathrm{d}v^2 + \mathrm{d}u^2\right) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

- **Robertson-Walker metric:**

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + R^2(t)\left[\mathrm{d}r^2 + S_k(r)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right].$$

Equivalent form under change of variables:

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right].$$

Here $R(t)$ is the ***cosmic scale factor***, given by $d = R(t)r$, where d is the so called ***proper distance***.[2] Also,

$$S_k(r) = \begin{cases} \sin r & \text{if } k = 1, \\ \sinh r & \text{if } k = -1, \\ r & \text{if } k = 0. \end{cases}$$

The time evolution of the cosmic scale factor $R(t)$ is given by ***Friedmann's Equation***:

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2,$$

while the ***acceleration equation*** reads

$$\ddot{R} = -\frac{4\pi G R}{3}\left(\rho + \frac{3p}{c^2}\right)$$

and the ***deceleration parameter*** is

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{4\pi G}{3H^2}\left(\rho + \frac{3p}{c^2}\right).$$

Accepting Friedmann's equation, there is always a ***critical density***

$$\rho_c = \frac{3H^2}{8\pi G},$$

that will yield $k = 0$, making the spatial part of the metric look Euclidean. A universe with density above this critical value will be ***spatially closed***, whereas a lower-density universe will be ***spatially open***.

It is common to define a dimensionless ***density parameter*** Ω as the ratio of density to critical density:

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}.$$

[2] The ***Hubble parameter*** $H(t)$ is given by taking the time derivative $\dot{d} = \dot{R}r = (\dot{R}/R)d = H(t)d$; that is, $H(t) = \dot{R}/R$.

In terms of this notation, the Friedmann equation is

$$\frac{kc^2}{H^2R^2} = \Omega - 1.$$

The relation between ***redshift*** z and the scale factor $R(t)$ reads

$$1 + z = \frac{R_0}{R(t)}.$$

2. Electrodynamics

Note: For this section we switch over to the obnoxious $(+---)$ metric signature.

- **Electromagnetic potentials:** In terms of the potentials, the electric and magnetic fields are

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

In Lorentz gauge (see 'Lorentz gauge condition' below), we have the following conditions:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\rho$$
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{1}{c}\vec{J}.$$

Using the ***wave operator*** $-\partial^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ and the ***current density*** $J^\mu = (\rho c, \vec{J})$, we can rewrite the above equations as

$$\partial^2\phi = \frac{1}{c}J^0$$
$$\partial^2\vec{A} = \frac{1}{c}\vec{J}.$$

Finally, defining the ***gauge field*** (or 4-***vector potential***) as $A^\mu = (\phi(\vec{x},t), \vec{A}(\vec{x},t))$, we end up with

$$\partial^2 A^\mu = \frac{1}{c}J^\mu.$$

Notice that the ***Lorentz gauge condition***

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0.$$

takes on a nice form in terms of A^μ. It now reads

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$
$$\implies \partial_i A^i + \partial_0 A^0 = 0$$
$$\implies \partial_\mu A^\mu = 0.$$

- **Lorentz force:** The ***Lorentz force*** is given by

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right),$$

while the ***Lorentz force density*** is

$$\vec{f} = \rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}.$$

The *relativistic equation of motion for a point charge* q with proper velocity $u^\mu = (\gamma c, \gamma \vec{v})$ and 4-momentum $p^\mu = (E/c, \vec{p})$ in an electromagnetic field is

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{q}{c} F^{\mu\nu} u_\nu,$$

whose spatial components ($\nu = i = 1, 2, 3$) are in fact the Lorentz force law

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right).$$

- **Maxwell Equations:** In Heaviside-Lorentz units:

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \text{(Gauss)} \tag{3}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{(Faraday)} \tag{5}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{\vec{J}}{c} \qquad \text{(Ampère)} \tag{6}$$

In covariant form, (the homogeneous) Maxwell's equations (4) and (5) (in Minkowski spacetime) become

$$\partial_{[\mu} F_{\nu\rho]} = 0,$$

while (the inhomogeneous) equations (3) and (6) become

$$\eta_{\mu\nu} \partial^\mu F^{\nu\rho} = \partial_\nu F^{\nu\rho} = \frac{J^\rho}{c}.$$

To see why the latter is true, note that

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = \frac{J^\nu}{c}.$$

(Note that we used the Lorentz gauge condition $\partial_\mu A^\mu = 0$ on the last step.)

These two covariant equations generalize in the obvious way to curved spacetimes:

$$\nabla_{[\mu} F_{\nu\rho]} = 0$$

$$g_{\mu\nu} \nabla^\mu F^{\nu\rho} = \nabla_\nu F^{\nu\rho} = \frac{J^\rho}{c}.$$

(Note that $\nabla_{[\mu} F_{\nu\rho]}$ reduces to $\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0$, due to the antisymmetry of $F_{\mu\nu}$.)

Here $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the antisymmetric ***electromagnetic tensor field***, given in covariant matrix form as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix},$$

and in contravariant ($F^{\mu\nu} = \eta^{\mu\alpha} F_{\alpha\beta} \eta^{\beta\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu$) matrix form as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

One last thing to mention on this section ... It turns out we can define a ***dual field strength tensor***

$$F^*_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}\partial^\alpha A^\beta,$$

and notice that

$$\partial^\mu F^*_{\mu\nu} = -\frac{1}{2}\partial^\mu\epsilon_{\mu\nu\alpha\beta}(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -\epsilon_{\mu\nu\alpha\beta}\partial^\mu\partial^\alpha A^\beta = 0.$$

This condition can be shown to be equivalent to $\partial_{[\mu}F_{\nu\rho]} = 0$; therefore we finally write ***Maxwell's equations*** in their most elegant, compact form:

$$\boxed{\begin{aligned} \partial_\mu F^{*\mu\nu} &= 0 \\ \partial_\mu F^{\mu\nu} &= \frac{J^\nu}{c} \end{aligned}}$$

- **Energy-Momentum Tensor Field:** The ***energy-momentum tensor field*** defined by the electro-magnetic field is

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu\sigma}\eta^{\rho\sigma} + \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma},$$

which satisfies $\partial^\mu T_{\mu\nu} = 0$ as a consequence of Maxwell's equations (in the sourceless $J^\mu = 0$ case of course!).

In curved spacetime,

The energy-momentum distribution of matter in spacetime is described (without considering sources) by a symmetric $(0,2)$ tensor field T_{ab} that obeys $\nabla^a T_{ab} = 0$.

In general, when sources are considered, the conservation of energy-momentum is expressed by

$$\partial^\mu T_{\mu\nu} = \frac{1}{c}J^\mu T_{\mu\nu}.$$

- **Hamiltonian:**
  - For a free non-relativistic particle:

$$H = \frac{|\vec{p}|^2}{2m}.$$

  - For a particle of charge q interacting with an EM field:

$$H = \frac{|\vec{p} - \frac{q}{c}\vec{A}|^2}{2m} + q\phi.$$

This Hamiltonian is chosen so that the force on the particle due to the EM field is the familiar *Lorentz force*

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right).$$

3. Quantum Theory

- **Klein-Gordon:** The relativistic expression for the total energy of a free particle is

(7)
$$E^2 = |\vec{p}|^2c^2 + m^2c^4.$$

Making the operator substitutions[3]

$$E \mapsto i\hbar\frac{\partial}{\partial t} \qquad \text{and} \qquad \vec{p} \mapsto -i\hbar\vec{\nabla},$$

we can rewrite (7) as

$$-\hbar^2\frac{\partial^2}{\partial t^2}\phi(\vec{r},t) = -\hbar^2c^2\nabla^2\phi(\vec{r},t) + m^2c^4\phi(\vec{r},t),$$

or, in covariant form,

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(x^\mu) = 0.$$

This is the so called ***Klein-Gordon equation***;[4] it has plane-wave solutions

$$\phi(\vec{r},t) = \exp\{i\vec{k}\cdot\vec{r} - i\omega t\},$$

provided that ω, $\vec{k}$, and m are related by

$$\hbar^2\omega^2 = \hbar^2c^2|\vec{k}|^2 + m^2c^4.$$

Defining the four-vector $k^\mu = \left(\frac{\omega}{c}, \vec{k}\right)$, we can write the solution in covariant form

$$\phi(x^\mu) = \exp\left(-ik^\mu x_\mu\right) = \exp\left(-ip^\mu x_\mu/\hbar\right),$$

and thus we interpret the ***four-momentum*** as $p^\mu = \hbar k^\mu$.

[3] In covariant form, this is

$$p^\mu \mapsto \hat{p}^\mu = i\hbar\frac{\partial}{\partial x_\mu} = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right).$$

[4] Note that for a massless particle $m = 0$, the KG equation reduces to the classical wave equation.