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Problem 1 (Exercise 7.8 (Schutz)). *Suppose that in some coordinate system the components of the metric $g_{\alpha\beta}$ are independent of some coordinate x^μ.*

(a) *Show that the conservation law ${T^\nu}_{\mu;\nu} = 0$ for any stress-energy tensor becomes*

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,{T^\nu}_\mu\right)_{,\nu} = 0. \tag{1}$$

(b) *Suppose that in these coordinates $T^{\alpha\beta} \neq 0$ only in some bounded region of each spacelike hypersurface $x^0 = const$. Show that Eq. (1) implies*

$$\int_{x^0=const} \sqrt{-g}\,{T^\nu}_\mu n_\nu \,\mathrm{d}^3x \tag{2}$$

is independent of x^0, if n_ν is the unit normal to the hypersurface.

(c) *Consider flat Minkowski space in a global inertial frame with spherical polar coordinates (t, r, θ, ϕ). Show from (b) that*

$$J = \int_{t=const} {T^0}_\phi \, r^2 \sin\theta \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{3}$$

is independent of t. This is the total angular momentum of the system.

(d) *Express the integral in (c) in terms of the components of $T^{\alpha\beta}$ on the Cartesian basis (t, x, y, z), showing that*

$$J = \int \left(xT^{y0} - yT^{x0}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{4}$$

This is the continuum version of the nonrelativistic expression $(\boldsymbol{r} \times \boldsymbol{p})_z$ for a particle's angular momentum about the z axis.

Solution to (a). Let's first expand Eq. (1):

$$\begin{aligned}\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\,{T^\nu}_\mu\right) &= \partial_\nu {T^\nu}_\mu + \frac{1}{\sqrt{-g}}{T^\nu}_\mu \partial_\nu\sqrt{-g}\\ &= \partial_\nu {T^\nu}_\mu + \frac{1}{2g}{T^\nu}_\mu \partial_\nu g.\end{aligned} \tag{5}$$

Now recall that on a previous exercise (past homework) we showed that

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}\,\partial_\nu(\log|g|) = \frac{1}{2g}\,\partial_\nu g. \tag{6}$$

Moreover, note that

$$\begin{aligned}\Gamma^\sigma_{\mu\nu}{T^\nu}_\sigma &= \frac{1}{2}{T^\nu}_\sigma g^{\sigma\alpha}\left(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}\right)\\ &= \frac{1}{2}{T^\nu}_\sigma g^{\sigma\alpha}\partial_\mu g_{\alpha\nu}\\ &= 0.\end{aligned} \tag{7}$$

On the first equality the terms

$$\frac{1}{2}{T^\nu}_\sigma g^{\sigma\alpha}\left(\partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)$$

vanished by relabeling. On the last equality we get zero because, by assumption on this exercise, we are using a coordinate system where the components of the metric tensor are independent of some coordinate x^μ; therefore $\partial_\mu g_{\alpha\nu} = 0$.

Now, expanding ${T^\nu}_{\mu;\nu}$,

$$\begin{aligned}\nabla_\nu {T^\nu}_\mu &= \partial_\nu {T^\nu}_\mu + \Gamma^\nu_{\sigma\nu}{T^\sigma}_\mu - \overbrace{\Gamma^\sigma_{\mu\nu}{T^\nu}_\sigma}^{=\,0 \text{ by Eq. (7)}}\\ &= \partial_\nu {T^\nu}_\mu + \frac{1}{2g}{T^\sigma}_\mu\,\partial_\sigma g. && \text{(By Eq. (6))}\end{aligned} \tag{8}$$

Relabeling $\sigma \mapsto \nu$ we get the same result as (5). Thus we have proved the validity of (1). ☐

Solution to (b). Here we use Gauss's Law. However, let me clarify something that is quite confusing from the notation of Schutz: Eq. (2) should actually read something like

$$\int_{x^0=\text{const}} \sqrt{\gamma}\, T^{\nu}{}_{\mu}\, n_\nu\, \mathrm{d}^3x, \tag{9}$$

where γ is the detertminant of the 3-metric γ_{ij} of the hypersurface x^0 = constant. Schutz uses g for the determinant of both the 4-metric and the 3-metric, which can lead to some confusion …

Anyhow, proceeding from Eq. (1) and expanding,

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, T^{\nu}{}_{\mu}\right)_{,\nu} = T^{\nu}{}_{\mu;\nu} \\ \implies\ 0 &= \int_{\mathcal{M}} \sqrt{-g}\, T^{\nu}{}_{\mu;\nu}\, \mathrm{d}x^4 = \int_{\mathcal{M}} \left(\sqrt{-g}\, T^{\nu}{}_{\mu}\right)_{,\nu} \mathrm{d}x^4 \\ &= \int_{\partial\mathcal{M}} \sqrt{\gamma}\, T^{\nu}{}_{\mu}\, n_\nu\, \mathrm{d}x^3. \qquad \text{(By Gauss's Law)} \end{aligned} \tag{10}$$

Hence, in some coordinate system where the components of the metric $g_{\alpha\beta}$ are independent of some coordinate x^μ, the integral (10) vanishes regardless of whether $\partial\mathcal{M} = x^0$ constant or it's some other spacelike hyperfurface. □

Solution to (c). We have previously shown that such change of coordinates transforms the volume form as

$$\mathrm{d}^3x = r^2 \sin\theta\, \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi. \tag{11}$$

Moreover, on the hypersurface $x^0 = t$ = constant, the unit normal has components $n_\nu = (1, 0, 0, 0)$; thus

$$T^{\nu}{}_{\mu}\, n_\nu = T^{0}{}_{\mu}\, n_0 = T^{0}{}_{\mu}.$$

Putting all this into Eq. (2), with $\mu = \phi$, we recover Eq. (3). □

Solution to (d). Recall that the change of coordinates from Euclidean to spherical polar in $\mathbb{R}^3$ is given by

$$(x, y, z) \mapsto (r\cos\phi\sin\theta,\ r\sin\phi\sin\theta,\ r\cos\theta).$$

Now we transform $T^{0}{}_{\phi}$ to Cartesian:

$$\begin{aligned} T^{0}{}_{\phi} &= \Lambda^{\alpha}{}_{\phi} T^{0}{}_{\alpha} = \frac{\partial x}{\partial\phi} T^{0}{}_{x} + \frac{\partial y}{\partial\phi} T^{0}{}_{y} + \frac{\partial z}{\partial\phi} T^{0}{}_{z} \\ &= -r\sin\phi\sin\theta T^{0}{}_{x} + r\sin\phi\sin\theta T^{0}{}_{y} + 0\cdot T^{0}{}_{z} \\ &= -yT^{0}{}_{x} + xT^{0}{}_{y} \\ &= -yT^{0x} + xT^{0y}, \end{aligned}$$

where on the last equality we used the fact that the Euclidean metric in Cartesian coordinates is just δ_{ij}. Combining this with Eq. (11) the result follows. □