MATH 725 NOTES
ISOMORPHISM THEOREMS

MARIO L. GUTIERREZ ABED

Theorem 1. *Let S be a subspace of V. The binary relation*

$$u \equiv v \iff u - v \in S$$

is an equivalence relation on V, whose equivalence classes are the cosets

$$v + S = \{v + s \mid s \in S\}$$

of S in V. The set V/S of all cosets of S in V, called the ***quotient space of V modulo S****, is a vector space under the well-defined operations*

$$(u + S) + (v + S) = (u + v) + S$$
$$\alpha(u + S) = \alpha u + S.$$

The zero vector in V/S is the coset $0 + S = S$.

Definition. *If S is a subspace of V then we can define a map $\pi_S : V \to V/S$ by sending each vector to the coset containing it;*

$$\pi_S(v) = v + S.$$

This map is called the ***canonical projection*** *of V onto V/S.* ★

Theorem 2. *The canonical projection $\pi_S : V \to V/S$ defined above is a surjective linear transformation with $\ker(\pi_S) = S$.*

Theorem 3 (**The Correspondence Theorem**)**.** *Let S be a subspace of V. Then the function that assigns to each intermediate subspace T (with $S \subseteq T \subseteq V$) the subspace T/S of V/S, is an order preserving (with respect to set inclusion) one-to-one correspondence between the set of all subspaces of V containing S and the set of all subspaces of V/S.*

The Universal Property of Quotients and the First Isomorphism Theorem

Let S be a subspace of V. The pair $(V/S, \pi_S)$ has a very special property, known as the *universal property* –a term that comes from the world of category theory. Figure 1 below shows a linear transformation $\tau \in \mathcal{L}(V, W)$, along with the canonical projection π_S from V to the quotient space V/S.

The universal property then states that if $S \subseteq \ker(\tau)$, then there is a unique $\tau' : V/S \to W$ for which

$$\tau' \circ \pi_S = \tau.$$

Another way to say this is that any such $\tau \in \mathcal{L}(V, W)$ can be factored through the canonical projection π_S. We formally state this on the following theorem:

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[d, "\pi_S"'] & W \\ V/S \arrow[ur, dashed, "\tau'"'] & \end{tikzcd}$$

FIGURE 1. The universal property.

Theorem 4. *Let S be a subspace of V and let $\tau \in \mathcal{L}(V, W)$ satisfy $S \subseteq \ker(\tau)$. Then, as pictured above on* Figure 1*, there is a unique linear transformation $\tau' : V/S \to W$ with the property that*

$$\tau' \circ \pi_S = \tau.$$

Moreover, $\ker(\tau') = \ker(\tau)/S$ *and also* $\mathrm{Im}(\tau') = \mathrm{Im}(\tau)$*.*

Proof. We have no other choice but to define τ' by the condition $\tau' \circ \pi_S = \tau$; that is,

$$(\tau' \circ \pi_S)(v) = \tau'(\pi_S(v)) = \tau'(v + S) = \tau(v).$$

This function is well-defined if and only if

$$v + S = u + S \Longrightarrow \tau'(v + S) = \tau'(u + S),$$

which is equivalent to each of the following statements:

$$\begin{aligned} v + S = u + S &\Longrightarrow \tau(v) = \tau(u) \\ v - u \in S &\Longrightarrow \tau(u - v) = 0 \\ x \in S &\Longrightarrow \tau(x) = 0 \\ S &\subseteq \ker(\tau). \end{aligned}$$

Thus, $\tau' : V/S \to W$ is well defined. Also,

$$\mathrm{Im}(\tau') = \{\tau'(v + S) \mid v \in V\} = \{\tau(v) \mid v \in V\} = \mathrm{Im}(\tau)$$

and

$$\begin{aligned} \ker(\tau') &= \{v + S \mid \tau'(v + S) = 0\} \\ &= \{v + S \mid \tau(v) = 0\} \\ &= \{v + S \mid v \in \ker(\tau)\} \\ &= \ker(\tau)/S. \end{aligned}$$

The uniqueness of τ' is evident. $\square$

<u>*Remark*</u>: This theorem has a very important corollary, which is often called the *first isomorphism theorem* and is obtained by taking $S = \ker(\tau)$.

Corollary 1 (**The First Isomorphism Theorem**)**.** *Let $\tau : V \to W$ be a linear transformation. Then the linear transformation $\tau' : V/\ker(\tau) \to W$ defined by*

$$\tau'(v + \ker(\tau)) = \tau(v)$$

is injective and furthermore,

$$V/\ker(\tau) \cong \operatorname{Im}(\tau).$$

<u>*Remark*</u>: According to the *first isomorphism theorem*, the image of any linear transformation on V is isomorphic to a quotient space of V. Conversely, any quotient space V/S of V is the image of a linear transformation on V: the canonical projection π_S. Thus, up to isomorphism, quotient spaces are equivalent to homomorphic images.

Quotient Spaces, Complements and Codimension

The *first isomorphism theorem* gives some insight into the relationship between complements and quotient spaces. Let S be a subspace of V and let T be a complement of S, i.e., $V = S \oplus T$. Since every vector $v \in V$ has the form $v = s + t$, for unique vectors $s \in S$ and $t \in T$, we can define a linear operator $\rho : V \to V$ by setting

$$\rho(s + t) = t.$$

Because s and t are unique, ρ is well-defined. It is called the ***projection onto*** T ***along*** S. (Note the word onto (rather than modulo) in the definition; this is not the same as projection modulo a subspace.) It is clear that

$$\operatorname{Im}(\rho) = T$$

and

$$\ker(\rho) = \{s + t \in V \mid t = 0\} = S.$$

Hence, the first isomorphism theorem implies that

$$T \cong V/S.$$

In general we have the following theorem:

Theorem 5. *Let S be a subspace of V. All complements of S in V are isomorphic to V/S and hence to each other.*

<u>*Remark*</u>: The previous theorem can be rephrased by writing

$$A \oplus B = A \oplus C \Longrightarrow B \cong C.$$

On the other hand, quotients and complements do not behave as nicely with respect to isomorphisms as one might casually think:

- It is possible that
$$A \oplus B = C \oplus D$$
with $A \cong C$ but $B \ncong D$. Hence, $A \cong C$ does not imply that a complement of A is isomorphic to a complement of C.
- It is possible that $V \cong W$ and
$$V = S \oplus B \qquad \text{and} \qquad W = S \oplus D$$
but $B \ncong D$. Hence, $V \cong W$ does not imply that $V/S \ncong W/S$. (However, according to the previous theorem, if V equals W then $B \cong D$.)

Corollary 2. *Let S be a subspace of a vector space V. Then*

$$\dim(V) = \dim(S) + \dim(V/S).$$

Definition. *If S is a subspace of V then $\dim(V/S)$ is called the* ***codimension*** *of S in V and it is denoted by $\operatorname{codim}(S)$ or $\operatorname{codim}_V(S)$.* ★

<u>*Remark*</u>: Putting all this together, we have that the codimension of S in V is the dimension of any complement of S in V and when V is finite-dimensional, we have

$$\operatorname{codim}_V(S) = \dim(V) - \dim(S).$$

Additional Isomorphism Theorems

There are several other isomorphism theorems that are consequences of the *first isomorphism theorem*. As we have seen, if $V = S \oplus T$ then $V/T \cong S$. This can be written

$$(S \oplus T)/T \cong S/(S \cap T).$$

This applies to nondirect sums as well, as we shall see on the next theorem.

Theorem 6 (**The Second Isomorphism Theorem**)**.** *Let V be a vector space and let S and T be subspaces of V. Then*

$$(S + T)/T \cong S/(S \cap T).$$

Proof. Let $\tau : (S + T) \to S/(S \cap T)$ be defined by

$$\tau(s + t) = s + (S \cap T).$$

We leave it to the reader to show that τ is a well-defined surjective linear transformation, with kernel T. An application of the *first isomorphism theorem* then completes the proof. □

The following theorem demonstrates one way in which the expression V/S behaves like a fraction:

Theorem 7 (**The Third Isomorphism Theorem**)**.** *Let V be a vector space and suppose that $S \subseteq T \subseteq V$ are subspaces of V. Then*

$$\frac{V/S}{T/S} \cong \frac{V}{T}.$$

Proof. Let $\tau : V/S \to V/T$ be defined by $\tau(v + S) = v + T$. We leave it to the reader to show that τ is a well-defined surjective linear transformation whose kernel is T/S. The rest follows from the *first isomorphism theorem.* □

The following theorem demonstrates one way in which the expression V/S does not behave like a fraction:

Theorem 8. *Let V be a vector space and let S be a subspace of V. Suppose that $V = V_1 \oplus V_2$ and $S = S_1 \oplus S_2$, with $S_i \subseteq V_i$. Then*

$$\frac{V}{S} = \frac{V_1 \oplus V_2}{S_1 \oplus S_2} \cong \frac{V_1}{S_1} \times \frac{V_2}{S_2}.$$

Proof. Let $\tau : V \to (V_1/S_1) \times (V_2/S_2)$ be defined by

$$\tau(v_1 + v_2) = (v_1 + S_1, v_2 + S_2).$$

This map is well-defined, since the sum $V = V_1 \oplus V_2$ is direct. We leave it to the reader to show that τ is a surjective linear transformation, whose kernel is $S_1 \oplus S_2$. The rest follows from the *first isomorphism theorem.* $\square$