Math 3101 HW # 2

Mario L. Gutierrez Abed

(1) Prove that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

Proof:
For the base case $(n = 1)$ we have

$$1 = \frac{1\,(1+1)}{2} = 1\,. \qquad \checkmark$$

Next, assume that the equation holds for $n = k$:

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2}\,.$$

We then want to show that it also holds for $n = k + 1$:

$$1 + 2 + \ldots + k + k + 1 = \frac{(k+1)\,(k+2)}{2}.$$

But $1 + 2 + \ldots + k$ is exactly $\frac{k(k+1)}{2}$.

Hence

$$\begin{aligned} \frac{k(k+1)}{2} + k + 1 &= \frac{k(k+1)+2\,(k+1)}{2} = \frac{k^2+k+2\,k+2}{2} \\ &= \frac{k^2+3\,k+2}{2} = \frac{(k+1)\,(k+2)}{2}. \qquad \checkmark \end{aligned}$$

Thus, it follows by induction that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. ■

(2) Find integers $r,\ s$ such that $\gcd(45,\ 33) = r\,45 + s\,33$.

Solution:
We have

$$\begin{aligned} 45 &= 33 \cdot 1 + 12 \\ 33 &= 12 \cdot 2 + 9 \\ 12 &= 9 \cdot 1 + 3 \\ 9 &= 3 \cdot 3 + 0\,. \end{aligned}$$

Hence

$$\gcd(45,\ 33) = \gcd(45,\ 33) = \gcd(33,\ 12) = \gcd(12,\ 9) = \gcd(9,\ 3) = 3.$$

Now we go backwards

$$\begin{aligned} 3 &= 12 + (-1)\,(9) \\ &= 12 + (-1)[33 + (-2)\,(12)] \\ &= (-1)\,(33) + 3\,(12) \\ &= (-1)\,(33) + 3[45 + (-1)\,(33)] \end{aligned}$$

$$= (3)(45) + (-4)(33).$$

Hence,

$$\gcd(45, 33) = 3 = r\,45 + s\,33 = (3)(45) + (-4)(33) \Longrightarrow r = 3,\ s = -4.$$

(3) Let a, b be integers. It was proven in class that if $\gcd(a, b) = 1$ then there exist integers r, s such that $r\,a + s\,b = 1$. Prove that the converse is also true. Namely prove that if r, s are integers such that $r\,a + s\,b = 1$ then $\gcd(a, b) = 1$.

Proof:

Let $a, b, r, s \in \mathbb{Z}$ such that $r\,a + s\,b = 1$. Assume a, b are not relatively prime. Then there exists a $t \neq 1$ that a and b share as a factor. But this means that there is $j, k \in \mathbb{Z}$ such that $a = t\,j$, $b = t\,k$. We then have

$$\begin{aligned} & r\,a + s\,b = 1 \\ \Longrightarrow\ & r\,t\,j + s\,t\,k = 1 \\ \Longrightarrow\ & t(r\,j + s\,k) = 1 \\ \Longrightarrow\ & r\,j + s\,k = 1/t\ . \end{aligned}$$

But $1/t < 1$ while r, j, s, and k are all integers. We know that integers are well defined and closed under addition and multiplication, meaning that it's impossible to add and multiply integers and end up with a fraction that's not an integer. ($\Rightarrow\Leftarrow$)

This contradiction tells us that if r, s are integers such that $r\,a + s\,b = 1$ then a and b must necessarily be relatively prime, i.e. $\gcd(a, b) = 1$. ■

(4) Let $a, b, c \in \mathbb{Z}$. Prove without using the fundamental theorem of arithmetic that if $\gcd(a, b) = 1$ and $a \mid b\,c$, then we must have that $a \mid c$.

Proof:

Let $a, b, c \in \mathbb{Z}$. If $a \mid b\,c$, there exists $d \in \mathbb{Z}$ such that $a\,d = b\,c$. Also since $\gcd(a, b) = 1$, there are integers r and s such that $a\,r + s\,b = 1$.

Now, multiplying by c we have

$$(a\,r + s\,b)\,c = c \ \Longrightarrow\ a\,r\,c + s\,b\,c = c\ .$$

But $b\,c = a\,d$, so

$$\begin{aligned} a\,r\,c + s\,a\,d = c \ &\Longrightarrow\ a(r\,c + s\,d) = c \\ &\Longrightarrow\ a \mid c. \end{aligned}$$ ■

(5) Give the multiplication table for $U(12) = \{1, 5, 7, 11\}$, the group of elements of $\mathbb{Z}_{12}$ which are invertible relative to multiplication.

Solution:

×	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

(6) Write out the Cayley table for the group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Solution:

+	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 0)	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 1)	(0, 1)	(0, 0)	(1, 1)	(1, 0)
(1, 0)	(1, 0)	(1, 1)	(0, 0)	(0, 1)
(1, 1)	(1, 1)	(1, 0)	(0, 1)	(0, 0)