CORRECTIONS TO MATH 746 MIDTERM

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Problem 1

1A) Define what it means for a function $f\colon E \subset \mathbb{R}^d \to \mathbb{R}$ to be continuous.

Solution. For such function f to be continuous, we must have that $f^{-1}(\mathcal{O}) \subseteq E$ is open for any open set $\mathcal{O} \subset \mathbb{R}$. Analogously, f is continuous if $f^{-1}(\mathcal{F}) \subseteq E$ is closed for any closed set $\mathcal{F} \subset \mathbb{R}$. $\square$

1B) Define what it means for a function $f\colon E \subset \mathbb{R}^d \to \mathbb{R}$ to be (Lebesgue) measurable.

Solution. For such function f to be (Lebesgue) measurable we must have that, for all $a \in \mathbb{R}$, the set

$$f^{-1}\{[-\infty, a)\} = \{x \in E \colon f(x) < a\}$$

is measurable. Similar conclusions hold for whichever combination of strict or weak inequalities one chooses. $\square$

1C) In what sense, according to Littlewood's principles, is a measurable function "almost continuous"?

Solution. According to Littlewood's principles, a measurable function is "almost continuous" in the sense described by *Lusin's Theorem.* This theorem states that if we take any finite-valued measurable function f defined on a set E of finite measure, then for every $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subset E$ with $m(E \setminus F_\varepsilon) \leq \varepsilon$, and such that $f|_{F_\varepsilon}$ is continuous. $\square$

Problem 2

2A) State the definition of outer measure of a subset $E \subset \mathbb{R}^d$.

Solution. The outer measure of a subset $E \subset \mathbb{R}^d$ is given by

$$m_*(E) = \inf \sum_{k=1}^{\infty} |\mathcal{Q}_k|,$$

where the infimum is taken over all countable coverings by closed cubes $\cup_{k=1}^{\infty} \mathcal{Q}_k \supset E$. □

2B) State the definition of measure of a subset $E \subset \mathbb{R}^d$.

Solution. The (Lebesgue) measure of a set E is the same as its outer measure, provided that E is a measurable set. That is, if E is a measurable set, then $m(E) = m_*(E)$. We know that E is a measurable set if, for any $\varepsilon \geq 0$, there exists an open set $\mathcal{O} \supset E$ such that $m_*(\mathcal{O} \setminus E) \leq \varepsilon$. Similarly, E is measurable if, for any $\varepsilon \geq 0$, there exists a closed set $\mathcal{F} \subset E$ such that $m_*(E \setminus \mathcal{F}) \leq \varepsilon$. Another way to check whether E is measurable is detailed in the following problem. □

2C) Prove that if E is measurable in $\mathbb{R}^d$, then for any subset A of $\mathbb{R}^d$, we have

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c).$$

Solution. We know from a previous result that outer measure is countably sub-additive. We also know that for any sets A and E, $A = (A \cap E) \cup (A \cap E^c)$. Combining these results we have

$$m_*(A) \leq m_*(A \cap E) + m_*(A \cap E^c).$$

Therefore, E is measurable if and only if for each set A, we have

$$m_*(A) \geq m_*(A \cap E) + m_*(A \cap E^c). \tag{1}$$

This inequality trivially holds if $m_*(A) = \infty$. Thus it suffices to establish (1) for sets A that have finite outer measure.

We know that the definition of measurability is symmetric in E and E^c, and therefore a set is measurable if and only if its complement is measurable, as shown in *Problem 2D)*. Clearly $\emptyset$ and $\mathbb{R}$ are measurable. Hence we establish inequality (1) by proving the following proposition:

Proposition 1. *Any set of outer measure zero is measurable. In particular, any countable set is measurable.*

Proof of Proposition 1. Let the set E have outer measure zero and let A be any set. Since

$$A \cap E \subseteq E \text{ and } A \cap E^c \subseteq A,$$

by the monotonicity of outer measure,

$$m_*(A \cap E) \leq m_*(E) = 0 \text{ and } m_*(A \cap E^c) \leq m_*(A).$$

Thus,

$$m_*(A) \geq m_*(A \cap E^c) = 0 + m_*(A \cap E^c) = m_*(A \cap E) + m_*(A \cap E^c),$$

and therefore E is measurable. $\surd$

This shows that if E is measurable in $\mathbb{R}^d$, then for any subset A of $\mathbb{R}^d$, we have $m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$, as desired. $\square$

2D) Prove that if E is measurable in $\mathbb{R}^d$, then E^c is also measurable in $\mathbb{R}^d$.

Solution. On the original exam I wrote something like:

"If E is a measurable set, then E is a set which belongs to the (Lebesgue) σ-algebra of all measurable sets, which includes the Borel sets and the null sets. Since by definition, σ-algebras are closed under complements, if E is a Lebesgue set, then so is E^c"

While I don't think that this argument is wrong, I'm afraid that you may have been looking for a more rigorous explanation, so here it is:

If E is measurable, then for every positive integer n we may choose an open set $\mathcal{O}_n$ with $E \subset \mathcal{O}_n$ and $m_*(\mathcal{O}_n \setminus E) \leq 1/n$. The complement $\mathcal{O}_n^c$ is closed, hence measurable (since closed sets are measurable), which implies that the union $S = \bigcup_{n=1}^{\infty} \mathcal{O}_n^c$ is also measurable by the property that says that all countable unions of measurable sets are measurable.

Now we simply note that $S \subset E^c$, and

$$(E^c \setminus S) \subset (\mathcal{O}_n \setminus E),$$

such that $m_*(E^c \setminus S) \leq 1/n$ for all n. Therefore, $m_*(E^c \setminus S) = 0$, and $E^c \setminus S$ is measurable by the property that says that any set of outer measure zero is measurable. Therefore E^c is measurable since it is the union of two measurable sets, namely S and $(E^c \setminus S)$. $\square$

Problem 3

3A) Give an example or prove the impossibility of the existence of a non-measurable set in $\mathbb{R}$.

Solution. Let us take the interval $[0,1]$ and define the relation

$$\text{Let } x \sim y \text{ whenever } x - y \in \mathbb{Q}.$$

Note that this is an equivalence relation, since the reflexive, symmetric, and transitive properties hold. We know that equivalence classes partition a set into distinct cells, thus the interval $[0,1]$ is the disjoint union of all equivalence classes that are defined on this interval, i.e.

$$[0,1] = \bigcup_{\alpha} \mathcal{E}_\alpha,$$

where each $\mathcal{E}_\alpha$ represents a unique equivalence class.

Now we construct the (Vitali) set $\mathcal{N}$ by choosing exactly one element x_α from each $\mathcal{E}_\alpha$ (this is justified by using the axiom of choice), and setting $\mathcal{N} = \{x_\alpha\}$.

The important result is stated in the following proposition:

Proposition 2. *The Vitali set $\mathcal{N}$ constructed above is not measurable.*

Proof of Proposition 2. Assume that $\mathcal{N}$ is measurable. Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of all the rationals in $[-1,1]$, and consider the translates

$$\mathcal{N}_k = \mathcal{N} + r_k$$

Note that the sets $\mathcal{N}_k$ are disjoint. To see why this is true, suppose that the intersection $\mathcal{N}_k \cap \mathcal{N}_{k'}$ is nonempty. Then there exist rationals $r_k \neq r_{k'}$ and α and β with

$$x_\alpha + r_k = x_\beta + r_{k'}$$

which implies that

$$x_\alpha - x_\beta = r_{k'} - r_k.$$

But this means that $\alpha \neq \beta$ and $x_\alpha - x_\beta$ is rational, which in turn implies that $x_\alpha \sim x_\beta$. This contradicts the fact that $\mathcal{N}$ contains only one representative of each equivalence class. Now we make the claim that

$$[0,1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \subset [-1,2] \tag{2}$$

To see why, notice that if $x \in [0,1]$, then $x \sim x_\alpha$ for some α, and therefore $x - x_\alpha = r_k \implies x = x_\alpha + r_k$ for some k. Hence $x \in \mathcal{N}_k$ for some k and the first inclusion holds. The second inclusion above is straightforward since each $\mathcal{N}_k$ is contained in $[-1,2]$ by construction.

Now we may conclude the proof of the theorem. If $\mathcal{N}$ were measurable, then so would be $\mathcal{N}_k$ for all k, and since the union $\bigcup_{k=1}^{\infty} \mathcal{N}_k$ is disjoint, the inclusions in (2) yield

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq 3.$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we must have $m(\mathcal{N}_k) = m(\mathcal{N})$ for all k. Consequently,

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3.$$

This is the desired contradiction, since neither $m(\mathcal{N}) = 0$ nor $m(\mathcal{N}) > 0$ is possible. $(\Rightarrow\Leftarrow)$

In other words, $m(\mathcal{N}) = 0$ is not possible by the above inequality, and $m(\mathcal{N}) > 0$ is not possible either because we are trying to find the measure of a countable set, which would have measure zero if any. $\checkmark$

Thus we have constructed a non-measurable set in $\mathbb{R}$, as desired. $\square$

3B) Give an example or prove the impossibility of the existence of a function that is not Riemann integrable over a closed interval in $\mathbb{R}$ but whose absolute value is.

Solution. Let us define the function $\widehat{\chi}_{[0,1]} \colon [0,1] \to \{-1, 1\}$ by

$$\widehat{\chi}_{[0,1]}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ -1, & \text{otherwise.} \end{cases}$$

This function is discontinuous everywhere on the closed interval $[0,1]$. Hence it is not Riemann integrable, since the set of discontinuities has positive measure. However, the absolute value $|\widehat{\chi}_{[0,1]}|$ is Riemann integrable and has value $\int_0^1 |\widehat{\chi}_{[0,1]}|\, dx = \int_0^1 (1)\, dx = 1$. $\square$

3C) Give an example or prove the impossibility of the existence of a function that is not Lebesgue integrable over a closed interval in $\mathbb{R}$ but whose absolute value is.

Solution. I missed the explanation on Wednesday's class but after giving it some thought I think I now understand what I did wrong. I answered the question saying that no such example exists because we know from a previous result that if the absolute value of a measurable function f is integrable, then we are guaranteed that f is also integrable. This follows from the following definition and proposition:

Definition 1. *The Lebesgue integral of a measurable function f is defined by*

$$\int f = \int f^+ - \int f^-.$$

Proposition 3. *Let f be a measurable function on E. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.*

Proof of Proposition 3 Assume f^+ and f^- are integrable nonnegative functions. By the linearity of integration for nonnegative functions, $|f| = f^+ + f^-$ is integrable over E. Conversely, suppose $|f|$ is integrable over E. Since $0 \le f^+ \le |f|$ and $0 \le f^- \le |f|$ on E, we infer from the monotonicity of integration for nonnegative functions that both f^+ and f^- are integrable over E. $\surd$

The key here is that we have to come up with a function that is not measurable, since otherwise the function is necessarily integrable whenever its absolute value is, according to the above proposition. Hence, to find a function that is not Lebesgue integrable over a closed interval in $\mathbb{R}$ but whose absolute value is, we take $\widetilde{\chi}_{[-1,2]} \colon [-1,2] \to \{-1,1\}$, defined by

$$\widetilde{\chi}_{[-1,2]}(x) = \begin{cases} -1, & \text{if } x \in \mathcal{N}_k, \\ 1, & \text{otherwise,} \end{cases}$$

where the $\mathcal{N}_k$ are the Vitali translates defined on *Problem 3A)*, whose union is contained in $[-1,2]$, i.e. $\bigcup_{k=1}^{\infty} \mathcal{N}_k \subset [-1,2]$.

Now notice that $\widetilde{\chi}_{[-1,2]}$ is not a measurable function since $\widetilde{\chi}^{-1}_{[-1,2]}(-1) = \mathcal{N}_k$ is not measurable. Hence $\widetilde{\chi}_{[-1,2]}$ is not Lebesgue integrable either, while its absolute value is Lebesgue integrable with $\int_{[-1,2]} |\widetilde{\chi}_{[-1,2]}(x)|\, dx = \int_{[-1,2]} (1)\, dx = 3$. $\square$

3D) Give an example or prove the impossibility of the existence of a non-measurable set of outer measure zero.

Solution. No such example can possibly exist because any set of outer measure zero is by definition measurable. In fact, any subset of a set of outer measure zero is measurable. To see why this is true, recall a property of outer measure that says that if $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ containing E. It follows from this property that, for every $\varepsilon > 0$, there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(O) \le \varepsilon$. Since $(\mathcal{O} \setminus E) \subset \mathcal{O}$, monotonicity implies $m_*(\mathcal{O} \setminus E) \le \varepsilon$, as desired. $\square$

Problem 4

4A) Outline the specification of the Lebesgue integral in $\mathbb{R}$, making reference to the *Monotone Convergence Theorem* and to *Fatou's Lemma.*

Solution. We are going to proceed outlining some details of the Lebesgue integral in four stages, starting with simple functions, then bounded functions supported on a set of finite measure, then non-negative functions, and lastly we conclude with the general case of all integrable functions.

0.0.1. *Simple Functions.* A simple function φ is a finite sum of the form $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, where the a_k are constants and the E_k are measurable sets. To avoid ambiguities however, we want to define the canonical form of φ. Since φ can only take finitely many distinct and non-zero values, say $a_1, ..., a_M$, we may set

$$S_k = \{x \colon \varphi(x) = a_k\}$$

and note that the sets S_k are disjoint. Therefore

$$\varphi = \sum_{k=1}^{M} a_k \chi_{S_k}$$

is the desired canonical form of φ.

Now we define the Lebesgue integral for the class of simple functions as

$$\int \varphi \, dx = \int \sum_{k=1}^{M} a_k \, \chi_{S_k} \, dx = \sum_{k=1}^{M} a_k \, m(S_k),$$

where φ is in canonical form.

0.0.2. *Bounded Functions Supported on a Set of Finite Measure.* An important result for this class of functions is that if f is a function bounded by some bound M and supported on a set E, then there exists a sequence $\{\varphi_n\}$ of simple functions, with each φ_n bounded by M and supported on E, and such that $\varphi_n(x) \to f(x) \, \forall x$. As a consequence, for this class of functions we have a very important result known as the *Bounded Convergence Theorem*, which is stated on *Problem 4D).*

Here's a key lemma for this class of functions:

Lemma 1. *Let f be a bounded function supported on a set E of finite measure. If $\{\varphi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by M, supported on E, and with $\varphi_n(x) \to f(x)$ for a.e. x, then:*
(i) $\lim_{n\to\infty} \int \varphi_n$ *exists.*
(ii) if $f = 0$ a.e., then $\lim_{n\to\infty} \int \varphi_n = 0$.

Remark: Using the above lemma we can now turn to the integration of bounded functions that are supported on sets of finite measure. For such a function f we define its Lebesgue integral by

$$\int f(x)\,dx = \lim_{n\to\infty}\int \varphi_n(x)\,dx,$$

where $\{\varphi_n\}$ is any sequence of simple functions satisfying:
(i) $|\varphi_n| \le M$.
(ii) each φ_n is supported on the support of f.
(iii) $\varphi_n(x) \to f(x)$ for a.e. x as n tends to infinity (we know by the above lemma that this limit exists).

0.0.3. *Non-negative Functions.* This is the class of functions that are measurable and non-negative but not necessarily bounded. We define the Lebesgue integral of such functions by

$$\int f(x)\,dx = \sup_g \int g(x)\,dx,$$

where this supremum is taken over all measurable functions g such that $0 \le g \le f$, and where g is bounded and supported on a set of finite measure.

Remark: With the above definition of the integral, there are only two possible cases: the supremum is either finite, or infinite. In the case where $\int f(x)\,dx < \infty$, we shall say that f is (Lebesgue) integrable.

It is in this class of functions that we find the following two key results:

Lemma 2 (*Fatou's Lemma*)**.** *Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then*

$$\int \lim_{n\to\infty} f_n = \int f \le \liminf_{n\to\infty}\int f_n.$$

Before stating the next result, let us define the following notation:

Notation 1. *$f_n \nearrow f$ refers to a sequence $\{f_n\}$ of monotonically increasing functions that are converging to the limit f as $n \to \infty$ a.e. x.*

Now here's the theorem:

Theorem 1 (*Monotone Convergence Theorem*)**.** *Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$. Then*

$$\lim_{n\to\infty}\int f_n = \int f.$$

0.0.4. *Non-negative Functions.* We have arrived at the general case that includes all (Lebesgue) integrable functions. We start by defining

$$f^+(x) = \max(f(x), 0) \text{ and } f^-(x) = \max(-f(x), 0)$$

so that both f^+ and f^- are non-negative and

$$f^+ - f^- = f.$$

Since $f^\pm \leq |f|$, both functions f^+ and f^- are integrable whenever f is (by the result obtained above for the class of non-negative functions). Now we define the Lebesgue integral of f by

$$\int f = \int f^+ - \int f^-.$$

Now we conclude this short outline of the theory of the Lebesgue integral. □

4B) State some of the principal properties of the Lebesgue integral in $\mathbb{R}$.

Solution. The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality. In other words, the Lebesgue integral satisfies:

i) *Linearity*
If f and g are integrable, and $\alpha, \beta \in \mathbb{R}$, then

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g.$$

ii) *Additivity*
If E_1 and E_2 are disjoint, and f is integrable, then

$$\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f.$$

iii) *Monotonicity*
If $f \leq g$, where f and g are integrable, then

$$\int f \leq \int g.$$

iv) *Triangle Inequality*
If f is integrable, then so is $|f|$ and

$$\left| \int f \right| \leq \int |f| \,.$$

That concludes our short outline of some of the principal properties of the Lebesgue integral. □

4C) State the *Bounded Convergence Theorem.*

Solution. **Theorem)** Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by some bound M, are supported on a set E of finite measure, and $f_n(x) \to f(x)$ a.e. x as $n \to \infty$. Then f is measurable, bounded, supported on E for a.e. x, and

$$\int |f_n - f| \to 0 \text{ as } n \to \infty,$$

which in turn implies that

$$\int f_n \to \int f \text{ as } n \to \infty.$$

□