It's time to get real!

In this course we have concentrated almost exclusively on complex Lie algebras. (It is technically easier to work over algebraically closed fields.) However in many applications of Lie algebras to Physics, Geometry, Analysis,... it is the real Lie algebras which play a dominant rôle. The purpose of this (the final) workshop in the course is to explore real Lie algebras and their relation with the complex Lie algebras we have been studying.

The definition of a **real Lie algebra** mimics that of a complex Lie algebra, except we replace the complex numbers for the reals: so it is a real vector space and the bracket is real bilinear, but still obeying skewsymmetry and Jacobi.
Let L be a *complex* Lie algebra. By a **conjugation** on L we mean a map $c : \mathrm{L} \to \mathrm{L}$ satisfying the following conditions:
(C1) c is complex antilinear:

$$c(\alpha x + \beta y) = \bar{\alpha} c(x) + \bar{\beta} c(y) \qquad \forall \alpha, \beta \in \mathbb{C},\ x, y \in \mathrm{L},$$

(C2) c is involutive: $c(c(x)) = x$ for all $x \in \mathrm{L}$, and

(C3) c preserves the bracket: $c([x, y]) = [c(x), c(y)]$ for all $x, y \in \mathrm{L}$.
One often uses the notation $c(x) = \bar{x}$, so that the complex antilinearity condition is naturally written as $\overline{\alpha x} = \bar{\alpha}\bar{x}$.
Now let L be a *real* Lie algebra and let $\mathrm{L}_{\mathbb{C}} = \mathrm{L} \oplus i\mathrm{L}$. We write $x + iy \in \mathrm{L}_{\mathbb{C}}$ where $x, y \in \mathrm{L}$. Then we turn $\mathrm{L}_{\mathbb{C}}$ into a complex vector space by $i(x + iy) = -y + ix$ and we turn $\mathrm{L}_{\mathbb{C}}$ into a complex Lie algebra by extending the bracket $\mathbb{C}$-linearly:

$$[x + iy, u + iv] := [x, u] - [y, v] + i\left([x, v] + [y, u]\right).$$

The complex Lie algebra $\mathrm{L}_{\mathbb{C}}$ is called the **complexification** of the real Lie algebra L.

1. Let L be a real Lie algebra and $\mathrm{L}_{\mathbb{C}}$ its complexification. Show that a real basis for L gives a complex basis for $\mathrm{L}_{\mathbb{C}}$ and deduce that the complex dimension of $\mathrm{L}_{\mathbb{C}}$ equals the real dimension of L.

Solution: Let $\{e_a\}$ be a real basis for L so that every $x \in \mathrm{L}$ can be written in a unique way as $x = \sum_a x_a e_a$ for some $x_a \in \mathbb{R}$. Then let $x + iy \in \mathrm{L}_{\mathbb{C}}$. We may write it as

$$x + iy = \sum_a x_a e_a + i \sum_a y_a e_b = \sum_a (x_a + i y_a) e_a,$$

so that the $\{e_a\}$ span $\mathrm{L}_{\mathbb{C}}$. To show that they are linearly independent, suppose that $\sum_a z_a e_a = 0 \in \mathrm{L}_{\mathbb{C}}$ for some $z_a \in \mathbb{C}$. Then decompose $z_a = x_a + i y_a$ into real and imaginary parts, so that

$$0 = \sum_a x_a e_a + i \sum_a y_a e_a \in \mathrm{L} \oplus i\mathrm{L}.$$

This means that each term must be zero separately. Since the e_a are a basis for L, it follows that $x_a = y_a = 0$, so $z_a = 0$. The complex dimension of $\mathrm{L}_{\mathbb{C}}$ is the cardinality of any complex basis (e.g., the $\{e_a\}$); whereas the real dimension of L is the cardinality of any real basis (e.g., the $\{e_a\}$).

2. Let L be a complex Lie algebra with a conjugation c. Let $\mathrm{K} := \{x \in \mathrm{L} \mid c(x) = x\}$.
   (a) Show that K is a real Lie subalgebra of L; that is, show that it is a real subspace and that it is closed under the bracket.

Solution: First of all, we show that K is a real subspace, so that it is closed under real linear combinations. So let $\alpha, \beta \in \mathbb{R}$ and $x, y \in K$. Then

$$\overline{\alpha x + \beta y} = \overline{\alpha x} + \overline{\beta y} = \alpha\bar{x} + \beta\bar{y} = \alpha x + \beta y \implies \alpha x + \beta y \in K,$$

where we have used that α and β are real and that $x, y \in K$.
Next we show that K is closed under the bracket. Let $x, y \in K$, then

$$\overline{[x,y]} = [\bar{x}, \bar{y}] = [x,y] \implies [x,y] \in K.$$

(b) Show that $K_{\mathbb{C}} \cong L$ as complex Lie algebras.

Solution: Let $\{e_a\}$ be a basis for K. Each $e_a \in L$ and $\overline{e_a} = e_a$. The e_a are linearly independent over $\mathbb{R}$, but then they are also linearly independent over $\mathbb{C}$. Indeed, let $\sum_a z_a e_a = 0$ for $z_a \in \mathbb{C}$. Then conjugate that equation to arrive at

$$\sum_a \overline{z_a} e_a = 0 \implies \sum_a (z_a + \overline{z_a}) e_a = 0 \implies z_a + \overline{z_a} = 0,$$

but also

$$\sum_a i(z_a - \overline{z_a}) e_a = 0 \implies i(z_a - \overline{z_a}) = 0 \implies z_a - \overline{z_a} = 0.$$

Adding the two conditions we find that $z_a = 0$. Therefore $K_{\mathbb{C}} = L$ as vector spaces. The Lie brackets on $K_{\mathbb{C}}$ are obtained by extending complex linearly those for K, so they are determined by the structure constants of K relative to the e_a, but these are also the structure constants of L, since K is a (real) subalgebra. This shows, by the way, that a complex Lie algebra with a conjugation admits a basis where the structure constants are real. This is certainly not the case for an arbitrary complex Lie algebra.

3. Let L be a real Lie algebra and $L_{\mathbb{C}}$ its complexification. Show that if $L_{\mathbb{C}}$ is simple, then so is L.

Solution: Let $J \subset L$ be an ideal. Then J is a real Lie algebra in its own right which we can complexify to $J_{\mathbb{C}}$. One checks that it is an ideal of $L_{\mathbb{C}}$. Since $L_{\mathbb{C}}$ is simple, $J_{\mathbb{C}}$ is either 0 in which case $J = 0$ or else $J_{\mathbb{C}} = L_{\mathbb{C}}$ in which case $J = L$ (using the dimension count in Q1).

Let us define the following real vector spaces:

- $\mathfrak{sl}_2(\mathbb{R}) :=$ {traceless 2×2 real matrices}
- $\mathfrak{su}_2 :=$ {traceless 2×2 skewhermitian complex matrices}
- $\mathfrak{so}_3 :=$ {skewsymmetric 3×3 real matrices}
- $\mathfrak{so}_{2,1} := \{\text{real } 3 \times 3 \text{ matrices } M \mid M^T\eta + \eta M = 0\}$, where $\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. (a) Show that $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$ are real Lie algebras and that that they are not isomorphic. (*Hint*: compare their Killing forms.)

Solution: The space $\mathfrak{gl}_2(\mathbb{R})$ of real 2×2 matrices is a real vector space and a real Lie algebra under the commutator. (The proof is just as for the complex case.) The trace defines a (surjective) linear map $\mathrm{tr} : \mathfrak{gl}_2(\mathbb{R}) \to \mathbb{R}$, which is a Lie algebra homomorphism if we view $\mathbb{R}$ as an abelian Lie algebra: this follows from $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, as in the complex case. The kernel of tr (i.e., $\mathfrak{sl}_2(\mathbb{R})$) is therefore an ideal, so a Lie algebra in its own right. By rank-nullity it is a 3-dimensional real Lie algebra.

As we saw in the first lecture of the course, the space $\mathfrak{gl}_2(\mathbb{C})$ of complex 2×2 matrices is complex vector space and a complex Lie algebra under the commutator. Hermitian conjugation (=complex conjugation and transposition) defines a complex antilinear map $\dagger : \mathfrak{gl}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C})$, which reverses the order of matrix multiplication: $(xy)^\dagger = y^\dagger x^\dagger$. The subspace of skew-hermitian matrices (i.e., consisting of those $x \in \mathfrak{gl}_2(\mathbb{C})$ such that $x^\dagger = -x$) is a real vector space: if $\alpha, \beta \in \mathbb{R}$ and x, y are skewhermitian,

$$(\alpha x + \beta y)^\dagger = (\alpha x)^\dagger + (\beta y)^\dagger = -\alpha x - \beta y = -(\alpha x + \beta y),$$

hence so is any real linear combination. Similarly,

$$[x, y]^\dagger = (xy - yx)^\dagger = y^\dagger x^\dagger - x^\dagger y^\dagger = yx - xy = -[x, y].$$

Therefore the space of skewhermitian matrices is a real Lie algebra. As before, the trace defines a surjective Lie algebra homomorphism to $i\mathbb{R}$, and hence its kernel $\mathfrak{su}_2$ is an ideal and in particular a real Lie algebra.

To show that $\mathfrak{sl}_2(\mathbb{R}) \not\cong \mathfrak{su}_2$, we will calculate their Killing forms and show that they have different signatures. By Sylvester's Law of Inertia, they are not related by a change of basis, which would be the case if the Lie algebras were isomorphic.

As a basis for $\mathfrak{sl}_2(\mathbb{R})$ we can take the basis e, f, h for $\mathfrak{sl}_2$, since these matrices are real, traceless and linearly independent over $\mathbb{C}$, so in particular over $\mathbb{R}$. We calculated the Killing form in PS7 and its matrix relative to this basis is

$$\kappa_{\mathfrak{sl}_2(\mathbb{R})} = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

which has indefinite signature.

A matrix $x \in \mathfrak{su}_2$ is skewhermitian and traceless, so it takes the form

$$x = \begin{pmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & -i\alpha \end{pmatrix} = \alpha \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \alpha(ih) + \beta(e - f) + \gamma(i(e + f)),$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ and where e, f, h are the usual complex basis for $\mathfrak{sl}_2$. We can now calculate the Killing form relative to the basis $\{e - f, i(e + f), ih\}$ from that of $\mathfrak{sl}_2$, and we find

$$\kappa_{\mathfrak{su}_2} = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix},$$

which is (negative) definite.

(b) Show that their complexifications are isomorphic and, in fact, isomorphic to the complex Lie algebra $\mathfrak{sl}_2$. Deduce that $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$ are simple real Lie algebras.

Solution: To show this all we need to observe is that if we take complex linear combinations of the bases for $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$, we span $\mathfrak{sl}_2$. This is particularly evident for $\mathfrak{sl}_2(\mathbb{R})$, since the same basis works: $\{e, f, h\}$. But also for $\mathfrak{su}_2$, we have that

$$h = -i(ih) \qquad e = \tfrac{1}{2}(e - f) - \tfrac{i}{2}(i(e + f)) \qquad \text{and} \qquad f = -\tfrac{i}{2}(i(e + f)) - \tfrac{1}{2}(e - f).$$

Since $\mathfrak{sl}_2$ is simple, by Q3 so are $\mathfrak{su}_2$ and $\mathfrak{sl}_2(\mathbb{R})$.

5. (a) By considering the adjoint representation of $\mathfrak{su}_2$, show that $\mathfrak{su}_2 \cong \mathfrak{so}_3$ as real Lie algebras.

Solution: The adjoint representation of $\mathfrak{su}_2$ is faithful, since $\mathfrak{su}_2$ is simple. Relative to the basis $\{e - f, i(e + f), ih\}$, every $\mathrm{ad}(x) \in \mathrm{ad}(\mathfrak{su}_2)$ is a real 3×3 matrix. The associativity condition of the Killing form says that, as matrices,

$$\mathrm{ad}(x)^{\mathrm{T}}\kappa_{\mathfrak{su}_2} + \kappa_{\mathfrak{su}_2}\,\mathrm{ad}(x) = 0.$$

From the explicit form of $\kappa_{\mathfrak{su}_2}$ in Q4, this says that $\mathrm{ad}(x)$ is skewsymmetric. In other words, $\mathrm{ad} : \mathfrak{su}_2 \to \mathfrak{so}_3$, but by dimension count this is an isomorphism.

(b) By considering the adjoint representation of $\mathfrak{sl}_2(\mathbb{R})$, show that $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}$ as real Lie algebras.

Solution: This is a little trickier in that the adjoint representation of $\mathfrak{sl}_2(\mathbb{R})$, which can be read off from the adjoint representation of $\mathfrak{sl}_2$ worked out in PS6, obeys

$$\mathrm{ad}(x)^{\mathrm{T}}\kappa_{\mathfrak{sl}_2(\mathbb{R})} + \kappa_{\mathfrak{sl}_2(\mathbb{R})}\,\mathrm{ad}(x) = 0 \qquad \forall x \in \mathfrak{sl}_2(\mathbb{R})$$

and $\kappa_{\mathfrak{sl}_2(\mathbb{R})} \neq \eta$ in the definition of $\mathfrak{so}_{2,1}$. Nevertheless there is a change of basis in $\mathfrak{sl}_2(\mathbb{R})$ which makes the matrix of $\kappa_{\mathfrak{sl}_2(\mathbb{R})}$ equal to a scalar multiple of η, namely: the Killing form relative to $\{e-f, e+f, h\}$ has matrix 8η. Again a dimension count shows that the linear map $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{so}_{2,1}$ induced by ad is an isomorphism.