MATH 709 NOTES
TANGENT VECTORS

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJIAN

Geometric Tangent Vectors

Definition. *Given a fixed point $a \in \mathbb{R}^n$, let us define the* ***geometric tangent space to*** $\mathbb{R}^n$ ***at*** a*, denoted by $\mathbb{R}^n_a$, to be the set $\{a\} \times \mathbb{R}^n = \{(a, v) \mid v \in \mathbb{R}^n\}$. A* ***geometric tangent vector*** *in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$.* ★

Remark 1: As a matter of notation, we abbreviate (a, v) as v_a (or sometimes $v|_a$ if it is clearer, for example if v itself has a subscript). We think of v_a as the vector v with its initial point at a (see *Figure 1* below).

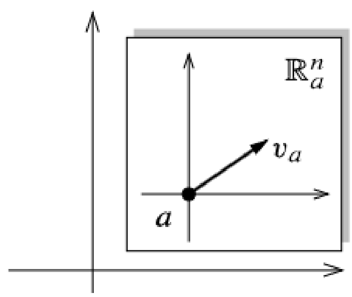


Figure 1. Geometric tangent space.

Remark 2: Note that the geometric tangent space $\mathbb{R}^n_a$ is a real vector space under the natural operations

$$v_a + w_a = (v + w)_a \qquad \text{and} \qquad c(v_a) = (cv)_a \qquad \forall\ v_a, w_a \in \mathbb{R}^n_a, \quad \forall\ c \in \mathbb{R}.$$

One thing that a geometric tangent vector provides is a means of taking directional derivatives of functions. For example, any geometric tangent vector $v_a \in \mathbb{R}^n_a$ yields a map $D_v|_a \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$, which takes the directional derivative in the direction v at a:

$$D_v\big|_a f = D_v f(a) = \frac{d}{dt}\Big|_{t=0} f(a + tv).$$

This operation is linear over $\mathbb{R}$ and satisfies the product rule:

$$D_v\big|_a (fg) = f(a) D_v\big|_a g + g(a) D_v\big|_a f.$$

If $v_a = v^i e_i\big|_a$ in terms of the standard basis, then by the chain rule $D_v\big|_a f$ can be written more concretely as

$$D_v\big|_a f = v^i \frac{\partial f}{\partial x^i}(a).$$

With this construction in mind, we make the following definition:

Definition. *If a is a point of $\mathbb{R}^n$, a map $\varpi\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a* ***derivation at*** *a if it is linear over $\mathbb{R}$ and satisfies the product rule $\varpi(fg)(a) = f(a)\varpi(g) + g(a)\varpi(f)$.* ★

Remark: Note that the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a, denoted $\mathfrak{D}_a\mathbb{R}^n$, is a real vector space under the operations

$$(\varpi_1 + \varpi_2)(f) = \varpi_1(f) + \varpi_2(f) \qquad \text{and} \qquad (c\,\varpi)(f) = c(\varpi(f)).$$

The most important (and perhaps somewhat surprising) fact about $\mathfrak{D}_a\mathbb{R}^n$ is that it is finite-dimensional, and in fact is naturally isomorphic to the geometric tangent space $\mathbb{R}^n_a$ that we defined above. The proof will be based on the following lemma:

Lemma 1 (**Properties of Derivations**)**.** *Suppose $a \in \mathbb{R}^n$, $\varpi \in \mathfrak{D}_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$. Then we have the following:*

a) If f is a constant function, then $\varpi(f) = 0$.

b) If $f(a) = g(a) = 0$, then $\varpi(fg) = 0$.

The next proposition shows that derivations at a are in one-to-one correspondence with geometric tangent vectors:

Proposition 1. *Let $a \in \mathbb{R}^n$. Then,*

a) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v\big|_a\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at a.

b) The map $v_a \mapsto D_v\big|_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $\mathfrak{D}_a\mathbb{R}^n$.

Corollary 1. *For any $a \in \mathbb{R}^n$, the n derivations*

$$\frac{\partial}{\partial x^1}\bigg|_a, \dots, \frac{\partial}{\partial x^n}\bigg|_a \quad \text{defined by} \quad \frac{\partial}{\partial x^i}\bigg|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a basis for $\mathfrak{D}_a\mathbb{R}^n$, which therefore has dimension n.

Proof. Apply the previous proposition and note that $\partial/\partial x^i\big|_a = D_{e_i}\big|_a$. □

Tangent Vectors on Manifolds

Now we are in a position to define tangent vectors on manifolds:

Definition. *Let M be a smooth manifold (with or without boundary), and let p be a point of M. A linear map. A linear map $\nu\colon C^\infty(M) \to \mathbb{R}$ is called a* ***derivation at*** *p if it satisfies the product rule*

$$\nu(fg)(p) = f(p)\nu(g) + g(p)\nu(f) \qquad \text{for all } f, g \in C^\infty(M).$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the ***tangent space to*** *M* ***at*** *p. An element of T_pM is called a* ***tangent vector at*** *p.* ★

The following lemma is the analogue of *Lemma 1* for manifolds:

Lemma 2 (Properties of Tangent Vectors on Manifolds). *Suppose M is a smooth manifold (with or without boundary), $p \in M$, $\nu \in T_pM$, and $f, g \in C^\infty(M)$. Then we have the following:*

a) If f is a constant function, then $\nu(f) = 0$.

b) If $f(p) = g(p) = 0$, then $\nu(fg) = 0$.

Note: To relate the abstract tangent spaces we have defined on manifolds to geometric tangent spaces in $\mathbb{R}^n$, we have to explore the way smooth maps affect tangent vectors. In the case of a smooth map between Euclidean spaces, the total derivative of the map at a point (represented by its Jacobian matrix) is a linear map that represents the "best linear approximation" to the map near the given point. In the manifold case there is a similar linear map, but it makes no sense to talk about a linear map between manifolds. Instead, it will be a linear map between tangent spaces.

Definition. *If M and N are smooth manifolds (with or without boundary) and $F\colon M \to N$ is a smooth map, then for each $p \in M$ we define a map*

$$dF_p\colon T_pM \to T_{F(p)}N,$$

*called the **differential of** F **at** p, as follows: Given $\nu \in T_pM$, we let $dF_p(\nu)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule $dF_p(\nu)(f) = \nu(f \circ F)$ (see* Figure 2 *below).* ★

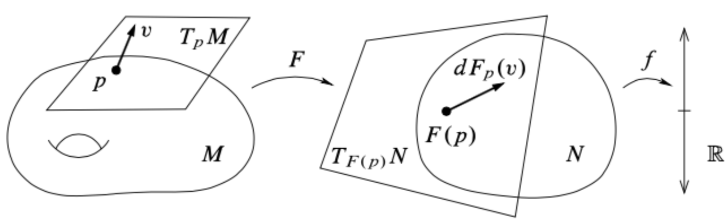


FIGURE 2. The differential.

Note that if $f \in C^\infty(N)$, then $f \circ F \in C^\infty(M)$ so $\nu(f \circ F)$ makes sense. The operator $dF_p(\nu)\colon C^\infty(N) \to \mathbb{R}$ is linear because ν is, and is a derivation at $F(p)$ because for any $f, g \in C^\infty(N)$ we have

$$\begin{aligned} dF_p(\nu)(fg) &= \nu\left((fg) \circ F\right) = \nu\left((f \circ F)(g \circ F)\right) \\ &= f \circ F(p)\, \nu(g \circ F) + g \circ F(p)\, \nu(f \circ F) \\ &= f\left(F(p)\right) dF_p(\nu)(g) + g\left(F(p)\right) dF_p(\nu)(f). \end{aligned}$$

Proposition 2 (Properties of Differentials). *Let M, N, and S be smooth manifolds (with or without boundary), let $F\colon M \to N$ and $G\colon N \to S$ be smooth maps, and let $p \in M$. Then,*

a) $dF_p\colon T_pM \to T_{F(p)}N$ is linear.

b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{G\circ F(p)}S$.

c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_p(M)}\colon T_pM \to T_pM$.

d) *If F is a diffeomorphism, then $dF_p\colon T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

The next proposition indicates that tangent vectors act locally.

Proposition 3. *Let M be a smooth manifold (with or without boundary), $p \in M$, and $\nu \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $\nu(f) = \nu(g)$.*

Using this proposition, we can identify the tangent space to an open submanifold with the tangent space to the whole manifold:

Proposition 4 (The Tangent Space to an Open Submanifold). *Let M be a smooth manifold (with or without boundary), let $U \subseteq M$ be an open subset, and let $\iota\colon U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p\colon T_pU \to T_pM$ is an isomorphism.*

Remark: Given an open subset $U \subseteq M$, the isomorphism $d\iota_p$ between T_pU and T_pM is canonically defined, independently of any choices. Hence from now on we identify T_pU with T_pM for any point $p \in U$.

Proposition 5 (Dimension of the Tangent Space). *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

Recall that every finite-dimensional vector space has a natural smooth manifold structure that is independent of any choice of basis or norm. The following proposition shows that the tangent space to a vector space can be naturally identified with the vector space itself. Suppose V is a finite-dimensional vector space and $a \in V$. Just as we did earlier in the case of $\mathbb{R}^n$, for any vector $v \in V$, we define a map $D_v\big|_a\colon C^\infty(V) \to \mathbb{R}$ by

$$D_v\big|_a f = \frac{d}{dt}\Big|_{t=0} f(a + tv). \tag{$\heartsuit$}$$

Proposition 6 (The Tangent Space to a Vector Space). *Suppose V and W are finite-dimensional vector spaces with their respective standard smooth manifold structures. For each point $a \in V$, the map $v \mapsto D_v|_a$ defined by ($\heartsuit$) is a canonical isomorphism from V to T_aV, such that for any linear map $L\colon V \to W$, the following diagram commutes:*

$$\begin{tikzcd}
V \arrow[r, "\cong"] \arrow[d, "L"'] & T_aV \arrow[d, "dL_a"] \\
W \arrow[r, "\cong"'] & T_{La}W
\end{tikzcd}$$

Remark: It is important to understand that each isomorphism $V \cong T_aV$ is canonically defined, independently of any choice of basis. Because of this result, we can routinely identify tangent vectors to a finite-dimensional vector space with elements of the space itself. More generally, if M is an open submanifold of a vector space V, we can combine our identifications $T_pM \leftrightarrow T_pV \leftrightarrow V$ to obtain a canonical identification of each tangent space to M with V. For example, since $\mathrm{GL}(n, \mathbb{R})$

is an open submanifold of the vector space $M(n,\mathbb{R})$, we can identify its tangent space at each point (i.e. matrix) $X \in \mathrm{GL}(n,\mathbb{R})$ with the full space of matrices $M(n,\mathbb{R})$.

There is another natural identification for tangent spaces to a product manifold:

Proposition 7 (**The Tangent Space to a Product Manifold**). *Let $M_1,\dots,M_k$ be smooth manifolds, and for each j, let $\pi_j\colon M_1\times\cdots\times M_k \to M_j$ be the projection onto the M_j factor. For any point $p=(p_1,\dots,p_k)\in M_1\times\cdots\times M_k$ and tangent vector $\nu\in T_p(M_1\times\cdots\times M_k)$, the map*

$$\alpha\colon T_p(M_1\times\cdots\times M_k) \longrightarrow T_{p_1}M_1\oplus\cdots\oplus T_{p_k}M_k$$

defined by

$$\alpha(\nu) = (d(\pi_1)_p(\nu),\dots,d(\pi_k)_p(\nu)) \tag{$\dagger$}$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Remark: Once again, because the isomorphism ($\dagger$) is canonically defined, independently of any choice of coordinates, we can consider it as a canonical identification, and we will always do so. Thus, for example, we identify $T_{(p,q)}(M\times N)$ with $T_pM\oplus T_qN$, and treat T_pM and T_qN as subspaces of $T_{(p,q)}(M\times N)$.

Computations in Coordinates

Suppose M is a smooth manifold and let (U,φ) be a smooth coordinate chart on M. Then φ is, in particular, a diffeomorphism from U to an open subset $\widehat{U}\subseteq\mathbb{R}^n$. Combining Propositions 4 and 2 part d) from above, we see that $d\varphi(p)\colon T_pM\to\mathfrak{D}_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. Then by Corollary 1, the derivations $\partial/\partial x^1|_{\varphi(p)},\dots,\partial/\partial x^n|_{\varphi(p)}$ form a basis for $\mathfrak{D}_{\varphi(p)}\mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $d\varphi_p$ form a basis for T_pM.

In keeping with our standard practice of treating coordinate maps as identifications whenever possible, we use the notation $\partial/\partial x^i|_p$ for these vectors, characterized by either of the following expressions:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = d(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right).$$

Unwinding the definitions, we see that $\partial/\partial x^i|_p$ acts on a function $f\in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}(f\circ\varphi^{-1}) = \frac{\partial\widehat{f}}{\partial x^i}(\widehat{p}),$$

where $\widehat{f} = f\circ\varphi^{-1}$ is the coordinate representation of f, and $\widehat{p} = (p^1,\dots,p^n) = \varphi(p)$ is the coordinate representation of p. In other words, $\partial/\partial x^i|_p$ is just the derivation that takes the i^{th} partial derivative of (the coordinate representation of) f at (the coordinate representation of) p.

The vectors $\partial/\partial x^i|_p$ are called the ***coordinate vectors at*** p associated with the given coordinate system. In the special case of standard coordinates on $\mathbb{R}^n$, the vectors $\partial/\partial x^i|_p$ are literally the partial derivative operators.

The following proposition summarizes the discussion so far:

Proposition 8. *Let M be a smooth n-manifold (with or without boundary), and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $\big(U,(x^i)\big)$ containing p, the coordinate vectors $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$ form a basis for T_pM.*

Thus, a tangent vector $\nu \in T_pM$ can be written uniquely as a linear combination

$$\nu = v^i \frac{\partial}{\partial x^i}\Big|_p,$$

(where we use the Einstein summation convention). The ordered basis $\big(\partial/\partial x^i|_p\big)$ is called a ***coordinate basis for*** T_pM, and the coefficients $(v^1, \dots, v^n)$ are called the ***components of*** ν with respect to the coordinate basis. If ν is known, its components can be computed easily from its action on the coordinate functions. For each j, the components of ν are given by $v^j = \nu(x^j)$ (where we think of x^j as a smooth real-valued function on U), because

$$\nu(x^j) = \left(v^i \frac{\partial}{\partial x^i}\Big|_p \right)(x^j) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j.$$

The Differential in Coordinates. Next we explore how differentials look in coordinates. We begin by considering the special case of a smooth map $F\colon U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets of Euclidean spaces. For any $p \in U$, we will determine the matrix of $dF_p\colon T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Using $(x^1, \dots, x^n)$ to denote the coordinates in the domain and $(y^1, \dots, y^m)$ to denote those in the codomain, and letting $f \in C^\infty(\mathbb{R}^m)$, we use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x^i}\Big|_p \right) f &= \frac{\partial}{\partial x^i}\Big|_p (f \circ F) \\ &= \frac{\partial f}{\partial y^j}\,(F(p))\,\frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\frac{\partial F^j}{\partial x^i}(p)\,\frac{\partial}{\partial y^j}\Big|_{F(p)} \right) f. \end{aligned}$$

Thus we have that

$$dF_p\left(\frac{\partial}{\partial x^i}\Big|_p \right) = \frac{\partial F^j}{\partial x^i}(p)\,\frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

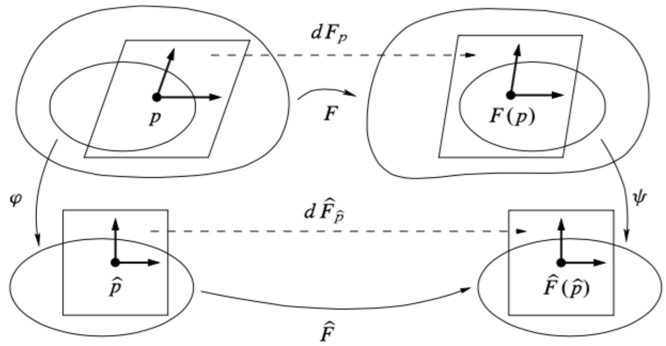
In other words, the matrix of dF_p in terms of the coordinate bases is

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}.$$

Recall that the columns of the matrix are the components of the images of the basis vectors. This matrix is none other than the Jacobian matrix of F at p, which is the matrix representation of the total derivative $DF(p)\colon \mathbb{R}^n \to \mathbb{R}^m$. Therefore, in this case, $dF_p\colon T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ corresponds to the total derivative under our usual identification of Euclidean spaces with their tangent spaces.

Now consider the more general case of a smooth map $F\colon M \to N$ between smooth manifolds (with or without boundary). Choosing smooth coordinate charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation

$$\widehat{F} = \psi \circ F \circ \varphi^{-1} \colon \varphi(U \cap F^{-1}(V)) \to \psi(V).$$



Let $\widehat{p} = \varphi(p)$ denote the coordinate representation of p. By the above computation, $d\widehat{F}_{\widehat{p}}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\widehat{F}$ at $\widehat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we compute

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dF_p\left(d(\varphi^{-1})_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right) \\ &= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(d\widehat{F}_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right) \\ &= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\widehat{p})}\right) \\ &= \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \end{aligned}$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representative of) F. In fact, the definition of the differential was cooked up precisely to give a coordinate-independent meaning to the Jacobian matrix. In the differential geometry literature, the differential is sometimes called the ***tangent map***, the ***total derivative***, or simply the ***derivative of*** F. Because it "pushes" tangent vectors forward from the domain manifold to the codomain, it is also called the ***(pointwise) pushforward***.

The Tangent Bundle

Definition. *Given a smooth manifold M (with or without boundary), we define the **tangent bundle of M**, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:*

$$TM = \bigsqcup_{p \in M} T_pM.$$

*We usually write an element of this disjoint union as an ordered pair (p, ν), with $p \in M$ and $\nu \in T_pM$. The tangent bundle comes equipped with a natural **projection map** $\pi\colon TM \to M$, which sends each vector ν in T_pM to the point p at which it is tangent: $\pi(p, \nu) = p$.* ★

Proposition 9. *For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi\colon TM \to M$ is smooth.*

Proof. See proof on Pg 66, *Lee's Smooth Manifolds* (this one is important!). □

Proposition 10. *If M is a smooth n-manifold (with or without boundary), and M can be covered by a single smooth chart, then TM is diffeomorphic to $M \times \mathbb{R}^n$.*

Definition. *By putting together the differentials of F at all points of M, we obtain a globally defined map between tangent bundles, called the* ***global differential*** *or* ***global tangent map*** *and denoted by $dF\colon TM \to TN$. This is just the map whose restriction to each tangent space $T_pM \subseteq TM$ is dF_p (when we apply the differential of F to a specific vector $\nu \in T_pM$, we can write either $dF_p(\nu)$ or $dF(\nu)$, depending on how much emphasis we wish to give to the point p).* ★

Proposition 11. *If $F\colon M \to N$ is a smooth map, then its global differential $dF\colon TM \to TN$ is a smooth map.*

The following properties of the global differential follow immediately from Proposition 2:

Corollary 2 (**Properties of the Global Differential**)**.** *Let M, N, and S be smooth manifolds (with or without boundary), let $F\colon M \to N$ and $G\colon N \to S$ be smooth maps, and let $p \in M$. Then,*

a) $d(G \circ F) = dG \circ dF$.

b) $d(\mathrm{Id}_M) = \mathrm{Id}_{TM}$.

c) *If F is a diffeomorphism, then $dF\colon TM \to TN$ is also a diffeomorphism, and $(dF)^{-1} = d(F^{-1})$.*