Algebraic Topology
Worksheet 1 Hand-In

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Problem 1. *Let X be a space. Let Y be the space obtained from X by adding a whisker at $a \in X$ – draw a picture! More formally, Y is the quotient space of the disjoint union $X \amalg I$ in which $a \in X$ is identified with $0 \in I$. Prove that the inclusion $\iota\colon X \hookrightarrow Y$ is a homotopy equivalence.*

Proof. The map $\widehat{\pi}\colon Y = (X \amalg I)/\{a \sim 0\} \to X$ given by

$$\begin{cases} x \mapsto x & \text{for } x \in X, \\ t \mapsto a & \text{for } t \in I, \end{cases}$$

is clearly well-defined and, moreover, it satisfies $\widehat{\pi} \circ \iota = \mathbb{1}_X$ ($\widehat{\pi} \circ \iota$ is not merely homotopic to the identity on X; it is equal to it). Now the map $h\colon Y \times I \to Y$ given by

$$\begin{cases} (x, s) \mapsto x & \text{for } (x, s) \in X \times I, \\ (t, s) \mapsto st & \text{for } (t, s) \in I \times I, \end{cases}$$

is also clearly well-defined and defines a homotopy $h\colon \iota \circ \widehat{\pi} \simeq \mathbb{1}_Y$. Thus we conclude that the inclusion ι is indeed a homotopy equivalence with homotopy inverse $\widehat{\pi}$. □

Problem 2. *There are four topologies $\mathcal{T}$ on $X = \{0, 1\}$. For which $\mathcal{T}$ is $(X, \mathcal{T})$ contractible? Give reasons for your answer.*

Solution. The four possible topologies for $X = \{0, 1\}$ are given by

$$\boxed{\begin{aligned} \mathcal{T}_1 &= \{\emptyset, \{0\}, \{1\}, \{0,1\}\}, \\ \mathcal{T}_2 &= \{\emptyset, \{0,1\}\}, \\ \mathcal{T}_3 &= \{\emptyset, \{0\}, \{0,1\}\}, \\ \mathcal{T}_4 &= \{\emptyset, \{1\}, \{0,1\}\}. \end{aligned}}$$

Now let $X_i = (X, \mathcal{T}_i)$ for $i = \{1, \ldots, 4\}$ and let Y be any other topological space throughout. Then,

(X_1 Case) A map $\varphi\colon Y \to X_1$ is continuous if and only if both $\varphi^{-1}(0) \subseteq Y$ and $\varphi^{-1}(1) \subseteq Y$ are open. Since the unit interval I is path-connected, there is no path $\lambda\colon I \to X_1$ from $\lambda(0) = 0$ to $\lambda(1) = 1$. Thus X_1 is not path-connected, and hence not contractible.

(X_2 Case) In this case every function $Y \to X_2$ is continuous. The maps f and g given by

$$\begin{aligned} f\colon X_2 &\to \{0\} \\ x &\mapsto 0 \\ g\colon \{0\} &\to X_2 \\ 0 &\mapsto 0 \end{aligned}$$

are inverse homotopy equivalences with $f \circ g = \mathbb{1}_{\{0\}}$. Meanwhile, the map

$$\begin{aligned} h\colon X_2 \times I &\to X_2 \\ (x,t) &\mapsto \begin{cases} 0 & \text{if } 0 \le t < 1, \\ x & \text{if } t = 1, \end{cases} \end{aligned}$$

defines a homotopy $h\colon g \circ f \simeq \mathbb{1}_{X_2}$. Hence we conclude that X_2 is contractible.

(X_3 Case) A map $\varphi\colon Y \to X_3$ is continuous if and only if $\varphi^{-1}(0) \subseteq Y$ is open. By a similar construction as above, the maps f and g given by

$$\begin{aligned} f\colon X_3 &\to \{0\} \\ x &\mapsto 0 \\ g\colon \{0\} &\to X_3 \\ 0 &\mapsto 0 \end{aligned}$$

are inverse homotopy equivalences with $f \circ g = \mathbb{1}_{\{0\}}$. Meanwhile, the map

$$\begin{aligned} h\colon X_3 \times I &\to X_3 \\ (x,t) &\mapsto \begin{cases} 0 & \text{if } 0 \le t < 1, \\ x & \text{if } t = 1, \end{cases} \end{aligned}$$

defines a homotopy $h\colon g \circ f \simeq \mathbb{1}_{X_3}$. Lastly, note that h is continuous, since

$$h^{-1}(0) = X_3 \times [0,1) \cup \{a\} \times I \subset X_3 \times I$$

is open in $X_3 \times I$. Thus we conclude that X_3 is also contractible.

(X_4 Case) This is identical to the latter case, merely swapping 0 and 1; i.e., X_4 is homeomorphic to X_3, and thus also contractible. □

Problem 3. *Let $f, g\colon X \to \mathbb{S}^1$ be maps such that $f(x) \neq -g(x)$ for all $x \in X$. Construct a homotopy between f and g.*

Solution. Consider $\mathbb{S}^1$ embedded in $\mathbb{R}^2$ (or $\mathbb{C}$ for that matter...). Then the line segment in $\mathbb{R}^2$ (or $\mathbb{C}$) joining $f(x)$ to $g(x)$ cannot pass through the origin because of the imposed condition $f(x) \neq -g(x)\, \forall x \in X$; that is,

$$(1-t)f(x) + tg(x) \neq 0 \qquad \forall x \in X,\ \forall t \in I = [0,1].$$

This shows that the following map, which turns out to be the desired homotopy between f and g, is well-defined:

$$\begin{aligned} h\colon X \times I &\to \mathbb{S}^1 \\ (x,t) &\mapsto \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}, \end{aligned}$$

where the norm is from $\mathbb{R}^2$ (or $\mathbb{C}$). □