Linear Algebra Notes

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Inner Product Spaces

INNER PRODUCTS AND NORMS

Inner products inject geometry into vector spaces in the form of lengths and angles.

Definition: Let V be a VS over $\mathbb{F}$. An inner product on V is a map $\langle -, - \rangle : V \times V \longrightarrow \mathbb{F}$ such that, $\forall$ $x, y, z \in V$ and $\forall$ $c \in \mathbb{F}$, the following properties hold:

a) Linearity in the first component:
* $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$.
* $\langle c\, x, y \rangle = c \langle x, y \rangle$.
From these two we have
$\langle c\, x + z, y \rangle = c \langle x, y \rangle + \langle z, y \rangle$.

b) Positive definitiveness:
$\langle x, x \rangle = \|x\|^2 \geq 0$.
Equality only occurs when $x = 0$.

c) Conjugate symmetry:
$\langle x, y \rangle = \overline{\langle y, x \rangle}$ (the complex conjugate).

Example:
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$.
Define $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$.
This inner product is called the standard inner product on $\mathbb{F}^n$. In the case that $\mathbb{F} = \mathbb{R}$ this is called a dot product.

Definition: A VS V endowed with a specific inner product is called an inner product space (IPS).

Definition: Let $A \in M_{n\times n}(\mathbb{F})$. Then the conjugate transpose of A (also known as the adjoint of A), is $A^* \in M_{n\times n}(\mathbb{F})$, such that $A = \overline{A}^{\mathrm{T}}$.

Example:
Let $\langle A, B\rangle = \text{trace}(B^* A)$, with $A, B \in M_{n\times n}(\mathbb{F})$.
** The trace of a square matrix is the sum of the diagonal entries in the matrix **
If $\mathbb{F} = \mathbb{R}$, V is a real inner product space.
If $\mathbb{F} = \mathbb{C}$, V is a complex inner product space.
The VS $M_{n\times n}(\mathbb{F})$ endowed with the defined inner product above is called the Frobenius inner product space.

Note: You can endow the same VS with different inner products and get different inner product spaces.

• Theorem:
Let V be an inner product space over $\mathbb{F}$. Then for $x, y, z \in V$ and $c \in \mathbb{F}$, we have
a) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
b) $\langle x, c\,y\rangle = \bar{c}\,\langle x, y\rangle$
c) $\langle x, 0\rangle = \langle 0, x\rangle = 0$
d) $\langle x, y\rangle = \langle x, z\rangle\ \forall\ x \Longrightarrow y = z$

Proof:
a) $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle}$ (by conjugate symmetry)
$= \overline{\langle y, x\rangle + \langle z, x\rangle}$
$= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle}$
$= \langle x, y\rangle + \langle x, z\rangle$ (by conjugate symmetry again) ✓

b) $\langle x, c\,y\rangle = \overline{\langle c\,y, x\rangle}$ (by conjugate symmetry)
$= \overline{c\,\langle y, x\rangle}$
$= \bar{c}\,\overline{\langle y, x\rangle}$
$= \bar{c}\,\langle x, y\rangle$ (by conjugate symmetry again) ✓

c) Consider $\langle x, y\rangle = \langle x, y\rangle$.
$\Longrightarrow \langle x, y\rangle - \langle x, y\rangle = 0$
$\Longrightarrow \langle x, y - y\rangle = 0$
$\Longrightarrow \langle x, 0\rangle = 0$
Similarly $\langle 0, y\rangle = 0$ ✓

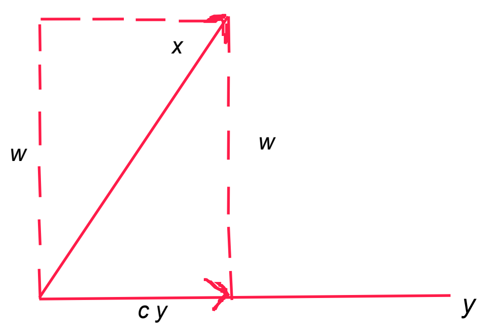
d) Suppose $\langle x, y\rangle = \langle x, z\rangle\ \forall\ x$.
That implies that $\langle x, y-z\rangle = 0$.
Since this is true for all x, it's true when $x = y - z$.
$\Longrightarrow \langle y-z, y-z\rangle = 0$
Then, by positive definitiveness, $y - z = 0 \Longrightarrow y = z$ ✓ ■

Definition: Let $x, y \in V$, where V is an inner product space. Then x and y are said to be orthogonal to each other if the inner product is zero.

• <u>Proof of the Pythagorean theorem:</u>
Let V be an IPS, and let $x,\ y \in V^*$ be orthogonal. Then,

$$\begin{aligned} \| x+y \|^2 &= \langle x+y,\ x+y\rangle \\ &= \langle x,\ x+y\rangle + \langle y,\ x+y\rangle \\ &= \langle x,\ x\rangle + \underbrace{\langle x,\ y\rangle}_{=0} + \underbrace{\langle y,\ x\rangle}_{=0} + \langle y,\ y\rangle \\ &= \langle x,\ x\rangle + \langle y,\ y\rangle \\ &= \| x \|^2 + \| y \|^2 \qquad \blacksquare \end{aligned}$$

Let's use the following figure to define orthogonal decomposition:



Following the figure above, let $x,\ y \in V$, where V is an IPS.
Then the orthogonal decomposition is given by $x = c\, y + w = c\, y + (x - c\, y)$.
If this is indeed the orthogonal decomposition, then it must be true that $\langle x - c\, y,\ y\rangle = 0$. The goal is to find a scalar c such that the orthogonal decomposition described above breaks down into a scalar multiple of y plus an orthogonal component of x.

Thus we set $\langle x - c\, y,\ y\rangle = 0$.
This implies that

$$\begin{aligned} &\langle x,\ y\rangle - \langle c\, y,\ y\rangle = 0. \\ \Longrightarrow\ &\langle x,\ y\rangle - c\,\langle y,\ y\rangle = 0 \\ \Longrightarrow\ &\langle x,\ y\rangle = c\,\langle y,\ y\rangle \\ \Longrightarrow\ &c = \frac{\langle x,\ y\rangle}{\langle y, y\rangle} = \frac{\langle x,\ y\rangle}{\| y \|^2} \end{aligned}$$

Now we can write the orthogonal decomposition as

$$x = \frac{\langle x,\ y\rangle}{\| y \|^2}\, y + \underbrace{\left(x - \frac{\langle x,\ y\rangle}{\| y \|^2}\, y\right)}_{\text{We call this term } w}$$

(Orthogonal Decomposition)

The term on the orthogonal decomposition that we call w (see figure above) is the orthogonal component of x (we are going to use this term w in the proof of the Cauchy-Schwarz inequality below).

• Theorem:
Let V be an IPS, and let $x, y \in V$. Then the following are true:
a) $\|c\,x\| = |c|\,\|x\|$
b) $\|x\| \geq 0$, with equality only when $x = 0$.
c) Cauchy-Schwarz inequality:
$$|\langle x, y\rangle| \leq \|x\|\;\|y\|$$
d) Triangle inequality:
$$\|x+y\| \leq |\,\|x\| + \|y\|\,|$$

Proof:
a)
$$\begin{aligned}\|c\,x\|^2 &= \langle c\,x, c\,x\rangle\\ &= c\,\langle x, c\,x\rangle\\ &= c\,\bar{c}\,\langle x, x\rangle\\ &= |c|^2\,\|x\|^2 \Longrightarrow \|c\,x\| = |c|\,\|x\| \quad ✓\end{aligned}$$

b) $\|x\| = \sqrt{\langle x, x\rangle}$, where $\langle x, x\rangle \geq 0$ (by positive definitiveness).
Then $\|x\| \geq 0$. (with equality only when $x = 0$ by positive definitiveness again) ✓

c) Note that if $y = 0$, then $0 = |\langle x, y\rangle| = \|x\|\;\|y\| = 0$.
Suppose $x, y \in V^*$ (that is $V\setminus\{0\}$), and w is defined as in the orthogonal decomposition described above.
Then by orthogonal decomposition, $x = \frac{\langle x, y\rangle}{\|y\|^2}\,y + w$.
Now, taking the norm on both sides we have
$$\|x\| = \left\|\frac{\langle x, y\rangle}{\|y\|^2}\,y + w\right\|$$
$$\begin{aligned}\Longrightarrow \|x\|^2 &= \left\|\frac{\langle x, y\rangle}{\|y\|^2}\,y + w\right\|^2\\ &= \left\|\frac{\langle x, y\rangle}{\|y\|^2}\,y\right\|^2 + \|w\|^2 \quad \text{(by the Pythagorean theorem)}\\ &= \left|\frac{\langle x, y\rangle}{\|y\|^2}\right|^2 \|y\|^2 + \|w\|^2\\ &= \frac{|\langle x, y\rangle|^2}{\|y\|^4}\,\|y\|^2 + \|w\|^2\\ &= \frac{|\langle x, y\rangle|^2}{\|y\|^2} + \|w\|^2\\ &\geq \frac{|\langle x, y\rangle|^2}{\|y\|^2}\end{aligned}$$
$$\Longrightarrow \|x\|^2\;\|y\|^2 \geq |\langle x, y\rangle|^2$$
$$\Longrightarrow \|x\|\;\|y\| \geq |\langle x, y\rangle| \quad ✓$$

d)
$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle\\ &= \langle x, x+y\rangle + \langle y, x+y\rangle\\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\end{aligned}$$

$$= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$$
$$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle$$
$$\leq \|x\|^2 + \|y\|^2 + 2\,|\langle x, y\rangle|$$
$$\leq \|x\|^2 + \|y\|^2 + 2\,\|x\|\,\|y\| \quad \text{(by the Cauchy-Schwarz inequality)}$$
$$= (\|x\| + \|y\|)^2 \Longrightarrow \|x+y\| \leq \|x\| + \|y\| \quad \checkmark \qquad \blacksquare$$

Definition: Let V be an IPS, then a subset $S \subseteq V$ is said to be an orthogonal set if $\langle u, v\rangle = 0 \;\forall$ $u, v \in S$ with $u \neq v$. A set is called orthonormal if it's orthogonal and it only consists of unit vectors.

GRAM – SCHMIDT ORTHOGONALIZATION

• Theorem:

Let $S = \{e_1, \ldots, e_n\}$ be an orthonormal set in V and let $a_i \in \mathbb{F}$. Then, for any element $x \in \mathrm{span}(S)$,

$$\langle x, x\rangle = \|x\|^2 = |a_1|^2 + \ldots + |a_n|^2,$$

where $x = a_1 e_1 + \ldots + a_n e_n$.

Proof:

Let $x \in \mathrm{span}(S)$. Then $x = a_1 e_1 + \ldots + a_n e_n$ for $a_i \in \mathbb{F}$.

Then, $\|x\|^2 = \|a_1 e_1 + \ldots + a_n e_n\|^2$

$$= \|a_1 e_1 + \ldots a_{n-1} e_{n-1}\|^2 + \|a_n e_n\|^2 \quad \text{(by Pythagorean theorem)}$$
$$= \|a_1 e_1\|^2 + \ldots + \|a_{n-1} e_{n-1}\|^2 + \|a_n e_n\|^2 \quad \text{(by repeatedly applying the Pythagorean theorem)}$$
$$= |a_1|^2 \underbrace{\|e_1\|^2}_{=1} + \ldots + |a_{n-1}|^2 \underbrace{\|e_{n-1}\|^2}_{=1} + |a_n|^2 \underbrace{\|e_n\|^2}_{=1}$$
$$= |a_1|^2 + \ldots + |a_n|^2 \qquad \blacksquare$$

• Corollary:

S is linearly independent.

Proof:

Let $0 = b_1 e_1 + \ldots + b_n e_n$ for $b_i \in \mathbb{F}$.

Then,

$$\|0\|^2 = \|b_1 e_1 + \ldots + b_n e_n\|^2$$
$$\Longrightarrow 0 = |b_1|^2 + \ldots + |b_n|^2$$
$$\Longrightarrow b_i = 0 \quad \forall\, i \qquad \blacksquare$$

Note: Even if S is only orthogonal (that is, not necessarily orthonormal), S is still linearly independent.

• Theorem:

Let V be an IPS and $S = \{v_1, \ \ldots, \ v_k\} \subseteq V$ be $\perp$. Then if $y \in \text{span}(S)$, we have that

$$y = \sum_{i=1}^{k} a_i \, v_i = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\| v_i \|^2} \, v_i \,.$$

So span(S) is a subspace of V and S is a basis for span(S).

Proof:

Let $y \in \text{span}(S)$.

$\Longrightarrow y = a_1 \, v_1 + \ldots + a_k \, v_k$ for $a_i \in \mathbb{F}$.

Now consider $\langle y, v_i \rangle = \langle a_1 \, v_1 + \ldots + a_k \, v_k, \, v_i \rangle$

$$\begin{aligned}
&= \langle a_1 \, v_1, \, v_i \rangle + \langle a_2 \, v_2, \, v_i \rangle + \ldots + \langle a_k \, v_k, \, v_i \rangle \\
&= a_1 \, \langle v_1, \, v_i \rangle + a_2 \, \langle v_2, \, v_i \rangle + \ldots + a_k \, \langle v_k, \, v_i \rangle \\
&= a_i \, \langle v_i, \, v_i \rangle \quad \text{(all the other terms are zero because they are orthogonal)}
\end{aligned}$$

$$\Longrightarrow a_i = \frac{\langle y, v_i \rangle}{\langle v_i, v_i \rangle} = \frac{\langle y, v_i \rangle}{\| v_i \|^2} . \qquad \blacksquare$$

• Corollary:

If S is $\perp_n$, then $y = \sum_{i=1}^{k} \langle y, \, v_i \rangle \, v_i$ (since $\| v_i \|^2 = 1$).

Example:

For $V \in \mathbb{R}^3$ with the standard dot product endowed, let $S = \{(1, \ 0, \ 0), \ (0, \ 1, \ 0)\}$.
Note that S is $\perp$, so we have that $\langle (1, \ 0, \ 0), \ (0, \ 1, \ 0) \rangle = 0$.

Let $(3, 4, 0) \in \text{span}(S)$. Then,

$$(3, \ 4, \ 0) = a_1 \, (1, \ 0, \ 0) + a_2 \, (0, \ 1, \ 0).$$

Now we are going to use the theorem above to solve for a_1 and a_2 (even though in this simple case we could easily determine these values by quick inspection):

$$a_1 = \frac{\langle (3,4,0),(1,0,0) \rangle}{\|(1,0,0)\|^2} = \frac{3}{1} = 3 \quad \text{and} \quad a_2 = \frac{\langle (3,4,0),(0,1,0) \rangle}{\|(0,1,0)\|^2} = 4$$

$$\begin{aligned}
\text{Hence } (3, \ 4, \ 0) &= \sum_{i=1}^{2} \frac{\langle (3,4,0), v_i \rangle}{\| v_i \|^2} \, v_i \\
&= \frac{\langle (3,4,0),(1,0,0) \rangle}{\|(1,0,0)\|^2} \, (1, \ 0, \ 0) + \frac{\langle (3,4,0),(0,1,0) \rangle}{\|(0,1,0)\|^2} \, (0, \ 1, \ 0) \\
&= 3 \, (1, \ 0, \ 0) + 4 \, (0, \ 1, \ 0) \quad \checkmark
\end{aligned}$$

Example:

In $\mathbb{R}^2$ endowed with the standard dot product, we have the set $S = \left\{ \left(\frac{3}{\sqrt{10}}, \ \frac{1}{\sqrt{10}} \right), \ \left(-\frac{1}{\sqrt{10}}, \ \frac{3}{\sqrt{10}} \right) \right\}$,

which contains only unit vectors.

First, let's check if the set is also orthogonal:

$$\left\langle \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) \right\rangle = -\frac{3}{\sqrt{10}} + \frac{3}{\sqrt{10}} = 0 \qquad \checkmark$$

Now, given $(1, 5) \in \mathbb{R}^2$, we have

$$(1, 5) = a_1\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) + a_2\left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$\Longrightarrow a_1 = \left\langle (1, 5), \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) \right\rangle = \frac{3}{\sqrt{10}} + \frac{5}{\sqrt{10}} = \frac{8}{\sqrt{10}}$$

$$a_2 = \left\langle (1, 5), \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) \right\rangle = -\frac{1}{\sqrt{10}} + \frac{15}{\sqrt{10}} = \frac{14}{\sqrt{10}}$$

Hence

$$(1, 5) = \frac{8}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) + \frac{14}{\sqrt{10}}\left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) \qquad \checkmark$$

Gram-Schmidt process:

1) Orthogonalization.
2) Normalization.

• Gram-Schmidt theorem:

Let V be an IPS, and let $S = \{w_1, \ldots, w_n\}$ be a linearly indepedent subset of V. Then we define a new set $S' = \{v_1, \ldots, v_n\}$, where $v_1 = w_1$, and for every vector afterwards we have

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\| v_j \|^2} v_j \qquad \text{for } k = 2, \ldots, n.$$

The result is that S' is $\perp$. Moreover, $\text{span}(S') = \text{span}(S)$.

Proof:

We use induction on n.

Let $S = \{w_1, \ldots, w_n\}$.

Then,

⟜ Base case:

• $n = 1$:

If $n = 1$, then we have $S = \{w_1\}$ and $S' = \{v_1\}$. Since $v_1 = w_1$, we have that $S' = S$, and thus $\text{span}(S) = \text{span}(S')$. ✓

• $n = 2$:

$S = \{w_1, w_2\}$

$$S' = \left\{ w_1, w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \right\}$$

Now let's show that S' is $\perp$..

Check $\left\langle w_1, w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 \right\rangle = 0$

$\Longrightarrow \left\langle w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1, w_1 \right\rangle = 0$ (By applying the complex conjugate property twice)

$\Longrightarrow \langle w_2, v_1\rangle - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} \langle v_1, v_1\rangle = 0 \Longrightarrow 0 = 0$ ✓

• Now we show that span(S') = span(S):

First we show that span(S') ⊆ span(S).

Let $x \in$ span(S')

$\Longrightarrow x = a_1 w_1 + b_1\left(w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1\right)$

$= a_1 w_1 + b_1 w_2 - \frac{b_1 \langle w_2, v_1\rangle}{\|v_1\|^2} w_1$ (since $w_1 = v_1$)

$= \left(a_1 - \frac{b_1 \langle w_2, v_1\rangle}{\|v_1\|^2}\right) w_1 + b_1 w_2 \in$ span(S)

This also proves the other direction, since $|S| = |S'|$ and

span(S') ⊆ span(S) ⟹ span(S) ⊆ span(S') . ✓

•→ Assumption Step:

Let $S_k = \{w_1, \ldots, w_k\} \subseteq S$ and $S'_k = \{v_1, \ldots, v_k\}$. Suppose that S'_k is $\perp$ and span(S'_k) = span(S_k).

Making this assumption we move on to the induction (final) step.

•→ Induction step:

We want to show that $S'_{k+1} = S'_k \bigcup \{v_{k+1}\}$ is $\perp$ and span(S'_{k+1}) = span(S_{k+1}) = $S_k \bigcup \{w_{k+1}\}$.

First we show that S'_{k+1} is $\perp$.

$S'_{k+1} = S'_k \bigcup \{v_{k+1}\}$ is $\perp \Longleftrightarrow \langle v_{k+1}, v_i\rangle = 0 \;\; \forall\, i$, where $1 \le i \le k$.

Thus,

$$\langle v_{k+1}, v_i\rangle = \left\langle w_{k+1} - \sum_{j=1}^{k} \frac{\langle w_{k+1}, v_j\rangle}{\| v_j\|^2} v_j, v_i \right\rangle$$

$$= \langle w_{k+1}, v_i\rangle - \sum_{j=1}^{k} \frac{\langle w_{k+1}, v_j\rangle}{\| v_j\|^2} \langle v_j, v_i\rangle$$

$$= \langle w_{k+1}, v_i\rangle - \frac{\langle w_{k+1}, v_i\rangle}{\| v_i\|^2} \langle v_i, v_i\rangle$$

$$= \langle w_{k+1}, v_i\rangle - \frac{\langle w_{k+1}, v_i\rangle}{\| v_i\|^2} \| v_i \|^2 = 0$$

Thus S'_{k+1} is $\perp$. ✓

Now we show that span(S'_{k+1}) = span(S_{k+1}):

We show first that span(S'_{k+1}) ⊆ span(S_{k+1}).

Let $y \in$ span(S'_{k+1}).

$\Longrightarrow y = a_1 v_1 + \ldots + a_{k+1} v_{k+1} = \sum b_i w_i \Longrightarrow y \in$ span(S_{k+1}). ✓

(⊇) is similar to this.... ■

• Corollary:
Every finite-dimensional IPS has an orthonormal base.

Proof:
By applying the Gram-Schmidt process to any basis of the IPS and then normalizing it, we get an orthonormal base. ■

Example:
Let $V = P_2(\mathbb{R})$ with $\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$. Find an orthonormal basis for V.

Solution:
Start with the standard basis for $P_2(\mathbb{R})$, $\{1, x, x^2\}$. Then we use Gram-Schimdt:

⟼ $v_1 = w_1 = 1$ ✓

⟼ $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$ where

$$\langle w_2, v_1 \rangle = \langle x, 1 \rangle = \int_{-1}^{1} t\, dt = \left.\frac{t^2}{2}\right|_{-1}^{1} = 1.$$

Hence
$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = x - \frac{1}{1} 1 = x - 1$ ✓

⟼ $v_3 = w_3 - \sum_{j=1}^{2} \frac{\langle w_3, v_j \rangle}{\|v_j\|^2} v_j = x^2 - \left(\frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 \right)$

where

$$\langle w_3, v_1 \rangle = \langle x^2, 1 \rangle = \int_{-1}^{1} t^2\, dt = \left.\frac{t^3}{3}\right|_{-1}^{1} = \frac{2}{3}$$

and

$$\langle w_3, v_2 \rangle = \langle x^2, x - 1 \rangle = \int_{-1}^{1} (t^3 - t^2)\, dt = \left.\left(\frac{t^4}{4} - \frac{t^3}{3} \right)\right|_{-1}^{1} = -\frac{2}{3} .$$

Hence,

$$\begin{aligned} v_3 &= w_3 - \sum_{j=1}^{2} \frac{\langle w_3, v_j \rangle}{\|v_j\|^2} v_j \\ &= x^2 - \left(\frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 \right) \\ &= x^2 - \left(\frac{2}{3} (1) + \frac{-\frac{2}{3}}{\frac{8}{3}} (x - 1) \right) \\ &= x^2 + \frac{1}{4} (x - 1) - \frac{2}{3} = x^2 + \frac{1}{4} x - \frac{11}{12} \; ✓ \end{aligned}$$

Thus we have that

$\left\{1,\ x-1,\ x^2+\frac{1}{4}x-\frac{11}{12}\right\}$ is an orthogonal basis. ✓

Now we normalize each of the vectors to get an orthonormal basis...

$\longmapsto \frac{1}{\|1\|} = 1$ ✓

$\longmapsto \frac{x-1}{\|x-1\|} = \frac{x-1}{\sqrt{\langle x-1, x-1\rangle}} = \frac{x-1}{\sqrt{\int_{-1}^{1}(t-1)^2\,dt}} = \frac{x-1}{\sqrt{\frac{8}{3}}} = \frac{\sqrt{3}}{2\sqrt{2}}(x-1)$ ✓

$$\longmapsto \frac{x^2+\frac{1}{4}x-\frac{11}{12}}{\left\|x^2+\frac{1}{4}x-\frac{11}{12}\right\|} = \frac{x^2+\frac{1}{4}x-\frac{11}{12}}{\sqrt{\left\langle x^2+\frac{1}{4}x-\frac{11}{12},\ x^2+\frac{1}{4}x-\frac{11}{12}\right\rangle}}$$

$$= \frac{x^2+\frac{1}{4}x-\frac{11}{12}}{\sqrt{\int_{-1}^{1}\left(t^2+\frac{1}{4}t-\frac{11}{12}\right)^2 dt}}$$

$$= \frac{x^2+\frac{1}{4}x-\frac{11}{12}}{\sqrt{\frac{9}{10}}} = \frac{\sqrt{10}}{3}\left(x^2+\frac{1}{4}x-\frac{11}{12}\right) \quad ✓$$

Thus, (FINALLY!!) our orthonormal basis is

$\left\{1,\ \frac{\sqrt{3}}{2\sqrt{2}}(x-1),\ \frac{\sqrt{10}}{3}\left(x^2+\frac{1}{4}x-\frac{11}{12}\right)\right\}$.

Definition: Let $S \subseteq V$ be nonempty. We define $S^{\perp} = \{x \in V : \langle x,\ y\rangle = 0\ \ \forall\ y \in S\}$. This set $S^{\perp}$ is a subspace of V.

• Theorem:

Let U be a subspace of a finite dimensional VS V. Then, $V = U \oplus U^{\perp}$.

Proof:

We need to show that

i) $V = U + U^{\perp}$ and

ii) $U \bigcap U^{\perp} = \{0\}$.

i) We want to show that $V = U + U^{\perp}$.

Let $U \subseteq V$ be a subspace, and let $\beta = \{e_1,\ \ldots,\ e_m\}$ be an orthonormal basis for U.

Then let $v \in V$.

Now consider the following sum

$$V = \underbrace{\langle v,\ e_1\rangle e_1 + \ldots + \langle v,\ e_m\rangle e_m}_{=u} + \underbrace{V - \langle v,\ e_1\rangle e_1 - \ldots - \langle v,\ e_m\rangle e_m}_{=w}.$$

We define $u = \langle v,\ e_1\rangle e_1 + \ldots + \langle v,\ e_m\rangle e_m$ and $w = V - \langle v,\ e_1\rangle e_1 - \ldots - \langle v,\ e_m\rangle e_m$.

Then we have that $V = u + w$, with $u \in U$ and we want to prove that $w \in U^{\perp}$.

It suffices to show that $\langle w,\ e_j\rangle = 0$, for $j = 1,\ \ldots,\ m$.

$$\begin{aligned}\langle w, e_j\rangle &= \langle v - \langle v, e_1\rangle e_1 - \ldots - \langle v, e_m\rangle e_m, e_j\rangle \\ &= \langle v, e_j\rangle - \underbrace{\langle v, e_1\rangle \langle e_1, e_j\rangle}_{=0} - \ldots - \langle v, e_j\rangle \langle e_j, e_j\rangle - \underbrace{\langle v, e_m\rangle \langle e_m, e_j\rangle}_{=0} \\ &= \langle v, e_j\rangle - \langle v, e_j\rangle \langle e_j, e_j\rangle \\ &= \langle v, e_j\rangle - \langle v, e_j\rangle = 0 \\ &\Longrightarrow w \perp U. \\ &\Longrightarrow w \in U^{\perp}. \qquad \checkmark\end{aligned}$$

ii) Now we want to show that $U \cap U^{\perp} = \{0\}$

Let $x \in U \cap U^{\perp}$.

Then

$x \in U \ \wedge \ x \in U^{\perp}$.

$\Longrightarrow \langle x, x\rangle = 0$.

$\Longrightarrow x = 0$ (by positive definitiviness). ✓ ■

• Corollary:

$U = (U^{\perp})^{\perp}$.

• Theorem:

Let $S = \{v_1, \ \ldots, v_k\}$ be an orthonormal set in V^n.

a) S can be extended to $\{v_1, \ \ldots v_k, v_{k+1}, \ \ldots v_n\}$, which is an orthonormal basis of V.

b) Let $U = \text{span}(S)$. Then $\{v_{k+1}, \ \ldots, v_n\}$ is a basis of $U^{\perp}$.