MATH 709 HW # 2

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Problem 1 (Problem 2-1)**.** *Define $f\colon \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth (in the sense we have defined in chapter 2.)

Proof. Clearly, f is smooth on $\mathbb{R} \smallsetminus \{0\}$. Now, if $x = 0$, then $\varphi = \mathrm{Id}_{(-\varepsilon,\varepsilon)}$ is a coordinate map containing x and $\psi = \mathrm{Id}_{(1-\varepsilon,1+\varepsilon)}$ is a coordinate map containing $f(x) = 1$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth on

$$\varphi\Big((-\varepsilon,\varepsilon) \bigcap f^{-1}\left((1-\varepsilon, 1+\varepsilon)\right)\Big) = \varphi\left([0,\varepsilon)\right) = [0,\varepsilon).$$

However, f is clearly not smooth since it is not continuous at 0. □

Problem 2 (Problem 2-3)**.** *For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.*

a) *$p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the n^{th} **power map** for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.*

b) *$\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is the **antipodal map** $\alpha(x) = -x$.*

c) *$F\colon \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w,z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$, where we think of $\mathbb{S}^3$ as the subset $\{(w,z) \in \mathbb{C}^2 \colon |w|^2 + |z|^2 = 1\} \subset \mathbb{C}^2$.*

Proof of a). Identifying $\mathbb{S}^1$ as a subset of $\mathbb{C}$ (i.e. $\mathbb{S}^1 = \{z \in \mathbb{C} : |z|^2 = 1\}$), every point in $\mathbb{S}^1$ can be written as $\cos\theta + i\sin\theta$ for $\theta \in [0, 2\pi)$. Thus we can rewrite the n^{th} power map as

$$p_n(\cos\theta + i\sin\theta) = p_n(e^{i\theta}) = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

We use the stereographic projection (and its inverse) on $\mathbb{S}^1 \smallsetminus \{(0,1)\}$ given by

$$\sigma_1(e^{i\theta}) = \frac{\cos\theta}{1-\sin\theta} \qquad \text{and}$$
$$\sigma_1^{-1}(x) = \frac{2x + i(x^2-1)}{x^2+1},$$

respectively. We also use the stereographic projection on the south pole $\widetilde{\sigma}_1\colon \mathbb{S}^1 \smallsetminus \{(0,-1)\} \to \mathbb{R}$ given by $\widetilde{\sigma}_1(\mathbf{x}) = -\sigma_1(-\mathbf{x})$ and its inverse $\widetilde{\sigma}_1^{-1}(\mathbf{u}) = -\sigma_1^{-1}(-\mathbf{u})$, that is,

$$\widetilde{\sigma}_1(e^{i\theta}) = \frac{\cos\theta}{1+\sin\theta} \qquad \text{and}$$
$$\widetilde{\sigma}_1^{-1}(x) = \frac{2x + i(1-x^2)}{1+x^2}.$$

Now let P_n be the coordinate representation of p_n, which is of the form $\ddot{\sigma}_1 \circ p_n \circ \dot{\sigma}_1$, where $\ddot{\sigma}_1$ is either σ_1 or $\widetilde{\sigma}_1$ and $\dot{\sigma}_1$ is either σ_1^{-1} or $\widetilde{\sigma}_1^{-1}$. We proceed to show that $P_n \colon \mathbb{R} \to \mathbb{R}$ is a smooth function (which will in turn show smoothness of p_n).

Thus for $x \in \mathbb{R}$, we have

$$\dot{\sigma}_1 = \frac{2x \pm i(1-x^2)}{1+x^2},$$

where the $\pm$ sign depends on whether $\dot{\sigma}_1$ is σ_1^{-1} or $\widetilde{\sigma}_1^{-1}$. Now we write

$$\cos\vartheta = \frac{2x}{1+x^2} \qquad \text{and} \qquad \sin\vartheta = \pm\frac{1-x^2}{1+x^2}.$$

Choosing appropriate domains for the inverse trigonometric function, we have that $\vartheta = \arccos x$. Therefore

$$\begin{aligned} P_n(x) = \ddot{\sigma}_1 \circ p_n \circ \dot{\sigma}_1(x) &= \ddot{\sigma}_1(p_n(\dot{\sigma}_1(x))) \\ &= \ddot{\sigma}_1(p_n(\cos\vartheta + i\sin\vartheta)) \\ &= \ddot{\sigma}_1(\cos(n\vartheta) + i\sin(n\vartheta)) \\ &= \frac{\cos(n\vartheta)}{1 \pm \sin(n\vartheta)} \\ &= \frac{\cos(n\arccos x)}{1 \pm \sin(n\arccos x)}. \end{aligned}$$

Since sin, cos, and arccos are all smooth functions, it follows that P_n (and hence p_n) is also a smooth function. □

Proof of b). We start by using the stereographic projection (and its inverse) on $\mathbb{S}^n \smallsetminus \{(0,\dots,1)\}$ given by

$$\begin{aligned} \sigma_n(x^1,\dots,x^{n+1}) &= \frac{(x^1,\dots,x^n)}{1-x^{n+1}} \\ \text{and} \qquad \sigma_n^{-1}(x^1,\dots,x^n) &= \frac{\big(2x^1,\dots,2x^n,(x^1)^2+\cdots+(x^n)^2-1\big)}{(x^1)^2+\cdots+(x^n)^2+1}, \end{aligned}$$

respectively. We also use the stereographic projection on the south pole $\widetilde{\sigma}_n \colon \mathbb{S}^n \smallsetminus \{(0,\dots,-1)\} \to \mathbb{R}^n$ given by $\widetilde{\sigma}_n(\mathbf{x}) = -\sigma_n(-\mathbf{x})$ and its inverse $\widetilde{\sigma}_n^{-1}(\mathbf{u}) = -\sigma_n^{-1}(-\mathbf{u})$, that is,

$$\begin{aligned} \widetilde{\sigma}_n(x^1,\dots,x^{n+1}) &= -\frac{(x^1,\dots,x^n)}{1+x^{n+1}} \\ \text{and} \qquad \widetilde{\sigma}_n^{-1}(x^1,\dots,x^n) &= \frac{\big(2x^1,\dots,2x^n,1-((x^1)^2+\cdots+(x^n)^2)\big)}{(x^1)^2+\cdots+(x^n)^2+1}, \end{aligned}$$

Now to simplify notation a bit let $x = (x^1, \dots, x^n)$ and $|x|^2 = (x^1)^2 + \cdots + (x^n)^2$, and then let us look at the coordinate representations of α:

$$\begin{aligned}
\sigma_n \circ \alpha \circ \sigma_n^{-1}(x) &= \sigma_n \left(\alpha \left(\frac{2x^1}{|x|^2+1}, \dots, \frac{2x^n}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1} \right) \right) \\
&= \sigma_n \left(-\frac{2x^1}{|x|^2+1}, \dots, -\frac{2x^n}{|x|^2+1}, \frac{1-|x|^2}{|x|^2+1} \right) \\
&= \frac{\left(-\frac{2x^1}{|x|^2+1}, \dots, -\frac{2x^n}{|x|^2+1} \right)}{1 - \frac{1-|x|^2}{|x|^2+1}} \\
&= -\frac{(2x^1, \dots, 2x^n)}{|x|^2+1-1+|x|^2} \\
&= -\frac{2(x^1, \dots, x^n)}{2|x|^2} \\
&= -\frac{x}{|x|^2}.
\end{aligned}$$

$$\begin{aligned}
\sigma_n \circ \alpha \circ \widetilde{\sigma}_n^{-1}(x) &= \sigma_n \left(\alpha \left(\frac{2x^1}{|x|^2+1}, \dots, \frac{2x^n}{|x|^2+1}, \frac{1-|x|^2}{|x|^2+1} \right) \right) \\
&= \sigma_n \left(-\frac{2x^1}{|x|^2+1}, \dots, -\frac{2x^n}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1} \right) \\
&= \frac{\left(-\frac{2x^1}{|x|^2+1}, \dots, -\frac{2x^n}{|x|^2+1} \right)}{1 - \frac{|x|^2-1}{|x|^2+1}} \\
&= -\frac{(2x^1, \dots, 2x^n)}{|x|^2+1-|x|^2+1} \\
&= -\frac{2(x^1, \dots, x^n)}{2} \\
&= -x.
\end{aligned}$$

$$\begin{aligned}
\widetilde{\sigma}_n \circ \alpha \circ \widetilde{\sigma}_n^{-1}(x) &= \widetilde{\sigma}_n \left(\alpha \left(\frac{2x^1}{|x|^2+1}, \dots, \frac{2x^n}{|x|^2+1}, \frac{1-|x|^2}{|x|^2+1} \right) \right) \\
&= \widetilde{\sigma}_n \left(-\frac{2x^1}{|x|^2+1}, \dots, -\frac{2x^n}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1} \right) \\
&= -\frac{\left(\frac{2x^1}{|x|^2+1}, \dots, \frac{2x^n}{|x|^2+1} \right)}{1 - \frac{|x|^2-1}{|x|^2+1}} \\
&= -\frac{(2x^1, \dots, 2x^n)}{|x|^2+1-|x|^2+1} \\
&= -\frac{2(x^1, \dots, x^n)}{2} \\
&= -x.
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\sigma}_n \circ \alpha \circ \sigma_n^{-1}(x) &= \widetilde{\sigma}_n \left(\alpha \left(\frac{2x^1}{|x|^2+1}, \ldots, \frac{2x^n}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1} \right) \right) \\
&= \widetilde{\sigma}_n \left(-\frac{2x^1}{|x|^2+1}, \ldots, -\frac{2x^n}{|x|^2+1}, \frac{1-|x|^2}{|x|^2+1} \right) \\
&= -\frac{\left(\frac{2x^1}{|x|^2+1}, \ldots, \frac{2x^n}{|x|^2+1} \right)}{1 - \frac{1-|x|^2}{|x|^2+1}} \\
&= -\frac{(2x^1, \ldots, 2x^n)}{|x|^2+1-1+|x|^2} \\
&= -\frac{2(x^1, \ldots, x^n)}{2|x|^2} \\
&= -\frac{x}{|x|^2}.
\end{aligned}
$$

Since there are all smooth rational functions, we have that α must be smooth (by definition). □

Proof of c). We can start by identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ via $(x^1 + ix^2, x^3 + ix^4) \leftrightarrow (x^1, x^2, x^3, x^4)$, so that

$$
\begin{aligned}
f(x^1, x^2, x^3, x^4) &= F(x^1 + ix^2, x^3 + ix^4) \\
&= \left(2x^1x^3 + 2x^2x^4, 2x^2x^3 - 2x^1x^4, (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 \right).
\end{aligned}
$$

Then since $\mathbb{C}^2$ and $\mathbb{R}^4$ are diffeomorphic, it suffices to show that f is smooth. We use the stereographic projection (and its inverse) on $\mathbb{S}^3 \setminus \{(0,0,0,1)\}$ given by

$$
\sigma_3(x^1, x^2, x^3, x^4) = \frac{(x^1, x^2, x^3)}{1 - x^4}
$$

$$
\text{and} \qquad \sigma_3^{-1}(x^1, x^2, x^3) = \frac{\left(2x^1, 2x^2, 2x^3, (x^1)^2 + (x^2)^2 + (x^3)^2 - 1\right)}{(x^1)^2 + (x^2)^2 + (x^3)^2 + 1},
$$

respectively. We also use the stereographic projection on the south pole $\widetilde{\sigma}_3 \colon \mathbb{S}^3 \setminus \{(0,0,0,-1)\} \to \mathbb{R}^3$ given by $\widetilde{\sigma}_3(\mathbf{x}) = -\sigma_3(-\mathbf{x})$ and its inverse $\widetilde{\sigma}_3^{-1}(\mathbf{u}) = -\sigma_3^{-1}(-\mathbf{u})$. We have similar functions on $\mathbb{S}^2$ which will be denoted by σ_2 and $\widetilde{\sigma}_2$. Now computing compositions of these functions, we have

$$
\begin{aligned}
\sigma_2 \circ f \circ \sigma_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2-1)^2 + (x^3)^2}, x^2 \right) \\
\sigma_2 \circ f \circ \widetilde{\sigma}_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2-1)^2 + (x^3)^2}, \frac{x^2}{(x^1)^2 + (x^2)^2 + (x^3)^2} \right) \\
\widetilde{\sigma}_2 \circ f \circ \widetilde{\sigma}_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2+1)^2 + (x^3)^2}, x^2 \right) \\
\widetilde{\sigma}_2 \circ f \circ \sigma_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2+1)^2 + (x^3)^2}, \frac{x^2}{(x^1)^2 + (x^2)^2 + (x^3)^2} \right).
\end{aligned}
$$

Since there are all smooth rational functions, we have that f must be smooth (by definition). Thus F is also smooth, as desired. □

Problem 3 (Problem 2-14)**.** *Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.*

Proof. This is almost *Urhyson's Lemma*, but not quite! Our manifold M is both paracompact and Hausdorff (the latter by definition), and we know from topology that a space that has these two properties is a normal topological space. *Urhyson's Lemma* then guarantees the existence of a continuous function $g\colon M \to [0,1]$ that satisfies the conditions on our problem. The issue is that "continuous" is not enough for us, we want our function to be C^∞.

But hold on a second! We know by a previous theorem on the text that there are functions $F_A, F_B\colon M \to \mathbb{R}$ such that $F_A^{-1}(0) = A$ and $F_B^{-1}(0) = B$. [1] Thus if we define f as

$$f(x) = \frac{F_A(x)}{F_A(x) + F_B(x)},$$

then this rational smooth function satisfies the required properties. It is clear that $0 \leq f(x) \leq 1$. Moreover, if $f(x) = 1$, then $F_B(x) = 0$. But $F_B^{-1}(0) = B$, thus $f^{-1}(1) = B$. Similarly, if $f(x) = 0$, then $F_A(x) = 0$. But $F_A^{-1}(0) = A$, thus $f^{-1}(0) = A$. □

[1] Here's the theorem, for reference:

Theorem (**Level Sets of Smooth Functions**)**.** *Let M be a smooth manifold. If C is any closed subset of M, there is a smooth nonnegative function $f\colon M \to \mathbb{R}$ such that $f^{-1}(0) = C$.*