Math 3101 HW # 1

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Section 2

In exercises 7-11, determine whether the binary operation $*$ defined is commutative and whether $*$ is associative:

(#7) $*$ defined on $\mathbb{Z}$ by letting $a * b = a - b$.

Solution:

‣ We check for commutativity:

Let $a,\ b \in \mathbb{Z}$. Then

$$\begin{aligned} a * b &= a - b \neq b - a \quad \forall\ a,\ b \in \mathbb{Z} \\ &= b * a \qquad\qquad\qquad \text{(commutativity fails)} \end{aligned}$$

‣ We check for associativity:

Let $a,\ b,\ c \in \mathbb{Z}$. Then

$$\begin{aligned} a * (b * c) = a * (b - c) &= a - (b - c) \\ &= a - b + c \\ &= (a - b) - (-c) \\ &= (a * b) * (-c) \neq (a * b) * c \quad \forall\ c \in \mathbb{Z} \quad \text{(associativity fails)} . \end{aligned}$$

(#8) $*$ defined on $\mathbb{Q}$ by letting $a * b = a\,b + 1$.

Solution:

‣ We check for commutativity:

Let $a,\ b \in \mathbb{Q}$. Then

$$\begin{aligned} a * b &= a\,b + 1 = b\,a + 1 \quad \forall\ a,\ b \in \mathbb{Q} \\ &= b * a \qquad\qquad\qquad \text{(commutativity holds)} \end{aligned}$$

‣ We check for associativity:

Let $a,\ b,\ c \in \mathbb{Q}$. Then

$$\begin{aligned} a * (b * c) = a * (b\,c + 1) &= a\,(b\,c + 1) + 1 \\ &= a\,b\,c + a + 1 \end{aligned}$$

Now

$$\begin{aligned} (a * b) * c = (a\,b + 1) * c &= (a\,b + 1)\,c + 1 \\ &= a\,b\,c + c + 1 \end{aligned}$$

But we have that

$$a\,b\,c + a + 1 = a\,b\,c + c + 1$$

only when $a = c$, which is not always the case. (associativity fails) .

(#9) $*$ defined on $\mathbb{Q}$ by letting $a * b = a\,b/2$.

<u>Solution:</u>

‣ We check for commutativity:

Let $a,\ b \in \mathbb{Q}$. Then

$$\begin{aligned} a * b &= a\,b/2 = b\,a/2 \ \ \forall\ a,\ b \in \mathbb{Q} \\ &= b * a \qquad\qquad \text{(commutativity holds)} \end{aligned}$$

‣ We check for associativity:

Let $a,\ b,\ c \in \mathbb{Q}$. Then

$$\begin{aligned} a * (b * c) &= a * (b\,c/2) = (a\,b\,c/2)/2 \\ &= a\,b\,c/4 \end{aligned}$$

Now

$$\begin{aligned} (a * b) * c &= (a\,b/2) * c = (a\,b/2)\,c/2 \\ &= a\,b\,c/4 \end{aligned}$$

Thus we have that

$$a * (b * c) = (a * b) * c \ \ \forall\ a,\ b,\ c \in \mathbb{Q} \qquad \text{(associativity holds)} .$$

(#10) $*$ defined on $\mathbb{Z}^+$ by letting $a * b = 2^{a\,b}$.

<u>Solution:</u>

‣ We check for commutativity:

Let $a,\ b \in \mathbb{Z}^+$. Then

$$\begin{aligned} a * b &= 2^{a\,b} = 2^{b\,a} \ \ \forall\ a,\ b \in \mathbb{Z}^+ \\ &= b * a \qquad\qquad \text{(commutativity holds)} \end{aligned}$$

‣ We check for associativity:

Let $a,\ b,\ c \in \mathbb{Z}^+$. Then

$$\begin{aligned} a * (b * c) &= a * (2^{b\,c}) = 2^{a\,2^{b\,c}} \\ &= (2^a)^{2^{b\,c}} \end{aligned}$$

Now

$$\begin{aligned} (a * b) * c &= (2^{a\,b}) * c = 2^{2^{a\,b}\,c} \\ &= (2^c)^{2^{a\,b}} \end{aligned}$$

Thus we have that

$$a * (b * c) \neq (a * b) * c \ \ \forall\ a,\ b,\ c \in \mathbb{Z}^+ \qquad \text{(associativity fails)} .$$

(#11) $*$ defined on $\mathbb{Z}^+$ by letting $a * b = a^b$.

Solution:

- We check for commutativity:

Let $a,\ b \in \mathbb{Z}^+$. Then

$$\begin{aligned} a * b &= a^b \neq b^a \ \ \forall\ a,\ b \in \mathbb{Z}^+ \\ &= b * a \qquad\qquad \text{(commutativity fails)} \end{aligned}$$

- We check for associativity:

Let $a,\ b,\ c \in \mathbb{Z}^+$. Then

$$a * (b * c) = a * (b^c) = a^{b^c}$$

Now

$$\begin{aligned} (a * b) * c = (a^b) * c &= \left(a^b\right)^c \\ &= a^{b\,c} \end{aligned}$$

Thus we have that

$$a * (b * c) \neq (a * b) * c \ \ \forall\ a,\ b,\ c \in \mathbb{Z}^+ \quad \text{(associativity fails)}\ .$$

Section 4

In exercises 1,2,4, determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in the order $\mathcal{G}_1,\ \mathcal{G}_2,\ \mathcal{G}_3$ from definition 4.1 that does not hold:

(#1) Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = a\,b$.

Solution:

For any $a,\ b \in \mathbb{Z}$, we have that $a * b = a\,b \in \mathbb{Z}$, hence $\mathbb{Z}$ is clearly closed under $*$.

Now we test for the group axioms:

- $\mathcal{G}_1$: Let $a,\ b,\ c \in \mathbb{Z}$. Then,

$$a * (b * c) = a * (b\,c) = a\,b\,c = (a\,b)\,c = (a * b) * c. \qquad \checkmark$$

- $\mathcal{G}_2$: Let $a \in \mathbb{Z}$. Now by letting $e = 1$, we have

$$1 * a = a * 1 = a. \qquad \checkmark$$

- $\mathcal{G}_3$: Only $-1,\ 1 \in \mathbb{Z}$ have inverses in $\mathbb{Z}$. Hence $\mathcal{G}_3$ fails and therefore $\langle \mathbb{Z},\ * \rangle$ is not a group.

(#2) Let $*$ be defined on $2\,\mathbb{Z} = \{2\,n : n \in \mathbb{Z}\}$ by letting $a * b = a + b$.

Solution:
Let $a = 2s$, $b = 2t$ with $s, t \in \mathbb{Z}$. Then for any $a, b \in 2\mathbb{Z}$, we have
$$a * b = a + b = 2s + 2t = 2(s+t) \in 2\mathbb{Z} \qquad (\text{since } s+t \in \mathbb{Z}).$$
Hence $2\mathbb{Z}$ is closed under $*$.

Now we test for the group axioms:

- $\mathcal{G}_1$: Let a, b be defined as above and put $c = 2r$. Then for any $a, b, c \in 2\mathbb{Z}$, we have
$$\begin{aligned} a * (b * c) = a * (b + c) &= a + (b + c) \\ &= 2s + (2t + 2r) = (2s + 2t) + 2r \\ &= (a + b) + c = (a * b) + c = (a * b) * c . \quad ✓ \end{aligned}$$

- $\mathcal{G}_2$: Let $a \in 2\mathbb{Z}$. Now by letting $e = 0$, we have
$$e * a = e + a = a + e = a * e = a. \qquad ✓$$

- $\mathcal{G}_3$: For each $a \in 2\mathbb{Z}$, we let $a' = -a$. Then we have have
$$a * a' = a + (-a) = 0 = e \qquad ✓$$

Since all axioms hold, we conclude that $\langle 2\mathbb{Z}, *\rangle$ is a group.

(#4) Let $*$ be defined on $\mathbb{Q}$ by letting $a * b = a\,b$.

Solution:
For any $a, b \in \mathbb{Q}$, we have that $a * b = a\,b \in \mathbb{Q}$, hence $\mathbb{Q}$ is clearly closed under $*$.
Now we test for the group axioms:

- $\mathcal{G}_1$: Let $a, b, c \in \mathbb{Q}$. Then,
$$a * (b * c) = a * (b\,c) = a\,b\,c = (a\,b)\,c = (a * b) * c. \qquad ✓$$

- $\mathcal{G}_2$: Let $a \in \mathbb{Q}$. Now by letting $e = 1$, we have
$$e * a = a * e = a. \qquad ✓$$

- $\mathcal{G}_3$: Since $0 \in \mathbb{Q}$ has no inverse, we see that $\mathcal{G}_3$ fails and therefore $\langle \mathbb{Q}, *\rangle$ is not a group.

(#31) If $*$ is a binary operation on a set S, an element $x \in S$ is said to be an idempotent for $*$ if $x * x = x$. Prove that a group has exactly one idempotent element.

Proof:
Let G be a group. We assume to the contrary that there exists two such elements $x, x' \in G$, such that

$$x' * x' = x' \quad \text{and} \quad x * x = x\,.$$

But then, since both x and x' are assumed to be in G, we must have two distinct identity elements in G, ie. $x = e = x'$, which is impossible since the identity element of a group must be unique. Therefore we have that $x = x'$, contrary to our assumption that they were distinct elements. $(\Rightarrow\Leftarrow)$ ■

(#32) Show that every group G with identity e and such that $x * x = e \;\forall\, x \in G$ is abelian. [Hint: Consider $(a * b) * (a * b)$.]

Proof:
Let G be a group and assume $x * x = e \;\forall\, x \in G$.
Then, for all $a, b \in G$ we have

$$e = (a * b) * (a * b) \quad \text{and} \quad (a * a) * (b * b) = e * e = e\,.$$

Thus,

$$a * b * a * b = a * a * b * b.$$

Using left and right cancellation, we have $b * a = a * b \;\forall\, a, b \in G$. Hence G is abelian. ■