MATH 752 NOTES

MARIO L. GUTIERREZ ABED

Some Preliminaries

Definition. *If $\pi\colon X \to Y$ is a map, a subset $U \subseteq X$ is said to be* ***saturated with respect to*** π *if U is the entire preimage of its image under π, i.e., if $U = \pi^{-1}(\pi(U))$. Given $y \in Y$, the* ***fiber of*** π ***over*** y *is the set $\pi^{-1}(y)$. (Thus, a subset of X is saturated if and only if it is a union of fibers).* ★

Definition. *If X and Y are topological spaces, a map $F\colon X \to Y$ (continuous or not) is said to be* ***proper*** *if for every compact set $K \subseteq Y$, the preimage $F^{-1}(K)$ is compact as well.* ★

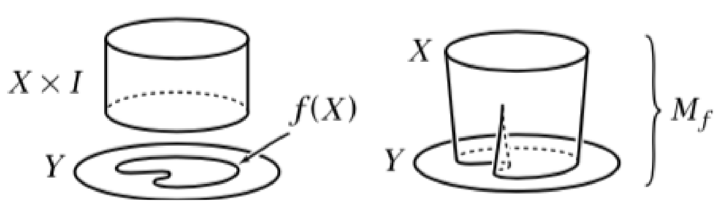
Here are some useful sufficient conditions for a map to be proper:

Proposition 1 (**Sufficient Conditions for Properness**)**.** *Suppose X and Y are topological spaces, and $F\colon X \to Y$ is a continuous map.*

- *a) If X is compact and Y is Hausdorff, then F is proper.*
- *b) If F is a closed map with compact fibers, then F is proper.*
- *c) If F is a topological embedding with closed image, then F is proper.*
- *d) If Y is Hausdorff and F has a continuous left inverse (i.e., a continuous map $G\colon Y \to X$ such that $G \circ F = \mathrm{Id}_X$), then F is proper.*
- *e) If F is proper and $A \subseteq X$ is a subset that is saturated with respect to F, then $F|_A\colon A \to F(A)$ is proper.*

Definition. *A* ***deformation retraction*** *of a space X onto a subspace A is a family of maps $f_t\colon X \to X$, for $t \in I$, such that $f(0) = \mathbb{1}$ (the identity map), $f_1(X) = A$, and $f_t|_A = \mathbb{1}$ for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X$ given by $(x,t) \mapsto f_t(x)$, is continuous.* ★

Definition. *For a map $f\colon X \to Y$, the* ***mapping cylinder*** *M_f is the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by identifying each $(x,1) \in X \times I$ with $f(x) \in Y$.* ★



Definition. *A* ***homotopy*** *is any family of maps $f_t\colon X \to Y$, for $t \in I$, such that the associated map $F\colon X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous. One says that two maps $f_0, f_1\colon X \to Y$ are* ***homotopic*** *if there exists a homotopy f_t connecting them, in which case we write $f_0 \simeq f_1$.* ★

Remark 1: In these terms, a deformation retraction of X onto a subspace A is a homotopy from the identity map of X to a ***retraction*** of X onto A (a map $r\colon X \to X$ such that $r(X) = A$ and $r|_A = \mathbb{1}$. Equivalently, we may regard a retraction as a map $X \to A$ restricting to the identity on the subspace $A \subset X$). From a more formal viewpoint a retraction is a map $r\colon X \to X$ with $r^2 = r$, since this equation says exactly that r is the identity on its image. Retractions are the topological analogs of projection operators in other parts of mathematics.

Remark 2: A homotopy $f_t\colon X \to X$ that gives a deformation retraction of X onto a subspace A has the property that $f_t|_A = \mathbb{1}$ for all t. In general, a homotopy $f_t\colon X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is called a ***homotopy relative to*** A (or more concisely, a homotopy rel A). Thus, a deformation retraction of X onto A is a homotopy rel A from the identity map of X to a retraction of X onto A.

Remark 3: If a space X deformation retracts onto a subspace A via $f_t\colon X \to X$, then if $r\colon X \to A$ denotes the resulting retraction and $\iota\colon A \to X$ the inclusion, we have $r\iota = \mathbb{1}$ and $\iota r \simeq \mathbb{1}$, the latter homotopy being given by f_t. Generalizing this situation, a map $f\colon X \to Y$ is called a ***homotopy equivalence*** if there is a map $g\colon Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$. The spaces X and Y are said to be ***homotopy equivalent*** or to have the same ***homotopy type***, which we denote by $X \simeq Y$. It is true in general that two spaces X and Y are homotopy equivalent if and only if there exists a third space Z containing both X and Y as deformation retracts.

Definition. *A space having the homotopy type of a point is called* ***contractible****. This amounts to requiring that the identity map of the space be* ***nullhomotopic****, that is, homotopic to a constant map.* ★

Remark: In general, this is slightly weaker than saying the space deformation retracts to a point; see the exercises at the end of chapter 0, *Hatcher's*, for an example distinguishing these two notions.

Covering Maps

Definition. *Suppose E and X are topological spaces. A map $\pi\colon E \to X$ is called a* ***covering map*** *if E and X are connected and locally path-connected, π is surjective and continuous, and each point $p \in X$ has a neighborhood U that is* ***evenly covered by*** *π, meaning that each component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π. In this case, X is called the* ***base of the covering****, and E is called a* ***covering space of*** *X. If U is an evenly covered subset of X, the components of $\pi^{-1}(U)$ are called the* ***sheets of the covering over*** *U.* ★

Definition. *If $\pi\colon E \to X$ is a covering map and $F\colon B \to X$ is a continuous map, a* ***lift of*** F *is a continuous map $\widetilde{F}\colon B \to E$ such that $\pi \circ \widetilde{F} = F$:*

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ B \arrow[ur, dashed, "\widetilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

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Proposition 2 (**Lifting Properties of Covering Maps**). *Suppose $\pi\colon E \to X$ is a covering map.*

a) UNIQUE LIFTING PROPERTY*: If B is a connected space and $F\colon B \to X$ is a continuous map, then any two lifts of F that agree at one point are identical.*

b) PATH LIFTING PROPERTY*: If $f\colon I \to X$ is a path, then for any point $e \in E$ such that $\pi(e) = f(0)$, there exists a unique lift $\widetilde{f}\colon I \to E$ of f such that $\widetilde{f}(0) = e$.*

c) MONODROMY THEOREM*: If $f, g\colon I \to X$ are path-homotopic paths and $\widetilde{f}_e, \widetilde{g}_e\colon I \to E$ are their lifts starting at the same point $e \in E$, then $\widetilde{f}_e$ and $\widetilde{g}_e$ are path-homotopic and $\widetilde{f}_e(1) = \widetilde{g}_e(1)$.*

Proposition 3 (**Lifting Criterion**). *Suppose $\pi\colon E \to X$ is a covering map, Y is a connected and locally path-connected space, and $F\colon Y \to X$ is a continuous map. Let $y \in Y$ and $e \in E$ be such that $\pi(e) = F(y)$. Then there exists a lift $\widetilde{F}\colon Y \to E$ of F satisfying $\widetilde{F}(y) = e$ if and only if $F_*\left(\pi_1(Y, y)\right) \subseteq \pi_*\left(\pi_1(E, e)\right)$.*

Proposition 4 (**Coverings of Simply Connected Spaces**). *If X is a simply connected space, then every covering map $\pi\colon E \to X$ is a homeomorphism.*

Definition. *A topological space is said to be* ***locally simply connected*** *if it admits a basis of simply connected open subsets.* ★

Proposition 5 (**Existence of a Universal Covering Space**). *If X is a connected and locally simply connected topological space, there exists a simply connected topological space $\widetilde{X}$ and a covering map $\pi\colon \widetilde{X} \to X$. If $\widehat{\pi}\colon \widehat{X} \to X$ is any other simply connected covering of X, then there is a homeomorphism $\varphi\colon \widetilde{X} \to \widehat{X}$ such that $\widehat{\pi} \circ \varphi = \pi$.*

Definition. *The simply connected covering space $\widetilde{X}$ whose existence and uniqueness (up to homeomorphism) are guaranteed by this last proposition is called the* ***universal covering space of*** *X.* ★