MATH 710 HW # 10

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJIAN

Exercise 1. *Let V be an n-dimensional real vector space. Find a basis for $\Lambda^k(V)$ and show that* $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Given bases (E_i) for V and their dual counterparts (φ^i) for V^*, our basis for $\Lambda^k(V)$ is given by the set

$$\mathcal{B} = \{\varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

Let $\omega \in \Lambda^k(V) \subset T^k(V)$, so we can write

$$\omega = \sum_{\{I\colon i_1 < \cdots < i_k\}} \alpha_{i_1,\ldots,i_k}\, \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}.$$

But then,

$$\omega = \mathrm{Alt}(\omega) = \sum_{\{I\colon i_1 < \cdots < i_k\}} \alpha_{i_1,\ldots,i_k}\, \mathrm{Alt}(\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}).$$

Then since each $\mathrm{Alt}(\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k})$ is a constant times one of the $\varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k}$, these elements span $\Lambda^k(V)$.

Now to show linear independence of $\mathcal{B}$, suppose that the identity $\sum_{\{I\colon i_1 < \cdots < i_k\}} \alpha_{i_1,\ldots,i_k}\, \varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k} = 0$ holds for some coefficients $\alpha_{i_1,\ldots,i_k}$. Let $J = \{j_i \mid 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}$ be any other increasing multi-index. Since the result of evaluating $\varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k}$ on a sequence of basis vectors is $\varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k}(E_{j_1}, \ldots, E_{j_k}) = \delta^I_J$, by applying both sides of the identity to the vectors $(E_{j_1}, \ldots, E_{j_k})$ we get

$$0 = \sum_{\{I\colon i_1 < \cdots < i_k\}} \alpha_{i_1,\ldots,i_k}\, \varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k}(E_{j_1}, \ldots, E_{j_k}) = \alpha_{j_1,\ldots,j_k}.$$

Thus each coefficient $\alpha_{j_1,\ldots,j_k}$ is zero, and we are done. Note that a strictly ascending multi-index $I = (i_1 < \cdots < i_k)$ is obtained by choosing a subset of k letters from $1, \ldots, n$, and this can be done in $\binom{n}{k}$ ways; hence we have the desired dimension for $\Lambda^k(V)$. □

Exercise 2. *Prove the following properties of the wedge product:*

i) Bilinearity. $(\alpha\omega + \alpha'\omega') \wedge \eta = \alpha(\omega \wedge \eta) + \alpha'(\omega' \wedge \eta)$.

ii) Associativity. *If $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^\ell(V)$, and $\theta \in \Lambda^m(V)$, then*

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta).$$

iii) Anticommutativity. *If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, then*

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

Solution of i). This follows rather trivially from the definition of the wedge product. Recall that

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\,\ell!}\,\mathrm{Alt}(\omega \otimes \eta).$$

Thus, since the tensor product is bilinear and Alt is linear, the bilinearity of the wedge product must hold as well. □

Solution of ii). Unwinding the definition of the wedge product, we have

$$\begin{aligned}
(\omega \wedge \eta) \wedge \theta &= \frac{(k+\ell+m)!}{(k+\ell)!\,m!}\,\mathrm{Alt}((\omega \wedge \eta) \otimes \theta) \\
&= \frac{(k+\ell+m)!}{(k+\ell)!\,m!}\,\mathrm{Alt}\left(\frac{(k+\ell)!}{k!\,\ell!}\,\mathrm{Alt}(\omega \otimes \eta) \otimes \theta\right) \\
&= \frac{(k+\ell+m)!}{(k+\ell)!\,m!}\,\frac{(k+\ell)!}{k!\,\ell!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta) \qquad (\dagger) \\
&= \frac{(k+\ell+m)!}{k!\,\ell!\,m!}\,\mathrm{Alt}(\omega \otimes \eta \otimes \theta).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\omega \wedge (\eta \wedge \theta) &= \frac{(k+\ell+m)!}{k!\,(\ell+m)!}\,\mathrm{Alt}(\omega \otimes (\eta \wedge \theta)) \\
&= \frac{(k+\ell+m)!}{(k+\ell)!\,m!}\,\mathrm{Alt}\left(\omega \otimes \left(\frac{(\ell+m)!}{\ell!\,m!}\,\mathrm{Alt}(\eta \otimes \theta)\right)\right) \\
&= \frac{(k+\ell+m)!}{k!\,(\ell+m)!}\,\frac{(\ell+m)!}{\ell!\,m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta) \qquad (\dagger') \\
&= \frac{(k+\ell+m)!}{k!\,\ell!\,m!}\,\mathrm{Alt}(\omega \otimes \eta \otimes \theta).
\end{aligned}$$

Note that ($\dagger$) and ($\dagger'$) hold because of the property of the alternating map:

$$\begin{aligned}
\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \theta) &= \mathrm{Alt}(\omega \otimes \eta \otimes \theta) \\
&= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta)),
\end{aligned}$$

which can be easily checked with a quick computation. □

Solution of iii). Define $\tau \in S_{k+\ell}$ to be the permutation

$$\tau = \begin{bmatrix} 1 & \cdots & \ell & \ell+1 & \cdots & \ell+k \\ k+1 & \cdots & k+\ell & 1 & \cdots & k \end{bmatrix}.$$

This is pretty standard notation for permutations. It means that

$$\tau(1) = k+1, \quad \ldots, \quad \tau(\ell) = k+\ell, \quad \tau(\ell+1) = 1, \quad \ldots, \quad \tau(\ell+k) = k.$$

It is not hard to show that $\mathrm{sgn}(\tau) = (-1)^{k\ell}$. Now we have

$$\begin{aligned}
\sigma(1) = \sigma\tau(\ell+1), \ldots, \sigma(k) = \sigma\tau(\ell+k), \\
\sigma(k+1) = \sigma\tau(1), \ldots, \sigma(k+\ell) = \sigma\tau(\ell).
\end{aligned}$$

Then, for any $v_1, \ldots, v_{k+\ell} \in V$, we have

$$\begin{aligned}
\mathrm{Alt}(\omega \otimes \eta)(v_1, \ldots, v_{k+\ell}) &= \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
&= \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\, \omega(v_{\sigma\tau(\ell+1)}, \ldots, v_{\sigma\tau(\ell+k)})\, \eta(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(\ell)}) \\
&= (\operatorname{sgn} \tau)\, \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma\tau)\, \eta(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(\ell)})\, \omega(v_{\sigma\tau(\ell+1)}, \ldots, v_{\sigma\tau(\ell+k)}) \\
&= (\operatorname{sgn} \tau)\, \mathrm{Alt}(\eta \otimes \omega)(v_1, \ldots, v_{k+\ell}).
\end{aligned}$$

The last equality follows from the fact that as σ runs through all permutations in $S_{k+\ell}$, so does $\sigma\tau$. Thus we have proven that

$$\mathrm{Alt}(\omega \otimes \eta) = (\operatorname{sgn} \tau)\, \mathrm{Alt}(\eta \otimes \omega).$$

Thus multiplying both sides by $\frac{(k+\ell)!}{k!\,\ell!}$, we get

$$\begin{aligned}
\omega \wedge \eta &= (\operatorname{sgn} \tau)\, \eta \wedge \omega \\
&= (-1)^{k\ell} \eta \wedge \omega.
\end{aligned}$$

□

Exercise 3. *Show by an example that there exist k-alternating tensors (for some $k > 1$) that are not decomposable.*[1]

Solution. Note that, if ω is decomposable, then we must have $\omega \wedge \omega = 0$. The reason is that if we assume that ω is indeed decomposable, we should be able to express it as $\omega = \varphi^1 \wedge \cdots \wedge \varphi^k$ for some $\varphi^1, \ldots, \varphi^k \in \Lambda^1(V)$, and therefore $\omega \wedge \omega = \varphi^1 \wedge \cdots \wedge \varphi^k \wedge \varphi^1 \wedge \cdots \wedge \varphi^k = 0$. An example of a form that is not decomposable is, for instance, $\omega = \varphi^1 \wedge \varphi^2 + \varphi^3 \wedge \varphi^4 \in \Lambda^2(\mathbb{R}^4)$. In this case, we have

$$\omega \wedge \omega = 2\varphi^1 \wedge \varphi^2 \wedge \varphi^3 \wedge \varphi^4 \neq 0 \implies \omega \text{ is not decomposable.}$$

(Note, however, that $\omega \wedge \omega = 0$ is a necessary but not sufficient condition for ω to be decomposable. For instance, if ω is any arbitrary odd alternating tensor, say of dimension $2k+1$, then $\omega \wedge \omega = (-1)^{(2k+1)^2} \omega \wedge \omega = 0$.) □

Exercise 4. *Recall the space $\Lambda(V) = \oplus_{k=0}^{\infty} \Lambda^k(V)$ which, when equipped with the wedge product, becomes an anticommutative graded algebra known as the **exterior algebra** or **Grasmann algebra**. Show that $\dim \Lambda(V) = 2^n$.*

Proof. We already have from Exercise 1 that $\dim \Lambda^k(V) = \binom{n}{k}$. Since for $k > n = \dim V$, $\Lambda^k(V) = \emptyset$, we have

$$\Lambda(V) = \oplus_{k=0}^{\infty} \Lambda^k(V) = \oplus_{k=0}^{n} \Lambda^k(V).$$

[1] Recall that a k-tensor $\omega \in \Lambda^k(V)$ is said to be ***decomposable*** if there exists $\varphi^1, \ldots, \varphi^k \in \Lambda^1(V)$ such that $\omega = \varphi^1 \wedge \cdots \wedge \varphi^k$.

We note that the scalars Λ^0 are one-dimensional; $\dim \Lambda^0 = 1$. Thus the total dimension of the entire Grassmann algebra is

$$\begin{aligned}\sum_{k=0}^{n} \dim \Lambda^k(V) &= 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \\ &= (1+1)^n \\ &= 2^n.\end{aligned}$$

□

Exercise 5. *Prove that for any covectors $\omega^1, \ldots, \omega^k \in V^*$ and vectors $v_1, \ldots, v_k \in V$, we have*

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) = \det\left(\omega^j(v_i)\right)_{i,j \leq k}.$$

Proof. A generalization of the result from Exercise 2 part ii) above (associativity of the wedge product) shows that for $\theta^i \in \Lambda^{d_i}(V)$,

$$\theta^1 \wedge \cdots \wedge \theta^r = \frac{(d_1 + \cdots + d_r)!}{d_1! \cdots d_r!} \operatorname{Alt}(\theta^1 \otimes \cdots \otimes \theta^r). \tag{1}$$

Now, by equation (1), for 1-forms ω^i we have

$$\begin{aligned}\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) &= \frac{(1 + \cdots + 1)!}{1! \cdots 1!} \operatorname{Alt}(\omega^1 \otimes \cdots \otimes \omega^k)(v_1, \ldots, v_k) \\ &= k! \operatorname{Alt}(\omega^1 \otimes \cdots \otimes \omega^k)(v_1, \ldots, v_k) \\ &= \frac{k!}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\omega^1 \otimes \cdots \otimes \omega^k)(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \omega^1(v_{\sigma(1)}) \cdots \omega^k(v_{\sigma(k)}) \\ &= \det\left(\omega^j(v_i)\right)_{i,j \leq k}.\end{aligned}$$

□