Math 351 Assignment 5

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(1) Decide whether the series converges or diverges in $\mathbb{R}$.

a) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

Convergent. ✓

b) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$

Convergent. ✓

c) $\sum_{n=1}^{\infty} \frac{\cos(3n)}{1+(1.2)^n}$

Convergent. ✓

d) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{5^n n!}$

Convergent. ✓

e) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

Divergent. ✓

(2) Find the exact sum of the series $\sum_{n=1}^{\infty} \left(\sin\left(\frac{\pi}{2} - \frac{\pi}{n+1}\right) - \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right)$.

Solution:

$$\sum_{n=1}^{\infty} \left(\sin\left(\frac{\pi}{2} - \frac{\pi}{n+1}\right) - \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right) = 1 + \frac{3}{2} + \left(1 + \frac{1}{\sqrt{2}}\right) + \left(1 + \frac{1}{4}\left(1+\sqrt{5}\right)\right) + \ldots = 2$$

(3) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by $a_n = \begin{cases} \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{k}\right) & \text{if } n = 2k - 1 \\ 2 - \frac{1}{2}\left(1 - \frac{1}{k}\right) & \text{if } n = 2k \end{cases}$

where $k \geq 1$. Calculate $\limsup \{a_n\}$ and $\liminf \{a_n\}$.

Solution:

$a_n = \left\{\frac{1}{2},\ 2,\ \frac{3}{4},\ \frac{7}{4},\ \frac{5}{6},\ \frac{5}{3},\ \frac{7}{8},\ \frac{13}{8},\ \ldots\right\}$.

We can see that $\lim\limits_{k\to\infty}\left(\frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{k}\right)\right) = 1$ while $\lim\limits_{k\to\infty}\left(2 - \frac{1}{2}\left(1 - \frac{1}{k}\right)\right) = \frac{3}{2}$. Therefore, since a_n oscillates indefinitely between 1 and $\frac{3}{2}$, we have that $\limsup(a_n) = \frac{3}{2}$ while $\liminf(a_n) = 1$.

(4) Compute $\limsup \{a_n\}$ and $\liminf \{a_n\}$ for the sequence $a_n = \sin\left(\frac{\pi}{2}\, n\right) \frac{n+2}{2\, n}$.

Solution:

$a_n = \left\{\frac{3}{2},\ 0,\ -\frac{5}{6},\ 0,\ \frac{7}{10},\ 0,\ -\frac{9}{14},\ 0,\ \frac{11}{18},\ 0,\ -\frac{13}{22},\ 0,\ \ldots\right\}$

Since $\lim\limits_{n\to\infty}\left(\frac{n+2}{2\, n}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{2}\, n\right)$ will oscillate between 0 (when n is even) and -1, and 1 (when n is odd), we can see that a_n ends up oscillating between $-\frac{1}{2}$ and $\frac{1}{2}$ indefinitely. Therefore, $\liminf(a_n) = -\frac{1}{2}$ and $\limsup(a_n) = \frac{1}{2}$.