Math 353 HW 9

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Section 4.1

(1) Evaluate the integrals $\frac{1}{2\pi i}\oint_C f(z)\,dz$, where C is the unit circle centered at the origin and $f(z)$ is given by the following:

a) $\dfrac{z+1}{2z^3-3z^2-2z}$

Solution:

$$\frac{z+1}{2z^3-3z^2-2z} = \frac{z+1}{z(2z^2-3z-2)} = \frac{z+1}{z\left(z+\frac{1}{2}\right)(z-2)}$$

We can see that $f(z)$ has two simple poles at $z = 0, -\frac{1}{2}$ ($z = 2$ is another singularity but it lies outside of the enclosed region so we can ignore it for our purpose). In order to solve the integral we want to find the residues of $f(z)$ at these singularities first:

- $\text{Res}((f(z);\,0)) = \left.\dfrac{z+1}{6z^2-6z-2}\right|_{z=0} = -\dfrac{1}{2}$
- $\text{Res}\left(\left(f(z);\,-\frac{1}{2}\right)\right) = \left.\dfrac{z+1}{6z^2-6z-2}\right|_{z=-\frac{1}{2}} = \dfrac{1}{5}$

Note: We could've also used the limit formula in this case but I chose to use $\text{Res}(f(z);\,z_0) = \frac{N(z_0)}{D'(z_0)}$ instead. Either way we get the same result.

Hence,

$\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\left[2\pi i\left(\text{Res}((f(z),\,0)) + \text{Res}\left(\left(f(z),\,-\frac{1}{2}\right)\right)\right)\right] = \left(\frac{1}{5} - \frac{1}{2}\right) = -\frac{3}{10}.$

b) $\dfrac{\cosh\frac{1}{z}}{z}$

Solution:

$$\cosh\frac{1}{z} = \frac{e^{1/z}+e^{-1/z}}{2} = \frac{1}{2}\left[\left(1+\frac{1}{z}+\frac{1}{2!\,z^2}+\ldots+\frac{1}{n!\,z^n}\right)+\left(1-\frac{1}{z}+\frac{1}{2!\,z^2}-\ldots+\frac{(-1)^n}{n!\,z^n}\right)\right]$$

Then,

$$\frac{\cosh \frac{1}{z}}{z} = \frac{1}{2z}\left[\left(1+\frac{1}{z}+\frac{1}{2!\,z^2}+\ldots+\frac{1}{n!\,z^n}\right)+\left(1-\frac{1}{z}+\frac{1}{2!\,z^2}-\ldots+\frac{(-1)^n}{n!\,z^n}\right)\right]$$
$$= \frac{1}{2z}+\frac{1}{2z}+\text{higher powers of } z\ldots$$

So we have $\frac{1}{2z}+\frac{1}{2z}=\frac{1}{z}$. Thus the residue of $f(z)$ is 1. Hence $\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\,2\pi i\,(1) = 1$.

c) $\dfrac{e^{-\cosh z}}{4z^2+\pi^2}$

Solution:

$$\frac{e^{-\cosh z}}{4z^2+\pi^2} = \frac{e^{-\cosh(z)}}{4\left(z^2+\frac{\pi^2}{4}\right)}$$

We can see that $f(z)$ has two simple poles at $z=-\frac{\pi}{2}\,i,\ \frac{\pi}{2}\,i$. However, these singularities are not enclosed in the unit circle centered around the origin. Hence, since the function is analytic inside this region, by Cauchy's theorem we have that $\frac{1}{2\pi i}\oint_C f(z)\,dz = 0$.

e) $\dfrac{z+\frac{1}{z}}{z\left(2z-\frac{1}{2z}\right)}$

Solution:

$$\frac{z+\frac{1}{z}}{z\left(2z-\frac{1}{2z}\right)} = \frac{z^2+1}{z}\ \frac{1}{\left(2z^2-\frac{1}{2}\right)} = \frac{z^2+1}{z}\ \frac{1}{2\left(z^2-\frac{1}{4}\right)} = \frac{z^2+1}{2z\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}$$

We can see that $f(z)$ has three simple poles at $z=0,\ \frac{1}{2},\ -\frac{1}{2}$ and they are located inside the enclosed region so we have to find the residues at these points:

- $\text{Res}((f(z);0)) = \lim_{z\to 0}\left(z\,\frac{z^2+1}{2z\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\right) = \lim_{z\to 0}\left(\frac{z^2+1}{2\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\right) = \frac{1}{-\frac{1}{2}} = -2$
- $\text{Res}\left(\left(f(z);\frac{1}{2}\right)\right) = \lim_{z\to \frac{1}{2}}\left(\left(z-\frac{1}{2}\right)\frac{z^2+1}{2z\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\right) = \lim_{z\to \frac{1}{2}}\left(\frac{z^2+1}{2z\left(z+\frac{1}{2}\right)}\right) = \frac{5}{4}$
- $\text{Res}\left(\left(f(z);-\frac{1}{2}\right)\right) = \lim_{z\to -\frac{1}{2}}\left(\left(z+\frac{1}{2}\right)\frac{z^2+1}{2z\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\right) = \lim_{z\to -\frac{1}{2}}\left(\frac{z^2+1}{2z\left(z-\frac{1}{2}\right)}\right) = \frac{5}{4}$

Hence we have

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\,2\pi i\left[\text{Res}((f(z);0))+\text{Res}\left(\left(f(z);\frac{1}{2}\right)\right)+\text{Res}\left(\left(f(z);-\frac{1}{2}\right)\right)\right]$$

$$= -2 + \frac{5}{4} + \frac{5}{4} = \frac{1}{2}.$$

(2) Evaluate the integrals $\frac{1}{2\pi i} \oint_C f(z)\, dz$, where C is the unit circle centered at the origin and $f(z)$ is given below. Do these problems by enclosing the singular points inside C.

a) $\frac{z^2+1}{z^2 - a^2}$, $a^2 < 1$

Solution:

$$\frac{z^2+1}{z^2 - a^2} = \frac{z^2+1}{(z+a)(z-a)}$$

We can see that $f(z)$ has two simple poles at $z = a, -a$, which lie inside the enclosed region. Thus we want to find the residues at these points:

- $\text{Res}((f(z); a)) = \underset{z \to a}{\text{limit}} \left((z-a) \frac{z^2+1}{(z+a)(z-a)} \right) = \underset{z \to a}{\text{limit}} \left(\frac{z^2+1}{(z+a)} \right) = \frac{a^2+1}{2a}$
- $\text{Res}((f(z); -a)) = \underset{z \to -a}{\text{limit}} \left((z+a) \frac{z^2+1}{(z+a)(z-a)} \right) = \underset{z \to -a}{\text{limit}} \left(\frac{z^2+1}{(z-a)} \right) = \frac{(-a)^2+1}{-2a} = -\frac{a^2+1}{2a}$

Hence we have

$$\frac{1}{2\pi i} \oint_C f(z)\, dz = \frac{1}{2\pi i} 2\pi i [\text{Res}((f(z); a)) + \text{Res}((f(z); -a))]$$
$$= \frac{a^2+1}{2a} - \frac{a^2+1}{2a} = 0.$$

b) $\frac{z^2+1}{z^3}$

Solution:

In this case we have that $f(z)$ has a pole of order 3 at $z = 0$, so we calculate the residue at this singularity as follows:

$$\text{Res}((f(z); 0)) = \frac{1}{2!} \frac{d^2}{dz^2} \left(z^3 \frac{z^2+1}{z^3} \right) \Big|_{z=0} = \frac{1}{2} \frac{d^2}{dz^2} (z^2+1) \Big|_{z=0} = \frac{1}{2} \cdot 2 = 1$$

Hence we have

$$\frac{1}{2\pi i} \oint_C f(z)\, dz = \frac{1}{2\pi i} 2\pi i (\text{Res}(f(z); 0)) = 1.$$

c) $z^2 e^{-1/z}$

Solution:

We can see that $f(z)$ has a singularity at $z=0$. However this is an essential singularity, thus we have to use a Laurent expansion to find the residue:

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \ldots + \frac{(-1)^n}{n!\,z^n}.$$

Then

$$z^2\, e^{-1/z} = z^2\left(1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \ldots + \frac{(-1)^n}{n!\,z^n}\right).$$

From here we can see that the residue is $-\frac{1}{3!}$.

Hence

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\,2\pi i \cdot \mathrm{Res}(f(z);\,0) = -\frac{1}{6}.$$

(Problem B) Evaluate $\oint_C \frac{e^{\pi z}}{z^2(z^2+1)}\,dz$; where C is the circle $|z| = 2$.

Solution:

We can see that $f(z)$ has two simple poles at $z = i,\ -i$, and a pole of order 2 at $z=0$. All three singularities lie inside the circle $|z|=2$ so we have to find the residues at these points:

- $\mathrm{Res}((f(z);\,0)) = \frac{1}{1!}\,\frac{d}{dz}\left(z^2\,\frac{e^{\pi z}}{z^2(z^2+1)}\right)\Big|_{z=0} = \frac{d}{dz}\left(\frac{e^{\pi z}}{(z^2+1)}\right)\Big|_{z=0}$
  $= \left(\frac{\pi e^{\pi z}(z^2+1) - 2\,z\,e^{\pi z}}{(z^2+1)^2}\right)\Big|_{z=0} = \pi$
- $\mathrm{Res}((f(z);\,i)) = \frac{e^{\pi z}}{\frac{d}{dz}(z^4+z^2)}\Big|_{z=i} = \frac{e^{\pi z}}{4\,z^3+2\,z}\Big|_{z=i} = \frac{e^{i\pi}}{-4\,i+2\,i} = \frac{\cos\pi + i\sin\pi}{-2\,i} = -\frac{i}{2}$
- $\mathrm{Res}((f(z);\,-i)) = \frac{e^{\pi z}}{\frac{d}{dz}(z^4+z^2)}\Big|_{z=-i} = \frac{e^{\pi z}}{4\,z^3+2\,z}\Big|_{z=-i} = \frac{e^{-i\pi}}{4\,i-2\,i} = \frac{\cos\pi - i\sin\pi}{2\,i} = \frac{i}{2}$

Hence we have

$$\oint_C f(z)\,dz = 2\pi i\,[(\mathrm{Res}(f(z);\,0) + \mathrm{Res}((f(z);\,i)) + \mathrm{Res}((f(z);\,-i)))]$$

$$= 2\,\pi i\left(\pi + \frac{i}{2} - \frac{i}{2}\right) = 2\,\pi^2\,i.$$