Math 353 HW 7

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Section 3.1

(1) In the following we are given sequences. Discuss their limits and whether the convergence is uniform, in the region $\alpha \le |z| \le \beta$, for finite $\alpha,\ \beta > 0$.

a) $\left\{ \dfrac{1}{n\, z^2} \right\}_{n=1}^{\infty}$

Solution:

$\underset{n\to\infty}{\text{limit}} \dfrac{1}{n\, z^2} = 0 \ \ \forall \ z \in [\alpha,\ \beta]$.

Now we need to determine whether the convergence is uniform or just pointwise.
To show uniform convergence we must show that for each $z \in [\alpha,\ \beta]$, given any $\varepsilon > 0$ there exists an N depending on ε (but not on z), such that whenever $n > N$, $\left| \dfrac{1}{n\, z^2} - 0 \right| < \varepsilon$ holds.

We have

$$\begin{aligned}
\left| \frac{1}{n\, z^2} - 0 \right| < \varepsilon \Longrightarrow \left| \frac{1}{n\, z^2} \right| < \varepsilon \Longrightarrow \frac{1}{|n|} &< \varepsilon\, |z^2| \\
\Longrightarrow |n| &> \frac{1}{\varepsilon\, |z^2|} \\
\Longrightarrow n &> \frac{1}{\varepsilon\, |z^2|} \quad \text{(since } n > 0\text{)}
\end{aligned}$$

Now since $|z| \ge \alpha > 0$ we have that $\dfrac{1}{|z|} \le \dfrac{1}{\alpha} \Longrightarrow \dfrac{1}{|z^2|} \le \dfrac{1}{\alpha^2}$.

So if we choose $N > \dfrac{1}{\varepsilon \alpha^2}$, it should force the ε-statement to work.

Let's show that if $n > N > \dfrac{1}{\varepsilon \alpha^2}$ then,

$$\begin{aligned}
n > \frac{1}{\varepsilon \alpha^2} > \frac{1}{\varepsilon\, |z^2|} \Longrightarrow \frac{1}{n} &< \varepsilon\, |z^2| \\
\Longrightarrow \frac{1}{n\, |z^2|} &< \varepsilon
\end{aligned}$$

$$\Longrightarrow \left| \frac{1}{n\,z^2} \right| < \varepsilon \Longrightarrow \left| \frac{1}{n\,z^2} - 0 \right| < \varepsilon\,.$$

Thus the sequence $\left\{ \frac{1}{n\,z^2} \right\}_{n=1}^{\infty}$ is uniformly convergent.

b) $\left\{ \frac{1}{z^n} \right\}_{n=1}^{\infty}$

Solution:

$\underset{n\to\infty}{\text{limit}}\ \frac{1}{z^n} = 0$ for $1 < \alpha \le |z| \le \beta$.

Now we have

$$\left| \frac{1}{z^n} - 0 \right| < \varepsilon \Longrightarrow \left| \frac{1}{z^n} \right| < \varepsilon \Longrightarrow \frac{1}{z^n} < \varepsilon$$

$$\Longrightarrow z^n > \frac{1}{\varepsilon} \Longrightarrow \log z^n > \log \frac{1}{\varepsilon}$$

$$\Longrightarrow n > \log \frac{1}{\varepsilon}\ \frac{1}{\log z} = (\log 1 - \log \varepsilon)\ \frac{1}{\log z}$$

$$\Longrightarrow n > -\frac{\log \varepsilon}{\log z} < \log \varepsilon\,.$$

So we make $\mathcal{N} > \log \varepsilon$ (we were able to define $\mathcal{N}$ exclusively in terms of ε).
Then since $n > \mathcal{N}$, we have

$$n > \log \varepsilon > -\frac{\log \varepsilon}{\log z} \Longrightarrow (\log z) \cdot n > -\log \varepsilon$$

$$\Longrightarrow \log z^n > \log 1 - \log \varepsilon \Longrightarrow \log z^n > \log \frac{1}{\varepsilon}$$

$$\Longrightarrow z^n > \frac{1}{\varepsilon} \Longrightarrow \frac{1}{z^n} < \varepsilon \Longrightarrow \left| \frac{1}{z^n} - 0 \right| < \varepsilon\,.$$

Thus the sequence $\left\{ \frac{1}{z^n} \right\}_{n=1}^{\infty}$ is uniformly convergent.

(2) For the sequence in 1a), what can be said if

a) $\alpha = 0$

Solution:

If $\alpha = 0$, then $\left\{ \frac{1}{z^n} \right\}_{n=1}^{\infty}$ converges to $f(z) = 0$ for $0 < |z|$ but the convergence is pointwise and not uniform this time since we saw that $n > \frac{1}{\varepsilon\,|z^2|}$ and in this case as $|z| \to 0$, n gets very large, i.e. n goes to infinity.

b) $\alpha > 0$

Solution:
As we saw on 1a), the sequence is uniformly convergent.

c) $\beta = \infty$

Solution:
Also as we saw on 1a), the sequence is uniformly convergent.

(5) Show that the following series converge uniformly in the given regions:

a) $\sum_{n=1}^{\infty} z^n$, $0 \leq |z| \leq R$, $R < 1$

Solution:
Since $|z| \leq R$, we have that $|z|^j \leq R^j$. We let $M_j = R^j$, then $\sum_{j=1}^{\infty} M_j = \sum_{j=1}^{\infty} R^j$.

This is a geometric series, which is convergent since $R < 1$. Hence by Weierstrass's M test the series $\sum_{n=1}^{\infty} z^n$ converges uniformly.

b) $\sum_{n=1}^{\infty} e^{-n z}$, $R < |\text{Re}(z)| \leq 1$, $R > 0$

Solution:
We have

$$|e^{-j z}| = |e^{-j(x+i y)}| = |e^{-j x} e^{-i j y}| = |e^{-j x}| \leq 1 \quad (\text{since } 0 < R < |x| = |\text{Re}(z)| \leq 1).$$

We know that the largest $e^{-j x}$ can be is when $x = R$, hence we let $M_j = e^{-j R}$ so that

$|e^{-j z}| < e^{-j R} = M_j$. Then we have that $\sum_{j=1}^{\infty} e^{-jR}$ is a geometric series that converges for $|e^{-R}| < 1$.

Since we are given that $R > 0$, we are certain that this series converges. Hence by Weierstrass's M

test the series $\sum_{n=1}^{\infty} e^{-n z}$ converges uniformly.

Section 3.2

(1) Obtain the radius of convergence of the series $\sum_{n=1}^{\infty} s_n(z)$, where $s_n(z)$ is given by the following :

b) $\frac{z^n}{(n+1)!}$

Solution:

$$\lim_{j\to\infty}\left|\frac{z^{j+1}}{(j+2)!}\;\frac{(j+1)!}{z^j}\right| = \lim_{j\to\infty}\left|\frac{z}{(j+2)}\right| = 0 < 1.$$

Hence the radius of convergence is $R = \infty$.

c) $n^n z^n$

Solution:

$$\begin{aligned}\lim_{j\to\infty}\left|\frac{(j+1)^{j+1} z^{j+1}}{j^j z^j}\right| &= \lim_{j\to\infty}\left|\frac{(j+1)^j (j+1) z}{j^j}\right| \\ &= \lim_{j\to\infty}\left|\left(\frac{j+1}{j}\right)^j (j+1)\, z\right| \\ &= |e \cdot \infty \cdot z| = \infty .\end{aligned}$$

Hence the series only converges when $z = 0$ and thus the radius of convergence is $R = 0$.

d) $\frac{z^{2n}}{2\,n!}$

Solution:

$$\lim_{j\to\infty}\left|\frac{z^{2j+2}}{(2j+2)!}\;\frac{2j!}{z^{2j}}\right| = \lim_{j\to\infty}\left|\frac{z^2}{(2j+2)(2j+1)}\right| = 0 < 1$$

Hence the radius of convergence is $R = \infty$.

e) $\dfrac{n!}{n^n}\ z^n$

<u>Solution:</u>

$$\underset{j\to\infty}{\text{limit}}\left|\ \frac{(j+1)!\ z^{j+1}}{(j+1)^{j+1}}\ \ \frac{j^j}{j!\ z^j}\ \right| = \underset{j\to\infty}{\text{limit}}\left|\ \frac{z\,j^j}{(j+1)^j}\ \right|$$

$$= \underset{j\to\infty}{\text{limit}}\left|\left(\frac{j}{j+1}\right)^j z\right| = \left|\frac{1}{e}\ z\right| < 1.$$

Thus we have that $\left|\frac{1}{e}\ z\right| < 1 \Longrightarrow |z| \leq e$, which indicates that the radius of convergence is $R = e$.

(2) Find Taylor series expansions around $z = 0$ of the following functions in the given regions:

b) $\dfrac{z}{1+z^2}$, $|z| < 1$

<u>Solution:</u>

We know that $\dfrac{1}{1-z}$ can be expanded as $\sum_{j=0}^{\infty} z^j$. So in this case we have

$$\frac{z}{1+z^2} = z\sum_{j=0}^{\infty}(-z^2)^j = \sum_{j=0}^{\infty}(-1)^j z^{2j+1}.$$

d) $\dfrac{\sin z}{z}$, $0 < |z| < \infty$

<u>Solution:</u>

Letting $b_j = \sin(z)$, we have

$f(0) = 0$

$f\,'(0) = \cos(0) = 1$

$f\,''(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$

$f^{(4)}(0) = \sin(0) = 0$ ....

We can see that all the even terms are zero so we are left with $\sin z = \sum_{j=0}^{\infty}\dfrac{(-1)^j z^{2j+1}}{(2j+1)!}$. Hence

$$\frac{\sin z}{z} = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!}.$$

f) $\dfrac{e^{z^2}-1-z^2}{z^3}$, $0 < |z| < \infty$

Solution:

Expanding e^{z^2} we have $\sum_{j=0}^{\infty} \frac{z^{2j}}{j!} = 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \ldots + \frac{z^{2j}}{j!}$.

Thus

$$e^{z^2} - 1 - z^2 = \frac{z^4}{2!} + \frac{z^6}{3!} + \ldots + \frac{z^{2j}}{j!},$$

which implies

$$\frac{e^{z^2}-1-z^2}{z^3} = \frac{1}{z^3} \sum_{j=2}^{\infty} \frac{z^{2j}}{j!} = \frac{1}{z^3} \sum_{j=0}^{\infty} \frac{z^{2j+4}}{(j+2)!} = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(j+2)!}.$$

(4) Show that about any point $z = x_0$, the equality $e^z = e^{x_0} \sum_{n=0}^{\infty} \frac{(z-x_0)^n}{n!}$ is true:

Solution:

The function e^z can be expanded about a point $z = x_0$ as follows :

$$f(z)\,|_{z=x_0} = e^z\,|_{z=x_0}$$
$$= f(x_0) + f'(x_0)(z - x_0) + \frac{f''(x_0)(z-x_0)^2}{2!} + \ldots + \frac{f^{(n)}(x_0)(z-x_0)^n}{n!}.$$

But notice that $f(x_0) = e^{x_0}$ and also all derivatives $f^{(i)}(x_0) = e^{x_0}\ \forall\ i$. Hence we can factor the e^{x_0} term out of the series and we have

$$e^z = e^{x_0}\left(1 + (z - x_0) + \frac{(z-x_0)^2}{2!} + \frac{(z-x_0)^3}{3!} + \ldots + \frac{(z-x_0)^n}{n!}\right)$$
$$= e^{x_0} \sum_{n=0}^{\infty} \frac{(z-x_0)^n}{n!}.$$