Math 351 DNHI 2

Mario L. Gutierrez Abed

(1) Check that the relation "is equivalent to" defines an equivalence relation. That is, show that

(i) $A \sim A$
Solution:
Let $\zeta : A \longrightarrow A$ be defined by $\zeta(x) = x$. Then clearly ζ is bijective. Hence $A \sim A$. ✓

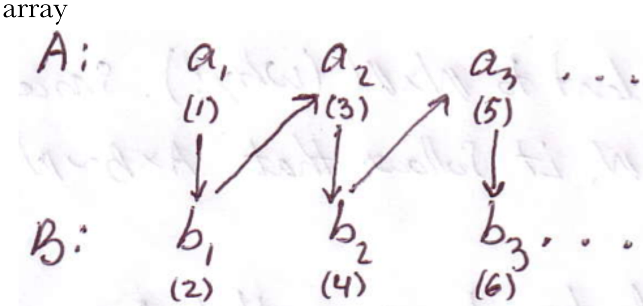
(ii) $A \sim B$ iff $B \sim A$
Solution:
Suppose $A \sim B$. Then, by hypothesis, there is some $f : A \longrightarrow B$ that is bijective .Thus f is invertible and $f^{-1} : B \longrightarrow A$ is also necessarily bijective. In particular $B \sim A$. ✓

(iii) if $A \sim B$ and $B \sim C$, then $A \sim C$.
Solution:
If $A \sim B$ and $B \sim C$, then by hypothesis, there are functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ that are both bijective. Clearly the function $h : A \longrightarrow C$ defined by $h(x) = g(f(x))$ (or $(g \circ f)\, x$) is also a bijection. Hence $A \sim C$, as desired. ✓

(2) Given finitely many countable sets $A_1, \ \ldots, \ A_n$, show that

a) $A_1 \cup \ \ldots \cup \ A_n$ is countable.

Proof:
First observe that for any countable sets A and B, $A \bigcup B$ is countable. To show this, we arrange A and B into an array



and then we define $f : \mathbb{N} \longrightarrow A \bigcup B$ by

$$f(n) = \begin{cases} a_k & \text{if } n = 2\,k - 1 \\ b_k & \text{if } n = 2\,k \end{cases}.$$

Observe now that f is bijective. Hence $\mathbb{N} \sim A \bigcup B$ and the result follows.

We will now use induction to prove that $A_1 \cup \ \ldots \cup \ A_n$ is countable. Clearly, by the argument we just proved, $A_1 \bigcup A_2$ is countable, so assume $A_1 \cup \ \ldots \cup \ A_{n-1}$ is countable. Let $A = A_1 \cup \ \ldots \cup \ A_{n-1}$ and $B = A_n$. Then, A and B are countable, from which follows that $A \bigcup B = (A_1 \cup \ \ldots \cup \ A_{n-1}) \bigcup A_n = A_1 \bigcup \ \ldots \ \bigcup A_n$ is countable as well. ■

b) $A_1 \times \ \ldots \ \times A_n$ is countable.

<u>Proof:</u>

First observe that if A and B are countable, then so is $A \times B$. To show this, we arrange $A \times B$ into an array

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \ldots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \ldots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Clearly $A \times B$ is equivalent to $\mathbb{N} \times \mathbb{N}$. Then, since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, it follows that $A \times B \sim \mathbb{N}$ and thus $A \times B$ is countable.

We will now prove by induction that $A_1 \times \ \ldots \ \times A_n$ is countable. Clearly, by the argument we just proved, $A_1 \times A_2$ is countable, so assume $A_1 \times \ \ldots \ \times A_{n-1}$ is countable. Let $A = A_1 \times \ \ldots \ \times A_{n-1}$ and $B = A_n$. Then, A and B are countable, from which follows that $A \times B = (A_1 \times \ \ldots \ \times A_{n-1}) \times A_n = A_1 \times \ \ldots \ \times A_n$ is countable. ■

(3) Prove that a set is infinite iff it is equivalent to a proper subset of itself.

(Hint: If A is infinite and $x \in A$, show that A is equivalent to $A \backslash \{x\}$)

<u>Proof:</u>

($\Leftarrow$)

Suppose A is finite and $B \subset A$. We can write $A = \{x_1, \ \ldots, \ x_n\}$, where, WLOG, $x_1 \notin B$. Therefore $B \subseteq A \backslash \{x_1\}$. It is easily seen from the pigeonhole principle that no map from $A \backslash \{x_1\}$ to A can be onto. Hence, no map from B to A is onto. In other words, A is not equivalent to any of its proper subsets.

(⇒)
Suppose on the other hand that A is infinite. Then, for some $x \in A$, $A \setminus \{x\}$ is infinite as well. Now, since every infinite set contains a countable subset, it follows that there is a sequence of distinct elements $\{y_n\}_{n=1}^{\infty} \subset A \setminus \{x\}$.
Define $f : A \setminus \{x\} \longrightarrow A$ by

$$f(y) = \begin{cases} x & \text{if } y = y_1 \\ y_{n-1} & \text{if } y = y_n\,;\, n \geq 2 \\ y & \text{otherwise} \end{cases}$$

It's easy to verify that f is bijective.
We can therefore conclude that $A \sim A \setminus \{x\}$. ■

(4) If A is infinite and B is countable, show that A and $A \cup B$ are equivalent.
(Hint: No containment relation between A and B is assumed here.)

Proof:
Suppose A is infinite and B is countable. Then A contains a countably infinite subset that we can list in sequential order $\{a_n\}_{n=1}^{\infty}$. Since B is countable, we can list its elements in a sequence $\{b_n\}_{n=1}^{\infty}$.
Now let's define $f : A \longrightarrow A \bigcup B$ by

$$f(x) = \begin{cases} a_n & \text{if } x = a_{2\,n} \\ b_n & \text{if } x = a_{2\,n-1} \\ x & \text{otherwise} \end{cases}$$

Clearly, f is bijective. Thus, $A \sim A \bigcup B$ as desired. ■

(5) Let A be countable. If $f : A \longrightarrow B$ is onto, show that B is countable. If $g : C \longrightarrow A$ is one-to-one, show that C is countable. [Hint: Be careful!]

Proof:
Let A be countable and suppose $f : A \longrightarrow B$ is onto. Then $A = f^{-1}(B) = \bigcup_{b \in B} A_b$ where $A_b = \{x \in A : f(x) = b\}$. Notice that since f is a function, $A_{b_1} \bigcap A_{b_2} = \emptyset$ whenever $b_1 \neq b_2$. Furthermore, since f is onto B, each $A_b \neq \emptyset$. By the axiom of choice, we get a subset $K \subset A$ that contains one element from each A_b. Clearly the restricted function $\gamma : K \longrightarrow B$ is bijective. Therefore $K \sim B$. Since A is countable, every subset of A is at most countable. It follows that K and therefore B are at most countable.
Now suppose $g : C \longrightarrow A$ is injective. Then $g : C \longrightarrow g(C)$ is bijective. It follows that $C \sim g(C) \subset A$. This implies that $g(C)$ is at most countable. Therefore C is at most countable. ■

(6) Show that $(0,\ 1)$ is equivalent to $[0,\ 1]$ and to $\mathbb{R}$.

Proof:

To see that $\mathbb{R} \sim (0,\ 1)$, observe that $f : \mathbb{R} \longrightarrow (0,\ 1)$ given by $f(x) = \dfrac{\tan^{-1}(x) + \frac{\pi}{2}}{\pi}$ is bijective. To prove that $(0,\ 1) \sim [0,\ 1]$, we take $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty} \subset (0,\ 1)$ and then define $g : (0,\ 1) \longrightarrow [0,\ 1]$ by

$$g(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{3} \\ \frac{1}{n-1} & \text{if } x = \frac{1}{n+1} : n \geq 3 \\ x & \text{otherwise} \end{cases}$$

Then g is bijective and the result $(0,\ 1) \sim [0,\ 1]$ follows. ■

(7) Show that $(0,\ 1)$ is equivalent to the unit square $(0,\ 1) \times (0,\ 1)$.
(Hint: "Interlace" decimals—but carefully!)

Proof:
Let $y \in (0,\ 1) \times (0,\ 1)$. Then $y = (y_1,\ y_2)$ where $y_1 = 0.\, a_1\, a_2\, a_3 \ldots.$ and $y_2 = 0.\, b_1\, b_2\, b_3 \ldots$ are the unique infinite base-10 decimal representations of $y_1$and y_2.
Now we define $f : (0,\ 1) \longrightarrow (0,\ 1) \times (0,\ 1)$ by

$$f(x) = (0.\, x_1\, x_2\, x_3\, x_5 \ldots x_{2n-1} \ldots,\ 0.\, x_2\, x_4\, x_6 \ldots x_{2n} \ldots)$$

whenever $x = 0.\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 \ldots x_{2n-1}\, x_{2n} \ldots$ is the unique infinite base-10 representation of x.
To see that f is onto, observe that if $y = (0.\, a_1\, a_2\, a_3 \ldots,\ 0.\, b_1\, b_2\, b_3 \ldots)$, then $x = 0.\, a_1\, b_1\, a_2\, b_2\, a_3\, b_3 \ldots$ will be mapped by f to y (i.e. $f(x) = y$). And since we can recover the input x from the output y, we see that f is invertible and therefore 1-1. We have thus shown that $(0,\ 1) \sim (0,\ 1) \times (0,\ 1)$. ■

(8) Prove that $(0,\ 1)$ can be put into one-to-one correspondence with the set of all functions $f : \mathbb{N} \longrightarrow \{0,\ 1\}$.

Proof:
Let $L_\infty\{0,\ 1\}$ be the set of all functions $f : \mathbb{N} \longrightarrow \{0,\ 1\}$. Since the set of all functions from $\mathbb{N}$ to any set M is just the set of sequences with range in M, we see that each f can be uniquely represented by a string of 0's and 1's. For example, $(0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 1,\ 1,\ \ldots)$ refers to the function $f \in L_\infty\{0,\ 1\}$ defined by $f(1) = 0$, $f(2) = 0$, $f(3) = 1$, etc. In general, $f(n) = 1$ whenever $n = 3\,k$

or $4k$ and $f(n) = 0$ otherwise.
Notice that each $x \in (0, 1)$ has a unique infinite base-2 decimal approximation. In other words, we can write $x = 0.x_1 x_2 x_3 \ldots$ (base 2) where each $x_i = 0$ or 1.
Now we define $\varphi : (0, 1) \longrightarrow L_\infty \{0, 1\}$ by $\varphi(x) = f_x$, where $f_x(n) = x_n$. In other words, if $x = 0.x_1 x_2 x_3 \ldots$ (base 2), then $f_x = (x_1, x_2, x_3, \ldots)$.
Clearly φ is bijective. Thus $(0, 1) \sim L_\infty \{0, 1\}$ as desired. ■

(9) Show that the set of all functions $f : A \longrightarrow \{0, 1\}$ is equivalent to $\mathcal{P}(A)$, the power set of A.

<u>Proof:</u>
Define $L_A \{0, 1\}$ to be the set of all functions $f : A \longrightarrow \{0, 1\}$. Now we define $\varphi : \mathcal{P}(A) \longrightarrow L_A \{0, 1\}$ by $\varphi(S) = f_S$, where

$$f_S(x) = \begin{cases} 0 & \text{if } x \notin S \\ 1 & \text{if } x \in S \end{cases}.$$

To see that φ is onto, pick any $f : A \longrightarrow \{0, 1\}$ and set $S = f^{-1}\{1\}$. Then clearly $\varphi(S) = f_S = f$. In particular, each $f \in L_A \{0, 1\}$ can be identified with the set $f^{-1}\{1\}$. Now to see that φ is 1-1, notice that if $\varphi(S) = \varphi(T)$, then $f_S = f_T$. In other words, $f_S(x) = f_T(x) \quad \forall\, x \in A$. This means that $S = \{x \in A : f_S(x) = 1\} = \{x \in A : f_T(x) = 1\} = T$.
Hence, we conclude that $\mathcal{P}(A) \sim L_A \{0, 1\}$. ■

(10) Show that $\mathbb{N}$ contains infinitely many pairwise disjoint infinite subsets.

<u>Proof:</u>
Let p_n be the n^{th} prime. Then we define $\mathbb{N}_{p_n} = \{m \in \mathbb{N} : m = p_n^k \quad \forall\, k \in \mathbb{N}\}$. Thus, for example $\mathbb{N}_{p_1} = \mathbb{N}_2 = \{2, 4, 8, 16, 32, \ldots\}$.
Clearly, each $\mathbb{N}_{p_n}$ is infinite and $\mathbb{N}_{p_n} \cap \mathbb{N}_{p_t} = \emptyset$ whenever $n \neq t$. Thus, the sequence of sets $\{\mathbb{N}_{p_n}\}_{n=1}^{\infty}$ consists of infinite pairwise disjoint subsets of $\mathbb{N}$. ■

(11) Show that any collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$ is at most countable. (Hint: Each one contains a rational!)

<u>Proof:</u>
The hint pretty much sums up the argument. Let $\{q_n\}$ be some list of all the rational numbers and let $\mathcal{L}$ be a collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$. Then each $I \in \mathcal{L}$ contains some rational r_I that is not found in any other open interval $J \in \mathcal{L}$.
Let $K = \{r_I \in \mathbb{Q} : r_I \in I\}$ be a collection of rational representatives of the intervals of $\mathcal{L}$ (one representative from each $I \in \mathcal{L}$). Then we can label $I = I_1$ if $r_I = q_{n_1}$, where n_1 is the smallest integer $n \in \mathbb{N}$

such that $q_n = r_I$ for any $I \in \mathcal{L}$.
More generally, $I = I_k$ if the rational chosen in I, r_I, is equal to q_{n_k}, where n_k is the k^{th} smallest integer $n \in \mathbb{N}$ for which q_n is among the elements of K. ■

(12) Prove that $\mathbb{N}$ contains uncountably many infinite subsets $\{\mathbb{N}_\alpha\}_{\alpha \in \mathbb{R}}$ such that $\mathbb{N}_\alpha \cap \mathbb{N}_\beta$ is finite if $\alpha \neq \beta$.
(This problem appeared on the Putnam Mathematical Competition. It is considered very hard).

<u>Proof:</u>
Since $\mathbb{R} \sim (0,\ 1)$, we might as well show that a collection $\{\mathbb{N}_\alpha\}_{\alpha \in (0,1)}$ exists for which $\mathbb{N}_\alpha$ is infinite for each $\alpha \in (0,\ 1)$, but $\mathbb{N}_\alpha \bigcap \mathbb{N}_\beta$ is finite whenever $\alpha \neq \beta$. To do this, recall that each $\alpha \in (0,\ 1)$ can be written uniquely as an infinite binary (base-2) decimal expansion $\alpha = 0.\, a_1\, a_2\, a_3 \ldots$ where each $a_i = 0$ or 1. Now we let $P_0 = 2$ and $P_1 = 3$ and define

$$\mathbb{N}_\alpha = \left\{P_{a_1},\ {P_{a_1}}^{P_{a_2}},\ {P_{a_1}}^{{P_{a_2}}^{P_{a_n}}},\ {P_{a_1}}^{{P_{a_2}}^{{P_{a_n}}^{P_{a_n}}}},\ \ldots\right\}.$$

For example, if $\alpha = 0.010011 \ldots$ then

$$\mathbb{N}_\alpha = \left\{2,\ 2^3,\ 2^{3^2},\ 2^{3^{2^2}},\ 2^{3^{2^{2^3}}},\ 2^{3^{2^{2^{3^3}}}},\ \ldots\right\} = \{2,\ 8,\ 512,\ \ldots\}.$$

Observe that the n^{th} largest number of $\mathbb{N}_\alpha$ encodes the first n terms of the binary expansion of α.
Since we are using the unique infinite representation of α in binary code, $\mathbb{N}_\alpha$ is infinite.
If $\alpha,\ \beta \in (0,\ 1)$ with $\alpha \neq \beta$, then WLOG, $\alpha < \beta$ and $\alpha = 0.\, a_1\, a_2\, a_3 \ldots a_n \ldots$ and $\beta = 0.\, b_1\, b_2\, b_3 \ldots b_n \ldots$ will first be the different in the n^{th} digit, where $n = 1,\ 2,\ 3,\ \ldots$
That is $a_i = b_i$ for $i = 1,\ 2,\ 3,\ \ldots,\ n-1$, while $a_n = 0$ and $b_n = 1$. Clearly then $\mathbb{N}_\alpha \bigcap \mathbb{N}_\beta$ have $n-1$ terms in common. ■