MATH 725 HW#1

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Exercise (Exercise 1)**.** *Prove that every linearly independent subset of a nonzero vector space V can be extended to a basis of V.*

Let us approach this problem in two parts. First we restate the theorem that we covered in class which states that every nonzero vector space contains a basis. Then we use this result to prove the statement on the exercise.

Theorem 1. *Every nonzero vector space contains a basis.*

Proof. The idea is that a basis can be constructed as a maximal linearly independent set, and this maximal set will be found by using Zorn's lemma.[1]
Let V be a nonzero vector space and let S be the set of linearly independent sets in V. Since a single nonzero $v \in V$ is a linearly independent set, we have that $\{v\} \in S$, which indicates that S is nonempty.
Now for two linearly independent sets L and L' in V, we declare that $L \leq L'$ if $L \subset L'$, where $\leq$ represents the partial ordering on S by inclusion. It is easy to see that any subset of a linearly independent set is also a linearly independent set, so if $L \in S$, then any subset of L is also in S.
Now that we have defined a partial order on S, let $\{L_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered subset of S, i.e. a chain on S. That is, every L_λ is a linearly independent set in V and for any L_α and L_β in our chain we have $L_\alpha \subset L_\beta$ or $L_\beta \subset L_\alpha$. An upper bound for the L_λ's in S is the union

$$L = \bigcup_{\lambda\in\Lambda} L_\lambda. \tag{$\clubsuit$}$$

The next step is to check whether L is indeed a linearly independent set, so that L is an element of S; once that is settled then L would be an upper bound in S since $L_\lambda \subset L \ \forall \lambda \in \Lambda$.

Let us take any finite set of vectors $v_1, \dots, v_n \in L$, and show that they are linearly independent. Each v_k is in some L_λ, say $v_1 \in L_{\lambda_1}, \dots, v_n \in L_{\lambda n}$. Since the L_λ's are totally ordered, one of the sets $L_{\lambda_1}, \dots, L_{\lambda_n}$ contains the others. That means $v_1, \dots, v_n$ are all in a common L_λ, so they are linearly independent, as desired.

Zorn's lemma now tells us that S contains a maximal element: there is a linearly independent set $L \in V$ that is not contained in any larger linearly independent set in V. We will show that L spans V, so it is a basis.

If span(L) does not span V, then span$(L) \neq V$, so we can pick $v \in V$ with $v \notin$ span(L). Then L is a proper subset of $L \cup \{v\}$. We will show $L \cup \{v\}$ s linearly independent, which contradicts the maximality of L and thus proves that span$(L) = V$.

[1] Here's Zorn's Lemma, for reference:
Zorn's Lemma: If P is a partially ordered set in which every chain has an upper bound, then P has a maximal element.

To prove that $L \cup \{v\}$ is linearly independent, assume otherwise. That is, take a sum

$$\sum_{i=1}^{k} c_i v_i = 0$$

where the c_i's are not all 0 and the v_i's are taken from $L \cup \{v\}$. Since the elements of L are linearly independent, one of the v_i's with a nonzero coefficient must be v. We can re-index and suppose $v_k = v$, so $c_k \neq 0$. Then we must have $k \geq 2$, since otherwise $c_1 v = 0$, which is impossible since $v \neq 0$ and the coefficient of v is nonzero. Consequently we have

$$0 = c_k v + \sum_{i=1}^{k-1} c_i v_i$$

$$\implies c_k v = -\sum_{i=1}^{k-1} c_i v_i. \tag{$\dagger$}$$

Multiplying both sides of ($\dagger$) by $1/c_k$, we get

$$v = \sum_{i=1}^{k-1} \left(-\frac{c_i}{c_k}\right) v_i,$$

which shows that $v \in \operatorname{span}(L)$. ($\Rightarrow\Leftarrow$)

This is the contradiction we wanted because by assumption $v \notin \operatorname{span}(L)$. Hence $L \cup \{v\}$ is a linearly independent set, and we are done. $\square$

Proof of Exercise 1. Let Γ be a linearly independent subset of V. A basis of V (which exists by Theorem 1) containing Γ will be found as a maximal linearly independent subset containing Γ.
Take S to be the set of linearly independent sets in V that contain Γ, so that $\Gamma \in S$, which indicates that S is nonempty. The same argument as in the proof of Theorem 1 shows that every chain of S has an upper bound (if the L_λ's are linearly independent sets in V that each contain Γ, then their union L (defined on equation (♣) on the proof of Theorem 1) also contains Γ, and $L \in S$ because the L_λ's are totally ordered.)
Now by Zorn's lemma we have a maximal element of S. This is a linearly independent set in V that contains Γ and is maximal with respect to inclusion among all linearly independent sets in S containing Γ. The proof that a maximal element of S is a basis of V follows just as in the proof of Theorem 1 above. $\square$