MATH 750 NOTES
DIFFERENTIATION & INTEGRATION

MARIO L. GUTIERREZ ABED

Some Preliminaries

If $f\colon A \to \mathbb{R}$ is bounded, the extent to which f fails to be continuous at $a \in A$ can be measured in a precise way:

For $\delta > 0$, let

$$M(a, f, \delta) = \sup\{f(x) \mid x \in A, |x - a| < \delta\},$$
$$m(a, f, \delta) = \inf\{f(x) \mid x \in A, |x - a| < \delta\}.$$

Definition. *The **oscillation** $o(f, a)$ of f at a is defined by $o(f, a) = \lim_{\delta\to 0}[M(a, f, \delta) - m(a, f, \delta)]$.* ★

Remark: This limit always exists, since $M(a, f, \delta) - m(a, f, \delta)$ decreases as δ decreases. There are two important facts about $o(f, a)$, given in the following two theorems:

Theorem 1. *The bounded function f is continuous at a iff $o(f, a) = 0$.*

Theorem 2. *Let $A \subset \mathbb{R}^n$ be closed. If $f\colon A \to \mathbb{R}$ is any bounded function, and $\varepsilon > 0$, then $\{x \in A \mid o(f, x) \geq \varepsilon\}$ is closed.*

Differentiation

Definition. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **differentiable** at $a \in \mathbb{R}^n$ if there exists a linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$(\clubsuit) \qquad \lim_{h\to 0} \frac{|f(a + h) - f(a) - \lambda(h)|}{|h|} = 0.$$

★

Remark: Note that h is a point of $\mathbb{R}^n$ while $f(a+h) - f(a) - \lambda(h)$ is a point of $\mathbb{R}^m$, hence the norm signs on ($\clubsuit$) are essential. The linear transformation λ is usually denoted $Df(a)$ and it is called the **derivative** of f at a. Also, the matrix associated with such linear transformation is called the **Jacobian matrix** of f at a.

The justification for the phrase "the linear transformation λ" is given by the following theorem:

Theorem 3. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, there is a unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that equation (♣) is satisfied.*

Proof. Suppose $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \mu(h)|}{|h|} = 0.$$

If $d(h) = f(a+h) - f(a)$, then

$$\begin{aligned}
\lim_{h\to 0} \frac{|\lambda(h) - \mu(h)|}{|h|} &= \lim_{h\to 0} \frac{|\lambda(h) - d(h) + d(h) - \mu(h)|}{|h|} \\
&\leq \lim_{h\to 0} \frac{|\lambda(h) - d(h)|}{|h|} + \lim_{h\to 0} \frac{|d(h) - \mu(h)|}{|h|} \\
&= 0.
\end{aligned}$$

If $x \in \mathbb{R}^n$, then $tx \to 0$ as $t \to 0$. Hence for $x \neq 0$ we have

$$0 = \lim_{t\to 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} = \frac{|\lambda(x) - \mu(x)|}{|x|}.$$

Therefore $\lambda(x) = \mu(x)$, and this concludes our proof. □

Definition. *Let $f \colon \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}$. If the limit*

$$\lim_{h\to 0} \frac{f(a^1, \ldots, a^i + h, \ldots, a^n) - f(a^1, \ldots, a^n)}{h}$$

*exists, then it is called the i^{th} **partial derivative** of f at a, and it is denoted by $D_i f(a)$.* ★

Theorem 4. *Let $A \subset \mathbb{R}^n$. If an extremum of $f \colon A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof. Let $g_i(x) = f(a^1, \ldots, x, \ldots, a^n)$. Clearly g_i has a maximum (or minimum) at a^i, and g_i is defined in an open interval containing a^i. Hence $0 = g_i'(a^i) = D_i f(a)$. □

Theorem 5. *If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at a, then $D_j f^i(a)$ exists for $1 \le i \le m$, $1 \le j \le n$, and $f'(a)$ is the $m \times n$ matrix $(D_j f^i(a))$.*

Remark: There are several examples in the problems to show that the converse of the above theorem is false. It is true, however, if one hypothesis is added, as shown by the following theorem:

Theorem 6. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f^i(x)$ exist in an open set containing a and if each function $D_j f^i$ is continuous at a. Such a function f is said to be* ***continuously differentiable*** *at a.*

Lemma 1. *Let $A \subset \mathbb{R}^n$ be a rectangle and let $f\colon A \to \mathbb{R}^n$ be continuously differentiable. If there is a number M such that $|D_j f^i(x)| \le M$ for all x in the interior of A, then*

$$|f(y) - f(x)| \le n^2 M |y - x| \qquad \forall\ x, y \in A.$$

Proof. We have that

$$f^i(y) - f^i(x) = \sum_{j=1}^{n} [f^i(y^1, \ldots, y^j, x^{j+1}, \ldots, x^n) - f^i(y^1, \ldots, y^{j-1}, x^j, \ldots, x^n)].$$

Now, applying the mean value theorem, there exists some z_{ij} such that

$$\begin{aligned} f^i(y^1, \ldots, y^j, x^{j+1}, \ldots, x^n) - f^i(y^1, \ldots, y^{j-1}, x^j, \ldots, x^n) &= (y^j - x^j) \cdot D_j f^i(z_{ij}) \\ &\le M \cdot |y^j - x^j|. \end{aligned}$$

Thus,

$$|f^i(y) - f^i(x)| \le \sum_{j=1}^{n} |y^j - x^j| \cdot M \le nM|y - x| \qquad \text{(Since each } |y^j - x^j| \le |y - x|\text{)}.$$

Finally,

$$|f(y) - f(x)| \le \sum_{j=1}^{n} |f^i(y) - f^i(x)| \le n^2 M \cdot |y - x|.$$

□

Theorem 7 (Inverse Function Theorem). *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a, and $\det f'(a) \neq 0$. Then there is an open set V*

containing a and an open set W containing $f(a)$ such that $f\colon V \longrightarrow W$ has a continuous inverse $f^{-1}\colon W \to V$ which is differentiable and for all $y \in W$ satisfies

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Proof. See page 35, Spivak's. □

Remark: It should be noted that an inverse function f^{-1} may exist even if $\det f'(a) = 0$. For example, if $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, then $f'(0) = 0$ but f has the inverse function $f^{-1}(x) = \sqrt[3]{x}$. One thing is certain however: if $\det f'(a) = 0$, then f^{-1} cannot be differentiable at $f(a)$. To prove this, note that $f \circ f^{-1}(x) = x$. If f^{-1} were differentiable at $f(a)$, then the chain rule would give

$$\begin{aligned} f'(a) \cdot (f^{-1})'(f(a)) &= I \\ \Longrightarrow \det f'(a) \cdot \det (f^{-1})'(f(a)) &= 1, \end{aligned}$$

contradicting the assumption that $\det f'(a) = 0$.

Implicit Function Theorem

Now we want to ask ourselves the following question:

If $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $f(a^1, \ldots, a^n, b) = 0$, when can we find, for each $(x^1, \ldots, x^n)$ near $(a^1, \ldots, a^n)$, a unique y near b such that $f(x^1, \ldots, x^n, y) = 0$?

Even more generally, we can ask about the possibility of solving m equations, depending upon parameters $x^1, \ldots, x^n$, in m unknowns:

If

$$f_i\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \qquad \text{for } i = 1, \ldots, m$$

and

$$f_i(a^1, \ldots, a^n, b^1, \ldots, b^m) = 0 \qquad \text{for } i = 1, \ldots, m$$

when can we find, for each $(x^1, \ldots, x^n)$ near $(a^1, \ldots, a^n)$ a unique $(y^1, \ldots, y^m)$ near $(b^1, \ldots, b^m)$ which satisfies $f_i(x^1, \ldots, x^n, y^1, \ldots, y^m) = 0$? The answer is provided by the following theorem:

Theorem 8 (**Implicit Function Theorem**). *Suppose that $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable in an open set containing (a, b), and $f(a, b) = 0$. Let M be the $m \times m$ matrix*

$$D_{n+j} f^i(a, b) \qquad \textit{for } 1 \le i, j \le m.$$

If $\det M \neq 0$*, there is an open set* $A \subset \mathbb{R}^n$ *containing* a *and an open set* $B \subset \mathbb{R}^m$ *containing* b*, with the following property: for* $x \in A$ *there is a unique* $g(x) \in B$ *such that* $f(x, g(x)) = 0$*. The function* g *is differentiable.*

Proof. Define $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by $F(x,y) = (x, f(x,y))$. Then $\det F'(a,b) = \det M \neq 0$. By the *Inverse Function Theorem* there is an open set $W \subset \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a,b) = (a,0)$ and an open set in $\mathbb{R}^n \times \mathbb{R}^m$ containing (a,b), which we may take to be of the form $A \times B$, such that $F\colon A \times B \to W$ has a differentiable inverse $h\colon W \to A \times B$. Clearly h is of the form $h(x,y) = (x, k(x,y))$ for some differentiable function k (since F is of this form). Now let $\pi_2\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be defined by $\pi_2(x,y) = y$; then $\pi_2 \circ F = f$. Therefore

$$\begin{aligned} f(x, k(x,y)) &= f \circ h(x,y) = (\pi_2 \circ F) \circ h(x,y) \\ &= \pi_2 \circ (F \circ h)(x,y) = \pi_2(x,y) = y. \end{aligned}$$

Thus, $f(x, k(x,0)) = 0$; in other words we can define $g(x) = k(x,0)$. $\square$

Integration

Definition. *The* ***volume*** *of a closed* k*-cube* $[a_1, b_1] \times \cdots \times [a_k, b_k]$*, and also of an open* k*-cube* $(a_1, b_1) \times \cdots \times (a_k, b_k)$*, is given by* $(b_1 - a_1) \cdot \cdots \cdot (b_k - a_k)$*.* ★

Definition. *Suppose that* A *is a rectangle,* $f\colon A \to \mathbb{R}$ *is bounded, and* P *is a partition of* A*. For each subrectangle* S *of the partition, let*

$$\begin{aligned} m_S(f) &= \inf\{f(x) : x \in S\}, \\ M_S(f) &= \sup\{f(x) : x \in S\}, \end{aligned}$$

and let $v(S)$ *be the volume of* S*. Then the* ***lower sum*** *and* ***upper sum*** *of* f *for* P *are defined by*

$$L(f,P) = \sum_S m_S(f) \cdot v(S) \qquad \text{and} \qquad U(f,P) = \sum_S M_S(f) \cdot v(S),$$

respectively. ★

Remark: Clearly, $L(f,P) \le U(f,P)$, and an even stronger assertion (given in the corollary to the following lemma) is true.

Lemma 2. *Suppose the partition* P' *refines* P *(that is, each subrectangle of* P' *is contained in a subrectangle of* P*). Then*

$$L(f,P) \le L(f,P') \qquad \text{and} \qquad U(f,P') \le U(f,P).$$

Corollary 1. *If P and P' are any two partitions, then it is always true that $L(f, P') \leq U(f, P)$.*

Definition. *A function $f\colon A \to \mathbb{R}$ is said to be **integrable** (in the Riemann sense) on the rectangle A if f is bounded and if*

$$\sup\{L(f,P)\} = \inf\{U(f,P)\}.$$

★

Theorem 9. *A bounded function $f\colon A \to \mathbb{R}$ is Riemann integrable iff, for every $\varepsilon > 0$, there is a partition P of A such that $U(f,P) - L(f,P) < \varepsilon$.*

Example 1: Here's an example of a Riemann integrable function. Let $f\colon A \to \mathbb{R}$ be a constant function, $f(x) = c$. Then for any partition P and subrectangle S we have $m_S(f) = M_S(f) = c$, so that

$$L(f,P) = U(f,P) = \sum_S c \cdot v(S) = c \cdot v(A).$$

Hence we have $\int_A f = c \cdot v(A)$. ▲

Example 2: Here's an example of a nonintegrable function (in the Riemann sense). Let $f\colon [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

If P is any partition, then every subrectangle S will contain points (x,y) with x rational, and also points (x,y) with x irrational. Hence, $m_S(f) = 0$ and $M_S(f) = 1$, so

$$L(f,P) = \sum_S 0 \cdot v(S) = 0,$$

while

$$U(f,P) = \sum_S 1 \cdot v(S) = v([0,1] \times [0,1]) = 1,$$

Therefore, f is not Riemann integrable. ▲

Lemma 3. *Let A be a closed rectangle and let $f\colon A \to \mathbb{R}$ be a bounded function such that $o(f,x) < \varepsilon$ for all $x \in A$. Then there is a partition P of A with $U(f,P) - L(f,P) < \varepsilon \cdot v(A)$.*

Definition. *A subset $A \subset \mathbb{R}^n$ is said to have **measure zero** if for every $\varepsilon > 0$, there is a cover $\{U_1, U_2, \dots\}$ of A by closed (or open) rectangles such that $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$.* ★

Definition. *A subset $A \subset \mathbb{R}^n$ is said to have* ***content zero*** *if for every $\varepsilon > 0$, there is a finite cover $\{U_1, \dots, U_n\}$ of A by closed (or open) rectangles such that $\sum_{i=1}^n v(U_i) < \varepsilon$.* ★

Theorem 10. *If $a < b$, then $[a, b] \subset \mathbb{R}$ does not have content 0. In fact, if $\{U_1, \dots, U_n\}$ is a finite cover of $[a, b]$ by closed intervals, then $\sum_{i=1}^n v(U_i) \geq b - a$*

If $a < b$, it is also true that $[a, b]$ does not have measure 0. This follows from the following theorem:

Theorem 11. *If A is compact and has measure 0, then A also has content 0.*

Remark: The conclusion of *Theorem 11* is false if A is not compact. For example, let A be the set of rational numbers between 0 and 1; then A has measure 0. Suppose, however, that $\{[a_1, b_1], \dots, [a_n, b_n]\}$ covers A. Then A is contained in the closed set $[a_1, b_1] \bigcup \cdots \bigcup [a_n, b_n]$ and therefore,

$$[0, 1] \subset [a_1, b_1] \bigcup \cdots \bigcup [a_n, b_n].$$

It follows from *Theorem 10* that $\sum_{i=1}^n (b_i - a_i) \geq 1$ for any such cover, and consequently A does not have content 0.

Theorem 12. *Let A be a closed rectangle and $f\colon A \to \mathbb{R}$ be a bounded function. Let $B = \{x \mid f \text{ is not continuous at } x\}$. Then f is integrable iff B is a set of measure 0.*

Partitions of Unity

Theorem 13. *Let $A \subset \mathbb{R}^n$ and let $\mathcal{O}$ be an open cover of A. Then there is a collection Φ of C^∞ junctions φ defined in an open set containing A , with the following properties:*

1) *For each $x \in A$ we have $0 \leq \varphi(x) \leq 1$.*
2) *For each $x \in A$ there is an open set V containing x such that all but finitely many $\varphi \in \Phi$ are 0 on V.*
3) *For each $x \in A$ we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$ (note that by part 2), for each x this sum is finite in some open set containing x).*
4) *For each $\varphi \in \Phi$ there is an open set $U \in \mathcal{O}$ such that $\varphi = 0$ outside of some closed set contained in U, i.e. if V is a closed set contained in U, then $\varphi(x) = 0$ for all $x \in U \setminus V$.*

Definition. *A collection satisfying 1)–3) on* Theorem 14 *is called a C^∞* ***partition of unity*** *for A. If Φ also satisfies 4), then Φ is said to be* ***subordinate*** *to the cover $\mathcal{O}$.* ★

Remark: An important consequence of condition 2) of *Theorem 14* should be noted. Let $K \subset A$ be compact. Then for each $x \in K$ there is an open set V_x containing x such that only finitely many $\varphi \in \Phi$ are not 0 on V_x. Now, since K is compact, only finitely many such V_x cover K. Thus only finitely many $\varphi \in \Phi$ are not 0 on K.

Definition. *An open cover $\mathcal{O}$ of an open set $A \subset \mathbb{R}^n$ is said to be* ***admissible*** *if each $U \in \mathcal{O}$ is contained in A.* ★

Definition. *If Φ is subordinate to $\mathcal{O}$, the function $f \colon A \to \mathbb{R}$ is bounded in some open set around each point of A, and the set of discontinuities of f has measure 0, then each $\int_A \varphi \cdot |f|$ exists and we define f to be* ***integrable*** *(in the extended sense) if*

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot |f|$$

converges. ★

Theorem 14. *We have the following conditions:*

1) *If Ψ is another partition of unity, which happens to be subordinate to an admissible cover $\mathcal{O}'$ of A, then $\sum_{\psi \in \Psi} \int_A \psi \cdot |f|$ also converges, and*
$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\psi \in \Psi} \int_A \psi \cdot f.$$
2) *If A and f are bounded, then f is integrable in the extended.*
3) *If A is Jordan-measurable and f is bounded, then this definition of $\int_A f$ agrees with the definition of integrability in the Riemann sense.*