# MATH 4106 HOMEWORK ASSIGNMENT

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PROF. A. CLEMENT

**1)** Solve the following two problems. Show all your work.

a) Show that $x^4 - 22x^2 + 1$ is irreducible over $\mathbb{Q}$.

b) The polynomial $2x^3 + 3x^2 - 7x - 5$ can be factored into linear factors in $\mathbb{Z}_{11}[x]$. Find this factorization.

*Solution to Part a).* In order to solve this problem we are going to make use of the following theorem and its corollary:

**Theorem 1.** *If $f(x) \in \mathbb{Z}[x]$, then $f(x)$ factors into a product of polynomials of lower degrees $r$ and $s$ in $\mathbb{Q}[x]$ iff it has such a factorization with polynomials of the same degrees $r$ and $s$ in $\mathbb{Z}[x]$.*

**Corollary 1.** *If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is in $\mathbb{Z}[x]$ with $a_0 \neq 0$, and if $f(x)$ has a zero in $\mathbb{Q}$, then it has a zero $m$ in $\mathbb{Z}$, and $m$ must divide $a_0$.*

Now with this theorem and corollary at hand, we are going to show that $f(x) = x^4 - 22x^2 + 1$, viewed in $\mathbb{Q}[x]$, is irreducible over $\mathbb{Q}$. Note that since $f(x)$ is of degree 4, if it were reducible, it would have to factor into either two quadratic factors or into one linear and one cubic factor. We treat both of these cases one at a time:

• **Case 1:** If we assume that $f(x)$ has a linear factor in $\mathbb{Q}[x]$, then it has a zero in $\mathbb{Z}$, and by *Corollary 1*, this zero would have to be a divisor in $\mathbb{Z}$ of 1, i.e. $\pm 1$. But $f(1) = f(-1) = -20$, thus such factorization is impossible.

• **Case 2:** If, on the other hand, $f(x)$ factors into quadratic factors in $\mathbb{Q}[x]$, then by *Theorem 1*, it has a factorization of the form

$$(x^2 + ax + b)(x^2 + cx + d)$$

in $\mathbb{Z}[x]$. Expanding we get

$$x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd.$$

Now, equating coefficients of powers of $x$, we find that we must have

$$bd = 1, \quad ad + bc = 0, \quad ac + b + d = -22, \quad \text{and} \quad a + c = 0$$

for integers $a, b, c, d \in \mathbb{Z}$. From $bd = 1$, we see that either $b = d = 1$ or $b = d = -1$ (in any case, $b = d$). Also from $a + c = 0$, we see that $a = -c$. Now putting all this together, we see that from $ac + b + d = -22$ we must have either $-a^2 = -24$, in which case $a = \sqrt{24}$, or $-a^2 = -20$, in which case $a = \sqrt{20}$. In both cases we see that $a \notin \mathbb{Z}$, contrary to assumption. Thus we may conclude that a factorization into quadratic polynomials is also impossible and thus $f(x)$ is irreducible over $\mathbb{Q}$. $\square$

*Solution to Part b).* By inspection, we can see that 3 is a zero of $2x^3 + 3x^2 - 7x - 5 = 2x^3 + 3x^2 + 4x + 6 \in \mathbb{Z}_{11}[x]$. That is,

$$2(3)^3 + 3(3)^2 + 4(3) + 6 = 99 = 0 \in \mathbb{Z}_{11}.$$

Now, dividing by $x - 3$, we get

$$
\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & & & 2x^2 & + & 9x & + & 20 \\ \cline{2-8}
x-3 & 2x^3 & + & 3x^2 & - & 7x & - & 5 \\
 & 2x^3 & - & 6x^2 & & & & \\ \cline{2-4}
 & & & 9x^2 & - & 7x & & \\
 & & & 9x^2 & - & 27x & & \\ \cline{4-6}
 & & & & & 20x & - & 5 \\
 & & & & & 20x & - & 60 \\ \cline{6-8}
 & & & & & & & 55
\end{array}
$$

Since $55 = 0 \mod 11$, we have the factorization

$$
\begin{aligned}
x^3 + 3x^2 - 7x - 5 &= (x-3)(2x^2 + 9x + 20) \\
&= (x-3)(2x^2 - 2x - 2) \\
&= (x-3)(2(x^2 - x - 1)) \\
&= (x-3)2((x-4)(x+3)) \\
&= (x-3)(2x-8)(x+3) \\
&= (x+8)(2x+3)(x+3) \in \mathbb{Z}_{11}[x].
\end{aligned}
$$

$\square$