MATH 750 HW # 3

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Ex 3-22) If A is a Jordan-measurable set and $\varepsilon > 0$, show that there is a compact Jordan-measurable set $C \subset A$ such that $\int_{A \setminus C} 1 < \varepsilon$.

Proof. Before proceeding with our proof let us prove the following lemma:

Lemma 1. *If A is a closed rectangle, then $C \subset A$ is Jordan measurable iff for every $\varepsilon > 0$, there is a partition P of A such that*

$$\sum_{S \in \mathcal{S}_1} v(S) - \sum_{S \in \mathcal{S}_2} v(S) < \varepsilon$$

where $\mathcal{S}_1$ consists of all subrectangles intersecting C and $\mathcal{S}_2$ consists of all subrectangles contained in C.

Proof of Lemma 1. ($\Rightarrow$) Suppose C is Jordan measurable, so that its boundary has measure 0 (and hence content 0). Let $\varepsilon > 0$ and choose a finite set S_i for $i = 1, \ldots, n$ of open rectangles the sum of whose volumes is less than ε and such that the S_i form a cover of the boundary of C. Let P be a partition of A such that every subrectangle of P is either contained within each S_i or does not intersect it. This P satisfies the condition in the statement of the problem.

($\Leftarrow$) Suppose that for every $\varepsilon/2 > 0$, there is a partition P as in the statement of the problem. Then by replacing the rectangles with slightly larger ones, one can obtain the same result except now we will have ε in place of $\varepsilon/2$ and the S_i will be open rectangles. This shows that the boundary of C is of content 0; hence C is Jordan measurable, as desired. $\surd$

Now we are ready to provide our proof:

Let B be a closed rectangle containing A and apply *Lemma 1* with A as the Jordan measurable set. Let P be the partition as in the lemma and define

$$C = \bigcup_{S \in \mathcal{S}_2} S.$$

Then $C \subset A$ and clearly C is Jordan measurable. Moreover, we have

$$\int_{A \setminus C} 1 < \sum_{S \in \mathcal{S}_1} v(S) - \sum_{S \in \mathcal{S}_2} v(S) < \varepsilon,$$

which yields our desired result. □

Ex 3-32) Let $f\colon [a,b]\times[c,d]\to\mathbb{R}$ be continuous and suppose $D_2 f$ is continuous. Define $F(y)=\int_a^b f(x,y)\,dx$. Prove *Leibnitz's rule*: $F'(y)=\int_a^b D_2 f(x,y)\,dx$.

[*Hint:* $F(y)=\int_a^b f(x,y)dx=\int_a^b \left(\int_c^y D_2 f(x,y)\,dy+f(x,c)\right)\,dx$.]

Proof. Using the hint provided, we have the following:

$$\begin{aligned}
F'(y) &= \lim_{h\to 0}\frac{F(y+h)-F(y)}{h}\\
&= \lim_{h\to 0}\frac{\int_a^b\int_y^{y+h} D_2 f(x,y)\,dy\,dx}{h}\\
&= \lim_{h\to 0}\frac{\int_y^{y+h}\int_a^b D_2 f(x,y)\,dx\,dy}{h}\\
&= \lim_{h\to 0}\frac{\int_c^{y+h}\int_a^b D_2 f(x,y)\,dx\,dy-\int_c^{y}\int_a^b D_2 f(x,y)\,dx\,dy}{h}\\
&= \int_a^b D_2 f(x,y)\,dx.
\end{aligned}$$

□