TRRT Final Hand-In (PQ4)

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December 1, 2017

Mathematical Physics
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2017

The Casimir Element and the Whitehead Lemmas

Let L be a Lie algebra and $\rho\colon L \to \mathfrak{gl}(V)$ a (finite-dimensional, complex) representation. Associated to this representation there is an associated symmetric bilinear form $B_\rho\colon L \times L \to \mathbb{C}$, defined by

$$B_\rho(x, y) = \operatorname{Tr}\left(\rho(x) \circ \rho(y)\right).$$

Recall that a bilinear form B_ρ is said to be ***nondegenerate*** if the only $x \in L$ for which $B_\rho(x, y) = 0$ for all $y \in L$ is $x = 0$.

In this problem we will see how to use B_ρ, at least when L is semisimple and ρ is faithful, in order to prove two very useful results, known as the Whitehead lemmas.

Problem 1. *Show that B_ρ is **associative**, so that for all $x, y, z \in L$,*

$$B_\rho([x, y], z) = B_\rho(x, [y, z]).$$

Proof. This is a straightforward calculation:

$$\begin{aligned}
B_\rho([x, y], z) &= \operatorname{Tr}\left(\rho([x, y]) \circ \rho(z)\right) \\
&= \operatorname{Tr}\left([\rho(x), \rho(y)] \circ \rho(z)\right) && (\text{Since } \rho \text{ is a Lie homomorphism}) \\
&= \operatorname{Tr}\left(\rho(x)\rho(y)\rho(z) - \rho(y)\rho(x)\rho(z)\right) \\
&= \operatorname{Tr}\left(\rho(x)\rho(y)\rho(z)\right) - \operatorname{Tr}\left(\rho(y)\rho(x)\rho(z)\right) && (\text{Since } \operatorname{Tr}(A + B) = \operatorname{Tr} A + \operatorname{Tr} B) \\
&= \operatorname{Tr}\left(\rho(x)\rho(y)\rho(z)\right) - \operatorname{Tr}\left(\rho(x)\rho(z)\rho(y)\right) && (\text{Since } \operatorname{Tr}(AB) = \operatorname{Tr}(BA)) \\
&= \operatorname{Tr}\left(\rho(x)\rho(y)\rho(z) - \rho(x)\rho(z)\rho(y)\right) \\
&= \operatorname{Tr}\left(\rho(x) \circ [\rho(y), \rho(z)]\right) \\
&= \operatorname{Tr}\left(\rho(x) \circ \rho([y, z])\right) \\
&= B_\rho(x, [y, z]). \qquad \boxed{\text{Victoria!}}
\end{aligned}$$

Problem 2. *Let L be semisimple and let ρ be faithful. Then show that B_ρ is nondegenerate as follows:*

a) *Show that*

$$J = \{x \in L \mid B_\rho(x, y) = 0\ \forall\, y \in L\}$$

is an ideal.

b) *Show that J is solvable and hence, since L is semisimple, that $J = 0$.*

*Proof of **a)***. We take $x \in J$ and $y \in L$, so that we need to show $[x, y] \in J$. Let $z \in L$ also; then

$$\begin{aligned} B_\rho([x, y], z) &= B_\rho(x, [y, z]) && \text{(By Q1)} \\ &= 0 && \text{(Since } x \in J \text{ and } [y, z] \in L) \\ &\Rightarrow [x, y] \in J, \end{aligned}$$

thus showing that J is indeed an ideal. $\boxed{\text{Victoria!}}$

*Proof of **b)***. Let $x, y, z \in J$, and consider the image $\rho(J)$, which is a subalgebra of $\mathfrak{gl}(V)$. Then, by *Cartan's Criterion*, we have that since $\operatorname{Tr}(\rho([x, y]) \circ \rho(z)) = \operatorname{Tr}([\rho(x), \rho(y)]) \circ \rho(z)) = 0$ for all $[\rho(x), \rho(y)] \in \rho(J)'$ and $z \in \rho(J)$, then $\rho(J)$ must be solvable. But, since ρ is faithful, we have that $\rho(J) \cong J$, and thus J is also solvable. Then, as remarked on the problem, since L is semisimple, J must be trivial. $\boxed{\text{Victoria!}}$

From now on let L be semisimple and $\rho \colon L \to \mathfrak{gl}(V)$ ***any*** representation. Let I be the (unique) ideal complementary to $\ker \rho$. Then $\rho|_I$ is a faithful representation of I. Let $x_1, \ldots, x_n$ be any basis for I and let $y_1, \ldots, y_n$ be the dual basis relative to B_ρ; that is,

$$B_\rho(x_i, y_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3. *Let $x \in L$ be a fixed element and let us define complex numbers α_{ij} and β_{ij} by*

$$[x, x_i] = \sum_j \alpha_{ij} x_j \qquad \textit{and} \qquad [x, y_i] = \sum_j \beta_{ij} y_j.$$

Show that $\beta_{ij} = -\alpha_{ji}$.

Proof. We have

$$\begin{aligned} B_\rho([x, x_i], y_j) &= B_\rho\left(\sum_j \alpha_{ij} x_j, y_j\right) \\ &= \sum_j \alpha_{ij} \underbrace{B_\rho(x_j, y_j)}_{=1\,\forall\, j}. \end{aligned}$$

But B_ρ is symmetric; therefore we must have $B_\rho([x,x_i],y_j) = B_\rho(y_j,[x,x_i])$. Now,

$$\begin{aligned} B_\rho(y_j,[x,x_i]) &= B_\rho([y_j,x],x_i]) && \text{(By associativity established in Q1)} \\ &= B_\rho(-[x,y_j],x_i]) \\ &= B_\rho\left(-\sum_i \beta_{ji} y_i, x_i\right) \\ &= -\sum_i \beta_{ji} \underbrace{B_\rho(y_i,x_i)}_{=1\,\forall i}. \end{aligned}$$

This establishes that, for all $i,j \in \{1,\dots,n\}$, we have $\alpha_{ij} = -\beta_{ji}$ (or, equivalently, $-\alpha_{ji} = \beta_{ij}$), as desired. Victoria!

Problem 4. *Define the* ***Casimir element*** $C_\rho := \sum_i \rho(x_i) \circ \rho(y_i)$ *and prove that it satisfies the following properties:*

a) C_ρ *is independent of the basis.*

b) $C_\rho \circ \rho(x) = \rho(x) \circ C_\rho$ *for all* $x \in L$.

c) $\operatorname{Tr} C_\rho = \dim I = \dim L - \dim\ker\rho$, *and hence if* ρ *is irreducible (and* $V \neq 0$*), that*

$$C_\rho = \frac{\dim L - \dim\ker\rho}{\dim V}\,\mathrm{Id}_V.$$

Proof of ***a)***. Let $\widetilde{x}_1,\dots,\widetilde{x}_n$ be any other basis for I with dual basis $\widetilde{y}_1,\dots,\widetilde{y}_n$. Then there are rotation matrices $\Lambda = (\lambda_{ij})$ and $\Gamma = (\gamma_{ij})$ such that

$$\widetilde{x}_i = \sum_j \lambda_{ij} x_j \qquad \text{and} \qquad \widetilde{y}_i = \sum_j \gamma_{ij} y_j. \tag{$\spadesuit$}$$

But then, since x_j is dual to y_j, ($\spadesuit$) shows that the condition that $\widetilde{y}_i$ is the basis dual $\widetilde{x}_i$ is precisely that $\Lambda^{-1} = \Gamma^T$:

$$\begin{aligned} 1 &= B_\rho(\widetilde{x}_i,\widetilde{y}_i) \\ &= B_\rho\left(\sum_j \lambda_{ij} x_j, \sum_j \gamma_{ij} y_j\right) \\ &= \sum_j \lambda_{ij}\,\gamma_{ij} \underbrace{B_\rho(x_j,y_j)}_{=1} \\ &= \sum_j \lambda_{ij}\,\gamma_{ij} \\ &\Rightarrow (\lambda_{ij})^{-1} = (\gamma_{ij})^T. \end{aligned}$$

It follows then that the Casimir operator is independent of the chosen basis, i.e., $\sum_i \rho(\widetilde{x}_i) \circ \rho(\widetilde{y}_i) = \sum_i \rho(x_i) \circ \rho(y_i)$, as desired. Victoria!

*Proof of **b)**.* This is equivalent tp showing that the bracket $[\rho(x), C_\rho(x)]$ vanishes. We have

$$\begin{aligned}
[\rho(x), C_\rho] &= [\rho(x), \sum_i \rho(x_i) \circ \rho(y_i)] \\
&= \sum_i ([\rho(x), \rho(x_i)]\rho(y_i) + \rho(x_i)[\rho(x), \rho(y_i)]) && (\dagger) \\
&= \sum_i (\rho([x, x_i])\rho(y_i) + \rho(x_i)\rho([x, y_i])) \\
&= \sum_i \left(\rho\left(\sum_j \alpha_{ij} x_j\right)\rho(y_i) + \rho(x_i)\,\rho\left(\sum_j \beta_{ij} y_j\right)\right) \\
&= \sum_{ij} (\alpha_{ij}\rho(x_j)\,\rho(y_i) + \beta_{ij}\rho(x_i)\rho(y_j)) \\
&= \sum_{ij} (\alpha_{ij}\rho(x_j)\,\rho(y_i) - \alpha_{ji}\rho(x_i)\rho(y_j)) && \text{(By Q3)} \\
&= \sum_{ij} (\alpha_{ij}\rho(x_j)\,\rho(y_i) - \alpha_{ij}\rho(x_j)\rho(y_i)) \\
&= 0.
\end{aligned}$$

Note that (†) comes from the fact that $[x, yz] = [x, y]z + y[x, z]$ for $x, y, z \in \mathfrak{gl}(V)$. Thus we have shown that $\rho(x)$ and C_ρ commute for all $x \in L$, as desired. Victoria!

*Proof of **c)**.* We have

$$\begin{aligned}
\operatorname{Tr} C_\rho &= \operatorname{Tr}\left(\sum_{i=1}^n \rho(x_i) \circ \rho(y_i)\right) \\
&= \sum_{i=1}^n \operatorname{Tr}\,(\rho(x_i) \circ \rho(y_i)) && \text{(By linearity of Tr)} \\
&= \sum_{i=1}^n B_\rho(x_i, y_i) \\
&= n = \dim I = \operatorname{codim}\,\ker \rho.
\end{aligned}$$

Now, since we are dealing with an algebraically closed field ($\mathbb{C}$), by *Schur's Lemma* we must have that any endomorphism of an irreducible representation is a scalar multiple of the identity map. Thus, if (V, ρ) is irreducible, the Casimir operator must satisfy $C_\rho = \lambda\,\mathrm{Id}_V$ for some scalar λ. Now,

$$\begin{aligned}
\operatorname{Tr}(\lambda \mathrm{Id}_V) &= \lambda \operatorname{Tr}\,\mathrm{Id}_V && \text{(By linearity of Tr)} \\
&= \lambda\,\dim V,
\end{aligned}$$

and by the result above we have that $\operatorname{Tr} C_\rho = \dim I$.

Hence,

$$\begin{aligned} C_\rho &= \lambda \operatorname{Id}_V \\ \operatorname{Tr} C_\rho &= \operatorname{Tr}(\lambda \operatorname{Id}_V) \\ \dim I &= \lambda \dim V \\ \lambda &= \frac{\dim I}{\dim V}. \end{aligned}$$

$\boxed{\text{Victoria!}}$

Problem 5. *Let $f\colon L \to V$ be a linear map satisfying the "co-cycle condition"*

$$f([x,y]) = x \cdot f(y) - y \cdot f(x), \tag{1}$$

where, here and in what follows, we use the shorthand $x \cdot v$ to mean $\rho(x)(v)$, for $x \in L$ and $v \in V$. Show that there exists $v \in V$ such that $f(x) = x \cdot v$ for all $x \in L$, as follows:

a) *First assume that V has a proper submodule $U \subsetneq V$ with quotient $W = V/U$. Show that if the result holds for U and W, then it also holds for V.*

b) *By induction on $\dim V$, we are done if we prove the result when V is irreducible. Since there is nothing to prove when the representation is trivial (the co-cycle condition says $f(x) = 0$, so we can take $v = 0$), assume that ρ is irreducible and nontrivial. By Q4(c), C_ρ is invertible. Show then that v, defined by $C_\rho \cdot v = \sum_i x_i \cdot f(y_i)$, does the job.*

*Proof of **a)**.* We start by composing the canonical projection $\pi\colon V \to V/U := W$ that sends $v \mapsto v + U$ with f:

$$L \xrightarrow{f} V \xrightarrow{\pi} W,$$

and we now show that this composition $\pi \circ f$ does satisfy the co-cycle condition (1):

$$\begin{aligned} (\pi \circ f)([x,y]) &= \pi(x \cdot f(y) - y \cdot f(x)) \\ &= (x \cdot f(y) - y \cdot f(x)) + U \\ &= (x \cdot f(y) + U) - (y \cdot f(x) + U) \\ &= x \cdot (f(y) + U) - y \cdot (f(x) + U) \\ &= x \cdot (\pi \circ f)(y) - y \cdot (\pi \circ f)(x). \end{aligned}$$

Then, since $\pi \circ f\colon L \to W$ satisfies (1), by assumption there exists a $w + U \in W$ such that $(\pi \circ f)(x) = x \cdot (w + U)$ for all $x \in L$. Now choose an element $\overline{w} \in \pi^{-1}(w + U) \subset V$ and define a new linear map

$$\begin{aligned} g\colon L &\longrightarrow V \\ x &\longmapsto f(x) - x \cdot \overline{w}, \end{aligned}$$

so that the image of the composite map $\pi \circ g$ (and, consequently, the image of g also) is entirely contained in U:

$$
\begin{aligned}
(\pi \circ g)(x) &= \pi(f(x) - x \cdot \overline{w}) \\
&= \pi(f(x)) - \pi(x \cdot \overline{w}) \\
&= x \cdot (w + U) - x \cdot \pi(\overline{w}) \\
&= x \cdot (w + U) - x \cdot (w + U) \\
&= U.
\end{aligned}
$$

Now define a new function $\Xi\colon L \to U$ sending $x \mapsto g(x)$ (we showed above that $\mathrm{Im}(g) \subseteq U$), and we show that this map also satisfies the co-cycle condition:

Let $x, y \in L$; then,

$$
\begin{aligned}
\Xi([x,y]) &= g([x,y]) \\
&= f([x,y]) - [x,y] \cdot \overline{w} \\
&= x \cdot f(y) - y \cdot f(x) - x \cdot (y \cdot \overline{w}) + y \cdot (x \cdot \overline{w}) \\
&= x \cdot (f(y) - y \cdot \overline{w}) - y \cdot (f(x) - x \cdot \overline{w}) \\
&= x \cdot g(y) - y \cdot g(x) \\
&= x \cdot \Xi(y) - y \cdot \Xi(x).
\end{aligned}
$$

As Ξ satisfies the cocycle condition, by assumption there exists a $u \in U$ such that $\Xi(x) = x \cdot u$ for all $x \in L$. However, this means that $\Xi(x) = x \cdot u = f(x) - x \cdot \overline{w}$, so

$$
\begin{aligned}
f(x) &= x \cdot \underbrace{(u + \overline{w})}_{= v \text{ for some } v \in V} \\
&= x \cdot v.
\end{aligned}
$$

Therefore there does exists an element $v \in V$ such that $f(x) = x \cdot v$ for all $x \in L$. $\boxed{\text{Victoria!}}$

*Proof of **b)**.* The goal is to show that there exists a $v \in V$ that satisfies $f(x) = x \cdot v$ or, equivalently, $(x \cdot v - f(x)) = 0$ for all $x \in L$. Consider $C_\rho \circ (x \cdot v - f(x)) = C_\rho \circ (x \cdot v) - C_\rho \circ f(x)$. Then,

$$
\begin{aligned}
C_\rho \circ (x \cdot v) &= C_\rho \circ (\rho(x)(v)) \\
&= (C_\rho \circ \rho(x))(v) \\
&= (\rho(x) \circ C_\rho)(v) && \text{(By 4\textbf{\textit{b)}})} \\
&= x \cdot (C_\rho(v)).
\end{aligned}
$$

In addition, for all $x, y, z \in L$ we have

$$
\begin{aligned}
\rho(x) \circ \rho(y) \circ f(z) &= \rho(x) \circ (y \cdot f(z)) \\
&= x \cdot (y \cdot f(z)).
\end{aligned}
$$

Thus, combining these results, we get

$$
\begin{aligned}
C_\rho \circ (x \cdot v - f(x)) &= x \cdot (C_\rho(v)) - C_\rho \circ f(x) \\
&= x \cdot \left(\sum_i x_i \cdot f(y_i) \right) - \sum_i (\rho(x_i) \circ \rho(y_i)) \circ f(x) \\
&= \sum_i x \cdot (x_i \cdot f(y_i)) - \sum_i x_i \cdot (y_i \cdot f(x) \\
&= \sum_i (x \cdot (x_i \cdot f(y_i)) - x_i \cdot (y_i \cdot f(x)) + x_i \cdot (x \cdot (f(y_i))) - x_i \cdot (x \cdot (f(y_i))) \\
&= \sum_i (x \cdot (x_i \cdot f(y_i)) - x_i \cdot (x \cdot (f(y_i))) - x_i \cdot (y_i \cdot f(x)) + x_i \cdot (x \cdot (f(y_i))) \\
&= \sum_i ([x, x_i] \cdot f(y_i) - x_i \cdot (y_i \cdot f(x) - x \cdot f(y_i)) \\
&= \sum_i \left([x, x_i] \cdot f(y_i) - x_i \cdot f([y_i, x]) \right) \\
&= \sum_i \left([x, x_i] \cdot f(y_i) + x_i \cdot f([x, y_i]) \right), \\
&= \sum_i \left(\left(\sum_j \alpha_{ij} x_j \right) \cdot f(y_i) + x_i \cdot f\left(\sum_j \beta_{ij} y_j \right) \right), \\
&= \sum_i \left(\sum_j \alpha_{ij} \left(x_j \cdot f(y_i) \right) + \sum_j \beta_{ij} x_i \cdot f(y_j) \right), \\
&= \sum_{ij} (\alpha_{ij}(x_j \cdot f(y_i)) + \beta_{ij}(x_i \cdot f(y_j))), \\
&= \sum_{ij} (\alpha_{ij}(x_j \cdot f(y_i)) - \alpha_{ji}(x_i \cdot f(y_j))) = 0.
\end{aligned}
$$

Therefore $C_\rho \circ (x \cdot v - f(x)) = 0$. We know from Q4(c) that C_ρ is invertible, so composing both sides by $(C_\rho)^{-1}$ on the left gives $(x \cdot v - f(x)) = 0$, so that $f(x) = x \cdot v$. Then, by induction on $\dim(V)$, the proof is complete. Victoria!

Problem 6. *The following are two easy corollaries of the above result:*

a) *Show that every derivation $D\colon L \to L$ of a semisimple Lie algebra L is inner.*

b) *Show that if $\widehat{L}$ is a central extension of a semisimple Lie algebra L, then $\widehat{L} \cong L \oplus \mathbb{C}$ as Lie algebras. (Hint: The bilinear form $\omega\colon L \times L \to \mathbb{C}$ in the central extension defines a linear map $f\colon L \to L^*$ by $f(x)(y) = \omega(x, y)$ which obeys the co-cycle condition in Q5 for the co-adjoint representation L^*.)*

*Proof of **a)**.* This is the same as saying that the map $\mathrm{ad}\colon L \to \mathrm{Der}(L)$ is an isomorphism. Let $\mathrm{ad}(L)^\perp$ denote the orthogonal complement of $\mathrm{ad}(L)$ for the Killing form κ on $\mathrm{Der}(L)$, so that it suffices to show that $\mathrm{ad}(L)^\perp = 0$. We have

$$[\mathrm{ad}(L)^\perp, \mathrm{ad}(L)] \subset \mathrm{ad}(L)^\perp \cap \mathrm{ad}(L) = 0,$$

since $\mathrm{ad}(L)$ and $\mathrm{ad}(L)^\perp$ are both ideals in $\mathrm{Der}(L)$ and $\kappa|_{\mathrm{ad}(L)}$ is nondegenerate. Therefore, for $x \in L$ and $D \in \mathrm{ad}(L)^\perp$, we have

$$\begin{aligned} \mathrm{ad}(Dx) &= [D, \mathrm{ad}(x)] && \text{(I show this on PQ3, Q5\textit{b)})} \\ &= 0. \end{aligned}$$

But since $\mathrm{ad}\colon L \hookrightarrow \mathrm{Der}(L)$ is an injection, we get

$$\mathrm{ad}(Dx) = 0\ \forall\, x \in L \implies Dx = 0\ \forall\, x \in L \implies D = 0.$$

Therefore $\mathrm{ad}(L)^\perp = 0$, and we have the desired result. Victoria!

*Proof of **b)***. From PQ3, Q4a), we know that ω defines a central extension if and only if $\omega \in Z^2(L; \mathbb{C})$, which is the same as saying that it satisfies the Jacobi identity. Let f be as in the hint and $x, y, z \in L$; then we have

$$\begin{aligned} \omega([x,y], z) &= (f([x,y]))(z) \\ &= (x \cdot f(y))(z) - (y \cdot f(x))(z) \\ &= -(f(y))(x \cdot z) + (f(x))(y \cdot z) \\ &= -\omega(y, x \cdot z) + \omega(x, y \cdot z). \end{aligned}$$

But, since ω must satisfy the co-cyclic condition

$$\omega(x, [y,z]) + \omega(y, [z,x]) + \omega(z, [x,y]) = 0,$$

this result shows that $f \in Z^1(L; L^*)$ if and only if $\omega \in Z^2(L; \mathbb{C})$.

Recall that the bracket on a general central extension $\hat{L}$ of L is given by $[x,y] = [x,y]_L + \omega(x,y)Z$. Thus, treating $L \oplus \mathbb{C}$ as a central extension, it has bracket $[x,y] = [x,y]_L + 0Z$, as $\mathbb{C}$ is the trivial module.

Now, as ω defines a central extension, f must then satisfy the co-cyclic condition and hence there is a $g \in L^*$ such that $f(x)(-) = x \cdot g(-)$, for all $x \in L$. In PQ3, Q3b) we proved that, if $\omega_1 - \omega_2 = \partial_1 \zeta$ for some $\zeta \in C^1(L; \mathbb{C})$, then the two central extensions defined by ω_1 and ω_2 must be isomorphic. Note that we have

$$\omega(x,y) = (f(x))(y) = x \cdot g(y) = -g([x,y]) = (\partial_1 g)(x,y) \qquad \forall\, x, y \in L.$$

Then it is true that there exists a ζ (namely, g in this case) such that $\omega - 0 = (\partial_1 \zeta)$. So the extension by ω and 0 are isomorphic, proving that $\hat{L} \cong L \oplus \mathbb{C}$, as desired. Victoria!

Problem 7. *Let $\omega\colon L\times L\to V$ be a bilinear map satisfying the "co-cycle conditions": $\omega(x,x)=0$ for all $x\in L$ and*

$$x\cdot\omega(y,z)+\omega(x,[y,z])+\text{cyclic}(x,y,z)=0, \tag{2}$$

for all $x,y,z\in L$. Show that there exists a linear map $\theta\colon L\to V$ such that

$$\omega(x,y)=x\cdot\theta(y)-y\cdot\theta(x)-\theta([x,y]),$$

as follows:

a) *First assume that V has a proper submodule $U\subsetneq V$ with quotient $W=V/U$. Show that if the result holds for U and W, then it also holds for V.*

b) *By induction on $\dim V$, we are done if we prove the result when V is irreducible. Since Q6(b) takes care of the case when the representation is trivial, we can assume that ρ is irreducible and nontrivial. By Q4(c), C_ρ is invertible. If we write the (second) co-cycle condition as $(\partial\omega)(x,y,z)=0$, expand the equation $\sum_i x_i\cdot(\partial\omega)(x,y,y_i)=0$ and show that θ defined by $C_\rho\cdot\theta(x)=\sum_i\cdot\omega(y_i,x)$ for all $x\in L$ does the job.*

Draft of sketch of proof of ***a)*** As on Q5a), we start by composing the canonical projection $\pi\colon V\to V/U:=W$ that sends $v\mapsto v+U$ with ω:

$$L\times L\xrightarrow{\omega}V\xrightarrow{\pi}W,$$

and we now show that this composition $\pi\circ\omega$ does satisfy the co-cycle condition (2):

$$\begin{aligned}(\pi\circ\omega)(x,[y,z])&=\pi(-x\cdot\omega(y,z)-\text{cyclic}(x,y,z))\\&=-x\cdot\omega(y,z)-\text{cyclic}(x,y,z)+U\\&=(-x\cdot\omega(y,z)+U)-(\text{cyclic}(x,y,z)+U)\\&=-x\cdot(\omega(y,z)+U)-(\text{cyclic}(x,y,z)+U)\\&=-x\cdot(\pi\circ\omega)(y,z)-(\pi\circ\text{cyclic}(x,y,z)).\end{aligned}$$

Then, since $\pi\circ\omega\colon L\times L\to W$ satisfies (2), by assumption there exists a linear map θ_W and $w+U\in W$ such that

$$w+U=\pi\circ\omega(x,y)=x\cdot\theta_W(y)-y\cdot\theta_W(x)-\theta_W([x,y])\ \forall\,x,y\in L.$$

Now choose an element $\overline{w}\in\pi^{-1}(w+U)\subset V$ and define a new bilinear map

$$\begin{aligned}\varpi\colon L\times L&\longrightarrow V\\(x,y)&\longmapsto\omega(x,y)-x\cdot\theta_W(y)-y\cdot\theta_W(x)-\theta_W([x,y]).\end{aligned}$$

I must apologize, but I just ran out of time before I finished working through this problem ☹. I'm literally typing this with just a few minutes left until we hit the deadline, so the clock wins this battle … but not the war! $\boxed{\text{No Victoria ☹}}$