Math 260 HW # 6

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Section 2.5

(2) For the following pair of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinates matrix that changes β'-coordinates into β-coordinates.

d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

Solution:

$\bullet\!\!\rightarrow (2, 1) = a(-4, 3) + b(2, -1)$

$$\Longrightarrow \begin{aligned} -4a + 2b &= 2 \\ 3a - b &= 1 \end{aligned}$$

$\Longrightarrow a = 2, \quad b = 5$

$\bullet\!\!\rightarrow (-4, 1) = a(-4, 3) + b(2, 1)$

$$\Longrightarrow \begin{aligned} -4a + 2b &= -4 \\ 3a - b &= 1 \end{aligned}$$

$\Longrightarrow a = -1, \quad b = -4$

Hence $Q = [I]_{\beta'}^{\beta} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$ ✓

Let's check this...

$[(-4, 1)]_\beta = Q\,[(-4, 1)]_{\beta'} = Q\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$ ✓ All good in the hood :)

Section 3.2

(3) Prove that for any $m \times n$ matrix A, rank $A = 0$ iff A is the zero matrix.

Proof:

$(\Rightarrow)$

Assume that rank $A = 0$. Then, rank A = rank L_A = dim $R(L_A) = 0$. Thus, $R(L_A)$ is trivial, implying $L_A(x) = 0\ \forall\, x \in \mathbb{F}^n$. Thus, L_A is the zero map. Then, A must be the zero matrix as A is the matrix representation of L_A with respect to the usual bases. ✓

($\Leftarrow$)

Suppose that A is the zero matrix. Then we have n m-tuples containing nothing but zeroes. By definition each one of these tuples is linearly dependent. Hence each row/column in A is linearly dependent, which implies that the rank of the matrix is zero. ✓ ■

(5) For the following matrix, compute the rank and the inverse if it exists:

h) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$

⟼ First let's find the rank...

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -3 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 1 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

At this point it is easy to see that we have three linearly independent rows. However we can simplify even more to see more clearly that the rank of this matrix is 3...

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now not only do we have three linearly independent rows but also the same number of linearly indepedendent columns. Hence we have that the rank of the matrix is 3. ✓

This matrix is clearly not invertible since it has a linearly dependent row/column, i.e. the rank of the matrix is less than the number of rows/columns of the matrix. ✡

(8) Let A be an $m \times n$ matrix. Prove that if c is any nonzero scalar, then rank($c\,A$) = rank A.

Proof:

Given that A is any $m \times n$ matrix, we know that multiplying any nonzero scalar c with A is the same as multiplying m type-2 $m \times m$ elementary matrices with A. However this will only scale each of the m rows of A by some nonzero c, which doesn't affect the linear independence/dependence of the rows (or columns). Hence multiplying A with any nonzero scalar is a rank-preserving operation, which means that rank($c\,A$) = rank A . ■