Math 353 HW 1

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Section 1.1

(1) Express each of the following complex numbers in polar exponential form:

b) $-i$

$r = \sqrt{(-1)^2 + 0^2} = 1; \quad \theta = \frac{3\pi}{2} + 2\pi n$

Thus we have $-i = e^{i\left(\frac{3\pi}{2} + 2\pi n\right)}$; $n = 0, \pm 1, \pm 2, \ldots$

c) $1 + i$

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$; $\theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} + 2\pi n$

Thus we have $1 + i = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + 2\pi n\right)}$; $n = 0, \pm 1, \pm 2, \ldots$

d) $\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1; \; \theta = \arctan\left(\frac{\sqrt{3}}{2} \cdot 2\right) = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3} + 2\pi n$

Thus we have $\frac{1}{2} + \frac{\sqrt{3}}{2} i = e^{i\left(\frac{\pi}{3} + 2\pi n\right)}$; $n = 0, \pm 1, \pm 2, \ldots$

(2) Express each of the following in the form $a + b\, i$, where a and b are real:

a) $e^{2 + i\frac{\pi}{2}}$

$e^{2 + i\frac{\pi}{2}} = e^2 \cdot e^{\frac{\pi}{2} i} = e^2\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = e^2\, i$.

b) $\dfrac{1}{1+i}$

$$\frac{1}{1+i}\cdot\frac{1-i}{1-i} = \frac{1\cdot(1-i)}{1^2-i^2} = \frac{1-i}{2} = \boxed{\frac{1}{2}-\frac{1}{2}\,i}$$

(3) Solve for the roots of the following equations:

a) $z^3 = 4$

$x = 4;\quad y = 0;\quad r = 4;\quad \theta = 0 + 2\pi n$

$z^3 = 4\,e^{i(0+2\pi n)} = 4\,e^{i\,2\pi n}$

$z = \sqrt[3]{4\,e^{i\,2\pi n}} = \sqrt[3]{4}\;e^{i\,\frac{2\pi n}{3}}$

For $n = 1:\ z = \sqrt[3]{4}\;e^{i\,\frac{2\pi}{3}}$

For $n = 2:\ z = \sqrt[3]{4}\;e^{i\,\frac{4\pi}{3}}$

For $n = 3:\ z = \sqrt[3]{4}\;e^{i2\pi}$

For $n = 4:\ z = \sqrt[3]{4}\;e^{i\,\frac{8\pi}{3}}$, which is the same as $\sqrt[3]{4}\;e^{i\,\frac{2\pi}{3}}$.
Therefore for $n \geq 4$, the roots are repeated. Hence our problem has three unique roots:

$$z = \boxed{\left\{\sqrt[3]{4}\;e^{i\,\frac{2\pi}{3}},\ \sqrt[3]{4}\;e^{i\,\frac{4\pi}{3}},\ \sqrt[3]{4}\;e^{i\,2\pi}\right\}}$$

b) $z^4 = -1$

$x = -1;\quad y = 0;\quad r = 1;\quad \theta = \pi + 2\pi n$

$z^4 = e^{i(\pi+2\pi n)}$

$z = \sqrt[4]{e^{i(\pi+2\pi n)}} = e^{i\left(\frac{\pi}{4}+\frac{2\pi}{4}\,n\right)}$

For $n = 1:\ z = e^{i\,\frac{3\pi}{4}}$

For $n = 2:\ z = e^{i\,\frac{5\pi}{4}}$

For $n = 3:\ z = e^{i\,\frac{7\pi}{4}}$

For $n = 4:\ z = e^{i\,\frac{9\pi}{4}}$

For $n = 5:\ z = e^{i\,\frac{11\pi}{4}}$, which is the same as $e^{i\,\frac{3\pi}{4}}$.

Therefore for $n \geq 5$, the roots are repeated. hence our problem has four unique roots:

$$z = \left\{e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}, e^{i\frac{9\pi}{4}}\right\}$$

Section 1.2

(4) Use the power series representation for e^z to determine series representations for the following series. Use the results to deduce where the power series for $\sin^2 z$ and sech z would converge. What can be said about $\tan z$?

a) $\sin z$

Since $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $e^{iz} = \sum_{j=0}^{\infty} \dfrac{(iz)^j}{j!}$, we have

$$\sin(z) = \frac{\sum_{j=0}^{\infty} \frac{(iz)^j}{j!} - \sum_{j=0}^{\infty} \frac{(-iz)^j}{j!}}{2i}$$

$$= \frac{\sum_{j=0}^{\infty} \left(\frac{(iz)^j}{j!} - \frac{(-1)^j (iz)^j}{j!}\right)}{2i} = \frac{1}{2i} \sum_{j=0}^{\infty} \frac{(iz)^j \left(1-(-1)^j\right)}{j!}$$

Now we have that

$1-(-1)^j = 0$ if j is even, and

$1-(-1)^j = 2$ if j is odd. Hence we only need to include the odd j's in our power series since the even j's are all 0.

$$\frac{1}{2i} \sum_{j=0}^{\infty} \frac{2(iz)^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} i^{-1} (i)^{2j+1} \frac{z^{2j+1}}{(2j+1)!}$$

$$= \sum_{j=0}^{\infty} (i)^{2j} \frac{z^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!} \quad ✓$$

Now to determine where the power series for $\sin^2 z$ would converge we look at the radius of convergence of the power series for $\sin z$:

$$R = \lim_{j\to\infty} \left| \frac{(-1)^j}{(2j+1)!} \cdot \frac{(2j+3)!}{(-1)^{j+1}} \right|$$

$$= \lim_{j\to\infty} \left| \frac{1}{(2\,j+1)!} \cdot \frac{(2\,j+3)\,(2\,j+2)\,(2\,j+1)!}{(-1)} \right|$$
$$= \lim_{j\to\infty} |-(2\,j+3)\,(2\,j+2)| = \infty$$

Therefore, $\sin z$ converges for all $z \in \mathbb{C}$. Since $\sin^2 z$ can be expressed as $\left(\sum_{j=0}^{\infty} (-1)^j \frac{z^{2\,j+1}}{(2\,j+1)!}\right)^2 = \left(\sum_{j=0}^{\infty} (-1)^j \frac{z^{2\,j+1}}{(2\,j+1)!}\right) \cdot \left(\sum_{j=0}^{\infty} (-1)^j \frac{z^{2\,j+1}}{(2\,j+1)!}\right)$, we can conclude that $\sin^2 z$ also converges for all $z \in \mathbb{C}$ (since we are basically multiplying a function that converges for all values of z times the same function which also converges for all z). ✓

b) $\cosh z$

$$\frac{\sum_{j=0}^{\infty} \frac{z^j}{j!} + \sum_{j=0}^{\infty} \frac{(-z)^j}{j!}}{2} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{z^j}{j!} + \frac{(-1)^j\,(z)^j}{j!} \right)$$
$$= \frac{1}{2} \sum_{j=0}^{\infty} \frac{z^j\left(1+(-1)^j\right)}{j!}$$

Now we have that
$1+(-1)^j = 0$ if j is odd, and
$1-(-1)^j = 2$ if j is even. Hence we only need to include the even j's in our power series since the odd-j's are all 0. Thus,

$$\frac{1}{2} \sum_{j=0}^{\infty} \frac{2\,z^{2\,j}}{(2\,j)!} = \sum_{j=0}^{\infty} \frac{z^{2\,j}}{(2\,j)!} \qquad ✓$$

- We can express $\operatorname{sech} z$ as $\frac{1}{\cosh z}$. Therefore, the radius of convergence for $\operatorname{sech} z$ will be all the values of z such that $\cosh z \neq 0$. We can write $\cosh z$ as $\cosh z = \frac{e^z+e^{-z}}{2}$.

Then,

$$\frac{e^z+e^{-z}}{2} = 0 \Longrightarrow e^z + e^{-z} = 0$$
$$\Longrightarrow e^z = -\frac{1}{e^z} \Longrightarrow e^{2\,z} = -1$$
$$\Longrightarrow e^{2\,(x+i\,y)} = -1 \Longrightarrow e^{2\,x}[\cos(2\,y) + i\sin(2\,y)] = -1$$
$$\Longrightarrow x = 0;\ \ 2\,y = \pi \Longrightarrow y = \frac{\pi}{2};\ \ z = 0 + \frac{\pi}{2}\,i = \frac{\pi}{2}\,i$$

Hence $\operatorname{sech} z$ converges for all $z \in \mathbb{C} \setminus z = \frac{\pi}{2}\,i + \pi\,n$, for $n = 1, 2, 3 \ldots$ ✓

‣ $\tan z$ can be written as $\dfrac{\sin z}{\cos z}$. Since we have proved that $\sin z$ converges for all $z \in \mathbb{C}$ and by a similar argument it can be proven that $\cos z$ also converges for all $z \in \mathbb{C}$, then $\tan z$ converges for all values of z as long as $\cos z \neq 0$, since $\tan z$ is not defined at those points.

Therefore we write $\cos z = \dfrac{e^{i z}+e^{-i z}}{2}$. Then,

$$\frac{e^{i z}+e^{-i z}}{2} = 0 \Longrightarrow e^{i z} + e^{-i z} = 0 \Longrightarrow e^{i z} = -\frac{1}{e^{i z}}$$
$$\Longrightarrow e^{2 i z} = -1 \Longrightarrow e^{2 i (x+i y)} = -1$$
$$\Longrightarrow e^{2 i x - 2 y} = -1 \Longrightarrow \frac{1}{e^{2 y}}[\cos(2 x) + i\sin(2 x)] = -1$$
$$\Longrightarrow y = 0; \;\; 2 x = \pi \Longrightarrow x = \frac{\pi}{2}; \;\; z = \frac{\pi}{2} + 0\, i = \frac{\pi}{2}$$

Hence $\tan z$ converges for all $z \in \mathbb{C} \setminus z = \frac{\pi}{2} + \pi\, n$, for $n = 0,\ 1,\ 2,\ 3,\ \ldots.$ ✓

(5) Use any method to determine series expansions for the following functions :

a) $\dfrac{\sin z}{z}$

$$= \frac{1}{z} \sum_{j=0}^{\infty} (-1)^j \frac{z^{2 j+1}}{(2 j+1)!} = \boxed{\sum_{j=0}^{\infty} (-1)^j \frac{z^{2 j}}{(2 j+1)!}}$$

b) $\dfrac{\cosh z - 1}{z^2}$

$$= \frac{\left(\sum_{j=0}^{\infty} \frac{z^{2 j}}{(2 j)!}\right) - 1}{z^2} = \frac{1 + \left(\sum_{j=1}^{\infty} \frac{z^{2 j}}{(2 j)!}\right) - 1}{z^2}$$
$$= \frac{1}{z^2} \sum_{j=1}^{\infty} \frac{z^{2 j}}{(2 j)!} = \boxed{\sum_{j=1}^{\infty} \frac{z^{2 j-2}}{(2 j)!}}$$

c) $\dfrac{e^z - 1 - z}{z}$

$$= \frac{\sum_{j=0}^{\infty} \frac{z^j}{j!} - 1 - z}{z} = \frac{1 + \left(\sum_{j=1}^{\infty} \frac{z^j}{j!}\right) - 1 - z}{z}$$

$$= \frac{\left(\sum_{j=1}^{\infty} \frac{z^j}{j!}\right) - z}{z} = \frac{1}{z} \sum_{j=1}^{\infty} \frac{z^j}{j!} - 1 = \sum_{j=1}^{\infty} \frac{z^{j-1}}{j!} - 1$$

$$= 1 + \sum_{j=2}^{\infty} \frac{z^{j-1}}{j!} - 1 = \sum_{j=2}^{\infty} \frac{z^{j-1}}{j!}$$

(6) Let $z_1 = x_1$ and $z_2 = x_2$, with $x_1,\ x_2$ real, and the relationship $e^{i(x_1+x_2)} = e^{i x_1} e^{i x_2}$ to deduce the known trig formulae:

a) $\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2$

Solution:

We know that $\sin z = \frac{e^{i z} - e^{-i z}}{2 i}$. Thus, $\sin(z_1 + z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2 i}$. Letting $z_1,\ z_2 = x_1,\ x_2 \in \mathbb{R}$, we have

$$\sin(x_1 + x_2) = \frac{e^{i(x_1+x_2)} - e^{-i(x_1+x_2)}}{2 i} = \frac{e^{i x_2} e^{i x_1} - e^{-i x_1} e^{-i x_2}}{2 i}$$

$$= \frac{1}{2 i} \left([\cos x_2 + i \sin x_2] \cdot [\cos x_1 + i \sin x_1] - [\cos x_1 - i \sin x_1] \cdot [\cos x_2 - i \sin x_2]\right)$$

$$= \frac{1}{2 i} \Big[\cos x_2 \cos x_1 + i \cos x_2 \sin x_1 + i \cos x_1 \sin x_2 -$$
$$\sin x_2 \sin x_1 - \cos x_2 \cos x_1 + i \cos x_1 \sin x_2 + i \cos x_2 \sin x_1 + \sin x_1 \sin x_2\Big]$$

$$= \frac{2 i \cos x_1 \sin x_2 + 2 i \cos x_2 \sin x_1}{2 i} = \frac{2 i[\cos x_1 \sin x_2 + \cos x_2 \sin x_1]}{2 i}$$

$$= \cos x_1 \sin x_2 + \cos x_2 \sin x_1$$

b) $\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2$

Solution:

We also know that $\cos z = \frac{e^{i z} + e^{-i z}}{2}$, then by letting $z_1,\ z_2 = x_1,\ x_2 \in \mathbb{R}$ we have

$\cos x = \frac{e^{i x} + e^{-i x}}{2}$. Hence,

$$\cos(x_1 + x_2) = \frac{e^{i(x_1+x_2)} + e^{-i(x_1+x_2)}}{2} = \frac{e^{i\,x_1}\, e^{i\,x_2} + e^{-i\,x_1}\, e^{-i\,x_2}}{2}$$

$$= \frac{1}{2}\left([\cos x_2 + i \sin x_2]\cdot[\cos x_1 + i \sin x_1] + [\cos x_1 - i \sin x_1]\cdot[\cos x_2 - i \sin x_2]\right)$$

$$= \frac{1}{2}[\cos x_2 \cos x_1 + i \cos x_2 \sin x_1 + i \cos x_1 \sin x_2 -$$

$$\sin x_2 \sin x_1 + \cos x_2 \cos x_1 - i \cos x_1 \sin x_2 - i \cos x_2 \sin x_1 - \sin x_1 \sin x_2]$$

$$= \frac{2 \cos x_2 \cos x_1 - 2 \sin x_1 \sin x_2}{2} = \frac{2[\cos x_2 \cos x_1 - \sin x_1 \sin x_2]}{2}$$

$$= \cos x_1 \cos x_2 - \sin x_1 \sin x_2$$

c) $\sin(2\,x) = 2 \sin x \cos x$

Solution:

$$2 \sin(x) \cos(x) = 2\, \frac{e^{i\,x} - e^{-i\,x}}{2\,i} \cdot \frac{e^{i\,x} + e^{-i\,x}}{2} = \frac{(e^{i\,x} - e^{-i\,x})\,(e^{i\,x} + e^{-i\,x})}{2\,i}$$

$$= \frac{e^{i2\,x} - e^{-i2\,x}}{2\,i} = \frac{[\cos(2\,x) + i \sin(2\,x)] - [\cos(2\,x) - i \sin(2\,x)]}{2\,i}$$

$$= \frac{2\,i \sin(2\,x)}{2\,i} = \sin(2\,x)$$

d) $\cos(2\,x) = \cos^2 x - \sin^2 x$

Solution:

$$\cos^2(x) - \sin^2(x) = \left(\frac{e^{i\,x} + e^{-i\,x}}{2}\right)^2 - \left(\frac{e^{i\,x} - e^{-i\,x}}{2\,i}\right)^2$$

$$= \frac{e^{i2\,x} + 2\,e^{i\,x - i\,x} + e^{-i\,2\,x}}{4} + \frac{e^{i2\,x} - 2\,e^{i\,x - i\,x} + e^{-i\,2\,x}}{4}$$

$$= \frac{2\,e^{i2\,x} + 2\,e^{-i2\,x}}{4} = \frac{2[e^{i2\,x} + e^{-i2\,x}]}{4}$$

$$= \frac{e^{i2\,x} + e^{-i2\,x}}{2} = \cos(2\,x)$$