MATH 710 HW # 2

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Problem 1 (Problem 8-1)**.** *Let M be a smooth manifold (with or without boundary), and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, show that there exists a smooth global vector field $\widetilde{X}$ on M such that $\widetilde{X}|_A = X$ and $\operatorname{supp} \widetilde{X} \subseteq U$.*

Proof. For each $p \in A$, choose a neighborhood W_p of p and a smooth vector field $\widetilde{X}_p$ on W_p that agrees with X on $W_p \cap A$. Replacing W_p by $W_p \cap U$, where U is an open set of M, we may assume that $W_p \subseteq U$. The family of sets $\{W_p : p \in A\} \cup \{M \smallsetminus A\}$ is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \smallsetminus A$.

Now for each $p \in A$, the product $\psi_p \widetilde{X}_p$ is smooth on W_p by Proposition 8.8,[1] and has a smooth extension to all of M, being zero on $M \smallsetminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open subset $W_p \smallsetminus \operatorname{supp} \psi_p$ where they overlap.) Then we can define $\widetilde{X} \colon M \to TM$ by

$$\widetilde{X}_x = \sum_{p \in A} \psi_p(x) \widetilde{X}_p|_x.$$

Because the collection of supports $\{\operatorname{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of M, and therefore defines a smooth function. If $x \in A$, then $\psi_0(x) = 0$ and $\widetilde{X}_p|_x = X_x$ for each p such that $\psi_p(x) \neq 0$. Thus,

$$\widetilde{X}_x = \sum_{p \in A} \psi_p(x) X_x = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) X_x = X_x,$$

so $\widetilde{X}$ is indeed an extension of X. It follows that

$$\operatorname{supp} \widetilde{X} = \overline{\bigcup_{p \in A} \operatorname{supp} \psi_p} = \bigcup_{p \in A} \operatorname{supp} \psi_p \subseteq U. \qquad \square$$

Problem 2 (Problem 8-16)**.** *For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$, compute the Lie bracket $[X, Y]$.*

a) $X = y \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial y}$; $\quad Y = \frac{\partial}{\partial y}$.

b) $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$; $\quad Y = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}$.

c) $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$; $\quad Y = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}$.

[1] Here's Proposition 8.8, for reference:
Let M be a smooth manifold (with or without boundary). Then,

- if X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.
- $\mathfrak{X}(M)$ is a module over the ring $C^\infty(M)$.

Remark: Note that the value of the vector field $[X,Y]$ at a point $p \in M$ is the derivation at p given by the formula

$$[X,Y]_p f = X_p(Yf) - Y_p(Xf).$$

However, this formula is of limited usefulness for computations, because it requires one to compute terms involving second derivatives of f that will always cancel each other out. Instead, we use the following proposition,[2] which gives an extremely useful coordinate formula for the Lie bracket, in which the cancellations have already been accounted for:

Proposition (**Coordinate Formula for the Lie Bracket**). *Let X, Y be smooth vector fields on a smooth manifold M (with or without boundary), and let $X = X^i\, \partial/\partial x^i$ and $Y = Y^j\, \partial/\partial x^j$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X,Y]$ has the following coordinate expression:*

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

or more concisely,

$$(\clubsuit) \qquad [X,Y] = \left(XY^j - YX^j \right) \frac{\partial}{\partial x^j}.$$

Solution of a). Using ($\clubsuit$), we get

$$\begin{aligned} [X,Y] &= XY^j \frac{\partial}{\partial x^j} - YX^j \frac{\partial}{\partial x^j} \\ &= X(1)\frac{\partial}{\partial y} - Y(-2xy^2)\frac{\partial}{\partial y} - Y(y)\frac{\partial}{\partial z} \\ &= 0 + 4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z} \\ &= 4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}. \end{aligned}$$

$\square$

Solution of b). Using ($\clubsuit$), we get

$$\begin{aligned} [X,Y] &= XY^j \frac{\partial}{\partial x^j} - YX^j \frac{\partial}{\partial x^j} \\ &= X(-z)\frac{\partial}{\partial y} + X(y)\frac{\partial}{\partial z} - Y(-y)\frac{\partial}{\partial x} - Y(x)\frac{\partial}{\partial y} \\ &= 0 \cdot \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} - 0 \cdot \frac{\partial}{\partial y} \\ &= -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}. \end{aligned}$$

$\square$

[2] The proof of this proposition can be found on Lee's *Smooth Manifolds*, page 187.

Solution of c). Using (♣), we get

$$\begin{aligned}[X,Y] &= XY^j\frac{\partial}{\partial x^j} - YX^j\frac{\partial}{\partial x^j}\\ &= X(y)\frac{\partial}{\partial x} + X(x)\frac{\partial}{\partial y} - Y(-y)\frac{\partial}{\partial x} - Y(x)\frac{\partial}{\partial y}\\ &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\\ &= 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}.\end{aligned}$$

□

Problem 3 (Problem 8-25). *Prove that if G is an abelian Lie group, then* $\mathrm{Lie}(G)$[3] *is abelian. [Hint: show that the inversion map $i\colon G \to G$ is a group homomorphism, and use Problem 7-2.]*

Preliminaries of proof. Since we are going to use results from Problem 7-2 in our proof, for the sake of completion I am stating and proving that problem here. The grader may skip to the actual proof of 8-25 below.

Problem. *Let G be a Lie group.*

- *a) Let $m\colon G\times G \to G$ denote the multiplication map. Using Proposition 3.14*[4] *to identify $T_{(e,e)}(G\times G)$ with $T_eG \oplus T_eG$, show that the differential $\mathrm{d}m_{(e,e)}\colon T_eG\oplus T_eG \to T_eG$ is given by $\mathrm{d}m_{(e,e)}(X,Y) = X+Y$. (Hint: compute $\mathrm{d}m_{(e,e)}(X,0)$ and $\mathrm{d}m_{(e,e)}(0,Y)$ separately.)*
- *b) Let $i\colon G\to G$ denote the inversion map. Show that $\mathrm{d}i_e\colon T_eG \to T_eG$ is given by $\mathrm{d}i_e(X) = -X$.*

To show part a), consider the maps $\dot m, \ddot m\colon G \to G\times G$ given by $x\mapsto (x,e)$ and $y \mapsto (e,y)$, respectively. Note that $m\circ \dot m = m \circ \ddot m = \mathrm{Id}_G$. Thus,

$$\begin{aligned}\mathrm{d}m_{(e,e)}(X,Y) &= \mathrm{d}m_{(e,e)}(X,0) + \mathrm{d}m_{(e,e)}(0,Y)\\ &= \mathrm{d}(m\circ\dot m)_e(X) + \mathrm{d}(m\circ \ddot m)_e(Y)\\ &= \mathrm{d}\,\mathrm{Id}_G\big|_e(X) + \mathrm{d}\,\mathrm{Id}_G\big|_e(Y)\\ &= X+Y.\end{aligned}$$

[3]Recall that $\mathrm{Lie}(G)$ is the Lie algebra of all smooth left-invariant vector fields on a Lie group G.

[4]Here's the proposition, for reference:

Proposition (The Tangent Space to a Product Manifold). *Let $M_1,\dots,M_k$ be smooth manifolds, and for each j, let $\pi_j\colon M_1\times\cdots\times M_k \to M_j$ be the projection onto the M_j factor. For any point $p=(p_1,\dots,p_k)\in M_1\times\cdots\times M_k$ and tangent vector $\nu \in T_p(M_1\times\cdots\times M_k)$, the map*

$$\alpha\colon T_p(M_1\times\cdots\times M_k) \longrightarrow T_{p_1}M_1\oplus\cdots\oplus T_{p_k}M_k$$

defined by

$$\alpha(\nu) = (\mathrm{d}(\pi_1)_p(\nu),\dots,\mathrm{d}(\pi_k)_p(\nu))$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Now to show part b), let $\varphi = m \circ (\mathrm{Id}_G \times i) \colon G \to G$. Note that $(m \circ (\mathrm{Id}_G \times i))(x) = m((x, x^{-1})) = xx^{-1} = e$. Therefore,

$$\begin{aligned} 0 &= \mathrm{d}\varphi_e(X) \\ &= \mathrm{d}m_{(e,e)} \circ \mathrm{d}(\mathrm{Id}_G \times i)_e(X) \\ &= \mathrm{d}m_{(e,e)} \circ (\mathrm{Id}_{T_eG} \times \mathrm{d}i_e)(X) \\ &= \mathrm{d}m_{(e,e)}(\mathrm{Id}_{T_eG}(X), \mathrm{d}i_e(X)) \\ &= \mathrm{d}m_{(e,e)}(X, \mathrm{d}i_e(X)) \\ &= X + \mathrm{d}i_e(X). && \text{(by part a)).} \end{aligned}$$

Hence it follows that $\mathrm{d}i_e(X) = -X$, as desired. □

Proof of Problem 8-25. If G is abelian then the inversion map $i \colon G \to G$ is a Lie group homomorphism (in fact an isomorphism, since it is bijective):

$$i(g_1g_2) = (g_1g_2)^{-1} = \underbrace{g_2^{-1}g_1^{-1} = g_1^{-1}g_2^{-1}}_{\text{Since } G \text{ is abelian.}} = i(g_1)i(g_2).$$

Then the induced Lie algebra isomorphism is given by

$$\begin{aligned} (i_*X)_g &= \mathrm{d}(L_g)_e(\mathrm{d}i_e(X_e)) \\ &= \mathrm{d}(L_g)_e(-X_e) && \text{(by part b) of Problem 7-2)} \\ &= -\mathrm{d}(L_g)_eX_e \\ &= -X_g, \end{aligned}$$

so that $i_*X = -X$ for all $X \in \mathrm{Lie}(G)$.

Thus,

$$[X, Y] = [-X, -Y] = \underbrace{[i_*X, i_*Y] = i_*[X, Y]}_{\text{By Corollary 8.31}} = -[X, Y] \implies [X, Y] = 0 \ \ \forall\, X, Y \in \mathrm{Lie(G)}.$$

This proves that Lie(G) is abelian, as desired. □