Algebraic Topology
HW Set # 2

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Problem 1. *Show that the groups G and H given by their presentations*

$$G \cong \langle a, b \mid aba^{-1}b \rangle \qquad H \cong \langle c, d \mid c^2d^2 \rangle$$

are isomorphic. [Hint: Show that both G and H are equal to π_1 of the Klein bottle. The presentation of G arises when the sets U and V (from the Seifert-VanKampen theorem) are chosen to be a disk and the Klein bottle minus a (slightly smaller) disk. The presentation of H comes about when the Klein bottle is viewed as $\mathbb{RP}^2 \# \mathbb{RP}^2$ and both U and V are taken to be $\mathbb{RP}^2 \setminus \mathbb{D}^2$.]

Aside: Before getting to our proof, for study purposes, I will restate the VanKampen theorem from the point of view of category theory as a *pushout*. It's not that I'm trying to kill a mosquito with a shotgun, but category theory is just too elegant and I like to think categorically as much as possible. First, here's the definition of a pushout as a side note:

Pushout

A ***pushout*** (also called a ***fibered coproduct*** or ***fibered sum*** or ***cocartesian square*** or ***amalgamated sum***) is the colimit of a diagram consisting of two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ with a common domain: it is the colimit of the span $X \leftarrow Z \to Y$.

Explicitly, the pushout of the morphisms f and g consists of an object P and two morphisms $i_1\colon X \to P$ and $i_2\colon Y \to P$ such that the diagram

$$\begin{tikzcd} P & Y \arrow[l, "i_2"'] \\ X \arrow[u, "i_1"] & Z \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

commutes and such that (P, i_1, i_2) is universal with respect to this diagram. That is, for any other such set $(\widetilde{P}, j_1, j_2)$ for which the following diagram commutes, there must exist a unique $\varphi\colon P \to \widetilde{P}$ also making the diagram commute:

$$
\begin{tikzcd}
\widetilde{P} & & \\
& P \arrow[ul, dashed, "\varphi"] & Y \arrow[l, "i_2"] \arrow[ull, bend right, "j_2"'] \\
& X \arrow[u, "i_1"] \arrow[uul, bend left, "j_1"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

As with all universal constructions, the pushout, if it exists, is unique up to a unique isomorphism.

Back to our situation at hand, suppose we have a path-connected space X, covered by path-connected open subspaces U and V whose intersection is also path-connected and nonempty with $x_0 \in U \cap V$. We take x_0 to be the base point for all fundamental groups under consideration. Then if we know the fundamental groups of U, V, and their intersection $U \cap V$, we can recover the fundamental group of X provided we also know the induced homomorphisms $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$. VanKampen's theorem then says that $\pi_1(X)$ is the pushout of these two induced homomorphims in **Grp**, the category of groups. Note that X is the pushout of the two (basepoint-preserving) inclusion maps of $U \cap V$ into U and V in $\mathbf{Top}_*$, the category of pointed topological spaces; thus we may interpret the theorem as confirming that the fundamental group functor preserves pushouts of inclusions. In a picture:

$$
\begin{tikzcd}
H & & \\
& \pi_1(X) \arrow[ul, dashed, "\varphi"] & \pi_1(V) \arrow[l, "j_V"] \arrow[ull, bend right, "\psi_V"'] \\
& \pi_1(U) \arrow[u, "j_U"] \arrow[uul, bend left, "\psi_U"] & \pi_1(U \cap V) \arrow[l, "i_{U,V}"'] \arrow[u, "i_{V,U}"]
\end{tikzcd}
$$

Here H is any arbitrary group. It follows from the statement of the theorem that the (unique) homomorphism $\Phi\colon \pi_1(U) * \pi_1(V) \to \pi_1(X)$ is surjective (see discussion on *coproducts* on Problem 5) and, furthermore, the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{U,V}(\omega)\, i_{V,U}(\omega)^{-1}$, where $\omega \in \pi_1(U \cap V)$ and $i_{U,V} : \pi_1(U \cap V) \to \pi_1(U)$

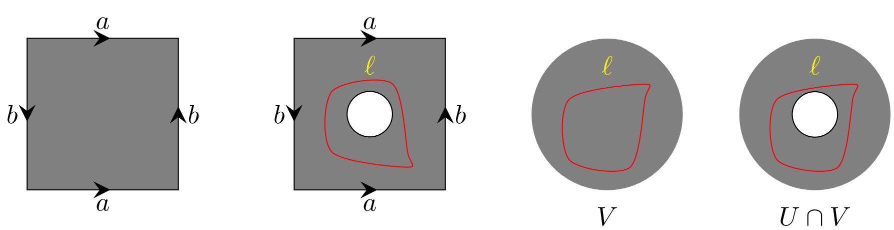
is the homomorphism induced by the inclusion $U \cap V \hookrightarrow U$ (note that $j_U\, i_{U,V} = j_V\, i_{V,U}$, both of these compositions being induced by the inclusion $U \cap V \hookrightarrow X$). Thus, in general, we have

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \frac{\pi_1(U) * \pi_1(V)}{N},$$

where N is the normal subgroup described above and $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$ is what is called an ***amalgamated free product***. The amalgamation has forced an identification between the image of $i_{U,V}$ in $\pi_1(U)$ and the image of $i_{V,U}$ in $\pi_1(V)$, element by element. This is the construction needed to compute the fundamental group of two connected spaces U and V joined along a path-connected subspace $U \cap V$.

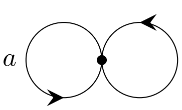
In particular, note that if $U \cap V$ is simply connected, then we have that $\pi_1(X)$ is just the free product $\pi_1(U) * \pi_1(V)$. However, in more general cases we will be dealing with amalgamated free products instead.

Proof of Problem 1. $(\mathbb{K} \cong G)$ We represent the Klein bottle $\mathbb{K}$ with its usual identification polygon, and we let $\mathbb{K} = U \cup V$, where $U = \mathbb{K} \setminus \mathbb{D}^2$ and V is homeomorphic to an open disk $\mathring{\mathbb{D}}^2$. Now we divide up these spaces:



$\mathbb{K} = U \cup V$ $\quad U \quad V \quad U \cap V$

- We see that the hole in U is homotopic to just the outside boundary of the identification space by means of deformation retract. Since all corners of the identification polygon are the same point, the boundary can be represented as a bouquet of two loops . The distinct loops are a and b:

  Hence, a and b are the generators for U, which generate infinite cyclic groups and thus have no relations.

- Now we turn to V, which is just a disk so it is contractible, which means that there are no generators or relations in V.

- Lastly, to see the relations of $U \cap V$, we consider a loop ℓ in $U \cap V$. The same loop ℓ in U is homotopic to the outside boundary; thus $\ell \simeq aba^{-1}b$, and the same loop in V is homotopic to the identity. Hence $\ell = aba^{-1}b = e$, and so $aba^{-1}b$ is our relation. Hence we conclude, by the VanKampen Theorem, that $\pi_1(\mathbb{K}) \cong G \cong \langle a, b \mid aba^{-1}b \rangle$, as desired.

($\mathbb{K} \cong H$) We know that $\mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{RP}^2$, and the computation for the group presentation of the fundamental theorem of the connected sum of n projective planes is already detailed on our textbook (Massey's). It is given to be $\langle a_1, \ldots, a_n \mid a_1^2 \cdots a_n^2 \rangle$. For $n = 2$, we have $H \cong \mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \cong \langle a_1, a_2 \mid a_1^2 a_2^2 \rangle$, as desired.

Note that, intuitively, we can easily swap between these two group presentations by representing the Klein bottle is as a square with opposite sides identified via the word $aba^{-1}b$ and then cutting the square along a diagonal and reassemble the resulting two triangles as shown in Figure1.

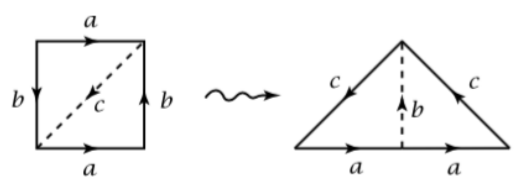


Figure 1: *Representations of the Klein Bottle.*

□

Problem 2. *i) What surface is represented by $a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_{n-1}^{-1} a_n$?*
ii) What surface is represented by $a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_{n-1}^{-1} a_n^{-1}$?

Solution of i). All the vertices are identified, since the end vertex of a_n is the beginning of both a_1 and a_1^{-1}, which then equals the beginning of a_2 and the beginning of a_2^{-1}, which in turn equals the beginning of a_3 and the beginning of a_3^{-1}, and so on .... Since there are n edges, one vertex, and a type-2 edge,[1] this is a non-orientable surface of Euler characteristic $1 + 1 - n = 2 - n$, which is $\#_{i=1}^{n} \mathbb{RP}^2$, the connected sum of n projective planes. □

Solution of ii). Note that all edges are of type 1, hence this surface is orientable. Each vertex is identified with a vertex $n+1$ steps around the $2n$-gon (in other words, the starting vertex of a_1 equals the end vertex of a_1^{-1}). Then if n is even, this means that all the vertices are identified. If n is odd however, this means that there are two classes of vertices. So the Euler characteristic for n even is $1 - n + 1 = 2 - n$, in which case we have the connected sum of $n/2$ tori. On the other hand, for n odd, the Euler characteristic is $2 - n + 1 = 3 - n$, so that we have the connected sum of $(n-1)/2$ tori. □

Problem 3. *Let G be a topological group[2] with identity element e. Prove that $\pi_1(G, e)$ ($= \pi(G)$) is an abelian group.*

Proof. Let $\Omega(G, e)$ denote the set of all loops in G based at e. If $f, g \in \Omega(G, e)$, let us define a loop $f \odot g$ by the rule

$$(f \odot g)(s) = f(s) \cdot g(s),$$

[1] Recall that if the letter designating a certain pair of edges occurs with both exponents, $+1$ and -1, in the symbol, then we call that pair of edges a ***pair of the first kind***; otherwise we cal it a ***pair of the second kind***.

[2] Recall that a ***topological group*** is a topological space G that also happens to be a group, such that the multiplication map $\mu \colon G \times G \to G$ is continuous, and the map $G \to G$ which sends g to g^{-1} is also continuous.

where $s \in [0,1]$ and "$\cdot$" is the group operation defined on the topological group G. Now, before proving that $\pi_1(G,e)$ is abelian, we make (and prove) the following three claims:

Claim I This operation $\odot$ makes the set $\Omega(G,e)$ into a group: To see this, note that since f and g are continuous, so is $f \odot g$, and $(f \odot g)(0) = (f \odot g)(1) = e$, i.e., $f \odot g \in \Omega(G,e)$. Moreover, since "$\cdot$" is associative, so is $\odot$, and further $e \odot f = f \odot e = f$, and the inverse of f is defined by $f^{-1}(s) = (f(s))^{-1}$.

Claim II This operation $\odot$ induces a group operation $\odot$ on $\pi_1(G,e)$: To see this note that $[f] \odot [g] = [f \odot g]$ is well defined, because $F(s,t) \cdot G(s,t)$ is a homotopy between $F|_{t=0} \odot G|_{t=0}$ and $F|_{t=1} \odot G|_{t=1}$, and it satisfies group properties induced by $\odot$.

Claim III The two operations $*$ (where $*$ is the usual product that we defined for the fundamental group) and $\odot$ are the same on $\pi_1(G,e)$: Recall that for any space X and any point $x \in X$, we denote by $\mathscr{E}_x$ the equivalence class of the constant map of an interval I into the point x of X. We have

$$\begin{aligned} [f] \odot [g] &= [f * \mathscr{E}_e] \odot [\mathscr{E}_e * g] \\ &= [(f * \mathscr{E}_e) \odot (\mathscr{E}_e * g)] \\ &= [f * g] \qquad (\clubsuit) \\ &= [f] * [g]. \end{aligned}$$

Note that (♣) holds because

$$\begin{aligned} ((f * \mathscr{E}_e) \odot (\mathscr{E}_e * g))(s) &= (f * \mathscr{E}_e)(s) \cdot (\mathscr{E}_e * g)(s) \\ &= \begin{cases} f(s) & \text{if } s \in [0, \frac{1}{2}], \\ g(s) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases} \\ &= (f * g)(s). \end{aligned}$$

Now, with these results at hand, we show that $\pi_1(G,e)$ is abelian:

$$[f] * [g] = [f] \odot [g] = [\mathscr{E}_e * f] \odot [g * \mathscr{E}_e] = [g * f] = [g] * [f]. \qquad \square$$

Remark: The following two group theoretic propositions will be applied when we study torus knots. For both of these group theory problems, use the fact that every element in $G * H$ is uniquely represented as a reduced word, i.e. a word

$$g_1 h_1 g_2 h_2 \cdots g_m h_m,$$

where $g_i \in G$ and $h_i \in H$ are such that $g_i, h_i \neq 1$ and g_1 and h_m may be the empty element.

Problem 4. *Let $X = \{G_i\}$ be a collection of more than one non-trivial group. Prove that their free product is non-abelian, contains elements of infinite order, and that its center is trivial.*

Proof. Let $x, y \in X$ be nontrivial elements. Then,

$$x \neq y \implies x^{-1}y^{-1}xy \text{ is reduced} \implies x^{-1}y^{-1}xy \neq e \implies xy \neq yx.$$

Furthermore, for any arbitrary $n \in \mathbb{Z}^+$,

$$(xy)^n = \underbrace{xy \cdots xy}_{n \text{ times}} \neq e \implies xy \text{ has infinite order.}$$

Now let $Z(X) = \{z \in X \mid zx = xz \; \forall \, x \in X\}$ be the center of $X = \{G_i\}$, and let $w = g_1 \cdots g_n$ be an arbitrary nonempty word, with $g_1 \in G_i$ and $g_n \in G_j$. If $G_i \neq G_j$, then

$$\begin{aligned} g_1^{-1}w &= g_2 \cdots g_n \\ &\neq wg_1^{-1} \qquad \text{(Since } \ast_i G_i \text{ is not abelian, as shown above)} \\ &= g_1 \cdots g_n g_1^{-1} \\ &\implies w \notin Z(X). \end{aligned}$$

Im the case when $G_i = G_j$, we pick a nontrivial element $g_k \in G_k$ with $k \neq i$. Then, since $\ast_i G_i$ is nonabelian, we have $g_k w \neq w g_k$. Thus w is not in the center of the group in this situation either. Hence we must have that $Z(X) = \{e\}$, as desired. □

Problem 5. *Let G, H, G', and H' be cyclic groups of orders m, n, m', and n', respectively. If $G * H$ is isomorphic to $G' * H'$, show that $m = m'$ and $n = n'$ or else $m = n'$ and $n = m'$.*

Proof by categorical nonsense. I will use some machinery from category theory to prove this proposition. Again, we have not covered any category theory in class, but I find it too elegant not to use! (☺) It is well known(!) that the free product of groups is the *coproduct* in **Grp**. For the sake of completeness and clarity, I will include as an aside the exact definition of a coproduct:

Coproducts

Let **C** be any category and let $\{X_i\}$ a collection of objects in that category. An object is called the ***coproduct*** of these objects, usually written $\amalg_i X_i$ if there exist morphisms $\iota_i \colon X_i \to \amalg_i X_i$ (called ***canonical injections***) satisfying a universal property:

For any object Y and morphisms $f_i \colon X_i \to Y$, there exists a unique morphism $f \colon \amalg_{i \in I} X_i \to Y$ such that $f_i = f \circ \iota_i$. That is, for all $i, j \in I$ the following diagram commutes:

$$\begin{tikzcd} & Y & \\ X_i \arrow[ur, "f_i"] \arrow[r, "\iota_i"] & \amalg_{i \in I} X_i \arrow[u, dashed, "\exists! f"] & X_j \arrow[ul, "f_j"'] \arrow[l, "\iota_j"'] \end{tikzcd}$$

Back to our situation at hand, our proof will follow from the following diagram:

$$\begin{tikzcd}
G \arrow[dr, bend right, "\iota_G"'] \arrow[ddd, dashed, "\psi"'] & & H \arrow[dl, bend left, "\iota_H"] \arrow[ddd, dashed, "\psi'"] \\
& G * H \arrow[ul, dashed, "\pi_G"'] \arrow[ur, dashed, "\pi_H"] \arrow[d, dashed, "\varphi"] & \\
& G' * H' \arrow[dl, dashed, "\pi_{G'}"] \arrow[dr, dashed, "\pi_{H'}"'] & \\
G' \arrow[ur, bend left, "\iota_{G'}"] & & H' \arrow[ul, bend right, "\iota_{H'}"']
\end{tikzcd}$$

Since $G * H$ is a coproduct, by definition for any homomorphism $\psi\colon G \to G'$ there is a unique homomorphism $\pi_{G'}\colon G' * H' \cong G * H \to G'$ such that $\psi = \pi_{G'} \circ \varphi \circ \iota_G$. (Note that in this case the intermediate morphism φ is merely an isomorphism, so nothing has been violated. In addition, it is important to stress that the π's are <u>NOT</u> canonical projections(!), as those are not characteristic of a coproduct. I am simply abusing notation here.) Similarly, for any homomorphism $\bar{\psi}\colon G' \to G$ there is a unique homomorphism $\pi_G\colon G * H \cong G' * H' \to G$ such that $\bar{\psi} = \pi_G \circ \varphi^{-1} \circ \iota_{G'}$. In particular, $\bar{\psi} = \psi^{-1}$, the inverse of ψ. We can proceed similarly for H and H' so that we have $G \cong G'$ and $H \cong H'$, and so $m = m'$ and $n = n'$.

As a different scenario, we could consider homomorphisms $G \to H'$ and $H \to G'$ and determine that these maps also have inverses and so $m = n'$ and $n = m'$. The reason why we can only have one of the two scenarios is the uniqueness of the homomorphisms of the coproduct, since having both scenarios simultaneously would clearly violate this uniqueness property. This shows that $m = m'$ and $n = n'$ or else $m = n'$ and $n = m'$, as desired. $\square$