Math Analysis Notes

Mario L. Gutierrez Abed

Finite, Countable, and Uncountable Sets

Definition: If there exists a one-to-one mapping of A onto B (that is, a map from A to B that is bijective), then we say that A and B can be put in 1-1 correspondence, or that A and B have the same cardinal number, or briefly, that A and B are equivalent, and we write $A \sim B$. This relation clearly has the following properties:
⟜ It is reflexive: $A \sim A$.
⟜ It is symmetric: If $A \sim B$, then $B \sim A$.
⟜ It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.
Any relation with these three properties is called an equivalence relation.

Definition: For any positive integer n, let $\mathcal{J}_n$ be the set whose elements are the integers $1, 2, \ldots, n$; let $\mathcal{J}$ be the set consisting of all positive integers (same as $\mathbb{N}$). Then, for any set A we say:
i) A is finite if $A \sim \mathcal{J}_n$ for some n (the empty set is also considered to be finite).
ii) A is infinite if A is not finite. (Duh!)
iii) A is countable if $A \sim \mathcal{J}$.
iv) A is uncountable if A is neither finite nor countable.
v) A is at most countable if A is finite or countable.
Countable sets are sometimes called enumerable, or denumerable.

Note: For two finite sets A and B, we evidently have $A \sim B$ iff A and B contain the same number of elements. For infinite sets however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

Example:
a) The set of all integers $\mathbb{Z}$ is countable.
To see this we can define a function $f : \mathbb{Z} \longrightarrow \mathcal{J}$ such that

$$f(n) = \begin{cases} 2n & \text{if } n \geq 1 \\ -2n+1 & \text{if } n \leq 0 \end{cases}$$

This function sets up the 1-1 correspondence

$$\begin{array}{llllll} \mathbb{Z}: & \ldots -3, & -2, & \ldots, & 2, & 3, \ \ldots \\ & \ \ \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\ \mathcal{J}: & \ldots \ \ 7, & 5, & \ldots, & 4, & 6, \ \ldots \end{array}$$

That is, the negative integers in $\mathbb{Z}$ are mapped to the odd numbers on $\mathcal{J}$ while the positive integers in $\mathbb{Z}$ are mapped to the even numbers in $\mathcal{J}$.

Note that usually there are multiple bijective maps capable of establishing a 1-1 correspondence between two sets. For instance, we could've used a map from $\mathcal{J}$ to $\mathbb{Z}$ instead, say $f : \mathcal{J} \longrightarrow \mathbb{Z}$ such that

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

This function sets up the 1-1 correspondence

$$\begin{array}{lccccc} \mathcal{J}: & \ldots\, 1, & 2, & 3, & 4, & \ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z}: & \ldots\, 0, & 1, & -1, & 2, & \ldots \end{array}$$

That f in both cases is bijective is easy to check. Notice that $\mathbb{Z}$ is equivalent to a proper subset of itself! This is typical of infinite sets whereas it's impossible for finite sets.

b) The set of all cartesian products on $\mathbb{N}$ is equivalent to $\mathbb{N}$ itself, i.e. $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.
A quick proof is supplied by the fundamental theorem of arithmetic:
Each positive integer $k \in \mathbb{N}$ can be uniquely written as $k = 2^{m-1}(2n-1)$ for some $m, n \in \mathbb{N}$. Define $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ by $f(m, n) = 2^{m-1}(2n-1)$. It is obvious that this f is bijective.

c) The set of all real numbers is equivalent to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, i.e. $\mathbb{R} \sim \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
To see this, define $f : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by $f(x) = \tan^{-1}(x)$. Recall from calculus that f is a strictly increasing (hence one-to-one) function from $\mathbb{R}$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and it's also onto.
Note: As a matter of fact, $\mathbb{R}$ is equivalent to any interval of real numbers (a, b).

2.7 Definition By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in* A, or a *sequence of elements of* A.

Note that the terms $x_1, x_2, x_3, \ldots$ of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Sometimes it is convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

• Theorem:
Every infinite subset of a countable set A is countable.

Proof:
Suppose $E \subset A$ and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements.
Then construct a sequence $\{n_k\}$ as follows:
Let n be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \ldots, n_{k-1}$ $(k = 2, 3, 4, \ldots)$,

let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.
Putting $f(k) = x_{n_k}$ for $k = 1, 2, 3, \ldots$, we obtain a 1-1 correspondence between E and $\mathbb{N}$. ■

Notation: Let A and B be sets. then $A \backslash B = \{x \in A : x \notin B\}$.

• Theorem:
Every infinite set has a countable subset.

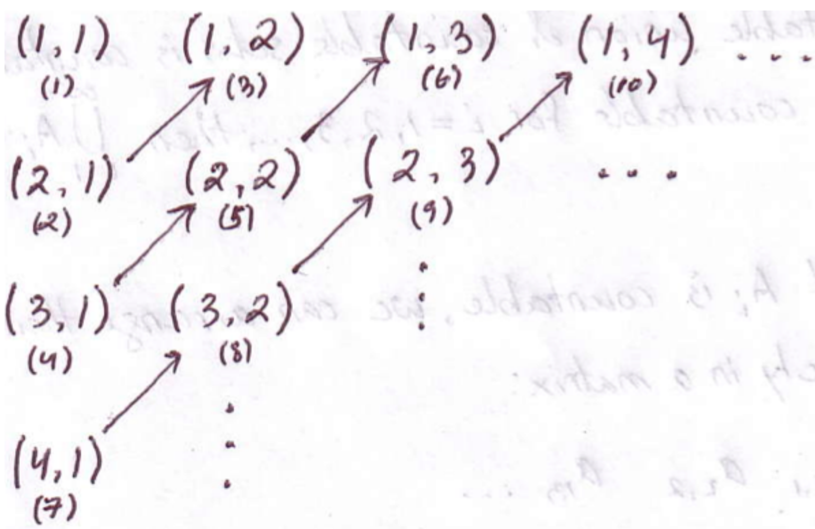
Proof:
Let A be an infinite set. Then $A \neq \emptyset$, because $\emptyset$ is considered to be finite. Let $x_1 \in A$ be any element of A. Then $A \backslash \{x_1\} \neq \emptyset$ (otherwise $A = \{x_1\}$ and A is finite). Pick $x_2 \in A \backslash \{x_1\}$ to be any element of $A \backslash \{x_1\}$.
Having chosen $x_1, \ldots, x_{n-1}$, observe that $A \backslash \{x_1, \ldots, x_{n-1}\} \neq \emptyset$ (otherwise $A = \{x_1, \ldots, x_{n-1}\}$, making A finite). Hence we are free to select $x_n \in A \backslash \{x_1, \ldots, x_{n-1}\}$.
Let $E = \{x_n\} \subset A$. Then E is countable. ■

Note: This last theorem shows that a countable infinity is the smallest type of infinity. That is, no uncountable set can be a subset of a countable set, while every infinite set has a countable subset.

To motivate our next several results, we now present a second proof that $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$.

• Theorem:
$\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$.

Proof:
Arrange $\mathbb{N} \times \mathbb{N}$ in a matrix:



The arrows and number marks indicate the order in which we will count the elements of $\mathbb{N} \times \mathbb{N}$.
Each diagonal that is traced by the arrows contains all ordered pairs whose components add up to the same number. Notice also that the first diagonal contains one element, the second diagonal contains two elements, and so on.
These observations allow us to construct a bijective map $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ explicitly:

$$f(m,\ n) = \frac{(m+n-2)\,(m+n-1)}{2} + n$$

Thus, we have a constructed an invertible map from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$, which implies that $\mathbb{N}\times\mathbb{N}\sim\mathbb{N}$, and this proves our theorem. ■

The above theorem gives us a ton of new information. We can see this materialize in the following theorem:

• Theorem:
The countable union of countable sets is countable. That is, if A_i is countable for $i = 1,\ 2,\ 3,\ ..$, then $\bigcup_{i=1}^{\infty} A_i$ is countable.

Proof:
Since each A_i is countable, we can arrange their elements collectively in a matrix:

$$\begin{array}{cccccc} A_1: & a_{11} & a_{12} & a_{13} & . & . \\ A_2: & a_{21} & a_{22} & a_{23} & . & . \\ A_3: & a_{31} & a_{32} & a_{33} & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}$$

So $\bigcup_{i=1}^{\infty} A_i$ is the range of some invertible map on $\mathbb{N}\times\mathbb{N}$ (just as the one constructed on the previous theorem). That is, $\bigcup_{i=1}^{\infty} A_i$ is equivalent to $\mathbb{N}\times\mathbb{N}$ and hence to $\mathbb{N}$. ■

Note that proof of the above theorem can be used to show that, given any two countable sets A and B, the set $A\times B$ is also countable.

• Corollary:
$\mathbb{Q}$ is countable.

Note: Recall that between any two real numbers there is a rational number. This means, in fact, that between any two real numbers, there are infinitely many rational numbers (since $\mathbb{R}$ is infinite and we know that $(a,\ b)\sim\mathbb{R}\ \ \forall\ a,\ b\in\mathbb{R}$). Surprisingly, $\mathbb{N}$ is as large as $\mathbb{Q}$ even though $\mathbb{N}\subset\mathbb{Q}$ and there are infinitely many rationals between any two rational numbers.

So far we have shown that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are all countable. Now we show the shocking result that $\mathbb{R}$ is not a countable set.

• Theorem:
$\mathbb{R}$ is uncountable.

Proof:
To prove that $\mathbb{R}$ is uncountable, it is enough to show that some subset in $\mathbb{R}$ is uncountable (since no countable set can have an uncountable subset). Therefore, we can use the subset $(0, 1)$ and prove that it's uncountable. To accomplish this, we will show that any countable subset of $(0, 1)$ is proper. Given any sequence $\{a_n\}$ in $(0, 1)$, we construct an element x in $(0, 1)$ with $x \neq a_n$ for any n. We begin by listing the decimal expansions of the a_n; for example:

$$a_1 = 0.\boxed{3}1572\ldots$$

$$a_2 = 0.0\boxed{4}268\ldots$$

$$a_3 = 0.91\boxed{5}36\ldots$$

$$a_4 = 0.759\boxed{9}9\ldots$$

....

(If any a_n has two representations, just use the infinite one)
Now let $x = 0.533353\ldots$, where the n^{th} digit in the expansion for x is taken to be 3, unless a_n happens to have 3 as its n^{th} digit, in which case we replace it with 5 (this is why we "boxed" the n^{th} digit in the expansion of a_n above. Note that the choices of 3 and 5 are more or less arbitrary, in truth we just want to avoid the troublesome digits 0 and 9 but any other digits would do).
Using this procedure, the decimal representation of x is unique because it does not end in all 0's or all 9's, and $x \neq a_n$ for any n because the decimal expansions for x and a_n differ in the n^{th} place. Thus we have shown that $\{a_n\}$ is a proper subset of $(0, 1)$, and hence $(0, 1)$ is uncountable, which in turn implies that $\mathbb{R}$ is uncountable. ■

Note: The proof that we just produced is known as Cantor's diagonalization method. It gives insight into the differences between countable and uncountable sets.

• Corollary:
The set of all irrationals $\mathbb{R} \setminus \mathbb{Q}$ (or simply $\mathbb{I}$), is uncountable.

Proof:
We know that $\mathbb{R} = \mathbb{Q} \bigcup \mathbb{I}$. We also know that the union of countable sets must be countable. Since $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable, it follows that $\mathbb{I}$ must be uncountable. ■

• Cantor's Theorem:
No map $F : A \longrightarrow \mathcal{P}(A)$ can be onto.

Proof:
Let's assume that $F : A \longrightarrow \mathcal{P}(A)$ is an onto function.
Then consider $S_F = \{x \in A : x \notin F(x)\} \in \mathcal{P}(A)$. Since F is assumed to be onto, there must exists an element $y \in A$ such that $F(y) = S_F$. We claim that $S_F \neq F(y)$ for any $y \in A$. Indeed, if $S_F = F(y)$, then we are faced with the following alternatives:

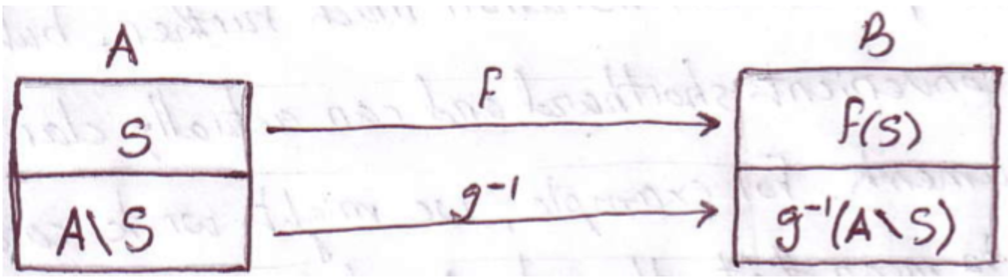
$$y \in F(y) = S_F \qquad\qquad y \notin F(y) = S_F.$$
$$\Longrightarrow y \notin F(y) \qquad \text{or} \qquad \Longrightarrow y \in F(y).$$

and both lead to contradictions! ■

• Bernstein's Theorem:
Let A and B be nonempty sets. If there exist one-to-one maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$, then there is a map $h : A \longrightarrow B$ that is both one-to-one and onto. Informally, this implies that if two cardinalities are both less than or equal to each other, then they are equal.

Proof:
We would like to find a set S that will allow us to define $h : A \longrightarrow B$ as a piecewise function

$$h(x) = \begin{cases} f(x) & \text{if } x \in S \\ g^{-1}(x) & \text{if } x \in A \backslash S \end{cases}$$



What conditions must the set S satisfy? Since h must be onto B, we must have $B = f(S) \bigcup g^{-1}(A \backslash S)$ or equivalently, $A \backslash S = g(B \backslash f(S))$. This last equation may be converted to $S = A \backslash g(B \backslash f(S))$.
Define $H : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ by $H(S) = A \backslash g(B \backslash f(S))$. We then have to find a solution to the "fixed point" equation $S = H(S)$.

To do this, observe that

i) H is increasing:
Suppose $S \subset T$, then $f(S) \subset f(T)$. Consequently,
$B \backslash f(S) \supset B \backslash f(T)$, $g(B \backslash f(S)) \supset g(B \backslash f(T))$, and $A \backslash g(B \backslash f(S)) \subset A \backslash g(B \backslash f(T))$.
Thus $H(S) \subset H(T)$.

ii) Let $\partial = \{S \in \mathcal{P}(A) : S \subset H(S)\}$. Then $\emptyset \in \partial$ and ∂ is not empty. Let $S^* = \bigcup_{S \in \partial} S$, then $S^* \subset H(S^*)$.

To see this, observe that for any $S \in \partial$, $S \subset S^*$, and $S \subset H(S)$.
Since H is increasing, it follows that $H(S) \subset H(S^*)$. Thus, $S \subset H(S) \subset H(S^*)$ for all $S^* \in \partial$. Hence $\bigcup_{S \in \partial} S \subset H(S^*)$.
Notice now that $H(S^*) \subset H(H(S^*))$. Thus $H(S^*) \in \partial$. It follows that $S^* = H(S^*)$. S^* is therefore the desired set. ■

(Alternate) Proof:
We call an element b of B *lonely* if there is no element $a \in A$ such that $f(a) = b$. We say that an element b_1 of B is a descendent of an element b_0 of B if there is a natural number n (possibly zero)

such that $b_1 = (f \circ g)^n (b_0)$.
We define the function $h : A \longrightarrow B$ as follows:

$$h(a) = \begin{cases} g^{-1}(a) & \text{if } f(a) \text{ is the descendent of a lonely point} \\ f(a) & \text{otherwise} \end{cases}$$

Note that if $f(a)$ is the descendent of a lonely point, then $f(a) = f \circ g(b)$ for some b; since g is injective, the element $g^{-1}(a)$ is well defined. Thus our function h is well defined. We claim that it is a bijection from A to B.
We first prove that h is surjective. Indeed, if $b \in B$ is the descendent of a lonely point, then $h(g(b)) = b$; and if b is not the descendent of a lonely point, then b is not lonely, so there is some $a \in A$ such that $f(a) = b$; by our definition, then, $h(a) = b$. Thus is surjective.
Next, we prove that h is injective. We first note that for any $a \in A$, the point $h(a)$ is a descendent of a lonely point if and only if $f(a)$ is a descendent of a lonely point. Now suppose that we have two elements $a_1, a_2 \in A$ such that $h(a_1) = h(a_2)$. We consider two cases.
If $f(a_1)$ is the descendent of a lonely point, then so is $f(a_2)$.
Then,

$$g^{-1}(a_1) = h(a_1) = h(a_2) = g^{-1}(a_2) .$$

Since g is a well defined function, it follows that $a_1 = a_2$.
On the other hand, if $f(a_1)$ is not a descendent of a lonely point, then neither is $f(a_2)$. It follows that

$$f(a_1) = h(a_1) = h(a_2) = f(a_2) .$$

Since is injective, $a_1 = a_2$.
Thus h is injective. Since h is surjective and injective, it is bijective, as desired. ■

To appreciate how incredible Bernstein's result truly is, consider the following example.

Example:
Let $\mathbb{R}^\infty$ be the set of all real-valued sequences. That is, if $x \in \mathbb{R}^\infty$, then $x = (x_1, x_2, ..., x_n, ...)$, where each $x_i \in \mathbb{R}$. Then $\mathbb{R}^\infty \sim (0, 1)$.
To show this, first observe that $\mathbb{R}^\infty \sim (0, 1)^\infty$
(Define $f : \mathbb{R}^\infty \longrightarrow (0, 1)^\infty$ by $f(x_1, x_2, ...) = \left(\frac{\tan^{-1}(x_1) + \frac{\pi}{2}}{\pi}, \frac{\tan^{-1}(x_2) + \frac{\pi}{2}}{\pi}, ... \right)$).
Thus, it is enough to show that $(0, 1) \sim (0, 1)^\infty$ (Note that $(0, 1)^\infty$ is the set of all sequences $\{x_n\}$ with $x_n \in (0, 1)$).
To do this, observe that $f : (0, 1) \longrightarrow (0, 1)^\infty$ given by $f(x) = (x, 0, 0, ...)$ (the choice of zeroes is arbitrary, what's important is to fix the first element) is an injective map from $(0, 1)$ into $(0, 1)^\infty$.
Thus,

$$\text{card}(0, 1) \leq \text{card}(0, 1)^\infty.$$

To prove the other direction, let $x \in (0, 1)^\infty$. Then $x = (x_1, x_2, ..., x_n, ...)$, where $x_n \in (0, 1)$ for all $n \in \mathbb{N}$. Represent each x_n by its unique finite decimal expansion

$$x_n = 0.\, x_{n_1} x_{n_2} x_{n_3} ....$$

In addition, let p_n be the n^{th} prime and define $g : (0, 1)^\infty \longrightarrow (0, 1)$ by $g(x) = 0.\, y_1 y_2 y_3 ...$, where

$$y_k = \begin{cases} x_{n_i} & \text{if } k = p_n^i \\ 0 & \text{otherwise} \end{cases}$$

Then g is injective. In particular,

$$\text{card}(0,\ 1)^{\infty} \leq \text{card}(0,\ 1).$$

Thus it follows, by Bernstein's theorem, that $\mathbb{R}^{\infty} \sim (0,\ 1)^{\infty} \sim (0,\ 1)$.

• Theorem:
The rational numbers ($\mathbb{Q}$) have measure 0 (i.e. occupy no space) on the real number line.

Proof:
Since $\mathbb{Q}$ is a countable set, we can list all of its elements in a sequence $\{x_n\}$. We will show that $\mathbb{Q}$ has measure 0 by proving that for any $\varepsilon > 0$, there is a collection of open intervals which cover $\mathbb{Q}$ and whose combined length is less than ε.
To do this, for each $x_n \in \mathbb{Q}$, define I_n by

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}},\ x_n + \frac{\varepsilon}{2^{n+1}}\right).$$

In other words, I_n is just an interval of length $L(I_n) = \frac{\varepsilon}{2^n}$ centered at x_n. Clearly, it is true that $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$. Now we have

$$L\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} L(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon\,. \qquad \blacksquare$$

Note: The above theorem can be interpreted as saying that the likelihood of selecting a rational number at random in the set of real numbers is 0. To put it in more colorful terms, having selected one object, the chance that another randomly selected object can be described in terms of the first is 0.

Definition: A number is said to be algebraic if there exist integers $a_0,\ a_1,\ \ldots,\ a_n \in \mathbb{Z}$ such that $a_0 + a_1\,x + \ldots + a_n\,x^n = 0$.

• Theorem:
The set of all algebraic numbers is countable.

Proof:
Let A_n be the set of all polynomials of degree n with integer coefficients.
The map $a_0 + a_1\,x + \ldots + a_n\,x^n \ \longmapsto\ (a_0,\ a_1,\ \ldots,\ a_n)$ shows that $A_n \sim \mathbb{Z}^{n+1}$, which implies that A_n is countable. Now the set of all polynomials with integer coefficients can be written as the countably infinite union $A = \bigcup_{n=1}^{\infty} A_n$, which must therefore be countable. Thus, each polynomial in A can be assigned a natural number that uniquely identifies it.

Let $k \in \mathbb{N}$ be the unique positive integer corresponding to $p(x) = a_0 + a_1\,x + \ldots + a_n\,x^n$. Observe that this polynomial can have at most n distinct complex roots. We can arrange these roots in lexicographic order from smallest to largest and associate $k.1$ with the smallest root of p, $k.01$ with the next smallest root of p, $k.001$ with the third smallest root, etc.
Clearly, each algebraic number is thus paired with at least one rational number. This implies that

algebraic numbers are countable. ■

Note: Notice that all countable sets have measure 0 in $\mathbb{R}$ or $\mathbb{C}$. Thus, the probability that a number is algebraic is 0, which implies that almost all numbers are transcendental.