# Math 746 Notes

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## Integration Theory

The general notion of the Lebesgue integral on $\mathbb{R}^d$ will be defined in a step-by-step fashion, proceeding successively to increasingly larger families of functions. At each stage we shall see that the integral satisfies elementary properties such as linearity and monotonicity, and we show appropriate convergence theorems that amount to interchanging the integral with limits. At the end of the process we shall have achieved a general theory of integration that will be decisive in the study of further problems. We emphasize from the onset that all functions are assumed to be measurable.

STAGE ONE

SIMPLE FUNCTIONS

Recall that a simple function $\varphi$ is a finite sum

$$\text{(I)} \qquad \varphi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x)\,,$$

where the $E_k$ are measurable sets of finite measure and the $a_k$ are constants. A complication that arises from this definition is that a simple function can be written in a multitude of ways as such finite linear combinations; for example, $0 = \chi_E - \chi_E$ for any measurable set $E$ of finite measure. Fortunately, there is an unambiguous choice for the representation of a simple function, which is natural and useful in applications:

Definition: The canonical form of $\varphi$ is the unique decomposition as in (I), where the numbers $a_k$ are distinct and non-zero, and the sets $E_k$ are disjoint.

Finding the canonical form of $\varphi$ is straightforward:
Since $\varphi$ can take only finitely many distinct and non-zero values, say $c_1, \ \ldots, \ c_M$, we may set

$$F_k = \{x : \varphi(x) = c_k\},$$

and note that the sets $F_k$ are disjoint. Therefore

$$\varphi = \sum_{k=1}^{M} c_k \chi_{F_k}$$

is the desired canonical form of $\varphi$.

Definition: If $\varphi$ is a simple function with canonical form $\varphi = \sum_{k=1}^{M} c_k \chi_{F_k}$, then we define the Lebesgue integral of $\varphi$ by

$$\int_{\mathbb{R}^d} \varphi(x)\, dx = \int_{\mathbb{R}^d} \sum_{k=1}^{M} c_k\, \chi_{F_k}(x)\, dx = \sum_{k=1}^{M} c_k\, m(F_k)\,.$$

If $E$ is a measurable subset of $\mathbb{R}^d$ with finite measure, then $\varphi(x)\, \chi_E(x)$ is also a simple function, and we define

$$\int_E \varphi(x)\, dx = \int \varphi(x)\, \chi_E(x)\, dx\,.$$

Remark: To emphasize the choice of the Lebesgue measure $m$ in the definition of the integral, one sometimes writes $\int_{\mathbb{R}^d} \varphi(x)\, dm(x)$ for the Lebesgue integral of $\varphi$. However, as a matter of convenience, we shall often write $\int \varphi(x)\, dx$ or simply $\int \varphi$ for the integral of $\varphi$ over $\mathbb{R}^d$.

• Proposition:
The integral of simple functions defined above satisfies the following properties:

(i) (Independence of the representation)
If $\varphi = \sum_{k=1}^{N} a_k\, \chi_{E_k}$ is any representation of $\varphi$, then

$$\int \varphi = \sum_{k=1}^{N} a_k\, m(E_k).$$

(ii) (Linearity).
If $\varphi$ and $\psi$ are simple, and $a,\ b \in \mathbb{R}$, then

$$\int (a\, \varphi + b\, \psi) = a \int \varphi + b \int \psi\,.$$

(iii) (Additivity)
If $E$ and $F$ are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi.$$

(iv) (Monotonicity)
If $\varphi \le \psi$ are simple, then

$$\int \varphi \le \int \psi\,.$$

(v) (Triangle inequality)
If $\varphi$ is a simple function, then so is $|\varphi|$, and

$$\left| \int \varphi \right| \le \int |\varphi|.$$

Proof:
(See pg 51, Stein's) ■

# STAGE TWO

## BOUNDED FUNCTIONS SUPPORTED ON A SET OF FINITE MEASURE

Definition: The support of a measurable function $f$ is defined to be the set of all points where $f$ does not vanish, that is

$$\operatorname{supp} f = \{x : f(x) \neq 0\}.$$

We shall also say that $f$ is supported on a set $E$, if $f(x) = 0$ whenever $x \notin E$.

It follows that, since $f$ is measurable, so is the set $\operatorname{supp} f$. We shall next be interested in those bounded measurable functions that have $m(\operatorname{supp} f) < \infty$.

An important result is that if $f$ is a function bounded by $M$ and supported on a set $E$, then there exists a sequence $\{\varphi_n\}$ of simple functions, with each $\varphi_n$ bounded by $M$ and supported on $E$, and such that

$$\varphi_n(x) \to f(x) \qquad \forall\, x\,.$$

The key lemma that follows allows us to define the integral for the class of bounded functions supported on sets of finite measure:

• Lemma:
Let $f$ be a bounded function supported on a set $E$ of finite measure. If $\{\varphi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by $M$, supported on $E$, and with $\varphi_n(x) \to f(x)$ for a.e. $x$, then:
(i) $\lim\limits_{n\to\infty} \int \varphi_n$ exists.
(ii) if $f = 0$ a.e., then $\lim\limits_{n\to\infty} \int \varphi_n = 0$.

Proof:
The assertions of the lemma would be nearly obvious if we had that $\varphi_n$ converges to $f$ uniformly on $E$. Instead, we recall one of Littlewood's principles, which states that the convergence of a sequence of measurable functions is "nearly" uniform. The precise statement lying behind this principle is Egorov's theorem, which we apply here.

For part (i), since the measure of $E$ is finite, given $\varepsilon > 0$ Egorov's theorem guarantees the existence of a (closed) measurable subset $A_\varepsilon \subset E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$, and $\varphi_n \to f$ uniformly on $A_\varepsilon$. Therefore, setting $I_n = \int \varphi_n$ we have that

$$\begin{aligned} |I_n - I_m| &\leq \int_E |\varphi_n(x) - \varphi_m(x)|\, dx \\ &= \int_{A_\varepsilon} |\varphi_n(x) - \varphi_m(x)|\, dx + \int_{E\setminus A_\varepsilon} |\varphi_n(x) - \varphi_m(x)|\, dx \\ &\leq \int_{A_\varepsilon} |\varphi_n(x) - \varphi_m(x)|\, dx + 2\, M\, m(E \setminus A_\varepsilon) \end{aligned}$$

$$\leq \int_{A_\varepsilon} |\varphi_n(x) - \varphi_m(x)|\, dx + 2\, M\, \varepsilon\,.$$

By the uniform convergence, one has, for all $x \in A_\varepsilon$ and all large $n$ and $m$, the estimate $|\varphi_n(x) - \varphi_m(x)| < \varepsilon$, so we deduce that

$$|I_n - I_m| \leq m(E)\, \varepsilon + 2\, M\, \varepsilon \qquad \text{for all large } n \text{ and } m.$$

Since $\varepsilon$ is arbitrary and $m(E) < \infty$, this proves that $\{I_n\}$ is a Cauchy sequence and hence converges, as desired.

To show part (ii), we note that if $f = 0$, we may repeat the argument above to find that $|I_n| \leq m(E)\, \varepsilon + M\, \varepsilon$, which yields $\lim_{n\to\infty} I_n = 0$, as was to be shown. ■

Remark: Using the above lemma we can now turn to the integration of bounded functions that are supported on sets of finite measure. For such a function $f$ we define its Lebesgue integral by

$$\int f(x)\, dx = \lim_{n\to\infty} \int \varphi_n(x)\, dx \,,$$

where $\{\varphi_n\}$ is any sequence of simple functions satisfying:

i) $|\varphi_n| \leq M$.
ii) each $\varphi_n$ is supported on the support of $f$.
iii) $\varphi_n(x) \to f(x)$ for a.e. $x$ as $n$ tends to infinity (we know by the above lemma that this limit exists).

Next, we must first show that $\int f$ is independent of the limiting sequence $\{\varphi_n\}$ used, in order for the integral to be well-defined. Therefore, suppose that $\{\psi_n\}$ is another sequence of simple functions that is bounded by $M$, supported on $\operatorname{supp} f$, and such that $\psi_n(x) \to f(x)$ for a.e. $x$ as $n$ tends to infinity. Then, if $\eta_n = \varphi_n - \psi_n$, the sequence $\{\eta_n\}$ consists of simple functions bounded by $2\, M$, supported on a set of finite measure, and such that $\eta_n \to 0$ a.e. as $n$ tends to infinity. We may therefore conclude, by the second part of the above lemma, that $\int \eta_n \to 0$ as $n$ tends to infinity. Consequently, the two limits

$$\lim_{n\to\infty} \int \varphi_n(x)\, dx \qquad \text{and} \qquad \lim_{n\to\infty} \int \psi_n(x)\, dx$$

(which exist by the lemma) are indeed equal.

Remark: If $E$ is a subset of $\mathbb{R}^d$ with finite measure, and $f$ is bounded with $m(\operatorname{supp} f) < \infty$, then it is natural to define

$$\int_E f(x)\, dx = \int f(x)\, \chi_E(x)\, dx\,.$$

Clearly, if $f$ is itself simple, then $\int f$ as defined above coincides with the integral of simple functions presented earlier. This extension of the definition of integration also satisfies all the basic properties of the integral of simple functions:

• Proposition:
Suppose $f$ and $g$ are bounded functions supported on sets of finite measure. Then the following properties hold:

(i) (Linearity).
If $a,\ b \in \mathbb{R}$, then

$$\int (a\, f + b\, g) = a \int f + b \int g\,.$$

(ii) (Additivity)
If $E$ and $F$ are disjoint subsets of $\mathbb{R}^d$, then

$$\int_{E \cup F} f = \int_E f + \int_F f\,.$$

(iii) (Monotonicity)
If $f \leq g$, then

$$\int f \leq \int g\,.$$

(iv) (Triangle inequality)
$|f|$ is also bounded, supported on a set of finite measure, and

$$\left| \int f \right| \leq \int |f|.$$

We are now in a position to prove the first important convergence theorem:

• Bounded Convergence Theorem:
Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by $M$, are supported on a set $E$ of finite measure, and $f_n(x) \to f(x)$ a.e. $x$ as $n \to \infty$. Then $f$ is measurable, bounded, supported on $E$ for a.e. $x$, and

$$\int |f_n - f| \to 0 \quad \text{as} \quad n \to \infty\,.$$

Consequently,

$$\int f_n \to \int f \qquad \text{as} \quad n \to \infty\,.$$

Proof:
From the assumptions one sees at once that $f$ is bounded by $M$ almost everywhere and vanishes outside $E$, except possibly on a set of measure zero. Clearly, the triangle inequality for the integral implies that it suffices to prove that $\int |f_n - f| \to 0$ as $n$ approaches infinity.
Given $\varepsilon > 0$, we may find, by Egorov's theorem, a measurable subset $A_\varepsilon \subset E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$

and $f_n \to f$ uniformly on $A_\varepsilon$. Then, we know that for all sufficiently large $n$ and for all $x \in A_\varepsilon$, we have

$$|f_n(x) - f(x)| \leq \varepsilon.$$

Putting these facts together yields

$$\begin{aligned} \int_E |f_n(x) - f(x)|\, dx &= \int_{A_\varepsilon} |f_n(x) - f(x)|\, dx + \int_{E \setminus A_\varepsilon} |f_n(x) - f(x)|\, dx \\ &\leq \varepsilon\, m(E) + 2\, M\, m(E \setminus A_\varepsilon) \end{aligned}$$

for all large $n$. Since $\varepsilon$ is arbitrary, the proof of the theorem is complete. ■

Remark: We note that the above convergence theorem is a statement about the interchange of an integral and a limit, since its conclusion simply says

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n .$$

A useful observation that we can make at this point is the following: if $f \geq 0$ is bounded and supported on a set of finite measure $E$ and $\int f = 0$, then $f = 0$ almost everywhere. Indeed, if for each integer $k \geq 1$ we set

$$E_k = \{x \in E : f(x) \geq 1/k\},$$

then the fact that $k^{-1}\, \chi_{E_k}(x) \leq f(x)$ implies

$$k^{-1}\, m(E_k) \leq \int f ,$$

by monotonicity of the integral. Thus $m(E_k) = 0$ for all $k$, and since

$$\{x : f(x) > 0\} = \bigcup_{k=1}^{\infty} E_k ,$$

we see that $f = 0$ almost everywhere.

## Return to Riemann integrable functions:

• Theorem:

Suppose $f$ is Riemann integrable on the closed interval $[a, b]$. Then $f$ is measurable, and

$$\int_{[a,b]}^{\mathcal{R}} f(x)\, dx = \int_{[a,b]}^{\mathcal{L}} f(x)\, dx ;$$

where the integral on the left-hand side is the standard Riemann integral, and that on the right-hand side is the Lebesgue integral.

<u>Proof:</u>
By definition, a Riemann integrable function is bounded, say $\lfloor f(x)| \leq M$, so we need to prove that $f$ is measurable, and then establish the equality of integrals.
Again, by definition of Riemann integrability, we may construct two sequences of step functions $\{\varphi_k\}$ and $\{\psi_k\}$ that satisfy the following properties:

(*) $|\varphi_k(x)| \leq M$ and $|\psi_k(x)| \leq M$ for all $x \in [a, b]$ and $k \geq 1$.
(**) $\varphi_1(x) \leq \varphi_2(x) \leq \ldots \leq f \leq \psi_2(x) \leq \psi_1(x)$.
(***) $\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \varphi_k(x)\, dx = \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \psi_k(x)\, dx = \int_{[a,b]}^{\mathcal{R}} f(x)\, dx$ .

Several observations are in order. First, it follows immediately from their definition that for step functions the Riemann and Lebesgue integrals agree; therefore

$$(\spadesuit) \qquad \int_{[a,b]}^{\mathcal{R}} \varphi_k(x)\, dx = \int_{[a,b]}^{\mathcal{L}} \varphi_k(x)\, dx \quad \text{and} \quad \int_{[a,b]}^{\mathcal{R}} \psi_k(x)\, dx = \int_{[a,b]}^{\mathcal{L}} \psi_k(x)\, dx\ ,$$

for all $k \geq 1$.

Next, if we let

$$\tilde{\varphi}(x) = \lim_{k\to\infty} \varphi_k(x) \quad \text{and} \quad \tilde{\psi}(x) = \lim_{k\to\infty} \psi_k(x)\ ,$$

we have $\tilde{\varphi} < f < \tilde{\psi}$. Moreover, both $\tilde{\varphi}$ and $\tilde{\psi}$ are measurable (being the limit of step functions), and the bounded convergence theorem yields

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \varphi_k(x)\, dx = \int_{[a,b]}^{\mathcal{L}} \tilde{\varphi}(x)\, dx$$

and

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \psi_k(x)\, dx = \int_{[a,b]}^{\mathcal{L}} \tilde{\psi}(x)\, dx\ .$$

This together with property (***) and ($\spadesuit$) yields

$$\int_{[a,b]}^{\mathcal{L}} \left[\tilde{\psi}(x) - \tilde{\varphi}(x)\right] dx = 0\ ,$$

and since $\psi_k - \varphi_k \geq 0$, we must have $\tilde{\psi} - \tilde{\varphi} \geq 0$.

By the remark following the proof of the bounded convergence theorem, we conclude that $\tilde{\psi} - \tilde{\varphi} = 0$ a.e., and therefore $\tilde{\psi} = \tilde{\varphi} = f$ a.e., which proves that $f$ is measurable. Finally, since $\varphi_k \to f$ almost everywhere, we have (by definition)

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \varphi_k(x)\, dx = \int_{[a,b]}^{\mathcal{L}} f(x)\, dx\ ,$$

and by (***) and ($\spadesuit$) we see that $\int_{[a,b]}^{\mathcal{R}} f(x)\, dx = \int_{[a,b]}^{\mathcal{L}} f(x)\, dx$ as desired. ■

## STAGE THREE

## NONNEGATIVE FUNCTIONS

We proceed with the integrals of functions that are measurable and non-negative but not necessarily bounded. It will be important to allow these functions to be extended-valued, that is, these functions may take on the value $+\infty$ (on a measurable set). We recall in this connection the convention that one defines the supremum of a set of positive numbers to be $+\infty$ if the set is unbounded.
In the case of such a function $f$ we define its (extended) Lebesgue integral by

$$\int f(x)\,dx = \sup_g \int g(x)\,dx\,,$$

where the supremum is taken over all measurable functions $g$ such that $0 \leq g \leq f$, and where $g$ is bounded and supported on a set of finite measure.

Remark: With the above definition of the integral, there are only two possible cases: the supremum is either finite, or infinite. In the first case, when $\int f(x)\,dx < \infty$, we shall say that $f$ is Lebesgue integrable (or simply integrable).
Clearly, if $E$ is any measurable subset of $\mathbb{R}^d$, and $f \geq 0$, then $f\,\chi_E$ is also positive, and we define

$$\int_E f(x)\,dx = \int f(x)\,\chi_E(x)\,dx\,.$$

Example:
Let

$$f(x) = \begin{cases} |x|^{-\alpha} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} |x|^{-\alpha} & \text{if } |x| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ is integrable on $\mathbb{R}^d$ iff $\alpha < d$ and $g$ is integrable on $\mathbb{R}^d$ iff $\alpha > d$.

• Proposition:
The integral of nonnegative measurable functions enjoys the following properties:

(i) (Linearity).
If $f, g \geq 0$ and $a, b \in \mathbb{R}$, then

$$\int (a\,f + b\,g) = a \int f + b \int g\,.$$

(ii) (Additivity)

If $E$ and $F$ are disjoint subsets of $\mathbb{R}^d$, and $f \geq 0$, then

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(iii) (Monotonicity)
If $0 \leq f \leq g$, then

$$\int f \leq \int g.$$

(iv) If $g$ is integrable and $0 \leq f \leq g$, then $f$ is integrable.

(v) If $f$ is integrable, then $f(x) < \infty$ for almost every $x$.

(vi) If $\int f = 0$, then $f(x) = 0$ for almost every $x$.

Proof:
Of the first four assertions, only (i) is not an immediate consequence of the definitions, and to prove it we argue as follows.
We take $a = b = 1$ and note that if $\varphi \leq f$ and $\psi \leq g$, where both $\varphi$ and $\psi$ are bounded and supported on sets of finite measure, then $\varphi + \psi \leq f + g$, and $\varphi + \psi$ is also bounded and supported on a set of finite measure.
Consequently

$$\int f + \int g \leq \int (f + g).$$

To prove the reverse inequality, suppose $\eta$ is bounded and supported on a set of finite measure, and $\eta \leq f + g$.
If we define

$$\eta_1(x) = \min(f(x), \eta(x)) \quad \text{and} \quad \eta_2 = \eta - \eta_1,$$

we note that

$$\eta_1 \leq f \quad \text{and} \quad \eta_2 \leq g.$$

Moreover both $\eta_1$, $\eta_2$ are bounded and supported on sets of finite measure.
Hence

$$\int \eta = \int (\eta_1 + \eta_2) = \int \eta_1 + \int \eta_2 \leq \int f + \int g.$$

Taking the supremum over $\eta$ yields the required inequality.

To prove the conclusion (v) we argue as follows.
Suppose

$$E_k = \{x : f(x) \geq k\} \quad \text{and} \quad E_\infty = \{x : f(x) = \infty\}$$

Then

$$\int f \geq \int \chi_{E_k} f \geq k\, m(E_k)\ ,$$

hence $m(E_k) \to 0$ as $k \to \infty$. Since $E_k \searrow E_\infty$, by a previous corollary we have that $m(E_\infty) = 0$. ■

We now turn our attention to some important convergence theorems for the class of non-negative measurable functions. To motivate the results that follow, we ask the following question:

Suppose $f_n \geq 0$ and $f_n(x) \to f(x)$ for almost every $x$. Is it true that $\int f_n\, dx \to \int f\, dx$?

Unfortunately, the example that follows provides a negative answer to this, and shows that we must change our formulation of the question to obtain a positive convergence result:

Example:
Let

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n(x) \to 0\ \ \forall\, x$, yet $\int f_n(x)\, dx = 1\ \ \forall\, n$.

In this particular example, the limit of the integrals is greater than the integral of the limit function. This turns out to be the case in general, as we shall see now:

• Fatou's Lemma:
Suppose $\{f_n\}$ is a sequence of measurable functions with $f_n \geq 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x$, then $\int f \leq \liminf_{n\to\infty} \int f_n$ .

Proof:
Suppose $0 \leq g \leq f$, where $g$ is bounded and supported on a set $E$ of finite measure. If we set $g_n(x) = \min(g(x), f_n(x))$, then $g_n$ is measurable, supported on $E$, and $g_n(x) \to g(x)$ a.e., so by the bounded convergence theorem

$$\int g_n \to \int g\ .$$

By construction, we also have $g_n \leq f_n$, so that $\int g_n \leq \int f_n$ by the monotonicity of the integral. Thus,

$$\int g \leq \liminf_{n\to\infty} \int f_n\ .$$

Taking the supremum over all $g$ yields the desired inequality. ■

Remark: In particular, we do not exclude the cases $\int f = \infty$, or $\liminf_{n\to\infty} f_n = \infty$.

We can now immediately deduce the following series of corollaries:

• Corollary:
Suppose $f$ is a non-negative measurable function, and $\{f_n\}$ a sequence of non-negative measurable functions with $f_n(x) \leq f(x)$ and $f_n(x) \to f(x)$ for almost every $x$.
Then

$$\lim_{n\to\infty} \int f_n = \int f\ .$$

Proof:
Since $f_n(x) \leq f(x)$ a.e. $x$, we necessarily have $\int f_n \leq \int f \ \ \forall\ n$.
Hence,

$$\liminf_{n\to\infty} \int f_n \leq \int f.$$

This inequality combined with Fatou's lemma proves the desired limit. ■

In particular, we can now obtain a basic convergence theorem for the class of non-negative measurable functions. Its statement requires the following notation:

Notation: $f_n \nearrow f$ refers to a sequence $\{f_n\}$ of monotonically increasing functions that are converging to the limit $f$ as $n \to \infty$ a.e. $x$. We denote the decreasing analogues by $f_n \searrow f$.

• Corollary (Monotone convergence theorem):
Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$.
Then

$$\lim_{n\to\infty} \int f_n = \int f\ .$$

Proof:
The proof follows from the preceding corollary and its proof. ■

The monotone convergence theorem has the following useful consequence:

• Corollary:
Consider a series $\sum_{k=1}^{\infty} a_k(x)$, where $a_k \geq 0$ is measurable for every $k \geq 1$.
Then

$$\int \sum_{k=1}^{\infty} a_k(x)\, dx = \sum_{k=1}^{\infty} \int a_k(x)\, dx .$$

If $\sum_{k=1}^{\infty} \int a_k(x)\, dx$ is finite, then the series $\sum_{k=1}^{\infty} a_k(x)$ converges for a.e. $x$.

Proof:
Let $f_n(x) = \sum_{k=1}^{n} a_k(x)$ and $f(x) = \sum_{k=1}^{\infty} a_k(x)$. The functions $f_n$ are measurable, $f_n(x) \le f_{n+1}(x)$, and $f_n(x) \to f(x)$ as $n$ tends to infinity. Since

$$\int f_n = \sum_{k=1}^{n} \int a_k(x)\, dx ,$$

the monotone convergence theorem implies

$$\sum_{k=1}^{\infty} \int a_k(x)\, dx = \int \sum_{k=1}^{\infty} a_k(x)\, dx .$$

If $\sum \int a_k < \infty$, then the above implies that $\sum_{k=1}^{\infty} a_k(x)$ is integrable, and by our earlier observation, we conclude that $\sum_{k=1}^{\infty} a_k(x)$ is finite almost everywhere. ■

• Borel-Cantelli Lemma:
If $E_1, E_2, \ldots$ is a collection of measurable subsets with $\sum m(E_k) < \infty$, then the set of points that belong to infinitely many sets $E_k$ has measure zero.

Proof:
Let $a_k(x) = \chi_{E_k}(x)$, and note that a point $x$ belongs to infinitely many sets $E_k$ iff $\sum_{k=1}^{\infty} a_k(x) = \infty$. Our assumption on $\sum m(E_k)$ says precisely that $\sum_{k=1}^{\infty} \int a_k(x)\, dx < \infty$, and the preceding corollary implies that $\sum_{k=1}^{\infty} a_k(x)$ is finite except possibly on a set of measure zero. ■

## STAGE FOUR

## GENERAL CASE

If $f$ is any real-valued measurable function on $\mathbb{R}^d$, we say that $f$ is Lebesgue integrable (or just integrable) if the non-negative measurable function $|f|$ is integrable in the sense discussed on stage three. If $f$ is Lebesgue integrable, we give a meaning to its integral as follows:

First, we may define

$$f^+(x) = \max(f(x), 0) \qquad \text{and} \qquad f^-(x) = \max(-f(x), 0) ,$$

so that both $f^+$ and $f^-$ are non-negative and

$$f^+ - f^- = f .$$

Since $f^\pm \le |f|$, both functions $f^+$ and $f^-$ are integrable whenever $f$ is, and we then define the Lebesgue integral of $f$ by

$$\int f = \int f^+ - \int f^- .$$

In practice one encounters many decompositions $f = f_1 - f_2$, where $f_1$, $f_2$ are both non-negative integrable functions, and one would expect that regardless of the decomposition of $f$, we always have

$$\int f = \int f_1 - \int f_2 .$$

In other words, the definition of the integral should be independent of the decomposition $f = f_1 - f_2$.

Simple applications of the definition and the properties shown previously yield all the elementary properties of the integral:

• Proposition:
The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality.

• Proposition:
Suppose $f$ is integrable on $\mathbb{R}^d$. Then for every $\varepsilon > 0$:
i) There exists a set of finite measure $B$ (a ball, for example) such that $\int_{B^c} |f| < \varepsilon$ .
ii) There is a $\delta > 0$ such that $\int_E |f| < \varepsilon$ whenever $m(E) < \delta$.
Condition ii) is known as absolute continuity.

Proof:
By replacing $f$ with $|f|$ we may assume WLOG that $f \geq 0$.
To prove condition i), let $B_N$ denote the ball of radius $N$ centered at the origin, and note that if $f_N(x) = f(x)\,\chi_{B_N}(x)$, then $f_N \geq 0$ is measurable, $f_N(x) \leq f_{N+1}(x)$, and $\lim_{N\to\infty} f_N(x) = f(x)$. By the monotone convergence theorem, we must have

$$\lim_{N\to\infty} \int f_N = \int f .$$

In particular, for some large $N$,

$$0 \leq \int f - \int f\,\chi_{B_N} < \varepsilon ,$$

and since $1 - \chi_{B_N} = \chi_{B_N^c}$, this implies $\int_{B_N^c} f < \varepsilon$, as we set out to prove.

Now to prove condition ii), we assume again that $f \geq 0$ and we let $f_N(x) = f(x)\,\chi_{E_N}$, where $E_N = \{x : f(x) \leq N\}$.
Once again, $f_N \geq 0$ is measurable, $f_N(x) \leq f_{N+1}(x)$, and given $\varepsilon > 0$ there exists (by the monotone convergence theorem) an integer $N > 0$ such that

$$\int (f - f_N) < \frac{\varepsilon}{2}.$$

We now pick $\delta > 0$ so that $N\,\delta < \varepsilon / 2$. If $m(E) < \delta$, then

$$\begin{aligned}\int_E f &= \int_E (f - f_N) + \int_E f_N \\ &\leq \int (f - f_N) + \int_E f_N \\ &\leq \int (f - f_N) + N\, m(E) \\ &\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This concludes the proof of the proposition. ■

Intuitively, integrable functions should in some sense vanish at infinity since their integrals are finite. Condition i) of the above proposition attaches a precise meaning to this intuition. One should observe, however, that integrability need not guarantee the more naive pointwise vanishing as $|x|$ becomes large (see exercise 6, chpt 2, Stein).

We are now ready to prove a cornerstone of the theory of Lebesgue integration, the dominated convergence theorem. It can be viewed as a culmination of our efforts, and is a general statement about the interplay between limits and integrals.

• Dominated Convergence Theorem:

Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. $x$ as $n \to \infty$. If $|f_n(x)| \leq g(x)$, where $g$ is integrable, then

$$\int |f_n - f| \to 0 \quad \text{as} \quad n \to \infty\,,$$

and consequently,

$$\int f_n \to \int f \qquad \text{as} \quad n \to \infty\,.$$

Proof:

For each $N \geq 0$ let $E_N = \{x : |x| \leq N,\ g(x) \leq N\}$. Given $\varepsilon > 0$, we may argue (as in the first part of above proposition) that there exists $N$ so that

$$\int_{E_N^c} |f| < \varepsilon\,.$$

Then the functions $f_n\, \chi_{E_N}$ are bounded (by $N$) and supported on a set of finite measure, so that by the bounded convergence theorem, we have

$$\int_{E_N} |f_n - f| < \varepsilon \qquad \text{for all large } n.$$

Hence, we obtain the estimate

$$\begin{aligned}\int |f_n - f| &= \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f| \\ &\leq \int_{E_N} |f_n - f| + 2 \int_{E_N^c} g \\ &\leq \varepsilon + 2\,\varepsilon = 3\,\varepsilon\end{aligned}$$

for all large $n$. This proves the theorem. ■