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Problem Set 3
Numerical Analysis I

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Problem 1. *Derive a Newton-Cotes formula for* $\int_0^1 f(x)\,\mathrm{d}x$ *based on the nodes* $\{0,\ 1/3,\ 2/3,\ 1\}$.

Solution. Our starting point is the Lagrange interpolation of the function $f(x)$, given by

$$f(x) \approx \sum_{i=0}^{n} f_i \ell_i(x), \tag{1}$$

where $f_i \equiv f(x_i)$ and

$$\ell_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \qquad \text{for } i = 0, \dots, n. \tag{2}$$

Then, letting the *weights* w_i be given by

$$w_i := \int_a^b \ell_i(x)\,\mathrm{d}x, \tag{3}$$

we have the ***quadrature formula***

$$\boxed{\int_a^b f(x)\,\mathrm{d}x = \sum_{i=0}^{n} w_i f_i + \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x - x_i)\, f^{(n+1)}(\xi(x))\,\mathrm{d}x} \tag{4}$$

where $\xi \in (a, b)$. This formula is of the form

$$\int_a^b f(x)\,\mathrm{d}x = P + E,$$

where P is the *quadrature approximation*

$$P = \sum_{i=0}^{n} w_i f_i, \tag{5}$$

and E is the *error term*

$$E = \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x - x_i)\, f^{(n+1)}(\xi(x))\,\mathrm{d}x. \tag{6}$$

If we let $n = 1$ in (4), the approximation is linear and we get the *Trapezoidal Rule*, while for $n = 2$ the approximation is quadratic and yields *Simpson's* $1/3$ *rule*. Now, in our case, we have the four nodes $\{x_0 = 0,\ x_1 = 1/3,\ x_2 = 2/3,\ x_3 = 1\}$. A more accurate approximation to the integral of $f(x)$ over the interval $[0, 1]$ would entail a *composite* Newton-Cotes formula (for instance, break up the interval into four subintervals using the provided four nodes), but this problem is asking to write a generic, non-composite formula, so we will accomplish that by using $n = 3$ in (4) and get a cubic polynomial to integrate in place of $f(x)$:

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{i=0}^{3} w_i f_i + \frac{1}{4!}\int_0^1 \prod_{i=0}^{3}(x - x_i)\, f^{(4)}(\xi(x))\,\mathrm{d}x.$$

Let's look at the approximation and error terms, one at a time. Starting with the approximation,

$$\begin{aligned}
P &= w_0 f_0 + w_1 f_1 + w_2 f_2 + w_3 f_3 \\
&= \int_0^1 \frac{x - x_1}{x_0 - x_1}\frac{x - x_2}{x_0 - x_2}\frac{x - x_3}{x_0 - x_3} f_0\,\mathrm{d}x + \int_0^1 \frac{x - x_0}{x_1 - x_0}\frac{x - x_2}{x_1 - x_2}\frac{x - x_3}{x_1 - x_3} f_1\,\mathrm{d}x \\
&+ \int_0^1 \frac{x - x_0}{x_2 - x_0}\frac{x - x_1}{x_2 - x_1}\frac{x - x_3}{x_2 - x_3} f_2\,\mathrm{d}x + \int_0^1 \frac{x - x_0}{x_3 - x_0}\frac{x - x_1}{x_3 - x_1}\frac{x - x_2}{x_3 - x_2} f_3\,\mathrm{d}x.
\end{aligned} \tag{7}$$

Instead of looking for a tricky change of variables here or just doing a straightforward, but LONG antiderivation, we make our lives easier by summoning Mathematica to the rescue:

```
Integrate[ (x - x1)/(x0 - x1) * (x - x2)/(x0 - x2) * (x - x3)/(x0 - x3),
        {x, a, b}] /. {x0 -> 0, x1 -> 1/3, x2 -> 2/3, x3 -> 1}
Integrate[ (x - x0)/(x1 - x0) * (x - x2)/(x1 - x2) * (x - x3)/( x1 - x3),
        {x, a, b}] /. {x0 -> 0, x1 -> 1/3, x2 -> 2/3, x3 -> 1}
Integrate[ (x - x0)/(x2 - x0) * (x - x1)/(x2 - x1) * (x - x3)/(x2 - x3),
        {x, a, b}] /. {x0 -> 0, x1 -> 1/3, x2 -> 2/3, x3 -> 1}
Integrate[ (x - x0)/(x3 - x0) * (x - x1)/(x3 - x1)*(x - x2)/( x3 - x2),
        {x, a, b}] /. {x0 -> 0, x1 -> 1/3, x2 -> 2/3, x3 -> 1}
```

The above yields

```
-(3/8) ((8a)/3 - (22 a^2)/3 + 8a^3 - 3a^4 +  b(-(8/3) + (22b)/3 - 8b^2 + 3b^3))

9/8 (-4a^2 + (20a^3)/3 - 3a^4 + b(4b - (20b^2)/3 + 3b^3))

-(9/8) (-2a^2 + (16a^3)/3 - 3a^4 + b(2b - (16b^2)/3 + 3b^3))

3/8 (-((4a^2)/3) + 4a^3 - 3a^4 + b((4 b)/3 - 4b^2 + 3b^3))
```

That shows the result for a general $[a, b]$, but it simplifies greatly once we substitute actual values…In our case, with $a = 0$, $b = 1$, the results simplify to

$$\{w_0, w_1, w_2, w_3\} = \left\{\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right\},$$

so that all the weights add up to 1. Hence, the quadrature approximation is

$$P = \sum_{i=0}^{3} w_i f_i = \frac{1}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right). \tag{8}$$

In fact, for a general $[a, b]$, this result takes the form

$$P = \sum_{i=0}^{3} w_i f_i = \frac{b-a}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right). \tag{9}$$

Moreover, can also rewrite this in terms of $\mathrm{h} = x_{i+1} - x_i$. Since there are four interpolating nodes in the interval $[a, b]$ in this approach, $[a, b]$ splits into three subintervals $\{[a = x_0, x_1],\ [x_1, x_2],\ [x_2, x_3 = b]\}$. Thus, $\mathrm{h} = (b - a)/3$, and we have

$$P = \sum_{i=0}^{3} w_i f_i = \frac{3\mathrm{h}}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right). \tag{10}$$

(Whence the name for this approach: *Simpson's* $3/8$ *rule*.)

Similarly, for the error term:

```
Integrate[
  1/4! * Product[x - Subscript[x, i], {i, 0, 3}], {x, 0, 1}]
      /. {Subscript[x, 0] -> 0, Subscript[x, 1] -> 1/3,
          Subscript[x, 2] -> 2/3, Subscript[x, 3] -> 1}
```

which yields $-1/6480$. Hence,

$$\begin{aligned} E &= \frac{1}{4!}\int_0^1 \prod_{i=0}^{3}(x - x_i)\, f^{(4)}(\xi(x))\,\mathrm{d}x \\ &= -\frac{1}{6480} f^{(4)}(\xi(x)). \end{aligned} \tag{11}$$

Note that here we were able to pull the factor $f^{(4)}(\xi(x))$ out of the integral because $\prod_{i=0}^{3}(x - x_i)$ is a fourth-order polynomial which does not change sign in the entire interval $[0, 1]$; thus we were able to use the following theorem:

Weighted Mean Value Theorem for Integrals

Suppose that f is continuous on $[a, b]$. Let g be another function such that its Riemann integral exists, and g does not change sign on $[a, b]$. Then, there exists a number $c \in (a, b)$ that satisfies

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x. \tag{12}$$

The error term (11) generalizes to any interval $[a, b]$, by using $\mathrm{h} = (b - a)/3$ as before:

$$E = -\frac{3}{80}\,\mathrm{h}^5 f^{(4)}(\xi(x)). \qquad \square \tag{13}$$

Problem 2. *Apply the composite Simpson's rule with $m = 4$ to the integral $\int_0^{\pi} x \cos x \, \mathrm{d}x$.*

Solution. In *Simpson's $(1/3)$ composite rule*, an integral

$$I = \int_a^b f(x) \, \mathrm{d}x$$

is approximated by partitioning the interval $[a, b]$ into N evenly spaced segments $a = x_0 < x_1 < \cdots < x_N = b$ with spacing $\mathrm{h} \equiv (b - a)/N$, and then putting

$$I \approx \frac{\mathrm{h}}{3} \left[f_0 + 4 \sum_{\substack{i=1 \\ i \text{ is odd}}}^{N-1} f_i + 2 \sum_{\substack{i=2 \\ i \text{ is even}}}^{N-2} f_i + f_N \right]. \tag{14}$$

(Note that N must be even in order for (14) to work.) In the case at hand, x and $\cos x$ have the same sign on the whole interval $[0, \pi]$, so we may exploit this symmetry and integrate instead the function $f(x) = 2x \cos x$ on the interval $[a, b] = [0, \pi/2]$. In this interval we must work with $m = 4$ panels (subintervals) and, moreover, Simpson's $1/3$ rule uses three neighboring points, so each of these panels contain two further sub-subintervals. That makes a total of $N = 2 \times 4 = 8$ evenly spaced segments, and thus $\mathrm{h} = (\pi/2 - 0)/8 = \pi/16$. Thus our nine nodes are

$$\left\{ x_0 = 0,\ x_1 = \frac{\pi}{16},\ x_2 = \frac{\pi}{8},\ x_3 = \frac{3\pi}{16},\ x_4 = \frac{\pi}{4},\ x_5 = \frac{5\pi}{16},\ x_6 = \frac{3\pi}{8},\ x_7 = \frac{7\pi}{16},\ x_8 = \frac{\pi}{2} \right\}.$$

Evaluating $f(x) = 2x \cos x$ at these nodes,

```
In[1]:= Table[2*i*Cos[i], {i, 0, Pi/2, Pi/16}]

Out[1]= {0, 1/8 \[Pi] Cos[\[Pi]/16], 1/4 \[Pi] Cos[\[Pi]/8],
 3/8 \[Pi] Cos[(3 \[Pi])/16], \[Pi]/(2 Sqrt[2]),
 5/8 \[Pi] Sin[(3 \[Pi])/16], 3/4 \[Pi] Sin[\[Pi]/8],
 7/8 \[Pi] Sin[\[Pi]/16], 0}
```

We plug these values back into (14),

$$\begin{aligned}
\int_0^{\pi} x \cos x \, \mathrm{d}x &= 2 \int_0^{\pi/2} x \cos x \, \mathrm{d}x \\
&\approx \frac{\pi}{16 \cdot 3} [0 + 4 \left(\frac{\pi}{8} \cos \frac{\pi}{16} + \frac{3\pi}{8} \cos \frac{3\pi}{16} + \frac{5\pi}{8} \sin \frac{3\pi}{16} + \frac{7\pi}{8} \sin \frac{\pi}{16} \right) \\
&\quad + 2 \left(\frac{\pi}{4} \cos \frac{\pi}{8} + \frac{\sqrt{2}\pi}{4} + \frac{3\pi}{4} \sin \frac{\pi}{8} \right) + 0] \\
&= \frac{\pi}{48} \left[\frac{\pi}{2} \cos \frac{\pi}{16} + \frac{3\pi}{2} \cos \frac{3\pi}{16} + \frac{5\pi}{2} \sin \frac{3\pi}{16} + \frac{7\pi}{2} \sin \frac{\pi}{16} + \frac{\pi}{2} \cos \frac{\pi}{8} + \frac{\sqrt{2}\pi}{2} + \frac{3\pi}{2} \sin \frac{\pi}{8} \right] \\
&= 1.14167.
\end{aligned}$$

The error term is given by

$$-\frac{\mathrm{h}^5}{90} f^{(4)}(\xi) = -\frac{\pi^5}{94371840} f^{(4)}(\xi),$$

for some $\xi \in (0, \pi/2)$. □

Problem 3. *Develop a composite version of the following formula and give the error term*

$$\int_{x_0}^{x_4} f(x) \, \mathrm{d}x = \frac{4\mathrm{h}}{3} [2f(x_1) - f(x_2) + 2f(x_3)] + \frac{14\mathrm{h}^2}{45} f^{(4)}(c), \tag{15}$$

where

$$\mathrm{h} = \frac{x_4 - x_0}{4}, \qquad x_1 = x_0 + \mathrm{h}, \qquad x_2 = x_0 + 2\mathrm{h}, \qquad x_3 = x_0 + 3\mathrm{h}, \qquad \textit{and} \qquad c \in (x_0, x_4).$$

Solution. Let's extend the integral in (15) to a larger, general interval $[a, b] = [x_0, x_n]$, and let us use, say, m panels. In order to mimic Eq. (15), in each of these panels $[x_i, x_{i+4}]$ we want three interior nodes, besides the two panel endpoints x_i and x_{i+4}. So, we have

$$\int_{x_0}^{x_n} f(x)\,\mathrm{d}x = \underbrace{\int_{x_0}^{x_4} f(x)\,\mathrm{d}x}_{1^{\text{st}}\text{ panel}} + \cdots + \underbrace{\int_{x_{n-4}}^{x_n} f(x)\,\mathrm{d}x}_{m^{\text{th}}\text{ panel}}.$$

Since we are using evenly spaced nodes, h remains unchanged for all i:

$$\mathrm{h} := \frac{x_{i+4} - x_i}{4} \qquad \text{for } i = 0, \dots, n-4.$$

Putting all this together, we get

$$\int_{x_0}^{x_n} f(x)\,\mathrm{d}x = \frac{4\mathrm{h}}{3}\left[2\sum_{\substack{i=1\\ i\text{ is odd}}}^{n-1} f_i - \sum_{i\in I} f_i\right] + \frac{14\mathrm{h}^2}{45}\sum_{j=1}^{m} f^{(4)}(c_j), \tag{16}$$

where $I = \{2, 6, 10, \dots, n-2\}$ and c_j is located in the j^{th} panel. □

Problem 4. *Show, by induction or otherwise, that for* $0 \le k \le n$,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1-x^2)^n = (1-x^2)^{n-k}\, q_k(x), \tag{17}$$

where q_k is a polynomial of degree k. Deduce that all the derivatives of the function $(1-x^2)^n$ of order less than n vanish at $x = \pm 1$. Define

$$\psi_j(x) := \frac{\mathrm{d}^j}{\mathrm{d}x^j}(1-x^2)^j, \tag{18}$$

and show by repeated integration by parts that

$$\int_{-1}^{1} \psi_k(x)\psi_j(x)\,\mathrm{d}x = 0, \quad 0 \le k < j. \tag{19}$$

Hence obtain the expressions for the Legendre polynomials of degrees $0, 1, 2,$ and 3.

Solution. Let's show Eq. (17) by induction. For $k = 0$,

$$\frac{\mathrm{d}^0}{\mathrm{d}x^0}(1-x^2)^n = (1-x^2)^n = (1-x^2)^{n-0}\, q_0(x) \qquad \text{with } q_0(x) = 1. \qquad \checkmark \tag{20a}$$

For $k = 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^n = (1-x^2)^{n-1}\cdot(-2nx) = (1-x^2)^{n-1}\, q_1(x) \qquad \text{with } q_1(x) = -2nx. \qquad \checkmark \tag{20b}$$

Now assume (17) holds for any k; then we show that it must hold for $k+1$:

$$\begin{aligned}
\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(1-x^2)^n &= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^n\right) \\
&= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left((1-x^2)^{n-1}\, q_1(x)\right) && \text{(By (20b))} \\
&= (1-x^2)^{n-k-1}\, q_k(x) q_1(x) && \text{(By (17))} \\
&= (1-x^2)^{n-(k+1)}\, q_{k+1}(x). \qquad \checkmark
\end{aligned}$$

On the last equality we used the fact that a degree-i polynomial multiplying a degree-j polynomial results in a degree-$(i+j)$ polynomial.

Now, it is obvious that plugging in $x = \pm 1$ will make all k-derivatives vanish, except when $k = n$, since in that case we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^2)^n = (1-x^2)^{n-n}\, q_n(x) = q_n(x).$$

Integrating by parts now, we show the orthogonality condition (19). From the calculations above, note that for $k < j$, the k^{th}-derivative of $(1-x^2)^j$ is divisible by $(1-x^2)$; that is,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1-x^2)^j = (1-x^2)^{j-k}\, q_k(x) = \frac{(1-x^2)^j}{(1-x^2)^k}\, q_k(x) \qquad \text{for } k < j. \tag{21}$$

Note also that since $(1-x^2)^k$ is a polynomial of degree $2k$, its $(2k+1)^{\text{st}}$ derivative must vanish; i.e.,

$$\frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}(1-x^2)^k = 0. \tag{22}$$

(These last two expressions are crucial to our calculation below.) Now define the integration-by-parts variables $\{u, v\}$:

$$u = \frac{\mathrm{d}^k}{\mathrm{d}x^k}(1-x^2)^k, \qquad \mathrm{d}v = \frac{\mathrm{d}^j}{\mathrm{d}x^j}(1-x^2)^j\,\mathrm{d}x,$$
$$\mathrm{d}u = \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(1-x^2)^k\,\mathrm{d}x, \qquad v = \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}(1-x^2)^j.$$

We are now ready to integrate:

$$\begin{aligned}
\int_{-1}^{1}\psi_k(x)\psi_j(x)\,\mathrm{d}x &= \int_{-1}^{1}\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1-x^2)^k\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}(1-x^2)^j\,\mathrm{d}x \\
&= uv|_{-1}^{1} - \int_{-1}^{1} v\,\mathrm{d}u \\
&= \underbrace{\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1-x^2)^k\,\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}(1-x^2)^j\right)\Bigg|_{-1}^{1}}_{=\,0\text{ by (21)}} - \int_{-1}^{1}\left(\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}(1-x^2)^j\,\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(1-x^2)^k\right)\mathrm{d}x \\
&= -\int_{-1}^{1}\left(\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}(1-x^2)^j\,\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(1-x^2)^k\right)\mathrm{d}x.
\end{aligned}$$

Repeating the integration $k+1$ times, we get

$$(-1)^{k+1}\int_{-1}^{1}\left(\frac{\mathrm{d}^{j-k-1}}{\mathrm{d}x^{j-k-1}}(1-x^2)^j\,\frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}(1-x^2)^k\right)\mathrm{d}x \overset{\text{by Eq. (22)}}{=\!=\!=} 0.$$

□

Problem 5. *For what value of α is the formula $\int_0^2 f(x)\,\mathrm{d}x \approx f(\alpha) + f(2-\alpha)$ exact on Π_3?*

Solution. The integration is exact, i.e.,

$$\int_0^2 f(x)\,\mathrm{d}x = f(\alpha) + f(2-\alpha) \tag{23}$$

on Π_3, whenever $f(x)$ is a polynomial of degree ≤ 3 (that is, the quadrature has degree of precision 3). Thus,

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 \tag{24}$$

for some undetermined coefficients $\{A_0, A_1, A_2, A_3\}$. Now, expanding the LHS of (23) with f given as in (24),

$$\begin{aligned}
\int_0^2 f(x)\,\mathrm{d}x &= \int_0^2 (A_0 + A_1 x + A_2 x^2 + A_3 x^3)\,\mathrm{d}x \\
&= \left(A_0 x + \frac{x^2}{2}A_1 + \frac{x^3}{3}A_2 + \frac{x^4}{4}A_3\right)\Bigg|_0^2 \\
&= 2A_0 + 2A_1 + \frac{8}{3}A_2 + 4A_3.
\end{aligned} \tag{25}$$

On the other hand, expanding the RHS of (23) with f given as in (24),

$$\begin{aligned}
f(\alpha)+f(2-\alpha) &= A_0+A_0+A_1(\alpha)+A_1(2-\alpha)+A_2(\alpha^2)+A_2([2-\alpha]^2)+A_3(\alpha^3)+A_3([2-\alpha]^3)\\
&= 2A_0+A_1(\alpha+2-\alpha)+A_2(\alpha^2+(2-\alpha)^2)+A_3(\alpha^3+(2-\alpha)^3)\\
&= 2A_0+2A_1+A_2(\alpha^2+4-4\alpha+\alpha^2)+A_3(\alpha^3-\alpha^3+6\alpha^2-12\alpha+8)\\
&= 2A_0+2A_1+(2\alpha^2-4\alpha+4)A_2+(6\alpha^2-12\alpha+8)A_3. \qquad (26)
\end{aligned}$$

Hence, comparing Eqs. (25) and (26), we see that in order for (23) to hold, the following equations must be satisfied

$$2\alpha^2-4\alpha+4=\frac{8}{3} \qquad (27a)$$
$$6\alpha^2-12\alpha+8=4. \qquad (27b)$$

From either of these two equations we get $\alpha = 1 \pm \sqrt{3}/3$, so this the value that satisfies Eq. (23) on Π_3. □

Problem 6. *A quadrature formula on $[-1,1]$ uses the quadrature points $x_0=-\alpha$ and $x_1=\alpha$, where $0<\alpha\leq 1$:*

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx w_0 f(-\alpha)+w_1 f(\alpha). \qquad (28)$$

The formula is required to be exact whenever f is a polynomial of degree 1. Show that $w_0=w_1=1$, independent of the value of α. Show also that there is one particular value of α for which the formula is exact also for all polynomials of degree 2. Find this α, and show that, for this value, the formula is also exact for all polynomials of degree 3.

Proof. Assume that Eq. (28) is exact on Π_1; that is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = w_0 f(-\alpha)+w_1 f(\alpha) \qquad (29)$$

on Π_1. Then

$$f(x)=A_0+A_1x \qquad (30)$$

for some undetermined coefficients $\{A_0, A_1\}$. Now, expanding the LHS of (29) with f given as in (30),

$$\begin{aligned}
\int_{-1}^{1} f(x)\,\mathrm{d}x &= \int_{-1}^{1}(A_0+A_1x)\,\mathrm{d}x\\
&= \left(A_0x+\frac{x^2}{2}A_1\right)\Bigg|_{-1}^{1}\\
&= 2A_0. \qquad (31)
\end{aligned}$$

On the other hand, expanding the RHS of (29) with f given as in (30),

$$\begin{aligned}
w_0f(-\alpha)+w_1f(\alpha) &= w_0\left(A_0+A_1(-\alpha)\right)+w_1\left(A_0+A_1(\alpha)\right)\\
&= (w_0+w_1)A_0+(w_1\alpha-w_0\alpha)A_1. \qquad (32)
\end{aligned}$$

Thus we have the system

$$w_0+w_1=2 \qquad (33a)$$
$$-\alpha w_0+\alpha w_1=0, \qquad (33b)$$

from which we get $w_0=w_1=1$, which is completely independent of α.

On Π_2, on the other hand, the exactness of this quadrature will depend on a specific value of α; we show this now. Let

$$f(x)=A_0+A_1x+A_2x^2 \qquad (34)$$

for some undetermined coefficients $\{A_0, A_1, A_2\}$. Now, expanding the LHS of (29) with f given as in (34),

$$\begin{aligned}\int_{-1}^{1} f(x)\,\mathrm{d}x &= \int_{-1}^{1}(A_0 + A_1 x + A_2 x^2)\,\mathrm{d}x\\ &= \left(A_0 x + \frac{x^2}{2}A_1 + \frac{x^3}{3}A_2\right)\Big|_{-1}^{1}\\ &= 2A_0 + \frac{2}{3}A_2. \end{aligned}\tag{35}$$

Expanding the RHS of (29) with f given as in (34),

$$\begin{aligned} w_0 f(-\alpha) + w_1 f(\alpha) &= w_0\left(A_0 + A_1(-\alpha) + A_2(-\alpha)^2\right) + w_1\left(A_0 + A_1\alpha + A_2\alpha^2\right)\\ &= (w_0 + w_1)A_0 + (w_1\alpha - w_0\alpha)A_1 + (w_0\alpha^2 + w_1\alpha^2)A_2. \end{aligned}\tag{36}$$

Thus we have the system

$$w_0 + w_1 = 2 \tag{37a}$$

$$-\alpha w_0 + \alpha w_1 = 0 \tag{37b}$$

$$\alpha^2 w_0 + \alpha^2 w_1 = \frac{2}{3}, \tag{37c}$$

from which we get $w_0 = w_1 = 1$, but now

$$2\alpha^2 = \frac{2}{3} \implies \alpha = \pm\frac{\sqrt{3}}{3}. \tag{38}$$

Since $w_0 = w_1$, it's clear from (29) that it doesn't matter whether we use + or $-\ \alpha$; thus set $\alpha = \sqrt{3}/3$. Then, in order for the quadrature to be exact on Π_3, f must be of the form

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3. \tag{39}$$

Similarly as before, upon expanding the LHS of (29), with f now given as in (39), we end up with

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 2A_0 + \frac{2}{3}A_2. \tag{40}$$

So there is no change when compared to Π_2. Also, expanding the RHS of (29) with f now given as in (39) and $\alpha = \sqrt{3}/3$, we get

$$\begin{aligned} w_0 f\left(-\frac{\sqrt{3}}{3}\right) + w_1 f\left(\frac{\sqrt{3}}{3}\right) &= w_0\left(A_0 + A_1\left(-\frac{\sqrt{3}}{3}\right) + A_2\left(-\frac{\sqrt{3}}{3}\right)^2 + A_3\left(-\frac{\sqrt{3}}{3}\right)^3\right)\\ &\quad + w_1\left(A_0 + A_1\left(\frac{\sqrt{3}}{3}\right) + A_2\left(\frac{\sqrt{3}}{3}\right)^2 + A_3\left(\frac{\sqrt{3}}{3}\right)^3\right)\\ &= (w_0 + w_1)A_0 + \left(w_1\frac{\sqrt{3}}{3} - w_0\frac{\sqrt{3}}{3}\right)A_1 + \left(w_0\frac{1}{3} + w_1\frac{1}{3}\right)A_2\\ &\quad + \left(w_1\frac{\sqrt{3}}{3} - w_0\frac{\sqrt{3}}{3}\right)A_3. \end{aligned}\tag{41}$$

This yields the same system (37) that we saw for Π_2. Hence, for the given value of $\alpha = \pm\sqrt{3}/3$ the quadrature (28) is also exact on Π_3. $\square$

Problem 7. *The function f has a continuous fourth derivative on the interval $[-1, 1]$. Construct the Hermite interpolation polynomial of degree 3 for f using the interpolation points $x_0 = -1$ and $x_1 = 1$. Deduce that*

$$\int_{-1}^{1} f(x)\,\mathrm{d}x - [f(-1) + f(1)] = \frac{1}{3}[f'(-1) - f'(1)] + E, \tag{42}$$

where

$$|E| \le \frac{2}{45}\max_{x\in[-1,1]}|f^{(4)}(x)|. \tag{43}$$

Solution. We use the following important theorem:

Hermite Interpolation Theorem

Given $n+1$ distinct nodes $x_0, \ldots, x_n$ and a differentiable function $f(x)$, there exists a unique polynomial $H_n \in \Pi_{2n+1}$ such that

$$H_n(x_i) = f(x_i), \qquad H_n'(x_i) = f'(x_i), \qquad 0 \le i \le n. \tag{44}$$

Defining

$$h_i(x) \equiv \ell_i^2(x)\left(1 - 2\ell_i'(x_i)\right)(x - x_i) \tag{45a}$$
$$\tilde{h}_i(x) \equiv \ell_i^2(x)(x - x_i), \tag{45b}$$

we get the ***Hermitian polynomial***

$$H_n(x) = \sum_{i=0}^{n} \left(f(x_i)h_i(x) + f'(x_i)\tilde{h}_i(x) \right), \tag{46}$$

which does satisfy Eq. (44).

Proceeding to compute Eq. (46) by direct calculation is highly inefficient though. A more time-saving approach is to tabulate the method, similarly to how we did with the Newton's Divided Differences coefficients. In the cubic Hermite interpolation we only need two nodes $x_0 = a$ and $x_1 = b$. The algorithm is as follows: We set up the table

$$\begin{array}{c|cccc}
a & f(a) & & & \\
 & & f'(a) & & \\
a & f(a) & & B & \\
 & & A & & D \\
b & f(b) & & C & \\
 & & f'(b) & & \\
b & f(b) & & &
\end{array} \tag{47}$$

where A, B, C, D are calculated as usual in finite difference tables,

$$A = \frac{f(b) - f(a)}{b - a}, \qquad B = \frac{A - f'(a)}{b - a},$$
$$C = \frac{f'(b) - A}{b - a}, \qquad D = \frac{C - B}{b - a}.$$

Then the cubic Hermite polynomial is given by

$$H_1(x) = f(a) + f'(a)(x - a) + B(x - a)^2 + D(x - a)^2(x - b). \tag{48}$$

In the case at hand, we have $x_0 = -1$ and $x_1 = 1$. Thus,

$$\begin{array}{c|cccc}
-1 & f(-1) & & & \\
 & & f'(-1) & & \\
-1 & f(-1) & & B & \\
 & & A & & D \\
1 & f(1) & & C & \\
 & & f'(1) & & \\
1 & f(1) & & &
\end{array} \tag{49}$$

where

$$A = \frac{f(1) - f(-1)}{2},$$
$$B = \frac{\frac{f(1)-f(-1)}{2} - f'(-1)}{2} = \frac{f(1) - f(-1) - 2f'(-1)}{4},$$
$$C = \frac{f'(1) - \frac{f(1)-f(-1)}{2}}{2} = \frac{2f'(1) - f(1) + f(-1)}{4},$$
$$D = \frac{\frac{2f'(1)-f(1)+f(-1)}{4} - \frac{f(1)-f(-1)-2f'(-1)}{4}}{2} = \frac{f'(1) - f(1) + f'(-1) + f(-1)}{4}.$$

Hence, plugging back into Eq. (48)

$$H_1(x) = f(-1) + f'(-1)(x+1) + \frac{f(1) - f(-1) - 2f'(-1)}{4}(x+1)^2 + \frac{f'(1) - f(1) + f'(-1) + f(-1)}{4}(x+1)^2(x-1). \tag{50}$$

Integrating this expression

```
In[1]:=
h[x_] :=
  f[-1] + f'[-1]*(x + 1) + (f[1] - f[-1] - 2 f'[-1])/4*(x + 1)^2
  + (f'[1] + f'[-1] - f[1] + f[-1])/4*(x + 1)^2*(x - 1);

Integrate[h[x], {x, -1, 1}]

Out[2]= f[-1] + f[1] + 1/3 Derivative[1][f][-1] - Derivative[1][f][1]/3
```

Thus we have found

$$\int_{-1}^{1} H_1(x)\,\mathrm{d}x = f(1) + f(-1) + \frac{1}{3}[f'(-1) - f'(1)]. \tag{51}$$

We now use the Gaussian formula with error term, for a Hermite polynomial $H(x)$,

$$\int_a^b w(x)\left[f(x) - H(x)\right]\,\mathrm{d}x = \int_a^b w(x)\frac{f^{(2n+2)}(\xi(x))}{(2n+2)!}\psi_n^2(x)\,\mathrm{d}x, \tag{52}$$

where $\xi \in [a,b]$ and

$$\psi_n = \prod_{i=0}^{n}(x - x_i).$$

In our case we have $(w(x) = 1$ and $[a,b] = [-1,1]$. Thus, putting together all our results, we have

$$\begin{aligned}
\int_{-1}^{1}\left(f(x) - H_1(x)\right)\,\mathrm{d}x &= \int_{-1}^{1} f(x)\,\mathrm{d}x - f(1) - f(-1) - \frac{1}{3}[f'(-1) - f'(1)] \\
&= \int_{-1}^{1}\frac{f^{(4)}(\xi(x))}{4!}\psi_1^2(x)\,\mathrm{d}x \\
&= \frac{f^{(4)}(\xi(x))}{24}\int_{-1}^{1}\left[(x+1)^2(x-1)^2\right]\,\mathrm{d}x \\
&= \frac{f^{(4)}(\xi(x))}{24}\cdot\frac{16}{25} \\
&= \frac{2}{45}f^{(4)}(\xi(x)) \leq \frac{2}{45}\max_{x\in[-1,1]}|f^{(4)}(x)|.
\end{aligned}$$

The last inequality holds because $\xi \in [-1,1]$. On the third equality we were able to pull the $f^{(4)}(\xi)$ out of the integral by applying the Weighted Mean Value Theorem for Integrals, which we previously used on Problem 1 (c.f., Eq. (12)). Hence, we have shown that Eq. (42) holds. □

Problem 8. *Approximate the integral* $\int_{-1}^{1}\cos(\pi x)\,\mathrm{d}x$ *using* $n = 3$ *Gaussian Quadrature.*[1]

Proof. For $n = 3$ Gaussian Quadrature we need to use as nodes the three roots of the degree-3 Legendre polynomial

$$\begin{aligned}
p_3(x) &= \frac{1}{2^3\,3!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[(x^2-1)^3\right] \\
&= \frac{1}{48}\left(120x^3 - 72x\right) \\
&= x\left(\frac{5}{2}x^2 - \frac{3}{2}\right).
\end{aligned} \tag{53}$$

[1]For this problem I switch to indexing starting at 1 in order to avoid confusion, since in my convention this problem is actually asking for $n = 2$ Gaussian Quadrature.

So one root is 0, and as for the other two,

$$\frac{5}{2}x^2 - \frac{3}{2} = 0 \implies x = \pm\sqrt{\frac{3}{5}}.$$

So the three roots are

$$x_1 = -\sqrt{\frac{3}{5}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{5}}.$$

Now we Lagrange-interpolate the function $f(x) = \cos(\pi x)$ as usual,

$$\cos(\pi x) \approx \sum_{i=1}^{3} \cos(\pi x_i)\ell_i(x), \tag{54}$$

except that now the Lagrange polynomials ℓ_i are applied on the Legendre roots we have just found. Start with ℓ_1:

$$\begin{aligned}\ell_1(x) &= \frac{x - x_2}{x_1 - x_2}\,\frac{x - x_3}{x_1 - x_3} \\ &= \frac{x - 0}{-\sqrt{\frac{3}{5}} - 0}\,\frac{x - \sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}} - \sqrt{\frac{3}{5}}} \\ &= \frac{5}{6}x\left(x - \sqrt{\frac{3}{5}}\right).\end{aligned} \tag{55}$$

Integrating, we get the first coefficient

$$c_1 = \int_{-1}^{1} \ell_1(x)\,\mathrm{d}x = \int_{-1}^{1} \left[\frac{5}{6}x\left(x - \sqrt{\frac{3}{5}}\right)\right]\,\mathrm{d}x = \frac{5}{9}.$$

An identical calculation shows that the remaining two coefficients are $c_2 = 8/9$ and $c_3 = c_1 = 5/9$. Hence, putting it all together and integrating Eq. (54), we get we have the quadrature

$$\begin{aligned}\int_{-1}^{1} \cos(\pi x)\,\mathrm{d}x &= \sum_{i=1}^{3} c_i \cos(\pi x_i) \\ &= \frac{5}{9}\cdot\cos\left(-\sqrt{\frac{3}{5}}\pi\right) + \frac{8}{9}\cdot 1 + \frac{5}{9}\cdot\cos\left(\sqrt{\frac{3}{5}}\pi\right) \\ &\approx 0.045.\end{aligned}$$

□

Problem 9. *Show how the Gaussian quadrature rule*

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right) \tag{56}$$

can be used for $\int_a^b f(x)\,\mathrm{d}x$. *Apply this result to evaluate* $\int_0^{\pi/2} x\,\mathrm{d}x$.

Solution. This is the same $n = 3$ Gaussian quadrature that we derived on the previous problem. If the quadrature is to take place on a more general interval $[a, b]$, the problem needs to be translated back to $[-1, 1]$. Using the substitution

$$t = \frac{2x - (b + a)}{b - a}, \tag{57}$$

a Gaussian quadrature rule of the form

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx \sum_{i=0}^{n} A_i f(t_i) \tag{58}$$

can be used over the interval $[a,b]$; i.e.,

$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{(b-a)t+b+a}{2}\right)\,\mathrm{d}t. \tag{59}$$

For this problem we have $[a,b]=[0,\pi/2]$. Hence, plugging into (59),

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\,\mathrm{d}x &= \frac{\frac{\pi}{2}-0}{2}\int_{-1}^{1} f\left(\frac{\left(\frac{\pi}{2}-0\right)t+\frac{\pi}{2}+0}{2}\right)\,\mathrm{d}t\\ &= \frac{\pi}{4}\int_{-1}^{1} f\left(\frac{\pi}{4}t+\frac{\pi}{4}\right)\,\mathrm{d}t\\ &= \frac{\pi}{4}\left[\frac{5}{9}f\left(\frac{\pi}{4}-\frac{\pi}{4}\sqrt{\frac{3}{5}}\right)+\frac{8}{9}f\left(\frac{\pi}{4}\right)+\frac{5}{9}f\left(\frac{\pi}{4}\sqrt{\frac{3}{5}}+\frac{\pi}{4}\right)\right]. \qquad \text{(Using (58))}\end{aligned} \tag{60}$$

Then, since $f(x)=x$ in this case,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\,\mathrm{d}x &= \frac{\pi}{4}\left[\frac{5}{9}\left(\frac{\pi}{4}-\frac{\pi}{4}\sqrt{\frac{3}{5}}\right)+\frac{8}{9}\left(\frac{\pi}{4}\right)+\frac{5}{9}\left(\frac{\pi}{4}\sqrt{\frac{3}{5}}+\frac{\pi}{4}\right)\right]\\ &= \frac{\pi^2}{8}\approx 1.2337.\end{aligned}$$

□

Problem 10. *Solve the following:*

- *a) Find $\{p_0,p_1,p_2\}$ such that p_i is polynomial of degree i and this set is orthogonal on $[0,\infty)$ with weight function $w(x)=e^{-x}$.*
- *b) Determine the nodes and the weights in the two-node Gaussian Quadrature formula:*

$$\int_0^{\infty} f(x)e^{-x}\,\mathrm{d}x = w_1 f(x_1)+w_2 f(x_2). \tag{61}$$

Solution to a). We need to find a set of polynomials that are w-orthogonal on $[0,\infty)$, with $w(x)=e^{-x}$; that is, we need to find a set $\{p_k\}$ that satisfies

$$\int_0^{\infty} e^{-x}p_i(x)p_j(x)\,\mathrm{d}x = 0 \qquad i\neq j. \tag{62}$$

In general, constructing a set of orthogonal polynomials entails using a Gram-Schmidt algorithm (c.f., (67)); we use this approach on the next problem. For this particular weight function $w(x)=e^{-x}$, however, the set of polynomials that satisfy Eq. (62) are well known; they go by the name of *Laguerre polynomials*, and can be easily derived from their *Rodrigues representation*

$$p_k(x) = \frac{e^x}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^k e^{-x}\right). \tag{63}$$

For $k=0,1,2$, we have

$$p_0(x) = 1, \tag{64a}$$

$$p_1(x) = 1-x, \tag{64b}$$

$$p_2(x) = \frac{1}{2}(x^2-4x+2). \tag{64c}$$

It can be easily checked that these polynomials are indeed w-orthogonal (with $w(x)=e^{-x}$, of course) in the non-negative real line $[0,\infty)$ by just plugging them into Eq. (62) and making sure the integrals do vanish. □

Solution to b). Just as in the case of a unit weight function the Gauss nodes were roots of the Legendre polynomial, in the case of the nontrivial weight function $w(x)=e^{-x}$ the nodes are the Laguerre roots. Thus, for the two-node Gaussian Quadrature, the nodes are the two roots of the degree-2 Laguerre polynomial

$$\frac{1}{2}(x^2-4x+2)=0 \qquad \Longrightarrow \qquad x = 2\pm\sqrt{2}.$$

Thus the two Gauss nodes are

$$x_1 = 2 - \sqrt{2}, \qquad x_2 = 2 + \sqrt{2}.$$

As for the weights, they are calculated as before; start with the Lagrange polynomials

$$\ell_1 = \frac{x - x_2}{x_1 - x_2} = \frac{x - 2 - \sqrt{2}}{-2\sqrt{2}};$$
$$\ell_2 = \frac{x - x_1}{x_2 - x_1} = \frac{x - 2 + \sqrt{2}}{2\sqrt{2}}.$$

Then, integrating, we get

$$w_1 = \int_0^\infty e^{-x}\, \frac{x - 2 - \sqrt{2}}{-2\sqrt{2}}\, \mathrm{d}x = \frac{1}{4}(2 + \sqrt{2});$$
$$w_2 = \int_0^\infty e^{-x}\, \frac{x - 2 + \sqrt{2}}{2\sqrt{2}}\, \mathrm{d}x = \frac{1}{4}(2 - \sqrt{2}).$$

Thus, we have

$$\int_0^\infty f(x) e^{-x}\, \mathrm{d}x = \frac{2 + \sqrt{2}}{4} f\left(2 - \sqrt{2}\right) + \frac{2 - \sqrt{2}}{4} f\left(2 + \sqrt{2}\right). \qquad \square$$

Problem 11. *Solve the following:*

a) Construct orthogonal polynomials of degrees $0, 1,$ *and* 2 *on the interval* $(0, 1)$ *with the weight function* $w(x) = -\log x$.

b) Determine the quadrature points and weights for the weight function $w\colon x \mapsto -\log x$ *on the interval* $(0, 1)$*, for* $n = 1$.

Solution to a). The general procedure for orthogonalizing polynomials is given by the Gram-Schmidt recurrence relation

$$p_{-1}(x) \equiv 0 \tag{67a}$$
$$p_0(x) \equiv 1 \tag{67b}$$
$$p_{i+1}(x) = (x - a_i) p_i(x) - b_i p_{i-1}(x) \qquad i = 0, \ldots, n, \tag{67c}$$

where

$$a_i = \frac{\langle x p_i, p_i \rangle}{\langle p_i, p_i \rangle} \qquad i = 0, \ldots, n, \tag{68a}$$
$$b_i = \frac{\langle p_i, p_i \rangle}{\langle p_{i-1}, p_{i-1} \rangle} \qquad i = 1, \ldots, n, \tag{68b}$$
$$b_0 = \text{constant} \quad \text{(can be set to 0)}. \tag{68c}$$

We could then divide each p_i by $\langle p_i, p_i \rangle^2$ if we wanted to normalize, but this is not necessary in our case; we are merely after an orthogonal set of polynomials. In the case at hand we want the first three w-orthogonal polynomials

$$p_0(x) = 1, \tag{69a}$$
$$p_1(x) = (x - a_0) p_0(x) = x - a_0 = x - \frac{\int_0^1 x(-\log x)\, \mathrm{d}x}{\int_0^1 (-\log x)\, \mathrm{d}x} = x - \frac{1}{4}, \tag{69b}$$

$$\begin{aligned} p_2(x) &= (x - a_1) p_1(x) - b_1 p_0(x) \\ &= \left[x - \frac{\int_0^1 x(-\log x)\left(x - \frac{1}{4}\right)^2 \mathrm{d}x}{\int_0^1 (-\log x)\left(x - \frac{1}{4}\right)^2 \mathrm{d}x} \right] \left[x - \frac{1}{4} \right] - \frac{\int_0^1 (-\log x)\left(x - \frac{1}{4}\right)^2 \mathrm{d}x}{\int_0^1 (-\log x)\, \mathrm{d}x} \\ &= \left[x - \frac{\frac{13}{576}}{\frac{7}{144}} \right] \left[x - \frac{1}{4} \right] - \frac{7}{144} \\ &= x^2 - \frac{5}{7} x + \frac{17}{252}. \end{aligned} \tag{69c}$$

A quick check to make sure that these three polynomials are indeed w-orthogonal:

```
In[1]:=
Integrate[(-Log[x])*(17/252 - (5 x)/7 + x^2)*(x - 1/4), {x, 0, 1}]
Integrate[(-Log[x])*1*(x - 1/4), {x, 0, 1}]
Integrate[(-Log[x])*(17/252 - (5 x)/7 + x^2)*1, {x, 0, 1}]

Out[1]= 0
Out[2]= 0
Out[3]= 0
```

□

Solution to b). For $n = 1$ the only Gauss node is the single root of p_1, namely $x_0 = 1/4$. Since we only have one node, the Lagrange polynomial $\ell_0 = 1$, and thus the (single) weight is

$$w_0 = \int_0^1 (-\log x)\ell_0 \, \mathrm{d}x = \int_0^1 (-\log x) \, \mathrm{d}x = 1.$$

□

Problem 12. *Find weights w_0, w_1, w_2 and nodes $x_0, x_1, x_2 \in [-1, 1]$ such that the quadrature*

$$\int_{-1}^1 f(x) \, \mathrm{d}x \approx \sum_i w_i f(x_i)$$

integrates all quintic polynomials exactly.

Solution. This is yet again the $n = 3$ Gaussian Quadrature rule that we found earlier (c.f. (56)),

$$\int_{-1}^1 f(x) \, \mathrm{d}x \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

Whence the weights and nodes are

$$\{w_0, w_1, w_2\} = \left\{\frac{5}{9}, \frac{8}{9}, \frac{5}{9}\right\} \tag{70a}$$

$$\{x_0, x_1, x_2\} = \left\{-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}\right\}. \tag{70b}$$

However, previously we saw that the rule is exact for polynomials up to degree three (as is expected from $n = 3$ Gaussian Quadrature). This time we will show that the rule is, in fact, exact for quintic polynomials as well, which was not expected. Let

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5. \tag{71}$$

Then, expanding the LHS of (56) with f given as in (71),

$$\begin{aligned}
\int_{-1}^1 f(x) \, \mathrm{d}x &= \int_{-1}^1 (A_0 + A_1 x + A_2 x^2 + A_3 x^3 A_4 x^4 + A_5 x^5) \, \mathrm{d}x \\
&= \left(A_0 x + \frac{x^2}{2} A_1 + \frac{x^3}{3} A_2 + \frac{x^4}{4} A_3 + \frac{x^5}{5} A_4 + \frac{x^6}{6} A_5\right)\Bigg|_{-1}^1 \\
&= 2A_0 + \frac{2}{3} A_2 + \frac{2}{5} A_4.
\end{aligned} \tag{72}$$

On the other hand, consider the RHS of (56), term by term, with f given as in (71),

$$\begin{aligned}
\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) &= \frac{5}{9}\left[A_0 + A_1\left(-\sqrt{\frac{3}{5}}\right) + A_2\left(-\sqrt{\frac{3}{5}}\right)^2 + A_3\left(-\sqrt{\frac{3}{5}}\right)^3 + A_4\left(-\sqrt{\frac{3}{5}}\right)^4 + A_5\left(-\sqrt{\frac{3}{5}}\right)^5\right] \\
&= \frac{5}{9}\left[A_0 - \sqrt{\frac{3}{5}} A_1 + \frac{3}{5} A_2 - \frac{3}{5}\sqrt{\frac{3}{5}} A_3 + \frac{9}{25} A_4 - \frac{9}{25}\sqrt{\frac{3}{5}} A_5\right].
\end{aligned} \tag{73}$$

Similarly,

$$\frac{8}{9} f(0) = \frac{8}{9} A_0, \tag{74}$$

and

$$\frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) = \frac{5}{9}\left[A_0 + \sqrt{\frac{3}{5}} A_1 + \frac{3}{5} A_2 + \frac{3}{5}\sqrt{\frac{3}{5}} A_3 + \frac{9}{25} A_4 + \frac{9}{25}\sqrt{\frac{3}{5}} A_5\right] \tag{75}$$

Thus, adding Eqs. (73)–(75), we get

$$\begin{aligned} \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) &= \frac{5}{9}\left[2A_0 + \frac{6}{5} A_2 + \frac{18}{25} A_4\right] + \frac{8}{9} A_0 \\ &= 2A_0 + \frac{2}{3} A_2 + \frac{2}{5} A_4. \end{aligned} \tag{76}$$

We see that this result coincides with (72). Thus we have proven that the $n = 3$ Gaussian quadrature given by Eq. (56) is, in fact, exact for quintic polynomials. □