$\mathfrak{sl}_2$

The aim of this workshop is to understand explicitly some of the representation theory of $\mathfrak{sl}_2$, since it turns out (see Chapter 10) that understanding this allows us to deduce many stronger results. Since $\mathfrak{sl}_2$ is only a three-dimensional algebra, how hard can its representation theory be?

Recall that $\mathfrak{sl}_2$ has basis $\{e, f, h\}$ subject to the relations

$$[e,f] = h, \qquad [h,e] = 2e, \qquad [h,f] = -2f.$$

"Recall" (from a future lecture!) the following notions:

- A **representation** of a Lie algebra L is Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$ for some finite-dimensional complex vector space V. We also say that V is an L-**module**.
- A subspace $W \subseteq V$ is said to be a **submodule** (or a **subrepresentation**) if $\rho(\ell)w \in W$ for all $w \in W$ and $\ell \in L$.
- An L-module V is said to be **simple** if it has no proper submodules: so that if $W \subseteq V$ is a submodule, then either $W = 0$ or $W = V$.

Thus a representation of $\mathfrak{sl}_2$ is just a (finite-dimensional, complex) vector space V together with three linear maps $E, F, H \colon V \to V$ satisfying

$$\begin{aligned} EF - FE &= H \\ HE - EH &= 2E \\ HF - FH &= -2F. \end{aligned}$$

We will call the eigenvalues of $H \colon V \to V$ the **weights** of V. Further, if $a \in \mathbb{C}$ is a weight, then

$$V_a := \{v \in V \mid Hv = av\}$$

is called the **weight space of weight** a. Note that $0 \in V_a$.

We will see in the next couple of lectures (as a summary of Chapter 8 in the book) that simple representations of $\mathfrak{sl}_2$ are given by some explicit countable list. This is really nice, but their definition in the book is completely unmotivated, and it looks like magic! In this workshop, we will construct all simple representations of $\mathfrak{sl}_2$ using linear algebra; the construction should be natural and beautiful.

Throughout, we suppose that $V \neq 0$ is a finite dimensional representation of $\mathfrak{sl}_2$, we consider the linear maps E, F, H as above, and let $a \in \mathbb{C}$ be a weight.

1. Show that if $v \in V_a$, then $Ev \in V_{a+2}$, $Fv \in V_{a-2}$ and $Hv \in V_a$.

In terms of maps, this means $E \colon V_a \to V_{a+2}$, $F \colon V_a \to V_{a-2}$ and $H \colon V_a \to V_a$. We summarise this in the diagram

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "E"] & V_{a-2} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_a \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_{a+2} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & \cdots \arrow[l, bend left, "F"]
\end{tikzcd}$$

2. Prove that there exists a weight $a \in \mathbb{C}$ such that $Ev = 0$ for all $v \in V_a$ (visually: far enough to the right, keep on applying E and we get zero).

Denote the complex number a from Q2 by m, and call any non-zero element $v \in V_m$ a **highest weight vector**. Now choose a highest weight vector, and denote it w_0. For each $i \geq 1$, define

$$w_i := \frac{1}{i!} F^i w_0.$$

3. By induction or otherwise, show that for all $i \geq 1$
   (a) $\mathrm{F}w_i = (i+1)w_{i+1}$.
   (b) $\mathrm{H}w_i = (m-2i)w_i$.
   (c) $\mathrm{E}w_i = (m-i+1)w_{i-1}$.
4. Observe that $w_i \in \mathrm{V}_{m-2i}$, and further (by adapting Q2) prove that there exists some $j \in \mathbb{N}$ such that $w_j \neq 0$ but $w_{j+1} = 0$.
5. By combining your answers to Q3 and Q4, deduce that $m = j \in \mathbb{N}$. (This shows that V has only integer weights.)

Thus, inside any $\mathfrak{sl}_2$-module V, we can find vectors

$$\{w_0, \dots, w_m\},$$

as above.

6. Prove that $w_0, \dots, w_m$ are linearly independent, and hence

$$\mathrm{V}(m) := \mathrm{Span}\{w_0, \dots, w_m\}$$

   is an $m+1$ dimensional subspace of V.
7. Using the formulae in Q3, prove that $\mathrm{V}(m)$ is a subrepresentation (i.e., a submodule) of V, and further each $\mathrm{V}(m)$ is simple.
8. Prove that if V is itself simple, then $\mathrm{V} = \mathrm{V}(m)$. This shows that every simple $\mathfrak{sl}_2$-module is isomorphic to one of the form $\mathrm{V}(m)$.

Please hand in your solution to Q6, Q7 and Q8 by the start of lecture on Monday 23 October.