TRRT
Workshop 2 Hand-In

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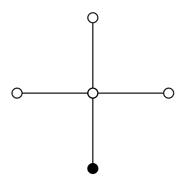
Problem 1 (Solution to Exercise 8). *a) We are assuming that Γ contains no cycles, and no double or triple edges. Since there are no double or triple edges, condition (B) tells us that the number of edges connecting any two nodes in Γ is either 0 or 1. This, combined with the assumption that Γ has no branch point and the fact that Γ is connected (by property (A)), leaves us with $\Gamma = A_n$ as the only possibility.* □

b) Note that if Γ contained more than one branch point, we would have that either $\Gamma = \widetilde{D}_n$ -which has zero determinant, and therefore violates condition (C) (by Sylvester's criterion)- or that $\widetilde{D}_n$ is a subgraph of Γ.[1] *In the latter case, from Q4, we have that any graph satisfying conditions (A), (B), and (C) cannot have an affine Dynkin diagram as a subgraph. Thus Γ is restricted to one branch point at most.* □

c) Assume instead that there are n branches, for $n > 3$. Then either $\Gamma = \widetilde{D}_4$,



which, again, has zero determinant, and therefore violates condition (C) (by Sylvester's criterion),- or $\widetilde{D}_4$ is a subgraph of Γ. Once again, by Q4, any graph satisfying conditions (A), (B), and (C) cannot have an affine Dynkin diagram as a subgraph, so Γ can only have exactly three branches emerging from a branch point. □

d) Let the number of vertices of the three branches be n_1, n_2, and n_3, with $n_1 \geq n_2 \geq n_3 \geq 1$. Then,

i. *Note that if $n_3 > 1$, we get that either $\Gamma = \widetilde{E}_6$ or we get that $\widetilde{E}_6$ is a subgraph of Γ. In either case the arguments we have previously made apply, and thus n_3 is forced to have value 1.*

ii. *By the same token, having established that the smallest branch must have $n_3 = 1$, note that the second smallest one must have $n_2 < 3$, for if $n_2 \geq 3$ then either $\Gamma = \widetilde{E}_7$ or $\widetilde{E}_7$ is a subgraph of Γ, which, again, violates the result obtained in Q4. Now for $n_2 = 1$ we get D_n, while for $n_2 = 2$ we get E_6, E_7, or E_8, all of which do satisfy conditions (A), (B), and (C) (we proved this ($\Leftarrow$) on the previous workshop). Now why are D_n, E_6, E_7, and E_8 the only game in town? That's because...*

iii. *Because(!), for $n_2 = 2$, if $n_1 > 4$ we have either $\Gamma = \widetilde{E}_8$ or $\widetilde{E}_8$ is a subgraph of Γ (the result from Q4 comes to the rescue again! *wipes sweat off forehead*). Thus, for $n_2 = 2$, n_1 can only take on values 2, 3, and 4, which correspond to diagrams E_6, E_7, and E_8, respectively.*

[1] There is yet another possibility, in which the branch is directed upwards instead of downwards as in $\widetilde{D}_n$, but in practice this makes no difference, as the determinant will still be zero (cf. Q2).

Lastly, note that for $n_2 = 1$, D_n *is the only free lady in the party, for we are not allowed to have more than one branch point.* □