Math Analysis Notes

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Perfect Sets and The Cantor Set

Before we start our discussion of perfect sets, let's present a very important theorem, called the nested interval theorem, which is an essential tool that we'll be using in the next few topics.

Yet before presenting this theorem let's make the following observation:

• Theorem:
A monotone, bounded sequence of real numbers converges in $\mathbb{R}$.

Proof:
Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be monotone and bounded.
We first suppose that $\{x_n\}$ is increasing (that is, $x_m \leq x_n$ whenever $m < n$). Now, since $\{x_n\}$ is bounded, we may set $x = \sup_{n \in \mathbb{N}} x_n$ (a real number). We will show that $x = \operatorname{limit}_{n \to \infty} x_n$.
Let $\varepsilon > 0$. Since $x - \varepsilon < x = \sup_{n \in \mathbb{N}} x_n$, we must have $x_N > x - \varepsilon$ for some N. But then, for any $n \geq N$, we have $x - \varepsilon < x_N \leq x_n \leq x$. That is, $|x - x_n| < \varepsilon \quad \forall\, n \geq N$. Consequently, $\{x_n\}$ converges and $x = \sup_{n \in \mathbb{N}} x_n = \operatorname{limit}_{n \to \infty} x_n$.
Finally, if $\{x_n\}$ is decreasing, consider the increasing sequence $\{-x_n\}$. From the first part of the proof, $\{-x_n\}$ converges to $\sup_{n \in \mathbb{N}} (-x_n) = -\inf_{n \in \mathbb{N}} x_n$. It then follows that $\{x_n\}$ converges to $\inf_{n \in \mathbb{N}} x_n$. ■

• The Nested Interval Theorem:
If $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed, bounded, nonempty intervals in $\mathbb{R}$ with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots,$$

then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. If, in addition, $\operatorname{length}(I_n) \longrightarrow 0$, then $\bigcap_{n=1}^{\infty} I_n$ contains precisely one point.

Proof:
Write $[a_n, b_n]$. Then $I_n \supset I_{n+1}$ means that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad \forall\, n$.
Thus,

$$a = \operatorname{limit}_{n \to \infty} a_n = \sup_{n \in \mathbb{N}} a_n \quad \text{and} \qquad b = \operatorname{limit}_{n \to \infty} b_n = \inf_{n \in \mathbb{N}} b_n$$

both exist (as finite real numbers) and satisfy $a \leq b$. Thus, we must have $\bigcap_{n=1}^{\infty} I_n = [a, b]$. Indeed, if $x \in I_n \ \ \forall\, n$, then $a_n \leq x \leq b_n \ \ \forall\, n$, and hence $a \leq x \leq b$. Conversely, if $a \leq x \leq b$, then $a_n \leq x \leq b_n \ \ \forall\, n$.

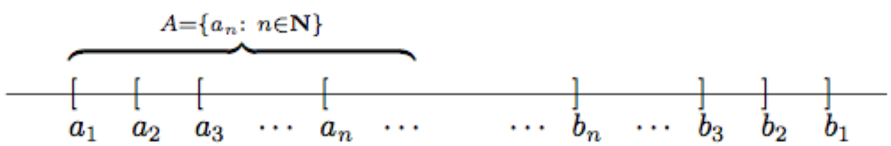
That is, $x \in I_n \ \forall n$. Finally, if $b_n - a_n = \text{length}(I_n) \longrightarrow 0$, then $a = b$ and so $\bigcap_{n=1}^{\infty} I_n = \{a\}$. ■

(Alternate) Proof:

Proof. In order to show that $\bigcap_{n=1}^{\infty} I_n$ is not empty, we are going to use the Axiom of Completeness (AoC) to produce a single real number x satisfying $x \in I_n$ for every $n \in \mathbf{N}$. Now, AoC is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in \mathbf{N}\}$$

of left-hand endpoints of the intervals.



Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = \sup A.$$

Now, consider a particular $I_n = [a_n, b_n]$. Because x is an upper bound for A, we have $a_n \le x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \le b_n$.

Altogether then, we have $a_n \le x \le b_n$, which means $x \in I_n$ for every choice of $n \in \mathbf{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is not empty. □

Example:

a) Note that it is essential that the intervals used in the nested interval theorem be both closed and bounded.

Indeed, $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$ and $\bigcap_{n=1}^{\infty} (0, 1/n] = \emptyset$.

b) Suppose that $\{I_n\}$ is a sequence of closed intervals with $I_n \supset I_{n+1}$ for all n and with $\text{length}(I_n) \longrightarrow 0$ as $n \longrightarrow \infty$. If $\bigcap_{n=1}^{\infty} I_n = \{x\}$, then any sequence of points $\{x_n\}_{n=1}^{\infty}$, with $x_n \in I_n$ for all n, must converge to x.

PERFECT SETS

Definition: A set P is a perfect set if it is empty or if it is a closed set and every point of P is a limit point of P.

Example:

a) The sets

• $\mathbb{R}$ • $(-\infty, a]$ • $[a, \infty)$

as well as any closed and bounded intervals $[a, b]$ $(a < b)$, are perfect sets.

b) The sets

• (a, b) • $[a, b] \bigcup \{c\}$ $(b < c)$ • $\mathbb{Q}$ • $\mathbb{R} \setminus \mathbb{Q}$

are not perfect sets. The sets (a, b), $\mathbb{Q}$, and $\mathbb{R} \setminus \mathbb{Q}$ fail to be closed, even though every point in each of these sets is a limit point of the set. The set $[a, b] \bigcup \{c\}$ on the other hand, fails to be a perfect set because c is not a limit point of $[a, b] \bigcup \{c\}$.

c) Let $\{x_n\}_{n=1}^{\infty}$ be convergent in (M, d), that is, $x_n \xrightarrow{d} x$ in M. Then the set $\{x_n : n \geq 1\} \bigcup \{x\}$ is not perfect. Although the set is closed, only x is a limit point. ✵

Notice that in all of the above listed examples of perfect subsets of $\mathbb{R}$, the perfect sets turned out to be uncountable sets. The next theorem shows that this must always be the case.

• Theorem:

Let P be a perfect subset of $\mathbb{R}$. Then P is uncountable.

Proof:

Suppose to the contrary that $P = \{x_1, x_2, \ldots, x_n, \ldots\}$ is countable. Let I_1 be any closed interval centered at x_1 of $\text{length}(I_1) \leq 1$. Then, since $x_1 \in P$ and P is perfect, it follows that x_1 is a limit point of P. In particular, $(I_1 \setminus \{x_1\}) \bigcap P \neq \emptyset$.

Let n_2 be the smallest integer for which $x_{n_2} \in (I_1 \setminus \{x_1\}) \bigcap P$ and let I_2 be any closed interval centered at x_{n_2} of $\text{length}(I_2) \leq \frac{1}{2}$ such that $I_2 \subset I_1$ and $x_1 \notin I_2$. Observe that by the minimality of n_2, $x_k \notin I_2$ for any $k < n_2$.

Since $x_{n_2} \in P$, it is a limit point of P and therefore $(I_2 \setminus \{x_{n_2}\}) \bigcap P \neq \emptyset$. Now let n_3 be the smallest integer for which $x_{n_3} \in (I_2 \setminus \{x_{n_2}\}) \bigcap P$. Set I_3 to be any closed interval centered at x_{n_3} of $\text{length}(I_3) \leq \frac{1}{3}$ such that $I_3 \subset I_2$ and $x_{n_2} \notin I_3$.

Continuing in this fashion, we obtain a nested sequence of closed intervals $I_1 \supset I_2 \supset I_3 \supset \ldots$ such that $\text{length}(I_n) \to 0$ as $n \to \infty$ and $x_k \in I_m$ for all $k < n_m$. By the Nested Interval Theorem, $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x \in \mathbb{R}$. Notice, however, that x is a limit point of P because it is the limit point of the center points of the intervals I_n. Thus, as P is closed, we must have $x \in P$. $(\Rightarrow\Leftarrow)$

This is a contradiction: x cannot be any of the x_m, since $m \leq n_m$ and $x_m \notin I_{m+1}$.

Thus P must be uncountable, as desired. ■

Example:

a) Although we are still lacking the means to prove it, it can be shown that if P is a nonempty perfect subset of (M, d) in which every Cauchy sequence converges, P must be an uncountable set.

b) Let (M, d) be a discrete metric space. Suppose $P \subset M$ is not empty. Then P is not perfect; $B_1(x) = \{x\}$ for any $x \in P$. Hence x cannot be a limit point of P.
What went wrong? Every Cauchy sequence in (M, d) must converge in (M, d). Why don't we have perfect sets other than $\emptyset$?

c) Let $M = \mathbb{Q}$ under the usual metric of $\mathbb{R}$. Then $P = [0, 1] \bigcap \mathbb{Q}$ is perfect in $\mathbb{Q}$. Notice, however, that $P \subset \mathbb{Q}$ and must therefore be countable. Does this contradict a)? Absolutely not! Lots of Cauchy sequences of elements of $\mathbb{Q}$ fail to converge in $\mathbb{Q}$.

Note: It appears that nonempty perfect subsets are rather large. One would expect these sets to occupy some space on the real number line, for instance. However, reality is stronger than intuition. It seems that perfect subsets of $\mathbb{R}$ can be so constructed as to include almost all of $\mathbb{R}$ and yet be so thin that they fail to contain a single interval, no matter how small this interval may be.
We will see shortly how this materializes but before we can present our argument, a few definitions will prove useful.

Definition: Let A be a subset of a metric space (M, d). If $x \in A$ and x is not a limit point of A, then x is called an isolated point of A.
Remark: Note that with this definition, we say that a set is perfect if it is closed and has no isolated points.

Definition: A set A is said to be dense in $(M.d)$ (or, as some authors say, everywhere dense) if $\overline{A} = M$. That is, every point of M is either in A or is a limit point of A if A is dense in M.

Alternatively, we can say that $A \subset M$ is dense in M if A meets every nonempty open subset $W \subset M$, i.e. $A \bigcap W \neq \emptyset$.
Note that the intersection of two dense sets need not be dense; it can be empty, as it's the case with $\mathbb{Q}$ and $\mathbb{Q}^c = \mathbb{R} \backslash \mathbb{Q} = \mathbb{I}$. On the other hand if U, V are open dense sets in M, then $U \bigcap V$ is open dense in M. For if W is any nonempty open subset of M then $U \bigcap W$ is a nonempty open subset of M as well, and by denseness of V, it is true that V meets $U \bigcap W$, i.e. $U \bigcap V \bigcap W$ is nonempty and $U \bigcap V$ meets W.

Moral: Open dense sets do a good job of being dense.

Definition: A set (or space) X is said to be separable if it has a countable dense subset.
(For instance, $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$, hence $\mathbb{R}^n$ is separable.)

Definition: A set A is said to be nowhere dense in (M, d) if $(\overline{A})^o = \emptyset$.

Definition: The countable intersection $G = \bigcap G_n$ of open dense sets is called a thick subset of M.

Extending our vocabulary in a natural way we say that the complement of a thick set is thin (or meager). A subset H of M is thin iff it is a countable union of nowhere dense closed sets, $H = \bigcup H_n$.

Clearly, thickness and thinness are topological properties. A thin set is the topological analog of a zero set (a set whose outer measure is zero).

• Baire's Theorem:
Every thick subset of a complete metric space M is dense in M. A nonempty, complete metric space is not thin: if M is the union of countably many closed sets, then at least one has nonempty interior.

Example:
a) The sets $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense subsets of $\mathbb{R}$.

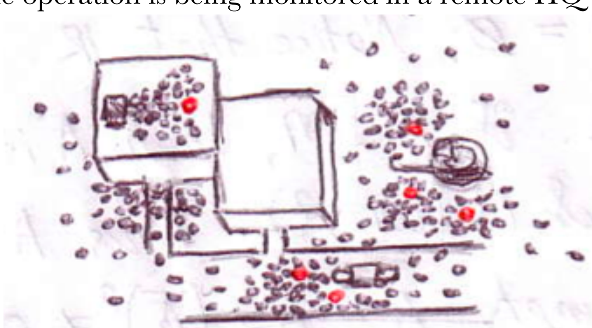
b) Let $A = \left\{\frac{1}{n} : n \geq 1\right\}$. Then A is nowhere dense in $\mathbb{R}$, because $\overline{A} = A \bigcup \{0\}$ and $(\overline{A})^o = \emptyset$.

c) Let (M, d) be discrete. If $A \subset M$, then A is both a closed and an open subset of M. Thus, $A = \overline{A}$ and $A = A^o$. Thus, if A is not empty, $A = \overline{A} = (\overline{A})^o \neq \emptyset$. Hence, A is not nowhere dense in M. Notice, however, that the statement "not nowhere dense" is not equivalent to the phrase "dense". In fact, the only dense subset of M is M itself. Every point of a discrete space is an isolated point.

d) In general, "not nowhere dense" is not the same as "dense". In $\mathbb{R}$, the set $A = (0, 1)$ is not dense because $\overline{A} = [0, 1] \neq \mathbb{R}$. However, A is not nowhere dense, as $(\overline{A})^o = A = (0, 1) \neq \emptyset$. The term "nowhere dense" is an unfortunate choice of language that is sadly too common in the literature to avoid. We should think of nowhere dense sets as "thin" sets or sets that are far from having even a single neighborhood.

e) While $\mathbb{R}$ is everywhere dense in itself, it is nowhere dense when it is considered as a subset of $\mathbb{R}^2$ (why?).

Example:
Suppose that a special forces unit has been sent into a zombie-infested area to search for survivors. The progress of the operation is being monitored in a remote HQ (see fig)



If the red dots represent the positions of the special-forces soldiers and the black dots are positions of detected zombies, what can you say about the outcome of this operation?

Solution:
Every point on the screen is either a zombie or a limit point of zombies. Thus, topologically speak-

ing, the set of all zombies is dense in the set of points. Notice also that the set of all red points is nowhere dense; that is, all neighborhoods of the red points have been breached with no chance for their recovery. Sadly, the operation is a fiasco. All of our brave soldiers will soon join the ranks of the undead. ✵

Now we are finally ready to present the argument that we alluded to previous to the above definitions:

• Lemma 1:
Let E be a closed subset of (M, d) and $E^{(i)}$ be the set of all isolated points of E. Then $E \backslash E^{(i)}$ is a closed subset of (M, d).

Proof:
Let $\{x_n\}_{n=1}^{\infty} \subset E \backslash E^{(i)}$ be a sequence that converges to $x \in M$. To show that $E \backslash E^{(i)}$ is closed, we must prove that $x \in E \backslash E^{(i)}$. Notice, however, that since E is closed, $x_n \longmapsto x \Longrightarrow x \in E$. Furthermore, this means that x is a limit point of E so x is not an isolated point, i.e. $x \notin E^{(i)}$. Hence it follows that $x \in E \backslash E^{(i)}$ as desired. ■

• Lemma 2:
Let E be a closed subset of $\mathbb{R}$. Then the set of all isolated points of E, $E^{(i)}$, is at most countable.

Proof:
Every isolated point of E is contained in an open interval that has no other points of E. In other words, if $x \in E^{(i)}$, then $x \in I_x$ such that $I_x \bigcap E = \{x\}$ and I_x is open. Thus, $E^{(i)} \subset \bigcup_{x \in E^{(i)}} I_x$ where $I_x \bigcap I_y = \emptyset$ if $x \neq y$.
Since each interval contains a rational, this union is countable. This indicates that $E^{(i)}$ is also countable as each I_x contains only one point of $E^{(i)}$. ■

• Theorem:
There exist perfect subsets of $\mathbb{R}$ that contain nearly all of $\mathbb{R}$ and yet fail to have even a single rational number. Such sets must be nowhere dense.

Proof:
Let $\varepsilon > 0$. List all the rational numbers in a sequence $\{r_n\}_{n=1}^{\infty}$ and put each r_n at the center of an open interval I_n of length$(I_n) = \frac{\varepsilon}{2^n}$.
If $G = \bigcup_{n=1}^{\infty} I_n$, then G is open and

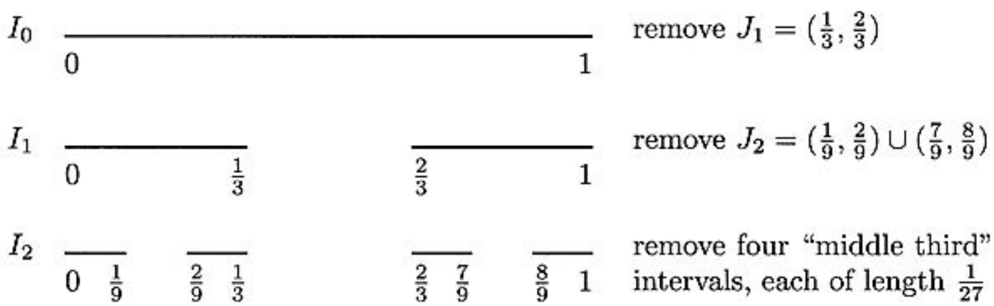
$$\text{length}(G) \leq \sum_{n=1}^{\infty} \text{length}(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon .$$

Setting $E = G^c$, we see that E is a closed set whose size must be infinite. That is, length$(E) = \infty$.
Thus, E must be uncountable.
By lemma 1, $E \backslash E^{(i)}$ is also a closed set and by lemma 2, $E \backslash E^{(i)}$ must be uncountable (because we

delete at most countably many points). Finally, observe that $P = E \setminus E^{(i)}$ is a perfect subset of $\mathbb{R}$. ■

THE CANTOR SET

Consider the process of successively removing "middle thirds" from the interval $[0, 1]$.



We continue this process inductively. At the n^{th} stage we construct I_n from I_{n-1} by removing 2^{n-1} disjoint, open, "middle thirds" intervals from I_{n-1}, each of length 3^{-n}; we will call this discarded set $\mathcal{J}_n$.
Thus, I_n is the union of 2^n closed subintervals of I_{n-1}, and the complement of I_n in $[0, 1]$ is $\mathcal{J}_1 \bigcup \ldots \bigcup \mathcal{J}_n$. The Cantor set Δ is defined as the set of points that still remain at the end of this process, in other words, the "limit" of the sets I_n.

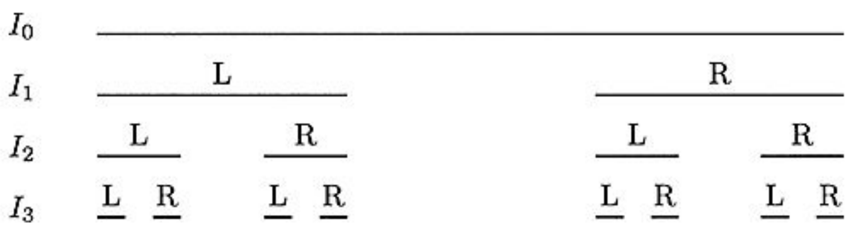
More precisely, $\Delta = \bigcap_{n=1}^{\infty} I_n$.

It follows from the nested interval theorem that $\Delta \neq \emptyset$, but notice that Δ is at least countably infinite. The endpoints of each I_n are in Δ:

$$0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \ldots \in \Delta .$$

We will refer to these points as the endpoints of Δ, that is, all of the points in Δ of the form $\frac{a}{3^n}$ for some integers a and n.

As we shall see presently, Δ is actually uncountable! This is more than a little surprising. Just try to imagine how terribly sparse the next few levels of the "middle thirds" diagram would look on the page. Adding even a few more levels defies the limits of typesetting!
For good measure we will give two proofs that Δ is uncountable, the first being somewhat combinatorial. Notice that each subinterval of I_{n-1} results in two subintervals of I_n (after discarding a middle third). We label these two new intervals L and R (for left and right) :



As we progress down through the levels of the diagram toward the Cantor set (somewhere far below), imagine that we "step down" from one level to the next by repeatedly choosing either a step to the left (landing on an L interval in the next level below) or a step to the right (landing on an R interval). At each stage we are only allowed to step down to a subinterval of the interval we are presently on – jumping across "gaps" is not allowed! Thus, each string of choices, *LRLRRLLRL-LLR*…, describes a unique "path" from the top level I_0 down to the bottom level Δ.

The Cantor set, then, is quite literally the "dust" at the end of the trail. Said another way, each such "path" determines a unique sequence of nested subintervals, one from each level, whose intersection is a single point of Δ. Conversely, each point $x \in \Delta$ lies at the end of exactly one such path, because at any given level there is only one possible subinterval of I_n on our diagram, call it $\tilde{I}_n$, that contains x. The resulting sequence of intervals is clearly nested (why?).

Thus, the Cantor set Δ is in one-to-one correspondence with the set of all paths, that is, the set of all sequences of L's and R's. Of course, any two choices would have done just as well, so we might also say that Δ is equivalent to the set of all sequences of 0's and 1's – a set we already know to be uncountable.
Here is what this means:

$$\text{card}(\Delta) = 2^{\aleph_0} = \text{card}([0,\ 1])\ .$$

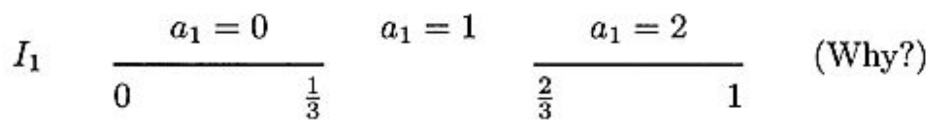
Absolutely amazing! The Cantor set is just as "big" as [0, 1] and yet it strains the imagination to picture such a sparse set of points. Before we give our second proof that Δ is uncountable, let's see why Δ is "small" (in at least one sense). We will show that Δ has "measure zero"; that is, the "measure" or "total length" of all of the intervals in its complement $[0,\ 1]\backslash\Delta$ is 1.
Here's why:
By induction, the total length of the 2^{n-1} disjoint intervals comprising $\mathcal{J}_n$ (the set we discard at the n^{th} stage) is $\frac{2^{n-1}}{3^n}$.
So the total length of $[0,\ 1]\backslash\Delta$ must be

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \frac{1}{1-\frac{2}{3}} = 1\ .$$

We have discarded everything!? And left uncountably many points behind!? How bizarre! This simultaneous "bigness" and "smallness" is precisely what makes the Cantor set so intriguing.

Our second proof that Δ is uncountable is based on an equivalent characterization of Δ in terms of ternary (base 3) decimals. Recall that each x in $[0, 1]$ can be written, in possibly more than one way, as: $x = 0.\, a_1\, a_2\, a_3 \ldots$ (base 3), where each $a_n = 0, 1,$ or 2. This three-way choice for decimal digits (base 3) corresponds to the three-way splitting of intervals that we saw earlier.

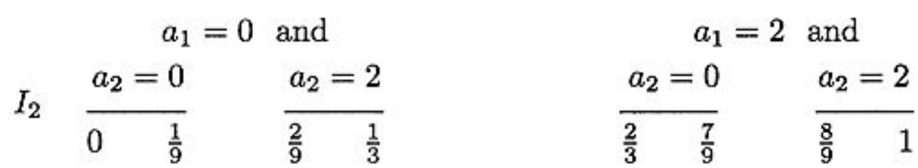
To see this, let us consider a few specific examples.
For instance, the three cases $a_1 = 0, 1,$ or 2 correspond to the three intervals $[0, 1/3]$, $(1/3, 2/3)$, and $[2/3, 1]$:



There is some ambiguity at these endpoints:

$$\begin{aligned} 1/3 &= 0.1 \text{ (base 3)} = 0.0222\ldots \text{ (base 3)}, \\ 2/3 &= 0.2 \text{ (base 3)} = 0.1222\ldots \text{ (base 3)}, \\ 1 &= 1.0 \text{ (base 3)} = 0.2222\ldots \text{ (base 3)}, \end{aligned}$$

but each of these ambiguous cases has at least one representation with a_1 in the proper range. Next, the figure below shows the situation for I_2 (but this time ignoring the discarded intervals):



Again, some confusion is possible at the endpoints:

$$\begin{aligned} 1/9 &= 0.01 \text{ (base 3)} = 0.00222\ldots \text{ (base 3)}, \\ 8/9 &= 0.22 \text{ (base 3)} = 0.21222\ldots \text{ (base 3)}. \end{aligned}$$

We take these few examples as proof of the following theorem.

• Theorem:
$x \in \Delta$ iff x can be written as $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where each a_n is either 0 or 2.

Thus the Cantor set consists of those points in $[0, 1]$ having some base 3 decimal representation that excludes the digit 1. Knowing this we can list all sorts of elements of Δ. For example, $\frac{1}{4} \in \Delta$ because $\frac{1}{4} = 0.020202\ldots$ (base 3).

The above theorem also leads to another proof that Δ is uncountable; or, rather, it gives us a new way of writing the old proof. The first proof used sequences of 0's and 1's, and now we find ourselves with sequences of 0's and 2's; the connection isn't hard to guess.

• Corollary:
Δ is uncountable. In fact, Δ is equivalent to $[0, 1]$.

Proof:
By altering our notation we can easily display a correspondence between Δ and $[0, 1]$. Each $x \in \Delta$ may be written $x = \sum_{n=1}^{\infty} \frac{2\,b_n}{3^n}$, where $b_n = 0$ or 1 , and now we define the Cantor function $f : \Delta \longrightarrow [0, 1]$ by

$$f\left(\sum_{n=1}^{\infty} \frac{2\,b_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{b_n}{2^n} \qquad (\text{where } b_n = 0, 1)$$

That is,

$$f\big(0.a_1a_2a_3\cdots \text{ (base 3)}\big) = 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\cdots \text{ (base 2)} \qquad (a_n = 0, 2).$$

Now f is clearly onto, and hence we have a second proof that Δ is uncountable (why?). But f isn't one-to-one; here's why:

$$f(1/3) = f\big(0.0222\ldots \text{ (base 3)}\big) = 0.0111\ldots \text{ (base 2)}$$
$$= 0.1 \text{ (base 2)} = f\big(0.2 \text{ (base 3)}\big) = f(2/3).$$

The same phenomenon occurs at each pair of endpoints of any discarded "middle third" interval (i.e., a subinterval of $\mathcal{J}_n$):

$$f(1/9) = f\big(0.00222\ldots \text{ (base 3)}\big) = 0.00111\ldots \text{ (base 2)}$$
$$= 0.01 \text{ (base 2)} = f\big(0.02 \text{ (base 3)}\big) = f(2/9).$$

It is easy to see that f is increasing; that is, if with $x < y$, then $f(x) \le f(y)$. We leave it as an exercise to check that $f(x) = f(y)$ iff x and y are endpoints of a discarded "middle third" interval (see Exercise 26 on Carother's (Cantor set section)). Thus, f is one-to-one except at the endpoints of Δ (a countable set), where it's two-to-one. It follows that Δ is equivalent to $[0, 1]$. (How?) ■