Algebraic Topology HW Set # 1

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Problem 1. *Let $\mathcal{C} \subset [0,1]$ be the Cantor set.*

a) *Prove that every element $x \in \mathcal{C}$ has a unique representative*

$$x = \sum_{n \geq 1} \frac{a_n}{3^n} \qquad \text{for } a_n = 0 \text{ or } 2.$$

b) *There is the map*

$$f \colon X = \prod \{0,2\} \to \mathcal{C}$$

defined by

$$f(\langle a_1, \dots, a_n, \dots \rangle) = \sum_{n \geq 1} \frac{a_n}{3^n}.$$

With $\{0,2\}$ given the discrete topology and X given the product topology,[1] *prove that f is a homeomorphism. [HINT: To show that f is continuous, use the ε, δ definition of continuity, i.e., let $p = \langle a_1, \dots \rangle \in X$ and let $\varepsilon > 0$. Then you want to show that there is an open set $B \subset X$ containing p such that $x \in B \implies |f(x) - f(p)| < \varepsilon$.]*

c) *The Cantor ternary function $F \colon \mathcal{C} \to [0,1]$ is defined by*

$$F\left(\sum_{n \geq 1} \frac{a_n}{3^n} \right) = \sum \frac{\frac{a_n}{2}}{2^n} \in [0,1].$$

Use this function to show that the Cantor set has the cardinality of the reals.

d) *Prove the Cantor ternary function is continuous.*

e) *Prove that F extends to a map of $[0,1] \to [0,1]$. [HINT: Notice that F has the same value at the two ends of each deleted interval, so one can define the extension to a constant on the deleted interval. To see this do some examples. The first step in defining the Cantor set is to delete $(1/3, 2/3)$. The point $1/3 \in \mathcal{C}$ is represented in the Cantor set by $0.0222222\cdots$. The point $2/3 \in \mathcal{C}$ is represented by 0.2. We have $F(0.0222\cdots) = 0.0111\cdots = 0.1 = F(0.2)$. So the extension of F would take the value 0.1 on $(1/3, 2/3)$. Now try to prove this for each omitted interval.]*

f) *Use part b) to construct a space filling curve, i.e., a surjective, continuous map $[0,1] \to [0,1] \times [0,1]$. [HINT: Use part b) to prove $\mathcal{C} = \mathcal{C} \times \mathcal{C}$.]*

[1] Elements of X are infinite sequences of 0's and 2's. So $(0,0,0,2)$ and $(0,2,2,2,2,2,2)$ are not elements of X; however if you pad these with an infinite string of 0's to get $(0,0,0,2,0,0,0,0,\dots)$ and $(0,2,2,2,2,2,2,0,0,0,0,\dots)$, then you do get points of X. (A more interesting point of X is the sequence $(p_n)_n$, where $p_n = 1$ if n is prime, and $p_n = 0$ otherwise.)

Proof of a). Every element $x \in \mathcal{C}$ has a unique ternary representation because, if otherwise it had two different representations, e.g.

$$\frac{1}{3} = 0.1000\cdots = 0.0222\cdots ,$$

then at most one of these can be written without any 1's in it. Therefore this representation of points of $\mathcal{C}$ is unique. $\square$

Proof of b). It is clear that f is bijective. To show that it is continuous, given an $\varepsilon > 0$ and $c = \sum a_i/3^i \in \mathcal{C}$, choose $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} \frac{2}{3^i} < \varepsilon.$$

Now let $\mathcal{U} = \pi_1^{-1}(a_1) \cap \cdots \cap \pi_N^{-1}(a_N) \subset X$, where the π_i's are the projection maps of the product space X. Note that $\mathcal{U}$ is open since the projection map is continuous and a finite intersection of open sets is open. Then for $a \in \mathcal{U}$,

$$|f(a) - c| \le \sum_{N+1}^{\infty} \frac{2}{3^i} < \varepsilon.$$

Thus $f(\mathcal{U}) \subset \mathbb{B}_\varepsilon(c)$, and f is continuous.

Now to show that f^{-1} is also continuous, for a given $c \in \mathcal{C}$, let $a = \langle a_i \rangle = f^{-1}(c) \in X$. Then for a basic open set $\mathcal{V} = \pi_{i_1}^{-1}(a_{i_1}) \cap \cdots \cap \pi_{i_n}^{-1}(a_{i_n})$ containing a (where $i_1 < \cdots < i_n = N$), we have

$$f^{-1}\left(\mathbb{B}_{\left(\frac{1}{3}\right)^{N+1}}(c)\right) \subset \mathcal{V} \subset X.$$

Indeed, if $b \in \mathbb{B}_{\left(\frac{1}{3}\right)^{N+1}}(c)$, then

$$b_1 = a_1, \ldots, b_N = a_N.$$

From this we get that $f^{-1}(b)$ is in $\mathcal{V}$. Hence f^{-1} is continuous as well, and it follows that f is a homeomorphism. $\square$

Proof of c). It follows from the definition of the Cantor ternary function that any element $y \in [0,1]$ may be written as $y = \sum_{n=1}^{\infty} b_n 2^{-n}$, where $b_n \in \{0,1\}\ \forall n$. It is obvious now that $y = F(x)$, where $x \in \mathcal{C}$ is defined as $x = \sum_{n=1}^{\infty} 2b_n 3^{-n}$. Hence it follows that F is surjective and thus $|\mathcal{C}| = |[0,1]| = |\mathbb{R}| = 2^{\aleph_0}$. $\square$

Proof of d). Given $x, y \in \mathcal{C}$, we wish to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|F(x) - F(y)| < \varepsilon$. Given ε, let n be such that $2^{-n} < \varepsilon$. Then $\delta = 3^{-n}$ suffices. To see this, suppose that $F(x) \neq F(y)$; then there must be some binary digit where they first differ; let n be the index of this digit. Now suppose that $|x - y| < 3^{-n}$; then the ternary expansions of both x and y are the same up to the first $n-1$ digits. Then $F(x)$ and $F(y)$ in binary are the same up to the first $n-1$ digits, and thus they are separated by no more than 2^{-n}. $\square$

Proof of e). Now we extend F so that we get a continuous function $F : [0,1] \to [0,1]$ that is increasing with $F(0) = 0$ and $F(1) = 1$. Recall that $\mathcal{C} = \cap_{k=0}^{\infty} \mathcal{C}_k$ and each $\mathcal{C}_k$ is a disjoint union of 2^k closed intervals (for example, $\mathcal{C}_1 = [0,1/3] \cup [2/3,1]$). Let $F_1(x)$ be the continuous increasing function on $[0,1]$ (and linear on $\mathcal{C}_1$) that satisfies

$$F_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/2 & \text{if } 1/3 \le x \le 2/3, \\ 1 & \text{if } x = 1. \end{cases}$$

Similarly, let $F_2(x)$ (see Figure 1) be the continuous increasing function on $[0,1]$ (and linear on $\mathcal{C}_2$) that satisfies

$$F_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/4 & \text{if } 1/9 \le x \le 2/9, \\ 1/2 & \text{if } 1/3 \le x \le 2/3, \\ 3/4 & \text{if } 7/9 \le x \le 8/9, \\ 1 & \text{if } x = 1. \end{cases}$$

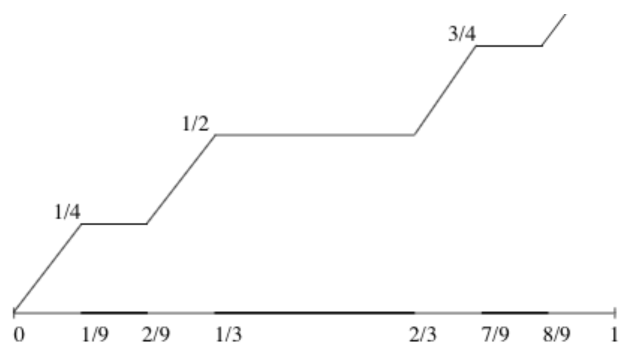

Figure 1: *Here's a visualization of the construction of F_2.*

This process yields a sequence of continuous increasing functions $\{F_n\}_{n=1}^{\infty}$ such that clearly

$$|F_{n+1}(x) - F_n(x)| \le \frac{1}{2^{n+1}}\,.$$

Hence $\{F_n\}_{n=1}^{\infty}$ converges uniformly to a continuous limit F, which is called the ***Cantor-Lebesgue function*** (see Figure 2 below). By construction F is increasing, $F(0) = 0$, $F(1) = 1$, and we see that F is constant on each interval of the complement of the Cantor set. $\square$

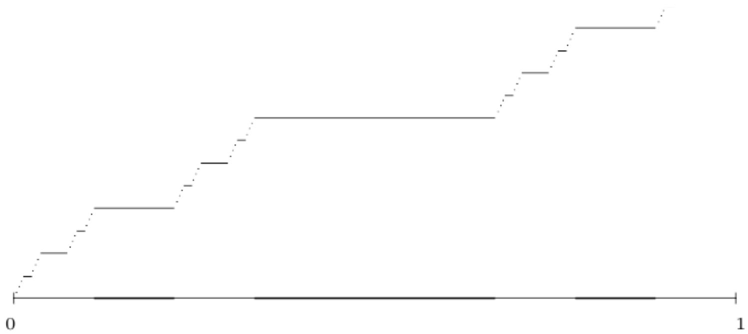

Figure 2: *Here's a visualization of the* Cantor-Lebesgue *function.*

Proof of f). Building on our previous results, consider the map $g = F \circ f\colon X \to [0,1]$ (see diagram) defined by $\langle a_i \rangle \mapsto \sum_i (a_i/2)/2^i$.

$$\begin{tikzcd}
& {[0,1]} \\
X \arrow[ur, "g"] \arrow[r, "f"'] & \mathcal{C} \arrow[u, "F"']
\end{tikzcd}$$

A composition of continuous functions is continuous, so g must be continuous. Now to show surjectivity, suppose $x \in [0,1]$. We know that there is a unique sequence $(b_i)_{i=1}^{\infty}$ (the binary expansion of x) such that $x = \sum_{i=1}^{\infty} b_i/2^i$. Letting $a_i = 2b_i$, we have $g(\langle a_i \rangle) = x$. Thus g is surjective and continuous, as desired.

Now, it is clear that $\prod\{0,2\} \times \prod\{0,2\} \cong \prod\{0,2\}$. Hence we have a surjective continuous map $H\colon X \to [0,1] \times [0,1]$ as illustrated in the diagram:

$$\begin{tikzcd}
X \times X \arrow[r, "g \times g"] \arrow[d, "\cong"'] & {[0,1] \times [0,1]} \\
X \arrow[ur, "H"'] &
\end{tikzcd}$$

This map H is a space-filling curve (i.e. a surjective and continuous map). We can factor this map as $H = h \circ g$ (see diagram) so that h is our desired map.

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[rr, dashed, bend right, "H"'] & {[0,1]} \arrow[r, "h"] & {[0,1] \times [0,1]}
\end{tikzcd}$$

$\square$

Problem 2. *Prove the product of two Hausdorff spaces is Hausdorff.*

Proof. Let X and Y be Hausdorff spaces. Let $x_1 \in U_1$ and $x_2 \in U_2$, where $x_1 \neq x_2$ and $U_1, U_2 \subset X$ are open sets in X. Similarly, let $y_1 \in V_1$ and $y_2 \in V_2$, where $y_1 \neq y_2$ and $V_1, V_2 \subset Y$ are open sets in Y. Our goal is to show that for $(x_1, y_1) \in (U_1 \times V_1)$ and $(x_2, y_2) \in (U_2 \times V_2)$, we have that $(U_1 \times V_1) \cap (U_2 \times V_2) = \emptyset$.

Since the projection map $\pi_1 \colon X \times Y \to X$ is continuous, we have that $\pi_1^{-1}(U_1) = U_1 \times Y$ must be open. Similarly, $\pi_1^{-1}(U_2)$ is open as well. Now,

$$\begin{aligned} \pi_1^{-1}(U_1) \cap \pi_1^{-1}(U_2) &= \pi_1^{-1}(U_1 \cap U_2) \\ &= \pi_1^{-1}(\emptyset) \\ &= \emptyset. \end{aligned}$$

Hence, since $\pi_1^{-1}(U_1) \cap \pi_1^{-1}(U_2) = (U_1 \times Y) \cap (U_2 \times Y)$ is empty, it follows that $(U_1 \times V_1) \cap (U_2 \times V_2) = \emptyset$, as desired. ☐

Problem 3. *Show that a retract[2] of a Hausdorff space is closed.*

Proof. Let $A \subset X$ be a retract of X. Consider a point $x \in \partial A$. If $x \notin A$, then $r(x) \neq x$ and there are disjoint neighborhoods U of x and V of $r(x)$. Then by continuity of r, there must be a neighborhood $W \subseteq U$ of x, so that $r(W) \subseteq V$. However, $r(W \cap A)$ is a nonempty subset of W,[3] so it cannot be in V. Hence we must have $r(x) = x \in A$, and hence A is closed. ☐

Alternate proof. Alternatively, we could have used the fact that for any two maps $f, g \colon X \to A$, the so-called ***equalizer*** $\{x \in X \mid f(x) = g(x)\}$ is a closed subspace of X if A is Hausdorff. This follows from the diagonal Δ_A being closed in $A \times A$ and $(f, g) \colon X \to A \times A$ being continuous. In this particular case we just take $f = \mathrm{Id}_A$ and $g = r$; the equalizer is then the retract A. ☐

[2]Recall that a subspace $A \subset X$ is called a ***retract*** of X if there is a map $r \colon X \to A$ such that $r|_A = \mathrm{Id}_A$; that is, $r(a) = a$ for all $a \in A$. (Such a map is called a ***retraction***.)
[3]$W \cap A$ is nonempty because, by assumption, $x \in \partial A$.