Modules for $\mathfrak{sl}_2$ are completely reducible

One of the purposes of representation theory (=the study of modules) is to detect properties of the underlying rings/algebras/Lie algebras. In this workshop we will show that the representation theory of $\mathfrak{sl}_2$ is as easy as it could possibly be. It will not be clear from this workshop, but it is nevertheless true, that the reason for this is that $\mathfrak{sl}_2$ is a simple Lie algebra.

Recall that $\mathfrak{sl}_2$ has basis $\{e, f, h\}$ subject to the relations

$$[e,f] = h, \qquad [h,e] = 2e, \qquad [h,f] = -2f.$$

Thus a representation of $\mathfrak{sl}_2$ is just a (finite-dimensional, complex) vector space V together with three maps $E, F, H \colon V \to V$ satisfying

$$\begin{aligned} EF - FE &= H \\ HE - EH &= 2E \\ HF - FH &= -2F. \end{aligned}$$

Recall that a module (or representation) V is called **completely reducible** if

$$V \cong S_1 \oplus \ldots \oplus S_t$$

for some simple modules $S_1, \ldots, S_t$.

At the last workshop, we worked out what the simple $\mathfrak{sl}_2$-modules are. For a representation $V \neq 0$, we found an eigenvector $0 \neq w_0 \in V$ for H such that $Ew_0 = 0$.

(A) Given such a w_0, we produced $w_i := \frac{1}{i!}F^i w_0$. We proved that at some stage $Fw_j = 0$, and that

$$V(m) := \operatorname{Span}\{w_0, \ldots, w_m\}$$

is a submodule of V, for some integer $m \geq 0$.

(B) Each $V(m)$ is a simple module with one-dimensional weight spaces with weights $m, m-2, \ldots, -m$.

(C) The $V(m)$ are the only simple modules.

Today we will show that an arbitrary $\mathfrak{sl}_2$-module V is completely reducible; that is, $V \cong V(m_1) \oplus \cdots \oplus V(m_t)$ for some non-negative integers $m_1, \ldots, m_t$.

1. Using induction in dim V show that it is enough to prove that if $W \subsetneq V$ is a proper submodule, there there exists another proper submodule W' such that $V = W \oplus W'$.

Solution: We use induction in dim V. The base step is clear, since if $\dim V = 1$, then V is simple. Assume that modules with any dimension $n < \dim V$ are completely reducible. Then if W is a proper submodule of V, $\dim W < \dim V$, and if W has a complement W', then $\dim W' < \dim V$ as well. By the induction, $W \cong V(\ell_1) \oplus \cdots \oplus V(\ell_r)$ and $W' \cong V(m_1) \oplus \cdots \oplus V(m_s)$ are completely reducible, and so is their direct sum $V \cong V(\ell_1) \oplus \cdots \oplus V(\ell_r) \oplus V(m_1) \oplus \cdots \oplus V(m_s)$.

Hence we search for such a W'.

2. Show that in order to prove the existence of W', it is enough to construct a submodule $U \neq 0$ such that $W \cap U = 0$.

> **Solution:** If we find such a U, then $U \oplus W$ is a submodule of V. If $V = U \oplus W$ we are done by Q1. If not, replace W by $U \oplus W$ (which is strictly larger) and repeat.

Recall that $v_0 \in V$ is called a **highest weight vector** if it is an eigenvector of $H\colon V \to V$, and further $Ev_0 = 0$. We will construct U in Q2 as follows:

- we will find a highest weight vector $v_0 \in V$ such that $v_0 \notin W$;
- by fact (A), using v_0 we can generate a submodule of V (denote it by U), which by fact (B) is simple;
- since $v_0 \notin W$, $W \cap U$ is a proper submodule of U; but U is simple, so we deduce $W \cap U = 0$.

The remaining questions construct the required v_0.

3. By considering the module V/W, use facts (A) and (B) to conclude that there exists $v_0 \in V$ such that

$$v_0 \notin W, \quad (H - m)v_0 \in W \quad \text{and} \quad Ev_0 \in W \tag{1}$$

for some $m \geq 0$.

> **Solution:** Consider the $\mathfrak{sl}_2$-module V/W. Then plugging this module into the chain of logic found under the blue box (the summary from last workshop!), we can find $0_{V/W} \neq v_0 + W$, which is an eigenvector of H (with eigenvalue m say), such that $E(v_0 + W) = 0_{V/W}$. Translating these three conditions, we see that they correspond to $v_0 \notin W$, $H(v_0 + W) = m(v_0 + W)$, and $Ev_0 + W = W$, respectively. Or, in other words, $v_0 \notin W$, $(H - m)v_0 \in W$, and $Ev_0 \in W$.

We will 'adjust' v_0 by adding elements of W (but preserving the three properties in (1)) until it becomes a highest weight vector of weight m.

4. Show that every element of W can be written uniquely as the sum of eigenvectors for H. (Remember that we are assuming by induction that W is completely reducible since $\dim W < \dim V$.)

> **Solution:** By the induction hypothesis $W \cong V(\ell_1) \oplus \cdots \oplus V(\ell_r)$. By the construction of the basis of $V(\ell_i)$ in Fact (A), each of the basis elements is an eigenvector of H. Thus an arbitrary element of W can be written as the sum of elements in some $V(\ell_i)$, and each of these elements in this sum can be written as the sum of eigenvectors. Thus every element of W can be written as the sum of eigenvectors of H.

5. Using Q4, how can we ensure that $(H - m)v_0$ has eigenvalue m? We make this modification, and so in what follows assume that $(H - m)v_0 \in W_m$.

> **Solution:** By Q3 $(H - m)v_0 \in W$, so by Q4 we can write
>
> $$(H - m)v_0 = u_1 + u_2 + \cdots + u_t$$
>
> where the u_i are eigenvectors of H (say with eigenvalues λ_i). If u_1 does not have eigenvalue m, then
>
> $$(H - m) \cdot \frac{1}{\lambda_1 - m} u_1 = \frac{\lambda_1 - m}{\lambda_1 - m} u_1 = u_1,$$
>
> thus
>
> $$(H - m)(v_0 - \frac{u_1}{\lambda_1 - m}) = u_2 + \cdots + u_t.$$
>
> Continuing in this way, by suitably subtracting things from v_0, we can assume that $(H - m)v_0$ has eigenvalue m.

6. (a) By computing $\mathrm{HE}v_0$ using the $\mathfrak{sl}_2$ relations, and using Q5, show that

$$(\mathrm{H} - (m+2))\mathrm{E}v_0 \in \mathrm{W}_{m+2}.$$

Solution: Observe that

$$\mathrm{HE}v_0 = \mathrm{EH}v_0 + 2\mathrm{E}v_0 = \mathrm{E}(\mathrm{H} - m)v_0 + (m+2)\mathrm{E}v_0,$$

thus

$$(\mathrm{H} - (m+2))\mathrm{E}v_0 = \mathrm{E}\underbrace{(\mathrm{H} - m)v_0}_{\in \mathrm{W}_m} \in \mathrm{W}_{m+2}, \tag{2}$$

where $\mathrm{E} \cdot \mathrm{W}_m \subseteq \mathrm{W}_{m+2}$ as in the last workshop.

(b) Further, using Q4 or otherwise, show that $\mathrm{E}v_0 \in \mathrm{W}_{m+2}$.

Solution: Since $\mathrm{E}v_0 \in \mathrm{W}$, again by Q4 we can write

$$\mathrm{E}v_0 = y_1 + \cdots + y_s$$

with each y_i an eigenvector of H (with eigenvalue μ_i say). Then

$$(\mathrm{H} - (m+2))\mathrm{E}v_0 = (\mu_1 - (m+2))y_1 + \cdots + (\mu_s - (m+2))y_s.$$

If $\mu_i \neq m+2$ then by definition $y_i \notin \mathrm{W}_{m+2}$. Hence the only way that the right hand side can belong to W_{m+2} is if $\mathrm{E}v_0 \in \mathrm{W}_{m+2}$.

7. Using the structure of the $\mathrm{V}(\ell_i)$ making up W, justify why $\mathrm{E}\colon \mathrm{W}_m \to \mathrm{W}_{m+2}$ is surjective. Hence we can find $w \in \mathrm{W}_m$ such that $\mathrm{E}w = \mathrm{E}v_0$.

Solution: By construction of the basis of the $\mathrm{V}(\ell_i)$ it is clear that

$$\mathrm{V}(\ell_i)_m \to \mathrm{V}(\ell_i)_{m+2}$$

is surjective. Since W is the sum of these, $\mathrm{W}_m \to \mathrm{W}_{m+2}$ is surjective.

After replacing v_0 by $v_0 - w$ (which is nonzero since $v_0 \notin \mathrm{W}$), we can thus assume that $\mathrm{E}v_0 = 0$. This replacement does not effect the fact that $v_0 \notin \mathrm{W}$, and also it does not effect the fact that $(\mathrm{H}-m)v_0 \in \mathrm{W}_m$. Set $w_0 := (\mathrm{H} - m)v_0$. If $w_0 = 0$ then indeed v_0 is an eigenvector of H satisfying $\mathrm{E}v_0 = 0$ and so we are done. Hence for the remainder of this workshop we assume that $w_0 \neq 0$ and aim for a contradiction.

8. Use your answer to Q6 to show that $\mathrm{E}w_0 = 0$.

Solution: Since $\mathrm{E}v_0 \in \mathrm{W}_{m+2}$ (by Q6(b)), the left hand side of (2) is zero. Hence $0 = \mathrm{E}(\mathrm{H} - m)v_0 = \mathrm{E}w_0$.

Define $v_i := \frac{1}{i!}\mathrm{F}^i v_0$ and $w_i := \frac{1}{i!}\mathrm{F}^i w_0$, so by last workshop $w_j = 0$ for $j > m$.

9. By induction (or otherwise), prove that

$$\mathrm{H}v_i = (m - 2i)v_i + w_i \qquad \text{for all } i \geq 0$$
$$\mathrm{E}v_i = (m - i + 1)v_{i-1} + w_{i-1} \qquad \text{for all } i \geq 1.$$

Solution: This is very similar to Workshop 3 Q3.

10. Deduce from the first formula that for $i > m$, we have $v_i \in V_{m-2i}$. As in the last workshop, there exists a j such that $v_j \neq 0$ and $v_{j+1} = 0$. By the first formula, deduce that $j \geq m$.

Solution: As in the statement of Q9, $w_i = 0$ for all $i > m$. Hence the first formula in Q9 shows that $Hv_i = (m-2i)v_i$ for all $i > m$, i.e., $v_i \in V_{m-2i}$ for all $i > m$. Since $w_m \neq 0$, the first formula clearly shows that $v_m \neq 0$.

11. Keeping the j from Q10, substitute $i = j+1$ into the second formula of Q9 and deduce that $0 = (m-j)v_j + w_j$. If $j = m$ reach a contradiction, and if $j > m$ reach another contradiction.

Solution: The j in Q10 satisfies $j \neq m$. If $j = m$ the formula $0 = (m-j)v_j + w_j$ in the question shows that $w_j = 0$, which is a contradiction. On the other hand, if $j > m$ then $0 = (m-j)v_j + 0$, so $v_j = 0$, which contradicts the fact $v_j \neq 0$.

Please hand in your solution to Q8, Q9, Q10 and Q11 by the start of lecture on Monday 6 November.