Analytic Functions
Exam # 3

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Schwarz's Lemma

Since we are going to use Schwarz's Lemma heavily on our proofs, I state it here for completeness:

Lemma (Schwarz's Lemma). *Suppose that $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ is holomorphic and $f(0) = 0$. Then,*

- $\|f(z)\| \leq \|z\|$ *for all* $z \in \mathbb{D}^2$ *and* $\|f'(0)\| \leq 1$.
- *if* $\|f(z)\| = \|z\|$ *or* $\|f'(0)\| = 1$ *for some nonzero* $z \in \mathbb{D}^2$, *then* f *is a rotation, i.e.* $f(z) = \beta z$ *for some constant* β *with* $\|\beta\| = 1$.

Problem 1. *Suppose f is a holomorphic automorphism of the unit disc $\mathbb{D}^2$ such that f has two fixed points. Show that f must be the identity.*

Proof. Let $p, q \in \mathbb{D}^2$ be two such distinct fixed points so that $f(p) = p$ and $f(q) = q$. Let us first consider the case $p = 0$. Since $\|f(q)\| = \|q\|$ and $q \neq 0$, we have $f(z) = \beta$ for some $\|\beta\| = 1$ by Schwarz's lemma. Note that β has to be 1 since $f(q) = q$.

For the general case, let

$$\varphi(z) = \frac{p - z}{1 - \overline{p}z}.$$

Then the analytic function $g = \varphi \circ f \circ \varphi\colon \mathbb{D}^2 \to \mathbb{D}^2$ satisfies

$$g(0) = \varphi(f(\varphi(0))) = \varphi(f(p)) = \varphi(p) = 0 \qquad \text{and} \qquad g(\varphi(q)) = \varphi(f(q)) = \varphi(q)$$

since $\varphi = \varphi^{-1}$. Since $p \neq q$, we have $\varphi(q) \neq 0$ and hence, by the first case, $g = \mathrm{Id}_{\mathbb{D}^2}$. Thus $f = \mathrm{Id}_{\mathbb{D}^2}$, as desired. □

Problem 2 (Schwarz-Pick Theorem). *Show that for any holomorphic function $f\colon \mathbb{D}^2 \to \mathbb{D}^2$*

$$\frac{\|f'(z)\|}{1 - \|f(z)\|^2} \leq \frac{1}{1 - \|z\|^2} \tag{$\clubsuit$}$$

for all z in the unit disc $\mathbb{D}^2$.

Proof. Throughout this problem, we use the fact that for any $w_0 \in \mathbb{D}^2$, the Möbius transformation

$$\varphi(z) = \frac{w_0 - z}{1 - \overline{w_0}\, z}$$

maps $\mathbb{D}^2$ to $\mathbb{D}^2$ and swaps w_0 and 0. For a given $z_0 \in \mathbb{D}^2$, let

$$g(z) = \frac{z_0 - z}{1 - \overline{z_0}\, z} \qquad \text{and} \qquad h(z) = \frac{f(z_0) - z}{1 - \overline{f(z_0)}\, z}.$$

Then $h \circ f \circ g$ is an analytic map from $\mathbb{D}^2$ to $\mathbb{D}^2$ which fixes 0, so we may apply Schwartz's lemma to obtain

$$\left\| \frac{f(z_0) - f(g(z_0))}{1 - f(g(z_0))\, \overline{f(z_0)}} \right\| \leq \|z\|.$$

Now, letting $w = g^{-1}(z)$, we get

$$\begin{aligned} \left\| \frac{f(z_0) - f(w)}{1 - \overline{f(z_0)}\, f(w)} \right\| &\leq \left\| \frac{z_0 - w}{1 - \overline{z_0}\, w} \right\| \\ \Longrightarrow \left\| \frac{f(z_0) - f(w)}{z_0 - w} \right\| &\leq \left\| \frac{1 - f(w)\, \overline{f(z_0)}}{1 - \overline{z_0}\, w} \right\| \end{aligned}$$

Taking the limit as $w \to z_0$ (which we can do since g is bijective on $\mathbb{D}^2$), we get

$$\|f'(z_0)\| \leq \frac{1 - \|f(z_0)\|^2}{1 - \|z_0\|^2} \qquad \Longrightarrow \qquad \frac{\|f'(z_0)\|}{1 - \|f(z_0)\|^2} \leq \frac{1}{1 - \|z_0\|^2}$$

for all $z_0 \in \mathbb{D}^2$, as desired. □

Problem 3. *Does there exist a holomorphic function $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ such that $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$?*

Solution. We use the estimate for the derivative (♣) that we computed on Problem 2 above. A straight computation shows that such a holomorphic function cannot possibly exist:

$$\begin{aligned} \frac{1 - \left\| f\left(\frac{1}{2}\right) \right\|^2}{1 - \left\| \frac{1}{2} \right\|^2} &= \frac{1 - \frac{9}{16}}{1 - \frac{1}{4}} = \frac{16 - 9}{16 - 4} = \frac{7}{12} \\ &< \frac{8}{12} = \frac{2}{3} = \left\| f'\left(\frac{1}{2}\right) \right\|. \quad (\Rightarrow\Leftarrow) \end{aligned}$$

□

Problem 4. *Suppose $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ is holomorphic. Show that*

$$\frac{\|f(0)\| - \|z\|}{1 + \|f(0)\|\, \|z\|} \leq \|f(z)\| \leq \frac{\|f(0)\| + \|z\|}{1 - \|f(0)\|\, \|z\|} \tag{♠}$$

for all $\|z\| < 1$.

Proof. Let us set $a = f(0)$ and

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a} z}.$$

Then $\varphi_a \circ f$ maps $\mathbb{D}^2$ to $\mathbb{D}^2$ and fixes zero. Thus, by Schwarz' lemma, we have

$$\|\varphi_a(f(z))\| = \left\| \frac{f(z)-a}{1-\overline{a}\, f(z)} \right\| \le \|z\|,$$

so that

$$\|f(z)-a\| \le \|z\| \, \|1-\overline{a}\, f(z)\| \le \|z\| + \|z\| \, \|a\| \, \|f(z)\|. \tag{$\heartsuit$}$$

Applying the triangle inequality, we get

$$\|f(z)\| \le \|z\| + \|z\| \, \|a\| \, \|f(z)\| + \|a\|.$$

Then,

$$\begin{aligned} \|f(z)\| - \|z\| \, \|a\| \, \|f(z)\| &\le \|z\| + \|a\| \\ \implies \|f(z)\| &\le \frac{\|z\| + \|a\|}{1 - \|a\| \, \|z\|} \\ &= \frac{\|z\| + \|f(0)\|}{1 - \|f(0)\| \, \|z\|}, \end{aligned}$$

This establishes the second desired inequality in ($\spadesuit$).

To obtain the first inequality, we begin with

$$\|a\| = \|a - f(z) + f(z)\| \le \|a - f(z)\| + \|f(z)\| \le \|z\| + \|z\| \, \|a\| \, \|f(z)\| + \|f(z)\|,$$

where the last inequality follows from ($\heartsuit$). Then finally we get

$$\begin{aligned} \|a\| - \|z\| &\le \|z\| \, \|a\| \, \|f(z)\| + \|f(z)\| \\ \implies \frac{\|a\| - \|z\|}{\|z\| \, \|a\| + 1} = \frac{\|f(0)\| - \|z\|}{1 + \|f(0)\| \, \|z\|} &\le \|f(z)\|. \end{aligned}$$

□

Problem 5. *If $f(z)$ is holomorphic and $\mathfrak{Im}\, f(z) \ge 0$ whenever $\mathfrak{Im}\, z > 0$ (i.e. whenever $z \in \mathbb{H}^2$), show that*

$$\frac{\|f(z) - f(z_0)\|}{\|f(z) - \overline{f(z_0)}\|} \le \frac{\|z - z_0\|}{\|z - \overline{z_0}\|}$$

and

$$\frac{\|f'(z)\|}{\mathfrak{Im}\, f(z)} \le \frac{1}{\mathfrak{Im}\, z}$$

Proof. We map the upper-half plane conformally to the unit disc $\mathbb{D}^2$ with appropriate holomorphic transformations. More precisely, we define

$$\Phi(z) = \frac{z - z_0}{z - \overline{z_0}}, \quad z \in \mathbb{H}^2, \qquad \Psi(w) = \frac{w - f(z_0)}{w - \overline{f(z_0)}}, \quad w \in \mathbb{H}^2.$$

Since for $z \in \mathbb{R}$, we have $\|z - z_0\| = \|z - \overline{z_0}\|$, then $\Phi(\mathbb{R}) = \{z, \|z\| = 1\}$. Notice that we mapped z_0 to the center of the disc while $\overline{z_0}$ (its symmetric point with respect to the real axis), was mapped to the point symmetric to the origin with respect to the circle, i.e. ∞. This shows that $\Phi\colon \mathbb{H}^2 \to \mathbb{D}^2$. Similar considerations apply for Ψ and, in particular, $\Psi(f(z_0)) = 0$. Now consider the function

$$g = \Psi \circ f \circ \Phi^{-1}\colon \mathbb{D}^2 \to \mathbb{D}^2,$$

and notice that

$$g(0) = \Psi(f(\Phi^{-1}(0))) = \Psi(f(z_0)) = 0,$$

so that we can apply Schwarz lemma. This gives $\|g(\xi)\| \le \|\xi\|$. We set $\Phi(z) = \xi \iff z = \Phi^{-1}(\xi)$. This gives

$$\|\Psi(f(\Phi^{-1}(\xi)))\| \;\le\; \|\xi\| \iff \|\Psi(f(z))\| \;\le\; \|\Phi(z)\| \iff \left\|\frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}}\right\| \le \left\|\frac{z-z_0}{z-\overline{z_0}}\right\|$$

for $z, z_0 \in \mathbb{H}^2$. Notice that for the second inequality we use the fact that $\|g'(0)\| \le 1$. First we note that

$$\Phi'(z) = \frac{z_0 - \overline{z_0}}{(z-\overline{z_0})^2} \implies \Phi'(z_0) = \frac{1}{2i\,\mathfrak{Im}\, z_0}.$$

Similarly,

$$\Psi'(w) = \frac{f(z_0) - \overline{f(z_0)}}{(w-\overline{f(z_0)})^2} \implies \Psi'(f(z_0)) = \frac{1}{2i\,\mathfrak{Im}(f(z_0))}.$$

By the chain rule then we get

$$\|\Psi'(f(\Phi^{-1}(0)))\|\,\|f'(\Phi^{-1}(0))\|\,\|(\Phi^{-1})'(0)\| \le 1 \iff \|\Psi'(f(z_0))\|\,\|f'(z_0)\|\,\frac{1}{\|\Phi'(z_0)\|} \le 1$$

since

$$(\Phi^{-1})'(z) = \frac{1}{\Phi'(\Phi^{-1}(z))}.$$

As a result,

$$\frac{\|f'(z_0)\|}{\|2i\,\mathfrak{Im}(f(z_0))\|} \le \frac{1}{\|2i\,\mathfrak{Im}\, z_0\|}$$

and this clearly implies the inequality for an arbitrary point $z_0 \in \mathbb{H}^2$. If equality holds in either of the inequalities, then $g(\xi) = c\,\xi$, where c is a constant with $\|c\| = 1$. This gives

$$g(\xi) = \Psi(f(\Phi^{-1}(\xi))) = c\,\xi \iff \Psi(f(z)) = c\,\Phi(z) \iff f(z) = \Psi^{-1}(c\,\Phi(z)). \qquad \square$$