Math 260 DNHI # 5

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Section 2.3

(12) Let V, W, and Z be VS's, and let $T : V \longrightarrow W$ and $U : W \longrightarrow Z$ be linear. Then

a) Prove that if UT is one to one, then T is one to one. Must U also be one to one?

<u>Proof:</u>
If UT is injective, we have that $UT(x) = 0$ implies $x = 0$. Thus we have that $UT(x) = U(T(x)) = 0 \Longrightarrow T(x) = 0$ and we already know that $x = 0$. So T is injective. But U does not have to be injective. ■

b) Prove that if UT is onto, then U is onto. Must T also be onto?

<u>Proof:</u>
If UT is surjective, we have that for all $z \in Z$ there is a vector $x \in V$ such that $UT(x) = z$. Thus for all $z \in Z$ we have $z = U(T(x))$ and hence U is surjective. But T does not have to be surjective. ■

c) Prove that if U and T are one to one and onto, then UT is also onto.

<u>Proof:</u>
For all $z \in Z$, we can find $z = U(y)$ for some $y \in W$ since U is surjective, and then find $y = T(x)$ for some $x \in V$ since T is surjective as well. Thus we have $z = UT(x)$ for some x and hence UT is surjective. On the other hand, if $UT(x) = 0$, this means $T(x) = 0$ since U is injective and $x = 0$ since T is injective, hence UT is injective as well. ■

(16) Let V be a finite dimensional VS, and let $T : V \longrightarrow V$ be linear. Then

a) If rank T = rank T^2, prove that $R(T) \bigcap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.

<u>Proof:</u>
Let $T \in \mathcal{L}(V)$ and suppose rank T = rank T^2. We will show that $R(T) \bigcap N(T) = \{0\}$ and that $V = R(T) \oplus N(T)$.

First note that $V \xrightarrow{T} V \xrightarrow{T} V$. Thus, $T^2 = T \circ T : V \longrightarrow V$ is linear.
We will show that $N(T) = N(T^2)$.
By the Rank-Nullity Theorem,

$$\dim(V) = \text{ nullity } T + \text{rank } T \quad \text{and} \quad \dim(V) = \text{nullity } T^2 + \text{rank } T^2.$$

Setting both sides equal to each other and using the fact that rank T = rank T^2 yields nullity T = nullity T^2. Thus, $N(T) \cong N(T^2)$.
To show equality, we need to show

$$N(T) \subseteq N(T^2) \quad \text{and} \quad N(T^2) \subseteq N(T) .$$

We first show that $N(T) \subseteq N(T^2)$:
Let $x \in N(T)$. Then,

$$T(x) = 0 \Longrightarrow T^2(x) = T(T(x)) = T(0) = 0 .$$

Thus, $x \in N(T^2)$ and, in turn, $N(T) \subseteq N(T^2)$.
Recall from Thm 1.11 (Friedberg's) that, since $N(T) \subseteq N(T^2)$ and $\dim(N(T)) = \dim(N(T^2))$, we have that $N(T) = N(T^2)$. Thus we don't need to prove that $N(T^2) \subseteq N(T)$. ✓

We now prove that $R(T) \bigcap N(T) = \{0\}$. Let $w \in R(T) \bigcap N(T)$ such that $w \neq 0$. Then, $w \in R(T)$ and $w \in N(T)$ and . Since $w \in R(T)$, there exists $x \in V$ such that $T(x) = w$. Since $w \in N(T)$, $T(w) = 0 = T(T(x)) = T^2(x)$. Thus, $x \in N(T^2)$. Since $N(T) = N(T^2)$, $x \in N(T)$. Then, $T(x) = 0 = w$, which yields a contradiction. ($\Rightarrow\Leftarrow$) ✓

The only thing left to prove is that $V = R(T) + N(T)$. We use the following formula: Let W_1 and W_2 be subspaces of a vector space V.
Then,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \bigcap W_2).$$

Since $R(T)$ and $N(T)$ are subspaces of V , we plug these into the equation above:

$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \bigcap N(T)).$$

Applying the Rank-Nullity Theorem to the above equation reduces to

$$\dim(R(T) + N(T)) = \dim(V) .$$

Since $R(T) + N(T) \subseteq V$ (this is an old homework problem), by the same Thm 1.11, $V = R(T) + N(T)$. Thus, $V = R(T) \oplus N(T)$. ■

Section 2.4

(2) For each of the following linear transformations T, determine whether T is invertible and justify your answer.

a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$

<u>Solution:</u>

T cannot possibly be invertible because the domain and codomain don't have the same dimension.
✓

c) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3\, a_1 - 2\, a_3, a_2, 3\, a_1 + 4\, a_2)$

Solution:
T is invertible because it bijective. By choosing $a_1, a_2, a_3 = 0$ we can see that
$T(0, 0, 0) = (3 \cdot 0 - 2 \cdot 0, 0, 3 \cdot 0 + 4 \cdot 0) = (0, 0, 0) \Longrightarrow N(T) = \{0\}$, hence T injective. Also since rank $T = 3 = \dim(\mathbb{R}^3)$, we have that T is also surjective. ✓

e) $T : M_{2\times 2}(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2\, b\, x + (c + d)\, x^2$

Solution:
T cannot possibly be invertible because the domain and codomain don't have the same dimension.
✓