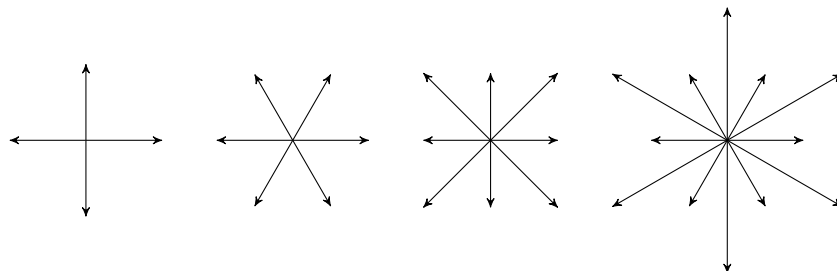


ROOT SYSTEMS

Root systems are very important objects, and encode many combinatorial properties that are used throughout mathematics. Since as humans we like to draw pictures to understand things, this workshop mainly focuses on the root systems inside \mathbb{R}^2 , and the purpose is to prove that the only such root systems are



It turns out that the above root systems correspond to the Dynkin diagrams $A_1 \times A_1$, A_2 , B_2 and G_2 from Workshop 1, although we will not prove this here.

Recall that a *euclidean space* E is a real vector space equipped with a positive definite symmetric bilinear form $(-, -)$ (the *inner product*). A subset R of E is called a *root system* if

(R1) R is finite, it spans E , and it does not contain 0.

(R2) If $\alpha \in R$, then the only scalar multiples of α also in R are $\pm\alpha$.

(R3) If $\alpha \in R$, then the reflection $s_\alpha : E \rightarrow E$ defined

$$s_\alpha(x) := x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha$$

permutes the elements of R .

(R4) If $\alpha, \beta \in R$, then $\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$. (Notice that $\langle -, - \rangle$ is only linear in the first slot!)

The standard example of a euclidean space is \mathbb{R}^n , with inner product (the so-called “dot product”)

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n x_i y_i.$$

The standard example of a root system comes from Lie algebras: if L is a semisimple Lie algebra, consider $E_{\mathbb{R}}$ (defined to be the \mathbb{R} -span of all the roots in H^*), equipped with the Killing form, together with the set R of roots.

Angles can be defined for any euclidean space E (motivated by \mathbb{R}^n !). Indeed, for $0 \neq u, v \in E$, the angle θ between u and v is defined to be

$$\cos \theta := \frac{(u, v)}{\sqrt{(u, u)}\sqrt{(v, v)}}.$$

Since we have angles, we will often just pretend that our euclidean space is \mathbb{R}^n , since this will allow us to draw pictures!

1. Suppose that $R \subset E$ is a root system. Just using the axioms of root systems:

(a) If $\alpha, \beta \in R$, deduce from the angle formula that

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3, 4\}.$$

(b) If $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$, deduce that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$.

In what follows, suppose that $R \subset E$ is a root system, and $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$. By changing the labels if necessary, we can further assume that $(\beta, \beta) \geq (\alpha, \alpha)$.

2. Deduce that $|\langle \beta, \alpha \rangle| \geq |\langle \alpha, \beta \rangle|$.
3. Use Q1 and Q2 to complete the following table of possibilities:

$\langle \beta, \alpha \rangle$	$\langle \alpha, \beta \rangle$	θ	$\frac{(\beta, \beta)}{(\alpha, \alpha)}$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1		
2			
-2			
3			
-3			

Given $\alpha, \beta \in R$, to classify root systems we need to know when $\alpha + \beta \in R$. The above table will be useful, and gives the following. As notation, let $\angle(\alpha, \beta)$ denote the angle between α and β .

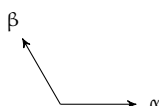
4. Using (R3) and the above table, prove that:
 - (a) If $\frac{\pi}{2} < \angle(\alpha, \beta) < \pi$, then $\alpha + \beta \in R$.
 - (b) If $0 < \angle(\alpha, \beta) < \frac{\pi}{2}$ and $(\beta, \beta) \geq (\alpha, \alpha)$, then $\alpha - \beta \in R$.

We now classify the root systems in \mathbb{R}^2 (with the standard inner product). For this, suppose R is a root system. We first *choose* a root α of smallest possible length. Since R spans \mathbb{R}^2 there must be another root, β , such that $\beta \neq \pm\alpha$. By considering $-\beta$ if necessary, we can assume that

$$\frac{\pi}{2} \leq \theta := \angle(\alpha, \beta) < \pi.$$

Moreover, by choosing a different β if necessary, we can further assume that this angle θ is as large as possible. From the table there are four options: $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$.

5. Consider the case $\theta = \frac{2\pi}{3}$. By the last column in the table, the length of β equals the length of α , so we begin



Further, since θ is as large as possible, there are no roots in the space between β and $-\alpha$. Using (R1) then Q4 (or (R3)), show that we obtain the second root system at the top of page 1.

6. Repeating this analysis, show that the cases $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ (in order) correspond to the root systems (in order) at the top of page 1.

Please hand in your solution to either Q5 or one of the parts of Q6 by the start of lecture on Monday 20 November.