MATH 752
OPTIONAL PROBLEM SET # 2

MARIO L. GUTIERREZ ABED

Problem 1. *Let $\mathbb{R}$ have the standard topology. Let*

$$p\colon \mathbb{R} \to \{a,b,c,d,e\} \qquad \textit{such that} \qquad x \mapsto \begin{cases} a & \textit{if } x > 2, \\ b & \textit{if } x = 2, \\ c & \textit{if } 0 \leq x < 2, \\ d & \textit{if } -1 < x < 0, \\ e & \textit{if } x \leq -1. \end{cases}$$

List the open sets in the quotient topology on $\{a,b,c,d,e\}$. (The quotient topology on $\{a,b,c,d,e\}$ is the collection of all subsets U of $\{a,b,c,d,e\}$ such that the inverse image $p^{-1}(U)$ is open in $\mathbb{R}$).

Proof. For every subset U of $\{a,b,c,d,e\}$, check whether the inverse image $p^{-1}(U)$ is open in $\mathbb{R}$. The open subsets of $\{a,b,c,d,e\}$ are exactly those subsets U for which $p^{-1}(U)$ is open. The open sets are $\emptyset$, $\{a\}$, $\{d\}$, $\{a,d\}$, $\{c,d\}$, $\{d,e\}$, $\{a,c,d\}$, $\{a,d,e\}$, $\{c,d,e\}$, $\{a,c,d,e\}$, $\{a,b,c,d\}$, and $\{a,b,c,d,e\}$. $\square$

Problem 2. *Define an equivalence relation on $\mathbb{R}^2$ by $(x_1,x_2) \sim (y_1,y_2)$ if $|x_1| = |y_1|$ and $|x_2| = |y_2|$. Equip the quotient space $\mathbb{R}^2/\sim$ with the quotient topology. Show that $\mathbb{R}^2/\sim$ is homeomorphic to the product $[0,\infty) \times [0,\infty)$.*

Proof. Let

$$g\colon \mathbb{R}^2 \to [0,\infty) \times [0,\infty) \qquad \text{such that} \qquad (x,y) \mapsto (|x|,|y|).$$

Then g is a continuous surjection. We show that g is an identification; it suffices to show that g is an open map. Let

$$h\colon \mathbb{R} \to [0,\infty) \qquad \text{such that} \qquad x \mapsto |x|.$$

Let (a,b) be an open interval in $\mathbb{R}$. If $a \geq 0$, then $h((a,b)) = (a,b)$ is open in $[0,\infty)$. If $b \leq 0$, then $h((a,b)) = (-b,-a)$ is open in $[0,\infty)$. If $a < 0 < b$, then $h((a,b)) = [0, \max\{|a|,|b|\}$ is again open in $[0,\infty)$. It thus follows that h is an open map and so is $g = h \times h$. The equivalence classes in $\mathbb{R}^2$ are exactly the inverse images $g^{-1}(u,v)$, where $(u,v) \in [0,\infty) \times [0,\infty)$. It follows then from a previous result that g induces a homeomorphism $f\colon \mathbb{R}^2/\sim \to [0,\infty) \times [0,\infty)$. $\square$

Problem 3. a) *Show that the sphere $\mathbb{S}^n$ is path connected for all $n \geq 1$.*
b) *Show that every contractible space is path connected.*

Proof of a). Let $x,y \in \mathbb{S}^n$. Assume that they are not antipodal points, i.e., that $y \neq -x$. Let $f\colon I \to \mathbb{R}^{n+1}$ such that $t \mapsto x + t(y-x)$. Then $L = f(I)$ is the line segment joining the points x and y. Since $y \neq x$, the segment does not go through the origin. Thus the map

$$\bar{f}\colon I \to \mathbb{S}^n \qquad \text{such that} \qquad t \mapsto \frac{f(t)}{\|f(t)\|}$$

is well defined. Clearly, $\bar{f}$ is continuous, $\bar{f}(0) = x$ and $\bar{f}(1) = y$, i.e., $\bar{f}$ is a path in $\mathbb{S}^n$ from x to y.

Assume then that $y = -x$. Let $z \in \mathbb{S}^n$, with $z \notin \{x, y\}$. Then there are paths in $\mathbb{S}^n$ from x to z and from z to y. Combining these paths gives a path from x to y. Thus $\mathbb{S}^n$ is path connected, as desired. □

Proof of b). Let X be a contractible space. Then there exist a point $x_0 \in X$ and a homotopy F such that $\mathbb{1}_X \simeq k$, where k is the constant map taking every point in X to x_0. Let $x_1 \in X$ and let $f\colon I \to X$ such that $t \mapsto F(x_1, t)$. Then $f(0) = F(x_1, 0) = x_1$ while $f(1) = F(x_1, 1) = x_0$. Thus f is a path from x_1 to x_0. Since x_1 was chosen arbitrarily, it follows that every point of X can be joined to x_0 by a path. Thus X is path connected. □

Problem 4. *a) Is a product of path connected spaces necessarily path connected?*
b) If $A \subset X$ and A is path connected, is the closure $\bar{A}$ necessarily path connected?
c) If $f\colon X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected?
d) If $\{A_\alpha\}$ is a collection of path connected subspaces of X and if $\cap A_\alpha \neq \emptyset$, is the union $\cup A_\alpha$ necessarily path connected?

Solution of a). A finite product of path connected spaces is always path connected. For example, assume that X and Y are path connected spaces. Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then there is a path $f_1\colon [a_1, b_1] \to X$ such that $f_1(a_1) = x_1$ and $f_1(b_1) = y_1$. Similarly, there is a path $f_2\colon [a_2, b_2] \to Y$ such that $f_2(a_2) = x_2$ and $f_2(b_2) = y_2$. Clearly, we may assume that $a_1 = a_2 = a$ and $b_1 = b_2 = b$. Then $(f_1, f_2)\colon [a, b] \to X \times Y$ that maps $t \mapsto (f_1(t), f_2(t))$, is a path from (x_1, x_2) to (y_1, y_2). Similarly, one can show that a finite product of path connected spaces X_i, $1 \le i \le n$, is path connected. (In fact, an infinite product of path connected spaces is path connected if it is equipped with the product topology. However, an infinite product of path connected spaces does not need to be path connected if it is equipped with the box topology.) □

Solution of b). No. The set $A = \{(x, \sin 1/x) \mid 0 < x < 1/(2\pi)\}$ is path connected but its closure is not path connected (the topologists's sine curve). □

Solution of c). Yes. Let $y_1, y_2 \in f(X)$, and let $x_1, x_2 \in X$ be such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is path connected, there is a path $\alpha\colon [a, b] \to X$ such that $\alpha(a) = x_1$ and $\alpha(b) = x_2$. Since f is continuous, the map $f \circ \alpha\colon [a, b] \to f(X)$ is continuous. Thus it is a path from y_1 to y_2. □

Solution of d). Yes. Let $x \in A_\beta$ and $y \in A_\gamma$. Let $w \in \cap A_\alpha$. Since A_β is path connected, there is a path $f\colon [a, b] \to A_\beta$ such that $f(a) = x$ and $f(b) = w$. Similarly, since A_γ is path connected, there is a path $g\colon [c, d] \to A_\gamma$ such that $g(c) = w$ and $g(d) = y$. Clearly, we may assume that $[a, b] = [0, 1]$ and that $[c, d] = [1, 2]$. Then $h\colon [0, 2] \to \cup A_\alpha$, where

$$h(t) = \begin{cases} f(t) & \text{if } 0 \le t \le 1, \\ g(t) & \text{if } 1 \le t \le 2 \end{cases}$$

is a path from x to y. □

Problem 5. *Let $f\colon X \to Y$ be a continuous map. Let $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ be the induced map. For each of the following claims, either prove it or give a counterexample:*

a) If f is surjective, then also $\pi_0(f)$ is surjective.

b) If f is injective, then also $\pi_0(f)$ is injective.

Solution of a). Yes. Let $C \in \pi_0(Y)$. Then C is a path component of Y. Since f is a surjection, there is $x \in X$ such that $f(x) \in C$. Thus $\pi_0(f)$ takes the path component of x to C. It follows that $\pi_0(f)$ is a surjection. $\square$

Solution of b). No. Consider the injection $\iota\colon \mathbb{Z} \to \mathbb{R}$ that takes $x \mapsto x$, where $\mathbb{R}$ has the standard topology and $\mathbb{Z}$ has the relative topology (i.e. subspace topology) from $\mathbb{R}$. Then ι is a continuous injection. Now, $\pi_0(\mathbb{Z}) = \mathbb{Z}$, while $\pi_0(\mathbb{R})$ has only one element. Thus $\pi_0(\iota)\colon \pi_0(\mathbb{Z}) \to \pi_0(\mathbb{R})$ cannot be an injection. $\square$