MATH 746 NOTES
HILBERT SPACES

MARIO L. GUTIERREZ ABED

Definition. *A set $\mathcal{H}$ is a **Hilbert space** if it satisfies the following:*

i) *$\mathcal{H}$ is a vector space over $\mathbb{C}$ (or $\mathbb{R}$).*

ii) *$\mathcal{H}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, so that*

- *$f \mapsto \langle f, g \rangle$ is linear on $\mathcal{H}$ for every $g \in \mathcal{H}$.*
- *$\langle f, g \rangle = \overline{\langle g, f \rangle}$.*
- *$\langle f, f \rangle \geq 0$ for all $f \in \mathcal{H}$.*

iii) *We let $\|f\| = \sqrt{\langle f, f \rangle}$. Then $\|f\| = 0$ if and only if $f = 0$.*

iv) *$\mathcal{H}$ is complete in the metric $d(f, g) = \|f - g\| = \sqrt{\langle f - g, f - g \rangle}$.* ★

Remark 1: Notice that the Cauchy-Schwarz and triangle inequalities

$$|\langle f, g \rangle| \leq \|f\| \|g\| \qquad \text{and} \qquad \|f + g\| \leq \|f\| + \|g\|$$

are in fact easy consequences of assumptions *(i)* and *(ii)* of our definition.

Remark 2: Notice that saying that $\mathcal{H}$ a *Hilbert space* is the same as saying that $\mathcal{H}$ is a *Banach space* (i.e. a *complete* normed linear space), with the norm induced by an inner product $\langle \cdot, \cdot \rangle$.

Definition. *If F is a function defined in the unit disc $\mathbb{D}$, we say that F has a **radial limit** at the point $-\pi \leq \theta \leq \pi$ on the circle, if the limit*

$$\lim_{\substack{r \to 1 \\ r < 1}} F(re^{i\theta})$$

exists. ★

Theorem 1. *A bounded holomorphic function $F(re^{i\theta})$ on the unit disc has radial limits at almost every θ.*

Definition. *The **Hardy space** $H^2(\mathbb{D})$ is the space that consist of all holomorphic functions F on the unit disc $\mathbb{D}$ that satisfy*

$$\sup_{0\leq r<1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \sup_{0\leq r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^2 \, d\theta < \infty,$$

where the a_n are the Fourier coefficients for $n \geq 0$ and 0 for $n < 0$.

We also define the "norm" for functions F in this class, $\|F\|_{H^2(\mathbb{D})}$, to be the square root of the above quantity. ★

Remark: Note that if F is bounded, then $F \in H^2(\mathbb{D})$, and moreover the conclusion of the existence of radial limits almost everywhere stated in the above theorem holds for any $F \in H^2(\mathbb{D})$.

Finally, we note that $F \in H^2(\mathbb{D})$ if and only if $F(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$; moreover, $\sum_{n=0}^{\infty} |a_n|^2 = \|F\|^2_{H^2(\mathbb{D})}$. This states in particular that $H^2(\mathbb{D})$ is in fact a Hilbert space that can be viewed as the "subspace" $\ell^2(\mathbb{Z}^+)$ of $\ell^2(\mathbb{Z})$, consisting of all $\{a_n\} \in \ell^2(\mathbb{Z})$, with $a_n = 0$ when $n < 0$.

<u>*Side Note:*</u> The **parallelogram law** states that in a Hilbert space $\mathcal{H}$, we have

$$\|A+B\|^2 + \|A-B\|^2 = 2[\|A\|^2 + \|B\|^2] \qquad \text{for all } A, B \in \mathcal{H}.$$

Definition. *Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces with respective inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}'}$ and the corresponding norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}'}$. A mapping $U\colon \mathcal{H} \longrightarrow \mathcal{H}'$ between these spaces is called **unitary** if:*

i) U is linear, i.e. $U(\alpha f + \beta g) = \alpha U(f) + \beta U(g)$.
ii) U is a bijection.
iii) $\|Uf\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$. ★

Definition. *If S is a subspace of a Hilbert space $\mathcal{H}$, we define the **orthogonal complement** of S by*

$$S^{\perp} = \{f \in \mathcal{H} \mid \langle f, g\rangle = 0 \quad \forall\, g \in S\}.$$

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Remark: Clearly, $S^\perp$ is also a subspace of $\mathcal{H}$, and moreover $S \cap S^\perp = \{0\}$. To see this, note that if $f \in S \cap S^\perp$, then f must be orthogonal to itself; thus $0 = \langle f, f \rangle = \|f\|$, and therefore $f = 0$. Moreover, $S^\perp$ is itself a closed subspace. Indeed, if $f_n \to f$, then $\langle f_n, g \rangle \to \langle f, g \rangle$ for every g by the Cauchy-Schwarz inequality. Hence if $\langle f_n, g \rangle = 0$ for all $g \in S$ and all n, then $\langle f, g \rangle = 0$ for all those g.

Proposition 1. *If S is a closed subspace of a Hilbert space $\mathcal{H}$, then*

$$\mathcal{H} = S \oplus S^\perp.$$

Remark: The notation in the proposition means that every $f \in \mathcal{H}$ can be written uniquely as $f = g + h$, where $g \in S$ and $h \in S^\perp$; we then say that $\mathcal{H}$ is the **direct sum** of S and $S^\perp$. This is equivalent to saying that any $f \in \mathcal{H}$ is the sum of two elements, one in S, the other in $S^\perp$, and that $S \cap S^\perp$ contains only 0.

With the decomposition $\mathcal{H} = S \oplus S^\perp$ one has the natural projection onto S defined by

$$P_s(f) = g, \qquad \text{where} \quad f = g + h \quad \text{and} \quad g \in S, h \in S^\perp.$$

The mapping P_s is called the **orthogonal projection** onto S and satisfies the following simple properties:

i) $f \mapsto P_s(f)$ is linear.

ii) $P_s(f) = f$ whenever $f \in S$.

iii) $P_s(f) = 0$ whenever $f \in S^\perp$.

iv) $\|P_s(f)\| \leq \|f\|$ for all $f \in \mathcal{H}$.

Property i) means that $P_s(\alpha f_1 + \beta f_2) = \alpha P_s(f_1) + \beta P_s(f_2)$, whenever $f_1, f_2 \in \mathcal{H}$ and α and β are scalars.

Now let us look at a very important result in the next example:

<u>*Example:*</u> Consider $L^2([-\pi, \pi])$ and let S denote the subspace that consists of all $F \in L^2([-\pi, \pi])$ with

$$F(\theta) \sim \sum_{n=0}^{\infty} a_n e^{in\theta}.$$

In other words, S is the space of square integrable functions whose Fourier coefficients a_n vanish for $n < 0$. From the proof of Fatou's theorem (see page 73, Stein's), this implies that S can be identified with the Hardy space $H^2(\mathbb{D})$, and so is a closed subspace unitarily isomorphic to $\ell^2(Z^+)$.

Therefore, using this identification, if P denotes the orthogonal projection from $L^2([-\pi,\pi])$ to S, we may also write $P(f)(z)$ for the element corresponding to $H^2(\mathbb{D})$, that is,

$$P(f)(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Now given $f \in L^2([-\pi,\pi])$, we define the **Cauchy integral** of f by

$$C(f)(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where γ denotes the unit circle and z belongs to the unit disc.

Then we have the identity

$$P(f)(z) = C(f)(z), \qquad \forall\, z \in \mathbb{D}.$$

Indeed, since $f \in L^2$ it follows by the Cauchy-Schwarz inequality that $f \in L^1([-\pi,\pi])$, and therefore we may interchange the sum and integral in the following calculation (recall $|z| < 1$):

$$\begin{aligned} P(f)(z) = \sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta \right) z^n \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \sum_{n=0}^{\infty} (e^{-i\theta} z)^n\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{1 - e^{-i\theta} z}\, d\theta \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{e^{i\theta} - z}\, i e^{i\theta}\, d\theta \\ &= C(f)(z). \end{aligned}$$

Linear Functionals and the Riesz Representation Theorem

It is a remarkable fact that every continuous linear functional on a Hilbert space arises as an inner product, as stated by the following theorem:

Theorem 2 (**Riesz representation theorem**). *Let ℓ be a continuous linear functional on a Hilbert space $\mathcal{H}$. Then, there exists a unique $g \in \mathcal{H}$ such that*

$$\ell(f) = \langle f, g \rangle \qquad \forall\, f \in \mathcal{H}.$$

Moreover, $\|\ell\| = \|g\|$.

Proof. Consider the subspace of $\mathcal{H}$ defined by

$$\mathcal{S} = \{f \in \mathcal{H} \mid \ell(f) = 0\}.$$

Since ℓ is continuous the subspace $\mathcal{S}$, which is called the nullspace of ℓ, is closed. If $\mathcal{S} = \mathcal{H}$, then $\ell = 0$ and we take $g = 0$. Otherwise $\mathcal{S}^{\perp}$ is non-trivial and we may pick any $h \in \mathcal{S}^{\perp}$ with $\|h\| = 1$. With this choice of h we determine g by setting $g = \overline{\ell(h)}h$. Thus if we let $u = \ell(f)h - \ell(h)f$, then $u \in \mathcal{S}$, and therefore $\langle u, h\rangle = 0$. Hence

$$0 = \langle \ell(f)h - \ell(h)f, h\rangle = \ell(f)\langle h, h\rangle - \langle f, \overline{\ell(h)}h\rangle.$$

Since $\langle h, h\rangle = 1$, we find that $\ell(f) = \langle f, g\rangle$ as desired. □

The first application of the Riesz representation theorem is to determine the existence of the "adjoint" of a linear transformation:

Proposition 2. *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded linear transformation. There exists a unique bounded linear transformation T^* on $\mathcal{H}$ so that*

- *i)* $\langle Tf, g\rangle = \langle f, T^*g\rangle$.
- *ii)* $\|T\| = \|T^*\|$.
- *iii)* $(T^*)^* = T$.

The linear operator $T^* : \mathcal{H} \longrightarrow \mathcal{H}$ satisfying the above conditions is called the ***adjoint*** *of T. In the special case when $T = T^*$ we say that T is* ***symmetric****.*