MATH 709 TAKE HOME EXAM

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Problem 1 (Problem 6-9)**.** *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map $F(x, y) = (e^y \cos x, e^y \sin x, e^{-y})$. For which positive numbers r is F transverse to the sphere $S_r(0) \subseteq \mathbb{R}^3$? For which positive numbers r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?*

Solution. Note that we have $\|F(x, y)\| = r$ only when

$$r^2 = e^{2y} \cos^2 x + e^{2y} \sin^2 x + e^{-2y} = e^{2y} + e^{-2y};$$

that is, when

$$y = \frac{1}{2} \log\left(\frac{1}{2}\left(r^2 \pm \sqrt{r^4 - 4}\right)\right).$$

Hence $\|F(x, y)\| \geq \sqrt{2}\ \forall (x, y) \in \mathbb{R}^2$, and therefore F is trivially transverse to S_r for $r \in [0, \sqrt{2})$ ($F^{-1}(S_r(0))$ is empty in this case). Now, for every $(x, y) \in \mathbb{R}^2$, we must have $F(x, y) \perp T_{F(x,y)}S_r$ if $F(x, y) \in S_r$. Then we compute the inner product of $F(x, y)$ with the columns of $DF(x, y)$:

$$(e^y \cos x, e^y \sin x, e^{-y})\cdot \begin{pmatrix} -e^y \sin x \\ e^y \cos x \\ 0 \end{pmatrix} = 0 \qquad \text{and} \qquad (e^y \cos x, e^y \sin x, e^{-y})\cdot \begin{pmatrix} e^y \cos x \\ e^y \sin x \\ -e^{-y} \end{pmatrix} = e^{2y} - e^{-2y}.$$

The second inner product is zero only when $y = 0$, which is the case only when $\|F(x, y)\| = \sqrt{2}$. Hence, since $T_{F(x,y)}S_r$ and $DF_{(x,y)}(T_{(x,y)}\mathbb{R}^2)$ span $T_{F(x,y)}\mathbb{R}^3$ only when $r \neq \sqrt{2}$, we conclude that F is transverse to S_r for $r \in [0, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

Now, by part a) of *Theorem 6.30*[1] from the text, we are guaranteed that $F^{-1}(S_r(0))$ is an embedded submanifold of $\mathbb{R}^2$ for $r \in [0, \sqrt{2}) \cup (\sqrt{2}, \infty)$; so we only need to check when $r = \sqrt{2}$. But note that in this case $F^{-1}(S_{\sqrt{2}}(0))$ is just a line in $\mathbb{R}^2$, so it is an embedded submanifold as well. ☐

Problem 2 (Problem 7-1)**.** *Show that for any Lie group G, the multiplication map $m\colon G \times G \to G$ is a smooth submersion. (Hint: use local sections.)*

Proof. Following the provided hint, note that for each $g \in G$, the map $\sigma_g\colon G \to G \times G$ given by $x \mapsto (g, g^{-1}x)$ (which is clearly well defined) is a smooth local section of m, since

$$m(\sigma_g(x)) = m((g, g^{-1}x)) = g(g^{-1}x) = (gg^{-1})x = ex = x.$$

Hence, since every point $(g_1, g_2) \in G \times G$ is in the image of σ_{g_1}, we know by *The Local Section Theorem*[2] that m is a smooth submersion. To see this, take any point $p = (g_1, g_2) \in G \times G$ and

[1] Here's the statement, for reference:
Suppose N and M are smooth manifolds and $S \subseteq M$ is an embedded submanifold. If $F\colon N \to M$ is a smooth map that is transverse to S, then $F^{-1}(S)$ is an embedded submanifold of N whose codimension is equal to the codimension of S in M.

[2] The theorem states that if M and N are smooth manifolds and $\pi\colon M \to N$ is a smooth map, then π is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.

let $\sigma_{g_1}\colon U \to G \times G$ be a smooth local section such that $\sigma_{g_1}(q) = p$. Here U is an open set of G containing q, where $q = m(\sigma_{g_1}(q)) = m(p) \in G$. Then,

$$m \circ \sigma_{g_1} = \mathrm{Id}_U \implies dm_p \circ d\sigma_{g_1}\big|_q = \mathrm{Id}_{T_q}G \implies dm_p \text{ is surjective.} \qquad \square$$

Problem 3 (Problem 7-2)**.** *Let G be a Lie group.*

a) *Let $m\colon G \times G \to G$ denote the multiplication map. Using Proposition 3.14[3] to identify $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$, show that the differential $dm_{(e,e)}\colon T_eG \oplus T_eG \to T_eG$ is given by $dm_{(e,e)}(X, Y) = X + Y$. (Hint: compute $dm_{(e,e)}(X, 0)$ and $dm_{(e,e)}(0, Y)$ separately.)*

b) *Let $i\colon G \to G$ denote the inversion map. Show that $di_e\colon T_eG \to T_eG$ is given by $di_e(X) = -X$.*

Proof of a). Consider the maps $\dot{m}, \ddot{m}\colon G \to G \times G$ given by $x \mapsto (x, e)$ and $y \mapsto (e, y)$, respectively. Note that $m \circ \dot{m} = m \circ \ddot{m} = \mathrm{Id}_G$. Thus,

$$\begin{aligned} dm_{(e,e)}(X, Y) &= dm_{(e,e)}(X, 0) + dm_{(e,e)}(0, Y) \\ &= d(m \circ \dot{m})_e(X) + d(m \circ \ddot{m})_e(Y) \\ &= d\,\mathrm{Id}_G\big|_e(X) + d\,\mathrm{Id}_G\big|_e(Y) \\ &= X + Y. \end{aligned} \qquad \square$$

Proof of b). Let $\varphi = m \circ (\mathrm{Id}_G \times i)\colon G \to G$. Note that $(m \circ (\mathrm{Id}_G \times i))(x) = m((x, x^{-1})) = xx^{-1} = e$. Therefore,

$$\begin{aligned} 0 &= d\varphi_e(X) \\ &= dm_{(e,e)} \circ d(\mathrm{Id}_G \times i)_e(X) \\ &= dm_{(e,e)} \circ (\mathrm{Id}_{T_eG} \times di_e)(X) \\ &= dm_{(e,e)}(\mathrm{Id}_{T_eG}(X), di_e(X)) \\ &= dm_{(e,e)}(X, di_e(X)) \\ &= X + di_e(X). && \text{(by part a)).} \end{aligned}$$

Hence it follows that $di_e(X) = -X$, as desired. $\square$

Problem 4 (Problem 7-6)**.** *Suppose G is a Lie group and U is any neighborhood of the identity. Show that there exists a neighborhood V of the identity such that $V \subseteq U$ and $gh^{-1} \in U$ whenever $g, h \in V$.*

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[3] Here's the proposition, for reference:

Proposition (The Tangent Space to a Product Manifold). *Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let $\pi_j\colon M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$ and tangent vector $\nu \in T_p(M_1 \times \cdots \times M_k)$, the map*

$$\alpha\colon T_p(M_1 \times \cdots \times M_k) \longrightarrow T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(\nu) = (d(\pi_1)_p(\nu), \ldots, d(\pi_k)_p(\nu))$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. Define $f\colon G \times G \to G$ by $f(g,h) = gh^{-1}$ and let $W = f^{-1}(U)$. By assumption $e \in U$, so $(e,e) \in W$ and there are neighborhoods W_1, W_2 of e such that $(e,e) \in W_1 \times W_2 \subseteq W$. Then $V = W_1 \cap W_2$ is the desired neighborhood of the identity. Note that by elementary set theory, we have $f(f^{-1}(U) \subseteq U$, so it holds that $W_1 \cap W_2 \subseteq U$ and $gh^{-1} \in U$ whenever $g, h \in W_1 \cap W_2$, as desired. $\square$

Problem 5. *Determine all of the Lie subgroups of the Lie group* $(\mathbb{R}^2, +)$.

Solution. Let us break down all of the Lie subgroups of $(\mathbb{R}^2, +)$ by dimensions:

<u>0 dimension</u>: For $\mathbb{Q}$ and $\mathbb{Z}$ with the discrete topology, we have $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{Z}^2, +)$, $(\mathbb{Q}^2, +)$, and $(\mathbb{Z} \times \mathbb{Q}, +)$ (similarly $(\mathbb{Q} \times \mathbb{Z}, +)$). Note that these groups are clearly subgroups of $(\mathbb{R}^2, +)$ (in the algebraic sense) and they are also immersed submanifolds of $\mathbb{R}^2$. in addition, they are countably infinite, so they are zero-dimensional (discrete) Lie groups.

<u>1 dimension</u>:

- If $\mathbb{Q}$ and $\mathbb{Z}$ are given the discrete topology, then $(\mathbb{R} \times \mathbb{Q}, +)$ and $(\mathbb{R} \times \mathbb{Z}, +)$ (similarly $(\mathbb{Q} \times \mathbb{R}, +)$ and $(\mathbb{Z} \times \mathbb{R}, +)$) are Lie subgroups of $(\mathbb{R}^2, +)$ having dimension 1.
- All lines through the origin.

<u>2 dimension</u>: All of the Lie subgroups of $(\mathbb{R}^2, +)$ of codimension 0 are exactly the open Lie subgroups of $(\mathbb{R}^2, +)$. But by *Lemma 7.12*[4], we have that these subgroups must also be closed. However, since $\mathbb{R}^2$ is connected, we have that $(\mathbb{R}^2, +)$ does not contain any proper nontrivial two-dimensional Lie subgroup.

I believe that these are (up to isomorphism) all of the Lie subgroups of $(\mathbb{R}^2, +)$. $\square$

[4] The lemma states that if G is a Lie group and $H \subseteq G$ is an open subgroup, then H is an embedded Lie subgroup. In addition, H is closed, so it is a union of connected components of G.