MATH 709 NOTES
SUBMERSIONS, IMMERSIONS, & EMBEDDINGS

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJIAN

Maps of Constant Rank

Definition. *Suppose M and N are smooth manifolds (with or without boundary). Given a smooth map $F\colon M \to N$ and a point $p \in M$, we define the* ***rank of*** F ***at*** p *to be the rank of the linear map $dF_p\colon T_pM \to T_{F(p)}N$; it is the rank of the Jacobian matrix of F in any smooth chart, or the dimension of $\operatorname{Im}(dF_p) \subseteq T_{F(p)}N$. If F has the same rank r at every point, we say that it has* ***constant rank****, and write* $\operatorname{rank} F = r$. ★

Remark: Because the rank of a linear map is never higher than the dimension of either its domain or its codomain, the rank of F at each point is bounded above by $\min\{\dim M, \dim N\}$. If the rank of dF_p is equal to this upper bound, we say that F ***has full rank at*** p, and if F has full rank everywhere, we say F ***has full rank***.

The most important constant-rank maps are those of full rank:

Definition. *A smooth map $F\colon M \to N$ is called a* ***smooth submersion*** *if its differential is surjective at each point (or equivalently, if* $\operatorname{rank} F = \dim N$*). It is called a* ***smooth immersion*** *if its differential is injective at each point (equivalently,* $\operatorname{rank} F = \dim M$*).* ★

Proposition 1. *Suppose $F\colon M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.*

Example 1. *a) Suppose $M_1, \dots, M_k$ are smooth manifolds. Then each of the projection maps $\pi_i\colon M_1 \times \cdots \times M_k \to M_i$ is a smooth submersion. In particular, the projection $\pi\colon \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a smooth submersion.*

b) If $\gamma\colon I \to M$ is a smooth curve in a smooth manifold M (with or without boundary), then γ is a smooth immersion iff $\gamma'(t) \neq 0$ for all $t \in I$.

c) If M is a smooth manifold and its tangent bundle TM is given the smooth manifold structure described in Proposition 3.18 (Pg 66, Lee's Smooth Manifolds), the projection $\pi\colon TM \to M$ is a smooth submersion. To verify this, just note that with respect to any smooth local coordinates (x^i) on an open subset $U \subseteq M$ and the corresponding natural coordinates (x^i, v^i) on $\pi^{-1}(U) \subseteq TM$ (see Proposition 3.18), the coordinate representation of π is $\widehat{\pi}(x, v) = x$.

d) The smooth map $X\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(u,v) = ((2+\cos 2\pi u)\cos 2\pi v, (2+\cos 2\pi u)\sin 2\pi v, \sin 2\pi u)$$

is a smooth immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is the doughnut-shaped surface obtained by revolving the circle $(y-2)^2+z^2=1$ in the (y,z)-plane about the z-axis (see Figure 1 *below).*

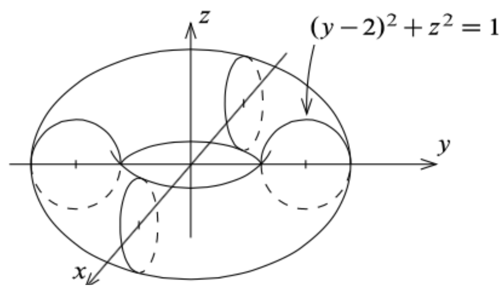

FIGURE 1. A torus of revolution in $\mathbb{R}^3$.

Definition. *If M and N are smooth manifolds (with or without boundary), a map $F\colon M \to N$ is called a **local diffeomorphism** if every point $p \in M$ has a neighborhood U such that $F(U)$ is open in N and $F|_U\colon U \to F(U)$ is a diffeomorphism.* ★

The next theorem is the key to the most important properties of local diffeomorphisms:

Theorem 1 (**Inverse Function Theorem for Manifolds**). *Suppose M and N are smooth manifolds (without boundary)* [1] *and $F\colon M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism.*

Proposition 2 (**Elementary Properties of Local Diffeomorphisms**). *We have the following properties for local diffeomorphisms:*

- *a) Every composition of local diffeomorphisms is a local diffeomorphism.*
- *b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.*
- *c) Every local diffeomorphism is a local homeomorphism and an open map.*
- *d) The restriction of a local diffeomorphism to an open submanifold (with or with out boundary) is a local diffeomorphism.*
- *e) Every diffeomorphism is a local diffeomorphism.*
- *f) Every bijective local diffeomorphism is a diffeomorphism.*
- *g) A map between smooth manifolds (with or without boundary) is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.*

Proposition 3. *Suppose M and N are smooth manifolds (without boundary), and $F\colon M \to N$ is a map. Then,*

[1] To see a case where the theorem fails for a map whose domain has nonempty boundary, see Problem 4-1, Pg 95, Lee's *Smooth Manifolds*.

a) *F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.*

b) *If $\dim M = \dim N$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.*

The most important fact about constant-rank maps is the following consequence of the inverse function theorem, which says that a constant-rank smooth map can be placed locally into a particularly simple canonical form by a change of coordinates. It is a nonlinear version of the canonical form theorem for linear maps (Theorem B.20, Pg 626, Lee's *Smooth Manifolds*):

Theorem 2 (**Rank Theorem**). *Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F\colon M \to N$ is a smooth map with constant rank r. For each $p \in M$, there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form*

$$\widehat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

In particular, if F is a smooth submersion, this becomes

$$\widehat{F}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n),$$

and if F is a smooth immersion, it is

$$\widehat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

The next corollary can be viewed as a more invariant statement of the rank theorem. It says that constant-rank maps are precisely the ones whose local behavior is the same as that of their differentials:

Corollary 1. *Let M and N be smooth manifolds, let $F\colon M \to N$ be a smooth map, and suppose M is connected. Then the following are equivalent:*

a) *For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.*

b) *F has constant rank.*

The rank theorem is a purely local statement. However, it has the following powerful global consequence.

Theorem 3 (**Global Rank Theorem**). *Let M and N be smooth manifolds and suppose $F\colon M \to N$ is a smooth map of constant rank. Then,*

a) *If F is surjective, it is a smooth submersion.*

b) *If F is injective, it is a smooth immersion.*

c) *If F is bijective, it is a diffeomorphism.*

Embeddings

One special kind of immersion is particularly important:

Definition. *If M and N are smooth manifolds (with or without boundary) a* ***smooth embedding of*** *M* ***into*** *N is a smooth immersion $F\colon M \to N$ that is also a topological embedding, i.e. a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.* $\bigstar$

Remark: Note that a smooth embedding is a map that is both a topological embedding and a smooth immersion, not just a topological embedding that happens to be smooth.

Example 2 (Smooth Embeddings). *a) If M is a smooth manifold (with or without boundary) and $U \subseteq M$ is an open submanifold, the inclusion map $U \hookrightarrow M$ is a smooth embedding.*

b) If $M_1, \dots, M_k$ are smooth manifolds and $p_i \in M_i$ are arbitrarily chosen points, each of the maps $\iota_j \colon M_j \to M_1 \times \cdots \times M_k$ given by

$$\iota_j(q) = (p_1, \dots, p_{j-1}, q, p_{j+1}, \dots, p_k)$$

is a smooth embedding. In particular, the inclusion map $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ given by

$$(x^1, \dots, x^n) \hookrightarrow (x^1, \dots, x^n, \underbrace{0, \dots, 0}_{k-n \text{ zeroes}})$$

is a smooth embedding.

To understand more fully what it means for a map to be a smooth embedding, it is useful to bear in mind some examples of injective smooth maps that are NOT smooth embeddings. The next three examples illustrate three rather different ways in which this can happen:

Example 3 (Smooth Topological Embedding). *The map $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^3, 0)$ is a smooth map and a topological embedding. However, since $\gamma'(0) = 0$, we have that γ is not a smooth immersion and thus not a smooth embedding.*

Example 4 (The Figure-Eight Curve). *Consider the curve $\beta \colon (-\pi, \pi) \to \mathbb{R}^2$ defined by*

$$\beta(t) = (\sin 2t, \sin t).$$

Its image is a set that looks like a figure-eight in the plane (see Figure 2*), sometimes called a* ***lemniscate****. (It is the locus of points (x, y) where $x^2 = 4y^2(1 - y^2)$, as you can check.) It is easy to*

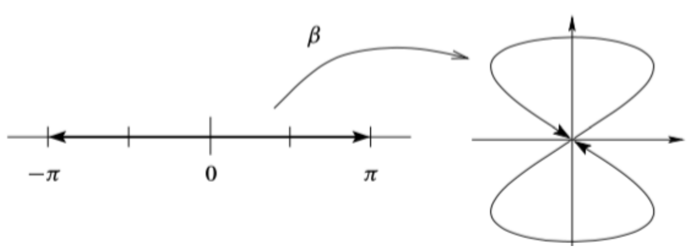

FIGURE 2. A figure-eight curve (lemniscate).

see that β is an injective smooth immersion because $\beta'(t)$ never vanishes; but it is not a topological embedding, because its image is compact in the subspace topology, while its domain (the open set $(-\pi, \pi)$) is not.

Example 5 (**A Dense Curve on the Torus**). *Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus, and let α be any irrational number. The map $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ given by*

$$\gamma(t) = \left(e^{2\pi i t}, e^{2\pi i \alpha t}\right)$$

is a smooth immersion because $\gamma'(t)$ never vanishes. It is also injective, because $\gamma(t_1) = \gamma(t_2)$ implies that both $t_1 - t_2$ and $\alpha t_1 - \alpha t_2$ are integers, which is impossible unless $t_1 = t_2$.

Consider the set $\gamma(\mathbb{Z}) = \{\gamma(n) \mid n \in \mathbb{Z}\}$. It follows from Dirichlet's approximation theorem (see below) that for every $\varepsilon > 0$, there are integers n, m such that $|\alpha n - m| < \varepsilon$. Using the fact that $|e^{it_1} - e^{it_2}| \leq |t_1 - t_2|$ for $t_1, t_2 \in \mathbb{R}$ (because the line segment from e^{it_1} to e^{it_2} is shorter than the circular arc of length $|t_1 - t_2|$), we have

$$|e^{2\pi i \alpha n} - 1| = |e^{2\pi i \alpha n} - e^{2\pi i m}| \leq |2\pi(\alpha n - m)| < 2\pi\varepsilon.$$

Therefore,

$$|\gamma(n) - \gamma(0)| = \left|\left(e^{2\pi i n}, e^{2\pi i \alpha n}\right) - (1,1)\right| = \left|\left(1, e^{2\pi i \alpha n}\right) - (1,1)\right| < 2\pi\varepsilon.$$

Thus, $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z})$. But this means that γ is not a homeomorphism onto its image, because $\mathbb{Z}$ has no limit point in $\mathbb{R}$. In fact, it is not hard to show that the image set $\gamma(\mathbb{R})$ is actually dense in $\mathbb{T}^2$ (see Problem 4-4, Pg 96, Lee's Smooth Manifolds*).*

The preceding example and Problem 4-4 depend on the following elementary result from number theory:

Lemma 1 (**Dirichlet's Approximation Theorem**). *Given $\alpha \in \mathbb{R}$ and any positive integer N, there exist integers n, m with $1 \leq n \leq N$ such that $|n\alpha - m| < 1/N$.*

The following proposition gives a few simple sufficient criteria for an injective immersion to be an embedding:

Proposition 4. *Suppose M and N are smooth manifolds (with or without boundary), and $F\colon M \to N$ is an injective smooth immersion. If any of the following holds, then F is a smooth embedding.*

a) F is an open or closed map.

b) F is a proper map.[2]

c) M is compact.

d) M has empty boundary and $\dim M = \dim N$.

Example 6. *Let $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map. We have previously shown that ι is smooth by computing its coordinate representation with respect to graph coordinates. It is easy to verify in the same coordinates that its differential is injective at each point, so it is an injective smooth immersion. Moreover, because $\mathbb{S}^n$ is compact, ι is a smooth embedding by part c) of the above proposition.*

Example 7. *We now give an example of a smooth embedding that is neither an open nor a closed map. Let $X = [0,1)$ and $Y = [-1,1]$, and let $f\colon X \to Y$ be the identity map on X. Then f is a smooth embedding, X is both open and closed (in X), but $f(X)$ is neither open nor closed (in Y).*

[2] Recall that if X and Y are topological spaces, a map $F\colon X \to Y$ (continuous or not) is said to be ***proper*** if for every compact set $K \subseteq Y$, the preimage $F^{-1}(K)$ is compact as well.

Theorem 4 (**Local Embedding Theorem**). *Suppose M and N are smooth manifolds (with or without boundary), and $F\colon M \to N$ is a smooth map. Then F is a smooth immersion iff every point in M has a neighborhood $U \subseteq M$ such that $F|_U\colon U \to N$ is a smooth embedding.*

Theorem 4 points the way to a notion of immersions that makes sense for arbitrary topological spaces:

Definition. *If X and Y are topological spaces, a continuous map $F\colon X \to Y$ is called a **topological immersion** if every point of X has a neighborhood U such that $F|_U$ is a topological embedding.* ★

Thus, every smooth immersion is a topological immersion; but, just as with embeddings, a topological immersion that happens to be smooth need not be a smooth immersion (see Example 3 above).

Submersions

Definition. *If $\pi\colon M \to N$ is any continuous map, a **section of** π is a continuous right inverse for π, i.e., a continuous map $\pi_S\colon N \to M$ such that $\pi \circ \pi_S = \mathrm{Id}_N$:*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \\ N \arrow[u, bend right, "\pi_S"'] \end{tikzcd}$$

*A **local section of** π is a continuous map $\pi_S\colon U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \pi_S = \mathrm{Id}_U$.* ★

Theorem 5 (**Local Section Theorem**). *Suppose M and N are smooth manifolds and $\pi\colon M \to N$ is a smooth map. Then π is a smooth submersion iff every point of M is in the image of a smooth local section of π.*

This theorem motivates the following definition:

Definition. *If $\pi\colon X \to Y$ is a continuous map, we say that π is a **topological submersion** if every point of X is in the image of a (continuous) local section of π. (The preceding theorem shows that every smooth submersion is a topological submersion).* ★

Example 8. *For an example of a smooth map that is a topological submersion but not a smooth submersion, consider $f(x) = x^3$ at $x = 0$.*

Proposition 5 (**Properties of Smooth Submersions**). *Let M and N be smooth manifolds, and suppose $\pi\colon M \to N$ is a smooth submersion. Then π is an open map, and if it is surjective it is a quotient map.*

The next three theorems provide important tools that we will use frequently when studying submersions. They demonstrate that surjective smooth submersions play a role in smooth manifold theory analogous to the role of quotient maps in topology:

Theorem 6 (**Characteristic Property of Surjective Smooth Submersions**)**.** *Suppose M and N are smooth manifolds, and $\pi\colon M \to N$ is a surjective smooth submersion. For any smooth manifold S (with or without boundary), a map $F\colon N \to S$ is smooth if and only if $F \circ \pi$ is smooth:*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F \circ \pi"] & \\ N \arrow[r, "F"'] & S \end{tikzcd}$$

Note (Side Note from Topology)**.** *Recall that if $\pi\colon X \to Y$ is a map, a subset $U \subseteq X$ is said to be **saturated with respect to** π if U is the entire preimage of its image under π, i.e., if $U = \pi^{-1}(\pi(U))$. Given $y \in Y$, the **fiber of** π **over** y is the set $\pi^{-1}(y)$. Thus, a subset of X is saturated iff it is a union of fibers.*

Theorem 7 (**Passing Smoothly to the Quotient**)**.** *Suppose M and N are smooth manifolds and $\pi\colon M \to N$ is a surjective smooth submersion. If S is a smooth manifold (with or without boundary) and $F\colon M \to S$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\widetilde{F}\colon N \to S$ such that $\widetilde{F} \circ \pi = F$:*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\widetilde{F}"'] & S \end{tikzcd}$$

Theorem 8 (**Uniqueness of Smooth Quotients**)**.** *Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ are surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F\colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:*

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "F"'] & & N_2 \end{tikzcd}$$

Smooth Covering Maps

Recall from topology the notion of covering maps:

Definition. *Suppose E and X are topological spaces. A map $\pi\colon E \to X$ is called a **covering map** if E and X are connected and locally path-connected, π is surjective and continuous, and each point $p \in X$ has a neighborhood U that is **evenly covered by** π, meaning that each component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π. In this case, X is called the **base of the covering**, and E is called a **covering space of** X. If U is an evenly covered subset of X, the components of $\pi^{-1}(U)$ are called the **sheets of the covering over** U.* ★

In the context of smooth manifolds, it is useful to introduce a slightly more restrictive type of covering map:

Definition. *If E and M are connected*[3] *smooth manifolds (with or without boundary), a map $\pi\colon E \to M$ is called a* ***smooth covering map*** *if π is smooth and surjective, and each point in M has a neighborhood U such that each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U by π. In this context we also say that U is* ***evenly covered by*** *π and that the space M is called the* ***base of the covering****, while E is called a* ***covering manifold of*** *M. If E is simply connected, it is called the* ***universal covering manifold of*** *M.* ★

Remark: To distinguish this new definition from the previous one, we often call an ordinary (not necessarily smooth) covering map a ***topological covering map***. A smooth covering map is, in particular, a topological covering map. But as with other types of maps we have studied in this chapter, a smooth covering map is more than just a topological covering map that happens to be smooth: the definition requires in addition that the restriction of π to each component of the preimage of an evenly covered set be a diffeomorphism, not just a smooth homeomorphism.

Proposition 6 (**Properties of Smooth Coverings**). *We have the following properties of smooth coverings:*

a) Every smooth covering map is a local diffeomorphism, a smooth submersion, an open map, and a quotient map.

b) An injective smooth covering map is a diffeomorphism.

c) A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

Remark: Because smooth covering maps are surjective smooth submersions, all of the previous results about smooth submersions can be applied to them. For example, Theorem 7 is a particularly useful tool for defining a smooth map out of the base of a covering space.

For smooth covering maps, the local section theorem can be strengthened:

Proposition 7 (**Local Section Theorem for Smooth Covering Maps**). *Suppose E and M are smooth manifolds (with or without boundary), and $\pi\colon E \to M$ is a smooth covering map. Given any evenly covered open subset $U \subseteq M$, any $q \in U$, and any p in the fiber of π over q, there exists a unique smooth local section $\pi_S\colon U \to E$ such that $\pi_S(q) = p$.*

Proposition 8 (**Covering Spaces of Smooth Manifolds**). *Suppose M is a connected smooth n-manifold, and $\pi\colon E \to M$ is a topological covering map. Then E is a topological n-manifold, and has a unique smooth structure such that π is a smooth covering map.*

Proposition 9 (**Covering Spaces of Smooth Manifolds with Boundary**). *Suppose M is a connected smooth n-manifold with boundary, and $\pi\colon E \to M$ is a topological covering map. Then E is a topological n-manifold with boundary such that $\partial E = \pi^{-1}(\partial M)$, and it has a unique smooth structure such that π is a smooth covering map.*

[3] Note that here we make no mention of path-connectedness, since, as you may well recall, connectedness $\Longrightarrow$ path-connectedness in manifolds.

Corollary 2 (**Existence of a Universal Covering Manifold**). *If M is a connected smooth manifold, there exists a simply connected smooth manifold $\widetilde{M}$, called the* ***universal covering manifold of*** *M, and a smooth covering map $\pi\colon \widetilde{M} \to M$. The universal covering manifold is unique in the following sense: if $\widetilde{M}'$ is any other simply connected smooth manifold that admits a smooth covering map $\pi'\colon \widetilde{M}' \to M$, then there exists a diffeomorphism $\Phi\colon \widetilde{M} \to \widetilde{M}'$ such that $\pi' \circ \Phi = \pi$.*

There are not many simple criteria for determining whether a given map is a smooth covering map, even if it is known to be a surjective local diffeomorphism. The following proposition gives one useful sufficient criterion. (It is not a necessary condition, however; see Problem 4-11, Pg 96, Lee's *Smooth Manifolds*.)

Proposition 10 (**Covering Spaces of Smooth Manifolds with Boundary**). *Suppose E and M are nonempty connected smooth manifolds (with or without boundary). If $\pi\colon E \to M$ is a proper local diffeomorphism, then π is a smooth covering map.*