The Initial Data Problem of Numerical Relativity

Bowen-York Initial Data

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Overview

Can't even trust black holes nowadays smh...

Introduction

Einstein Field Equations

1915 Relation between Geometry and Matter [Ein15]

$$\underbrace{G_{ab}}_{\text{Geometry}} = 8\pi \underbrace{T_{ab}}_{\text{Matter-Energy}}$$

1952 Can be posed as a Cauchy problem (with ***constraints***!) [Fou52]

- Not trivial. In GR space and time are on equal footing!

1969 Extension from local to “global” (MGHD) [CG69]

- Foliation is possible in ***globally hyperbolic*** spacetimes [strongest causality condition; physically relevant spacetimes must satisfy it]

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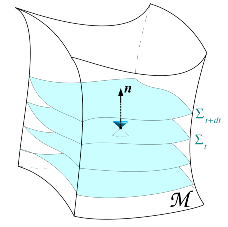
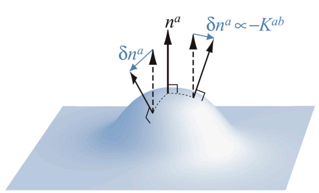
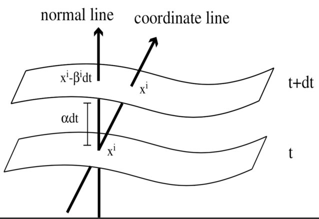
3+1 Approach to Numerical Relativity







ADM Equations

- **Evolution Equations:**

$$\boxed{\begin{aligned} \partial_t \gamma_{ij} &= 2D_{(i}\beta_{j)} - 2\alpha K_{ij} \\ \partial_t K_{ij} &= \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \\ &\quad - D_iD_j\alpha + \beta^k D_k K_{ij} + 2K_{k(j}D_{i)}\beta^k \end{aligned}}$$

- **Constraint Equations:**

$$\boxed{\begin{aligned} R + K^2 - K_{ij}K^{ij} &= 16\pi\rho \\ D_j\left(K^{ij} - \gamma^{ij}K\right) &= 8\pi S^i \end{aligned}}$$

Problems with ADM

- Straightforward and relatively easy...but practically useless
  - Numerical simulations violently unstable
  - Equations in this form are "weakly hyperbolic"
- "More hyperbolic" form needed
  - use conformal rescaling
  - messier, but effective

Initial Data

Initial Data Problem

Solving the constraints

$$\boxed{\begin{aligned} R + K^2 - K_{ij}K^{ij} &= 0 \\ D_j\left(K^{ij} - \gamma^{ij}K\right) &= 0 \end{aligned}}$$

is not so trivial.

- there's a total 12 DoF on the system ($\{\gamma_{ij}, K_{ij}\}$);
- the constraints are just 4 equations (thus removing 4 DoF);
- no a priori preference for which eight of the total data to use as ***free*** parameters and which four as ***constrained*** quantities.

York-Lichnerowicz Conformal Transformations

Re-scale the 3-metric γ_{ij} and the (traceless part of) the extrinsic curvature A_{ij}:

$$\boxed{\begin{aligned}\bar{\gamma}_{ij} &= \psi^{-4}\gamma_{ij}\\ \widetilde{A}_{ij} &= \psi^{2}A_{ij}\end{aligned}}$$

where

$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K.$$

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These conformal transformations lead to the ***constraints***

$$\boxed{\begin{aligned}\bar{D}^2\psi + \frac{1}{8}\left(\psi^{-7}\widetilde{A}_{ij}\widetilde{A}^{ij} - \psi\bar{R}\right) - \frac{1}{12}\psi^5K^2 &= 0\\ \bar{D}_j\widetilde{A}^{ij} - \frac{2}{3}\psi^6\bar{D}^iK &= 0\end{aligned}}$$

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Conformal Transverse-Traceless (CTT) method

$$\boxed{\widetilde{A}^{ij} = \widetilde{A}^{ij}_{\mathrm{L}} + \widetilde{A}^{ij}_{\mathrm{TT}}}$$

with the ***transverse-traceless*** (TT) part of $\widetilde{A}^{ij}$ satisfying

$$\bar{D}_j \widetilde{A}^{ij}_{\mathrm{TT}} = 0 \qquad \text{(Transverse)}$$

$$\bar{\gamma}_{ij} \widetilde{A}^{ij}_{\mathrm{TT}} = 0. \qquad \text{(Traceless)}$$

The ***longitudinal*** (L) part of $\widetilde{A}^{ij}$ is expressed in terms of the ***conformal Killing operator*** $(\bar{\mathbb{L}}X)^{ij}$:

$$\widetilde{A}^{ij}_{\mathrm{L}} \equiv (\bar{\mathbb{L}}X)^{ij} := 2\bar{D}^{(i}X^{j)} - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k X^k.$$

Relation between X^i and $\widetilde{A}^{ij}$ is given by the ***conformal vector Laplacian***

$$\bar{\Delta}_{\mathbb{L}} X^i \equiv \bar{D}_j(\bar{\mathbb{L}}X)^{ij} = \bar{D}_j \widetilde{A}^{ij}.$$

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Conformal Transverse-Traceless (CTT) method

- **CTT approach** (12 **DoF**)
  - `Constrained Data (4 DoF):` ψ, X^i
  - `Free Data (8 DoF):` $\widetilde{A}^{ij}_{\mathrm{TT}}, \bar{\gamma}_{ij}, K$

The physical solution is then constructed from

$$
\begin{aligned}
\gamma_{ij} &= \psi^4 \bar{\gamma}_{ij} \\
K_{ij} &= \psi^{-2} \widetilde{A}_{ij} + \frac{1}{3}\gamma_{ij} K
\end{aligned}
$$

The Model

Bowen-York Model

$$
\begin{aligned}
\widetilde{A}^{ij}_{\mathrm{TT}} &= 0 \\
K &= 0 && \text{(Maximal Slicing)} \\
\bar{\gamma}_{ij} &= f_{ij} \quad (= \delta_{ij}). && \text{(Conformal Flatness)}
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Two crucial solutions

$$
\begin{aligned}
X^i_{\vec{J}} &= \frac{1}{r^2}\left(\bar{\epsilon}^{ijk}\,\ell_j J_k\right) \\
X^i_{\vec{P}} &= -\frac{1}{4r}\left(7P^i + \ell^i \ell_j P^j\right).
\end{aligned}
$$

Bowen-York Model

With $X^i_{\vec{J}}$, $X^i_{\vec{P}}$ as above, plugging into

$$\widetilde{A}^{ij} = \widetilde{A}^{ij}_{\mathrm{L}} = (\bar{\mathbb{L}}\boldsymbol{X})^{ij} = 2\bar{D}^{(i}X^{j)} - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k X^k,$$

we end up with

$$\boxed{\begin{aligned} \widetilde{A}^{ij}_{\vec{J}} &= \frac{6}{r^3}\ell^{(i}\bar{\epsilon}^{j)kl}J_k\ell_l \\ \widetilde{A}^{ij}_{\vec{P}} &= \frac{3}{2r^2}\left[2P^{(i}\ell^{j)} + \ell_k P^k\left(\ell^i\ell^j - \delta^{ij}\right)\right] \end{aligned}}$$

Puncture Data

Using

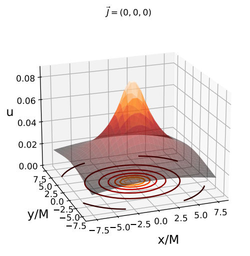
$$\psi = 1 + \frac{1}{\vartheta} + u, \qquad \text{where} \quad \frac{1}{\vartheta} \equiv \frac{M}{2r}$$

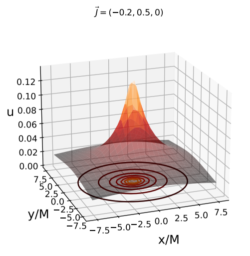
$$\varrho \equiv \frac{\vartheta^7}{8} \widetilde{A}_{ij} \widetilde{A}^{ij}$$

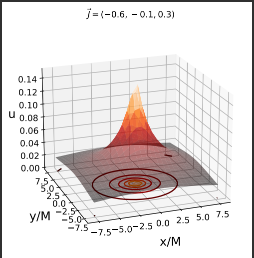
we end up with the final form of the Hamiltonian constraint:

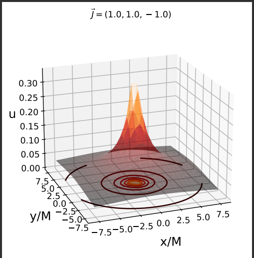
$$\boxed{\partial^2 u = -\varrho \left(\vartheta(1+u)+1\right)^{-7}}$$

Results


$\vec{J} = (0, 0, 0)$
u
x/M
y/M


$\vec{J} = (-0.2, 0.5, 0)$
u
x/M
y/M


$\vec{J} = (-0.6, -0.1, 0.3)$
u
x/M
y/M


$\vec{J} = (1.0, 1.0, -1.0)$
u
x/M
y/M

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THANK YOU!