Math 260 HW 1

Mario L. Gutierrez Abed

Section 1.2

(9) Prove the following two corollaries:

• Corollary 1:

The zero vector, $\vec{0}$, is unique in a vector space.

Proof:

Assume there exists another zero vector in V, denoted $\hat{0}$, that also satisfies (VS3). Namely, $x+\hat{0}=x$ for all $x\in V$, and $\hat{0}\neq\vec{0}$. Then we have

$$\begin{aligned} x+\hat{0} &= x = x+\vec{0} \\ x+\hat{0} &= x+\vec{0} \\ \hat{0} &= \vec{0} \quad \text{(by theorem 1.1)} \quad (\Rightarrow\Leftarrow) \end{aligned}$$

We have a contradiction, which means that the zero vector of a vector space is unique. ■

• Corollary 2:

Each vector in a vector space has a unique additive inverse.

Proof:

From (VS4) we know that for each element x in a vector space V, there exists an element y in V such that $x+y=0$. Now we assume that this element y, called the additive inverse of x, is not unique. Namely, there exists an element $\hat{y}$ in V such that $x+\hat{y}=0$, and $y\neq\hat{y}$. In this case we have

$$\begin{aligned} x+y &= 0 = x+\hat{y} \\ x+y &= x+\hat{y} \\ y &= \hat{y} \quad \text{(by theorem 1.1)} \quad (\Rightarrow\Leftarrow) \end{aligned}$$

We have a contradiction, which means that the additive inverse of each element $x\in V$ is unique. ■

(17) Let $V=\{(a_1,a_2): a_1, a_2\in\mathbb{F}\}$, where $\mathbb{F}$ is a field. Define addition of elements of V coordinate-wise, and for $c\in\mathbb{F}$ and $(a_1,a_2)\in V$, define

$$c(a_1,a_2) = (a_1, 0).$$

Is V a vector space over $\mathbb{F}$ with these operations? Justify your answer.

Solution:
V is not a vector space. We can easily see that, with the way scalar multiplication has been defined there is no multiplicative identity. In other words, by letting $c = 1$ we have
$1\,(a_1,\, a_2) \;=\; (a_1,\, 0) \;\neq\; (a_1,\, a_2)$. This violates one of the required properties of a vector space, hence we may conclude that V is not a vector space over $\mathbb{F}$. $\blacksquare$

(21) Let V and W be vector spaces over a field $\mathbb{F}$. Let

$$\mathcal{Z} = \{(v,\, w) : v \in V \text{ and } w \in W\}.$$

Prove that $\mathcal{Z}$ is a vector space over $\mathbb{F}$ with the operations

$$(v_1,\, w_1) \;+\; (v_2,\, w_2) \;=\; (v_1 + v_2,\, w_1 + w_2) \text{ and}$$
$$c(v_1,\, w_1) \;=\; (c\, v_1,\, c\, w_1).$$

Solution:
Let us test the conditions that $\mathcal{Z}$ must meet in order to be considered a vector space:

→ • Given two arbitrary vectors $(v_1,\, w_1),\, (v_2,\, w_2) \in \mathcal{Z}$, their sum is also in $\mathcal{Z}$:
$(v_1,\, w_1) \;+\; (v_2,\, w_2) \;=\; (v_1 + v_2,\, w_1 + w_2) \in \mathcal{Z}$ (closed under addition) ✓
• Given an arbitray vector $(v_1,\, w_1)$ and a scalar $c \in \mathbb{F}$, we have $c(v_1,\, w_1) \in \mathcal{Z}$, and
$c(v_1,\, w_1) = (c\, v_1,\, c\, w_1) \in \mathcal{Z}$ (closed under scalar multiplication) ✓

→ (VS1) $(v_1,\, w_1) \;+\; (v_2,\, w_2) \;=\; (v_1 + v_2,\, w_1 + w_2) \;=\; (v_2 + v_1,\, w_2 + w_1)$
$= \;(v_2,\, w_2) \;+\; (v_1,\, w_1)$ (commutativity of addition) ✓

→ (VS2) Given three arbitrary vectors $(v_1,\, w_1),\, (v_2,\, w_2),\, (v_3,\, w_3) \in \mathcal{Z}$, we need to prove that

$$(v_1,\, w_1) \;+\; ((v_2,\, w_2) \;+\; (v_3,\, w_3)) \;=\; ((v_1,\, w_1) \;+\; (v_2,\, w_2)) \;+\; (v_3,\, w_3)$$

First we start with the left hand side
$(v_1,\, w_1) \;+\; ((v_2 + v_3,\, w_2 + w_3)\,) = \;(v_1 +\; v_2 + v_3,\, w_1 + w_2 + w_3)$
Now the right hand side
$((v_1 + v_2,\, w_1 + w_2)) + \;(v_3,\, w_3) \;=\; (v_1 +\; v_2 + v_3,\, w_1 + w_2 + w_3)$ (associativity of addition) ✓

→ (VS3) $(v_1,\, w_1) \;+\; (0,\, 0) \;=\; (v_1 + 0,\, w_1 + 0) \;=\; (v_1,\, w_1)$ (existence of the additive identity) ✓

→ (VS4) $(v_1,\, w_1) \;+\; (-v_1,\, -w_1) \;=\; (v_1 + (-v_1),\, w_1 + (-w_1)) \;=\; (0,\, 0) \;=\; \vec{0}$
(existence of the additive inverse) ✓

→ (VS5) $1\,(v_1,\, w_1) \;=\; (1\, v_1,\, 1\, w_1) \;=\; (v_1,\, w_1)$ (existence of the multiplicative identity) ✓

→ (VS6) Given two arbitrary scalars $a,\ b \in \mathbb{F}$, we have
$(a\,b)\,(v_1,\ w_1)\ = (a)\,b(v_1,\ w_1)\ =\ a(b\,v_1,\ b\,w_1)$ (associativity of multiplication) ✓

→ (VS7) Given an arbitrary scalar $c \in \mathbb{F}$ and two arbitrary vectors
$(v_1,\ w_1)$, $(v_2,\ w_2) \in \mathcal{Z}$, we have
$c((v_1,\ w_1)\ +\ (v_2,\ w_2))\ =\ c(v_1 + v_2,\ w_1 + w_2)\ = (c(v_1 + v_2),\ c(w_1 + w_2))$
$= (c\,v_1 + c\,v_2,\ c\,w_1 + c\,w_2)\ =\ (c\,v_1,\ c\,w_1) + (c\,v_2,\ c\,w_2)\ =\ c(v_1,\ w_1)\ +\ c(v_2,\ w_2)$
(distributivity) ✓

→ (VS8) Given two arbitrary constants $a,\ b \in \mathbb{F}$ and any given vector $(v_1,\ w_1) \in \mathcal{Z}$, we have

$$\begin{aligned}(a+b)\,(v_1,\ w_1)\ &=\ ((a+b)\,v_1,\ (a+b)\,w_1)\ =\ (a\,v_1 + b\,v_1,\ a\,w_1 + b\,w_1)\\ &=\ (a\,v_1,\ a\,w_1)\ +\ (b\,v_1,\ b\,w_1)\ =\ a(v_1,\ w_1)\ +\ b(v_1,\ w_1)\end{aligned}$$

(distributivity) ✓

Since all the properties are satisfied, we can conclude that $\mathcal{Z}$ is a vector space over $\mathbb{F}$. ■