MATH 751 NOTES
COUNTABILITY & SEPARATION AXIOMS

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Countability Axioms

Definition. *A space X is said to have a* ***countable basis*** *at a point $x \in X$ if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the* ***first countability axiom****, or to be* ***first-countable****.* ★

Remark: Note that every metrizable space satisfies this axiom.

Theorem. *Let X be a topological space. Then,*

- *let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is first-countable.*
- *let $f\colon X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.*

Of greater importance than the first countability axiom is the following:

Definition. *If a space X has a countable basis for its topology, then X is said to satisfy the* ***second countability axiom****, or to be* ***second-countable****.* ★

Remark: Obviously, the second axiom implies the first: if $\mathcal{B}$ is a countable basis for the topology of X, then the subset of $\mathcal{B}$ consisting of those basis elements containing the point x is a countable basis at x. The second axiom is, in fact much stronger than the first; it is so strong that not even every metric space satisfies it.

Theorem. *Suppose that X has a countable basis.*

- *Every open covering of X contains a countable subcovering X. A space satisfying this condition is called a* ***Lindelöf space****.*
- *There exists a countable subset of X that is dense in X. In this case the space X is said to be* ***separable****.*

Remark: The two properties stated in the previous theorem are generally weaker than the second countability axiom, albeit they are equivalent when the space X is metrizable.

Separation Axioms

Definition 1. *Suppose that one-point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exists disjoint open sets containing x and B, respectively.* ★

Definition 2. *Suppose that one-point sets are closed in X. Then X is said to be **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.* ★

Remark: It is clear that a regular space is Hausdorff, and that a normal space is regular. However, note that we need to include the condition that one-point sets be closed as part of the definition of regularity and normality in order for this to be the case. To see why, take for instance a two-point space Y in the indiscrete topology. Then Y satisfies the other part of the definitions of regularity and normality, even though it is not Hausdorff. To see examples showing that regularity is stronger than Hausdorff, and normality is stronger than regularity, check examples 1-3 after *Theorem 31.2* below.

The three separation axioms are illustrated in *Figure 1* below:

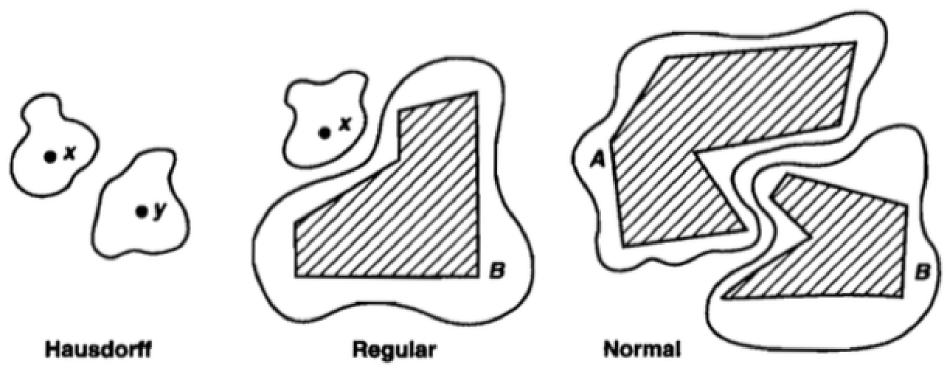


Figure 1. Normal $\Longrightarrow$ Regular $\Longrightarrow$ Hausdorff.

Remark: There are other ways to formulate the separation axioms. One formulation that is sometimes useful is given by the following lemma:

Lemma 1. *Let X be a topological space and let one-point sets in X be closed.*

a) X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.

b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof of a). ($\Rightarrow$) Suppose that X is regular, and suppose that the point x and the neighborhood U of x are given. Let $B = X \setminus U$, so that B is a closed set. By hypothesis, there exists disjoint open sets V and W containing x and B, respectively. The set $\overline{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V. Therefore, $\overline{V} \subset U$, as desired.

($\Leftarrow$) To prove the converse, suppose the point x and the closed set B not containing x are given. Let $U = X \setminus B$. By hypothesis, there is a neighborhood V of x such that $\overline{V} \subset U$. The open sets V and $X \setminus \overline{V}$ are disjoint open sets containing x and B, respectively. Thus, X is regular, as desired. □

Proof of b). This proof uses exactly the same argument as in part a). We only need to replace the point x by the set A throughout. □

Now we relate the separation axioms with the concepts previously introduced:

Theorem 31.2) We have the following properties:

a) A subspace of a Hausdorff space is Hausdorff. Also, a product of Hausdorff spaces is Hausdorff.

b) A subspace of a regular space is regular. Also, a product of regular spaces is regular.

Proof of a). Let X be Hausdorff. Let x and y be two points of the subspace Y of X. If U and V are disjoint neighborhoods in X of x and y, respectively, then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y in Y. Hence, a subspace of a Hausdorff space is Hausdorff.

Now let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $x = (x_\alpha)$ and $y = (y_\alpha)$ be distinct points of the product space $\prod X_\alpha$. Because we are assuming that $x \neq y$, there is some index β such that $x_\beta \neq y_\beta$. Choose disjoint open sets U and V in X_β containing x_β and y_β, respectively. Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ containing x and y, respectively. Thus, a product of Hausdorff spaces is Hausdorff, as desired. □

Proof of b). Let Y be a subspace of the regular space X. Then one-point sets are closed in Y. Let x be a point of Y and let B be a closed subset of Y disjoint from x. Now $\overline{B} \cap Y = B$, where $\overline{B}$ denotes the closure of B in X. Therefore $x \notin \overline{B}$, so using regularity of X, we can choose disjoint open sets U and V of X containing x and $\overline{B}$, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B, respectively. Hence, we have that a subspace of a regular space is regular.

Now let $\{X_\alpha\}$ be a family of regular spaces, and let $X = \prod X_\alpha$. By part a), X is Hausdorff, so that one-point sets are closed in X. We now use *Lemma 1* to prove regularity of X:
Let $x = (x_\alpha)$ be a point of X and let U be a neighborhood of x in X. Choose a basis element $\prod U_\alpha$ about x contained in U. Choose, for each α, a neighborhood V_α of x_α in X_α such that $\overline{V_\alpha} \subset U_\alpha$; if it happens that $U_\alpha = X_\alpha$, then choose $V_\alpha = X_\alpha$. Then $V = \prod V_\alpha$ is a neighborhood of x in X. Since $\bar{V} = \prod \overline{V_\alpha}$ by *Theorem 19.5*,[1] it follows at once that $\bar{V} \subset \prod U_\alpha \subset U$, so that X is regular. Thus, we have shown that a product of regular spaces is regular, and we are done. □

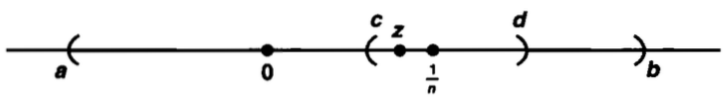
Remark: There is no analogous theorem for normal spaces, as we shall see shortly.

[1] Here's *Theorem 19.5*, for reference:
Theorem 19.5) Let $\{X_\alpha\}$ be an indexed family of spaces, and let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given by either the product or the box topology, then we have

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}.$$

Example 1: The space $\mathbb{R}_K$ is Hausdorff but not regular. It is Hausdorff because any two distinct points have disjoint open intervals containing them. However it is not regular, for the set K is closed in $\mathbb{R}_K$ and it does not contain the point $\{0\}$:

Suppose that there exist disjoint open sets U and V containing $\{0\}$ and K, respectively. Now choose a basis element containing $\{0\}$ and lying in U. It must be a basis element of the form $(a,b) \smallsetminus K$, since each basis element of the form (a,b) containing $\{0\}$ intersects K. Choose n large enough so that $1/n \in (a,b)$. Then choose a basis element about $1/n$ contained in V; it must be a basis element of the form (c,d). Finally, choose z such that $\max\{c, 1/(n+1)\} < z < 1/n$. Then z belongs to both U and V, so they are not disjoint. See the figure below:



Example 2: The space $\mathbb{R}_\ell$ is normal. It is immediate that one-point sets are closed in $\mathbb{R}_\ell$, since the topology of $\mathbb{R}_\ell$ is finer than that of $\mathbb{R}$. To check normality, suppose that A and B are disjoint closed sets in $\mathbb{R}_\ell$. For each point $a \in A$, choose a basis element $[a, x_a)$ not intersecting B; and for each point $b \in B$, choose a basis element $[b, x_b)$ not intersecting A. The open sets

$$U = \bigcup_{a\in A} [a, x_a) \qquad \text{and} \qquad V = \bigcup_{b \in B} [b, x_b)$$

are disjoint open sets of A and B, respectively.

Example 3: The *Sorgenfrey plane* $\mathbb{R}_\ell^2$ is not normal.
The space $\mathbb{R}_\ell$ is regular (in fact, normal), so the product space $\mathbb{R}_\ell^2$ is also regular. Thus this example serves two purposes: It shows that a regular space need not be normal, and it shows that the product of two normal spaces need not be normal:

We suppose that $\mathbb{R}_\ell^2$ is normal and then arrive at a contradiction. Let L be the subspace of $\mathbb{R}_\ell^2$ consisting of all points of the form $x \times (-x)$. Then L is closed in $\mathbb{R}_\ell^2$, and L has the discrete topology. Hence every subset A of L, being closed in L, is also closed in $\mathbb{R}_\ell^2$. Because $L \smallsetminus A$ is also closed in $\mathbb{R}_\ell^2$, this means that for every nonempty proper subset A of L, one can find disjoint open sets U_A and V_A containing A and $L \smallsetminus A$, respectively.

Let D denote the set of points of $\mathbb{R}_\ell^2$ having rational coordinates; it is dense in $\mathbb{R}_\ell^2$. We define a map θ that assigns, to each subset of the the line L, a subset of the set D, by setting

$$\begin{aligned} \theta(A) &= D \bigcap U_A \qquad \text{if } \emptyset \subsetneq A \subsetneq L. \\ \theta(\emptyset) &= \emptyset, \\ \theta(L) &= D. \end{aligned}$$

We now show that $\theta\colon \mathcal{P}(L) \to \mathcal{P}(D)$ is injective:

Let A be a proper nonempty subset of L. Then $\theta(A) = D \bigcap U_A$ is nether empty (since U_A is open and D is dense in $\mathbb{R}_\ell^2$) nor all of D (since $D \bigcap V_A$ is nonempty). It remains to show that if B is another proper nonempty subset of L, then $\theta(A) \neq \theta(B)$.
One of the sets A, B contains a point not in the other; suppose WLOG that $x \in A$ and $x \notin B$. Then $x \in L \smallsetminus B$, so that $x \in U_A \bigcap V_B$; since the latter set is open and nonempty, it must contain

points of D. These points belong to U_A and not to U_B; therefore, $D \bigcap U_A \neq D \bigcap U_B$, as desired. Thus, θ is injective.

Now we show that there exists an injective map $\varphi \colon \mathcal{P}(D) \to L$. Because D is countably infinite and L has the cardinality of $\mathbb{R}$, it suffices to define an injective map ψ of $\mathcal{P}(\mathbb{Z}^+)$ into $\mathbb{R}$. For that, we let ψ assign to the subset S of $\mathbb{Z}^+$ the infinite decimal $.a_1 a_2 \ldots$, where

$$a_i = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{if } i \notin S. \end{cases}$$

That is,

$$\psi(S) = \sum_{i=1}^{\infty} \frac{a_i}{10^i}.$$

Now the composite

$$\mathcal{P}(L) \xrightarrow{\theta} \mathcal{P}(D) \xrightarrow{\psi} L$$

is an injective map of $\mathcal{P}(L)$ into L. But *Theorem 7.8* [2] tells us that such a map does not exist! Thus we have reached the desired contradiction. ($\Rightarrow\Leftarrow$).

Normal Spaces

Theorem 32.1) Every regular space with a countable basis is normal.

Proof. Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. Each point $x \in A$ has a neighborhood U not intersecting B. Using regularity, choose a neighborhood V of x whose closure lies in U, and then choose an element of $\mathcal{B}$ containing x and contained in V. By choosing such a basis element for each $x \in A$, we construct a countable covering of A by open sets whose closures do not intersect B. Since this covering of A is countable, we can index it with the positive integers; let us denote it by $\{U_n\}$.

Similarly, choose a countable collection $\{V_n\}$ of open sets covering B, such that each set $\overline{V_n}$ is disjoint from A. The sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B, respectively, but they need not be disjoint. Thus, in order to construct two open sets that are disjoint, we perform the following simple trick:

Given n, define

$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i} \qquad \text{and} \qquad V_n' = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}.$$

Note that each set U_n' is open, being the difference of an open set U_n and a closed set $\bigcup_{i=1}^{n} \overline{V_i}$. Similarly, each set V_n' is open. It follows that the collection $\{U_n'\}$ covers A, because each $x \in A$ belongs to U_n for some n, and x belongs to none of the sets $\overline{V_i}$. Similarly, the collection $\{V_n'\}$ covers B. (See *Figure 2* below)

Finally, the open sets

$$U' = \bigcup_{n \in \mathbb{Z}^+} U_n' \qquad \text{and} \qquad V' = \bigcup_{n \in \mathbb{Z}^+} V_n'$$

[2]Here's *Theorem 7.8*, for reference:
Theorem 7.8) Let A be a set. There is no injective map $f \colon \mathcal{P}(A) \to A$, and there is no surjective map $g \colon A \to \mathcal{P}(A)$.

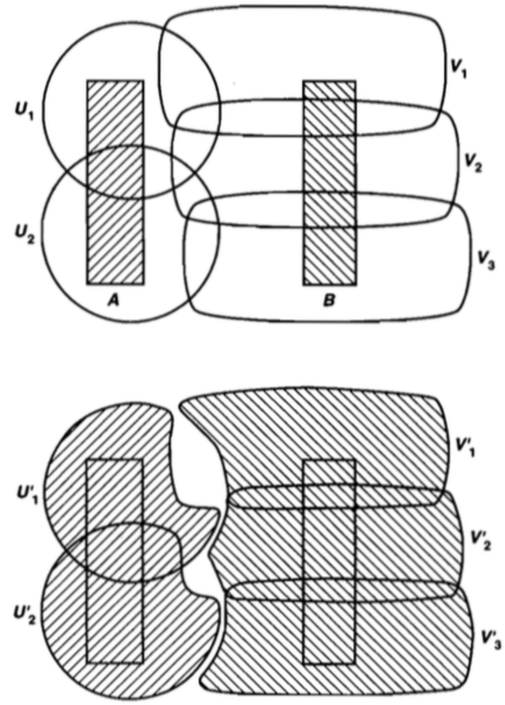


FIGURE 2.

are disjoint. For if $x \in U' \bigcap V'$, then $x \in U'_j \bigcap V'_k$ for some j and k. Suppose that $j \leq k$. It then follows from the definition of U'_j that $x \in U_j$; and since $j \leq k$, it follows from the definition of V'_k that $x \notin \overline{U_j}$. A similar contradiction arises if $j \geq k$. □

Theorem. *Every metrizable space is normal.*

Theorem. *Every compact Hausdorff space is normal.*

Theorem. *Every well-ordered set X is normal in the order topology.*

Remark: To see proofs of these three theorems, and to see related examples, see Pages 202 − 205, *Munkres's*.