1 Kinetics of Liquid Penetration

If you have ever spilled some water on your counter, and you just touch the puddle with the edge of a paper towel, you have seen the towel absorb the water rapidly. Capillary action pulls the water into the pores in the paper towel. A simple model of this basic phenomenon is based on the *Hagen-Poisseuille Equation* for the flow in a cylindrical pipe, and the *Young-Laplace Equation* for the pressure produced by surface tension at a curved interface.

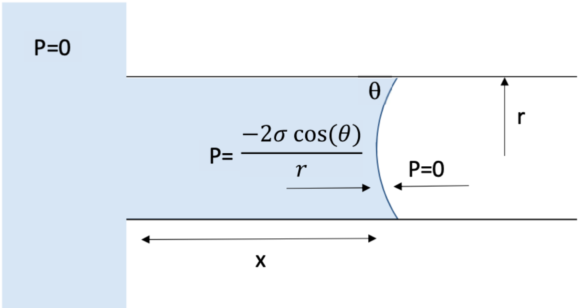

Figure 1: Liquid penetrating a distance x into a tube of radius r.

The simple model takes a porous medium to consist of a bunch of tubes, one of which is shown in the figure. Here we can see a slug of liquid having penetrated a distance x into a tube of radius r under the influence of the *Laplace Pressure*. The pressure in the reservoir at the left is 0, as is the pressure in the empty part of the tube to the right. Just inside the curved liquid interface the surface tension has produced a negative Laplace Pressure whose magnitude is

$$P = -\frac{2\sigma\cos\theta}{r}, \tag{1}$$

where σ is the surface tension at the interface and θ is the contact angle (as shown in the figure). The Hagen-Poisseuille equation tells us that this pressure will pull the liquid into the tube against the viscous drag according to this equation:

$$\frac{8\pi\mu x}{\pi r^2}\frac{dx}{dt} = \frac{2\sigma\cos\theta}{r}, \tag{2}$$

where μ is the viscosity of the fluid.

1.1 First Question

Find units for all of the parameters and variables and confirm that the dimensions of the two sides of the ODE are consistent. Moreover, do the units make physical sense?

Solution. We shall use SI units, where

$$\begin{aligned}
[x] = [r] &= m, \\
[t] &= s \\
[\sigma] &= N\cdot m^{-1} \\
[\mu] &= N\cdot s\cdot m^{-2}.
\end{aligned}$$

A quick dimensional analysis of our ODE (2) shows consistency:

$$\begin{aligned}
\frac{8\pi \overbrace{\mu}^{N\cdot s\cdot m^{-2}} \overbrace{x}^{m}}{\pi \underbrace{r^2}_{m^2}} \frac{\overbrace{dx}^{m}}{\underbrace{dt}_{s}} &= \frac{2\overbrace{\sigma}^{N\cdot m^{-1}}\cos\theta}{\underbrace{r}_{m}} \\
\frac{N\cdot s\cdot m}{m^4}\cdot\frac{m}{s} &= \frac{N}{m^2} \\
\frac{N}{m^2} &= \frac{N}{m^2}. \qquad \checkmark
\end{aligned}$$

The units do make physical sense: we have Newtons per squared meter on both sides, which is reassuring since these are SI units for pressure, and that is precisely what we are modeling with this ODE. □

1.2 Second Question

We can do a little algebra and simplify the ODE (2):

$$x\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sigma r \cos\theta}{4\mu}. \tag{3}$$

Solve this ODE. (Note that the right-hand side of this equation is a constant. (Also note that this solution is sometimes called the *Washburn Equation* after the person who first derived it). Again, you will find that you need a parameter that has not been provided. Please say what it is. Graph x as a function of t. How does x, and thus the graph, change if σ increases? If σ decreases? If μ increases? If μ decreases? If r increases? If r decreases? If θ increases? If θ decreases? Does this all make sense?

Solution. A straightforward separation of variables yields

$$\begin{aligned} \int x\mathrm{d}x &= \frac{\sigma r\cos\theta}{4\mu}\int \mathrm{d}t \\ \frac{x^2}{2} &= \frac{\sigma r t\cos\theta}{4\mu} + C \\ x &= \pm\sqrt{\frac{\sigma r t\cos\theta}{2\mu} + 2C}. \end{aligned}$$

The radius x cannot possibly be negative, so we have the only feasible solution to (3):

$$\boxed{x(t) = \sqrt{\frac{\sigma r t\cos\theta}{2\mu} + C}} \tag{4}$$

We also relabeled C as $2C$, but we can always do that since it is simply an undefined constant. It[1] gives us information about how far the liquid has already penetrated before we start our evolution ODE (i.e., it sets the initial distance $x(0)$). We now plot this solution and make sure the roles of the parameters make sense. Note that, mathematically, θ must be in the interval $[-\pi/2, \pi/2]$ in order for $\cos\theta$ to remain always nonnegative; this is a necessary requirement since time (t), radius of the tube (r), surface tension (σ), and viscosity (μ) are all nonnegative (on physical grounds). Moreover, $C \geq 0$ as well because at time $t = 0$ we must also have a nonegative root. With that being said, Fig. 1 makes it seem as though θ takes values on the interval $[0, \pi]$ (please do correct me if I'm wrong, but this what I can discern from the figure! It's a bit confusing to me in all honesty ...); of course, values in $(\pi/2, \pi]$ would make the cos term negative, which is not possible (as we explained above) unless we had negative surface tension σ, which is physically meaningless. Thus we instead take θ to be in $[-\pi/2, \pi/2]$.

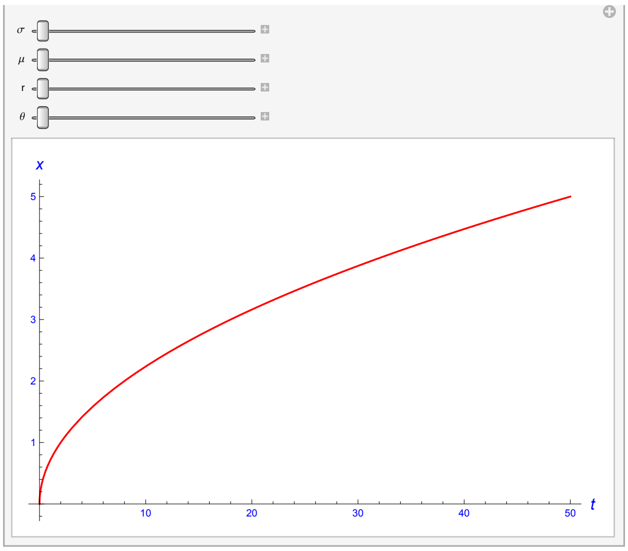

Figure 2: Graph of (4) (with $C = 0$). The sliders for the parameters are interactive in the accompanying `.cdf` file attached to the assignment.

As for the influence of the parameters on our solution, they behave –for the most part– just as you would expect from simply eyeballing Eq. (4) (this can be confirmed by moving the sliders on the accompanying `.cdf` file attached to this assignment): x

[1] or rather, $\sqrt{C}$, but again, we can always relabel.

is proportional to (the square root of) both σ and r, so (holding other parameters constant) a corresponding increase/decrease in either parameter brings about an increase/decrease in x. The opposite is true with μ and θ; this is made obvious by the fact that x is inversely proportional to (the square root of) μ, and as for θ it comes from the fact that an increasing angle means a smaller $\cos$ value (and thus in turn a smaller value for x, since $x \propto \sqrt{\cos\theta}$). This all makes sense, except for one big caveat: As r gets larger and larger we may very well end up with a contact angle that makes the $\cos$ term smaller and smaller, acting as a counter to the ever-expanding tube radius. We can deduce then, that there is a threshold for the tube radius beyond which the liquid stops penetrating. From a physical standpoint, the preceding discussion boils down to:

- a higher liquid viscosity μ would work to impede the liquid from penetrating further into the tube;
- a higher surface tension σ implies a higher pressure of the liquid pushing up against the tube, which is favorable for the liquid's penetration;
- the radius of the tube (r) and the angle of contact θ present a more complex mechanism, as they work against each other, effectively bringing balance to the model. □

1.3 Third Question

If, initially, $x = 0$; that is, if $x(0) = 0$, the ODE has an interesting quality. What is it? Does this singularity make sense physically?

Solution. If $x(0) = 0$, then from (3) we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sigma r \cos\theta}{0 \cdot 4\mu}, \tag{5}$$

which is undefined. This singularity makes no sense whatsoever because, as we see from Eq. (5), it would imply that the liquid rushes into the tube at an "infinitely fast" rate. □

1.4 Fourth Question

How do x and $\mathrm{d}x/\mathrm{d}t$ change as t increases? Does the slug of liquid speed up or slow down as it penetrates the tube? Does this make sense?

Solution. From (3) we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sigma r \cos\theta}{4x\mu}, \tag{6}$$

while from the solution (4),

$$x(t) \propto \sqrt{t}. \tag{7}$$

Combining these, we see that

$$\frac{\mathrm{d}x}{\mathrm{d}t} \propto \frac{1}{\sqrt{t}}. \tag{8}$$

Thus, as t increases, the rate of change of x, although positive, decreases; i.e., x keeps increasing (as we evidently saw in our solution (4)), but this increment gets slower and slower in time. This is telling us that the "acceleration" of the penetrating liquid is decreasing the more the liquid penetrates; we show this now:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) &\propto \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{t}}\right)\\ &= -\frac{1}{2\sqrt{t^3}}.\end{aligned}$$

This proves correct our suspicion of a decelerating liquid. Everything is consistent. □