MATH 709 HW # 1

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Problem 1 (Problem 1-1)**.** *Let X be the set of all points $z = (x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the* ***line with two origins****.)*

Proof. We start by showing that

$$M = X/^{(x,-1)\sim(x,1)}_{\forall\, x\neq 0} = (\mathbb{R} \smallsetminus \{0\}) \cup \{-1, 1\}$$

is locally Euclidean and second countable. Let

$$\begin{aligned} \mathcal{U}_1 &= \{(x, 1) \mid x \neq 0\} \cup (0, 1) \\ \mathcal{U}_2 &= \{(x, -1) \mid x \neq 0\} \cup (0, -1). \end{aligned}$$

But now both $\mathcal{U}_1$ and $\mathcal{U}_2$ are naturally homeomorphic to $\mathbb{R}$. To see this note that, for instance, in the case of $\mathcal{U}_1$, we can define a map $\varphi\colon \mathcal{U}_1 \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(z) = \begin{cases} x & z \in \{(x, 1) \mid x \neq 0\}, \\ 0 & z = (0, 1). \end{cases}$$

Note that this map is clearly a homeomorphism (you can check this!). Similarly, we can define such a map for $\mathcal{U}_2$ as well. Hence, since both $\mathcal{U}_1$ and $\mathcal{U}_2$ are homeomorphic to $\mathbb{R}$, they are locally Euclidean and second countable, and so is $M = \mathcal{U}_1 \cup \mathcal{U}_2$.

However M is not Hausdorff: for any two open sets U and V in M containing the two "origins" $(0, 1) \in U$ and $(0, -1) \in V$, the intersection $U \cap V$ is never empty. $\square$

Problem 2 (Problem 1-5)**.** *Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components. (Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.)*

Proof. ($\Rightarrow$) Let $x \in M$ be an arbitrary point and U_x be a corresponding neighborhood. We can assume that $U_x \cap U_y = \emptyset$ whenever $x \neq y$, since M is Hausdorff. Assume that M is paracompact so that, for any open cover χ of M, there exists an open refinement $\{S_i\}$ of χ such that each $U_x \subset M$ intersects with only finitely many S_i, i.e. $U_x \cap \{S_i\}$ is finite and open (being a finite intersection of open sets) for every $x \in M$. But then we are also assuming that M has countably many connected components $\{\mathcal{C}_\alpha\}_{\alpha \in A}$. Hence we define a collection $\{V_\alpha\}_{\alpha \in A}$ such that

$$V_\alpha = \bigcup_{\substack{x \in U_x \\ U_x \subseteq \mathcal{C}_\alpha}} \left(U_x \bigcap \{S_i\} \right).$$

Note that this collection $\{V_\alpha\}_{\alpha\in A}$ gives us a (countable) basis for M: Each V_α, being the arbitrary union of open sets, is open. For each $x \in M$, there is obviously at least one basis element V_α containing x (by construction). Moreover, we don't need to worry about intersections since, if $x \in V_\alpha \cap V_\beta$, then we must have that $\alpha = \beta$ (by construction). Thus, M is a second-countable space, as desired.

($\Leftarrow$) This direction is trivial, since if we assume that M locally Euclidean, Hausdorff, and second-countable, then M is a topological manifold by definition. Then by Theorem 1.15 from the text, we know that M is paracompact and from Proposition 1.11 part d), we have that M has countably many components. Notwithstanding, let's not be lazy and show some actual work instead of just calling on those propositions ☺:

First we show that M is paracompact. Since manifolds are locally compact, we can use the fact that second-countable, locally compact Hausdorff spaces admit an exhaustion by compact sets [1] (Proposition A.60 from the text). Now let χ and $\mathcal{B}$ be any arbitrary open cover and basis, respectively, for M, and let $(K_j)_{j=1}^{\infty}$ be an exhaustion of M by compact sets. For each j, let

$$V_j = K_{j+1} \smallsetminus \mathrm{Int}(K_j) \qquad \text{and} \qquad W_j = \mathrm{Int}(K_{j+2}) \smallsetminus K_{j-1},$$

(where we interpret K_j as $\emptyset$ if $j \le 1$). Then V_j is a compact set contained in the open subset W_j. Now, for each $x \in V_j$, there is some $U_x \in \chi$ containing x, and because $\mathcal{B}$ is a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U_x \cap W_j$. The collection of all such sets B_x as x ranges over V_j is an open cover of V_j, and thus has a finite subcover. The union of all such finite subcovers as j ranges over the positive integers is a countable open cover of M that refines χ. Because the finite subcover of V_j consists of sets contained in W_j, and $W_j \cap W_{j'} = \emptyset$ except when $j-2 \le j' \le j+2$, the resulting cover is locally finite. Thus we have shown that M is paracompact. In fact, we have demonstrated that given a topological manifold M, an open cover χ of M, and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite open refinement of χ consisting of elements of $\mathcal{B}$.

Lastly, to show that M has countably many components, note that (by a previous proposition) each component is open in M, so the collection of components is an open cover of M. Because M is assumed to be second-countable, this cover must have a countable subcover. But since the components are all disjoint, the cover must have been countable to begin with, which is to say that M has only countably many components, as desired. □

Problem 3 (Problem 1-6). *Let M be a nonempty topological manifold of dimension $n \ge 1$. If M has a smooth structure, show that it has uncountably many distinct ones. (Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.)*

Proof. Suppose M has a smooth structure $\mathcal{A}$ and fix a chart (U, φ) in $\mathcal{A}$. Assume WLOG that φ maps U onto the open unit ball $\mathbb{B}^n$. Let $p = \varphi^{-1}(0)$ and define, for $\alpha > 0$, $f_\alpha \colon \mathbb{B}^n \to \mathbb{B}^n$ and its inverse f_α^{-1} by

$$f_\alpha(x) = \frac{|x|^{1/\alpha}x}{|x|} \quad \text{and} \quad f_\alpha^{-1}(x) = \frac{|x|^{\alpha}x}{|x|},$$

[1] Recall that a sequence $(K_i)_{i=1}^{\infty}$ of compact subsets of a topological space X is called an **exhaustion of X by compact sets** if $X = \bigcup_i K_i$ and $K_i \subseteq \mathrm{Int}(K_{i+1})$ for each i.

so that both f_α and f_α^{-1} are everywhere continuous (i.e. f_α is a homeomorphism.) Also note that for $\alpha = 1$ this is just the identity, whereas for $\alpha > 1$ the map is not smooth at the origin. Hence f_α is a diffeomorphism if and only if $\alpha = 1$.

We now construct uncountably many smooth structures $\{\widehat{\mathcal{A}}_\alpha\}_{\alpha\in I}$. Let $\mathcal{A}_\alpha$ be an atlas containing the chart with coordinate map $f_\alpha \circ \varphi$. Observe that for any chart $(V, \psi) \in \mathcal{A}$, if $p = \varphi^{-1}(0) \notin V$ then ψ is smoothly compatible with $f_\alpha \circ \varphi$. With that in mind, we take a closed ball B about 0 in $\varphi(U)$. Then $\varphi^{-1}(B)$ is a closed set in M with p in its interior. Now for every chart $(V, \psi) \in \mathcal{A}$, let $\mathcal{A}_\alpha$ contain $\psi|_{(\varphi^{-1}(B))^c}$ (that is, we restrict all the other charts to the complement of our closed coordinate ball in M.) These maps $\psi|_{(\varphi^{-1}(B))^c}$ are then smoothly compatible with $f_\alpha \circ \varphi$ because they do not overlap at p. Thus $\mathcal{A}_\alpha$ is a smooth atlas, and it can be extended to a smooth structure. Let us call the induced smooth structure $\widehat{\mathcal{A}}_\alpha$. Thus we have uncountably many smooth structures $\{\widehat{\mathcal{A}}_\alpha\}_{\alpha\in I}$, one for each choice of α.

Now we show that our uncountably many smooth structures are distinct. Given any $\alpha, \beta \geq 1$, assume WLOG that $\alpha > \beta$. The charts with coordinate maps $f_\alpha \circ \varphi \in \mathcal{A}_\alpha$ and $f_\beta \circ \varphi \in \mathcal{A}_\beta$ have the transition map

$$f_\alpha \circ \varphi \circ (f_\beta \circ \varphi)^{-1} = f_\alpha \circ \varphi \circ \varphi^{-1} \circ f_\beta^{-1} = f_\alpha \circ f_\beta^{-1},$$

which is $x \mapsto (|x|^{\beta/\alpha} x)/|x|$. This map is not differentiable at 0, and both coordinate maps send p to 0, so they are not smoothly compatible and as a consequence they must induce distinct smooth structures. Hence all of the $\widehat{\mathcal{A}}_\alpha$ are distinct smooth structures, as desired. □

Problem 4 (Problem 1-7)**.** *Let N denote the* ***north pole*** $(0, \dots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$*, and let S denote the* ***south pole*** $(0, \dots, 0, -1)$*. Define the* ***stereographic projection*** $\sigma \colon \mathbb{S}^n \smallsetminus \{N\} \to \mathbb{R}^n$ *by*

$$\sigma(x^1, \dots, x^{n+1}) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}.$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \smallsetminus \{S\}$. Then,

- a) *For any $x \in \mathbb{S}^n \smallsetminus \{N\}$, show that $\sigma(x) = u$ where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (see* Figure 1 *below). Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\widetilde{\sigma}$ is called the* ***stereographic projection from the south pole****.)*

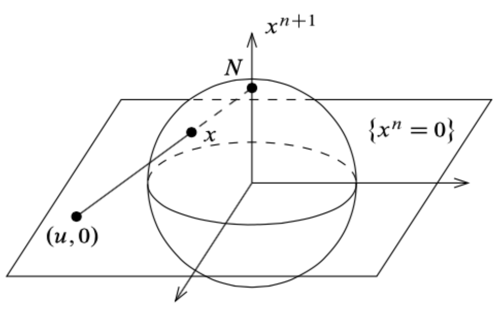

FIGURE 1. Stereographic projection

- b) *Show that σ is bijective, and*

$$(\clubsuit) \qquad \sigma^{-1}(u^1, \dots, u^n) = \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

c) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \smallsetminus \{N\}, \sigma)$ and $(\mathbb{S}^n \smallsetminus \{S\}, \widetilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called ***stereographic coordinates****.)*

d) Show that this smooth structure is the same as the ***standard smooth structure****, which is defined on Example 1.31, Page 20, Lee's Smooth Manifolds.*

Proof of a). The line through x and the north pole N is given by

$$\begin{aligned}\ell(t) &= (x - N)t + N \\ &= [(x^1, \ldots, x^{n+1}) - (0, \ldots, 0, 1)]t + (0, \ldots, 0, 1) \\ &= (x^1, \ldots, x^{n+1} - 1)t + (0, \ldots, 0, 1) \\ &= (x^1 \cdot t, \ldots, (x^{n+1} - 1) \cdot t + 1).\end{aligned}$$

The $(n+1)^{st}$ component $(x^{n+1} - 1) \cdot t + 1$ is equal to 0 when $t = t_0 = \frac{1}{1-x^{n+1}}$ so that

$$\ell(t_0) = \frac{1}{1 - x^{n+1}}(x^1, \ldots, x^n, 0) = \sigma(x).$$

Similarly the statement about $\widetilde{\sigma}$ follows in the same way. $\square$

Proof of b). By inspection we can see that σ is well defined on all of $\mathbb{S}^n \smallsetminus \{N\}$ as well as is σ^{-1} on all of $\mathbb{R}^n$. Hence showing that $\sigma \circ \sigma^{-1} = \mathrm{Id}_{\mathbb{R}^n}$ and $\sigma^{-1} \circ \sigma = \mathrm{Id}_{\mathbb{S}^n \smallsetminus \{N\}}$ is sufficient to establish that σ is a bijection and that the σ^{-1} given in ($\clubsuit$) is indeed its inverse. So here goes nothing:

$$\begin{aligned}\sigma \circ \sigma^{-1}(u) &= \sigma\left(\frac{2u^1}{|u|^2 + 1}, \ldots, \frac{2u^n}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1}\right) \\ &= \frac{\left(\frac{2u^1}{|u|^2+1}, \ldots, \frac{2u^n}{|u|^2+1}\right)}{1 - \frac{|u|^2-1}{|u|^2+1}} \\ &= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1 - |u|^2 + 1} \\ &= \frac{2(u^1, \ldots, u^n)}{2} \\ &= u \\ &= \mathrm{Id}_{\mathbb{R}^n}.\end{aligned}$$

But this holds for all $u \in \mathbb{R}^n$, thus $\sigma \circ \sigma^{-1} = \mathrm{Id}_{\mathbb{R}^n}$.

Similarly, we have

$$\begin{aligned}\sigma^{-1} \circ \sigma(x) &= \sigma^{-1}\left(\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}}\right) \\ &= \frac{\left(\frac{2x^1}{1-x^{n+1}}, \ldots, \frac{2x^n}{1-x^{n+1}}, \frac{(x^1)^2+\cdots+(x^n)^2}{(1-x^{n+1})^2} - 1\right)}{\frac{(x^1)^2+\cdots+(x^n)^2}{(1-x^{n+1})^2} + 1}\end{aligned}$$

But $x \in \mathbb{S}^n \smallsetminus \{N\}$, thus we can rewrite $(x^1)^2 + \cdots + (x^n)^2$ as $1 - (x^{n+1})^2$. So we continue the computation:

$$\begin{aligned}
\sigma^{-1} \circ \sigma(x) &= \frac{\left(2x^1, \ldots, 2x^n, \frac{1-(x^{n+1})^2-1+2x^{n+1}-(x^{n+1})^2}{1-x^{n+1}}\right)}{\frac{1-(x^{n+1})^2+1-2x^{n+1}+(x^{n+1})^2}{1-x^{n+1}}} \\
&= \frac{\left(2x^1, \ldots, 2x^n, \frac{2x^{n+1}(1-x^{n+1})}{1-x^{n+1}}\right)}{\frac{2(1-x^{n+1})}{1-x^{n+1}}} \\
&= \mathrm{Id}_{\mathbb{S}^n \smallsetminus \{N\}}(x).
\end{aligned}$$

Since this holds for all $x \in \mathbb{S}^n \smallsetminus \{N\}$ we may conclude that $\sigma^{-1} \circ \sigma = \mathrm{Id}_{\mathbb{S}^n \smallsetminus \{N\}}$. Therefore σ is bijective and has inverse σ^{-1} as defined on (♣), as desired. □

Proof of c). Let us compute $\widetilde{\sigma} \circ \sigma^{-1}$:

$$\begin{aligned}
\widetilde{\sigma} \circ \sigma^{-1}(u) &= \widetilde{\sigma}\left(\frac{2u^1}{|u|^2+1}, \ldots, \frac{2u^n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right) \\
&= \frac{\left(\frac{2u^1}{|u|^2+1}, \ldots, \frac{2u^n}{|u|^2+1}\right)}{1 + \frac{|u|^2-1}{|u|^2+1}} \\
&= \frac{u}{|u|^2},
\end{aligned}$$

which is a smooth rational function.

Now notice that

$$\widetilde{\sigma} \circ (-\sigma^{-1}(-x)) = -\sigma \circ \sigma^{-1}(-x) = -\mathrm{Id}_{\mathbb{R}^n}(-x) = -(-x) = x = \mathrm{Id}_{\mathbb{R}^n}(x)$$

for all $x \in \mathbb{R}^n$, so that $\widetilde{\sigma}^{-1}(x) = -\sigma^{-1}(-x)$. Therefore

$$\begin{aligned}
\sigma \circ \widetilde{\sigma}^{-1}(u) &= \sigma(-\sigma^{-1}(-u)) \\
&= -(-\sigma(-\sigma^{-1}(-u))) \\
&= -\widetilde{\sigma}(\sigma^{-1}(-u)) \\
&= -\left(\frac{-u}{|-u|^2}\right) \\
&= \frac{u}{|u|^2},
\end{aligned}$$

which is also a smooth rational function.

Finally, note that $\mathbb{S}^n = (\mathbb{S}^n \smallsetminus \{N\}) \bigcup (\mathbb{S}^n \smallsetminus \{S\})$. Thus the atlas $\mathcal{A} = \{(\mathbb{S}^n \smallsetminus \{N\}, \sigma), (\mathbb{S}^n \smallsetminus \{S\}, \widetilde{\sigma})\}$ defines a smooth structure on $\mathbb{S}^n$, as we set out to prove. □

Before I proceed to prove part d), I will derive the standard smooth structure for $\mathbb{S}^n$ for my own study purposes. The grader may want to skip this and jump right into the actual proof, which can be found below.

Preliminaries for proof of d). For each integer $n > 0$, the unit n-sphere $\mathbb{S}^n$ is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each index $i = 1, \ldots, n+1$ let U_i^+ denote the subset of $\mathbb{R}^{n+1}$ where the i^{th} coordinate is positive:

$$U_i^+ = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^i > 0\}.$$

Similarly, U_i^- is the set where $x^i < 0$ (see *Figure 2* below.)

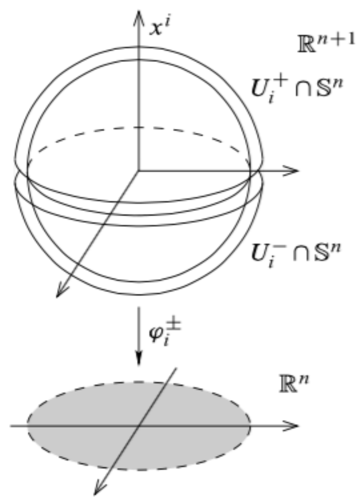

FIGURE 2. Charts for $\mathbb{S}^n$.

Now let $f\colon \mathbb{B}^n \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then for each $i = 1, \ldots, n+1$, it is easy to check that $U_i^+ \bigcap \mathbb{S}^n$ is the graph of the function

$$x^i = f(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}),$$

where the hat indicates that x^i is omitted. Similarly, $U_i^- \bigcap \mathbb{S}^n$ is the graph of

$$x^i = -f(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}).$$

Thus, each subset $U_i^\pm \bigcap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\varphi_i^\pm \colon U_i^\pm \bigcap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\varphi_i^\pm(x^1, \ldots, x^{n+1}) = (x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1})$$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n+2$ charts, $\mathbb{S}^n$ is a topological n-manifold.

Now we put a smooth structure on $\mathbb{S}^n$ as follows. For each $i = 1, \ldots, n+1$ let $(U_i^\pm, \varphi_i^\pm)$ denote the graph coordinate charts we constructed above. Then for any distinct indices i and j, the transition map $\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}$ can be easily computed. In the case when $i < j$, we get

$$\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}(u^1, \ldots, u^n) = (u^1, \ldots, \underbrace{\widehat{u^i}}_{i^{th}\text{ slot}}, \ldots, \underbrace{\pm\sqrt{1 - |u|^2}}_{j^{th}\text{ slot}}, \ldots, u^n),$$

and a similar formula holds when $i > j$. When $i = j$, an even simpler computation gives

$$\varphi_i^+ \circ (\varphi_i^-)^{-1} = \varphi_i^- \circ (\varphi_i^+)^{-1} = \mathrm{Id}_{\mathbb{B}^n}.$$

Thus, the collection of charts $\{(U_i^{\pm}, \varphi_i^{\pm})\}$ is a smooth atlas, and so defines a smooth structure on $\mathbb{S}^n$. We call this its ***standard smooth structure***. □

Proof of d). Now equipped with all the machinery from the above discussion, we can finally set out to prove that the standard smooth atlas on $\mathbb{S}^n$ $\mathcal{A}' = \{(U_i^{\pm}, \varphi_i^{\pm})\}$ is in fact the same as the atlas $\mathcal{A} = \{(\mathbb{S}^n \setminus \{N\}, \sigma), (\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma})\}$ defined on part c). But by a previous lemma we know that this is equivalent to showing that $\mathcal{A} \bigcup \mathcal{A}'$ is an atlas [2]. Since $\mathcal{A}$ and $\mathcal{A}'$ independently define smooth structures on $\mathbb{S}^n$, we know that $\mathcal{A} \bigcup \mathcal{A}'$ will cover $\mathbb{S}^n$ and that the transition functions from both $\mathcal{A}$ and $\mathcal{A}'$ are smooth. Thus it suffices to show that

$$\sigma \circ (\varphi_i^{\pm})^{-1}, \quad \widetilde{\sigma} \circ (\varphi_i^{\pm})^{-1}, \quad \varphi_i^{\pm} \circ \sigma^{-1}, \quad \text{and} \quad \varphi_i^{\pm} \circ \widetilde{\sigma}^{-1}$$

are smooth on their domain of definition. Thus let us roll up our sleeves and start computing:

$$\sigma \circ (\varphi_i^{\pm})^{-1}(u) = \begin{cases} \frac{u}{1 \mp \sqrt{1-|u|^2}} & \text{if } i = n, \\ \frac{\left(u^1, \ldots, u^{i-1}, \pm\sqrt{1-|u|^2}, \ldots, u^{n-1}\right)}{1-u^n} & \text{otherwise.} \end{cases}$$

$$\widetilde{\sigma} \circ (\varphi_i^{\pm})^{-1}(u) = \begin{cases} \frac{u}{1 \pm \sqrt{1-|u|^2}} & \text{if } i = n, \\ \frac{\left(u^1, \ldots, u^{i-1}, \pm\sqrt{1-|u|^2}, \ldots, u^{n-1}\right)}{1+u^n} & \text{otherwise.} \end{cases}$$

$$\varphi_i^{\pm} \circ \sigma^{-1} = \begin{cases} \frac{2u}{|u|^2+1} & \text{if } i = n+1, \\ \frac{(2u^1, \ldots, \widehat{2u^i}, \ldots, 2u^n, |u|^2-1)}{|u|^2+1} & \text{otherwise.} \end{cases}$$

$$\varphi_i^{\pm} \circ \widetilde{\sigma}^{-1} = \begin{cases} \frac{2u}{|u|^2+1} & \text{if } i = n+1, \\ \frac{(2u^1, \ldots, \widehat{2u^i}, \ldots, 2u^n, 1-|u|^2)}{|u|^2+1} & \text{otherwise.} \end{cases}$$

We can see that all of these transition maps are indeed smooth, hence $\mathcal{A}$ and $\mathcal{A}'$ determine the same smooth structure on $\mathbb{S}^n$, as desired. □

Problem 5 (Problem 1-8)**.** *By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An **angle function** on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.*

Proof. ($\Rightarrow$) We prove this direction by contradiction. Suppose that $U = \mathbb{S}^1$ and there exists a continuous $\theta\colon U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Then it is easy to check that θ is injective. Furthermore, since $\mathbb{S}^1$ is compact and $\mathbb{R}$ is Hausdorff, we must have that θ is actually a homeomorphism and $\theta(U)$ is compact and thus a closed bounded interval of $\mathbb{R}$. But θ cannot possibly be a homeomorphism since removing an interior point of $\theta(U) \subset \mathbb{R}$ gives a disconnected space while removing a point from $U = \mathbb{S}^1$ still results in a connected space. ($\Rightarrow\Leftarrow$)

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[2] Just to recall, the lemma states that two smooth atlases for M determine the same smooth structure iff their union is a smooth atlas.

($\Leftarrow$) Conversely, if $U \neq \mathbb{S}^1$, then there exists some $p \in \mathbb{S}^1 \smallsetminus U$. We can cut the complex plane through the origin and p and get a branch of a logarithmic function f which is continuous on U. Let $\theta(z) = -if(z)$ for all $z \in U$. Then we have

$$e^{i\theta(p)} = e^{f(z)} = z.$$

Lastly, we show that for any such angle function θ we have that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure. By problem 1-7 above, we know that the standard smooth structure on $\mathbb{S}^1$ is the same as that given by $\{(\mathbb{S}^1 \smallsetminus \{N\}, \sigma), (\mathbb{S}^1 \smallsetminus \{S\}, \widetilde{\sigma})\}$, where N and S are the north and the south pole, respectively, and σ and $\widetilde{\sigma}$ are their respective stereographic projections. Thus we proceed by rotating $\mathbb{R}^2$ appropriately, assuming that $N = (0, 1) \notin U$. Then let $\sigma\colon \mathbb{S}^1 \smallsetminus \{N\} \to \mathbb{R}$ be the stereographic projection given by $\sigma(x^1, x^2) = x^1/(1 - x^2)$. We can then compute

$$\sigma \circ \theta^{-1}(\alpha) = \frac{\cos\alpha}{1 - \sin\alpha},$$

which is a diffeomorphism on $\theta(U)$. Hence (U, θ) is a smooth coordinate chart, as desired. □

Problem 6 (Problem 1-10)**.** *Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \dots, e_k)$ and $(e_{k+1}, \dots, e_n)$, respectively, where e_i is the i^{th} standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed on Example 1.36 (Grassmann manifolds, Pages 22–24, Lee's Smooth Manifolds) is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\binom{I_k}{B}$, where I_k denotes the $k \times k$ identity matrix.*

Proof. Let $\mathcal{B} = \{e_1, \dots, e_k\}$. The matrix of $\varphi(S)$ represents the linear map $(\pi_Q|_S) \circ (\pi_P|_S)^{-1}$. Since $\pi_P|_S$ is an isomorphism, the vectors $(\pi_P|_S)^{-1}(\mathcal{B})$ form a basis for S. But then

$$\begin{aligned}\left(\mathrm{Id}_{\mathbb{R}^n}|_S \circ (\pi_P|_S)^{-1}\right)(\mathcal{B}) &= \left((\pi_P|_S + \pi_Q|_S) \circ (\pi_P|_S)^{-1}\right)(\mathcal{B}) \\ &= \left(\mathrm{Id}_P + \pi_Q|_S \circ (\pi_P|_S)^{-1}\right)(\mathcal{B})\end{aligned}$$

is a basis for S. This is precisely the desired result. □