Geometry of General Relativity
Workshop 4 Hand-In

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Problem (**WS4 Problem 6**). *Let $\bar{\nabla}$ and ∇ be two affine connections. In the previous workshop we showed that $H(X,Y) = \bar{\nabla}_X Y - \nabla_X Y$ defines a $(1,2)$ tensor field. Show that if ∇ is torsionless, then*

$$\bar{T}^a{}_{bc} = 2H^a{}_{[bc]} \tag{1}$$
$$\bar{R}^a{}_{bcd} = R^a{}_{bcd} + 2\nabla_{[c}H^a{}_{d]b} + 2H^e{}_{[d|b|}H^a{}_{c]e}\,, \tag{2}$$

where $\bar{T}$ and $\bar{R}$ are the torsion and Riemann curvature of $\bar{\nabla}$.

Proof. To show (1), we first note that

$$\begin{aligned} H^a{}_{bc} &= \langle f^a, H(e_b, e_c)\rangle \\ &= \langle f^a, \bar{\nabla}_b e_c - \nabla_b e_c\rangle \\ &= \langle f^a, \bar{\nabla}_b e_c\rangle - \langle f^a, \nabla_b e_c\rangle \\ &= \bar{\Gamma}^a_{cb} - \Gamma^a_{cb}. \end{aligned}$$

Thus,

$$\begin{aligned} 2H^a{}_{[bc]} &= 2\cdot\frac{1}{2}\left(\bar{\Gamma}^a_{cb} - \Gamma^a_{cb} - (\bar{\Gamma}^a_{bc} - \Gamma^a_{bc})\right) \\ &= \bar{\Gamma}^a_{cb} - \Gamma^a_{cb} - \bar{\Gamma}^a_{bc} + \Gamma^a_{bc} \\ &= \bar{\Gamma}^a_{cb} - \bar{\Gamma}^a_{bc}, \end{aligned}$$

where the last equality is due to the fact that, by assumption, ∇ is torsionless, and therefore $\Gamma^a_{bc} = \Gamma^a_{cb}$.

So we are left with $\bar{\Gamma}^a_{cb} - \bar{\Gamma}^a_{bc}$, which, as you may recall from equation (3.28) from our lecture notes, is precisely the components of a torsion tensor $\bar{T}^a{}_{bc}$, thus establishing (1). $\surd$

To prove (2), we are going to simplify our computations by using normal coordinates at a point p, where $\Gamma^a_{(bc)}(p) = 0$, combined with the fact that ∇ is torsionless, which yields $\Gamma^a_{bc}(p) = 0$ at this point (from now on we suppress the point p from our notation).

Now the Riemannian curvature of ∇ on the RHS reduces to

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc}, \tag{†}$$

while the last two terms of the RHS expand as

$$\begin{aligned} 2\nabla_{[c}H^a{}_{d]b} + 2H^e{}_{[d|b|}H^a{}_{c]e} &= 2\cdot\frac{1}{2}(\nabla_c H^a{}_{db} - \nabla_d H^a{}_{cb}) + 2\cdot\frac{1}{2}(H^e{}_{db}H^a{}_{ce} - H^e{}_{cb}H^a{}_{de}) \\ &= \nabla_c(\bar{\Gamma}^a_{bd} - \Gamma^a_{bd}) - \nabla_d(\bar{\Gamma}^a_{bc} - \Gamma^a_{bc}) + (\bar{\Gamma}^e_{bd} - \Gamma^e_{bd})(\bar{\Gamma}^a_{ec} - \Gamma^a_{ec}) - (\bar{\Gamma}^e_{bc} - \Gamma^e_{bc})(\bar{\Gamma}^a_{ed} - \Gamma^a_{ed}) \\ &= \partial_c\bar{\Gamma}^a_{bd} - \partial_c\Gamma^a_{bd} - \partial_d\bar{\Gamma}^a_{bc} + \partial_d\Gamma^a_{bc} + \bar{\Gamma}^e_{bd}\bar{\Gamma}^a_{ec} - \bar{\Gamma}^e_{bc}\bar{\Gamma}^a_{ed}, \end{aligned} \tag{††}$$

where on the last equation we used $\Gamma^a_{bc} = 0$ at the point p.

Now combining (†) with (††) yields (2). This is a tensor relation that must hold for any coordinates on any arbitrary point p, by the tensor transformation law, therefore we conclude our proof. $\boxed{\text{Victoria!}}$