Stokes's Theorem

Definition 1. *Let $\mathcal{M}$ be a smooth manifold and denote by $\Omega^k(\mathcal{M})$ the space of all smooth k-forms on $\mathcal{M}$. The **exterior derivative** is a map $\mathrm{d}\colon \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ that satisfies, for any k-form ω,*

$$\begin{aligned} \mathrm{d}\omega = \mathrm{d}(\omega_J \mathrm{d}x^J) &= \mathrm{d}\left(\omega_{j_1,\ldots,j_k}\, \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}\right) \\ &= \partial_i \omega_{j_1,\ldots,j_k} \mathrm{d}x^i \wedge \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}, \end{aligned} \tag{1}$$

where the above (Einstein) sum is done over a strictly-increasing sequence $J = j_1, \ldots, j_k$ with $j_i > j_{i-1}$.

Example 1. *If ω is a 1-form, (1) yields*[1]

$$\begin{aligned} \mathrm{d}\omega = \mathrm{d}(\omega_j \mathrm{d}x^j) &= \sum_{i,j} \partial_i \omega_j \,\mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \sum_{i<j} \partial_i \omega_j \,\mathrm{d}x^i \wedge \mathrm{d}x^j + \sum_{i>j} \partial_i \omega_j \underbrace{\mathrm{d}x^i \wedge \mathrm{d}x^j}_{=-\mathrm{d}x^j \wedge \mathrm{d}x^i} \\ &= \sum_{i<j} \left(\partial_i \omega_j - \partial_j \omega_i\right) \mathrm{d}x^i \wedge \mathrm{d}x^j. \end{aligned}$$

Theorem 1 (**Existence and Uniqueness of Exterior Differentiation**)**.** *Suppose $\mathcal{M}$ is a smooth manifold with or without boundary. There are unique operators $\mathrm{d}\colon \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ for all k, called **exterior differentiation**, satisfying the following four properties:*

- *d is linear over $\mathbb{R}$.*
- *If $\omega \in \Omega^k(\mathcal{M})$ and $\eta \in \Omega^\ell(\mathcal{M})$, then*
$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta. \tag{2}$$
- *For $f \in \Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$, $\mathrm{d}f$ is the differential of f, given by $\mathrm{d}f(X) = Xf$.*
- *$\mathrm{d} \circ \mathrm{d} \equiv 0$.*

In any smooth coordinate chart, d is given by (1).

Definition 2. *A k-form ω is said to be*

- ***exact** if ω is the exterior derivative of some $(k-1)$-form η, i.e., $\omega = \mathrm{d}\eta$;*
- ***closed** if the exterior derivative of ω vanishes, i.e., $\mathrm{d}\omega = 0$.*[2]

Note that, by the last property of Theorem 1, for any k-form θ we have $\mathrm{d}^2\theta = \mathrm{d}(\mathrm{d}\theta) = 0$; thus <u>every exact form is closed</u>. The question of whether every closed form is exact is answered by the *Poincaré Lemma*, which states that in a star-shaped (i.e., a contractible) domain every closed form is indeed exact (for general domains this result fails).

Theorem 2 (**Stokes's Theorem**)**.** *Let $\mathcal{M}$ be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on $\mathcal{M}$. Then*

$$\boxed{\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega} \tag{3}$$

[1] For clarity we use explicit (i.e., non-Einstein) summation, since in this case Einstein summation hides the fact that we're summing over a strictly-increasing sequence.

[2] Note that due to the result from Example 1, one often finds in the literature that a 1-form is closed if it satisfies $\partial_i \omega_j = \partial_j \omega_i$.

Example 2. *Let $\mathcal{M}$ be a smooth manifold and suppose $\gamma\colon [a,b] \to \mathcal{M}$ is a smooth embedding, so that the image $S = \gamma([a,b])$ is an embedded 1-submanifold with boundary in $\mathcal{M}$. If we give S the orientation such that γ is orientation-preserving, then for any smooth function $f \in C^\infty(\mathcal{M})$, Stokes's theorem says that*

$$\int_\gamma \mathrm{d}f = \int_{[a,b]} \gamma^* \mathrm{d}f = \int_S \mathrm{d}f = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a)). \tag{4}$$

The following corollaries are straightforward consequences of Stokes's Theorem:

Corollary 1 (Integrals of Exact Forms). *If $\mathcal{M}$ is a compact oriented smooth manifold without boundary, then the integral of every exact form over $\mathcal{M}$ vanishes:*

$$\int_{\mathcal{M}} \mathrm{d}\omega = 0 \quad \textit{if } \partial\mathcal{M} = \emptyset. \tag{5}$$

Corollary 2 (Integrals of Closed Forms over Boundaries). *Suppose $\mathcal{M}$ is a compact oriented smooth manifold with boundary. If ω is a closed form on $\mathcal{M}$, then the integral of ω over $\partial\mathcal{M}$ vanishes:*

$$\int_{\partial\mathcal{M}} \omega = 0 \quad \textit{if } \mathrm{d}\omega = 0 \textit{ on } \mathcal{M}. \tag{6}$$

Equation (3) showcases the elegance of differential forms; we illustrate this elegance/usefulness further in the following discussion:

The Euclidean metric ${}^E g_{ij}$ on $\mathbb{R}^3$ yields an index-lowering isomorphism $\flat\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ (usually called the *flat isomorphism* in the math literature) given by $\flat(X^j) = {}^E g_{ij} X^j = X_i$ for any vector field $X^i \in \mathfrak{X}(\mathbb{R}^3)$.[3] Just as exterior differentiation *increases* the rank of the differential form by one, there is another important operation on differential forms that *decreases* the rank by one, namely the ***interior multiplication*** $i_v\colon \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ by a vector field $\boldsymbol{v} \equiv v^i$; this operation is given by

$$i_v\omega(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k-1}) = \omega(\boldsymbol{v}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k-1}), \tag{7}$$

where $\omega \in \Omega^k(\mathcal{M})$ and $\boldsymbol{v}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k-1} \in \mathfrak{X}(\mathcal{M})$. In other words, $i_v\omega$ is obtained from ω by inserting $\boldsymbol{v}$ into the first slot.[4] We use this interior multiplication to construct another map $\beta\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ given by

$$\beta(\boldsymbol{X}) = i_{\boldsymbol{X}}(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3). \tag{8}$$

Lastly, we define another smooth bundle isomorphism $*\colon C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ by

$$*\,(f) = f\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{9}$$

The relationships amongst all of these operators and how they relate to d are summarized in the following commutative diagram:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{Id}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^3(\mathbb{R}^3).
\end{tikzcd}$$

[3] Of course there is also an an index-raising isomorphism $\sharp\colon \Omega^1(\mathbb{R}^3) \to \mathfrak{X}(\mathbb{R}^3)$ (usually called the *sharp isomorphism*) given by $\sharp(X_j) = {}^E g^{ij} X_j = X^i$. These two isomorphisms are called the *musical isomorphisms* in the math literature.

[4] By convention, we interpret $i_v\omega$ to be zero when ω is a 0-covector (i.e., a number).

The language of elementary vector calculus requires three different operators (grad, curl, and div) to represent operations that merely require one operator (d) in the language of differential forms. [5] This is illustrated in the vector calculus versions of Stokes's Theorem: For some smooth vector field $\boldsymbol{A} = A^i \in \mathfrak{X}(\mathbb{R}^3)$,

- for $n = 2$ we have
$$\int_{\mathcal{V}_2} \operatorname{curl}(\boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{\Sigma} = \int_{\partial\mathcal{V}_2} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}, \tag{10}$$
where $\mathcal{V}_2 \subseteq \mathbb{R}^2$ is a compact bounded region, $\partial\mathcal{V}_2$ is the 1-dimensional closed curve that bounds it, and the last integral is a line integral around that curve. Also, $\mathrm{d}\boldsymbol{\Sigma}$ is the infinitesimal vectorial surface area on $\mathcal{V}_2$.

- for $n = 3$ (this case is usually referred to in the physics literature as *Gauss's Theorem*) we have
$$\int_{\mathcal{V}_3} \operatorname{div}(\boldsymbol{A})\, \mathrm{d}V = \int_{\partial\mathcal{V}_3} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\Sigma}, \tag{11}$$
where $\mathrm{d}V = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ (usually just written $\mathrm{d}V = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$) is the volume element, $\mathcal{V}_3 \subseteq \mathbb{R}^3$ is a compact bounded region, $\partial\mathcal{V}_3$ is its closed 2-dimensional boundary surface, and $\mathrm{d}\boldsymbol{\Sigma}$ is the infinitesimal vectorial surface area on $\partial\mathcal{V}_3$.

We now derive (10) (and leave (11) as a trivial, muscle-flexing exercise to the reader) from the more succinct and elegant (3). Before doing so, we need to point out a nuisance that is usually encountered in both physics and elementary vector calculus texts. In these references it is quite common to ignore the difference between vectors and covectors (1-forms) thereby paying no heed to the placement of indices (up or down). This is partly justified by the fact that in Cartesian coordinates the Euclidean metric leaves intact the components of vectors and covectors, so the musical isomorphisms presented earlier do not have any effect whatsoever on these components. Despite this equality of components in the Euclidean case, such index-placement-agnostic behavior can be a slippery slope, and we do not encourage it. For instance, the infinitesimal vectorial surface area $\mathrm{d}\boldsymbol{\Sigma}$ can be written $\mathrm{d}\boldsymbol{\Sigma} = \boldsymbol{n}\mathrm{d}\Sigma = \boldsymbol{n}\mathrm{d}x^1\mathrm{d}x^2$, where "$\boldsymbol{n}$ is the unit normal to the infinitesimal surface area $\mathrm{d}\Sigma = \mathrm{d}x^1\mathrm{d}x^2$ of the parallelogram spanned by the legs $\mathrm{d}x^1$ and $\mathrm{d}x^2$" …this is what you would find in an elementary physics text; in reality the parallelogram is actually spanned by the vectors *dual* to $\mathrm{d}x^1$ and $\mathrm{d}x^2$, namely $\boldsymbol{\partial}_1$ and $\boldsymbol{\partial}_2$.

The starting point is to lower the index of $\boldsymbol{A} = A^i$ via the flat isomorphism $\flat$ to work exclusively with differential forms, thus obtaining the 1-form $\underline{\boldsymbol{A}} = A_i$. From Example 1, the exterior derivative of $\underline{\boldsymbol{A}}$ is

$$\begin{aligned}
\mathrm{d}\underline{\boldsymbol{A}} &= \mathrm{d}(A_j \mathrm{d}x^j) \\
&= \sum_{i<j} \left(\partial_i A_j - \partial_j A_i\right)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= \left(\partial_1 A_2 - \partial_2 A_1\right)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\partial_1 A_3 - \partial_3 A_1\right)\, \underbrace{\mathrm{d}x^1 \wedge \mathrm{d}x^3}_{=-\mathrm{d}x^3\wedge\mathrm{d}x^1} + \left(\partial_2 A_3 - \partial_3 A_2\right)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \left(\partial_1 A_2 - \partial_2 A_1\right)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\partial_2 A_3 - \partial_3 A_2\right)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(\partial_3 A_1 - \partial_1 A_3\right)\, \mathrm{d}x^3 \wedge \mathrm{d}x^1.
\end{aligned}$$

(We are color-coding for a reason; it will be evident soon) Thus, with $\mathcal{M} = \mathcal{V}_2$ and $\omega = \underline{\boldsymbol{A}}$, the LHS of (3) is

$$\begin{aligned}
\int_{\mathcal{V}_2} \mathrm{d}\underline{\boldsymbol{A}} &= \int_{\mathcal{V}_2} \left(\partial_1 A_2 - \partial_2 A_1\right)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\quad + \cancelto{0}{\int_{\mathcal{V}_2} \left(\partial_2 A_3 - \partial_3 A_2\right)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3} + \cancelto{0}{\int_{\mathcal{V}_2} \left(\partial_3 A_1 - \partial_1 A_3\right)\, \mathrm{d}x^3 \wedge \mathrm{d}x^1} \\
&= \int_{\mathcal{V}_2} \left(\partial_1 A_2 - \partial_2 A_1\right)\, \mathrm{d}x^1\mathrm{d}x^2,
\end{aligned} \tag{12}$$

where the last two integrals vanish because $\mathrm{d}x^3$ plays no part in the volume form of the 2-surface $\mathcal{V}_2$. (We also dropped the wedge $\wedge$ at the end, as it is customary when writing volume forms.)

[5] Also, curl only makes sense in three dimensions, whereas the generalization (d) applies to any arbitrary dimension.

On the other hand, from vector calculus and elementary physics we know that $\mathrm{curl}(\boldsymbol{A})$ expands as

$$\underbrace{\mathrm{curl}(\boldsymbol{A})}_{\text{vector calculus lingo}} = \underbrace{\epsilon_{ijk}A^{j;k}}_{\text{physics abstract index lingo}} = (\partial_2 A^3 - \partial_3 A^2)\boldsymbol{\partial}_1 + (\partial_3 A^1 - \partial_1 A^3)\boldsymbol{\partial}_2 + (\partial_1 A^2 - \partial_2 A^1)\boldsymbol{\partial}_3.$$

Note the striking similarity between $\mathrm{d}\underline{\boldsymbol{A}}$ and $\mathrm{curl}(\boldsymbol{A})$ by looking at their matched colors; they are essentially the same operation, although the former is purely in terms of smooth forms and the latter in terms of their dual vectors. In Cartesian coordinates on Euclidean space $A^i = A_i$ and, moreover, $\boldsymbol{\partial}_k$ is precisely the unit normal $\boldsymbol{n}$ to the infinitesimal surface $\mathrm{d}x^i \mathrm{d}x^j$ $(i \neq j \neq k)$; i.e., for $i \neq j \neq k$, $\boldsymbol{\partial}_k$ is the vector field dual to $\mathrm{d}x^i \wedge \mathrm{d}x^j$. Hence, all of the vector calculus gibberish can be entirely worked with the more elegant language of differential forms.

Tackling the LHS of (10),

$$\begin{aligned}
\int_{\mathcal{V}_2} \mathrm{curl}(\boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{\Sigma} &= \int_{\mathcal{V}_2} \mathrm{curl}(\boldsymbol{A}) \cdot \boldsymbol{n}\, \mathrm{d}\Sigma \\
&= \int_{\mathcal{V}_2} (\partial_2 A^3 - \partial_3 A^2, \partial_3 A^1 - \partial_1 A^3, \partial_1 A^2 - \partial_2 A^1) \cdot (0,0,1)\, \mathrm{d}x^1 \mathrm{d}x^2 \\
&= \int_{\mathcal{V}_2} (\partial_1 A^2 - \partial_2 A^1)\, \mathrm{d}x^1 \mathrm{d}x^2. \qquad (13)
\end{aligned}$$

This confirms the equality of the LHS of both (3) and (10). Straightforward calculations show the rest:

- (RHS of (3))
$$\int_{\partial\mathcal{V}_2} \underline{\boldsymbol{A}} = \int_{\partial\mathcal{V}_2} A_i \mathrm{d}x^i.$$
- (RHS of (10))
$$\int_{\partial\mathcal{V}_2} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \int_{\partial\mathcal{V}_2} {}^{E}g_{ij} A^i \mathrm{d}x^j = \int_{\partial\mathcal{V}_2} A_j \mathrm{d}x^j,$$
  where on the second equality we used the (geometric) definition of the dot product, which is merely a contraction given by the metric tensor. Equality of the RHS of both (3) and (10) has been established.

Showing the validity of (3) as a generalization of (11) is at this point a straightforward application of everything we have discussed, so it is left to the reader.