Math 260 HW # 2

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Section 1.3

(8) Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

Solution:

→ Given two arbitrary vectors $\vec{a} = (3a_2, a_2, -a_2)$ and $\vec{b} = (3b_2, b_2, -b_2)$ for $\vec{a}, \vec{b} \in W_1$, and an arbitrary scalar $c \in \mathbb{R}$, we test for closure under addition and scalar multiplication:

• $(3a_2, a_2, -a_2) + (3b_2, b_2, -b_2) = (3a_2 + 3b_2, a_2 + b_2, -a_2 + (-b_2))$
$= (3(a_2 + b_2), a_2 + b_2, -(a_2 + b_2))$

Hence $\vec{a} + \vec{b} \in W_1$. (closure under addition) ✓

• $c(3a_2, a_2, -a_2) = (3c\,a_2, c\,a_2, -c\,a_2)$

$c\,\vec{a} \in W_1$ (closed under scalar multiplication) ✓

→ Now we need to check whether the zero vector of $\mathbb{R}^3$ lies in W_1. We let $a_2 = 0$, then $a_1 = 3 \cdot (0) = 0$ and $a_3 = -0 = 0$. Hence $(a_1, a_2, a_3) = (0, 0, 0) = \vec{0} \in W_1$. ✓

Thus we conclude that W_1 is a subspace of $\mathbb{R}^3$ over the field $\mathbb{R}$.

f) $W_6 = \left\{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\right\}$

Solution:

From the given information we can see that W_6 is the subset of $\mathbb{R}^3$ that satisfies

$$(a_1, a_2, a_3) = \left(\sqrt{\frac{3a_2^2 - 6a_3^2}{5}}, \sqrt{\frac{5a_1^2 + 6a_3^2}{3}}, \sqrt{\frac{3a_2^2 - 5a_1^2}{6}}\right).$$

→ Clearly the zero vector of $\mathbb{R}^3$ lies in W_6, since by letting $(a_1, a_2, a_3) = (0, 0, 0)$ we get the zero vector $\vec{0}$. ✓

→ Now we need to test for closure under addition and scalar multiplication.

• We choose two arbitrary vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in W_6 and add them under the operations of addition defined on $\mathbb{R}^3$:

$$\left(\sqrt{\frac{3\,a_2^2 - 6\,a_3^2}{5}}, \sqrt{\frac{5\,a_1^2 + 6\,a_3^2}{3}}, \sqrt{\frac{3\,a_2^2 - 5\,a_1^2}{6}}\right) + \left(\sqrt{\frac{3\,b_2^2 - 6\,b_3^2}{5}}, \sqrt{\frac{5\,b_1^2 + 6\,b_3^2}{3}}, \sqrt{\frac{3\,b_2^2 - 5\,b_1^2}{6}}\right)$$

$$= \left(\sqrt{\frac{3\,a_2^2 - 6\,a_3^2}{5}} + \sqrt{\frac{3\,b_2^2 - 6\,b_3^2}{5}}, \sqrt{\frac{5\,a_1^2 + 6\,a_3^2}{3}} + \sqrt{\frac{5\,b_1^2 + 6\,b_3^2}{3}}, \sqrt{\frac{3\,a_2^2 - 5\,a_1^2}{6}} + \sqrt{\frac{3\,b_2^2 - 5\,b_1^2}{6}}\right)$$

This addition is not closed in W_6. (No closure under addition)

For instance let (a_1, a_2, a_3) take on the values $\left(\sqrt{3}, \sqrt{\frac{9}{5}}, 0\right)$ and (b_1, b_2, b_3) take on the same values $\left(\sqrt{3}, \sqrt{\frac{9}{5}}, 0\right)$. We can check that both $\vec{a}$ and $\vec{b}$ are in W_6 with these values, however their sum $\left(\sqrt{3} + \sqrt{3}, \sqrt{\frac{9}{5}} + \sqrt{\frac{9}{5}}, 0+0\right)$ does not satisfy the conditions on W_6.

Hence we conclude that W_6 is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$.

(19) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

<u>Proof:</u>

(⇒)

Suppose that $W_1 \cup W_2$ is a subspace of V and $W_1 \nsubseteq W_2$. We will show that $W_2 \subseteq W_1$.

Let $x \in W_2$ and $y \in W_1$ with $y \notin W_2$. Then $x \in W_1 \cup W_2$ and $y \in W_1 \cup W_2$. Since $W_1 \cup W_2$ is a subspace, $x + y \in W_1 \cup W_2$. Then we have that $x + y \in W_1$ or $x + y \in W_2$. Since $x \in W_2$, $-x \in W_2$. Then $x + y + (-x) \in W_2 \Longrightarrow y \in W_2$. (⇒⇐)

This contradicts our choice of y. Thus, $x + y \in W_1$. Since $y \in W_1$, $-y \in W_1$. Then, $x + y + (-y) \in W_1 \Longrightarrow x \in W_1$. Thus, $W_2 \subseteq W_1$.

(⇐)

Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. WLOG, suppose $W_1 \subseteq W_2$. We will show that $W_1 \cup W_2$ is a subspace. By subset properties, $W_1 \cup W_2 = W_2$, which is a subspace by assumption. Thus, $W_1 \cup W_2$ is a subspace of V. ■

(23) Let W_1 and W_2 be subspaces of a vector space V.
Note If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2 , denoted S_1+S_2, is the set $\{x+y: x\in S_1$ and $y\in S_2\}$.

a) Prove that W_1+W_2 is a subspace of V that contains both W_1 and W_2.

<u>Proof:</u>
We have that $W_1+W_2=\{w_1+w_2 : w_1\in W_1$ and $w_2\in W_2\}$. Since both W_1 and W_2 are subspaces they contain the zero vector from V. Therefore $\vec{0}+\vec{0}=\vec{0}\in(W_1+W_2)$.
Next we define two arbitrary a and b such that $a, b\in(W_1+W_2)$. Then we have $a=w_{1\,a}+w_{2\,a}$ and $b=w_{1\,b}+w_{2\,b}$, for $w_{1\,a}, w_{2\,a}\in W_1$ and $w_{1\,b}, w_{2\,b}\in W_2$, and so $a+b=(w_{1\,a}+w_{2\,a})+(w_{1\,b}+w_{2\,b})=(w_{1\,a}+w_{1\,b})+(w_{2\,a}+w_{2\,b})$.
Since W_1 and W_2 are subspaces, we have $(w_{1\,a}+w_{1\,b})\in W_1$ and $(w_{2\,a}+w_{2\,b})\in W_2$, and therefore $a+b\in(W_1+W_2)$.
Now we consider a scalar $c\in\mathbb{F}$ such that $c\,a=c(w_{1\,a}+w_{2\,a})=(c\,w_{1\,a})+(c\,w_{2\,a})$. Since W_1 and W_2 are subspaces we have that $c\,w_{1\,a}\in W_1$ and $c\,w_{2\,a}\in W_2$, and therefore $c\,a\in(W_1+W_2)$.
Hence we conclude that W_1+W_2 is a subspace. Now we only need to show that $W_1, W_2\subseteq(W_1+W_2)$. Let us consider an arbitrary element $w_1\in W_1$. Since $\vec{0}\in W_2$, we can write $w_1=w_1+\vec{0}\in W_1+W_2$. Hence we have proven that $W_1\subseteq(W_1+W_2)$. Since the zero vector also lies in W_1 , by a similar argument we conclude also that $W_2\subseteq(W_1+W_2)$. ■

b) Prove that any subspace of V that contains both W_1 and W_2 must also contain W_1+ W_2.

<u>Proof:</u>
Let W be some subspace of V containing W_1 and W_2. Let $a\in(W_1+W_2)$. Then $a=w_1+w_2$ for $w_1\in W_1$ and $w_2\in W_2$. However both w_1 and w_2 are in the larger subspace W, and since W is a subspace and it's closed under addition, we have that $a\in W$. This proves that $W_1+W_2\subseteq W$. ■