MATH 722 TAKE HOME EXAM

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Problem 1 (Exercise 1-4)**.** *Show that in the ring $A[x]$, the Jacobson radical $\mathfrak{J}$ is equal to the nilradical $\mathfrak{R}$.*

Proof. ($\mathfrak{J} \supseteq \mathfrak{R}$) Since all maximal ideals are prime in any ring, it is clear that the Jacobson radical $\mathfrak{J}$ contains the nilradical $\mathfrak{R}$.

($\mathfrak{J} \subseteq \mathfrak{R}$) Now on the other hand, if $f = a_0 + a_1 x + \cdots + a_n x^n$ is in the Jacobson radical $\mathfrak{J}$ of $A[x]$, then by a previous proposition we must have that $1 + gf$ is a unit in $A[x]$ for all $g \in A[x]$. Now we are going to use results from the following proposition:

Proposition 1. *Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x with coefficients in A. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Then,*

- *i) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent;*
- *ii) f is nilpotent $\iff a_0, a_1, \ldots, a_n$ are nilpotent;*
- *iii) f is a zero-divisor $\iff$ there exists $a \neq 0$ in A such that $af = 0$;*
- *iv) if $f, g \in A[x]$, then fg is primitive[1] $\iff$ f and g are primitive.*

Now letting $g = x$, note that $1 + xf = 1 + a_0 x + a_1 x^2 + \cdots + a_n x^{n+1}$. Hence we conclude that $a_0, a_1, \ldots, a_n$ are all nilpotent, since by Proposition 1 part i), if $1 + fx$ is a unit, every nonconstant term's coefficient is nilpotent. But then $f = a_0 + a_1 x + \cdots + a_n x^n$ is a nilpotent by Proposition 1, part ii), i.e. $f \in \mathfrak{R}$. Hence we conclude that $\mathfrak{J} = \mathfrak{R}$ in $A[x]$, as desired. $\square$

Problem 2 (Exercise 1-7)**.** *Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.*

Proof. Let $\mathfrak{p}$ be an arbitrary prime ideal of A. We need to show that the only ideal of A properly containing $\mathfrak{p}$ is $\langle 1 \rangle$. Let $\mathfrak{a}$ be an ideal such that $\mathfrak{p} \subsetneq \mathfrak{a}$. Then there exists an element $x \in a \setminus \mathfrak{p}$. But by assumption, $x^n = x$ for some $n > 1$. That is, $x - x^n = x(1 - x^{n-1}) = 0 \in \mathfrak{p}$, which implies that $(1 - x^{n-1}) \in \mathfrak{p} \subsetneq \mathfrak{a}$ since $\mathfrak{p}$ is prime and $x \notin \mathfrak{p}$. But then we have $1 = (1 - x^{n-1}) + x^{n-1} \in \mathfrak{a}$. Hence $\mathfrak{a} = \langle 1 \rangle$ is the unit ideal and thus $\mathfrak{p}$ must be maximal. Since $\mathfrak{p}$ was arbitrary, we conclude that every prime ideal in A is maximal, as desired. $\square$

Alternative proof. Let $\mathfrak{p}$ be an arbitrary prime ideal of A. Then to show that $\mathfrak{p}$ is maximal, it suffices to prove that $A/\mathfrak{p}$ is a field. Since there is a surjective ring homomorphism from A to $A/\mathfrak{p}$, every element $\bar{x} \in A/\mathfrak{p}$ satisfies $\bar{x}^n = \bar{x}$ for some $n > 1$ (depending on $\bar{x}$). But if $\bar{x} \neq 0 \in A/\mathfrak{p}$, then

$$\bar{x}^n = \bar{x} \implies \bar{x} - \bar{x}^n = \bar{x}(1 - \bar{x}^{n-1}) = 0 \in A/\mathfrak{p}.$$

[1] Recall that $f = a_0 + a_1 x + \cdots + a_n x^n$ is said to be ***primitive*** if $\langle a_0, a_1, \ldots, a_n \rangle = \langle 1 \rangle$.

But this implies that $1-\bar{x}^{n-1}=0$ since $A/\mathfrak{p}$ is an integral domain (it has no zero divisors). Hence we have that $\bar{x}^{n-1}=1\in A/\mathfrak{p}$ and thus $\bar{x}$ is a unit. Since x (and hence $\bar{x}$) was chosen arbitrarily, we have that every nonzero element of $A/\mathfrak{p}$ is a unit; thus $A/\mathfrak{p}$ is a field and every prime ideal $\mathfrak{p}$ is maximal. $\square$

Problem 3 (Exercise 1-11)**.** *A ring A is said to be* ***Boolean*** *if $x^2=x$ for all $x\in A$. In a Boolean ring A, show that*

i) $2x=0$ *for all $x\in A$;*

ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;

iii) every finitely generated ideal in A is principal.

Proof of i). Note that $(x+1)^2=x+1 \implies x^2+2x+1=x+1 \implies x+2x+1=x+1 \implies 2x=0$. $\square$

Proof of ii). Every prime ideal $\mathfrak{p}$ must be maximal since, if A is Boolean, we are dealing with a special case of Exercise 1-7 (see result above). Now to show that $A/\mathfrak{p}$ only has two elements, note that $x^2-x=x(x-1)=0$ holds for each $x\in A$. Hence (since there is a surjective ring homomorphism from A to $A/\mathfrak{p}$) we also have $\bar{x}(\bar{x}-1)=0$ for all $\bar{x}\in A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is an integral domain, each element $\bar{x}$ must be either 0 or 1, and we are done. $\square$

Proof of iii). We induct on the number of generators. The one-generator case is trivial. For two generators x and y, we claim that $\langle x,y\rangle=\langle xy+x+y\rangle$. This is clear since $x\langle xy+x+y\rangle = xy+x+xy=2xy+x=x$, and similarly for y. Now the more general result follows from induction: Suppose every ideal generated by n elements is principal, and $\mathfrak{a}=\langle x_1,\dots,x_n,y\rangle$. Let x generate $\langle x_1,\dots,x_n\rangle$, and let $z=x+y-xy$. Then $xz=x^2+xy-x^2y=x$ and similarly $yz=y$, so $\mathfrak{a}=\langle x,y\rangle=\langle z\rangle$. $\square$

Problem 4 (Exercise 1-13 (**Construction of an Algebraic Closure of a Field**))**.** *i) Let $\Bbbk$ be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in $\Bbbk$. Let A be the polynomial ring over $\Bbbk$ generated by indeterminates x_f, one for each $f\in\Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f\in\Sigma$. Show that $\mathfrak{a}\neq\langle 1\rangle$.*

ii) Now let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$, and let $\Bbbk_1=A/\mathfrak{m}$. Then $\Bbbk_1$ is an extension field of $\Bbbk$ in which each $f\in\Sigma$ has a root. Repeat the construction with $\Bbbk_1$ in place of $\Bbbk$, obtaining a field $\Bbbk_2$, and so on. Let $L=\bigcup_{n=1}^{\infty}\Bbbk_n$. Then L is a field in which each $f\in\Sigma$ splits completely into linear factors. Let $\overline{\Bbbk}$ be the set of all elements of L which are algebraic over $\Bbbk$. Then show that $\overline{\Bbbk}$ is an algebraic closure of $\Bbbk$.

Proof of i). If $\mathfrak{a}=\langle 1\rangle$, then there exist finitely many $y_f\in A$ such that $1=\sum y_f f(x_f)$. Then the set I of x_g occurring in this expression (not only those in the $f(x_f)$, but also those occurring in the y_f) is finite. Thus we may enumerate I as $x_1,\dots,x_n$, corresponding to irreducible polynomials f_i, and suppose n is minimal such that such an equation holds. Now let

$$B=\Bbbk[x_1,\dots,x_{n-1}],\qquad C=B[x_n],\qquad\text{and}\qquad \mathfrak{b}=\langle f_1(x_1),\dots,f_{n-1}(x_{n-1})\rangle\subsetneq B.$$

By minimality of n, the ideal $\mathfrak{b}$ is proper, so the extension $\mathfrak{b}^e=\mathfrak{b}[x_n]\subsetneq C$ is properly contained as well, while $\mathfrak{b}^e+\langle f_n(x_n)\rangle=C$. Since $\mathfrak{b}\neq B$, we know that $B/\mathfrak{b}\neq 0$. Now let g be the image of

$f_n(x_n)$ in $(B/\mathfrak{b})[x_n]$. Since f_n is irreducible in $\Bbbk[x_n]$, its degree $\deg_{x_n} f_n \geq 1$ and also $\deg_{x_n} g \geq 1$. Then putting all this together we have

$$0 = \frac{C}{\mathfrak{b}^e + \langle f_n(x_n)\rangle} \cong \frac{C/\mathfrak{b}[x_n]}{\langle g\rangle} = \frac{B[x_n]/\mathfrak{b}[x_n]}{\langle g\rangle} \cong \frac{(B/\mathfrak{b})[x_n]}{\langle g\rangle} \neq 0,$$

which is a contradiction. $(\Rightarrow\Leftarrow)$ $\square$

Proof of ii). We show that $\overline{\Bbbk}$ is closed under addition (subtraction) and multiplication. Let $a, b \in \overline{\Bbbk}$ have conjugates a_i, b_j over $\Bbbk$. Then $\prod_{i,j}(x - (a_i + b_j))$ is symmetric in the a_i and the b_j, and so it has coefficients in $\Bbbk$; thus $a - b \in \overline{\Bbbk}$. Similarly $\prod_{i,j}(x - a_i b_j)$ is symmetric, so $ab \in \overline{\Bbbk}$. $\square$

Problem 5 (Exercise 1-15)**.** *Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that*

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$;

ii) $V(0) = X$, $V(1) = \emptyset$;

iii) if $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i);$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals, $\mathfrak{a}, \mathfrak{b}$ of A.

Remark: These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the ***Zariski topology***. The topological space X is called the ***prime spectrum*** of A, and it is written $\operatorname{Spec}(A)$.

Proof of i). Since $E \subseteq \mathfrak{a} \subseteq r(\mathfrak{a})$, we have $V(r(\mathfrak{a})) \subseteq V(\mathfrak{a}) \subseteq V(E)$. Now for any prime ideal $\mathfrak{p}$ of A such that $E \subseteq \mathfrak{p}$, by definition we have that $\mathfrak{p} \in V(E)$ and $\mathfrak{a} \subseteq \mathfrak{p}$, that is, $\mathfrak{p} \in V(\mathfrak{a})$. Also since $\mathfrak{a} \subseteq \mathfrak{p}$, we have $r(\mathfrak{a}) \subseteq r(\mathfrak{p})$. But since $\mathfrak{p}$ is prime, we have that $r(\mathfrak{p}) = \mathfrak{p}$. Hence $r(\mathfrak{a}) \subseteq \mathfrak{p}$, that is, $\mathfrak{p} \in V(r(\mathfrak{a}))$. Thus we have concluded that $V(r(\mathfrak{a})) \supseteq V(\mathfrak{a}) \supseteq V(E)$ and hence $V(r(\mathfrak{a})) = V(\mathfrak{a}) = V(E)$, as desired. $\square$

Proof of ii). This part is trivial. For any prime ideal $\mathfrak{p}$ of A, we know that $0 \in \mathfrak{p}$, and thus $\mathfrak{p} \in V(0)$. Hence $V(0) = X$, as desired. For $V(1)$, we must have $V(1) = \emptyset$; otherwise there exists some prime ideal $\mathfrak{p}$ of A such that $1 \in \mathfrak{p}$, which implies that $\mathfrak{p} = A$, a contradiction. Hence $V(1) = \emptyset$, and we are done. $\square$

Proof of iii). $(\subseteq)$ Since for each $i \in I$ we have that $E_i \subseteq \bigcup_{i\in I} E_i$, then we must have that $V(\bigcup_{i\in I} E_i) \subseteq V(E_i)$ for all $i \in I$. Thus $V(\bigcup_{i\in I} E_i) \subseteq \bigcap_{i\in I} V(E_i)$, as desired.

$(\supseteq)$ On the other hand, notice that for all $\mathfrak{p} \in \bigcap_{i\in I} V(E_i)$ we have that $\mathfrak{p} \in V(E_i)$ for all $i \in I$, i.e., $E_i \subseteq \mathfrak{p}$ $\forall i \in I$. But this implies that $\bigcup_{i\in I} E_i \subseteq \mathfrak{p}$; that is, $\mathfrak{p} \in V(\bigcup_{i\in I} E_i)$. Therefore we conclude that $V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i)$, as desired. $\square$

Proof of iv). $(V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}))$ For the second equality, suppose that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{b} \not\subseteq \mathfrak{p}$. Then there exists an element $b \in \mathfrak{b} \setminus \mathfrak{p}$, and $ab \in \mathfrak{p}$ for all $a \in \mathfrak{a}$, so the primality of $\mathfrak{p}$ gives $a \in \mathfrak{p}$; thus $\mathfrak{a} \subseteq \mathfrak{p}$. Consequently, if $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, we have shown that either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{a}) \bigcup V(\mathfrak{b})$.

On the other hand, if $\mathfrak{p}$ contains either $\mathfrak{a}$ or $\mathfrak{b}$, then it is clear that it must contain the subset $\mathfrak{ab}$. Thus $V(\mathfrak{ab}) = V(\mathfrak{a}) \bigcup V(\mathfrak{b})$, as desired.

($V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab})$) Now for the first equality, note that $\mathfrak{ab} \subseteq \mathfrak{a} \bigcap \mathfrak{b}$; so if $\mathfrak{a} \bigcap \mathfrak{b} \subseteq \mathfrak{p}$, then $\mathfrak{ab} \subseteq \mathfrak{p}$. On the other hand, if $\mathfrak{ab} \subseteq \mathfrak{p}$ then, as we have shown for the second equality (see above), we must have that either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$; consequently, since $\mathfrak{a} \bigcap \mathfrak{b}$ is a subset of both of these we have that $\mathfrak{a} \bigcap \mathfrak{b} \subseteq \mathfrak{p}$. Thus $V(\mathfrak{a} \bigcap \mathfrak{b}) = V(\mathfrak{ab})$, as desired. □

Problem 6 (Exercise 1-17)**.** *For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \mathrm{Spec}(A)$. The sets X_f are open*[2]*. Show that they form a basis of open sets for the Zariski topology, and that*

i) $X_f \bigcap X_g = X_{fg}$;

ii) $X_f = \emptyset \iff f$ *is nilpotent;*

iii) $X_f = X \iff f$ *is a unit;*

iv) $X_f = X_g \iff r(\langle f \rangle) = r(\langle g \rangle)$;

v) *X is quasi-compact (that is, every open covering of X has a finite subcovering) [Hint: remark that it is enough to consider a covering of X by basic open sets X_{f_i} (for $i \in I$). Show that the f_i generate the unit ideal and hence that there is an equation of the form*

$$1 = \sum_{i \in J} g_i f_i \qquad (\textit{for } g_i \in A),$$

where J is some finite subset of I. Then the X_{f_i} (for $i \in J$) cover X.]

vi) *More generally, each X_f is quasi-compact.*

vii) *An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.*

To see that the collection $\{X_f\}$ forms a basis for the topology of X, we can show that it contains, for each $\mathfrak{p} \in X_f \cap X_g$, a set X_h with $\mathfrak{p} \in X_h \subseteq X_f \cap X_g$. It also includes $\emptyset$, and it covers X. These results follow, respectively, from i), ii), and iii) below.

Proof of i). Taking complements, this equality is the same as saying $V(f) \bigcup V(g) = V(fg)$, or that a prime contains fg if and only if it contains either f or g. But this is precisely in the definition of a prime ideal, so the equality checks out. □

Proof of ii). $X_f = \emptyset \iff V(f) = X \iff \forall\, \mathfrak{p} \in X$, we have $f \in \mathfrak{p} \iff f \in \mathfrak{R}$. □

Proof of iii). $X_f = X \iff V(f) = \emptyset \iff \forall\, \mathfrak{p} \in X$, we have $f \notin \mathfrak{p} \iff f \in A^*$, where A^* denotes the set of units of the ring A. Hence f is a unit, as desired. □

Proof of iv). $X_f = X_g \iff V(f) = V(g) \iff r(\langle f \rangle) = \bigcap_{\mathfrak{p} \in V(f)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in V(g)} \mathfrak{p} = r(\langle g \rangle)$. Note that the last equality holds by Proposition 1.4[3] from the text. □

Proof of v). This follows from the more general result given in vi) below, taking $f = 1$ ($X_f = X$ if $f = 1$). □

[2] These sets X_f are called ***basic open sets*** of $X = \mathrm{Spec}(A)$.

[3] The proposition states that the radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals which contain $\mathfrak{a}$.

Proof of vi). Since $(X_g)_{g\in A}$ forms a basis of open sets for X, it suffices to show that if $X_f \subseteq \bigcup_{g\in E} X_g$ for some subset E of A, there exist finitely many elements $g_1,\dots,g_n \in E$ such that $X_f \subseteq \bigcup_{i=1}^n X_{g_i}$. Since

$$\bigcup_{g\in E} X_g = \bigcup_{g\in E} (X \smallsetminus V(g)) = X \smallsetminus V(E),$$

we get that

$$X_f \subseteq \bigcup_{g\in E} X_g \implies V(E) \subseteq V(f) \implies V(\mathfrak{a}) \subseteq V(f),$$

where $\mathfrak{a}$ is the ideal generated by E (we know that $V(E) = V(\mathfrak{a})$ by Exercise 1.15, part i) above). But $V(\mathfrak{a}) \subseteq V(f)$ implies that $f \in \bigcap_{\mathfrak{p}\in V(f)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p}\in V(\mathfrak{a})} \mathfrak{p} = r(\mathfrak{a})$. Therefore $f^t \in \mathfrak{a}$ for some $t \in \mathbb{N}$; that is, there exist $g_1,\dots,g_n \in E$ and $h_1,\dots,h_n \in A$ such that $g_1h_1 + \cdots + g_nh_n = f^t$, which implies that $f \in r(\mathfrak{b})$, where $\mathfrak{b}$ is the ideal generated by the subset $F = \{g_1,\dots,g_n\} \subseteq E$. But then, since

$$\mathfrak{p} \supseteq F \iff \mathfrak{p} \supseteq \mathfrak{b} \implies \mathfrak{p} \ni f^t \iff \mathfrak{p} \ni f$$

for every prime ideal $\mathfrak{p}$ of A, we have $V(F) = V(\mathfrak{b}) \subseteq V(f)$, which in turn implies that

$$\bigcup_{i=1}^n X_{g_i} = X \setminus \left(\bigcap_{i=1}^n V(g_i)\right) = X \smallsetminus V(F) \supseteq X \smallsetminus V(f) = X_f. \qquad \square$$

Proof of vii). ($\Leftarrow$) If an open subset U of X is a finite union of X_f, then U is evidently quasi-compact since each X_f is quasi-compact.

($\Rightarrow$) Conversely, suppose that an open subset U of X is quasi-compact. Then, as all the X_f form a basis of open sets, there exist a subset E of A such that $U = \bigcup_{f\in E} X_f$. Therefore there exist finitely many $f_1,\dots,f_n \in E$ such that $U = \bigcup_{i=1}^n X_{f_i}$ by quasi-compactness. $\square$

Problem 7 (Exercise 1-21)**.** *Let $\phi\colon A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi_*\colon Y \to X$. Show that*

i) If $f \in A$, then $\phi_*^{-1}(X_f) = Y_{\phi(f)}$, and hence ϕ_* is continuous.

ii) If $\mathfrak{a}$ is an ideal of A, then $\phi_*^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

iii) If $\mathfrak{b}$ is an ideal of B, then $\overline{\phi_*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

iv) If ϕ is surjective, then ϕ_* is a homeomorphism of Y onto the closed subset $V(\operatorname{Ker}(\phi))$ of X. (In particular, $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A/\mathfrak{N})$ (where $\mathfrak{N}$ is the nilradical of A) are naturally homeomorphic.)

v) If ϕ is injective, then $\phi_*(Y)$ is dense in X. More precisely, $\phi_*(Y)$ is dense in X if and only if $\operatorname{Ker}(\phi) \subseteq \mathfrak{N}$.

vi) Let $\psi\colon B \to C$ be another ring homomorphism. Then $(\psi\circ\phi)_* = \phi_* \circ \psi_*$.

vii) Let A be an integral domain with just one nonzero prime ideal $\mathfrak{p}$, and let $\Bbbk$ be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times \Bbbk$. Define $\phi\colon A \to B$ by $\phi(x) = (\widetilde{x}, x)$, where $\widetilde{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ_* is bijective but not a homeomorphism.

Proof of i). For a prime ideal $\mathfrak{q} \in Y = \operatorname{Spec}(B)$, we have

$$\mathfrak{q} \in \phi_*^{-1}(X_f) \iff \phi_*(\mathfrak{q}) = \mathfrak{q}^c \in X_f \iff \mathfrak{q}^c \not\ni f = \phi^{-1}(\mathfrak{q}) \iff \mathfrak{q} \not\ni \phi(f) \iff \mathfrak{q} \in Y_{\phi(f)},$$

which proves that $\phi_*^{-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ_* is continuous since the X_f form a basis of open sets for X. $\square$

Proof of ii). Similarly, for a prime ideal $\mathfrak{q} \in Y = \operatorname{Spec}(B)$, we have

$$\begin{aligned} \mathfrak{q} \in \phi_*^{-1}(V(\mathfrak{a})) &\iff \phi_*(\mathfrak{q}) = \mathfrak{q}^c \in V(\mathfrak{a}) \\ &\iff \mathfrak{a} \subseteq \mathfrak{q}^c = \phi^{-1}(\mathfrak{q}) \\ &\iff \phi(\mathfrak{a}) \subseteq \mathfrak{q} \\ &\iff \mathfrak{a}^e \subseteq \mathfrak{q} \\ &\iff \mathfrak{q} \in V(\mathfrak{a}^e), \end{aligned}$$

which proves that $\phi_*^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$, and, once again, that ϕ_* is continuous. $\square$

Proof of iii). $(\overline{\phi_*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c))$ For any $\mathfrak{p} \in \phi_*(V(\mathfrak{b}))$, there exists a prime ideal $\mathfrak{q} \in V(\mathfrak{b}) \subseteq \operatorname{Spec}(B)$ (i.e., $\mathfrak{b} \subseteq \mathfrak{q}$) such that $\phi_*(\mathfrak{q}) = \mathfrak{q}^c = \mathfrak{p}$, which implies that $\mathfrak{b}^c \subseteq \mathfrak{q}^c = \mathfrak{p}$, i.e., $\mathfrak{p} \in V(\mathfrak{b}^c) \subseteq \operatorname{Spec}(A)$. Therefore $\phi_*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$, which implies that $\overline{\phi_*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$, as $V(\mathfrak{b}^c)$ is closed.

$(\overline{\phi_*(V(\mathfrak{b}))} \supseteq V(\mathfrak{b}^c))$ On the other hand, as $\overline{\phi_*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$ is closed, we have $\overline{\phi_*(V(\mathfrak{b}))} = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Using the above result of part ii), we get

$$V(\mathfrak{a}^e) = \phi_*^{-1}(V(\mathfrak{a})) = \phi_*^{-1}\left(\overline{\phi_*(V(\mathfrak{b}))}\right) \supseteq \phi_*^{-1}(\phi_*(V(\mathfrak{b}))) \supseteq V(\mathfrak{b}).$$

Therefore, we have

$$\mathfrak{a}^e \subseteq \bigcap_{\mathfrak{q} \in V(\mathfrak{a}^e)} \mathfrak{q} \subseteq \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q} = r(\mathfrak{b}).$$

So for any $x \in \mathfrak{a} \subseteq A$, we have $\phi(x) \in \mathfrak{a}^e \subseteq r(\mathfrak{b})$, which means that $\phi(x^n) = (\phi(x))^n \in \mathfrak{b}$ for some integer $n \in \mathbb{N}$; this in turn implies that $x^n \in \phi^{-1}(\mathfrak{b}) = \mathfrak{b}^c$. Therefore $\mathfrak{a} \subseteq r(\mathfrak{b}^c)$, which proves that

$$\overline{\phi_*(V(\mathfrak{b}))} = V(\mathfrak{a}) \supseteq V(r(\mathfrak{b}^c)) = V(\mathfrak{b}^c)$$

by our results from Exercise 1.15. Hence $\overline{\phi_*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$, as we set out to prove. $\square$

Proof of iv). By Proposition 1.1[4] from the text (generalized to the case of $\phi\colon A \to B$ where ϕ is surjective), we know that $\phi_*(Y) = V(\ker(\phi))$ and ϕ_* induces a bijective map from $Y = \operatorname{Spec}(B)$ to the closed subspace $V(\ker(\phi))$ of $X = \operatorname{Spec}(A)$ (which we still called ϕ_* by abuse of notation). We already know that ϕ_* is continuous by part i). To show that $\phi_*\colon Y \to V(\ker(\phi))$ is a homeomorphism, we only need to show that, for every closed subset K_Y of Y, we have that $\phi_*(K_Y)$ is closed in X (hence in $V(\ker(\phi))$). Now for every closed subset K_Y of Y, there exists an ideal $\mathfrak{b}$ of B such that $K_Y = V(\mathfrak{b})$ (see Exercise 1.15). Let $\mathfrak{a} = \phi^{-1}(\mathfrak{b})$. Then, for a prime ideal $\mathfrak{p}$ of A, we have

$$\begin{aligned} \mathfrak{p} \in \phi_*(K_Y) &= \phi_*(V(\mathfrak{b})) \\ &\iff \mathfrak{p} = \phi_*(\mathfrak{q}) \text{ for some } \mathfrak{q} \in V(\mathfrak{b}) \quad (\text{i.e., } \mathfrak{b} \subseteq \mathfrak{q}) \\ &\iff \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \supseteq \phi^{-1}(\mathfrak{b}) = \mathfrak{b}^c \\ &\iff \mathfrak{p} \in V(\mathfrak{b}^c), \end{aligned}$$

which means that $\phi_*(K_Y) = V(\mathfrak{b}^c)$ is closed in X (hence in $V(\ker(\phi))$). Therefore $\phi_*\colon Y \to V(\ker(\phi))$ is a homeomorphism, as desired.

[4] The proposition states that there is a 1-1 correspondence between the ideals $\mathfrak{b}$ containing the ideal $\mathfrak{a}$ and the ideals $\bar{\mathfrak{b}}$ of $A/\mathfrak{a}$, given by $\mathfrak{b} = \phi^{-1}(\bar{\mathfrak{b}})$.

In particular, let $\mathfrak{N}$ be the nilradical of A. Then there is a natural surjective homomorphism $\phi\colon A \to A/\mathfrak{N}$. Therefore $\phi_*\colon \operatorname{Spec}(A/\mathfrak{N}) \to V(\ker(\phi)) = V(\mathfrak{N}) \operatorname{Spec}(A)$ is a homeomorphism. □

Proof of v). We will prove the general statement that "$\phi_*(Y)$ is dense in $X \iff \operatorname{Ker}(\phi) \subseteq \mathfrak{N}$." (This more general statement does imply the first, because if ϕ is injective, then indeed $\ker(\phi) = 0 \subseteq \mathfrak{N}$). By our results from part iii), we have

$$\overline{\phi_*(Y)} = \overline{\phi_*(V(0))} = V(0^c) = V(\ker(\phi)).$$

Therefore,

$$\begin{aligned} \phi_*(Y) \text{ is dense in } X &\iff \overline{\phi_*(Y)} = V(\ker(\phi)) = X \\ &\iff \ker(\phi) \subseteq \mathfrak{p} \text{ for every prime ideal } \mathfrak{p} \text{ in } A \\ &\iff \ker(\phi) \subseteq \mathfrak{N}. \end{aligned}$$ □

Proof of vi). For any prime ideal $\mathfrak{q} \in \operatorname{Spec}(C)$, we have $(\psi \circ \psi)_*(\mathfrak{q}) = (\psi \circ \phi)^{-1}(\mathfrak{q})$ and $\phi_* \circ \psi_*(\mathfrak{q}) = \phi^{-1}(\psi^{-1}(\mathfrak{q}))$. Then the desired result $(\psi \circ \phi)_* = \phi_* \circ \psi_*$ follows immediately from the fact that $(\psi \circ \phi)_*(\mathfrak{q}) = \phi^{-1}(\psi^{-1}(\mathfrak{q}))$. □

Proof of vii). By assumption, A has exactly two prime ideals, namely 0 and $\mathfrak{p}$. Therefore $\mathfrak{p}$ is a maximal ideal of A, which implies $A/\mathfrak{p}$ must be a field. Hence we conclude that the ring $B = (A/\mathfrak{p}) \times \Bbbk$ also has exactly two ideals, namely $\mathfrak{q}_1 = \{(\overline{x}, 0) \mid x \in A\}$ and $\mathfrak{q}_2 = \{(\overline{0}, k) \mid k \in \Bbbk\}$. It is easy to check that $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals and there is no other prime ideal of B. Now we can see that $\phi\colon A \to B$, defined by $\phi(x) = (\overline{x}, x)$, is a ring homomorphism. A straight computation then shows that $\phi_*(\mathfrak{q}_1) = \phi^{-1}(\mathfrak{q}_1) = 0$ and $\phi_*(\mathfrak{q}_2) = \phi^{-1}(\mathfrak{q}_2) = \mathfrak{p}$. Therefore ϕ_* is bijective (and is always continuous), as desired.

However, ϕ_* is not a homeomorphism. Indeed, in the topological space $\operatorname{Spec}(B) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$, we have $\{\mathfrak{q}_1\} = V(\mathfrak{q}_1)$ is closed as $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$. But $\phi_*(\{\mathfrak{q}_1\}) = \{0\}$ is not closed in $\operatorname{Spec}(A)$ since 0 is not a maximal ideal of A. □

Problem 8 (Exercise 2-6)**.** *For any A-module M, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form*

$$m_0 + m_1 x + \cdots + m_r x^r \qquad (m_i \in M).$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module. Show that $M[x] \cong A[x] \otimes_A M$.

Proof. We first show that $M[x]$ is an $A[x]$-module. As an A-module, we have $M[x] \cong \oplus_{n \in \mathbb{N}} M x^n$. We define the action of $A[x]$ on $M[x]$ by $\left(\sum a_i x^i\right)\left(\sum m_j x^j\right) = \sum c_k x^k$, where $c_k = \sum_{i+j=k} a_i m_j$. It is easy to see that $M[x]$ is an additive group, and the above scalar multiplication by $A[x]$ is well defined. Hence we only need to check distributivity and associativity. Let $f(x) = \sum_i a_i x^i$ and $g(x) = \sum_j b_j x^j$ (where $f, g \in A[x]$), and let $\mathfrak{f}(x) = \sum_k m_k x^k$ and $\mathfrak{g}(x) = \sum_k n_k x^k$ (where

$\mathfrak{f}, \mathfrak{g} \in M[x]$). Associativity is then given by

$$\begin{aligned}
[f(x)g(x)]\mathfrak{f}(x) &= \left[\sum_{\ell}\left(\sum_{i+j=k} a_i b_j\right) x^k\right]\left(\sum_{\ell} m_\ell x^\ell\right) \\
&= \sum_p \left(\sum_{k+\ell=p}\left(\sum_{i+j=k} a_i b_j\right) m_k\right) x^p \\
&= \sum_p \left(\sum_{i+j+k=p} a_i b_j m_k\right) x^p;
\end{aligned}$$

$$\begin{aligned}
f(x)[g(x)\mathfrak{f}(x)] &= \left(\sum_i a_i x^i\right)\left[\sum_\ell\left(\sum_{j+k=\ell} b_j m_k\right) x^\ell\right] \\
&= \sum_p \left(\sum_{i+\ell=p} a_i \left(\sum_{j+k=\ell} b_j m_k\right)\right) x^p \\
&= \sum_p \left(\sum_{i+j+k=p} a_i b_j m_k\right) x^p.
\end{aligned}$$

Now that associative checks out, we check distributivity:

$$\begin{aligned}
[f(x)+g(x)]\mathfrak{f}(x) &= \left(\sum_i (a_i+b_i)x^i\right)\left(\sum_k m_k x^k\right) \\
&= \sum_\ell \left(\sum_{i+k=\ell} (a_i m_k + b_i m_k)\right) x^\ell \\
&= \left(\sum_i a_i x^i\right)\left(\sum_k m_k x^k\right) + \left(\sum_i b_i x^i\right)\left(\sum_k m_k x^k\right) \\
&= f(x)\mathfrak{f}(x) + g(x)\mathfrak{f}(x);
\end{aligned}$$

$$\begin{aligned}
f(x)[\mathfrak{f}(x)+\mathfrak{g}(x)] &= \left(\sum_i a_i x^i\right)\left(\sum_k (m_k+n_k)x^k\right) \\
&= \sum_\ell \left(\sum_{i+k=\ell} (a_i m_k + a_i n_k)\right) x^\ell \\
&= \left(\sum_i a_i x^i\right)\left(\sum_k m_k x^k\right) + \left(\sum_i a_i x^i\right)\left(\sum_k n_k x^k\right) \\
&= f(x)\mathfrak{f}(x) + f(x)\mathfrak{g}(x).
\end{aligned}$$

Hence we have that $M[x]$ is an $A[x]$-module, as desired.

Now to show that $M[x] \cong A[x] \otimes_A M$, define $\phi\colon M[x] \to A[x] \otimes_A M$ by $\mathfrak{f}(x) = \sum m_j x^j \mapsto \sum (x^j \otimes m_j)$. It is obviously additive, and is $A[x]$-linear, for if $f(x) = \sum a_i x^i \in A[x]$, then

$$\begin{aligned}
\phi(f(x)\mathfrak{f}(x)) &= \sum_k \sum_{i+j=k} \phi(a_i m_j x^k) \\
&= \sum_k \sum_{i+j=k} (x^k \otimes a_i m_j) \\
&= \sum_i \sum_j (x^i x^j \otimes a_i m_j) \\
&= \sum_j \left(\left(\sum_i a_i x^i \right) x^j \otimes m_j \right) \\
&= \left(\sum_i a_i x^i \right) \left(\sum_j x^j \otimes m_j \right) \\
&= f(x)\, \phi(\mathfrak{f}(x)).
\end{aligned}$$

Now define $\psi\colon A[x] \times M \to M[x]$ by $\psi(\sum a_i x^i, m) = \sum (a_i m) x^i$. It is clearly bi-additive and A-bilinear, and so it induces a linear map $\Psi\colon A[x] \otimes_A M \to M[x]$ sending $(\sum a_i x^i) \otimes m \mapsto \sum (a_i m) x^i$. Now ϕ and Ψ are inverse, for

$$\Psi(\phi(m_i x^i)) = \Psi(x^i \otimes m_i) = m_i x^i$$

and

$$\phi(\Psi(a_i x^i \otimes m)) = \phi((a_i m) x^i) = x^i \otimes a_i m = a_i x^i \otimes m.$$

Hence the map $\phi\colon M[x] \to A[x] \otimes_A M$ is an isomorphism and thus we have that $M[x] \cong A[x] \otimes_A M$, as we set out to prove. □

Problem 9 (Exercise 2-7)**.** *Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?*

Proof. We denote the quotient ring $A/\mathfrak{p}$ by $\bar{A}$ and denote an element $a + \mathfrak{p} \in \bar{A}$ by $\bar{a}$. Then there is a ring homomorphism $\phi\colon A[x] \to \bar{A}[x]$ defined by $\phi(c_0 + \cdots + c_r x^r) = \overline{c_0} + \cdots + \overline{c_r} x^r$. Now notice that $\bar{A} = A/\mathfrak{p}$ is an integral domain since $\mathfrak{p}$ is a prime ideal in A. Now in general,we know that if $\mathcal{R}$ is an integral domain, then $\mathcal{R}[x]$ is also an integral domain. Therefore $\ker(\phi)$ is a prime ideal in $A[x]$ since $\bar{A}[x]$ is an integral domain. But it is easy to check that $\ker(\phi)$ is exactly $\mathfrak{p}[x]$. Hence $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. Also notice that ϕ is surjective, so that $A[x]/\mathfrak{p}[x] \cong \bar{A}[x]$.

Now suppose that $\mathfrak{m}$ is a maximal ideal in A. Let $\Bbbk = A/\mathfrak{m}$ (which is a field). Applying the above discussion to the case of $\mathfrak{p} = \mathfrak{m}$, we get that $A[x]/\mathfrak{m}[x] \cong \Bbbk[x]$. As $\Bbbk[x]$ is never a field (for example, $x \neq 0$ is never a unit in $\Bbbk[x]$), we conclude that $\mathfrak{m}[x]$ is never a maximal ideal in $A[x]$. □