Math 351 DNHI 4

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(1) The set $C_r(x) = \{y \in M : d(x, y) \leq r\}$ is called the closed ball about x of radius r. Show that $C_r(x)$ is a closed set, but give an example showing that $C_r(x)$ need not equal the closure of the open ball $B_r(x)$.

Proof:
To prove that $C_r(x)$ is closed, it suffices to show that $[C_r(x)]^c$ is open. Let $y \in [C_r(x)]^c$ so that $d(x, y) = p > r$.
Let $\varepsilon = p - r > 0$. We now show that $B_\varepsilon(y) \subset [C_r(x)]^c$. With that in mind, pick $z \in B_\varepsilon(y)$, then,

$$d(z, x) > |d(z, y) - d(x, y)| = |\varepsilon - p| = r$$

which implies that $z \in [C_r(x)]^c$. Thus $B_\varepsilon(y) \subset [C_r(x)]^c$, which proves that $[C_r(x)]^c$ is open and therefore $C_r(x)$ is closed. ■

(2) Show that A is open iff $A^0 = A$ and that A is closed iff $\overline{A} = A$.

Proof:
Let $O_A = \{\mathcal{U} : \mathcal{U} \subseteq A\,,\ \mathcal{U}$ is open in $M\}$ (We are assuming that $A \subset (M, d)$).
Then $A^0 = \bigcup_{\mathcal{U} \in O_A} \mathcal{U}$. Clearly $A^0 \subseteq A$ by definition.
If $A^0 = A$, then A is the union of open sets, which means that A is open. If A is open then $A \in O_A$ and for any $\mathcal{U} \in O_A$, $\mathcal{U} \subseteq A$.
Thus,

$$\bigcup_{\mathcal{U} \in O_A} \mathcal{U} \subseteq A \subseteq \bigcup_{\mathcal{U} \in O_A} \mathcal{U} = A^0.$$

Hence $A^0 = A$ iff A is open.

We now verify that A is closed iff $A = \overline{A}$. Define $\mathcal{L}_A = \{F : A \subseteq F\,,\ F$ is closed in $M\}$. Then $\overline{A} = \bigcap_{F \in \mathcal{L}_A} F$. Clearly $A \subseteq \overline{A}$ by definition.
If $A = \overline{A}$, then A is equal to the intersection of closed sets, which is closed. This implies that A is closed. On the other hand, if A is closed then $A \in \mathcal{L}_A$ and $\overline{A} = \bigcap_{F \in \mathcal{L}_A} F \subseteq A$. Since A is always a subset of $\overline{A}$, we get $\overline{A} \subseteq A \subseteq \overline{A}$, which implies that $A = \overline{A}$. ■

(3) Given a nonempty bounded subset E of $\mathbb{R}$, show that $\sup E$ and $\inf E$ are

elements of $\overline{E}$. Thus $\sup E$ and $\inf E$ are elements of E whenever E is closed.

Proof:
Let $E \subset \mathbb{R}$ be nonempty and bounded. Set $\alpha = \sup E$. Then for any $\varepsilon > 0$, $B_\varepsilon(\alpha) = (\alpha - \varepsilon, \alpha + \varepsilon)$ contains some $x \in E$ in the segment $(\alpha - \varepsilon, \alpha) \subset B_\varepsilon(\alpha)$.
In other words, α is a limit point of E.
If $E \subset F$, where F is a closed set, then α is also a limit point of F. Therefore, $\alpha \in F$. Since $E \subseteq \overline{E}$, it is true that $\alpha \in \overline{E}$, which shows that $\sup E$ is an element of $\overline{E}$.
The proof that $\inf E$ is also an element of $\overline{E}$ is similar. ■

(4) Show that diam $(A) = \text{diam}(\overline{A})$.

Proof:
First observe that if $A \subset B$, then $\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}$.
Therefore

$$\text{diam}(A) = \sup \{d(a, b) : a, b \in A\} \leq \sup \{d(a, b) : a, b \in B\} = \text{diam}(B) .$$

Since $A \subseteq \overline{A}$, it follows that $\text{diam}(A) \leq \text{diam}(\overline{A})$. Thus, $\alpha = \text{diam}(\overline{A})$ is an upper bound of $\{d(a, b) : a, b \in A\}$. To show that $\text{diam}(A) = \text{diam}(B)$ it therefore suffices to prove that α is the least upper bound of the set $\{d(a, b) : a, b \in A\}$.
Let $\varepsilon > 0$. Then $\alpha - \varepsilon$ is not an upper bound of $\left\{d(a, b) : a, b \in \overline{A}\right\}$ and there are points $x, y \in \overline{A}$ such that $d(x, y) > \alpha - \frac{\varepsilon}{2}$. Notice however that for any $x, y \in \overline{A}$ we have $a, b \in A$ such that $d(x, a) < \frac{\varepsilon}{4}$ and $d(y, b) < \frac{\varepsilon}{4}$.
Therefore,

$$d(x, y) \leq d(x, a) + d(a, b) + d(b, y) < \frac{\varepsilon}{2} + d(a, b) .$$

Thus,

$$\alpha - \frac{\varepsilon}{2} < d(x, y) < \frac{\varepsilon}{2} + d(a, b) \qquad \text{or} \qquad \alpha - \varepsilon < d(a, b) .$$

This means that $\alpha - \varepsilon$ is also not an upper bound of $\{d(a, b) : a, b \in A\}$. Hence, $\alpha = \sup \{d(a, b) : a, b \in A\}$ and $\text{diam}(A) = \text{diam}(\overline{A})$ as desired. ■

(5) If $A \subset B$, show that $\overline{A} \subset \overline{B}$. Does $\overline{A} \subset \overline{B}$ imply $A \subset B$? Explain.

Proof:
Recall that $B \subseteq \overline{B}$ for any set B. If $A \subset B$, then $A \subset B \subset \overline{B}$, which means that $\overline{B} \in \mathcal{L}_A = \{F : A \subset F,\ F \text{ is closed}\}$. Hence $\overline{A} = \bigcap_{F \in \mathcal{L}_A} F \subseteq \overline{B}$, from which $\overline{A} \subset \overline{B}$ follows.
Note that $\overline{A} \subset \overline{B}$ does not imply $A \subset B$. In fact $A \cap B$ could be empty.
For instance, if $A = \mathbb{Q} \cap [0, 1]$ and $B = \mathbb{R} \setminus \mathbb{Q} \cap [0, 2]$, then $\overline{A} = [0, 1] \subset [0, 2] = \overline{B}$, but clearly

$A \cap B = \emptyset$. ■

(6) If A and B are any sets in M, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Give an example showing that this last inclusion can be proper.

Proof:
Let $A, B \subset M$. Observe that $\overline{A}$ is the smallest closed set that contains A. That is, if $A \subset F$ and F is closed, then $A \subseteq \overline{A} \subseteq F$.
Notice that $A, B \subset A \cup B \subset \overline{A \cup B}$. Since $\overline{A \cup B}$ is closed, $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ by the above remark. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Similarly, since the union of two closed sets is again closed, we have $A, B \subseteq \overline{A} \cup \overline{B}$, which implies that $A \cup B \subseteq \overline{A} \cup \overline{B}$ from which $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ follows. We have thus shown that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
To prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, notice that $A \cap B \subset \overline{A} \cap \overline{B}$ and since $\overline{A} \cap \overline{B}$ is closed, we have $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. This time equality does not always occur: If $A = \mathbb{Q}$ and $B = \mathbb{R} \backslash \mathbb{Q}$, then $\overline{A \cap B} = \overline{\emptyset} = \emptyset$, whereas $\overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$. ■

(7) True or False? $(A \cup B)^0 = A^0 \cup B^0$.

Solution:
False. Observe that $A^0 \cup B^0$ is always a subset of $(A \cup B)^0$, but equality does not always occur: Let $A = \mathbb{Q}$, $B = \mathbb{R} \backslash \mathbb{Q}$.
Then $A^0 \cup B^0 = \emptyset \cup \emptyset = \emptyset$ but $(A \cup B)^0 = \mathbb{R}^0 = \mathbb{R}$.

(8) Show that $\overline{A} = [\text{int}(A^c)]^c$ and that $A^0 = [\text{cl}(A^c)]^c$.

Proof:
Observe that $x \notin \overline{A}$ iff $x \notin A$ and x is an isolated point of A.
That is,

$$[\overline{A}]^c = \bigcup_{x: B_\varepsilon(x) \subset A^c} B_\varepsilon(x).$$

A bit of thought should convince us that $[\overline{A}]^c = \text{int}(A^c)$. Hence $\overline{A} = [\text{int}(A^c)]^c$ as desired.

Now to show that $A^0 = [\text{cl}(A^c)]^c$, set $B = A^c$.
Then,

$$\text{cl}(A^c) = \overline{B} = [\text{int}(B^c)]^c = [A^0]^c$$

by the result obtained above.

Hence $[\text{cl}(A^c)]^c = A^0$. ■

(9) A set that is simultaneously open and closed is sometimes called a clopen set. Show that $\mathbb{R}$ has no nontrivial clopen sets.

Proof:

Suppose that $A \subseteq \mathbb{R}$ is a nonempty, proper, open subset of $\mathbb{R}$. We will show that A cannot be closed. Let $x \in A$ and then set $b = \sup\{y : [x, y) \subset A\}$ and $a = \inf\{z : (z, x] \subset A\}$. Then since A is a proper subset, either a or b must be a finite number.

Assume, WLOG, that $b < +\infty$. Since A is open, $b \notin A$ (otherwise $b \in (b - \varepsilon, b + \varepsilon) \subset A$ for some $\varepsilon > 0$, which would imply that $(a, b + \varepsilon) \subset A$, contrary to our choice of b). Hence we have that $b \in A^c$ (a closed set). If A were clopen, A^c would have also been an open set. But this is impossible because for every $\varepsilon > 0$, $B_\varepsilon(b) \bigcap A \neq \emptyset$. This implies that A^c does not contain an entire neighborhood of b. ■

(10) Let (M, d) be a metric space and $A \subset M$. Show that if x is a limit point of A, then every neighborhood of x contains infinitely many points of A.

Proof:

Suppose x is a limit point of $A \subset M$ and let $B_r(x)$ be any neighborhood of x. Then, by our hypothesis on x, we have $B_r(x) \backslash \{x\} \bigcap A \neq \emptyset$.

If $B_r(x) \backslash \{x\}$ were to contain only finitely many points $a_1, \ldots, a_n$ of A, we could then set $\varepsilon_1 = d(x, a_1), \ldots, \varepsilon_n = d(x, a_n)$, where we have that each $\varepsilon_i > 0$.

Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $B_\varepsilon(x) \subset B_r(x)$ and $B_\varepsilon(x) \backslash \{x\} \bigcap A = \emptyset$, contradicting the hypothesis that x is a limit point. $(\Rightarrow\Leftarrow)$

Thus, each neighborhhod of x must contain infinitely many points of A. ■

(11) Suppose that $x_n \overset{d}{\rightarrow} x \in M$, and let $A = \{x\} \cup \{x_n : n \geq 1\}$. Prove that A is closed.

Proof:

Suppose $x_n \overset{d}{\rightarrow} x$ and $A = \{x\} \cup \{x_n : n \geq 1\}$. We will show that A is closed by proving that A^c is open. Pick $y \in A^c$, so that $d(y, x) = 2r$. Furthermore, for some $N \in \mathbb{N}$, $x_n \in B_r(x) \;\; \forall\, n \geq N$. Let $r_1 = d(y, x), \ldots, r_N = d(y, x_n)$. Then it is true that $r_1, \ldots, r_N > 0$.

Next define $\varepsilon = \min\{r, r_1, \ldots, r_N\}$. Then $B_\varepsilon(y) \bigcap A = \emptyset$, implying that $B_\varepsilon(y) \subset A^c$, as we should verify.

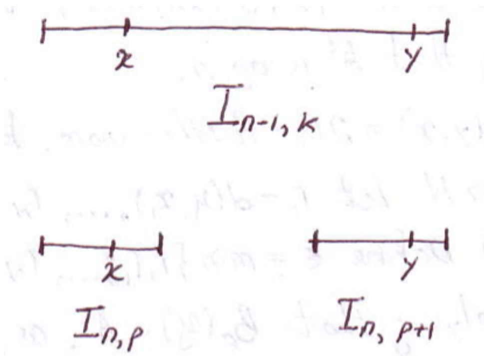
We have thus shown that A^c is open and that therefore A is closed. ■

(12) Show that any ternary decimal of the form $0.\, a_1\, a_2 \ldots a_n\, 11$ (base 3), i.e. any finite-length decimal ending in two (or more) 1's, is not an element of Δ.

Proof:
Recall that each point of Δ can be written using only the digits 0 and 2 in ternary (base 3) decimal expansion. Any number of the form $0.\, a_1\, a_2 \ldots a_n\, 11$ has only one other form, namely
$0.\, a_1\, a_2 \ldots a_n\, 10\,222 \ldots$.
Hence it is clear that a number of the form $0.\, a_1\, a_2 \ldots a_n\, 11$ cannot have the characteristic decimal expansion of elements in Δ. In particular, $0.\, a_1\, a_2 \ldots a_n\, 11$ cannot be an element of Δ. ■

(13) Show that Δ contains no (nonempty) open intervals. In particular, show that if $x,\ y \in \Delta$ with $x < y$, then there is some $z \in [0,\ 1]\backslash\Delta$ with $x < z < y$. (It follows from this that Δ is *nowhere dense*, which is another way of saying that Δ is "small")

Proof:
Every element in Δ is a limit of a sequence of nested closed subintervals. Let $x,\ y \in \Delta$ with $x < y$. Then $y - x = r$ and there is some n such that $3^{-(n-1)} \geq r$ while $3^{-n} < r$. This means that $x,\ y \in I_{n-1,k}$ where I_n is the "n^{th} level" and $I_{n,p}$ is the "n^{th}step" to the Cantor set. That is, $I_{n-1,k}$ is one of the 2^{n-1} subintervals of the $n-1$ st Cantor level of size $3^{-(n-1)}$.
Since $y - x > 3^{-n}$, we see that $x \in I_{n,p}$ and $y \in I_{n,p+1}$ for some integer $1 \leq p \leq 2^n - 1$.



Pick any point z in the omitted interval segment. Then $z \notin \Delta$ and $x < z < y$, proving that Δ contains no open interval. Since Δ is closed, we see that $\emptyset = \text{int}(\Delta) = \text{int}(\overline{\Delta})$, which establishes that Δ is nowhere dense. ■

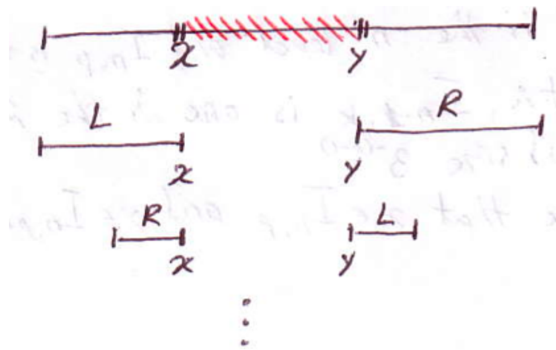
(14) The endpoints of Δ are those points in Δ having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all of the points in Δ of the form $a / 3^n$ for some integers n and $0 \leq a \leq 3^n$. Show that the endpoints of Δ other than 0 and 1 can be written as
$0.\, a_1\, a_2 \ldots a_{n+1}$ (base 3), where each a_k is 0 or 2, except
a_{n+1}, which is either 1 or 2. That is, the discarded "middle third" intervals are of the form

$(0.a_1 a_2 \ldots a_n 1,\ 0.a_1 a_2 \ldots a_n 2)$, where both entries are points of Δ written in base3.

Proof:

Since Δ is the set of limits of sequences of left "L" and right "R" nested intervals, it is clear that two successive endpoints $x < y$ are of the form

$$x = A_1 A_2 \ldots A_n LRRRR \ldots \ ; \quad y = A_1 A_2 \ldots A_n RLLLL \ldots$$



This means that $x = 0.a_1 a_2 \ldots a_n 0222\ldots$ and $y = 0.a_1 a_2 \ldots a_n 2000\ldots$ (base 3), where each a_i is either 0 or 2.

Hence $x = 0.a_1 a_2 \ldots a_n 1$ and $y = 0.a_1 a_2 \ldots a_n 2$. ■

(15) Show that Δ is perfect; that is, every point in Δ is the limit point of a sequence of distinct points from Δ. In fact, show that every point in Δ is the limit of a sequence of distinct endpoints.

Proof:

Each point $x \in \Delta$ is the limit of a sequence of nested, closed intervals $\{I_{n,k_n}\}$ with $\text{length}(I_{n,k_n}) = 3^{-n}$.

That is, $\{x\} = \bigcap_{n=1}^{\infty} I_{n,k_n}$.

Pick the right endpoints x_n of I_{n,k_n}. Then each $x_n \in \Delta$ and $x_n \to x$ (If x is itself a right endpoint of some interval, use the left endpoints of I_{n,k_n}).

Thus every point $x \in \Delta$ is a limit point of the endpoints of Δ. Furthermore, since Δ is closed, we may conclude that Δ is perfect. ■

(16) Let $f : \Delta \longrightarrow [0,\ 1]$ be the Cantor function and let $x,\ y \in \Delta$ with $x < y$. Show that $f(x) \le f(y)$. If $f(x) = f(y)$, show that x has two distinct binary decimal expansions.

Finally, show that $f(x) = f(y)$ iff x and y are "consecutive" endpoints of the form $x = 0.a_1 a_2 \ldots a_n 1$ and $y = 0.a_1 a_2 \ldots a_n 2$ (base 3).

Proof:

Suppose $x,\ y \in \Delta$ with $x < y$.

Then

$$x = 0.(2\,a_1)(2\,a_2)\ldots(2\,a_n)\ldots \quad \text{and} \quad y = 0.(2\,b_1)(2\,b_2)\ldots(2\,b_n)\ldots$$

where the a_i and b_i are either 0 or 1.

Let n be the smallest integer for which $a_i < b_i$. Then $a_n = 0$ and $b_n = 1$. Also, note that $a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_{n-1} = b_{n-1}$.

Now,

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{a_k}{2^k} = \sum_{k=1}^{n-1} \frac{a_k}{2^k} + a_n + \sum_{k=n+1}^{\infty} \frac{a_k}{2^k} \\ &= \sum_{k=1}^{n-1} \frac{b_k}{2^k} + \sum_{k=n+1}^{\infty} \frac{a_k}{2^k} \leq \sum_{k=1}^{n-1} \frac{b_k}{2^k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \\ &= \sum_{k=1}^{n-1} \frac{b_k}{2^k} + \frac{1}{2^n} = \sum_{k=1}^{n} \frac{b_k}{2^k} \leq \sum_{k=1}^{\infty} \frac{b_k}{2^k} = f(y) \end{aligned}$$

Thus $f(x) \leq f(y)$ with equality holding iff $a_{n+1} = a_{n+2} =.\,.. = 1$ and $b_{n+1} = b_{n+2} =.\,.. = 0$. That is, iff $x = 0.\,c_1\,c_2\ldots c_{n-1}\,1$ and $y = 0.\,c_1\,c_2\ldots c_{n-1}\,2$, where each c_i is either 0 or 2. That is, $f(x) = f(y)$ iff x and y are consecutive endpoints. ■