TRRT Final Hand-In (PQ3)

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Some Lie Algebra Cohomology

Let L be a Lie algebra and V an L-module. Recall that this means that we have a bilinear map $L \times V \to V$, sending $(x, v) \mapsto x \cdot v$, satisfying

$$x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, y] \cdot v. \tag{$*$}$$

We define vector spaces $C^p(L; V)$, for $p = 0, 1, 2, 3$, by declaring $C^0(L; V) = V$ and

$$\begin{aligned} C^1(L; V) &= \{\text{linear } L \to V\} \\ C^2(L; V) &= \{\text{alternating bilinear } L \times L \to V\} \\ C^3(L; V) &= \{\text{alternating trilinear } L \times L \times L \to V\}. \end{aligned}$$

(One can continue in the obvious way, but we will not need to go beyond trilinear maps in this question.) In particular, this means that if $\varphi \in C^2(L; V)$, then for all $x, y \in L$, $\varphi(x, y) = -\varphi(y, x) \in V$. Similarly if $\psi \in C^3(L; V)$, then for all $x, y, z \in L$, we have $\psi(x, y, z) = -\psi(x, z, y) = -\psi(y, x, z) \in V$.

We define linear maps $\partial_p \colon C^p(L; V) \to C^{p+1}(L; V)$ for $p = 0, 1, 2$ as follows:

$$\begin{aligned} (\partial_0 v)(x) &= x \cdot v \\ (\partial_1 \zeta)(x, y) &= x \cdot \zeta(y) - y \cdot \zeta(x) - \zeta([x, y]) \\ (\partial_2 \varphi)(x, y, z) &= -x \cdot \varphi(y, z) - \varphi(x, [y, z]) + \text{cyclic in } x, y, z, \end{aligned}$$

for all $x, y, z \in L$, $v \in V$, $\zeta \in C^1(L; V)$ and $\varphi \in C^2(L; V)$.

Problem 1. *Prove the following:*
a) $\partial_1 \circ \partial_0 = 0$.
b) $\partial_2 \circ \partial_1 = 0$.

*Proof of **a)***. We have

$$C^0(L; V) = V \xrightarrow{\partial_0} C^1(L; V) \xrightarrow{\partial_1} C^2(L; V),$$

where

$$v \xmapsto{\partial_0} \partial_0 v \xmapsto{\partial_1} \partial_1(\partial_0 v).$$

Then, for $x, y \in L$, we get

$$\begin{aligned}
\partial_1(\partial_0 v)(x,y) &= x \cdot (\partial_0 v)(y) - y \cdot (\partial_0 v)(x) - (\partial_0 v)([x,y]) \\
&= x \cdot (y \cdot v) - y \cdot (x \cdot v) - [x,y] \cdot v \\
&= 0. && \text{(By } (*)\text{)}
\end{aligned}$$

$\boxed{\text{Victoria!}}$

Proof of ***b)***. This time we have

$$C^1(L;V) \xrightarrow{\partial_1} C^2(L;V) \xrightarrow{\partial_2} C^3(L;V),$$

where

$$\zeta \xmapsto{\partial_1} \partial_1\zeta \xmapsto{\partial_2} \partial_2(\partial_1\zeta).$$

Then, for $x, y, z \in L$, we get the following mess:

$$\begin{aligned}
\partial_2(\partial_1\zeta)(x,y,z) &= -x \cdot (\partial_1\zeta)(y,z) - (\partial_1\zeta)(x,[y,z]) \dots \\
&\dots - y \cdot (\partial_1\zeta)(z,x) - (\partial_1\zeta)(y,[z,x]) \dots \\
&\dots - z \cdot (\partial_1\zeta)(x,y) - (\partial_1\zeta)(z,[x,y]) \\
&= -x \cdot \{y \cdot \zeta(z) - z \cdot \zeta(y) - \zeta([y,z])\} \dots \\
&\dots - \{x \cdot \zeta([y,z]) - [y,z] \cdot \zeta(x) - \zeta([x,[y,z]]\} \dots \\
&\dots + \text{cyclic mumbo jumbo} \\
&= -x \cdot y \cdot \zeta(z) + x \cdot z \cdot \zeta(y) + \underbrace{x \cdot \zeta([y,z]) - x \cdot \zeta([y,z])}_{=0} \dots \\
&\dots + [y,z] \cdot \zeta(x) + \underbrace{\zeta([x,[y,z]])}_{\mathrm{j}_1} \dots \\
&\dots - y \cdot z \cdot \zeta(x) + y \cdot x \cdot \zeta(z) + \underbrace{y \cdot \zeta([z,x]) - y \cdot \zeta([z,x])}_{=0} \dots \\
&\dots + [z,x] \cdot \zeta(y) + \underbrace{\zeta([y,[z,x]])}_{\mathrm{j}_2} \dots \\
&\dots - z \cdot x \cdot \zeta(y) + z \cdot y \cdot \zeta(x) + \underbrace{z \cdot \zeta([x,y]) - z \cdot \zeta([x,y])}_{=0} \dots \\
&\dots + [x,y] \cdot \zeta(z) + \underbrace{\zeta([z,[x,y]])}_{\mathrm{j}_3}
\end{aligned}$$

Note that the sum $\mathrm{j}_1 + \mathrm{j}_2 + \mathrm{j}_3$ (as I labeled on the above messy equation) vanishes, since ζ is linear and thus

$$\begin{aligned}
\zeta([x,[y,z]]) + \zeta([y,[z,x]]) + \zeta([z,[x,y]]) &= \zeta([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \\
&= \zeta(0) && \text{(By the Jacobi identity of the bracket)} \\
&= 0.
\end{aligned}$$

Hence we are left with

$$\begin{aligned}
\partial_2(\partial_1\zeta)(x,y,z) &= -x\cdot y\cdot\zeta(z) + x\cdot z\cdot\zeta(y) + [y,z]\cdot\zeta(x)\dots \\
&\quad \dots - y\cdot z\cdot\zeta(x) + y\cdot x\cdot\zeta(z) + [z,x]\cdot\zeta(y)\dots \\
&\quad \dots - z\cdot x\cdot\zeta(y) + z\cdot y\cdot\zeta(x) + [x,y]\cdot\zeta(z) \\
&= 0+0+0 && \text{(By }(*)\text{)} \\
&= 0. && \boxed{\text{Victoria!}}
\end{aligned}$$

Let $p = 0, 1, 2$ and define the subspaces $Z^p(L;V) = \ker\partial_p$, $B^0(L;V) = 0$, and $B^p(L;V) = \operatorname{im}\partial_{p-1}$ for $p = 1, 2$. By Q1, we have $B^p(L;V) \subset Z^p(L;V)$. Define $H^p(L;V) = Z^p(L;V)/B^p(L;V)$.

Problem 2. *Take $V = \mathbb{C}$ to be the trivial module. Prove that*
a) *$H^0(L;\mathbb{C}) \cong \mathbb{C}$.*
b) *$H^1(L;\mathbb{C}) \cong (L/L')^*$, where $L' = [L,L]$ is the derived ideal and $*$ denotes the dual vector space.*

*Proof of **a)**.* We have

$$\begin{aligned}
H^0(L;\mathbb{C}) &= Z^0(L;\mathbb{C})/B^0(L;\mathbb{C}) \\
&= \ker\partial_0/0 \\
&\cong \ker\partial_0,
\end{aligned}$$

where

$$\ker\partial_0 = \{v\in\mathbb{C} | x\cdot v = 0\ \forall x\in L\}.$$

But since we are using the trivial representation, $x\cdot v = 0$ not only for all $x\in L$, but also for all $v\in V = \mathbb{C}$. Thus we have that $\ker\partial_0 = \mathbb{C}$, and this proves the statement. $\boxed{\text{Victoria!}}$

*Proof of **b)**.* We have

$$\begin{aligned}
H^1(L;\mathbb{C}) &= Z^1(L;\mathbb{C})/B^1(L;\mathbb{C}) \\
&= \ker\partial_1/\operatorname{im}\partial_0.
\end{aligned}$$

But $\operatorname{im}\partial_0 = \{x\cdot v\}$ vanishes for all $x\in L$ and $v\in\mathbb{C}$, since we are using the trivial representation. Meanwhile the kernel of ∂_1 consists of all $\zeta\in C^1$ satisfying, for all $x, y\in L$,

$$x\cdot\zeta(y) - y\cdot\zeta(x) - \zeta([x,y]) = 0.$$

In the trivial representation $V = \mathbb{C}$, we have $\zeta\colon L\to\mathbb{C}$, i.e., $\zeta\in L^*$. Whence we have

$$\underbrace{x\cdot\zeta(y) - y\cdot\zeta(x)}_{x\cdot\zeta(-)-y\cdot\zeta(-)=\text{linear combination}\in L^*} \qquad - \qquad \underbrace{\zeta([x,y])}_{\zeta\in L'^*}.$$

Thus $\ker \partial_1$ consists of elements of the form $\xi + L'^*$, with $\xi \in L^*$, and so we have

$$\begin{aligned} H^1(L;\mathbb{C}) &= \ker \partial_1 / \operatorname{im} \partial_0 \\ &= (L^*/L'^*)/0 \\ &\cong (L/L')^*/0 \\ &\cong (L/L')^*. \end{aligned}$$

$\boxed{\text{Victoria!}}$

A ***central extension*** of a Lie algebra L is a Lie algebra on the vector space $\widehat{L} = L \oplus \operatorname{Span}\{Z\}$ with bracket

$$[x,y] = [x,y]_L + \omega(x,y)Z \qquad \text{and} \qquad [Z,x] = 0, \tag{1}$$

for all $x, y \in L$, and where $[-,-]_L$ denotes the bracket of L and $\omega \in C^2(L;\mathbb{C})$ (the quotient Lie algebra $\widehat{L}/\operatorname{Span}\{Z\}$ is isomorphic to L). A central extension $\widehat{L}$ of L is said to be ***trivial*** if $\widehat{L} = L \oplus \mathbb{C}$ as Lie algebras.

Problem 3. *Show that $H^2(L;\mathbb{C})$ classifies isomorphism classes of central extensions of L as follows:*

a) *Show that $\omega \in C^2(L;\mathbb{C})$ defines a central extension as in (1) if and only if $\omega \in Z^2(L;\mathbb{C})$.*

b) *Show that if $\omega_1, \omega_2 \in Z^2(L;\mathbb{C})$ are such that $\omega_1 - \omega_2 = \partial_1 \zeta$ for some $\zeta \in C^1(L;\mathbb{C})$, then the central extensions defined by ω_1 and ω_2 are isomorphic. (Hint: use $\zeta\colon L \to \mathbb{C}$ to build the isomorphism.)*

*Proof of **a)**.* If $\omega \in Z^2(L;\mathbb{C}) = \ker \partial_2$, then we have, for $x, y, z \in L$,

$$\begin{aligned} 0 = (\partial_2 \omega)(x,y,z) = \underbrace{-x \cdot \omega(y,z)}_{=0} &- \omega(x,[y,z]_L) \dots \\ \dots \underbrace{-y \cdot \omega(z,x)}_{=0} &- \omega(y,[z,x]_L) \dots \\ \dots \underbrace{-z \cdot \omega(x,y)}_{=0} &- \omega(z,[x,y]_L), \end{aligned}$$

where the vanishing terms are due to the fact that we are using the trivial representation $V = \mathbb{C}$, so that the L-action acts as an annihilator. Thus we are left with

$$0 = -\omega(x,[y,z]_L) - \omega(y,[z,x]_L) - \omega(z,[x,y]_L),$$

or equivalently,

$$\omega(x,[y,z]_L) + \omega(y,[z,x]_L) + \omega(z,[x,y]_L) = 0,$$

which is the co-cycle condition that needs to be satisfied for any one-dimensional central extension of L. It is also clear that if ω defines the central extension as in (1), then by applying the Jacobi identity to both sides of (1) we get that ω satisfies the co-cycle condition and therefore that it's an element of $Z^2(L;\mathbb{C})$. Thus the biconditional statement is satisfied. $\boxed{\text{Victoria!}}$

*Proof of **b)**.* Let $\widehat{L}_{\omega_1}$ and $\widehat{L}_{\omega_2}$ denote the central extensions defined by ω_1 and ω_2, respectively. Then we have a map

$$\begin{aligned}\psi\colon \widehat{L}_{\omega_1} &\longrightarrow \widehat{L}_{\omega_2}\\ (x,y) &\longmapsto (x, y+\zeta(x)),\end{aligned}$$

for all $x, y \in L$ with both y and $y+\zeta(x)$ lying in $\operatorname{Span}\{Z\}$. This map is clearly a vector space isomorphism, so we merely need to check that it respects the Lie bracket:

Consider two elements $(x,y),(x',y') \in \widehat{L}_{\omega_1}$.[1] Then

$$\begin{aligned}\psi([(x,y),(x',y')]) &= \psi([x,x']_L,\ x\cdot y' - x'\cdot y + \omega_1(x,x'))\\ &= ([x,x']_L,\ x\cdot y' - x'\cdot y + \omega_1(x,x') + \zeta([x,x'])),\end{aligned}$$

while

$$\begin{aligned}[\psi(x,y),\psi(x',y')] &= ([x,x']_L,\ x\cdot(y'+\zeta(x')) - x'\cdot(y+\zeta(x)) + \omega_2(x,x'))\\ &= ([x,x']_L,\ x\cdot y' - x'\cdot y + x\cdot\zeta(x') - x'\cdot\zeta(x) + \omega_2(x,x')).\end{aligned}$$

Note that these two expressions are equal precisely when $\omega_1 - \omega_2 = \partial_1\zeta$. Thus, under this condition, we have that ψ is indeed a Lie isomorphism between $\widehat{L}_{\omega_1}$ and $\widehat{L}_{\omega_2}$, which is the result we wanted. $\boxed{\text{Victoria!}}$

Now take $V = L$, the L-module corresponding to the adjoint representation.

Problem 4. *Show that $H^0(L;L) = Z(L)$, the centre of L.*

Proof. We have

$$\begin{aligned}H^0(L;L) &= Z^0(L;L)/B^0(L;L)\\ &= \ker\partial_0/0\\ &\cong \ker\partial_0.\end{aligned}$$

Now,

$$\begin{aligned}\partial_0\colon C^0(L;L) = L &\longrightarrow C^1(L;L)\\ x &\longmapsto \partial_0 x,\end{aligned}$$

[1] Note that I'm using slightly different notation for this exercise because I find the bracket defined in (1) a bit confusing for computational purposes.

where

$$(\partial_0 x)(y) = y \cdot x$$
$$= [y, x] \qquad \text{(Since the } L\text{-action is now given by the adjoint map)}$$

Thus,

$$\ker \partial_0 = \{x \in L | [y, x] = 0 \ \forall y \in L\} = Z(L).$$

This proves the result. Victoria!

A linear map $D\colon L \to L$ is a ***derivation*** if $D[x, y] = [Dx, y] + [x, Dy]$ for all $x, y \in L$.

Problem 5. *Show the following:*

a) *The set* $\mathrm{Der}(L)$ *of derivations of* L *is a Lie subalgebra of* $\mathfrak{gl}(L)$*;*
b) *The image* $\mathrm{ad}(L)$ *of the adjoint representation* $\mathrm{ad}\colon L \to \mathfrak{gl}(L)$ *is an ideal of* $\mathrm{Der}(L)$*;*
c) $H^1(L; L) = \mathrm{Der}(L)/\mathrm{ad}(L)$.

Proof of ***a)***. First of all, to see that $\mathrm{Der}(L)$ is a subspace of $\mathfrak{gl}(L)$, note that the derivations of L are those maps $D \in \mathfrak{gl}(L)$ which satisfy $D[x, y] - [Dx, y] - [x, Dy] = 0$ for all $x, y \in L$. For fixed x and y, the left hand side of this equation is linear in D, so that the set of endomorphisms satisfying the equation is a subspace. The set of dervations is the intersection over all $x, y \in L$ of these subspaces, which is itself a subspace.

Thus to show that $\mathrm{Der}(L)$ is a subalgebra of $\mathfrak{gl}(L)$, the only thing left to check is that the bracket of two derivations is also a derivation. Let $x, y \in L$ and $D_1, D_2 \in \mathrm{Der}(L)$. Then,

$$\begin{aligned}
[D_1, D_2]([x, y]) &= D_1(D_2([x, y])) - D_2(D_1([x, y])) \\
&= D_1([D_2(x), y] + [x, D_2(y)]) - D_2([D_1(x), y] + [x, D_1(y)]) \\
&= D_1([D_2(x), y]) + D_1([x, D_2(y)]) - D_2([D_1(x), y]) - D_2([x, D_1(y)]) \\
&= [D_1(D_2(x)), y] + \underbrace{[D_2(x), D_1(y)]}_{\mathfrak{d}_1} + \underbrace{[D_1(x), D_2(y)]}_{\mathfrak{d}_2} + [x, D_1(D_2(y))] \ldots \\
&\quad \cdots - [D_2(D_1(x)), y] \underbrace{-[D_1(x), D_2(y)]}_{-\mathfrak{d}_2} \underbrace{-[D_2(x), D_1(y)]}_{-\mathfrak{d}_1} - [x, D_2(D_1(y))] \\
&= [D_1(D_2(x)), y] - [D_2(D_1(x)), y] + [x, D_1(D_2(y))] - [x, D_2(D_1(y))] \\
&= [[D_1, D_2](x), y] + [x, [D_1, D_2](y)].
\end{aligned}$$

Victoria!

Proof of ***b)***. First of all, note that by the Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]], \qquad \text{where } x, y, z \in L,$$

it is clear that $\mathrm{ad}(x)$ is a derivation of L; i.e., $\mathrm{ad}(L)$ is a subspace of $\mathrm{Der}(L)$. Now to prove it is also an ideal, let $x \in L$ and $D \in \mathrm{Der}(L)$. We want to show that $[D, \mathrm{ad}(x)] \in \mathrm{ad}(L)$.

Take $z \in L$, and apply the bracket:

$$\begin{aligned} [D, \mathrm{ad}(x)](z) &= D(\mathrm{ad}(x)(z)) - \mathrm{ad}(x)(D(z)) \\ &= D([x, z]) - [x, D(z)] \\ &= [D(x), z] && \text{(Since } D \text{ is a derivation)} \\ &= \mathrm{ad}(D(x))(z). \end{aligned}$$

This shows that

$$[D, \mathrm{ad}(x)] = \mathrm{ad}(D(x)) \qquad \forall\, D \in \mathrm{Der}(L),\ \forall\, x \in L,$$

and thus $\mathrm{ad}(L)$ is an ideal of $\mathrm{Der}(L)$, as desired. Victoria!

Proof of ***c)***. We have

$$\begin{aligned} H^1(L; L) &= Z^1(L; L)/B^1(L; L) \\ &= \ker \partial_1 / \mathrm{im}\, \partial_0. \end{aligned}$$

Now, $\ker \partial_1$ consists of all $\zeta \in C^1(L; L)$ that satisfy

$$[x, \zeta(y)] - [y, \zeta(x)] - \zeta([x, y]) = 0,$$

or, equivalently,

$$\begin{aligned} \zeta([x, y]) &= -[y, \zeta(x)] + [x, \zeta(y)] \\ &= [\zeta(x), y] + [x, \zeta(y)] && \text{(By property of the Lie bracket)}. \end{aligned}$$

This shows that ζ is in fact a derivation; i.e., $\zeta \in \mathrm{Der}(L)$. The other inclusion is straightforward: every derivation $D \in \mathrm{Der}(L)$ by definition satisfies $D[x, y] = [Dx, y] + [x, Dy]$ for all $x, y \in L$. Then working our calculation above backwards we get that D must satisfy $[x, D(y)] - [y, D(x)] - D([x, y]) = 0$, which says precisely that $D \in \ker \partial_1$. Thus we conclude that $\mathrm{Der}(L) = \ker \partial_1$.

Now, with regards to $\mathrm{im}\, \partial_0$, note that, for $x, y \in L$, we have

$$(\partial_0 y)(x) = [x, y],$$

which shows that ∂_0 is actually the adjoint map, and thus $\mathrm{im}(\partial_0) = \mathrm{ad}(L)$. Thus we have that $H^1(L; L) = \mathrm{Der}(L)/\mathrm{ad}(L)$, as desired. Victoria!

From Q5(c), $H^1(L; L)$ is a Lie algebra. Let L be the 3-dimensional Lie algebra with basis p, q, r and only nonzero bracket $[p, q] = r$. (In particular, r is central.)

Problem 6. *Show that $H^1(L;L) \cong \mathfrak{gl}_2$ as Lie algebras.*

Proof. For an arbitrary $D \in \mathrm{Der}L$, we have

$$\begin{aligned} D(p) &= d_1p + d_2q + d_3r \\ D(q) &= d_4p + d_5q + d_6r \\ D(r) &= d_1r + d_5r, \end{aligned}$$

for arbitrary constants $d_i \in \mathbb{C}$ ($i = \{1, \cdots, 5\}$).

To see this, note

$$\begin{aligned} D(r) &= D([p,q]) \\ &= [Dp, q] + [p, Dq] \\ &= [d_1p + d_2q + d_3r, q] + [p, d_4p + d_5q + d_6r] \\ &= [d_1p, q] + [p, d_5q] \\ &= d_1[p,q] + d_5[p,q] \\ &= d_1r + d_5r. \end{aligned}$$

So, all outer derivations of L are linear maps of the form

$$D = \begin{pmatrix} d_1 & d_4 & 0 \\ d_2 & d_5 & 0 \\ d_3 & d_6 & d_1 + d_5 \end{pmatrix}.$$

Now we define a projection $\pi\colon \mathrm{Der}(L) \to \mathfrak{gl}_2$ by

$$\begin{pmatrix} d_1 & d_4 & 0 \\ d_2 & d_5 & 0 \\ d_3 & d_6 & d_1 + d_5 \end{pmatrix} \longmapsto \begin{pmatrix} d_1 & d_4 \\ d_2 & d_5 \end{pmatrix}.$$

Now, since $[p,q] = -[q,p] = r$ and $[p,r] = [q,r] = 0$ (and obviously $[p,p] = [q,q] = [r,r] = 0$ by properties of the bracket), we have that all the inner derivations $\mathrm{ad}(x)$ (for $x \in L$ and with ordered basis $\{p,q,r\}$) are linear combinations of

$$\mathrm{ad}(p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{ad}(q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}(r) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Note that the kernel of π consists precisely of linear combinations of these inner derivations; i.e. $\ker \pi = \mathrm{ad}(L)$. Therefore, since the projection π is clearly a surjective map, by the *First Isomorphism Theorem* and the result from Q5c) we have

$$H^1(L;L) = \mathrm{Der}(L)/\mathrm{ad}(L) \cong_\pi \mathfrak{gl}_2.$$

Victoria!