MATH 750 NOTES
INTEGRATION ON CHAINS

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Algebraic Preliminaries

Definition. *If V is a vector space (over $\mathbb{R}$), we will denote the k-fold product $V \times \cdots \times V$ by V^k. A function $T\colon V^k \to \mathbb{R}$ is called **multilinear** if it is linear in each coordinate, that is, if for each i with $1 \le i \le k$, we have*

$$T(v_1, \ldots, v_i + v_i', \ldots, v_k) = T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v_i', \ldots, v_k),$$
$$T(v_1, \ldots, \alpha v_i, \ldots, v_k) = \alpha T(v_1, \ldots, v_i, \ldots, v_k).$$

*A multilinear function $T\colon V^k \to \mathbb{R}$ is called a k**-tensor** on V and the set of all k-tensors, which we denote by $\mathfrak{J}^k(V)$, becomes a vector space (over $\mathbb{R}$) if for $S, T \in \mathfrak{J}^k(V)$ and $\alpha \in \mathbb{R}$ we define*

$$(S + T)(v_1, \ldots, v_k) = S(v_1, \ldots, v_k) + T(v_1, \ldots, v_k)$$
$$(\alpha S)(v_1, \ldots, v_k) = \alpha \cdot S(v_1, \ldots, v_k).$$

There is also an operation connecting the various spaces $\mathfrak{J}^k(V)$:

*If $S \in \mathfrak{J}^k(V)$ and $T \in \mathfrak{J}^\ell(V)$, then we define the **tensor product** $S \otimes T \in \mathfrak{J}^{k+\ell}(V)$ by*

$$S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = S(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+\ell}).$$

Note that the order of the factors S and T is crucial here since $S \otimes T$ and $T \otimes S$ are far from equal. ★

Remark: Note that $\mathfrak{J}^1(V)$ is just the algebraic dual space V^*. The operation $\otimes$ allows us to express the other vector spaces $\mathfrak{J}^k(V)$ in terms of $\mathfrak{J}^1(V)$. Note also that the inner product $\langle \cdot, \cdot \rangle \in \mathfrak{J}^2(\mathbb{R}^n)$ is a 2-tensor.

Theorem 1. *Let $v_1, \ldots, v_n$ be a basis for V, and let $\varphi_1, \ldots, \varphi_n$ be the dual basis $\varphi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products*

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \qquad \text{for} \quad 1 \le i_1, \ldots, i_k \le n$$

is a basis for $\mathfrak{J}^k(V)$*, which therefore has dimension* n^k*.*

Remark: One important construction, familiar for the case of dual spaces, can also be made for tensors. If $f\colon V \to W$ is a linear transformation, then we can define another linear transformation $f^* : \mathfrak{J}^k(W) \to \mathfrak{J}^k(V)$ by

$$f^*T(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$$

for $T \in \mathfrak{J}^k(W)$ and $v_1, \ldots, v_k \in V$. It is easy to verify that

$$f^*(S \otimes T) = f^*S \otimes f^*T.$$

Theorem 2. *If* T *is an inner product on* V*, then there is a basis* $v_1, \ldots, v_n$ *for* V *such that* $T(v_i, v_j) = \delta_{ij}$ *(such a basis is called* ***orthonormal*** *with respect to* T*). Consequently there is an isomorphism* $f\colon \mathbb{R}^n \to V$ *such that* $T(f(x), f(y)) = \langle x, y\rangle$ *for* $x, y \in \mathbb{R}^n$*. In other words,* $f^*T = \langle \cdot, \cdot \rangle$*.*

Definition. *A* k*-tensor* $\omega \in \mathfrak{J}^k(V)$ *is called* ***alternating*** *if*

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$*. (Note that in this equation* v_i *and* v_j *are interchanged and alI other* v*'s are left fixed.) The set of alI alternating* k*-tensors is clearly a subspace* $\Lambda^k(V)$ *of* $\mathfrak{J}^k(V)$*.* ★

How do we turn any tensor into an alternating tensor? The answer is in the following definition:

Definition. *If* $T \in \mathfrak{J}^k(V)$*, then we define the* ***alternator*** *of* T*, denoted Alt(*T*), by*

$$Alt(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where S_k *is the set of all permutations of the numbers from 1 to* k*.* ★

Theorem 3. *We have the following results:*

1) *If* $T \in \mathfrak{J}^k(V)$*, then* $Alt(T) \in \Lambda^k(V)$*.*
2) *If* $\omega \in \Lambda^k(V)$*, then* $Alt(\omega) = \omega$*.*
3) *If* $T \in \mathfrak{J}^k(V)$*, then* $Alt(Alt(T)) = Alt(T)$*.*

Remark: To determine the dimension of $\Lambda^k(V)$, we would like to have a theorem analogous to *Theorem 1*. Of course, note that if $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, then $\omega \otimes \eta$ is usually not in $\Lambda^{k+\ell}(V)$ (in other words, this tensor product may or may not result in an alternating tensor). Hence, we define a new product as follows:

Definition. *The* ***wedge product*** $\omega \wedge \eta \in \Lambda^{k+\ell}(V)$ *is defined by*

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\,\ell!} Alt(\omega \otimes \eta).$$

(The reason for the strange coefficient will appear later.) ★

Proposition 1. *Let $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^\ell(V)$, and let α be a scalar. Then the wedge product has the following properties:*

$$\begin{aligned}
(\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta, \\
\omega \wedge (\eta_1 + \eta_2) &= \omega \wedge \eta_1 + \omega \wedge \eta_2, \\
\alpha\omega \wedge \eta = \omega \wedge \alpha\eta &= \alpha(\omega \wedge \eta), \\
\omega \wedge \eta &= (-1)^{k\ell} \eta \wedge \omega_1, \\
f^*(\omega \wedge \eta) &= f^*(\omega) \wedge f^*(\eta).
\end{aligned}$$

Remark: The equation $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$ is also true but it requires more work. It is presented in the proposition below along with some other properties:

Proposition 2. *We have the following results:*

1) *If $S \in \mathfrak{J}^k(V)$, $T \in \mathfrak{J}^\ell(V)$, and $Alt(S) = 0$, then*

$$Alt(S \otimes T) = Alt(T \otimes S) = 0.$$

2) *For any tensors ω, η, θ, we have*

$$\begin{aligned}
Alt(Alt(\omega \otimes \eta) \otimes \theta) &= Alt(\omega \otimes \eta \otimes \theta) \\
&= Alt(\omega \otimes Alt(\eta \otimes \theta)).
\end{aligned}$$

3) *If $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^\ell(V)$, and $\theta \in \Lambda^m(V)$, then*

$$\begin{aligned}
(\omega \wedge \eta) \wedge \theta &= \omega \wedge (\eta \wedge \theta) \\
&= \frac{(k+\ell+m)!}{k!\,\ell!\,m!} Alt(\omega \otimes \eta \otimes \theta).
\end{aligned}$$

Remark: Now we have gathered the tools necessary to craft a theorem analogous to *Theorem 1* in order to determine the dimension of $\Lambda^k(V)$.

Theorem 4. *If $v_1, \dots, v_n$ is a basis for the vector space V, with dual basis $\varphi_1, \dots, \varphi_n$, then the set of all*

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \qquad \textit{for} \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$$

is a basis for $\Lambda^k(V)$, which therefore has dimension

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Remark: If V has dimension n, then it follows from *Theorem 4* that $\Lambda^n(V)$ has dimension 1. Thus alI alternating n-tensors on V are multiples of any nonzero one. Since the determinant is an example of such a member of $\Lambda^n(\mathbb{R}^n)$, it is not surprising to find it in the following theorem:

Theorem 5. *Let $v_1, \dots, v_n$ be a basis for the vector space V, and let $\omega \in \Lambda^n(V)$. If $w_i = \sum_{j=1}^n \alpha_{ij} v_j$ are n vectors in V, then*

$$\begin{aligned} \omega(w_1, \dots, w_n) &= \omega\left(\sum_{j=1}^n \alpha_{1j} v_j, \dots, \sum_{j=1}^n \alpha_{nj} v_j\right) \\ &= \det(\alpha_{ij}) \cdot \omega(v_1, \dots, v_n). \end{aligned}$$

Proof. Define $\eta \in \mathfrak{J}^n(\mathbb{R}^n)$ by

$$\eta((\alpha_{11}, \dots, \alpha_{1n}), \dots, (\alpha_{n1}, \dots, \alpha_{nn})) = \omega\left(\sum \alpha_{1j} v_j, \dots, \sum \alpha_{nj} v_j\right).$$

Clearly $\eta \in \Lambda^n(\mathbb{R}^n)$. Thus $\eta = \lambda \cdot \det$, for some $\lambda \in \mathbb{R}$, and furthermore,

$$\lambda = \eta(e_1, \dots, e_n) = \omega(v_1, \dots, v_n). \qquad \square$$

Remark: This theorem shows that a nonzero $w \in \Lambda^n(V)$ splits all the bases of V into two disjoint groups:

- those with $\omega(v_1, \dots, v_n) > 0$,
- and those for which $\omega(v_1, \dots, v_n) < 0$.

If $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases and $A = (\alpha_{ij})$ is defined by $w_i = \sum_j \alpha_{ij} v_j$, then $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are in the same group iff $\det(A) > 0$.

This criterion is independent of ω and can always be used to divide the bases of V into two disjoint groups. Either of these two groups is called an **orientation** for V. The orientation to which a basis $v_1, \ldots, v_n$ belongs is denoted $[v_1, \ldots, v_n]$ and the other orientation is denoted $-[v_1, \ldots, v_n]$. In $\mathbb{R}^n$ we define the **usual orientation** to be $[e_1, \ldots, e_n]$.

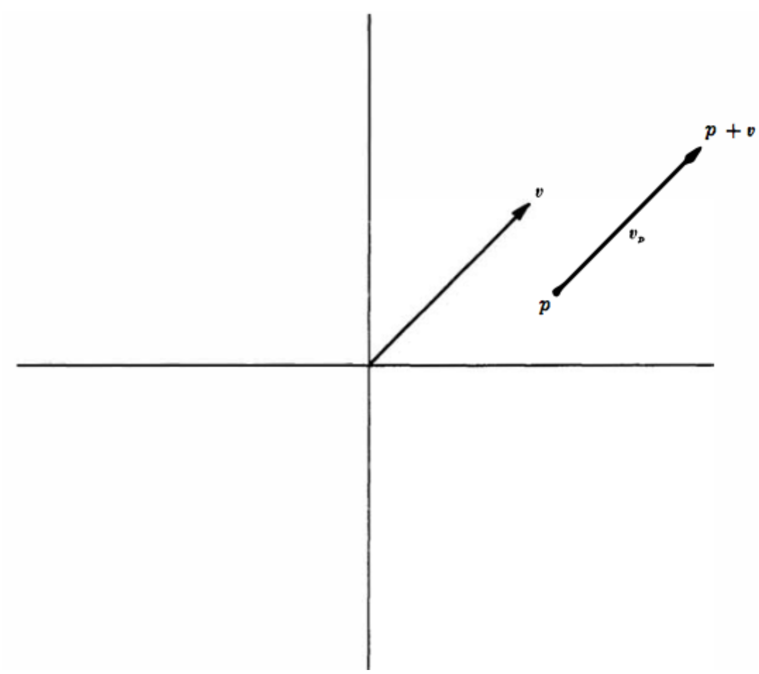
Fields & Forms

Definition. *If $p \in \mathbb{R}^n$, the set of all pairs (p, v), for $v \in \mathbb{R}^n$, is denoted $\mathbb{R}^n_p$, and called the **tangent space** of $\mathbb{R}^n$ at p.* ★

Remark 1: This set is made into a vector space in the most obvious way, by defining

$$(p, v) + (p, w) = (p, v + w),$$
$$\alpha \cdot (p, v) = (p, \alpha v).$$

A vector $v \in \mathbb{R}^n$ is often pictured as an arrow from 0 to v. The vector $(p, v) \in \mathbb{R}^n_p$ on the other hand may be pictured as an arrow with the same direction and length, but with initial point p (see figure below).



This arrow goes from p to the point $p+v$, and we therefore define $p+v$ to be the end point of (p, v). We will usually write (p, v) as v_p, which is read as "the vector v at p".

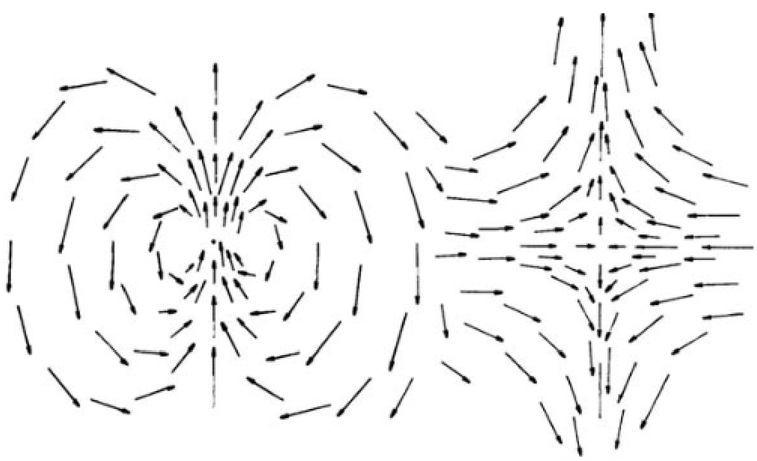
Remark 2: The vector space $\mathbb{R}^n_p$ is so closely allied to $\mathbb{R}^n$ that many of the structures on $\mathbb{R}^n$ have analogues on $\mathbb{R}^n_p$. In particular, the usual inner product $\langle\cdot,\cdot\rangle_p$ for $\mathbb{R}^n_p$ is defined by

$$\langle v_p, w_p\rangle_p = \langle v_p, w_p\rangle,$$

and the usual orientation for $\mathbb{R}^n_p$ is

$$[(e_1)_p, \ldots, (e_n)_p].$$

Remark 3: Any operation which is possible in a vector space may be performed in each $\mathbb{R}^n_p$, and most of this section is merely an elaboration of this theme. About the simplest operation in a vector space is the selection of a vector from it. If such a selection is made in each $\mathbb{R}^n_p$, then we obtain a *vector field* (see figure below).



To be precise, we give the following definition:

Definition. *A **vector field** is a function F such that $F(p) \in \mathbb{R}^n_p$ for each $p \in \mathbb{R}^n$. For each p, there are numbers $F^1(p), \ldots, F^n(p)$ such that*

$$F(p) = F^1(p)\cdot(e_1)_p + \cdots + F^n(p)\cdot(e_n)_p.$$

We thus obtain n component functions $F^i\colon \mathbb{R}^n \to \mathbb{R}$. ★

Remark: Operations on vectors yield operations on vector fields when applied at each point separately. For example, if F and G are vector fields and f is a function, then we define

$$\begin{aligned}(F+G)(p) &= F(p)+G(p),\\ \langle F,G\rangle(p) &= \langle F(p),G(p)\rangle,\\ (f\cdot F)(p) &= f(p)F(p).\end{aligned}$$

Definition. *We define the **divergence** of F, denoted div(F), as div(F) $=\sum_{i=1}^{n} D_iF^i$. Using standard notation, we define the operator*

$$\nabla=\sum_{i=1}^{n} D_i\cdot e_i.$$

Then we can write div(F) $=\langle\nabla,F\rangle$. ★

Definition. *For $n=3$, we have*

$$(\nabla\times F)(p)=(D_2F^3-D_3F^2)(e_1)_p+(D_3F^1-D_1F^3)(e_2)_p+(D_1F^2-D_2F^1)(e_3)_p.$$

*The vector field $\nabla\times F$ is called the **curl** of F, and it is denoted curl(F).* ★