A Brief Introduction to Topological Quantum Field Theories
SAMMS 2015

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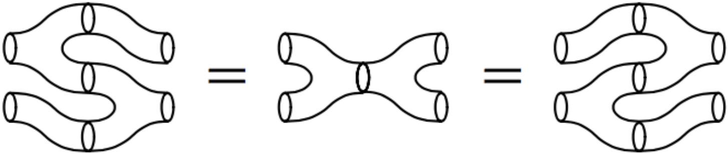
Overview

- Categories.
- Cobordisms.
- TQFT's.



Categories

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- *a class of* **objects**, *denoted* $\mathrm{Ob}(\mathcal{C})$.
- *given two objects* $x, y \in \mathcal{C}$, *a set* $\mathrm{Hom}(x, y)$ *of* **morphisms**.

Properties of Morphisms

- composition

$$\begin{tikzcd} & y \arrow[dl, "g"'] & \\ z & & x \arrow[ul, "f"'] \arrow[ll, "gf"] \end{tikzcd}$$

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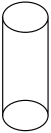
- identity morphism $1_x \colon x \to x$ such that, for any $f \colon x \to y$, we have $f1_x = f = 1_y f$.



$$\begin{tikzcd} S \arrow[d, "{[0,1]\times S}"] \\ S \end{tikzcd}$$

Examples of categories

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  - objects are (finite-dimensional) Hilbert spaces, and morphisms are linear operators.
- nCob.
  - objects are $(n-1)$-dimensional oriented compact manifolds, and morphisms are n-dimensional cobordisms.


COBOR
...What??

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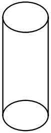

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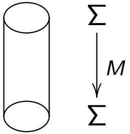

This notion of spacetime gives an intuitive representation of a **cobordism**.

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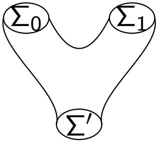

$$\begin{tikzcd} \Sigma_0 \sqcup \Sigma_1 \arrow[d, "M"] \\ \Sigma' \end{tikzcd}$$

In general spaces need not be connected.

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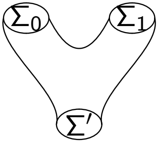

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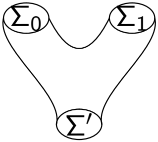

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Sooooo, what is this business of taking disjoint unions of objects in $n\mathrm{Cob}$??

Well, it turns out that $n\mathrm{Cob}$ is a **monoidal category**, i.e. a category equipped with a well defined operation $\otimes$.


WHAT'S THIS DUDE
TALKING ABOUT??

Examples of monoidal categories

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- Vect or Hilb: the operation $\otimes$ is the tensor product of vector (Hilbert, resp) spaces.
- $n\mathrm{Cob}$: the operation $\otimes$, both for objects and for morphisms, is the disjoint union of manifolds.

Disjoint union on $n\mathrm{Cob}$

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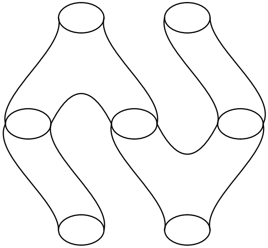
$$\begin{array}{c} S \sqcup S \\ \downarrow {\scriptstyle M^* \sqcup 1_S} \\ S \sqcup S \sqcup S \\ \downarrow {\scriptstyle 1_S \sqcup M} \\ S \sqcup S \end{array}$$



Time to talk about TQFT's...

Definition (Sort of...)

Given categories $\mathcal{C}$ *and* $\mathcal{D}$, *a* **functor** $F\colon \mathcal{C} \to \mathcal{D}$ *is a map that sends objects to objects, morphisms to morphisms, and preserves sources, targets, identities, and composition.*

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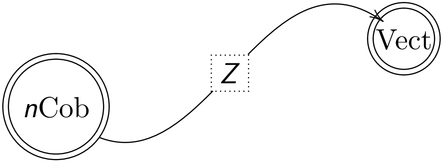
Definition

A topological quantum field theory is a monoidal functor $Z\colon n\mathrm{Cob} \to \mathrm{Vect}$.



Action of the TQFT

Let's take a closer look at the situation...

$$Z: \quad \mathrm{2Cob} \quad \longrightarrow \quad \mathrm{Vect}$$

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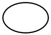 $\mapsto$ V

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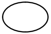
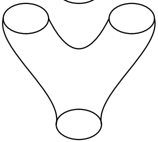
$$Z : \quad 2\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

 $\mapsto V$

 $\mapsto V \otimes V \to V$

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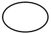
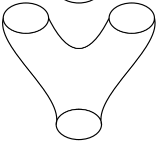
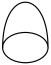


$$\mapsto \quad V$$



$$\mapsto \quad V \otimes V \to V$$



$$\mapsto \quad \mathbb{F} \to V$$

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$$
\begin{array}{cccc}
Z: & \mathrm{2Cob} & \longrightarrow & \mathrm{Vect} \\
 & & \mapsto & V \\
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 & & \mapsto & \mathbb{F} \to V \\
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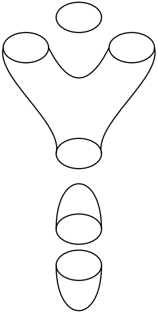
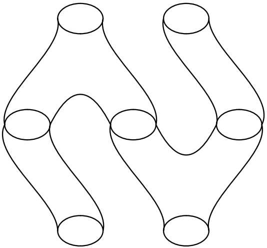
\end{array}
$$



Action of the TQFT

Monoidal "products" on $n\mathrm{Cob}$ and Vect:

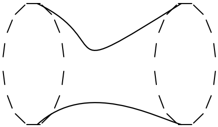
$$\begin{tikzcd} S \sqcup S \arrow[d, "M^* \sqcup 1_S"] \\ S \sqcup S \sqcup S \arrow[d, "1_S \sqcup M"] \\ S \sqcup S \end{tikzcd}$$



$$\begin{tikzcd} V \otimes V \arrow[d] \\ V \otimes V \otimes V \arrow[d] \\ V \otimes V \end{tikzcd}$$

Now things get interesting...

Consider an oriented closed n-manifold M.



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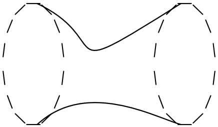
Consider an oriented closed n-manifold M.



Then we have

$$M\colon \emptyset_{n-1} \longrightarrow \emptyset_{n-1}$$

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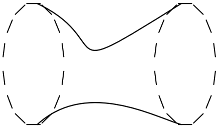

Then we have

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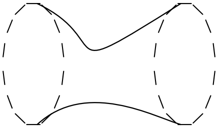

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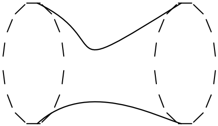

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$$Z(M)\colon Z(\emptyset_{n-1}) \longrightarrow Z(\emptyset_{n-1})$$
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$Z(M)$ is a topological invariant!

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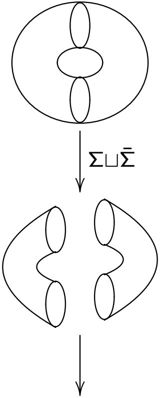

$\gamma \colon \mathbb{F} \to V \otimes V$ $\beta \colon V \otimes V \to \mathbb{F}$

Finishing up...

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Hence, $\beta \circ \gamma \colon \mathbb{F} \to \mathbb{F}$ gives us the invariant:

$$\begin{aligned}
\sum_{i,j} \gamma^{ij}\beta_{ij} &= \sum_{i,j} \gamma^{ij}\beta_{ji} && \text{(Since } (\beta_{ij}) \text{ is symmetric)} \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \gamma^{ij}\beta_{ji} \right) \\
&= \sum_{i=2}^{n} \left(\sum_{j=1}^{n} \gamma^{ij}\beta_{ji} \right) + (\gamma^{11}\beta_{11} + \gamma^{12}\beta_{21} + \cdots + \gamma^{1n}\beta_{n1}) \\
&= 1 + \sum_{i=2}^{n} \left(\sum_{j=1}^{n} \gamma^{ij}\beta_{ji} \right) \\
&= \mathrm{Tr}(I_d(V)) \\
&= n = \dim V.
\end{aligned}$$

THANK YOU!