MATH 751 MIDTERM REVIEW

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Lemma 13.3) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X.Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.
(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof. ((2) $\Rightarrow$ (1)) Given an element U of $\mathcal{T}$, we wish to show that $U \in \mathcal{T}'$. Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. Condition (2) tells us there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \mathcal{T}'$, by definition.

((1) $\Rightarrow$ (2)) We are given $x \in X$ and $B \in \mathcal{B}$, with $x \in B$. Now B belongs to $\mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (1); therefore, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. $\square$

Theorem 18.1) Let X and Y be topological spaces, and let $f\colon X \to Y$.Then the following are equivalent:

(1) f is continuous.
(2) For every subset A of X, we have $f(\bar{A}) \subset \overline{f(A)}$.
(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.
(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

(If condition (4) holds for the point $x \in X$, we say that f is **continuous** at the point x.)

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and that (1) $\Rightarrow$ (4) $\Rightarrow$ (1):
((1) $\Rightarrow$ (2)) Assume that f is continuous and let A be a subset of X. We want to show that if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set of X containing x; it must intersect A in some point $y \neq x$. Then V intersects $f(A)$ in the point $f(y)$, so that $f(x) \in \overline{f(A)}$, as desired.

((2) $\Rightarrow$ (3)) Let B be closed in Y and let $A = f^{-1}(B)$. We wish to prove that A is closed in X; thus we'll show that $\bar{A} = A$. By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$. Therefore, if $x \in \bar{A}$, then

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B,$$

so that $x \in f^{-1}(B) = A$. Thus $\bar{A} \subset A$, and hence $\bar{A} = A$, as desired.

((3) $\Rightarrow$ (1)) Let V be an open set of Y and set $B = Y \setminus V$. Then

$$f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V).$$

Now B is a closed set of Y. Then $f^{-1}(B)$ is closed in X by hypothesis, so that $f^{-1}(V)$ is open in X, as desired.

((1) $\Rightarrow$ (4)) Let $x \in X$ and let V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.

((4) $\Rightarrow$ (1)) Let V be an open set of Y, and let x be a point of $f^{-1}(V)$. Then $f(x) \in V$, so that by hypothesis there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open. □

Theorem 18.4) (Maps into Products) Let $f\colon A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous iff the functions

$$f_1\colon A \to X \qquad \text{and} \qquad f_2\colon A \to Y$$

are continuous.

(The maps f_1 and f_2 are called the **coordinate functions** of f.)

Proof. ($\Rightarrow$) Let $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ be projections onto the first and second factors, respectively. These maps are continuous. For $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$, and these sets are open if U and V are open. Note that for each $a \in A$, we have

$$f_1(a) = \pi_1(f(a)) \qquad \text{and} \qquad f_2(a) = \pi_2(f(a)).$$

If the function f is continuous, then f_1 and f_2 are composites of continuous functions and therefore continuous.

($\Leftarrow$) Conversely, suppose that f_1 and f_2 are continuous. We show that for each basis element $U \times V$ for the topology of $X \times Y$, its inverse image $f^{-1}(U \times V)$ is open. A point a is in $f^{-1}(U \times V)$ iff $f(a) \in U \times V$, that is, iff $f_1(a) \in U$ and $f_2(a) \in V$. Therefore,

$$f^{-1}(U \times V) = f_1^{-1}(U) \bigcap f_2^{-1}(V).$$

Since both of the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open, so is their intersection. Hence, f is continuous, as desired. □

Theorem 19.6) Let $f\colon A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha\colon A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous iff each function f_α is continuous.

Proof. ($\Rightarrow$) Let π_β be the projection of the product onto its β^{th} factor. The function π_β is continuous, for if U_β is open in X_β, then the set $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on $\prod X_\alpha$. Now suppose that $f\colon A \to \prod X_\alpha$ is continuous. The function f_β equals the composite $\pi_\beta \circ f$; being the composite of two continuous functions, it follows that is also continuous. Since this is true for any $\beta \in J$, we have proven this direction.

($\Leftarrow$) Conversely, suppose that each coordinate function f_α is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A. A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$, where β is some index and U_β is open in X_β. Now

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta),$$

because $f_\beta = \pi_\beta \circ f$. Since f_β is continuous, this set is open in A, as desired. □

To see why this theorem fails if we use the box topology instead, look at the following example:

Example: Take $\mathbb{R}^\omega$, the countable Cartesian product of the real line, and consider the function $f\colon \mathbb{R} \to \mathbb{R}^\omega$ given by

$$f(x) = (x, x, , x, \dots);$$

the n^{th} coordinate function of f is the function $f_n(x) = x$. Each of the coordinate functions $f_n\colon \mathbb{R} \to \mathbb{R}$ is continuous in the standard topology on $\mathbb{R}$, and thus f itself is continuous if $\mathbb{R}^\omega$ is given the product topology, but f is not continuous in the box topology.

Why? Consider the set

$$U = \prod_{n=1}^{\infty}(-1/n, 1/n).$$

This set U is open (it is a basis element) in the box topology, but not in the product topology. We assert that $f^{-1}(U)$ is not open in $\mathbb{R}$. If $f^{-1}(U)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. But this would mean that $f((-\delta, \delta)) \subset U$, so that, by applying the projection map on the n^{th} coordinate to both sides of this inclusion, we would get

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset (-1/n, 1/n) \qquad \text{for all } n.$$

This is a contradiction because the components of U get arbitrarily close to 0—any δ-neighborhood will eventually be outside some component of U. ▲

Theorem 26.7) The product of finitely many compact spaces is compact.

Proof. We shall prove that the product of two compact spaces is compact; the theorem then follows by induction for any finite number of products.

<u>*Step 1.*</u> Suppose that we are given spaces X and Y, with Y compact. Suppose that x_0 is a point of X, and N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$. We now prove the following:

There is a neighborhood W of x_0 in X such that N contains the entire set $W \times Y$ [1].

First let us cover $x_0 \times Y$ by basis elements $\{U_i \times V_i\}$ (for the topology of $X \times Y$) lying in N. The space $x_0 \times Y$ is compact, being homeomorphic to Y. Therefore, we can cover $x_0 \times Y$ by finitely many such basis elements

$$U_1 \times V_1 \dots, U_n \times V_n.$$

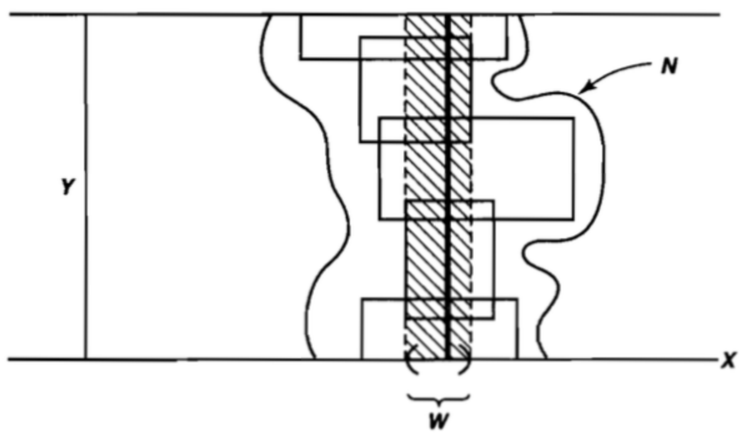
(We assume the each of the basis elements $U_i \times V_i$ actually intersects $x_0 \times Y$, since otherwise that basis element would be of no use to us; we could discard it from the finite collection and still have a covering of $x_0 \times Y$.)

Now define

$$W = U_1 \bigcap \cdots \bigcap U_n.$$

[1] The set $W \times Y$ is often called a **tube** about $x_0 \times Y$.

The set W is open, and it contains x_0 because each set $U_i \times V_i$ intersects $x_0 \times Y$. We assert that the sets $U_i \times V_i$, which were chosen to cover the slice $x_0 \times Y$, actually cover the tube $W \times Y$. Let $x \times y$ be a point of $W \times Y$. Consider the point $x_0 \times y$ of the slice $x_0 \times Y$ having the same y-coordinate as this point. Now $x_0 \times y$ belongs to $U_i \times V_i$ for some i, so that $y \in V_i$. But $x \in U_j$ for every j (because $x \in W$). Therefore, we have $x \times y \in U_i \times V_i$, as desired.
Since all the sets $U_i \times V_i$ lie in N, and since they cover $W \times Y$, the tube $W \times Y$ lies also in N (see figure below:)



Step 2. Now we are ready to prove the theorem. Let X and Y be compact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$. Their union $N = A_1 \cup \cdots \cup A_m$ is an open set containing $x_0 \times Y$; by *Step 1*, the open set N contains a tube $W \times Y$ about $x_0 \times Y$, where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$. Thus, for each $x \in X$, we can choose a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. The collection of all the neighborhoods W_x is an open covering of X; therefore by compactness of X, there exists a finite subcollection

$$\{W_1, \ldots, W_k\}$$

covering X. The union of the tubes

$$W_1 \times Y, \ldots, W_k \times Y$$

is all of $X \times Y$; since each may be covered by finitely many elements of $\mathcal{A}$, so may $X \times Y$ be covered. $\square$

Remark: The statement proved in *Step 1* of the preceding proof will be quite useful, so we formally state it as a lemma for referencing purposes:

Lemma. ***(The tube lemma)*** *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.*

Definition. *A collection $\mathcal{C}$ of subsets of X is said to have the* ***finite intersection property*** *if for every finite subcollection*

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 26.9) Let X be a topological space. Then X is compact iff for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} = \{X \smallsetminus A \mid A \in \mathcal{A}\}$$

be the collection of their components. Then the following statements hold:

(1) $\mathcal{A}$ is a collection of open sets iff $\mathcal{C}$ is a collection of closed sets.
(2) $\mathcal{A}$ covers X iff the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is empty.
(3) The finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ covers X iff the intersection of the corresponding elements $C_i = X \smallsetminus A_i$ of $\mathcal{C}$ is empty.

The first statement is trivial, while the second and third follow from DeMorgan's law:

$$X \setminus \left(\bigcup_{\alpha \in J} A_\alpha \right) = \bigcap_{\alpha \in J} (X \smallsetminus A_\alpha) .$$

The proof of the theorem now proceeds in two easy steps: taking the contrapositive (of the theorem), and then the complement (of the sets).

The statement that X is compact is equivalent to saying: "Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then some finite subcollection of $\mathcal{A}$ covers X." This statement is equivalent to its contrapositive, which is the following: "Given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X." Now letting $\mathcal{C}$ be as defined above

$$\mathcal{C} = \{X \smallsetminus A \mid A \in \mathcal{A}\}$$

and applying (1)-(3), we see that this statement is in turn equivalent to the following: "Given any collection $\mathcal{C}$ of closed sets, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection of all the elements of $\mathcal{C}$ is nonempty as well." This is just the condition of our theorem. □

Remark: A special case of this theorem occurs when we have a **nested sequence** $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \ldots$ of closed sets in a compact space X. If each of the sets C_n is nonempty, then the collection $\mathcal{C} = \{C_n\}_{n\in\mathbb{N}}$ automatically has the finite intersection property. Then the intersection

$$\bigcap_{n\in\mathbb{N}} C_n$$

is nonempty.

Theorem 27.1) Let X be a totally ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.

Proof. Given $a < b$, let $\mathcal{A}$ be a covering of $[a,b]$ by sets open in $[a,b]$ in the subspace topology (which is the same as the order topology because $[a,b]$ is convex). We wish to prove the existence of a finite subcollection of $\mathcal{A}$ covering $[a,b]$.

Step 1. First we prove the following:

If x is a point of $[a,b]$ different from b, then there is a point $y > x$ of $[a,b]$ such that the interval $[x,y]$ can be covered by at most two elements of $\mathcal{A}$.

If x has an immediate successor in X, let y be this immediate successor. Then $[x,y]$ consists of the two points x and y (i.e. $[x,y] = \{x,y\}$) so that it can be covered by at most two elements of $\mathcal{A}$. If x has no immediate successor in X, choose an element A of $\mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x,c)$, for some $c \in [a,b]$. Now choose a point $y \in (x,c)$; then the interval $[x,y]$ is covered by the single element A of $\mathcal{A}$.

Step 2. Let C be the set of all points $y > a$ of $[a,b]$ such that the interval $[a,y]$ can be covered by finitely many elements of $\mathcal{A}$. Now applying *Step 1* to the case $x = a$, we see that there exists at least one such y, so C is not empty. Let c be the least upper bound of the set C; then $a < c \leq b$.

Step 3. We now show that c belongs to C; that is, we show that the interval $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Choose an element A of $\mathcal{A}$ containing c. Since A is open, it contains an interval of the form $(d, c]$ for some d in $[a, b]$. If we assume that $c \notin C$, there must be a point $z \in C$ lying in the interval (d, c), because otherwise d would be a smaller upper bound on C than c (see *Figure 27.1* below). Since $z \in C$, the interval $[a, z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$. Now $[z, c]$ lies in the single element A of $\mathcal{A}$, hence $[a, c] = [a, z] \bigcup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$. Thus $c \in C$, contrary to assumption.

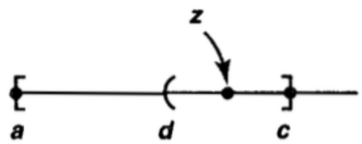


Figure 27.1

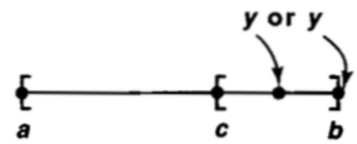


Figure 27.2

Step 4. Finally, we show that $c = b$ and our theorem is proved. Suppose that $c < b$. Applying *Step 1* to the case $x = c$, we conclude that there exists a point $y > c$ of $[a, b]$ such that the interval $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$ (see *Figure 27.2* above). We proved in *Step 3* that c is in C, so $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Therefore, the interval

$$[a, y] = [a, c] \cup [c, y]$$

can also be covered by finitely many elements of $\mathcal{A}$. This means that y is in C, contradicting the fact that c is an upper bound on C. $(\Rightarrow\Leftarrow)$ $\square$