Dynkin Diagrams (II)

The aim of this workshop is to prove the following result:

Theorem. A graph Γ satisfies properties (A),(B) and (C) below if and only if Γ is a Dynkin diagram:

(A) Γ is connected;

(B) the number of edges connecting any two nodes is either $0, 1, 2$ or 3; and

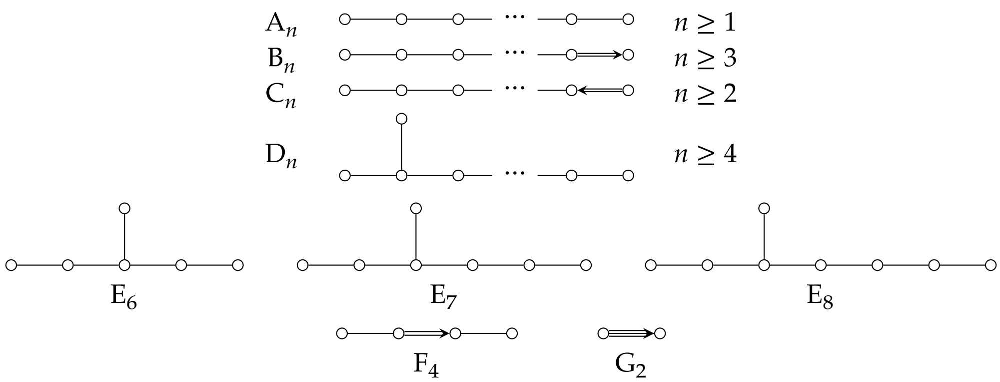
(C) the matrix B determined by Γ, namely $B = (b_{ij})$ where

$$b_{ij} := \begin{cases} 2 & \text{if } i = j \\ -\sqrt{n_{ij}} & \text{if } i \neq j \end{cases}$$

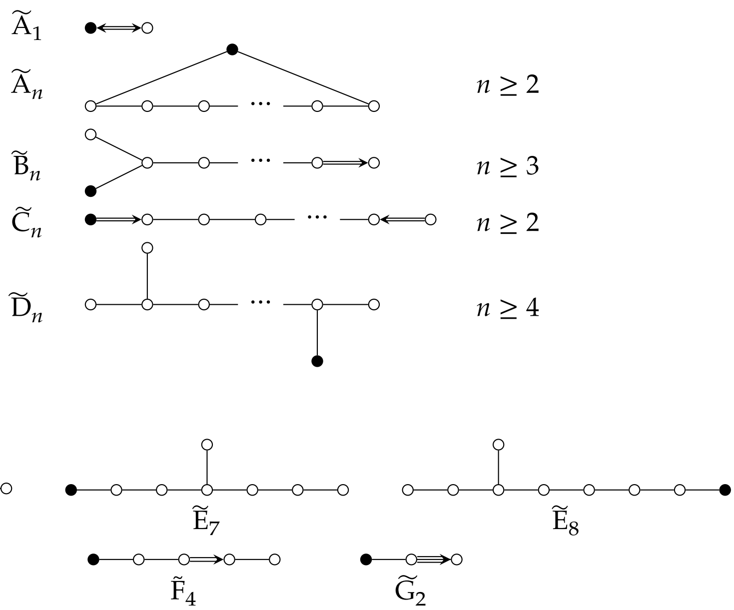
is positive definite.

At the last workshop we verified that ($\Leftarrow$) is true, and this involved a lot of calculations. Today we prove the other direction ($\Rightarrow$) through a series of reduction steps.

Recall from last workshop that the Dynkin diagrams are precisely



For this workshop, it will also be convenient to consider the *affine* Dynkin diagrams, which are obtained by adding a single extra vertex (denoted in black below) in the following way:



In all cases, the number of vertices is the subscript plus one. Again, ignore the arrows for today.

1. Show that the extended Dynkin diagrams have the following property: if we delete any vertex (and all edges adjacent to it), then the resulting graph is either a Dynkin diagram, or a disjoint union of Dynkin diagrams.
2. Prove that the matrix associated to every extended Dynkin diagram has determinant zero (don't check them all, share the burden!).
3. *(Important)* If Γ satisfies (A)(B)(C), and Γ' is a connected graph obtained from Γ by deleting vertices or edges (or both), show Γ' also satisfies (A)(B)(C).
4. Deduce from Q2, Q3 and the last workshop that any graph satisfying conditions (A)(B)(C) cannot have an extended Dynkin diagram as a subgraph.

We now use the observation in Q4 repeatedly. In what follows, let Γ be a graph satisfying conditions (A)(B)(C).

5. By using $\widetilde{\mathrm{A}}_n$ with $n \geq 2$, prove that Γ contains no cycles.
6. If Γ contains a triple edge, show that $\Gamma = \mathrm{G}_2$.
7. If Γ contains a double edge,
   (a) By using $\widetilde{\mathrm{C}}_n$ with $n \geq 2$, deduce that Γ cannot have more than one double edge.
   (b) In a similar way, deduce that Γ cannot have a branch point.
   (c) Conclude that Γ is a chain containing just one double edge.
   (d) Further,
      i. If the double edge occurs at the end of the chain, deduce that $\Gamma = \mathrm{B}_n$.
      ii. If not, deduce that $\Gamma = \mathrm{F}_4$.
8. By Q5, Q6 and Q7, we can assume that Γ contains no cycles, and no double or triple edges.
   (a) Show that if Γ contains no branch point, then $\Gamma = \mathrm{A}_n$.

   By (a), in what follows we can assume that Γ has a branch point.
   (b) Show that Γ contains at most one branch point.

   By (a) and (b) we can thus assume that Γ contains precisely one branch point.
   (c) Deduce that there must be exactly three branches emerging from this branch point.
   (d) Let the number of vertices of the three branches be n_1, n_2 and n_3, with $n_1 \geq n_2 \geq n_3 \geq 1$.
      i. Using $\widetilde{\mathrm{E}}_6$, deduce that $n_3 = 1$.
      ii. Hence using $\widetilde{\mathrm{E}}_7$, deduce that n_2 is either 1 or 2.
      iii. Deduce that Γ is either D_n, E_6, E_7 or E_8.

Combining Q5–Q8 proves the theorem, namely that the graphs satisfying conditions (A)(B)(C) are precisely the Dynkin diagrams.

Please hand in your solution to Q8 by the start of lecture on Monday 9 October.