Math 353 HW 2

Mario L. Gutierrez Abed

Section 1.3

(1) Evaluate the following limits :

a) $\lim_{z \to i} \left(z + \frac{1}{z} \right)$

$= \lim_{z \to i} \frac{z^2 + 1}{z} = \frac{-1+1}{i} = 0$

b) $\lim_{z \to z_0} \frac{1}{z^m}$

$= \frac{1}{z_0^m} = z_o^{-m}$; $m \in \mathbb{Z}$

f) $\lim_{z \to \infty} \frac{z^2}{(3\,z+1)^2}$

Here we make the substitution $z = \frac{1}{w}$. Then we have

$$\lim_{w \to 0} \frac{\left(\frac{1}{w}\right)^2}{\left(\frac{3}{w}+1\right)^2} = \lim_{w \to 0} \frac{\frac{1}{w^2}}{\frac{9}{w^2}+\frac{6}{w}+1} \cdot \frac{w^2}{w^2} = \lim_{w \to 0} \frac{1}{9+6\,w+w^2} = \frac{1}{9}$$

g) $\lim_{z \to \infty} \frac{z}{z^2+1}$

Here we also make the substitution $z = \frac{1}{w}$. Then we have

$$\lim_{w \to 0} \frac{\frac{1}{w}}{\left(\frac{1}{w}\right)^2 + 1} = \lim_{w \to 0} \frac{\frac{1}{w}}{\frac{1}{w^2} + 1} \cdot \frac{w^2}{w^2} = \lim_{w \to 0} \frac{w}{1+w^2} = 0$$

(4) Where are the following functions differentiable?

a) $\sin z = \dfrac{e^{i z} - e^{-i z}}{2 i}$

$$\lim_{h \to 0} \frac{\frac{e^{i(z+h)} - e^{-i(z+h)}}{2 i} - \frac{e^{i z} - e^{-i z}}{2 i}}{h} = \frac{1}{2 i} \lim_{h \to 0} \frac{e^{i h} e^{i z} - e^{-i h} e^{-i z} - e^{i z} + e^{-i z}}{h}$$

$$= \frac{1}{2 i} \lim_{h \to 0} \frac{e^{i z}(e^{i h} - 1) - e^{-i z}(e^{-i h} - 1)}{h}$$

$$= \frac{1}{2 i} \lim_{h \to 0} \frac{1}{h}\left[e^{i z}\left(1 + i h + \frac{(i h)^2}{2!} + \frac{(i h)^3}{3!} + \ldots - 1\right) - e^{-i z}\left(1 + (-i h) + \frac{(-i h)^2}{2!} + \frac{(-i h)^3}{3!} + \ldots - 1\right)\right]$$

$$= \frac{1}{2 i} \lim_{h \to 0} \frac{1}{h}\left[e^{i z}\, i h\left(1 + \frac{i h}{2!} + \frac{(i h)^2}{3!} + \ldots\right) - e^{-i z}\, i h\left((-1) + \frac{-i h}{2!} + \frac{(-i h)^2}{3!} + \ldots\right)\right]$$

$$= \frac{1}{2} (e^{i z}(1 + 0 + 0 \,..) - e^{-i z}(-1 + 0 + 0 + \ldots)) = \frac{e^{i z} + e^{-i z}}{2} = \cos z$$

The function $\sin z$ is differentiable for all $z \in \mathbb{C}$ because for any point z_o the limit as h approaches zero has the same value from all directions. Hence $\sin z$ is an entire function.

b) $\tan z = \dfrac{\sin z}{\cos z}$

We already know from part a) that $\sin z$ is differentiable everywhere. Similarly it can be shown that $\cos z$ is also analytic for all $z \in \mathbb{C}$. This means that $\tan z$ is differentiable everywhere except where $\cos z = 0$.

That is, where

$$\frac{e^{i z} + e^{-i z}}{2} = 0 \Longrightarrow e^{i z} + e^{-i z} = 0$$

$$\Longrightarrow e^{i z} = -\frac{1}{e^{i z}} \Longrightarrow e^{2 i z} = -1 \Longrightarrow e^{2 i (x + i y)} = -1 \Longrightarrow e^{2 i x - 2 y} = -1$$

$$\Longrightarrow \frac{1}{e^{2 y}} [\cos(2 x) + i \sin(2 x)] = -1$$

$$\Longrightarrow y = 0; \quad 2 x = \pi \Longrightarrow x = \frac{\pi}{2}, \quad z = \frac{\pi}{2} + 0 i = \frac{\pi}{2}$$

Hence $\tan z$ is differentiable for all $z \in \mathbb{C} \setminus z = \frac{\pi}{2} + \pi n$, for $n = 0, 1, 2, 3, \ldots$, because in order for the function to be differentiable it has to be defined in the first place and $\tan z$ is not defined at those points.

c) $\dfrac{z - 1}{z^2 + 1}$

We know that the derivative of a rational function $f(z) = \dfrac{g(z)}{q(z)}$ is defined for all z such that $q(z) \neq 0$.

Hence our function is differentiable for all z as long as $z^2 + 1 = 0$ holds. In other words, f is differentiable for all $z \in \mathbb{C} \setminus z = \pm i$ (which is the same as $z = e^{i\left(\frac{\pi}{2} + \pi n\right)}$, for $n = 0, 1, 2, 3, \ldots$)

d) $e^{1/z}$

$$\frac{d}{dz}(e^{1/z}) = -\frac{1}{z^2} e^{1/z}$$

We see that the derivative is defined $\forall z \in \mathbb{C} : z \neq 0$ and this means that the function is analytic in this domain.

e) $2\bar{z}$

$$\lim_{h\to 0} \frac{2(\overline{z+h}) - 2\bar{z}}{h} = \lim_{h\to 0} \frac{2\bar{z} + 2\bar{h} - 2\bar{z}}{h} = \lim_{h\to 0} \frac{2\bar{h}}{h}$$

Now if we choose h to approach 0 from the real axis $(h = x)$ we have $\bar{h} = h$. Hence $\lim_{h\to 0} \frac{2\bar{h}}{h} = 2$.
However if we choose h to approach 0 from the imaginary axis, then we have that $h = i\, y$ and $\bar{h} = -i\, y$. Hence $\bar{h} = -h$ and we have $\operatorname{limit}_{h\to 0} \frac{2\bar{h}}{h} = -2$. Since the limit of $2\bar{z}$ does not approach the same value from all directions, we can conclude that our function is nowhere differentiable.

(5) Show that the functions $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are nowhere differentiable.

Proof:

The limit of any complex function $f(z)$ at an arbitrary point z_o is written as $\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$. By letting $h = z - z_o$, we can alternatively write the limit as $\lim_{h\to 0} \frac{f(h+z_o) - f(z_0)}{h}$.

Now we write $z_0 = x + i\, y$, then $\mathrm{Re}(z_0) = x$ and $\mathrm{Im}(z_0) = y$.

- For $\mathrm{Re}(z_0)$ we have

If we choose h to be a real number :

$$\lim_{h\to 0} \frac{\mathrm{Re}(h+z_0) - \mathrm{Re}(z_0)}{h} = \operatorname{limit}_{h\to 0} \frac{x+h-x}{h} = \operatorname{limit}_{h\to 0} \frac{h}{h} = 1.$$

Hence $\mathrm{Re}(z_0)$ approaches 1 as h approaches 0.
On the other hand if we pick an h that is purely imaginary, we have $h = i\, y$, then :

$$\lim_{h\to 0} \frac{\mathrm{Re}(h+z_0) - \mathrm{Re}(z_0)}{h} = \lim_{i\, y\to 0} \frac{x + 0 - x}{i\, y} = 0.$$

Now we see that $\mathrm{Re}(z_o)$ approaches different values from different directions as h approaches zero. Hence $\mathrm{Re}\,(z)$ is nowhere differentiable.

- For $\mathrm{Im}(z_0)$ we have

If we choose h to be purely imaginary :

$$\lim_{h\to 0} \frac{\mathrm{Im}(h+z_0) - \mathrm{Im}(z_0)}{h} = \lim_{h\to 0} \frac{y + h - y}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

Hence $\mathrm{Im}(z_0)$ approaches 1 as h approaches 0.

Now if we choose h to be real :

$$\lim_{h\to 0} \frac{\mathrm{Im}(h+z_0) - \mathrm{Im}(z_0)}{h} = \lim_{h\to 0} \frac{0 + y - y}{h} = 0.$$

Now we see that $\mathrm{Im}(z_o)$ approaches different values from different directions as h approaches zero. Hence $\mathrm{Im}\,(z)$ is nowhere differentiable. ■

(10) Let $z = x$ be real. Use the relationship $\frac{d}{dx}(e^{ix}) = ie^{ix}$ to find the standard derivative formulae for trigonometric functions:

- $$\frac{d}{dx}\sin(x) = \frac{d}{dx}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) = \frac{1}{2i}\frac{d}{dx}(e^{ix} - e^{-ix})$$
$$= \frac{1}{2i}(i\,e^{ix} + i\,e^{-ix}) = \frac{1}{2i}\,i(e^{ix} + e^{-ix})$$
$$= \frac{e^{ix} + e^{-ix}}{2} = \cos x \quad ✓$$

- $$\frac{d}{dx}\cos x = \frac{d}{dx}\left(\frac{e^{ix} + e^{-ix}}{2}\right) = \frac{1}{2}\frac{d}{dx}(e^{ix} + e^{-ix})$$
$$= \frac{1}{2}(ie^{ix} - i\,e^{-ix}) = \frac{i}{2}(e^{ix} - e^{-ix}) \cdot \frac{i}{i}$$
$$= -\frac{e^{ix} - e^{-ix}}{2i} = -\sin x \quad ✓$$