MATH 710 HW # 9

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Exercise 1 (Exercise 4-1 [DoCarmo])**.** *Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$. Let $X, Y, Z \in \mathfrak{X}(G)$ be unit left-invariant vector fields on G.*

a) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$. [Hint: Use the symmetry of the connection and the fact that $\nabla_X X = 0$.]

b) Conclude from a) that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

c) Prove that, if X and Y are orthonormal, the sectional curvature $K(\sigma)$ of G with respect to the plane σ generated by X and Y is given by

$$K(\sigma) = \frac{1}{4}\|[X, Y]\|^2.$$

Therefore, the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative and is zero if and only if σ is generated by vectors X, Y which commute, that is, such that $[X, Y] = 0$.

Proof of a). To prove the existence and uniqueness of the Levi-Civita connection we showed that

$$2\langle Z, \nabla_Y X \rangle = X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle - \langle [X, Y], Z \rangle.$$

The inner product of two left-invariant vector fields is a left-invariant (and therefore constant) function; consequently, the first three terms on the right hand side must vanish. Putting $Y = X$, we find

$$2\langle Z, \nabla_X X \rangle = -\langle [X, Z], X \rangle - \langle [X, Z], X \rangle - \langle [X, X], Z \rangle = -2\langle [X, Z], X \rangle.$$

Because the metric is bi-invariant, we have

$$\langle [Y, X], Z \rangle + \langle Y, [Z, X] \rangle = 0$$

for all left-invariant vector fields. Again putting $Y = X$ yields

$$\langle X, [Z, X] \rangle = 0 \quad \text{and hence} \quad \langle Z, \nabla_X X \rangle = 0$$

for all left-invariant vector fields X and Z, i.e. $\nabla_X X = 0$ for all left-invariant vector fields X. This implies that

$$\nabla_X Y + \nabla_Y X = \nabla_{X+Y}(X + Y) - \nabla_X X - \nabla_Y Y = 0.$$

Since the connection is torsion-free (i.e. symmetric), we have

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Adding the last two equations gives

$$2\nabla_X Y = [X, Y].$$

□

Proof of b). From the definition and part a), we have

$$\begin{aligned} R(X,Y)Z &= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z \\ &= \nabla_Y\left(\frac{1}{2}[X,Z]\right) - \nabla_X\left(\frac{1}{2}[Y,Z]\right) + \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}\left([[Z,Z],Y] + [[Y,Z],X] + [[X,Y],Z]\right) + \frac{1}{4}[[X,Y],Z] \\ &= \frac{1}{4}[[X,Y],Z]. \end{aligned}$$

Note that the last equality follows from the Jacobi identity. □

Proof of c). Using part b) we find that

$$\begin{aligned} K(\sigma) &= \frac{\langle R(X,Y)X,Y\rangle}{\|X\wedge Y\|^2} \\ &= \frac{\langle \frac{1}{4}[[X,Y],X],Y\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}. \end{aligned}$$

The denominator equals 1 because X and Y are orthonormal. Moreover, since the metric is bi-invariant, we have

$$K(\sigma) = \frac{1}{4}\langle [[X,Y],X],Y\rangle = -\frac{1}{4}\langle [X,Y],[Y,X]\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle = \frac{1}{4}\|[X,Y]\|^2.$$

(*Remark*: Notice that the sectional curvature is always non-negative, and only zero if X and Y commute. We mentioned that every compact Lie group admits a bi-invariant metric. Some examples include the flat torus (clearly all X and Y commute since the torus is abelian), the orthogonal groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, and the unitary groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$. In low dimensions, $\mathrm{SO}(3) = \mathbb{RP}^3$ and $\mathrm{SU}(2) = \mathbb{S}^3$, both of which have strictly positive sectional curvature.) □

Exercise 2 (Exercise 4-4 [DoCarmo]). *Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p to q. Prove that the curvature of M is identically zero; that is, for all $X, Y, Z \in \mathfrak{X}(M)$, we have $R(X,Y)Z = 0$.*

Proof. Following the provided hint on the text, consider a parametrized surface $f\colon U \subset \mathbb{R}^2 \to M$, where

$$U = \{(s,t) \in \mathbb{R}^2 \mid -\varepsilon < s, t < 1+\varepsilon\}, \quad \varepsilon > 0, \quad \text{and} \quad f(s,0) = f(0,0) \text{ for all } s.$$

Let $V_0 \in T_{f(0,0)}M$ and define a field V along f by $V(s,0) = V_0$ and, if $t \neq 0$, $V(s,t)$ is the parallel transport of V_0 along the curve $t \mapsto f(s,t)$. Then, by definition of what it means to be parallel, $D/\partial t\, V \equiv 0$ and so, from Lemma 4.1, DoCarmo's,[1] we have

$$(\spadesuit) \qquad \frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V.$$

[1] The lemma simply states that

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V.$$

The proof is a long computation which can be found in pages 88-89, DoCarmo's.

Now, consider the vector $V(s,1) \in T_{f(s,1)}M$. Then $V(s,1)$ can be obtained by parallel transporting $V(0,1)$ down to $f(0,0)$ and then up to $f(s,1)$. On the other hand since, by hypothesis, the parallel transport from $f(0,1)$ to $f(s,1)$ does not depend on the curve we transport along, $V(s,1)$ is also the parallel transport of $V(s,0)$ along the curve $s \mapsto f(s,1)$. Therefore, again by what it means to be parallel, we have $D/\partial s\, V(s,1) = 0$ for all s. Hence,

$$\frac{D}{\partial t}\frac{D}{\partial s}V(s,1) = 0.$$

All of the above was independent of our choices of f and V_0, so, for any $X, Y, Z \in \mathfrak{X}(M)$, we can choose appropriate f and V_0 such that equation (♠) reduces to:

$$0 = R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1)\right)V(0,1) = R(X,Y)Z.$$

Thus, we conclude that the curvature of M is identically 0, as desired. □

Exercise 3 (Exercise 5-2 [DoCarmo]). *Let M be a Riemannian manifold, $\gamma\colon [0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$, where $f(t,0) = \gamma(t)$ and the curves $t \mapsto f(t,s)$ are geodesics, such that $J(t) = \partial f/\partial s(t,0)$.*

Proof. Let $\varsigma\colon (-\varepsilon,\varepsilon) \to M$ be a curve such that $\varsigma(0) = \gamma(0)$ and $\varsigma'(0) = J(0)$. Also, choose a vector field $W(s)$ along ς such that $W(0) = \gamma'(0)$ and $DW/\mathrm{d}s\,(0) = DJ/\mathrm{d}t\,(0)$ (we can certainly do this since we are just specifying initial conditions). Now, define $f(s,t) = \exp_{\varsigma(s)} tW(s)$. Note, first of all, that

$$f(t,0) = \exp_{\varsigma(0)} tW(0) = \exp_{\gamma(0)} t\gamma'(0) = \gamma(t).$$

Also, the curves $t \mapsto f(t,s)$ are geodesics by construction. Now, at $t = 0$, f is simply moving along ς, so

$$\frac{\partial f}{\partial s}(0,0) = \frac{\mathrm{d}\varsigma}{\mathrm{d}s}(0) = J(0) \quad \text{by construction.}$$

Note that at $s = 0$, we have $\partial f/\partial s = W$. Since we can switch the order of differentiation,

$$\frac{D}{\mathrm{d}t}\frac{\partial f}{\partial s}(0,0) = \frac{D}{\mathrm{d}s}\frac{\partial f}{\partial t}(0,0) = \frac{DW}{\mathrm{d}s}(0) = \frac{DJ}{\mathrm{d}t}(0) \qquad \text{by our choice of } W.$$

Now, $f(t,s)$ parametrizes a surface in M and

$$\frac{D}{\mathrm{d}s}\frac{D}{\mathrm{d}t}\frac{\partial f}{\partial t} - \frac{D}{\mathrm{d}t}\frac{D}{\mathrm{d}t}\frac{\partial f}{\partial t} = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}.$$

Since the curves $t \mapsto f(t,s)$ are geodesics, we have

$$\frac{D}{\mathrm{d}t}\frac{\partial f}{\partial t} = 0.$$

Making this simplification and swapping derivatives in the second term above, we see that

$$-\frac{D}{\mathrm{d}t}\frac{D}{\mathrm{d}t}\frac{\partial f}{\partial s} = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} \qquad \Longrightarrow \qquad \frac{D^2}{\mathrm{d}t^2}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} = 0,$$

so $\partial f/\partial s(t,0)$ is a Jacobi field along γ.

Since Jacobi fields are uniquely determined by their value and derivative at a point and

$$\frac{\partial f}{\partial s}(0,0) = J(0) \quad \text{and} \quad \frac{D}{\mathrm{d}t}\frac{\partial f}{\partial s}(0,0) = \frac{DJ}{\mathrm{d}t}(0),$$

we see that $\partial f/\partial s\,(t,0) = J(t)$ along the entire length of γ. $\square$

Exercise 4 (Exercise 5-6 [DoCarmo]). *Let M be a surface (i.e. a 2-Riemannian manifold. Let $B_\delta(p)$ be a normal ball around the point $p \in M$ and consider the parametrized surface*

$$f(\rho,\theta) = \exp_p \rho v(\theta), \qquad 0 < \rho < \delta, \qquad -\pi < \theta < \pi,$$

where $v(\theta)$ is a circle of radius δ in T_pM parametrized by the central angle θ.

a) *Show that (ρ,θ) are coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray $\exp_p(-\rho v(0))$, where $0 < \rho < \delta$. Such coordinates are called **polar coordinates** at p.*

b) *Show that the coefficients g_{ij} of the Riemannian metric in these polar coordinates are*

$$g_{12} = 0, \qquad g_{11} = \left\| \frac{\partial f}{\partial \rho} \right\|^2 = \|v(\theta)\|^2 = 1, \qquad g_{22} = \left\| \frac{\partial f}{\partial \theta} \right\|^2.$$

c) *Show that, along the geodesic $f(\rho,0)$, we have*

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(\rho), \qquad \text{where } \lim_{\rho\to 0} \frac{R(\rho)}{\rho} = 0$$

and $K(p)$ is the sectional curvature of M at p.

d) *Prove that*

$$\lim_{\rho\to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p).$$

(This last expression is the value of the Gaussian curvature of M at p given in polar coordinates. This fact from the theory of surfaces and d) show that, in dimension 2, the sectional curvature coincides with the Gaussian curvature.)

Proof of a). Since B_δ is a normal ball around p, the exponential map is bijective on U. Hence f is bijective and, since it is the composition of smooth maps with smooth inverses, it is a diffeomorphism. The inverse image $f^{-1}(U)$ is open in $T_pM \cong \mathbb{R}^2$, so (ρ,θ) are coordinates on U. $\square$

Proof of b). By definition, we have

$$g_{11} = \left\langle \frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \rho} \right\rangle = \left\| \frac{\partial f}{\partial \rho} \right\|^2.$$

Now,

$$\frac{\partial f}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\exp_p \rho v(\theta)\right) = (\mathrm{d}\exp_p)_{\rho v(\theta)}(v(\theta)) = v(\theta).$$

Thus, $g_{11} = \|v(\theta)\|^2 = 1$, as desired. Also,

$$g_{22} = \left\langle \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \theta} \right\rangle = \left\| \frac{\partial f}{\partial \theta} \right\|^2.$$

Finally, note that, since ρ and θ are orthogonal coordinates, we have that $g_{12} = 0$, as desired. $\square$

Proof of c). Note that $\partial f/\partial\theta(\rho,0)$ is a Jacobi field along the geodesic $f(\rho,0)$ given by $(\mathrm{d}\exp_p)_{\rho v}(\rho v(0))$. Therefore, by Corollary 2.10 (see this corollary and related results on page 115, DoCarmo's),

$$\sqrt{g_{22}} = \left\| \frac{\partial f}{\partial \theta}(\rho,0) \right\| = \rho - \frac{1}{6}\,K(p)\rho^3 + \widetilde{R}(\rho) \qquad \text{where } \lim_{\rho\to 0} \frac{\widetilde{R}(\rho)}{\rho^3} = 0.$$

Therefore, differentiating both sides twice with respect to ρ, we see that

$$\left(\sqrt{g_{22}}\right)_{\rho\rho} = -K(p)\rho + \widetilde{R}''(\rho).$$

Now, by applying the good old L'Hôpital's Rule (from the glory days!), we get

$$0 = \lim_{\rho\to 0} \frac{\widetilde{R}(\rho)}{\rho^3} = \lim_{\rho\to 0} \frac{\widetilde{R}'(\rho)}{3\rho^2} = \lim_{\rho\to 0} \frac{\widetilde{R}''(\rho)}{6\rho} = \frac{1}{6} \lim_{\rho\to 0} \frac{\widetilde{R}''(0)}{\rho}$$

$$(\clubsuit) \qquad \left(\sqrt{g_{22}}\right)_{\rho\rho} = -K(p)\rho + R(\rho),$$

where $\lim_{\rho\to 0} R(\rho)/\rho = 0$. □

Proof of d). Dividing both sides of ($\clubsuit$) by $\sqrt{g_{22}}$ we see that

$$(\heartsuit) \qquad \frac{\left(\sqrt{g_{22}}\right)_{\rho\rho}}{\sqrt{g_{22}}} = -\frac{K(p)\rho}{\sqrt{g_{22}}} + \frac{R(\rho)}{\sqrt{g_{22}}}.$$

Now, note that, from ($\spadesuit$), we have

$$\sqrt{g_{22}} \to \rho + H(\rho), \quad \text{where } \lim_{\rho\to 0} \frac{H(\rho)}{\rho^2} = 0.$$

Therefore,

$$\lim_{\rho\to 0} -\frac{K(p)\rho}{\sqrt{g_{22}}} = \lim_{\rho\to 0} -\frac{K(p)\rho}{\rho} = -K(p) \qquad \text{and} \qquad \lim_{\rho\to 0} \frac{R(\rho)}{\sqrt{g_{22}}} = \lim_{\rho\to 0} -\frac{R(\rho)}{\rho} = 0.$$

Hence, taking the limits on both sides of ($\heartsuit$), we see that

$$\lim_{\rho\to 0} \frac{\left(\sqrt{g_{22}}\right)_{\rho\rho}}{\sqrt{g_{22}}} = -K(p). \qquad \square$$