MATH 709 NOTES
SMOOTH MAPS

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Smooth Functions/Maps

Definition. *Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f\colon M \to \mathbb{R}^k$ is any function. We say that f is a* ***smooth function*** *if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (see* Figure 1 *below).* ★

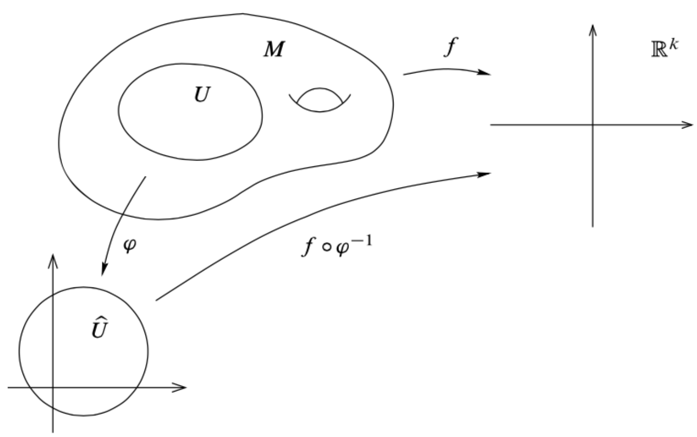

Figure 1. Definition of smooth functions.

Remark: The most important special case is that of smooth real-valued functions $f\colon M \to \mathbb{R}$; the set of all such functions is denoted by $C^\infty(M)$. Because sums and constant multiples of smooth functions are smooth, it turns out that $C^\infty(M)$ is a vector space over $\mathbb{R}$.

The definition of smooth functions generalizes easily to maps between manifolds:

Definition. *Let M and N be smooth manifolds, and let $F\colon M \to N$ be any map. We say that F is a* ***smooth map*** *if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$ (see* Figure 2 *below).* ★

Remark 1: Note that our previous definition of smoothness of real-valued or vector-valued functions can be viewed as a special case of this one, by taking $N = V = \mathbb{R}^k$ and $\psi = \mathrm{Id}\colon \mathbb{R}^k \to \mathbb{R}^k$.

Remark 2: In spite of the apparent complexity of the definition, it is usually not hard to prove that a particular map is smooth. There are basically only three common ways to do so:

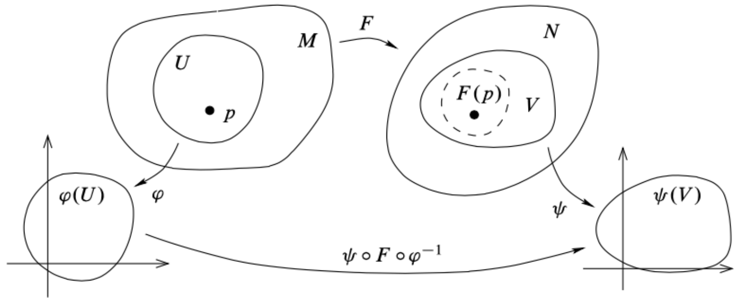

FIGURE 2. Definition of smooth maps.

- Write the map in smooth local coordinates and recognize its component functions as compositions of smooth elementary functions.
- Exhibit the map as a composition of maps that are known to be smooth.
- Use some special-purpose theorem that applies to the particular case under consideration.

Proposition 1. *Every smooth map is continuous.*

Proof. Suppose M and N are smooth manifolds (with or without boundary,) and $F\colon M \to N$ is smooth. Given $p \in M$, smoothness of F means there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is a smooth real/vector valued function, which is known to be continuous. Since $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are homeomorphisms, this implies in turn that

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi\colon U \to V,$$

which is a composition of continuous maps. Since F is continuous in a neighborhood of each point, it is continuous on M, as desired. $\square$

Proposition 2 (Equivalent Characterizations of Smoothness). *Suppose M and N are smooth manifolds (with or without boundary,) and $F\colon M \to N$ is a map. Then F is smooth iff either of the following conditions is satisfied:*

a) For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.

b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proposition 3 (Smoothness Is Local). *Suppose M and N are smooth manifolds (with or without boundary,) and let $F\colon M \to N$ be a map. Then*

a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

The next corollary is essentially just a restatement of the previous proposition, but it gives a highly useful way of constructing smooth maps:

Corollary 1 (**Gluing Lemma for Smooth Maps**). *Let M and N be smooth manifolds (with or without boundary,) and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha\colon U_\alpha \to N$ such that the maps agree on overlaps, that is*

$$F_\alpha|_{U_\alpha\cap U_\beta} = F_\beta|_{U_\alpha\cap U_\beta} \quad \forall\ \alpha,\beta\in A.$$

Then there exists a unique smooth map $F\colon M\to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha\in A$.

Definition. *If $F\colon M\to N$ is a smooth map, and (U,φ) and (V,ψ) are any smooth charts for M and N, respectively, we call the composite map $\widehat{F} = \psi\circ F\circ\varphi^{-1}$ the **coordinate representation** of F with respect to the given coordinates. It maps the set $\varphi(U\cap F^{-1}(V))$ to $\psi(V)$.* ★

Remark: As with real-valued or vector-valued functions, once we have chosen specific local coordinates in both the domain and codomain, we can often ignore the distinction between $\widehat{F}$ and F.

Proposition 4. *Let M, N, and P be smooth manifolds (with or without boundary.)*

a) Every constant map $c\colon M\to N$ is smooth.

b) The identity map of M is smooth.

c) If $U\subseteq M$ is an open submanifold (with or without boundary,) then the inclusion map $U\hookrightarrow M$ is smooth.

d) If $F\colon M\to N$ and $G\colon N\to P$ are smooth, then so is $G\circ F\colon M\to P$.

Proposition 5. *Suppose $M_1,\dots,M_k$ and N are smooth manifolds (with or without boundary,) such that at most one of $M_1,\dots,M_k$ has nonempty boundary. For each i, let $\pi_i\colon M_1\times\cdots\times M_k\to M_i$ denote the projection onto the M_i factor. Prove that a map $F\colon N\to M_1\times\cdots\times M_k$ is smooth iff each of the component maps $F_i = \pi_i\circ F\colon N\to M_i$ is smooth.*

Proof. See Problem 2-2 on HW set # 2. □

Partitions of Unity

The version of the gluing lemma for smooth maps that we presented on *Corollary 1* is well defined on open subsets, but we cannot expect to glue together smooth maps defined on closed subsets and obtain a smooth result. For example, the two functions $f_+\colon [0,\infty)\to\mathbb{R}$ and $f_-\colon(-\infty,0]\to\mathbb{R}$ defined by

$$\begin{aligned} f_+(x) &= +x, \qquad x\in[0,\infty),\\ f_-(x) &= -x, \qquad x\in(-\infty,0] \end{aligned}$$

are both smooth and agree at the point 0 where they overlap, but the continuous function $f\colon\mathbb{R}\to\mathbb{R}$ that they define, namely $f(x) = |x|$, is not smooth at the origin.

A disadvantage of this corollary is that in order to use it, we must construct maps that agree exactly on relatively large subsets of the manifold, which is too restrictive for some purposes. In this section we introduce partitions of unity, which are tools for "blending together" local smooth objects into global ones without necessarily assuming that they agree on overlaps.

All of our constructions in this section are based on the existence of smooth functions that are positive in a specified part of a manifold and identically zero in some other part. We begin by defining a smooth function on the real line that is zero for $t \leq 0$ and positive for $t > 0$:

Lemma 1. *The function* $f\colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(t) = \begin{cases} e^{-1/t} & t > 0, \\ 0 & t \leq 0 \end{cases}$$

is smooth (see Figure 3 *below).*

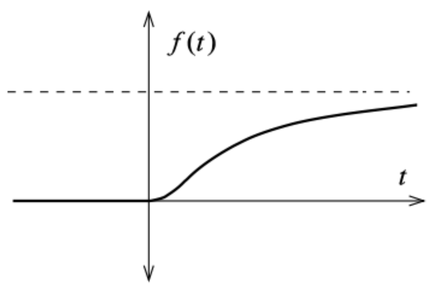

Figure 3. $f(t) = e^{-1/t}$.

Lemma 2. *Given any real numbers* r_1 *and* r_2 *such that* $r_1 < r_2$*, there exists a smooth function* $h\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$h(t) = \begin{cases} 1 & t \leq r_1, \\ 0 < h(t) < 1 & r_1 < t < r_2, \\ 0 & t \geq r_2. \end{cases}$$

Proof. Let f be the function of the previous lemma, and set

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

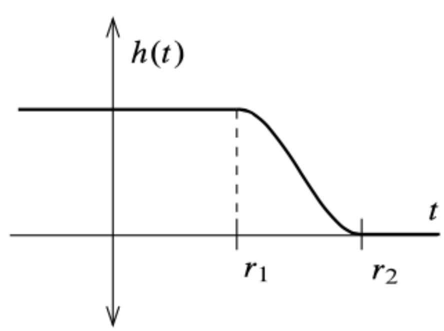

Figure 4. A cutoff function.

Such a function is usually called a **cutoff function**. Note that the denominator is positive for all t, because at least one of the expressions $r_2 - t$ and $t - r_1$ is always positive. The desired properties of h follow easily from those of f. $\square$

Lemma 3. *Given any positive real numbers $r_1 < r_2$, there is a smooth function $H\colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$H(x) = \begin{cases} 1 & x \in \overline{B}_{r_1}(0), \\ 0 < H(x) < 1 & x \in B_{r_2}(0) \smallsetminus \overline{B}_{r_1}(0), \\ 0 & x \in \mathbb{R}^n \smallsetminus B_{r_2}(0). \end{cases}$$

Proof. Just set $H(x) = h(|x|)$, where h is the function of the preceding lemma. Clearly, H is smooth on $\mathbb{R}^n \smallsetminus \{0\}$, because it is a composition of smooth functions there. Since it is identically equal to 1 on $B_{r_1}(0)$, it is smooth there too. □

Remark: The function H constructed in this lemma is an example of a ***smooth bump function***, a smooth real-valued function that is equal to 1 on a specified set and is zero outside a specified neighborhood of that set.

Definition. *Suppose M is a topological space, and let $\chi = (X_\alpha)_{\alpha \in A}$ be an arbitrary open cover of M, indexed by a set A. A* ***partition of unity subordinate to*** *χ is an indexed family $(\psi_\alpha)_{\alpha\in A}$ of continuous functions $\psi_\alpha\colon M \to \mathbb{R}$ with the following properties:*

- $0 \leq \psi_\alpha \leq 1$ *for all $\alpha \in A$ and all $x \in M$.*
- *supp*$(\psi_\alpha) \subseteq X_\alpha$ *for each $\alpha \in A$.*
- *The family of supports (supp$(\psi_\alpha))_{\alpha\in A}$ is locally finite.*
- $\sum_{\alpha\in A} \psi_\alpha(x) = 1$ *for all $x \in M$.*

If M is a smooth manifold with or without boundary (as opposed to just an arbitrary topological space,) a ***smooth partition of unity*** *is one for which each of the functions ψ_α is smooth.* ★

Theorem 1 (Existence of Partitions of Unity). *Suppose M is a smooth manifold (with or without boundary,) and $\chi = (X_\alpha)_{\alpha\in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to χ.*

Remark: There are basically two different strategies for patching together locally defined smooth maps to obtain a global one. If you can define a map in a neighborhood of each point in such a way that the locally defined maps all agree where they overlap, then the local definitions piece together to yield a global smooth map by *Corollary 1*. (This usually requires some sort of uniqueness result.) But if the local definitions are not guaranteed to agree, then you usually have to resort to a partition of unity. The trick then is showing that the patched-together objects still have the required properties.

Definition. *If M is a topological space, $A \subseteq M$ is a closed subset, and $U \subseteq M$ is an open subset containing A, a continuous function $\psi\colon M \to \mathbb{R}$ is called a* ***bump function for A supported in U*** *if $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A, and supp$(\psi) \subseteq U$.* ★

Proposition 6 (Existence of Smooth Bump Functions). *Let M be a smooth manifold (with or without boundary.) For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.*

Definition. *Suppose M and N are smooth manifolds (with or without boundary,) and $A \subseteq M$ is an arbitrary subset. If N has empty boundary, we say that a map $F\colon A \to N$ is* ***smooth on*** *A if it has a smooth extension in a neighborhood of each point: that is, if for every $p \in A$ there is an open subset $W \subseteq M$ containing p and a smooth map $\widetilde{F}\colon W \to N$ whose restriction to $W \cap A$ agrees with F. When $\partial N \neq \emptyset$, we say that $F\colon A \to N$ is* ***smooth on*** *A if for every $p \in A$ there exist an open subset $W \subseteq M$ containing p and a smooth chart (V, ψ) for N whose domain contains $F(p)$, such that $F(W \cap A) \subseteq V$ and $\psi \circ F|_{W \cap A}$ is smooth as a map into $\mathbb{R}^n$ in the sense defined above (i.e. it has a smooth extension in a neighborhood of each point.)* ★

Lemma 4 (**Extension Lemma for Smooth Functions**). *Suppose M is a smooth manifold (with or without boundary,) $A \subseteq M$ is a closed subset, and $f\colon A \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\widetilde{f}\colon M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and supp$(\widetilde{f}) \subseteq U$.*

Remark: The assumption in the extension lemma that the codomain of f is $\mathbb{R}^k$, and not some other smooth manifold, is needed: for other codomains, extensions can fail to exist for topological reasons. For example, the identity map $\mathbb{S}^1 \to \mathbb{S}^1$ is smooth, but does not have even a continuous extension to a map from $\mathbb{R}^2$ to $\mathbb{S}^1$. Later on in the course we will show that a smooth map from a closed subset of a smooth manifold into a smooth manifold has a smooth extension iff it has a continuous one.

Definition. *If M is a topological space, an* ***exhaustion function for*** *M is a continuous function $f\colon M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ (called a* ***sublevel set of*** *f) is compact for each $c \in \mathbb{R}$.* ★

Remark: The name "exhaustion function" comes from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}((-\infty, n])$ form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{B}^n \to \mathbb{R}$ given by

$$f(x) = |x|^2, \qquad g(x) = \frac{1}{1 - |x|^2}$$

are smooth exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 7 (**Existence of Smooth Exhaustion Functions**). *Every smooth manifold (with or without boundary) admits a smooth positive exhaustion function.*

The following theorem shows the remarkable fact that every closed subset of a manifold can be expressed as a level set of some smooth real-valued function:

Theorem 2 (**Level Sets of Smooth Functions**). *Let M be a smooth manifold. If C is any closed subset of M, there is a smooth nonnegative function $f\colon M \to \mathbb{R}$ such that $f^{-1}(0) = C$.*