MATH 709 HW # 5

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Problem 1 (Problem 4-6)**.** *Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F\colon M \to \mathbb{R}^k$ for any $k > 0$.*

Proof. Assume, to the contrary, that F is a smooth submersion. Then, by a previous result, we know that F must be an open map, and thus we have that $F(M)$ is open in $\mathbb{R}^k$. In addition, we must have that $F(M)$ is compact because M is compact and continuous images of compact sets are compact. Therefore $F(M) \subseteq \mathbb{R}^k$ (for $k > 0$) is open and also closed and bounded (by Heine-Borel). But this would imply that, since $\mathbb{R}^k$ is connected, the image $F(M)$ is either empty or it's the entire space $\mathbb{R}^k$. Since M is nonempty, we must have that $F(M) = \mathbb{R}^k$, which contradicts the boundedness (and compactness) of $F(M)$ since $\mathbb{R}^k$ is unbounded (and noncompact). $(\Rightarrow\Leftarrow)$ $\square$