MATH 751 TAKE HOME EXAM

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Section 13

Ex # 5) Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Proof. Let $(X, \mathfrak{T})$ be a topological space, and let $\mathcal{A}$ be the basis that generates the topology $\mathfrak{T}$. In addition, let $\{\mathfrak{T}_\alpha\}$ be the set of topologies on X that contain $\mathcal{A}$. We claim that $\mathfrak{T} = \bigcap \mathfrak{T}_\alpha$:

($\subseteq$) Let $U \in \mathfrak{T}$. Then we know that U is a union $U = \bigcup_\alpha A_\alpha$ for some collection $\{A_\alpha\}_\alpha \subseteq \mathcal{A}$ [1]. But then $U = \bigcup_\alpha A_\alpha \in \bigcap \mathfrak{T}_\alpha$, since each $A_\alpha \in \bigcap \mathfrak{T}_\alpha$.

($\supseteq$) This inclusion is obvious. It follows from the fact that $\mathfrak{T} \supseteq \mathcal{A}$, and so is one of the topologies that is intersected over in the construction of $\bigcap \mathfrak{T}_\alpha$.

Now let $\mathcal{A}$ be a subbasis. The proof that $\bigcap \mathfrak{T}_\alpha \subseteq \mathfrak{T}$ is identical; thus it remains to show that $\mathcal{T} \subseteq \bigcap \mathfrak{T}_\alpha$. Let $U \in \mathfrak{T}$. By definition of the topology generated by $\mathcal{A}$, U is the union of a finite intersection of elements $\{A_\alpha\}_\alpha \subseteq \mathcal{A}$. But then $U \in \bigcap \mathfrak{T}_\alpha$, since each $A_\alpha \in \bigcap \mathfrak{T}_\alpha$. $\square$

Ex # 7) Consider the following topologies on $\mathbb{R}$:

- $\mathfrak{T}_1 =$ the standard topology.
- $\mathfrak{T}_2 =$ the topology of $\mathbb{R}_K$.
- $\mathfrak{T}_3 =$ the finite complement topology.
- $\mathfrak{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as basis.
- $\mathfrak{T}_5 =$ the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

[1] This is a consequence of the lemma that states that if $\mathcal{A}$ is basis for a topology $\mathfrak{T}$ on a space X, then $\mathfrak{T}$ equals the collection of all unions of elements of $\mathcal{A}$.

Solution. We present the comparison between these five topologies in the following *Hasse* diagram:

$$\begin{tikzcd} & \mathcal{T}_4 \arrow[d, dash] & \\ & \mathcal{T}_2 \arrow[d, dash] & \\ & \mathcal{T}_1 \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{T}_3 & & \mathcal{T}_5 \end{tikzcd}$$

• $\mathcal{T}_3$ and $\mathcal{T}_5$ are not comparable. $\mathcal{T}_3 \nsubseteq \mathcal{T}_5$ since $\mathbb{R} \smallsetminus \{0\} \in \mathcal{T}_3$, but if we take $x > 0$, which is in this set, there is no basis element $(-\infty, a) \in \mathcal{T}_5$ that contains x but is contained in $\mathbb{R} \smallsetminus \{0\}$. $\mathcal{T}_5 \nsubseteq \mathcal{T}_3$ since $(-\infty, 0)^c$ is not finite.

• $\mathcal{T}_3 \subsetneq \mathcal{T}_1$. Inclusion is true since if $U \in \mathcal{T}_3$, then we have that U^c is finite, and so if we let $U^c = \{x_i\}_{i=1}^n$ with x_i in increasing order, then $U = \bigcup_{i=0}^n (x_i, x_{i+1})$ with $x_0 = -\infty$ and $x_{n+1} = \infty$. Inequality follows since for (a, b) such that $-\infty < a, b < \infty$, we have that $\mathbb{R} \smallsetminus (a, b)$ is not finite.

• $\mathcal{T}_5 \subsetneq \mathcal{T}_1$. Inclusion is clear since $(-\infty, a)$ is of the form (b, c). That is, $(-\infty, x) = \bigcup_{i=1}^\infty (x - i, x) \in \mathcal{T}_1$ for all $x \in \mathbb{R}$. Inequality follows since for $(b, c) \in \mathcal{T}_1$ and $x \in (b, c)$, there is no basis element $(-\infty, a) \in \mathcal{T}_5$ such that $x \in (-\infty, a) \subseteq (b, c)$.

• $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ This is given in *Lemma* 13.4 in our text. For inclusion, notice that if we take a basis element (a, b) for $\mathcal{T}_1$ and a point $x \in (a, b)$, this interval is also a basis element for $\mathcal{T}_2$ that contains x as well. Inequality follows because, given a basis element $B = (-1, 1) \smallsetminus K$ for $\mathcal{T}_2$ and the point 0 of B, there is no open interval that contains 0 and lies in B.

• $\mathcal{T}_2 \subsetneq \mathcal{T}_4$. Consider the interval (a, b) and the point $x \in (a, b)$. Then we have $(a, x] \in \mathcal{T}_4$ and $(a, x] \subseteq (a, b)$. For $(a, b) \smallsetminus K \in \mathcal{T}_2$ and $x \in (a, b) \smallsetminus K$, we note that $x \in (1/(n+1), c]$ where either $x < c < 1/n$, $x \in (a, 0]$, or $x \in (1, d]$ with $x < d < b$. In all three cases, these sets are subsets of $(a, b) \smallsetminus K$ and are members of $\mathcal{T}_4$. Inequality follows since for $(a, b] \in \mathcal{T}_4$, there is no basis element $U \in \mathcal{T}_2$ such that $b \in U$ and $U \subset (a, b]$. □

Section 16

Ex # 5) Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Also, let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.

a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

b) Does the converse of a) hold? Justify your answer.

Proof of a). This is straightforward. Since by assumption $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{U} \subset \mathcal{U}'$, then we must have that $\mathcal{T}'$ and $\mathcal{U}'$ contain every basis element of $\mathcal{T}$ and $\mathcal{U}$, respectively (and more). Hence we must have that $O_x \times O_y \in \mathcal{T}' \times \mathcal{U}'$ for every basis element $O_x \times O_y$ of $\mathcal{T} \times \mathcal{U}$. Thus, the topology on $X' \times Y'$ is finer than the topology on $X \times Y$, as desired. $\square$

Proof of b). The converse does hold. Assume that the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$. Then if U is open in X, with $x \in X$, and V is open in Y, with $y \in Y$, then $U \times V$ is open in $X \times Y$ and, therefore, open in $X' \times Y'$. Hence, there exists a basis element $O'_x \times O'_y$ in $X' \times Y'$ such that

$$x \times y \in O'_x \times O'_y \subset U \times V.$$

Thus, there are open sets $O'_x \in \mathcal{T}'$ and $O'_y \in \mathcal{U}'$ such that $x \in O'_x \subseteq U$ and $y \in O'_y \subseteq V$. So, U is open in X' and V is open in Y', as desired. $\square$

Ex # 8) If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. (In each case it is a familiar topology).

Solution. Note that a basis for $\mathbb{R}_\ell \times \mathbb{R}$ consists of elements of the form $[a, b) \times (c, d)$. Assume that L has zero slope with its first coordinate fixed. Then $L = \{(x, y) \mid x = x_0\}$ for some fixed x_0, and thus $L \cap ([a, b) \times (c, d))$ is either empty or is equal to $\{x_0\} \times (c, d)$. So we define the map $\varphi\colon L \cap (\mathbb{R}_\ell \times \mathbb{R}) \to \mathbb{R}$, given by

$$\{x_0\} \times (c, d) \mapsto (c, d).$$

This map is bijective, open, and continuous, and so the topology that L inherits is homeomorphic to $\mathbb{R}$ with the standard topology.

In the case that L has finite slope, say m, we first note that $L \cap (\mathbb{R}_\ell \times \mathbb{R}) = \{(x, mx + b) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, and that the basis for our topology are the sets of the form

$$\emptyset, \quad [(a, ma + b), (c, mc + b)), \quad \text{and} \quad ((a, ma + b), (c, mc + b)),$$

for $a, b, c \in \mathbb{R}$ and $a < c$.

We then define the function $\varphi\colon L \bigcap (\mathbb{R}_\ell \times \mathbb{R}) \to \mathbb{R}_\ell$ given by

$$(a, ma + b) \mapsto a.$$

This implies

$$((a, ma + b), (c, mc + b)) \mapsto (a, c),$$
$$[(a, ma + b), (c, mc + b)) \mapsto [a, c).$$

We claim that this defines a homeomorphism with $\mathbb{R}_\ell$. Clearly, it is continuous, for the basis elements of $\mathbb{R}_\ell$ have preimages that are basis elements in the topology on L, i.e. preimages of open sets are open, hence φ is continuous. Likewise, it is an open map since the basis elements of L map to sets that are open in $\mathbb{R}_\ell$. Finally this is a bijection since φ clearly has an inverse.

Finally, we need to check the case for $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Following the same steps as above, if $L = \{(x, y) \mid x = x_0\}$, then we have that $L \cap (\mathbb{R}_\ell \times \mathbb{R}_\ell)$ is homeomorphic to $\mathbb{R}_\ell$. For L with $|m| < \infty$, we must split it up into two cases. When $m \geq 0$, we have a similar situation as above, except we only have to consider basis elements of the form $[a, b)$; thus, $L \cap (\mathbb{R}_\ell \times \mathbb{R}_\ell)$ is homeomorphic to $\mathbb{R}_\ell$. When $m < 0$, notice that for every point $(x, y) \in L$, we can find a basis element $[x, a) \times [y, b) \in (\mathbb{R}_\ell \times \mathbb{R}_\ell)$ such that $L \cap ([x, a) \times [y, b)) = \{(x, y)\}$, and these form the open sets of our new topology. We see then that the topology on L is homeomorphic to the discrete topology on $\mathbb{R}$. $\square$

Ex # 10) Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Solution. Let us denote the product topology by $\mathcal{T}_1$, the dictionary order topology by $\mathcal{T}_2$, and the subspace topology by $\mathcal{T}_3$. Now notice that $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable. For instance, take $(0, 1) \in [0, 1] \times (1/2, 1]$, which has no open neighborhood in $\mathcal{T}_2$, and take $(0, 1/2) \in \{0\} \times (0, 1)$, which has no open neighborhood in $\mathcal{T}_1$.
Now we note that $\mathcal{T}_3$ is strictly finer than both $\mathcal{T}_1$ and $\mathcal{T}_2$. Indeed, $\mathcal{T}_3$ is generated by sets $\{x\} \times ((a, b) \bigcap [0, 1])$; every basis element $((a, b) \bigcap [0, 1]) \times ((c, d) \bigcap [0, 1])$ of $\mathcal{T}_1$ can be generated as the union of open sets in $\mathcal{T}_3$, and also every basis set $(a, b) < (x, y) < (c, d)$ in $\mathcal{T}_2$ can be generated as the union as well. The fact that $\mathcal{T}_3$ is strictly finer follows from the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable. $\square$

Section 17

Ex # 21) (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X \setminus A$ are functions from this collection to itself.

a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.

b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

Proof. a) Let $A_1 = A$ and set $B_1 = A_1^c$. Define $A_{2n} = \overline{A_{2n-1}}$ and $A_{2n+1} = A_{2n}^c$, for $n \in \mathbb{N}$. Also define $B_{2n} = \overline{B_{2n-1}}$ and $B_{2n+1} = B_{2n}^c$, for $n \in \mathbb{N}$.
Note that every set obtainable from A by repeatedly applying the closure and complement

operations is clearly one of the sets A_n or B_n. Now $A_7 = X \setminus \overline{X \setminus A_4} = A_4^o = (\overline{A_3})^o$. Since $A_3 = \overline{A_1}^c$, it follows that A_3 is open, hence $A_3 \subset A_7 \subset \overline{A_3}$, so $\overline{A_7} = \overline{A_3}$, i.e. $A_8 = A_4$, hence $A_{n+4} = A_n$ for $n \geq 4$. Similarly $B_{n+4} = B_n$ for $n \geq 4$. Thus every A_n or B_n is equal to one of the 14 sets $A_1, \ldots, A_7, B_1, \ldots, B_7$, and this proves the result. □

Solution. b) Let $A = ((-\infty, -1) \setminus \{-2\}) \bigcup ([-1, 1] \bigcap \mathbb{Q}) \bigcup \{2\}$. Then the 14 different sets are:

$$\begin{aligned}
A_1 &= ((-\infty, -1) \setminus \{-2\}) \cup ([-1, 1] \cap \mathbb{Q}) \cup \{2\} \\
A_2 &= (-\infty, 1] \cup \{2\} \\
A_3 &= (1, \infty) \setminus \{2\} \\
A_4 &= [1, \infty) \\
A_5 &= (-\infty, 1) \\
A_6 &= (-\infty, 1] \\
A_7 &= (1, \infty) \\
B_1 &= \{-2\} \cup ([-1, 1] \setminus \mathbb{Q}) \cup ((1, \infty) \setminus \{2\}) \\
B_2 &= \{-2\} \cup [-1, \infty) \\
B_3 &= (-\infty, -1) \setminus \{-2\} \\
B_4 &= (-\infty, -1] \\
B_5 &= (-1, \infty) \\
B_6 &= [-1, \infty) \\
B_7 &= (-\infty, -1).
\end{aligned}$$
□

Section 18

Ex # 12) Let $F \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0, \\ 0 & \text{if } x \times y = 0 \times 0. \end{cases}$$

a) Show that F is continuous in each variable separately.

b) Compute the function $g \colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

c) Show that F is not continuous.

Proof of a). Interchanging $x \leftrightarrow y$ leaves us in an identical situation, so it would certainly suffice to fix the first coordinate at x_0 and prove the result for all y in the second coordinate. Let $h(y) = F(x_0 \times y)$; we claim that h is continuous as a function $\mathbb{R} \to \mathbb{R}$. For $y = 0$, this is trivially true for the image of h is $(0,0)$ with preimage $\mathbb{R}$. Now suppose $y \neq 0$; then we have $h(y) = x_0 y/(x_0^2 + y^2)$. This is continuous since $x_0 y$ and $x_0^2 + y^2$ are both continuous, and so their quotient is also continuous (since also $x_0^2 + y^2 \neq 0$). □

Proof of b). Since $F(x \times x)$ for $x \neq 0$ equals $x^2/(x^2 + x^2) = x^2/2x^2 = 1/2$, we have

$$g(x) = \begin{cases} 1/2 & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$ □

Proof of c). We claim $F(x \times y)$ is not continuous along the line $L = \{(x, y) \mid x = y\}$ at $(0,0)$, i.e. $F|_L$ is not continuous at $(0,0)$. Note that the line L in the subspace topology is homeomorphic to $\mathbb{R}$, where the homeomorphism is given by either of the coordinate projection maps π_1 or π_2. Now the preimage of the closed set $\{1/2\} \subseteq \mathbb{R}$ is $L \setminus \{(0,0)\}$, which is not closed since $\mathbb{R} \setminus \{0\}$ is not closed, hence $F|_L$ is not continuous, and neither is F. □

Section 20

Ex # 10) Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \dots)$ such that $\sum x_i^2$ converges.

a) Show that if $x, y \in X$, then $\sum |x_i y_i|$ converges.

b)Let $c \in \mathbb{R}$. Show that if $x, y \in X$, then so are $x + y$ and cx.

c) Show that

$$d(x, y) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

is a well defined metric on X.

Proof of a). Notice that if $x, y \in X$, then we must have $\sum x_i^2 < \infty$ and $\sum y_i^2 < \infty$. But then by Cauchy-Schwarz, we have

$$\begin{aligned} \sum |x_i y_i| \leq \sum |x_i||y_i| \leq \sum |x_i|^2 |y_i|^2 \\ &= \sum x_i^2 y_i^2 \\ &= \sum (x_i y_i)^2 < \infty. \end{aligned}$$

Hence $\sum |x_i y_i|$ converges, as desired. □

Proof of b). To see why $x + y \in X$, notice that

$$\begin{aligned}\sum (x_i + y_i)^2 &= \sum (x_i^2 + 2x_i y_i + y_i^2)\\ &= \sum x_i^2 + 2\sum x_i y_i + \sum y_i^2\\ &\leq \sum x_i^2 + 2\sum |x_i y_i| + \sum y_i^2. \qquad (\dagger)\end{aligned}$$

In part a), we showed that $\sum |x_i y_i|$ converges, and thus every term in ($\dagger$) converges, which implies that $\sum (x_i + y_i)^2$ also converges and thus we have that $x + y \in X$.

The case for cx is trivial, since $\sum (cx_i)^2 = \sum c^2 x_i^2 = c^2 \sum x_i^2$. But c^2 is just a real number $< \infty$ multiplying a convergent sum, hence $\sum (cx_i)^2 < \infty$, which implies that $cx \in X$, as desired. □

Proof of c). For all $x, y \in X$, notice that the sum in d converges by our argument on part b) and it is also nonnegative, which implies that $0 \leq d(x, y) < \infty$. Note also that this sum is only zero iff $x = y$, which implies that $d(x, y) = 0 \iff x = y$. Now for the triangle inequality, notice that $d(x, y)$ is just the 2-norm $\|x - y\|$. Let us first apply Cauchy-Schwarz so that

$$\begin{aligned}\|x + y\|^2 = \langle x + y, x + y\rangle &= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2\\ &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2\\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \qquad \text{(By Cauchy Schwarz)}\\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

This gives us $\|x + y\| \leq \|x\| + \|y\|$.

Now it follows that for $x, y, z \in X$, we have

$$d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y).$$

Hence d is a metric on X, as desired. □

Section 22

Ex # 2) a) Let $p\colon X \to Y$ be a continuous map. Show that if there is a continuous map $f\colon Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

b) If $A \subset X$, a **retraction** of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Proof of a). Let $V \subset Y$ with $U = p^{-1}(V)$ open in X. Then,

$$f^{-1}(U) = f^{-1}(p^{-1}(V)) = (p \circ f)^{-1}(V) = \mathrm{Id}^{-1}(V) = \mathrm{Id}(V) = V,$$

so that V is open by the continuity of f. Hence we have that p is a quotient map, as desired. □

Proof of b). Let $\iota\colon A \to X$ be the inclusion map; then, $r \circ \iota$ is the identity map on A, hence it follows by $a)$ that r is a quotient map. □

Section 26

Ex # 12) Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$ (such a map is called a **perfect map**). Show that if Y is compact, then X is also compact. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, then there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.]

Proof. The goal of this exercise is to show that any perfect map is proper [2]. Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$. Then we are going to show that $p^{-1}(K)$ is compact for any compact subspace $K \subset Y$. In our proof we are going to use the provided hint, i.e. we use the fact that if $p^{-1}(\{y\}) \subset U$ (where U is an open subspace of X), then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of $\{y\}$.

Notice that this indeed holds because

$$\begin{aligned} p^{-1}(W) \subset U &\iff p(x) \in W \implies x \in U \\ &\iff x \notin U \implies p(x) \notin W \\ &\iff p(X \setminus U) \subset Y \setminus W \\ &\iff p(X \setminus U) \cap W = \emptyset. \end{aligned}$$

The point $\{y\}$ does not belong to the closed set $p(X \setminus U)$. Therefore a whole neighborhood $W \subset Y$ of $\{y\}$ is disjoint from $p(X \setminus U)$, i.e. $p^{-1}(W) \subset U$.

Now we are ready for the proof. Let $K \subset Y$ be compact. Consider a collection $\{U_\alpha\}_{\alpha \in J}$ of open sets covering of $p^{-1}(K)$. For each $y \in K$, the compact space $p^{-1}(\{y\})$ is contained in a the union of a finite subcollection $\{U_\alpha\}_{\alpha \in J(\{y\})}$. There is neighborhood W_y of $\{y\}$ such that $p^{-1}(W_y)$ is contained in this finite union. By compactness of K, finitely many $W_{y_1}, \ldots, W_{y_k}$ cover Y. Then the finite collection $\bigcup_{i=1}^{k} \{U_\alpha\}_{\alpha \in J(\{y_i\})}$ covers $p^{-1}(K)$. This shows that $p^{-1}(K)$ is compact, as desired. □

[2] A mapping $f\colon X \to Y$ between two topological spaces is said to be **proper** if the preimage of every compact set in Y is compact in X.

Section 27

Ex # 1) Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.

Proof. Let $A \subset X$ be bounded from above by $b \in X$, so that for any $a \in A$, the closed interval $[a, b]$ is compact. The set $C = \bar{A} \bigcap [a, b]$ is closed in $[a, b]$, hence compact. The inclusion map $j: C \to X$ is continuous [3]. By the extreme value theorem, C has a largest element $c \in C$. Clearly c is an upper bound for A. If $c \in A$, then clearly c is the least upper bound by definition. Then suppose $c \notin A$. If $d < c$, then (d, ∞) is an open set containing c, i.e. $A \bigcap (d, \infty) \neq \emptyset$, since c is a limit point for A (this is true because $c \in C \subset \bar{A}$ by definition). Thus d is not an upper bound for A, and we have that c is the least upper bound. Hence, X has the least upper bound property, as desired. □

Ex # 5) Let X be a compact Hausdorff space, and let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ also has empty interior in X (this is a special case of *Baire's Category Theorem*).

Proof. Let U_0 be any nonempty open set of X. We must find a point $x \in U_0$ that lies outside all the A_n. Consider the first set A_1. By hypothesis, we have $U_0 \not\subset A_1$, for A_1 has no interior. So the open set $U_0 \setminus A_1$ is nonempty. By regularity of X, along with the fact that A_1 is closed, we can find a nonempty open set U_1 such that

$$\overline{U_1} \bigcap A_1 = \emptyset$$

and

$$U_1 \subset \overline{U_1} \subset U_0 \setminus A_1 \subset U_0.$$

By the same reasoning, we have $U_1 \not\subset A_2$, for A_2 has no interior. So the open set $U_1 \setminus A_2$ is nonempty. By regularity of X, along with the fact that A_2 is closed, we can find a nonempty open set U_2 such that

$$\overline{U_2} \bigcap A_2 = \emptyset$$

and

$$U_2 \subset \overline{U_2} \subset U_1 \setminus A_2 \subset U_1.$$

Continuing in this fashion, we find a descending sequence of nonempty open sets U_n such that

$$U_n \subset \overline{U_n} \subset U_{n-1} \setminus A_n \subset U_{n-1} \qquad \text{for all } n.$$

[3] We know this from a previous proposition that says that if A is a subspace of a topological space X, then the inclusion map $j: A \to X$ is continuous.

Now we assert that the intersection $\bigcap \overline{U_n}$ is nonempty. From this fact, our proof will be concluded. For if x is a point of $\bigcap \overline{U_n}$, then x is in U_0 because $\overline{U_1} \subset U_0$. And for each n, the point x is not in A_n because $\overline{U_n}$ is disjoint from A_n.

Hence let us conclude the proof by showing that $\bigcap \overline{U_n}$ is nonempty. Since X is compact Hausdorff, consider the nested sequence $\overline{U_1} \supset \overline{U_2} \supset \dots$ of nonempty subsets of X. Then the collection $\{\overline{U_n}\}$ has the finite intersection property; since X is compact, the intersection $\bigcap \overline{U_n}$ must be nonempty, as desired. □

Section 31

Ex # 5) Let $f, g \colon X \to Y$ be continuous, and assume that Y is Hausdorff. Show that $\{x \mid f(x) = g(x)\}$ is closed in X.

Proof. We are going to use the result that a space X is Hausdorff iff the ***diagonal*** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

To see why this result holds, suppose Δ is closed in $X \times X$, i.e. the complement Δ^c is open. This is equivalent to saying that for all $(x, y) \in X \times X$ such that $x \neq y$, there exists a basis element $U \times V$ of $X \times X$ for U, V open in X such that $(x, y) \in U \times V$ but $(U \times V) \cap \Delta = \emptyset$. But then, by definition of Δ, this is equivalent to saying for all $x, y \in X$ such that $x \neq y$, there exist open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$, and so X is Hausdorff.

Hence, in our case we have that that since Y is Hausdorff, the diagonal $\Delta = \{y \times y \mid y \in Y\}$ must be closed in $Y \times Y$. Now by a previous theorem [4], since f and g are continuous, the map $(f, g) \colon X \to Y \times Y$ must be continuous as well. Thus

$$\{x \in X \mid f(x) = g(x)\} = (f, g)^{-1}(\Delta)$$

must also be closed, as desired. □

[4] Here's the theorem, for reference:
Let $f \colon A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then f is continuous iff the functions $f_1 \colon A \to X$ and $f_2 \colon A \to Y$ are continuous.

For the last two exercises we shall use the following lemma, which we partially used before on *Problem 26.12* above:

Lemma 1. *Let $p\colon X \to Y$ be a closed map. Then*

(1) If $p^{-1}(\{y\}) \subset U$ (where U is an open subspace of X), then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of $\{y\}$.
(2) If $p^{-1}(B) \subset U$ for some subspace B of Y and some open subspace U of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of B.

Proof. As we pointed out before in *Problem 26.12*, notice that

$$\begin{aligned} p^{-1}(W) \subset U &\iff p(x) \in W \Longrightarrow x \in U \\ &\iff x \notin U \Longrightarrow p(x) \notin W \\ &\iff p(X \smallsetminus U) \subset Y \smallsetminus W \\ &\iff p(X \smallsetminus U) \cap W = \emptyset. \end{aligned}$$

Now for part (1), note that the point $\{y\}$ does not belong to the closed set $p(X \smallsetminus U)$. Therefore a whole neighborhood $W \subset Y$ of $\{y\}$ is disjoint from $p(X \smallsetminus U)$, i.e. $p^{-1}(W) \subset U$.

For part (2), notice that each point $y \in B$ has a neighborhood W_y such that $p^{-1}(W_y) \subset U$. The union $W = \bigcup W_y$ is then a neighborhood of B with $p^{-1}(W) \subset U$. □

Ex # 6) Let $p\colon X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, show that there is a neighborhood W of y such that $p^{-1}(W) \subset U$.]

Proof. Since points are closed in X and the mapping p is closed, all points in $p(X)$ are closed. All fibres $p^{-1}(\{y\}) \subset X$ are therefore also closed. Let y_1 and y_2 be two distinct points in Y. Since X is normal we can separate the disjoint closed sets $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ by disjoint neighborhoods U_1 and U_2. Using *Lemma 1* part (1), choose neighborhoods W_1 of y_1 and W_2 of y_2 such that $p^{-1}(W_1) \subset U_1$ and $p^{-1}(W_2) \subset U_2$. Then W_1 and W_2 are disjoint. Thus Y is Hausdorff. Finally, using essentially the same argument, but this time making use of *Lemma 1* part (2), we have that we can separate disjoint closed sets in Y by disjoint open sets. Thus Y is normal, as desired. □

Ex # 7) Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$, i.e. p is a perfect map.

a) Show that if X is Hausdorff, then so is Y.

b) Show that if X is regular, then so is Y.

c) Show that if X is locally compact, then so is Y.

d) Show that if X is second-countable, then so is Y. [*Hint:* Let $\mathcal{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W an open set in Y, that are contained in the union of the elements of J.]

Proof. a) Before tackling our proof, let us recall a lemma that asserts that if K is a compact subspace of a Hausdorff space H and there is a point $x_0 \notin K$, then there exist disjoint open sets U and V containing x_0 and K, respectively.

A stronger assertion is the following:

Let K_1 and K_2 be disjoint compact subspaces of a Hausdorff space H. Then there exist disjoint open sets U and V containing K_1 and K_2, respectively.

To see why the latter assertion holds, for each $a \in K_1$ choose disjoint open sets $U_a \ni a$ and $V_a \supset K_2$. Since K_1 is compact, K_1 is contained in a finite union $U = U_1 \cup \cdots \cup U_n$ of the U_a's. Let $V = V_1 \cap \cdots \cap V_n$ be the intersection of the corresponding V_a's. Then U is an open set containing K_1, V is an open set containing K_2, and U and V are disjoint since $U \cap V = \cup(U_i \cap V) \subset \cup(U_i \cap V_i) = \emptyset$.

Now proceeding with our proof, let y_1 and y_2 be two distinct points in Y. Then, using the result we just proved, we can separate the two disjoint compact subspaces $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ by disjoint open subspaces $U_1 \supset p^{-1}(\{y_1\})$ and $U_2 \supset p^{-1}(\{y_2\})$ of the Hausdorff space X. Now, by *Lemma 1*, we may choose open sets $W_1 \ni y_1$ and $W_2 \ni y_2$ such that $p^{-1}(W_1) \subset U_1$ and $p^{-1}(W_2) \subset U_2$. Then W_1 and W_2 are disjoint, and we have that Y is Hausdorff, as desired. □

Proof. b) We know that Y is Hausdorff by part a). Now let $C \subset Y$ be a closed subspace and $y \in Y$ be a point outside C. It is enough to separate the compact fiber $p^{-1}(\{y\}) \subset X$ and the closed set $p^{-1}(C) \subset X$ by disjoint open sets (note that *Lemma 1* provides open sets in Y separating $\{y\}$ and C). Each $x \in p^{-1}(\{y\})$ can be separated by disjoint open sets from $p^{-1}(C)$ since X is regular. Then using compactness of $p^{-1}(\{y\})$ we obtain disjoint open sets $U \supset p^{-1}(\{y\})$ and $V \supset p^{-1}(C)$, as required. □

Proof. c) Using compactness of $p^{-1}(\{y\})$ and local compactness of X we construct an open subspace $U \subset X$ and a compact subspace $K \subset X$ such that $p^{-1}(\{y\}) \subset U \subset K$. In the process we need to use the fact that a finite union of compact subspaces is compact.

To see why this last remark is true, let $A_1, \dots, A_n$ be compact subspaces of X. Let $\mathcal{C}$ be an open covering of $\bigcup_{i=1}^n A_i$. Since $A_j \subset \bigcup_{i=1}^n A_i$ is compact for $1 \leq j \leq n$, there is a finite subcovering $\mathcal{C}_j$ of $\mathcal{C}$ covering A_j. Thus $\bigcup_{j=1}^n \mathcal{C}_j$ is a finite subcovering of $\mathcal{C}$, hence $\bigcup_{i=1}^n A_i$ is compact.

Now, to finish off our proof, notice that by *Lemma 1*, there is an open set $W \ni y$ such that $p^{-1}(\{y\}) \subset p^{-1}(W) \subset U \subset K$. Then $y \in W \subset p(K)$, where $p(K)$ is compact (because the image of a compact space under a continuous map is compact). Thus Y is locally compact, as desired. □

Proof. d) Let $\{B_j\}_{j\in\mathbb{N}}$ be a countable basis for X. For each finite subset $J \subset \mathbb{N}$, let $U_J \subset X$ be the union of all open sets of the form $p^{-1}(W)$ with open $W \subset Y$ and $p^{-1}(W) \subset \bigcup_{j\in J} B_j$. There are countably many open sets U_J. The image $p(U_J)$ is a union of open sets in Y, hence it is open. Now let $V \subset Y$ be any open subspace. The inverse image $p^{-1}(V) = \bigcup_{y\in V} p^{-1}(\{y\})$ is a union of fibers. Since each fiber $p^{-1}(\{y\})$ is compact, it can be covered by a finite union $\bigcup_{j\in J(y)} B_j$ of basis sets contained in $p^{-1}(V)$. Now by *Lemma 1*, there is an open set $W \subset Y$ such that

$$p^{-1}(\{y\}) \subset p^{-1}(W) \subset \bigcup_{j\in J(y)} B_j.$$

Taking the union of all these open sets W, we get

$$p^{-1}(\{y\}) \subset U_{J(y)} \subset \bigcup_{j\in J(y)} B_j \subset p^{-1}(V).$$

We now have $p^{-1}(V) = \bigcup_{y\in V} U_{J(y)}$ so that $V = pp^{-1}(V) = \bigcup_{y\in V} p(U_{J(y)})$ is a union of sets from the countable collection $\{p(U_J)\}$ of open sets. Thus Y is second countable, as desired. □