(1) Section 2.1

(#5) Prove that T is a linear transformation and find bases for both $N(T)$ and $R(T)$. Then compute nullity(T) and rank(T) and verify the dimension theorem. Finally determine whether T is one-to-one or onto.

$T : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $T(f(x)) = x\, f(x) + f'(x)$

<u>Solution:</u>

• First we want to prove that T is linear.

•⟶ $T(0) = x * 0 + 0 = 0$ ✓

•⟶ Given an arbitrary scalar $c \in \mathbb{R}$ and two arbitrary $g(x)$ and $h(x) \in P_2(\mathbb{R})$, we must show that $T\big(c\, g(x) + h(x)\big) = c\, T\, g(x) + T\, h(x)$.

Thus we have

$$T\big(c\, g(x) + h(x)\big) = x\big(c\, g(x) + h(x)\big) + \big(c\, g(x) + h(x)\big)'$$
$$= c\, x\, g(x) + x\, h(x) + c\, g'(x) + h'(x)$$
$$= c\, x\, g(x) + c\, g'(x) + x\, h(x) + h'(x)$$
$$= c\big(x\, g(x) + g'(x)\big) + x\, h(x) + h'(x)$$
$$= c\, T\, g(x) + T\, h(x) \qquad ✓$$

Hence we have proven that T is linear. ✓

• We need to find a basis for $N(T)$.

Given an arbitrary $f(x) \in P_2(\mathbb{R})$ with scalars $a_1, a_2, a_3 \in \mathbb{R}$,we map it to $P_3(\mathbb{R})$.

$$T(f(x)) = T\big(a_1 + a_2\, x + a_3\, x^2\big) = x\big(a_1 + a_2\, x + a_3\, x^2\big) + \big(a_1 + a_2\, x + a_3\, x^2\big)'$$
$$= x\, a_1 + a_2\, x^2 + a_3\, x^3 + a_1' + a_2\, x' + a_3\, x^{2\,\prime}$$
$$= x\, a_1 + a_2\, x^2 + a_3\, x^3 + 0 + a_2 + 2\, a_3\, x$$
$$= a_2 + (a_1 + 2\, a_3)\, x + a_2\, x^2 + a_3\, x^3$$

Now looking at the resulting equation we have that

$a_2 + (a_1 + 2\, a_3)\, x + a_2\, x^2 + a_3\, x^3 = 0 \Longrightarrow a_1, a_2, a_3 = 0$

This is because x, x^2, and x^3 are linearly independent. Hence in order to map this function to the zero vector in $P_3(\mathbb{R})$ we must let all the coefficients be equal to zero. That implies that the nullspace is trivial, i.e. $N(T) = \{0\}$. Hence the basis for $N(T)$ is $\emptyset$. ✓

• Now we need to find a basis for $R(T)$. We know from the R-N theorem that

$\dim(P_2(\mathbb{R})) = \text{nullity}(T) + \text{rank}(T)$

$\Longrightarrow 3 = 0 + \text{rank}(T) \Longrightarrow \text{rank}(T) = 3$

That means that a basis for $R(T)$ must have cardinality 3.

We pick an arbitrary $f(x) \in P_2(\mathbb{R})$ with scalars $a_1, a_2, a_3 \in \mathbb{R}$,and map it to $P_3(\mathbb{R})$.
$T(f(x)) = T\left(a_1 + a_2\,x + a_3\,x^2\right) = x\left(a_1 + a_2\,x + a_3\,x^2\right) + \left(a_1 + a_2\,x + a_3\,x^2\right)'$
$= x\,a_1 + a_2\,x^2 + a_3\,x^3 + a_1' + a_2\,x' + a_3\,x^2{}'$
$= x\,a_1 + a_2\,x^2 + a_3\,x^3 + 0 + a_2 + 2\,a_3\,x$
$= a_2 + (a_1 + 2\,a_3)\,x + a_2\,x^2 + a_3\,x^3 = R(T)$
Now we can rewrite $R(T)$ as
$R(T) = a_1\,x + a_2\left(1 + x^2\right) + a_3\left(x^3 + 2\,x\right)$

Hence $\left\{x, 1 + x^2, x^3 + 2\,x\right\}$ is a spanning set for $R(T)$. It is obvious that this set is also linearly independent (so I won t prove it!). Hence it is also a basis for $R(T)$ (and as we expected it has cardinality 3). ✓

• Lastly we need to determine whether T is one-to-one or onto.
Since $N(T) = \{0\}$, we know that T is one-to-one. However T is not onto because $R(T) \neq P_3(\mathbb{R})$. We know this because $\text{rank}(T) = 3 \neq 4 = \dim(P_3(\mathbb{R}))$. ✓