Linear Algebra Notes

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Vector Spaces

• Theorem:
Any intersection of subspaces of a vector space V is a subspace of V.

Proof:
Let C be the collection of subspaces of V, i.e. $C = \{W_1, \ \dots, \ W_n\}$, and let $W = \bigcap_{U \in C} U$. Then we wish to show that W is a subspace of V.

→ Since $\hat{0} \in W_i \ \ \forall \ W_i \in C$, then $\hat{0} \in W$. (existence of zero vector) ✓

→ Let $x, \ y \in W$. Then $x, \ y \in U \quad \forall \ U \in C$. Then $x + y \in U \quad \forall \ U \in C$ and thus $x + y \in W$. (closure under addition) ✓

→ Let $a \in \mathbb{F}$ and $x \in W$. Then $x \in U \ \ \forall \ U \in C$ and $a\,x \in U \ \ \forall \ U \in C$. Hence $a\,x \in W$. (closure under scalar multiplication) ✓

Hence W is *a* subspace of V. ■

• Theorem:
a) For $S \subseteq V$, span(S) is *a* subspace of V.
b) Any subspace of V that contains S must also contain span (S). (In other words, span (S) is the smallest subspace containing S).

Proof:
Let $S \subseteq V$.
a)
→ Suppose $S = \emptyset$. Then span(S) = $\{\hat{0}\} \subseteq V$, which is a subspace of V. Suppose $S \neq \emptyset$. Then we can observe that $\hat{0}$ can be written as a linear combination of vectors in S by simply letting all the coefficients be the zero scalar, i.e. $\hat{0} = \sum 0\,x_i \ \ \forall \ \ x_i \in S$. Thus $\hat{0} \in$ span(S). (existence of zero vector) ✓

→ Let $x, \ y \in$ span (S). We wish to show that $x + y \in$ span (S). Then

$\exists\ u_1, \ldots, u_n, v_1, \ldots, v_m \in S$ and $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{F}$ such that
$x = a_1 u_1 + \ldots + a_n u_n$ and $y = b_1 v_1 + \ldots + b_m v_m$.
Then $x + y = a_1 u_1 + \ldots + a_n u_n + b_1 v_1 + \ldots + b_m v_m$.
This is a linear combination of vectors in S and scalars in $\mathbb{F}$. Thus $x + y \in \text{span}(S)$. (closure under addition) ✓

→ Let $c \in \mathbb{F}$ and $x \in \text{span}(S)$. We wish to show that $c\,x \in \text{span}(S)$. Then
$\exists\ u_1, \ldots, u_n \in S$ and $a_1, \ldots, a_n \in \mathbb{F}$ such that $x = a_1 u_1 + \ldots + a_n u_n$.
Then $c\,x = c(a_1 u_1 + \ldots + a_n u_n) = (c\,a_1) u_1 + \ldots + (c\,a_n) u_n$.
Therefore $c\,x$ can be written as a linear combination of vectors in S,
i.e. $c\,x \in \text{span}(S)$. (closure under scalar multiplication) ✓

Hence we have proven that $\text{span}(S)$ is a subspace of V. ✓

b)
Let W be a subspace of V. Then we want to show that if $S \subseteq W$, then $\text{span}(S) \subseteq W$.
Let $x \in \text{span}(S)$. We will show that $x \in W$.
Then $\exists\ u_1, \ldots, u_n \in \mathrm{S}$ and $a_1, \ldots, a_n \in \mathbb{F}$ such that $x = a_1 u_1 + \ldots + a_n u_n$.
Since $S \subseteq W$, $u_1, \ldots, u_n \in W$. Then since W is a subspace, any linear combination $a_1 u_1 + \ldots + a_n u_n$ must also be in W. Since $x = a_1 u_1 + \ldots + a_n u_n$, we conclude that $x \in W$ which means that $\text{span}(S)$ is also in W. ✓ ■

• Theorem:
Let V be a vector space and let $S_1 \subseteq S_2 \subseteq V$. Then if S_1 is linearly dependent, so is S_2.

Proof:
Suppose S_1 is linearly dependent, then $\exists\ u_1, \ldots, u_n \in S_1$ and $a_1, \ldots, a_n \in \mathbb{F}$ not all $a_i = 0$ such that $\hat{0} = a_1 u_1 + .. + a_n u_n$ (nontrivial representation).
Since $S_1 \subseteq S_2$, $u_1, \ldots, u_n \in S_2$. Then $\hat{0} = a_1 u_1 + .. + a_n u_n$, where $a_i \in \mathbb{F}$, not all zero.
Thus S_2 is linearly dependent. ■

• Corollary:
If S_2 is linearly independent, then so is S_1.

Proof:
(It's just the contrapositive of the proof above). ■

• Linear dependence lemma:

Suppose $S \subseteq V$ is linearly dependent and contains at least one non-zero vector . Then $\exists\ v \in S$ such that

i) $v \in \text{span}(S \backslash \{v\})$

ii) $\text{span}(S \backslash \{v\}) = \text{span}\,(S)$.

Proof:

i) Let $S \subseteq V$ be linearly dependent. Then $\exists\, u_1, \ \ldots, u_n \in S$ and $a_1, \ \ldots, a_n \in \mathbb{F}$ such that $\hat{0} = a_1\, u_n\ +\ \ldots +\ a_n\, v_n$ with a_i not all 0. Let $a_k \neq 0$.

Then, $\hat{0} = a_1\, u_n + \ \ldots + a_k\, v_k + \ \ldots + a_n\, u_n$. Then $-a_k\, u_k = a_1\, u_n\ +\ \ldots +\ a_n\, v_n$.

Then we divide both sides by $-a_k$ and we have

(*) $u_k = \dfrac{-a_1}{a_k}\, u_1\ +\ \ldots\ +\ \dfrac{-a_n}{u_k}\, u_n$.

We let $v\ = u_k$. Then $v \in \text{span}(S \backslash \{v\})$. ✓

ii) We wish to show that $\text{span}(S \backslash \{v\}) = \text{span}(S)$.

(We have to show set containment from both sides. Namely for two given sets A and B, $A = B \Longrightarrow A \subseteq B$ and $B \subseteq A$.)

($\subseteq$)

Let $x \in \text{span}(S \backslash\, \{v\})$. Then $\exists\ w_1, \ \ldots, w_{n-1} \in S \backslash \{v\}$ and $b_1, \ \ldots, b_{n-1} \in \mathbb{F}$ such that $x = b_1\, w_1\ +\ \ldots +\ b_{n-1}\, w_{n-1}$. But $w_1, \ \ldots, w_{n-1} \in S$. So x has a linear combination representation with vectors from S. This means that $x \in \text{span}(S)$. ✓

($\supseteq$)

Let $y \in \text{span}\,(S)$. Then $\exists\ u_1, \ \ldots, u_n \in S$ and $c_1, \ \ldots, c_n \in \mathbb{F}$ such that $y\ = c_1\, u_1 + \ldots +\ c_n\, u_n$. Then we substitute u_k from (*).

Then $y = c_1\, u_1 + \ldots +\ c_k\left(\dfrac{-a_1}{a_k}\, u_1 + \ldots + \dfrac{-a_n}{u_k}\, u_n\right) + \ldots + c_n\, u_n$

$= \left(c_1 - \dfrac{c_k\, a_1}{a_k}\right) u_1 +\ \ldots + \left(c_n - \dfrac{c_k\, a_n}{a_n}\right) u_n$

Thus $y \in \text{span}(S \backslash \{v\})$. ✓

Hence we have proven that $\text{span}(S \backslash \{v\}) = \text{span}\,(S)$. ■

• Theorem:

Suppose V is finite dimensional and let S be a linearly independent subset of V and let T be a spanning set for V. Then $|S| \leq |T|$. (the cardinality of a linearly independent set is less than or equal to that of a spanning set)

<u>Proof:</u>

Let $S = \{v_1, \ldots, v_n\}$ and let $T = \{w_1, \ldots, w_m\}$. Then we need to show that $n \leq m$.
Suppose $|S| > |T|$, i.e $n > m$. Since T is a spanning set for V, $\operatorname{span}(T) = V$.
We start with $v_1 \in S$. Since any vector in V can be written as a linear combination of vectors in T, we know that $v_1 \in \operatorname{span}(T)$. Then, insert v_1 into T, i.e. create a new set $T' = T \bigcup \{v_1\}$. Hence $T' = \{w_1, w_2, \ldots, w_m, v_1\}$. Now T' is linearly dependent. By the linear dependence lemma we can substract any element w_i from T' and the span won't change. So

$$\operatorname{span}(T' \setminus \{w_1\}) = \operatorname{span}(T') = \operatorname{span}(T) = V.$$

Now we keep removing w's and adding v's to our spanning set until all w's have been removed and our spanning set consist of only v's. But since we are assuming that $|S| > |T|$, we will run out of w's while S will still have at least one v_i left over. Then repeating the above step once more will yield that a set of linearly independent vectors are linearly dependent. $(\Rightarrow\Leftarrow)$ ■

• <u>Theorem:</u>

Let V be a VS. Then $B \subseteq V$ is a basis for V iff every vector $x \in V$ has a unique linear combination representation with vectors from B.

<u>Proof:</u>

$(\Rightarrow)$

Let $B \subseteq V$ be a basis. Then suppose $\exists\ x \in V$ such that x has two linear combination representations in B. Let

$$a_1, \ldots, a_n \in \mathbb{F} \quad \text{and} \quad b_1, \ldots, b_n \in \mathbb{F} \quad \text{with } a_i \neq b_i.$$

Then $x = a_1 u_1 + \ldots + a_n u_n$, with $u_i \in B$ and also $x = b_1 u_1 + \ldots + b_n u_n$.
But then we know that the zero vector in V can be written as

$$x + (-x) = \hat{0} = (a_1 - b_1) u_1 + \ldots + (a_n - b_n) u_n.$$

Since B is a basis (and hence is linearly independent), all the coefficients $a_i - b_i$ have to equal to 0.
Hence $a_i = b_i\ \forall\ i$. $(\Rightarrow\Leftarrow)$
Thus $x \in V$ has a unique linear combination representation. ✓

$(\Leftarrow)$

Suppose every $x \in V$ has a unique linear combination representation with vectors from B, then B is a spanning set by assumption. So we only need to show that B is linearly independent.
By assumption, $\hat{0} \in V$ has a unique linear combination representation with vectors from B. Since $\hat{0}$ always has the trivial representation, this trivial representation is unique. Thus B is linearly independent and since it is a spanning set by definition, it is also a basis. ✓ ■

• <u>Theorem:</u>

In a finite dimensional VS V,

i) Every spanning set in V can be reduced to a basis for V.
ii) Every linearly independent set in V can be extended to be a basis for V.

Proof:
i) Let $S = \{v_1, \ldots, v_n\}$ be a spanning set for V. If S is already linearly independent, we're done. So let's suppose that S is linearly dependent. Then, by the linear dependence lemma, $\exists\ v_j \in S$ such that $v_j \in \text{span}\left(S \setminus \{v_j\}\right)$, i.e. v_j can be expressed as a linear combination of vectors in S. We can omit this v_j from S and by the LDL, $\text{span}\left(S \setminus \{v_j\}\right) = \text{span}(S) = V$. If this truncated set (call it S') is linearly independent we're done. If S' is linearly dependent we repeat the process until eventually S' becomes linearly independent and therefore a basis. ✓

ii) Let $T = \{v_1, \ldots, v_n\}$ be a linearly independent set in V, and let $U = \{w_1 \ldots., w_m\}$ be a spanning set for V. Then, WLOG, pick $w_1 \in U$ and check whether $w_1 \in \text{span}(v_1, \ldots v_n)$. If w_1 can be written as a linear combination of vectors in T, i.e. $w_1 \in \text{span}\,(T)$, throw w_1 out. But if $w_1 \notin \text{span}\,(T)$, then insert $w_1$into T and now we have a new set $T' = T \bigcup \{w_1\}$, and T' is still linearly independent. Repeat the process with each $w_i \in U$ until all w's that are not in the span of T' are inserted into T'. Once span (T') includes all the w's, T' is now a spanning set that is still linearly independent (by the way we constructed it). Thus, T' is a basis for V. ✓ ■

• Theorem:
In a finite dimensional VS V, every basis for V has the same order.

Proof:
Let B and B' be two distinct bases for V. Consider B to be a linearly independent set and B' to be a spanning set. Then by a previous theorem the order of B is less than or equal to the order of B' (i.e. $|B| \leq |B'|$). Now reverse the role of B and B', namely consider B to be the spanning set and B' to be the linearly independet set. Then by the same theorem, the order of B' is less than or equal to the order of B (i.e. $|B'| \leq |B|$). Thus $|B| = |B'|$. ■

• Theorem:
Let V be a finite dimensional VS. Then
i) Every spanning set whose order is equal to dim (V) is a basis for V.
ii) Every linearly independent set whose order is equal to dim (V) is also a basis for V.

Proof:
i) Let S be a spanning set for V and let $|S| = \dim(V) = n$. Then, by a previous theorem, S can be reduced to a basis of V (since every spanning set can be reduced to a basis). But removing just one vector from S will yield that $\dim(V) < n$, which is a contradiction. ($\Rightarrow\Leftarrow$)

Thus S is already a basis for V. ✓

ii) Let T be a linearly independent set in V and $|T| = \dim(V) = n$. By part ii) of a previous theorem, T can be extended to be a basis for V. But if we add another vector to T, we would have that $|T| = \dim(V) > n$. This is a contradiction. (⇒⇐)
Hence T is already a basis for V. ✓ ■

• Theorem:
Let $W \subseteq V$ be a subspace of V, and $\dim(V) < \infty$. Then $\dim(W) \leq \dim(V)$. Moreover, $W = V$ iff $\dim(W) = \dim(V)$.

Proof:
Let $W \subseteq V$ be a subspace for V with a finite basis $B_w = \{w_1, \ldots w_m\}$. Then $B_w \subseteq V$, and B_w is still linearly independent in V (although it doesn't necessarily span V). By a previous theorem, B_w can be extended to be a basis for V. Thus a basis for V by definition must have at least m vectors (i.e. $\dim(W) = m \leq \dim(V)$). ✓
Now we only need to show that $W = V$ iff $\dim(W) = \dim(V)$.
(⇒)
Let $W = V$. Then clearly $\dim(W) = \dim(V)$. ✓
(⇐)
Suppose $\dim(W) = \dim(V) = n$. Then let B_w be a basis for W, then $|B_w| = n$. Then $B_w \subseteq V$ is linearly independent. By the previous theorem, B_w is also a basis for V.
So $W = \text{span}(B_w) = V$. ✓ ■