Math 353 HW 8

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Section 3.2

(6) Evaluate the integrals $\oint_C f(z)\, dz$, where C is the unit circle centered at the origin and $f(z)$ is given by the following:

a) $\dfrac{\sin z}{z}$

Solution:

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + .. + \frac{(-1)^j z^{2j+1}}{(2j+1)!} + ..\right)$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + .. + \frac{(-1)^j z^{2j}}{(2j+1)!} + ...$$

Hence

$$\oint_C \frac{\sin z}{z}\, dz = \oint_C \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!}\, dz = \sum_{j=0}^{\infty} \oint_C \frac{(-1)^j z^{2j}}{(2j+1)!}\, dz$$
$$= \oint_C \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + .. + \frac{(-1)^j z^{2j}}{(2j+1)!}\right) dz$$

We can see now that the entire integrand above is analytic, thus by Cauchy's theorem we know that $\oint_C \dfrac{\sin z}{z}\, dz = 0$.

b) $\dfrac{\sin z}{z^2}$

Solution:

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + .. + \frac{(-1)^j z^{2j+1}}{(2j+1)!} + ..\right)$$
$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + .. + \frac{(-1)^j z^{2j-1}}{(2j+1)!} + ..$$

Hence

$$\oint_C \frac{\sin z}{z^2}\, dz = \oint_C \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j-1}}{(2j+1)!}\, dz = \sum_{j=0}^{\infty} \oint_C \frac{(-1)^j z^{2j-1}}{(2j+1)!}\, dz$$

$$= \oint_C \left(\frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + .. + \frac{(-1)^j z^{2j-1}}{(2j+1)!} + .. \right) dz$$

This time we have one term $(\frac{1}{z})$ that is not analytic in the region. We have that $\oint_C \frac{1}{z}\, dz = 2\pi i$, since $\oint_C z^n = 2\pi i$ if $n = -1$. All the remaining terms are zero by Cauchy's theorem. Thus we conclude that $\oint_C \frac{\sin z}{z^2}\, dz = 2\pi i + 0 = 2\pi i$.

c) $\dfrac{\cosh z - 1}{z^4}$

Solution:

$$\cosh z = \frac{e^z + e^{-z}}{2} = \frac{1}{2}\left(\sum_{j=0}^{\infty} \frac{z^j}{j!} + \sum_{j=0}^{\infty} \frac{(-z)^j}{j!} \right) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{z^j(1 + (-1)^j)}{j!}$$

Then, $1 + (-1)^j = 0$ when j is odd, and $1 + (-1)^j = 2$ when j is even.
Thus,

$$\frac{e^z + e^{-z}}{2} = \frac{1}{2} \sum_{j=0}^{\infty} \frac{2 z^{2j}}{(2j)!} = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}.$$

So we have

$$\frac{\cosh z - 1}{z^4} = \frac{1}{z^4} \left(\sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} - 1 \right)$$

$$= \frac{1}{z^4} \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots + \frac{z^{2j}}{(2j)!} - 1 \right)$$

$$= \frac{1}{z^4} \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots + \frac{z^{2j}}{(2j)!} \right)$$

$$= \frac{1}{z^2\, 2!} + \frac{1}{4!} + \frac{z^2}{6!} + \ldots + \frac{z^{2j-4}}{(2j)!} + ..$$

We can see that the only non-analytic term is $\frac{1}{z^2\, 2!}$. So we have $\oint_C \frac{1}{z^2\, 2!}\, dz = 0$, since $\oint_C z^n\, dz = 0$ if $n \neq -1$, and the remaining terms are zero by Cauchy's theorem (since they are all analytic in the region). Hence $\oint_C \frac{\cosh z - 1}{z^4}\, dz = 0$.

(7) Use the Taylor series for $\frac{1}{1+z}$ about $z = 0$ to find the Taylor series expansion of $\log(1+z)$ about

$z = 0$ for $|z| < 1$.

Solution:

$f(z) = \frac{1}{1+z}$; $f(0) = 1$

$f'(z) = -\frac{1}{(1+z)^2}$; $f'(0) = -1$

$f''(z) = \frac{2}{(1+z)^3}$; $f''(0) = 2$

$f^{(3)}(z) = \frac{-6}{(1+z)^4}$; $f^{(3)}(0) = -6$

Hence $\frac{1}{1+z} = \sum_{j=0}^{\infty} \frac{(-1)^j\, j!\, z^j}{j!} = \sum_{j=0}^{\infty} (-z)^j$.

Since $\int \frac{1}{1+z}\, dz = \log(1+z)$, we have that $\int \sum_{j=0}^{\infty} (-z)^j\, dz = \sum_{j=0}^{\infty} \int (-z)^j\, dz = \sum_{j=0}^{\infty} \frac{(-z)^{j+1}}{j+1}$.

Hence the Taylor expansion of $\log(1+z)$ about $z = 0$ for $|z| < 1$ is $\sum_{j=1}^{\infty} \frac{(-1)^j z^j}{j}$.

(8) Use the Taylor series representation of $\frac{1}{1-z}$ around $z = 0$ for $|z| < 1$ to find a series representation of $\frac{1}{1-z}$ for $|z| > 1$.

Solution:

The Taylor series expansion of $\frac{1}{1-z}$ for $|z| < 1$ is given by $\sum_{j=0}^{\infty} z^j$. If we rewrite $\frac{1}{1-z}$ as $-\frac{1}{z\left(1-\frac{1}{z}\right)}$ we can expand this as $-\frac{1}{z} \sum_{j=0}^{\infty} \left(\frac{1}{z}\right)^j$, which converges for $\left|\frac{1}{z}\right| < 1 \Leftrightarrow |z| > 1$.

Hence a Taylor series representation of $\frac{1}{1-z}$ for $|z| > 1$ is given by $-\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} = -\sum_{j=1}^{\infty} z^{-j}$.

(9) Use the Taylor series representation of $\frac{1}{1-z}$ around $z = 0$ for $|z| < 1$ to deduce the series representation of $\frac{1}{(1-z)^2}, \frac{1}{(1-z)^3}, \ldots, \frac{1}{(1-z)^m}$.

Solution:

The Taylor series representation of $\frac{1}{1-z}$ around $z = 0$ for $|z| < 1$ is given by $\sum_{j=0}^{\infty} z^j$.

Notice that

- $\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} = \frac{d}{dz}\left(\sum_{j=0}^{\infty} z^j\right) = \sum_{j=1}^{\infty} j z^{j-1}$
- $\frac{d^2}{dz^2}\left(\frac{1}{1-z}\right) = \frac{2}{(1-z)^3} = \frac{d}{dz}\left(\sum_{j=1}^{\infty} j z^j\right) = \sum_{j=2}^{\infty} j(j-1)\, z^{j-2}$

So $\frac{1}{(1-z)^3} = \sum_{j=2}^{\infty} \frac{1}{2}\, j(j-1)\, z^{j-2} = \sum_{j=0}^{\infty} \frac{1}{2}\,(j+2)\,(j+1)\, z^j$

Using induction we can deduce that

$$\frac{1}{(1-z)^m} = \sum_{j=m-1}^{\infty} \frac{j(j-1)\ldots(j-(m-2))\, z^{j-(m-1)}}{(m-1)!}.$$

Section 3.3

(1) Expand the function $f(z) = \frac{1}{1+z^2}$ in:

a) a Taylor series for $|z| < 1$.

Solution:

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{j=0}^{\infty} \frac{j!\,(-z^2)^j}{j!} = \sum_{j=0}^{\infty} (-1)^j\, z^{2j}.$$

b) a Laurent series for $|z| > 1$.

Solution:

We rewrite $\frac{1}{1+z^2}$ as $\frac{1}{z^2\left(1-\left(-\frac{1}{z^2}\right)\right)}$.

Then we have

$$\frac{1}{1+z^2} = \frac{1}{z^2} \sum_{j=0}^{\infty} \left(-\frac{1}{z^2}\right)^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{z^{2j+2}}.$$

(2) Given the function $f(z) = \frac{z}{a^2-z^2}$, $a > 0$, expand $f(z)$ in a Laurent series in powers of z in the

regions:

a) $|z| < a$

Solution:

We can rewrite $\frac{z}{a^2-z^2}$ as $\frac{z}{a^2\left(1-\frac{z^2}{a^2}\right)} = \frac{z}{a^2}\left(\frac{1}{1-\frac{z^2}{a^2}}\right)$.

Then we have

$$\frac{z}{a^2-z^2} = \frac{z}{a^2}\sum_{j=0}^{\infty}\left(\frac{z^2}{a^2}\right)^j = \sum_{j=0}^{\infty}\frac{z^{2j+1}}{a^{2j+2}}.$$

b) $|z| > a$

Solution:

We can rewrite $\frac{z}{a^2-z^2}$ as $-\frac{z}{z^2\left(1-\frac{a^2}{z^2}\right)} = -\frac{1}{z}\left(\frac{1}{1-\frac{a^2}{z^2}}\right)$.

Then we have

$$\frac{z}{a^2-z^2} = -\frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{a^2}{z^2}\right)^j = -\sum_{j=0}^{\infty}\frac{a^{2j}}{z^{2j+1}}.$$

(4) Evaluate the integral $\oint_C f(z)\, dz$, where C is the unit circle centered at the origin and $f(z)$ is given as follows:

a) $\frac{e^z}{z^3}$

Solution:

$$\frac{e^z}{z^3} = \frac{1}{z^3}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots+\frac{z^j}{j!}\right) = \underbrace{\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2!\,z}}_{\text{Principal Part}}+\frac{1}{3!}+\frac{z}{4!}+\ldots+\frac{z^{j-3}}{j!}$$

We can see that only the principal part of $f(z)$ is not analytic. Hence all the remaining terms will be zero when integrated by Cauchy's theorem.

So we just need to focus on the principal part. We know that $\oint_C \frac{1}{z^3}\, dz$ and $\oint_C \frac{1}{z^2}\, dz$ are also zero because $\oint_C z^n\, dz = 0$ if $n \neq -1$. We also have $\oint_C \frac{1}{2!\,z}\, dz = \pi i$, since $\oint_C z^n\, dz = 2\pi i$ if $n = -1$. This all means that $\oint_C \frac{e^z}{z^3}\, dz = \pi i$.

b) $\frac{1}{z^2 \sin z}$

<u>Solution:</u>

$$\frac{1}{z^2 \sin z} = \frac{1}{z^2}\, \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + .. + \frac{(-1)^j z^{2j+1}}{(2j+1)!}}$$

$$= \frac{1}{z^3}\, \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + .. + \frac{(-1)^j z^{2j}}{(2j+1)!}}$$

$$= \frac{1}{z^3}\, \frac{1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - .. + \frac{(-1)^j z^{2j}}{(2j+1)!}\right)}$$

$$= \frac{1}{z^3} \sum_{j=0}^{\infty} \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \ldots + \ldots\right)^j$$

$$= \frac{1}{z^3}\left[\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - .. + ..\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - .. + ..\right)^2 .. + .. \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - .. + .\right)^j\right]$$

$$= \frac{1}{3!\, z} - \frac{z}{5!} + \ldots \text{ higher power terms.}$$

Hence we can see that the only non-analytic term is $\frac{1}{3!\, z}$ and thus we have

$$\oint_C \frac{1}{z^2 \sin(z)} = \oint_C \frac{1}{3!\, z} = \frac{1}{6}\, 2\pi i = \frac{\pi i}{3}.$$

c) $\tanh z$

<u>Solution:</u>

$$\tanh z = \frac{\sinh z}{\cosh z} = \sinh z\, \frac{1}{1 - \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - .. - \frac{(-1)^j z^{2j}}{(2j)!}\right)}$$

$$= \sinh z \sum_{j=0}^{\infty} \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - .. - ..\right)^j$$

$$= \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + .. + ..\right)\left[\left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - .. - ..\right) + ..\right.$$

$$\left... + \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - .. - ..\right)^2 + \ldots + \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - .. - ..\right)^j\right]$$

We can see that after multiplying through all the terms are analytic, hence by Cauchy's theorem we have that $\oint_C \tanh z \, dz = 0$.

e) $e^{1/z}$

Solution:

$$e^{1/z} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{1}{z}\right)^j = \sum_{j=0}^{\infty} \frac{1}{z^j \, j!} = 1 + \frac{1}{z} + \frac{1}{2! \, z^2} + \frac{1}{3! \, z^3} + \ldots + \frac{1}{z^j \, j!} .$$

We can see that the first term (1) is analytic and every term from the third term on will be zero when integrated, since $\oint z^n \, dz = 0$ if $n \neq -1$. Also $\oint_C \frac{1}{z} \, dz = 2 \pi i$, since in this case $n = -1$. Hence we have $\oint_C e^{1/z} \, dz = 2 \pi i$.

(Problem A) Given $F(z) = \frac{z}{(z-1)(z-2)}$; find the Laurent series for $F(z)$ for :

**First we expand $F(z)$ using partial fractions:

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \Longrightarrow 1 = A(z-2) + B(z-1) \Longrightarrow 1 = (A+B)z - 2A - B$$

$A + B = 0$
$-2A - B = 1$
$\Longrightarrow A = -1, \;\; B = 1$

Hence $\frac{z}{(z-1)(z-2)} = z\left(-\frac{1}{z-1} + \frac{1}{z-2}\right)$. **

a) $|z| < 1$

Solution:

Using the partial fractions decomposition from above, we have

$$F(z) = z\left(-\frac{1}{z-1} + \frac{1}{z-2}\right) = z\left(\frac{1}{1-z} - \frac{1}{2} \frac{1}{1-\frac{z}{2}}\right)$$

$$= z\left(\sum_{j=0}^{\infty} z^j - \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j\right) = z\left(\sum_{j=0}^{\infty} z^j - \sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}}\right)$$

$$= z\sum_{j=0}^{\infty}\frac{z^j\left(2^{j+1}-1\right)}{2^{j+1}} = \sum_{j=0}^{\infty}\frac{z^{j+1}\left(2^{j+1}-1\right)}{2^{j+1}}.$$

b) $1 < |z| < 2$

<u>Solution:</u>

This time we have a different Laurent series for $\frac{1}{1-z}$...

$$F(z) = z\left(-\frac{1}{z-1} + \frac{1}{z-2}\right) = z\left(-\frac{1}{z}\,\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\,\frac{1}{1-\frac{z}{2}}\right)$$

$$= z\left(-\frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{1}{z}\right)^j - \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j\right) = z\left(-\frac{1}{z}\sum_{j=0}^{\infty}\frac{1}{z^j} - \sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}}\right)$$

$$= z\sum_{j=0}^{\infty} -\frac{2^{j+1} + z^{j+1}\,z^j}{z^{j+1}\,2^{j+1}} = z\sum_{j=0}^{\infty} -\frac{2^{j+1} + z^{2\,j+1}}{z^{j+1}\,2^{j+1}}$$

$$= \sum_{j=0}^{\infty} -\frac{z\,2^{j+1} + z^{2\,j+2}}{z^{j+1}\,2^{j+1}} = \sum_{j=0}^{\infty} -\frac{2^{j+1} + z^{2\,j+1}}{z^j\,2^{j+1}}.$$

c) $|z| > 2$

<u>Solution:</u>

This time we also have a different Laurent series for $\frac{1}{z-2}$:

$$F(z) = z\left(-\frac{1}{z-1} + \frac{1}{z-2}\right) = z\left(-\frac{1}{z}\,\frac{1}{1-\frac{1}{z}} + \frac{1}{z}\,\frac{1}{1-\frac{2}{z}}\right)$$

$$= z\left(-\frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{1}{z}\right)^j + \frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{2}{z}\right)^j\right) = z\left(-\frac{1}{z}\sum_{j=0}^{\infty}\frac{1}{z^j} + \frac{1}{z}\sum_{j=0}^{\infty}\frac{2^j}{z^j}\right)$$

$$= -\sum_{j=0}^{\infty}\frac{1}{z^j} + \sum_{j=0}^{\infty}\frac{2^j}{z^j} = \sum_{j=0}^{\infty}\frac{2^j-1}{z^j}.$$