MATH 710 HW # 4

MARIO L. GUTIERREZ ABED
PROF. A. BASMAJIAN

Problem 1 (Problem 9-9)**.** *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded hypersurface (not necessarily compact). Suppose further that there is a smooth vector field V defined on a neighborhood of S and nowhere tangent to S. Show that S has a neighborhood in M diffeomorphic to $(-1,1) \times S$, under a diffeomorphism that restricts to the obvious identification $\{0\} \times S \cong S$.*

Preliminaries of proof. This result relies heavily on the *Flowout Theorem*, which we state next for study purposes. The grader may skip to the actual proof of the problem below.

Theorem (**Flowout Theorem**)**.** *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded k-dimensional submanifold, and $V \in \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Let $\theta\colon \mathfrak{D} \to M$ be the flow of V, let $\mathcal{O} = (\mathbb{R} \times S) \cap \mathfrak{D}$, and let $\varphi = \theta|_{\mathcal{O}}$.*

i) *$\varphi\colon \mathcal{O} \to M$ is an immersion.*

ii) *$\partial/\partial t \in \mathfrak{X}(\mathcal{O})$ is φ-related to V.*

iii) *There exists a smooth positive function $\delta\colon S \to \mathbb{R}$ such that the restriction of φ to $\mathcal{O}_\delta$ is injective, where $\mathcal{O}_\delta \subseteq \mathcal{O}$ is the flow domain*

$$\mathcal{O}_\delta = \{(t,p) \in \mathcal{O} : |t| < \delta(p)\}.$$

Thus, $\varphi(\mathcal{O}_\delta)$ is an immersed submanifold of M containing S, and V is tangent to this submanifold.

iv) *If S has codimension 1, then $\varphi|_{\mathcal{O}_\delta}$ is a diffeomorphism onto an open submanifold of M.*

Remark: The submanifold $\varphi(\mathcal{O}_\delta) \subseteq M$ is called a ***flowout from** S **along** V* (see Figure 1). □

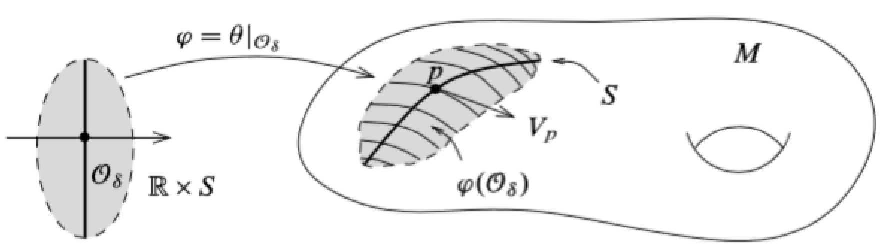

FIGURE 1. A flowout.

Proof of Problem 9-9. From the *Flowout Theorem* (part iv)), we have that $\varphi|_{\mathcal{O}_\delta}$ is a diffeomorphism onto an open submanifold of M (since S is assumed to have codimension 1). So it remains to show that the neighborhood $\mathcal{O}_\delta = \{(t,p) \in \mathcal{O} : |t| < \delta(p)\}$ of S is diffeomorphic to $(-1,1) \times S$. To this end, let $\psi\colon (-1,1) \times S \to \mathcal{O}_\delta$ be given by $(t,p) \mapsto (t/\delta(p), p)$ (note that $\delta(p)$ is a smooth positive function by definition; in particular, it is nonzero and $t/\delta(p)$ is well defined). Then ψ is clearly smooth, and so is its inverse $\psi^{-1}(t,p) = (t\delta(p), p)$. $\square$