MATH 725 HW#2

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Exercise (Exercise 1)**.** *Let $\mathcal{F} = \{V_i \mid i \in \Lambda\}$ be a family of vector spaces over some field of scalars $\mathbb{F}$. Then show that the direct product of $\mathcal{F}$*

$$\prod_{i\in\Lambda} V_i = \left\{ f\colon \Lambda \longrightarrow \bigcup_{i\in\Lambda} V_i \mid f(i) \in V_i \right\}$$

is a vector space.

Proof. ***(Edit: I have to fix this proof because I assumed countability!)*** To simplify notation, let $K = \prod_{i\in\Lambda} V_i$. We define the operations of addition and multiplication on K pointwise:

$$\begin{aligned}(u_1, \dots, u_k, \dots) + (v_1, \dots, v_k, \dots) &= (u_1 + v_1, \dots, u_k + v_k, \dots) \\ \alpha(v_1, \dots, v_k, \dots) &= (\alpha v_1, \dots, \alpha v_k, \dots),\end{aligned}$$

where $u_i, v_i = f_1(i), f_2(i) \in V_i$ and $\alpha \in \mathbb{F}$.

Now we are going to show that K satisfies all the axioms of a vector space:

- It is clear by the way our operations of addition and scalar multiplication are defined that K is closed under addition and scalar multiplication. Moreover, by letting $\alpha = 1$, we have that $1(v_1, \dots, v_k, \dots) = (1v_1, \dots, 1v_k, \dots) = (v_1, \dots, v_k, \dots) \in K$. (Multiplicative identity element)

- Since each V_i is a vector space, we know that $\theta \in V_i \;\; \forall i$ (where θ represents the zero vector). Hence

  $$K_\theta = \left\{ f_\theta\colon \Lambda \longrightarrow \bigcup_{i\in\Lambda} V_i \mid f_\theta(i) = \theta \in V_i \right\} = (\theta, \dots, \theta, \dots) \in K.$$

  Thus K contains the zero element K_θ. (Additive identity element)

- Similarly, since each V_i is a vector space, we know that for $f(i) \in V_i$ there exists an inverse element $-f(i) \in V_i$. Hence we have

  $$\begin{aligned}K_{\text{inv}} &= \left\{ f_{\text{inv}}\colon \Lambda \longrightarrow \bigcup_{i\in\Lambda} V_i \mid f_{\text{inv}}(i) = -f(i) \in V_i \right\} \\ &= (-f_1(i), \dots, -f_k(i), \dots) \\ &= (-1 \cdot f_1(i), \dots, -1 \cdot f_k(i), \dots) \\ &= -1(f_1(i), \dots, f_k(i), \dots) \in K.\end{aligned}$$

  Thus K contains the inverse element K_{inv}. (Inverse element)

- Take two elements $(u_1, \dots, u_k, \dots), (v_1, \dots, v_k, \dots) \in K$ and note that

$$\begin{aligned} &(u_1, \dots, u_k, \dots) + (v_1, \dots, v_k, \dots) \\ &= (u_1 + v_1, \dots, u_k + v_k, \dots) \\ &= (v_1 + u_1, \dots, v_k + u_k, \dots) && \text{(By commutativity on the } V_i\text{'s)} \\ &= (v_1, \dots, v_k, \dots) + (u_1, \dots, u_k, \dots). \end{aligned}$$

Thus we have that for any $a, b \in K$, $a + b = b + a$. (<u>Commutativity of addition</u>)

- Now we take three elements $(u_1, \dots, u_k, \dots), (v_1, \dots, v_k, \dots), (w_1, \dots, w_k, \dots) \in K$ and note that

$$\begin{aligned} &(u_1, \dots, u_k, \dots) + [(v_1, \dots, v_k, \dots) + (w_1, \dots, w_k, \dots)] \\ &= (u_1, \dots, u_k, \dots) + (v_1 + w_1, \dots, v_k + w_k, \dots) \\ &= (u_1 + v_1 + w_1, \dots, u_k + v_k + w_k, \dots) \\ &= (u_1 + v_1, \dots, u_k + v_k, \dots) + (w_1, \dots, w_k, \dots) \\ &= [(u_1, \dots, u_k, \dots) + (v_1, \dots, v_k, \dots)] + (w_1, \dots, w_k, \dots). \end{aligned}$$

Thus we have that for any $a, b, c \in K$, $a+(b+c) = (a+b)+c$. (<u>Associativity of addition</u>)

- It's time to test the multiplication axioms. Take an element $(v_1, \dots, v_k, \dots) \in K$ and two scalars $\alpha, \beta \in \mathbb{F}$ and note that

$$\begin{aligned} &(\alpha\beta)(v_1, \dots, v_k, \dots) \\ &= ((\alpha\beta)v_1, \dots, (\alpha\beta)v_k, \dots) \\ &= (\alpha(\beta v_1), \dots, \alpha(\beta v_k), \dots) && \text{(By associativity on the } V_i\text{'s)} \\ &= \alpha(\beta v_1, \dots, \beta v_k, \dots). \end{aligned}$$

Thus we have that for any $v \in K$ and $\alpha, \beta \in \mathbb{F}$, $(\alpha\beta)v = \alpha(\beta v)$. (<u>Associativity of multiplication</u>)

- Take two elements $(u_1, \dots, u_k, \dots), (v_1, \dots, v_k, \dots) \in K$ and a scalar $\alpha \in \mathbb{F}$ and note that

$$\begin{aligned} &\alpha[(u_1, \dots, u_k, \dots) + (v_1, \dots, v_k, \dots)] \\ &= \alpha(u_1 + v_1, \dots, u_k + v_k, \dots) \\ &= (\alpha(u_1 + v_1), \dots, \alpha(u_k + v_k), \dots) \\ &= (\alpha u_1 + \alpha v_1, \dots, \alpha u_k + \alpha v_k, \dots) && \text{(By distributivity on the } V_i\text{'s)} \\ &= (\alpha u_1, \dots, \alpha u_k, \dots) + (\alpha v_1, \dots, \alpha v_k, \dots) \\ &= \alpha(u_1, \dots, u_k, \dots) + \alpha(v_1, \dots, v_k, \dots). \end{aligned}$$

Thus we have shown that for any $v, w \in K$ and $\alpha \in \mathbb{F}$, $\alpha(v + w) = \alpha v + \alpha w$. (<u>Distributivity</u>)

- Finally (!!) let us take an element $(v_1, \dots, v_k, \dots) \in K$ and a pair of scalars $\alpha, \beta \in \mathbb{F}$ and note that

$$\begin{aligned} &(\alpha+\beta)(v_1, \dots, v_k, \dots) \\ &= ((\alpha+\beta)v_1, \dots, (\alpha+\beta)v_k, \dots) \\ &= (\alpha v_1 + \beta v_1, \dots, \alpha v_k + \beta v_k, \dots) && \text{(By distributivity on the } V_i\text{'s)} \\ &= (\alpha v_1, \dots, \alpha v_k, \dots) + (\beta v_1, \dots, \beta v_k, \dots) \\ &= \alpha(v_1, \dots, v_k, \dots) + \beta(v_1, \dots, v_k, \dots). \end{aligned}$$

Thus we have shown that for any $v \in K$ and $\alpha, \beta \in \mathbb{F}$, $(\alpha+\beta)v = \alpha v + \beta v$. (Distributivity)

After testing all these axioms, we may conclude that $K = \prod_{i \in \Lambda} V_i$ is indeed a vector space, as we set out to prove. □

Exercise (Exercise 2)**.** *Let V be a vector space, where V is the direct sum of a family $\mathcal{F} = \{S_i \mid i \in \Lambda\}$ of subspaces of V. Prove that $V \cong \oplus_{i \in \Lambda} S_i$.*

Proof. If $\{f_i\} \in \oplus_{i \in \Lambda} S_i$, then $f_i = 0$ for all but a finite number of $i \in \Lambda$. Let Λ_0 be the support of f_i, i.e. $\Lambda_0 = \{i \in \Lambda \mid f_i \neq 0\}$. Then $\oplus_{i \in \Lambda_0} f_i \subset \oplus_{i \in \Lambda} f_i$ is a well defined element of V. Consequently, we are going to define the map $\varphi\colon \oplus_{i \in \Lambda} S_i \to V$ by

$$\varphi(\{f_i\}) = \bigoplus_{i \in \Lambda_0} f_i \in V \qquad (\text{and } \{0\} \in \oplus_{i \in \Lambda} S_i \text{ maps to } 0 \in V).$$

This map φ is a homomorphism such that $\varphi\iota_i(f_i) = f_i$, for $f_i \in S_i$ and ι_i being the canonical i^{th} injection.

Now, since V is by hypothesis the direct sum of the subspaces S_i, we have that every element $v \in V$ is a finite sum of elements from various S_i, i.e. $v = f_1 + \cdots + f_k$, with $f_i \in S_i$. Thus, $\oplus_{i \in \Lambda_0} \iota_i(f_i) \in \oplus_{i \in \Lambda} S_i$ and

$$\varphi\left(\bigoplus_{i \in \Lambda_0} \iota_i(f_i)\right) = \bigoplus_{i \in \Lambda_0} \varphi\iota_i(f_i) = \bigoplus_{i \in \Lambda_0} f_i = v.$$

Hence we have that φ is a surjective linear map.

Now suppose that $\varphi(\{f_i\}) = \oplus_{i \in \Lambda_0} f_i = 0 \in V$. We may assume for convenience of notation that $\Lambda_0 = \{1, \dots, k\}$. Then $\oplus_{i \in \Lambda_0} f_i = f_1 + \cdots + f_k = 0$, with $f_i \in S_i$.
Hence,

$$-f_1 = f_2 + \cdots + f_k \in S_1 \bigcap \left(\bigcup_{i \neq 1} S_i\right) = \{0\}$$

and therefore $f_1 = 0$. Now by repeating this argument we get that $f_i = 0\ \forall i \in \Lambda$, so that φ is an injective linear map.

Hence we have shown that φ is a bijective linear map, which proves that V and $\oplus_{i \in \Lambda} S_i$ are isomorphic, as desired. □