MATH 751 NOTES
TOPOLOGICAL SPACES

MARIO L. GUTIERREZ ABED

Topologies

Definition. *If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that*

- *For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ containing x.*
- *If x belongs to the intersection of two basis elements $B_1, B_2 \in \mathcal{B}$, then there is another basis element $B_3 \in \mathcal{B}$ containing x such that $B_3 \subset B_1 \cap B_2$.*

*If $\mathcal{B}$ satisfies these two conditions, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$**[1] as follows: A subset $U \subset X$ is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.* ★

We now define three topologies on the real line $\mathbb{R}$, all of which are of interest to us:

Definition. *If $\mathcal{B}$ is the collection of all open intervals in the real line,*

$$(a, b) = \{x \mid a < x < b\},$$

*the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line.* ★

Remark: Whenever we consider $\mathbb{R}$, we shall suppose it is given with this topology, unless stated otherwise.

Definition. *If $\mathcal{B}'$ is the collection of all half-open intervals in the real line,*

$$[a, b) = \{x \mid a \leq x < b\},$$

*where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on the real line.* ★

Remark: When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_\ell$.

Example: Let $\mathbb{R}$ denote the set of real numbers in its usual topology, and let $\mathbb{R}_\ell$ denote the same set in the lower limit topology. Let

$$f : \mathbb{R} \to \mathbb{R}_\ell$$

be the identity function $f(x) = x$ for every real number x. Then f is not a continuous function; the inverse image of the open set $[a, b)$ of $\mathbb{R}_\ell$ equals itself, which is not open in $\mathbb{R}$.

[1] To see that this collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is indeed a topology, see pg 79, Munkres.

On the other hand, the identity function

$$g\colon \mathbb{R}_\ell \to \mathbb{R}$$

is indeed continuous, because the inverse image of (a, b) is itself, which is open in $\mathbb{R}_\ell$.

Definition. *Let $K = \{1/n \mid n \in \mathbb{N}\}$. Generate a topology on $\mathbb{R}$ by taking as basis $\mathcal{B}''$ the collection of all open intervals (a, b) and all the sets of the form $(a, b) \smallsetminus K$. The topology generated by $\mathcal{B}''$ is called the **K-topology** on the real line.* ★

Remark 1: When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$.

Remark 2: Relative to the set of all real numbers carrying the *standard topology*, the set $K = \{1/n \mid n \in \mathbb{N}\}$ is not closed since it doesn't contain its (only) limit point 0. Relative to the K-topology however, the set K is automatically decreed to be closed by adding 'more' basis elements to the standard topology on $\mathbb{R}$. Basically, the K-topology on $\mathbb{R}$ is strictly finer than the standard topology on $\mathbb{R}$, as we show in the lemma below.

The relation between these three topologies is the following:

Lemma 1. *The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but not comparable with one another.*

Proof. See page 82, *Munkres*'s. □

Definition. *A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.*[2] ★

Remark: The basic idea is that a basis is the collection of all finite intersections of subbasis elements (i.e. a subbasis generates a basis by taking finite intersections of its elements). Since the open sets in a topology are all possible unions of basis elements, we have that the open sets in a topology are all possible unions of finite intersections of subbasis elements.

Definition. *Let X be a set with a total order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:*

- *All open intervals (a, b) in X.*
- *All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.*
- *All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.*

*The collection $\mathcal{B}$ is a basis for a topology on X, which is called the **order topology** on X.* ★

[2] To see that this collection $\mathcal{T}$ is indeed a topology, see pg 82, Munkres.

Remark: The standard topology on $\mathbb{R}$, as previously defined, is just the order topology derived from the usual order on $\mathbb{R}$.

Example 1: The positive integers $\mathbb{Z}^+$ form an ordered set with a smallest element (i.e. 1). The order topology on $\mathbb{Z}^+$ is the discrete topology, for every one-point set is open:
If $n > 1$, then the one-point set $\{n\} = (n-1, n+1)$ is a basis element; and if $n = 1$, the one-point set $\{1\} = [1, 2)$ is a basis element.

Example 2: The set $X = \{1, 2\} \times \mathbb{Z}^+$ in the dictionary order is another example of an ordered set with a smallest element. Let $n \in \mathbb{Z}^+$. Then denoting $1 \times n$ by a_n and $2 \times n$ by b_n, we can represent X by

$$a_1, a_2, \ldots; b_1, b_2, \ldots$$

Note that the order topology on X is not the the discrete topology. The reason is that, even though most one-point sets on X are open, there is an exception—the one-point set $\{b_1\}$. Any open set containing b_1 must contain a basis element about b_1 (by definition), and any basis element containing b_1 contains points of the a_i sequence.

Definition. *Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U and V are open subsets of X and Y, respectively.* ★

Theorem 1. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Note that it is sometimes useful to express the product topology in terms of a subbasis. To do this, we first define certain functions called *projections*:

Definition. *Let $\pi_1 \colon X \times Y \longrightarrow X$ be defined by the equation*

$$\pi_1(x, y) = x$$

and let $\pi_2 \colon X \times Y \longrightarrow Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

*The maps π_1 and π_2 are called the **projections** of $X \times Y$ onto its first and second components, respectively.* ★

Remark: If U is an open subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then the set $\pi_2^{-1}(V)$ is precisely the set $X \times V$, which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$, as indicated in Figure 1 below:

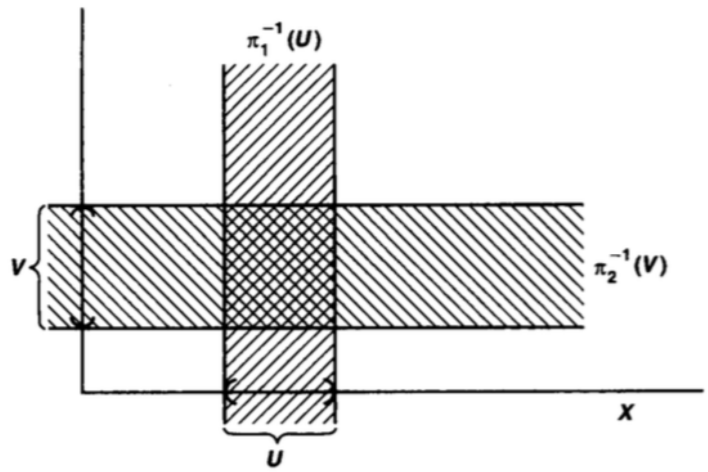

FIGURE 1. $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V$

This fact leads us to the following theorem:

Theorem 2. *The collection*

$$\mathcal{S} = \left\{\pi_1^{-1}(U) \mid U \text{ is open in } X\right\} \bigcup \left\{\pi_2^{-1}(V) \mid V \text{ is open in } Y\right\}$$

is a subbasis for the product topology on $X \times Y$*.*

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Because every element of $\mathcal{S}$ belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of $\mathcal{S}$. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since

$$U \times V = \pi_1^{-1}(U) \bigcap \pi_2^{-1}(V).$$

Therefore, $U \times V$ belongs to $\mathcal{T}'$, so that $\mathcal{T} \subset \mathcal{T}'$ as well. $\square$

Definition. *Let* $X = \prod_{i \in \mathcal{I}} X_i$*, where the* X_i*'s are topological spaces. Then the* ***box topology*** *on* X *is the topology generated by the basis*

$$\mathcal{B} = \left\{\prod_{i \in \mathcal{I}} U_i \mid U_i \text{ open in } X_i\right\}. \qquad \bigstar$$

Remark: While the *box topology* has a somewhat more intuitive definition than the *product topology*, it satisfies fewer desirable properties. In particular, if all the component spaces are compact, the box topology on their Cartesian product will not necessarily be compact, although the product topology on their Cartesian product will always be compact. In general, the box topology is finer than the product topology, although the two agree in the case when our index $\mathcal{I}$ is finite.

Example: Take $\mathbb{R}^\omega$, the countable Cartesian product of the real line, and consider the function $f\colon \mathbb{R} \to \mathbb{R}^\omega$ given by

$$f(x) = (x, x, , x, \dots);$$

the n^{th} coordinate function of f is the function $f_n(x) = x$. Each of the coordinate functions $f_n\colon \mathbb{R} \to \mathbb{R}$ is continuous in the standard topology on $\mathbb{R}$, and thus f itself is continuous if $\mathbb{R}^\omega$ is given the product topology, but f is not continuous in the box topology.

Why? Consider the set

$$U = \prod_{n=1}^{\infty} (-1/n, 1/n).$$

This set U is open (it is a basis element) in the box topology, but not in the product topology. We assert that $f^{-1}(U)$ is not open in $\mathbb{R}$. If $f^{-1}(U)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. But this would mean that $f((-\delta, \delta)) \subset U$, so that, by applying the projection map on the n^{th} coordinate to both sides of this inclusion, we would get

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset (-1/n, 1/n) \qquad \text{for all } n.$$

This is a contradiction because the components of U get arbitrarily close to 0—any δ-neighborhood will eventually be outside some component of U.

The product topology is the more “natural” topology in the sense that pointwise statements about its component spaces are transported to the Cartesian product and vice versa. A sequence in the product topology converges iff each of the component sequences converges in the component topology. A function is continuous in the product topology iff each of its component functions is continuous in the component topology. We formalize this by talking about projection maps, commutative diagrams, and natural transformations.

The following properties of the box topology are very important. Let $\mathbb{R}^\omega$ denote the countable cartesian product of $\mathbb{R}$ with itself. Then the box topology on $\mathbb{R}^\omega$ is:

- completely regular.
- neither compact nor connected.
- not first countable (hence not metrizable).
- not separable.
- paracompact (and hence normal and completely regular) if the continuum hypothesis is true.

Example: Let us show that $\mathbb{R}^\omega$ in the box topology is not metrizable. In order to do this we will show that the sequence lemma[3] does not hold for $\mathbb{R}^\omega$. Let A be the subset of $\mathbb{R}^\omega$ consisting of those points all of whose coordinates are positive:

$$A = \{(x_1, x_2, \dots) \mid x_i > 0 \ \ \forall\, i \in \mathbb{N}\}.$$

[3] Here's the sequence lemma for reference:
(The Sequence Lemma) Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is metrizable.

Let $\mathbf{0}$ be the "origin" in $\mathbb{R}^\omega$, that is, the point $(0, 0, \dots)$ each of whose coordinates is zero. In the box topology, $\mathbf{0}$ belongs to $\bar{A}$; for if

$$B = (a_1, b_1) \times (a_2, b_2) \times \dots$$

is any basis element containing $\mathbf{0}$, then B intersects A. For instance, the point

$$\left(\frac{1}{2}b_1, \frac{1}{2}b_2, \dots\right)$$

belongs to $B \cap A$.

But we assert that there is no sequence of points of A converging to $\mathbf{0}$. For let (a_n) be a sequence of points of A, where

$$a_n = (x_{1n}, x_{2n}, \dots, x_{in}, \dots).$$

Every coordinate x_{in} is positive, so we can construct a basis element B' for the box topology on $\mathbb{R}$ by setting

$$B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots .$$

Then B' contains the origin $\mathbf{0}$, but it contains no member of the sequence (a_n); the point a_n cannot belong to B' because the n^{th} coordinate x_{nn} does not belong to the interval $(-x_{nn}, x_{nn})$. Hence the sequence (a_n) cannot converge to $\mathbf{0}$ in the box topology.

We now give a more general definition for the *product topology*:

Definition. *Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let*

$$\pi_\beta \colon \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be the function (called the ***projection mapping*** *associated with the index β) assigning to each element of the product space $\prod_{\alpha \in J} X_\alpha$ its β^{th} coordinate,*

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta.$$

Now let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}.$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta.$$

Then the topology generated by the subbasis $\mathcal{S}$ is called the ***product topology****. In this topology, $\prod_{\alpha \in J} X_\alpha$ is called the* ***product space****.* $\bigstar$

In summary, we have the following theorem:

Theorem 3 (Comparison of the box and product topologies). *The* box topology *on $\prod X_\alpha$ has as basis all the sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The* product topology *on $\prod X_\alpha$ also has as basis all the sets of the form $\prod U_\alpha$, where U_α is open in X_α, <u>and</u> U_α equals X_α except for finitely many values of α.*

Definition. *Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, then the collection*

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the ***subspace topology****. Equipped with this topology, Y is called a* ***subspace*** *of X.* ★

Lemma 2. *If $\mathcal{B}$ is a basis for the topology of X, then the collection*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Now let us explore the relation between the subspace, order, and product topologies. For product topologies, the result is what one might expect; for order topologies it is not:

Theorem 4. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. Therefore $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are general open sets for the subspace topologies on A and B, respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.
The conclusion we draw is that the bases for the subspace topology on $A \times B$ and the product topology on $A \times B$ are the same. Hence the topologies are the same. □

Now let X be an ordered set in the order topology, and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. *However, the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X.*

► We now show an example where the subspace and order topologies on Y agree:

Example: Consider the subset $Y = [0, 1] \subset \mathbb{R}$, in the subspace topology. The subspace topology has as basis all sets of the form $(a, b) \cap Y$, where (a, b) is an open interval in $\mathbb{R}$. Such a set is one of the following types:

$$(a,b) \cap Y = \begin{cases} (a,b) & \text{if } a \text{ and } b \text{ are in } Y, \\ [0,b) & \text{if only } b \text{ is in } Y, \\ (a,1] & \text{if only } a \text{ is in } Y, \\ Y \text{ or } \emptyset & \text{if neither } a \text{ nor } b \text{ is in } Y. \end{cases}$$

By definition, each of these sets is open in Y, but the ones of type $[0, b)$ and $(a, 1]$ are not open in the ambient space $\mathbb{R}$.
Note that these sets form a basis for the order topology on Y. Thus, we see that in the case of the set $Y = [0, 1]$, its subspace topology (as a subspace of $\mathbb{R}$) and its order topology are the same.

► We now show an example where the subspace and order topologies on Y don't agree:

Example: Let Y be the subset $[0,1) \cup \{2\}$ of $\mathbb{R}$. In the subspace topology on Y, the one-point set $\{2\}$ is open, because it is the intersection of the open set $(3/2, 5/2)$ with Y. But in the order topology on Y, the set $\{2\}$ is not open. Any basis element for the order topology on Y that contains 2 is of the form

$$\{x \mid x \in Y \text{ and } a < x \leq 2\}$$

for some $a \in Y$. Such a set necessarily contains points of Y less than 2. Hence we see that the subspace and order topologies on Y do not agree.

We have however, that if Y is convex in X, then the subspace and order topologies do always agree:

Theorem 5. *Let X be an ordered set in the order topology, and let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. See page 91, *Munkres*'s. □

Theorem 6. *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Example: Consider the subspace $Y = (0,1]$ of the real line $\mathbb{R}$. The set $A = (0, 1/2)$ is a subset of Y; its closure in $\mathbb{R}$ is the set $[0, 1/2]$, while its closure in Y is the set $[0, 1/2] \cap Y = (0, 1/2]$.

Hausdorff Spaces

Definition. *A topological space X is called a* ***Hausdorff space*** *if for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.* ★

Remark: Note that the Hausdorff axiom is satisfied for every metric topology. If x and y are distinct points of a metric space (X, d), we let $\varepsilon = \frac{1}{2}d(x, y)$; then the triangle inequality implies that $B_d(x, \varepsilon)$ and $B_d(y, \varepsilon)$ are disjoint.

Theorem 7. *Every finite point set in a Hausdorff space X is closed.*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V, respectively. Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$. As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that it is closed. □

Remark: Note, however, that the condition that a finite point set is closed does not guarantee the Hausdorff condition. For example, $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. This condition that finite point sets be closed has a name of its own: it is called the T_1 **axiom**.

Here's the formal definition:

Definition (T_1 **Axiom**). *For any two points $x, y \in X$ there exists two open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$. A space satisfying this axiom is known as a T_1* ***space****.* ★

Theorem 8. *Let X be a space satisfying the T_1 axiom and let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. See page 99, *Munkres*'s. □

Theorem 9. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Suppose that x_n is a sequence of points of X that converges to x. If $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore, x_n cannot converge to y. □

Theorem 10. *Every totally ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

Lemma 3 (**Closed Map Lemma**). *Suppose F is a continuous map from a compact space to a Hausdorff space. Then,*

- *a) F is a closed map.*
- *b) If F is surjective, it is a quotient map.*
- *c) If F is injective, it is a topological embedding.*
- *d) If F is bijective, it is a homeomorphism.*

Theorem 11 (**The Pasting Lemma**). *Let $X = A \cup B$, where A and B are closed in X. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous, If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give another continuous map $h\colon X \to Y$, defined by setting*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

Proof. Let C be a closed subset of Y. Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C),$$

by elementary set theory. Since f is continuous, $f^{-1}(C)$ is closed in A and, therefore, closed in X. Similarly, $g^{-1}(C)$ is closed in B and therefore closed in X. Their union $h^{-1}(C)$ is thus closed in X. $\square$