• Lemma 1.2:

If $R$, $R_1$, ..., $R_N$ are rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$, then $|R| \leq \sum_{k=1}^{N} |R_k|$.

• Theorem 1.3:

Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

• Theorem 1.4:

Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, with $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.

## Properties of Exterior Measure

▸ Observation 1 (Monotonicity):

If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

▸ Observation 2 (Countable sub-additivity):

If $E = \bigcup_{n=1}^{\infty} E_n$, then $m_*(E) \leq \sum_{n=1}^{\infty} m_*(E_n)$.

▸ Observation 3:

If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ containing $E$.

▸ Observation 4:

If $E = E_1 \bigcup E_2$ and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.

▸ Observation 5:

If a set $E$ is the countable union of almost disjoint cubes $E = \bigcup_{n=1}^{\infty} Q_n$, then $m_*(E) = \sum_{n=1}^{\infty} |Q_n|$.

# Properties of Measurable Sets

▸ Property 1:

Every open set in $\mathbb{R}^d$ is measurable.

▸ Property 2:

If $m_*(E) = 0$, then $E$ is measurable. In particular, if $F$ is a subset of a set of exterior measure 0, then $F$ is measurable.

▸ Property 3:

A countable union of measurable sets is measurable.

▸ Property 4:

Closed sets are measurable.

• Lemma 3.1: If $F$ is closed, $K$ is compact, and these sets are disjoint, then $d(F, K) > 0$.

Proof:

Since $F$ is closed, for each point $x \in K$, there exists $\delta_x > 0$ so that $d(x, F) > 3\,\delta_x$. Since $\bigcup_{x \in K} B_{2\,\delta_x}(x)$ covers $K$, and $K$ is compact, we may find a subcover, which we denote by $\bigcup_{j=1}^{N} B_{2\,\delta_x}(x_j)$. If we let $\delta = \min(\delta_1, \ldots, \delta_N)$, then we must have $d(F, K) \geq \delta > 0$. Indeed, if $x \in K$ and $y \in F$, then for some $j$ we have $|x_j - x| \leq 2\,\delta_j$, and by construction $|y - x_j| \geq 3\,\delta_j$. Therefore

$$|y - x| \geq |y - x_j| - |x_j - x| \geq 3\,\delta_j - 2\,\delta_j \geq \delta,$$

and the lemma is proved. ■

▸ Property 5:

The complement of a measurable set is measurable.

▸ Property 6:

A countable intersection of measurable sets is measurable.

• Theorem 3.2:

If $E_1, E_2, \ldots$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then $m(E) = \sum_{j=1}^{\infty} m(E_j)$.

• Corollary 3.3:
Suppose $E_1, E_2, \ldots$ are measurable subsets of $\mathbb{R}^d$.
(i) If $E_k \nearrow E$, then $m(E) = \lim_{N \to \infty} m(E_N)$.
(ii) If $E_k \searrow E$ and $m(E_k) < \infty$ for some $k$, then $m(E) = \lim_{N \to \infty} m(E_N)$.

• Theorem 3.4:
Suppose $E$ is a measurable subset of $\mathbb{R}^d$. Then, for every $\varepsilon > 0$:
(i) There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} \setminus E) \leq \varepsilon$.
(ii) There exists a closed set $F$ with $F \subset E$ and $m(E \setminus F) \leq \varepsilon$.
(iii) If $m(E)$ is finite, there exists a compact set $K$ with $K \subset E$ and $m(E \setminus K) \leq \varepsilon$.
(iv) If $m(E)$ is finite, there exists a finite union $F = \bigcup_{n=1}^{N} Q_n$ of closed cubes such that $m(E \triangle F) \leq \varepsilon$.

• Corollary 3.5:
A subset $E$ of $\mathbb{R}^d$ is measurable
(i) iff $E$ differs from a $G_\delta$ by a set of measure zero,
(ii) iff $E$ differs from an $F_\sigma$ by a set of measure zero.

## Construction of a non-measurable set

The construction of a non-measurable (Vitali) set $\mathcal{N}$ uses the axiom of choice, and rests on a simple equivalence relation among real numbers in $[0, 1]$:

$$\text{We write } x \sim y \text{ whenever } x - y \in \mathbb{Q}\,.$$

Note that this is an equivalence relation since it satisfies the reflexive, symmetric, and transitive properties.

Since equivalence classes partition a set into distinct cells, we know that two equivalence classes either are disjoint or coincide; thus the interval $[0, 1]$ is the disjoint union of all equivalence classes that live in this interval, that is

$$[0,\ 1] = \bigcup_{\alpha} \mathcal{E}_\alpha ,$$

where each $\mathcal{E}_\alpha$ represents a unique equivalence class.

Now we construct the (Vitali) set $\mathcal{N}$ by choosing exactly one element $x_\alpha$ from each $\mathcal{E}_\alpha$ (this is justified by using the axiom of choice), and setting $\mathcal{N} = \{x_\alpha\}$.

Here's the important result:

• Theorem 3.6:
The Vitali set $\mathcal{N}$ constructed above is not measurable.

Proof:
Assume that $\mathcal{N}$ is measurable. Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of all the rationals in $[-1,\ 1]$, and consider the translates

$$\mathcal{N}_k = \mathcal{N} + r_k.$$

Note that the sets $\mathcal{N}_k$ are disjoint. To see why this is true, suppose that the intersection $\mathcal{N}_k \bigcap \mathcal{N}_{k'}$ is nonempty. Then there exist rationals $r_k \neq r_{k'}$ and $\alpha$ and $\beta$ with

$$x_\alpha + r_k = x_\beta + r_{k'}$$

which implies that

$$x_\alpha - x_\beta = r_{k'} - r_k .$$

But this means that $\alpha \neq \beta$ and $x_\alpha - x_\beta$ is rational, which in turn implies that $x_\alpha \sim x_\beta$. This contradicts the fact that $\mathcal{N}$ contains only one representative of each equivalence class.

We also claim that

$$[0,\ 1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \subset [-1,\ 2]. \qquad \text{(I)}$$

To see why, notice that if $x \in [0,\ 1]$, then $x \sim x_\alpha$ for some $\alpha$, and therefore $x - x_\alpha = r_k$ for some

$k$. Hence $x \in \mathcal{N}_k$ for some $k$ and the first inclusion holds. The second inclusion above is straightforward since each $\mathcal{N}_k$ is contained in $[-1, 2]$ by construction.

Now we may conclude the proof of the theorem. If $\mathcal{N}$ were measurable, then so would be $\mathcal{N}_k$ for all $k$, and since the union $\bigcup_{k=1}^{\infty} \mathcal{N}_k$ is disjoint, the inclusions in (I) yield

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq 3.$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we must have $m(\mathcal{N}_k) = m(\mathcal{N})$ for all $k$.
Consequently,

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3.$$

This is the desired contradiction, since neither $m(\mathcal{N}) = 0$ nor $m(\mathcal{N}) > 0$ is possible. $(\Rightarrow\Leftarrow)$

In other words, $m(\mathcal{N}) = 0$ is not possible by the above inequality, and $m(\mathcal{N}) > 0$ is not possible either because we are trying to find the measure of a countable set, which would have measure zero if any. ■

## Properties of Measurable Functions

▸ Property 1:

The finite-valued function $f$ is measurable iff $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$, and iff $f^{-1}(F)$ is measurable for every closed set $F$.

**Remark:** Note that this property also applies to extended-valued functions, if we make the additional hypothesis that both $f^{-1}(-\infty)$ and $f^{-1}(\infty)$ are measurable sets.

▸ Property 2:

If $f$ is continuous on $\mathbb{R}^d$, then $f$ is measurable. If $f$ is measurable and finite-valued, and $\Phi$ is continuous, then $\Phi \circ f$ is measurable.

**Remark:** In fact, $\Phi$ is continuous, so $\Phi^{-1}(-\infty, a)$ is an open set $\mathcal{O}$, and hence $(\Phi \circ f)^{-1}((-\infty, a)) = f^{-1}(\mathcal{O})$ is measurable. It should be noted, however, that in general it is not true that $f \circ \Phi$ is measurable whenever $f$ is measurable and $\Phi$ is continuous (see Exercise

35,Chpt 2, Stein).

‣ Property 3:

Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions.
Then

$$\sup_n f_n(x)\ , \qquad \inf_n f_n(x)\ , \qquad \limsup_{n\to\infty} f_n(x)\ , \quad \text{and} \quad \liminf_{n\to\infty} f_n(x)$$

are measurable.

Remark: Proving that $\sup_n f_n$ is measurable requires noting that $\{\sup_n f_n > a\} = \bigcup_n \{f_n > a\}$.
This also yields the result for $\inf_n f_n$, since this quantity equals $-\sup_n(-f_n(x))$.
The result for the limsup and liminf also follows from the two observations

$$\limsup_{n\to\infty} f_n = \inf_k \left\{\sup_{n\geq k} f_n\right\} \qquad \text{and} \qquad \liminf_{n\to\infty} f_n = \sup_k \left\{\inf_{n\geq k} f_n\right\}.$$

‣ Property 4:

If $\{f_n\}_{n=1}^{\infty}$ is a collection of measurable functions, and $\lim_{n\to\infty} f_n(x) = f(x)$, then $f$ is measurable.

Remark: Since $f(x) = \lim\sup_{n\to\infty} f_n(x) = \lim\inf_{n\to\infty} f_n(x)$, this property is a consequence of property 3.

‣ Property 5:

If $f$ and $g$ are measurable, then
(i) The integer powers $f^k$, for $k \geq 1$ are measurable.
(ii) $f + g$ and $f\, g$ are measurable if both $f$ and $g$ are finite-valued.

Remark: For (i) we simply note that if $k$ is odd, then $\{f^k > a\} = \{f > a^{1/k}\}$, and if $k$ is even and $a \geq 0$, then $\{f^k > a\} = \{f > a^{1/k}\} \bigcup \{f > -a^{1/k}\}$.
For (ii), we first see that $f + g$ is measurable because

$$\{f + g > a\} = \bigcup_{r\in\mathbb{Q}} \{f > a - r\} \bigcap \{g > r\}\ .$$

Finally, $f\, g$ is measurable because of the previous results and the fact that

$$f\, g = \tfrac{1}{4}\left[(f + g)^2 - (f - g)^2\right].$$

• Theorem 4.1:

Suppose $f$ is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that converges pointwise to $f$, namely,

$$\varphi_k(x) \le \varphi_{k+1}(x) \qquad \text{and} \qquad \lim_{k\to\infty} \varphi_k(x) = f(x),$$

for all $x$.

• Theorem 4.2:
Suppose $f$ is measurable on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that satisfies

$$|\varphi_k(x)| \le |\varphi_{k+1}(x)| \qquad \text{and} \qquad \lim_{k\to\infty} \varphi_k(x) = f(x),$$

for all $x$.
In particular, we have $|\varphi_k(x)| \le |f(x)|$ for all $x$ and $k$.

• Theorem 4.3:
Suppose $f$ is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges pointwise to $f(x)$ for almost every $x$.

Littlewood's three principles:
Although the notions of measurable sets and measurable functions represent new tools, we should not overlook their relation to the older concepts they replaced. Littlewood aptly summarized these connections in the form of three principles that provide a useful intuitive guide in the initial study of the theory:

(i) Every set is nearly a finite union of intervals.
(ii) Every function is nearly continuous.
(iii) Every convergent sequence is nearly uniformly convergent.

The sets and functions referred to above are of course assumed to be measurable. The catch is in the word "nearly," which has to be understood appropriately in each context.

• Egorov's Theorem:
Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set $E$ with $m(E) < \infty$, and assume that $f_k \to f$ a.e on $E$. Given $\varepsilon > 0$, we can find a closed set $A_\varepsilon \subset E$ such that $m(E \setminus A_\varepsilon) \le \varepsilon$ and $f_k \to f$ uniformly on $A_\varepsilon$.

Proof:
We may assume WLOG that $f_k(x) \to f(x)$ for every $x \in E$. For each pair of nonnegative

integers $n$ and $k$, let

$$E_k^n = \left\{x \in E : \left|f_j(x) - f(x)\right| < 1/n, \quad \text{for all } j > k\right\}.$$

Now fix $n$ and note that $E_k^n \subset E_{k+1}^n$, and $E_k^n \nearrow E$ as $k$ tends to infinity. By a previous corollary, we find that there exists $k_n$ such that $m\left(E \backslash E_{k_n}^n\right) < 1/2^n$.
By construction, we then have

$$\left|f_j(x) - f(x)\right| < 1/n \quad \text{whenever} \quad j > k_n \quad \text{and} \quad x \in E_{k_n}^n.$$

We choose $\mathcal{N}$ so that $\sum_{n=\mathcal{N}}^{\infty} 2^{-n} < \varepsilon/2$, and let

$$\tilde{A}_\varepsilon = \bigcap_{n \geq \mathcal{N}} E_{k_n}^n.$$

We first observe that

$$m\left(E \backslash \tilde{A}_\varepsilon\right) \leq \sum_{n=\mathcal{N}}^{\infty} m\left(E \backslash E_{k_n}^n\right) < \frac{\varepsilon}{2}.$$

Next, if $\delta > 0$, we choose $n \geq \mathcal{N}$ such that $1/n < \delta$, and note that $x \in \tilde{A}_\varepsilon$ implies $x \in E_{k_n}^n$. We see therefore that $\left|f_j(x) - f(x)\right| < \delta$ whenever $j > k_n$. Hence $f_k$ converges uniformly to $f$ on $\tilde{A}_\varepsilon$.

Finally, by a previous theorem we can choose a closed subset $A_\varepsilon \subset \tilde{A}_\varepsilon$ with $m\left(\tilde{A}_\varepsilon \backslash A_\varepsilon\right) < \varepsilon/2$.
As a result, we have $m(E \backslash A_\varepsilon) < \varepsilon$ and the theorem is proved. ■

- Lusin's Theorem:

Suppose $f$ is measurable and finite valued on a set $E$ of finite measure. Then for every $\varepsilon > 0$ there exists a closed set $F_\varepsilon$, with

$$F_\varepsilon \subset E \quad \text{and} \quad m(E \backslash F_\varepsilon) \leq \varepsilon$$

and such that $f|_{F_\varepsilon}$ is continuous.

Proof:
Let $f_n$ be a sequence of step functions so that $f_n \to f$ a.e. . Then we may find sets $E_n$ so that $m(E_n) < 1/2^n$ and $f_n$ is continuous outside $E_n$. By Egorov's theorem, we may find a set $A_{\varepsilon/3}$ on which $f_n \to f$ uniformly and $m(E \backslash A_{\varepsilon/3}) \leq \varepsilon/3$. Then we consider

$$F' = A_{\varepsilon/3} \backslash \bigcup_{n \geq \mathcal{N}} E_n$$

for $N$ so large that $\sum_{n \geq N} 1/2^n < \varepsilon/3$. Now for every $n \geq N$ the function $f_n$ is continuous on $F'$; thus $f$ (being the uniform limit of $\{f_n\}$ by Egorov's theorem) is also continuous on $F'$.

To finish the proof, we merely need to approximate the set $F'$ by a closed set $F_\varepsilon \subset F'$ such that $m(F' \setminus F_\varepsilon) < \varepsilon/3$. ■

## Integration Theory

• Give a prose of the four stages of Lebesgue integration from chpt 2:
  -Specify classes
  - State lemmas
  - State properties

We proceed in four stages, by progressively integrating:
1. Simple functions
2. Bounded functions supported on a set of finite measure
3. Non-negative functions
4. Integrable functions (the general case).

• Bounded Convergence Theorem:
Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by $M$, are supported on a set $E$ of finite measure, and $f_n(x) \to f(x)$ a.e. $x$ as $n \to \infty$. Then $f$ is measurable, bounded, supported on $E$ for a.e. $x$, and

$$\int |f_n - f| \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently,

$$\int f_n \to \int f \qquad \text{as} \quad n \to \infty.$$

Proof:
From the assumptions one sees at once that $f$ is bounded by $M$ almost everywhere and vanishes outside $E$, except possibly on a set of measure zero. Clearly, the triangle inequality for the integral implies that it suffices to prove that $\int |f_n - f| \to 0$ as $n$ approaches infinity.
Given $\varepsilon > 0$, we may find, by Egorov's theorem, a measurable subset $A_\varepsilon \subset E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ and $f_n \to f$ uniformly on $A_\varepsilon$. Then, we know that for all sufficiently large $n$ and

for all $x \in A_\varepsilon$ we have $|f_n(x) - f(x)| \leq \varepsilon$. Putting these facts together yields

$$\begin{aligned}\int_E |f_n(x) - f(x)|\, dx &= \int_{A_\varepsilon} |f_n(x) - f(x)|\, dx + \int_{E\setminus A_\varepsilon} |f_n(x) - f(x)|\, dx \\ &\leq \varepsilon\, m(E) + 2\, M\, m(E\setminus A_\varepsilon)\end{aligned}$$

for all large $n$. Since $\varepsilon$ is arbitrary, the proof of the theorem is complete. ■

• Fatou's Lemma:

Suppose $\{f_n\}$ is a sequence of measurable functions with $f_n \geq 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x$, then $\int f \leq \liminf_{n\to\infty} \int f_n$ .

Proof:

Suppose $0 \leq g \leq f$, where $g$ is bounded and supported on a set $E$ of finite measure. If we set $g_n(x) = \min(g(x), f_n(x))$, then $g_n$ is measurable, supported on $E$, and $g_n(x) \to g(x)$ a.e., so by the bounded convergence theorem

$$\int g_n \to \int g \,.$$

By construction, we also have $g_n \leq f_n$, so that $\int g_n \leq \int f_n$ by the monotonicity of the integral. Thus,

$$\int g \leq \liminf_{n\to\infty} \int f_n \,.$$

Taking the supremum over all $g$ yields the desired inequality. ■

• Corollary:

Suppose $f$ is a non-negative measurable function, and $\{f_n\}$ a sequence of non-negative measurable functions with $f_n(x) \leq f(x)$ and $f_n(x) \to f(x)$ for almost every $x$.

Then

$$\lim_{n\to\infty} \int f_n = \int f \,.$$

Proof:

Since $f_n(x) \leq f(x)$ a.e. $x$, we necessarily have $\int f_n \leq \int f \;\; \forall\, n$.

Hence,

$$\liminf_{n\to\infty} \int f_n \leq \int f.$$

This inequality combined with Fatou's lemma proves the desired limit. ■

• Corollary (Monotone convergence theorem):
Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$.
Then
$$\lim_{n\to\infty} \int f_n = \int f \,.$$

Proof:
The proof follows from the preceding corollary and its proof. ■

• Dominated Convergence Theorem:
Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. $x$ as $n \to \infty$. If $|f_n(x)| \leq g(x)$, where $g$ is integrable, then
$$\int |f_n - f| \to 0 \quad \text{as } n \to \infty \,,$$
and consequently,
$$\int f_n \to \int f \qquad \text{as } n \to \infty \,.$$

Proof:
For each $N \geq 0$ let $E_N = \{x : |x| \leq N,\ g(x) \leq N\}$. Given $\varepsilon > 0$, we may argue that there exists $N$ so that
$$\int_{E_N^c} |f| < \varepsilon \,.$$

Then the functions $f_n \chi_{E_N}$ are bounded (by $N$) and supported on a set of finite measure, so that by the bounded convergence theorem, we have
$$\int_{E_N} |f_n - f| < \varepsilon \quad \text{for all large } n.$$
Hence, we obtain the estimate
$$\begin{aligned} \int |f_n - f| &= \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f| \\ &\leq \int_{E_N} |f_n - f| + 2 \int_{E_N^c} g \\ &\leq \varepsilon + 2\,\varepsilon = 3\,\varepsilon \end{aligned}$$

for all large $n$.

This proves the theorem. ■