Math 353 HW 5

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Section 2.4

(1) From the basic definition of complex integration, evaluate the integral $\oint_C f(z)\, dz$, where C is the parametrized unit circle enclosing the origin, $C: x(t) = \cos t,\ y(t) = \sin t$, or $z = e^{i t}$, and where $f(z)$ is given by:

b) $\overline{z}^2$

Solution:

$$z(t) = e^{i t}\ ;\ z'(t) = i\, e^{i t}\ ;\quad 0 \le t \le 2\pi$$

So we have

$$\oint_C \overline{z}^2\, dz = \int_0^{2\pi} e^{-2 i t}\,(i\, e^{i t})\, dt = i \int_0^{2\pi} e^{-i t}\, dt = -\, e^{-i t}\Big|_0^{2\pi} = -1 + 1 = 0\ .$$

c) $\frac{z+1}{z^2}$

Solution:

$$\oint_C \frac{z+1}{z^2}\, dz = \int_0^{2\pi} \frac{e^{i t}+1}{e^{i\, 2 t}}\; i\, e^{i t}\, dt$$
$$= i \int_0^{2\pi} \frac{e^{i t}+1}{e^{i t}}\, dt = i \int_0^{2\pi} \left(1 + e^{-i t}\right) dt = i\, t - e^{-i t}\Big|_0^{2\pi}$$
$$= 2\pi i - 1 + 1 = 2\pi i$$

(2) Evaluate the integral $\oint_C f(z)\, dz$, where C is the unit circle enclosing the origin, and $f(z)$ is given as follows :

a) $1 + 2z + z^2$

Solution:
$z(t) = e^{i t}\ ;\ \ z'(t) = i\, e^{i t}\ ;\ 0 \le t \le 2\pi$
So we have

$$\oint_C (1+2z+z^2)\,dz = \int_0^{2\pi}\left(1+2e^{it}+e^{i2t}\right) i e^{it}\,dt$$
$$= i\int_0^{2\pi}\left(e^{it}+2e^{i2t}+e^{i3t}\right)dt$$
$$= e^{it}+e^{i2t}+\frac{1}{3}e^{i3t}\Big|_0^{2\pi}$$
$$= 1+1+\frac{1}{3}-\left(1+1+\frac{1}{3}\right)=0$$

We could've also noticed that $1+2z+z^2=(z+1)^2$. Then by letting $w=z+1$, $dw=dz$, we would have $\oint_C f(z)\,dz = \oint_C f(w)\,dw = \oint_C w^2\,dw$. Then since the exponent is $\neq -1$, this integral will always be zero.

b) $\dfrac{1}{\left(z-\frac{1}{2}\right)^2}$

Solution:
For this problem I'm going to use the rationale discussed above, i.e. I let $w=z-\frac{1}{2}$; $dw=dz$. Then $\oint_C f(z)\,dz = \oint_C f(w)\,dw = \oint_C w^{-2}\,dw$. Since the exponent $-2\neq -1$, we know that this integral is just zero.

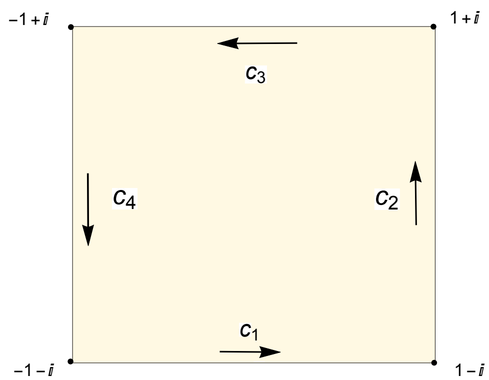
c) $\dfrac{1}{z}$

Solution:
$$\oint_C f(z)\,dz = \oint_C \frac{1}{z}\,dz = \int_0^{2\pi}\frac{1}{e^{-it}}\,i e^{it}\,dt = \int_0^{2\pi} i e^{i2t}\,dt$$
$$= \frac{i}{2i}e^{i2t}\Big|_0^{2\pi} = \frac{1}{2}e^{i2t}\Big|_0^{2\pi} = \frac{1}{2}-\frac{1}{2}=0$$

(3) Let C be the unit square with diagonal corners at $-1-i$ and $1+i$.
Evaluate $\oint_C f(z)$ where $f(z)$ is given by the following:

d) $\mathrm{Re}(z)$

Solution:



We have $\oint_C f(z)\,dz = \sum_{i=1}^{4} \int_{c_i} f(z)\,dz$. Now let's parametrize C, evaluate all four integrals, and then add the results:

- $c_1(t) = -1 - i + t\,[1 - i - (-1 - i)] = -1 - i + 2\,t$

$\Longrightarrow c_1'(t) = 2 \;\;;\;\; 0 \le t \le 1$

$$\int_{c_1} \text{Re}(z)\,dz = \int_0^1 (-1 + 2\,t)\,2\,dt = -2\,t + 2\,t^2\Big|_0^1$$
$$= -2 + 2 = 0$$

- $c_2(t) = 1 - i + t\,[1 + i - (1 - i)] = 1 - i + 2\,i\,t$

$\Longrightarrow c_2'(t) = 2\,i \;\;;\;\; 0 \le t \le 1$

$$\int_{c_2} \text{Re}(z)\,dz = \int_0^1 1\,(2\,i)\,dt = 2\,i\,t\Big|_0^1 = 2\,i$$

- $c_3(t) = 1 + i + t\,[-1 + i - (1 + i)] = 1 + i - 2\,t$

$\Longrightarrow c_3'(t) = -2 \;\;;\;\; 0 \le t \le 1$

$$\int_{c_3} \text{Re}(z)\,dz = \int_0^1 (1 - 2\,t)\,(-2)\,dt = -2\,t + 2\,t^2\Big|_0^1$$
$$= -2 + 2 = 0$$

- $c_4(t) = -1 + i + t\,[-1 - i - (-1 + i)] = -1 + i - 2\,i\,t$

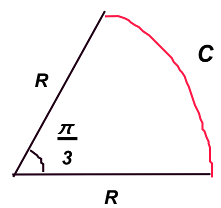
$\Longrightarrow c_4'(t) = -2\,i \;\;;\;\; 0 \le t \le 1$

$$\int_{c_4} \text{Re}(z)\,dz = \int_0^1 (-1)\,(-2\,i)\,dt = 2\,i\,t\Big|_0^1 = 2\,i$$

Hence $\oint_C f(z)\,dz = \sum_{i=1}^{4} \int_{c_i} f(z)\,dz = 0 + 2\,i + 0 + 2\,i = 4\,i$

(8) Let C be an arc of the circle $|z| = R$ $(R > 1)$ of angle $\pi/3$. Show that $\left|\int_C \frac{dz}{z^3+1}\right| \leq \frac{\pi}{3}\left(\frac{R}{R^3-1}\right)$ and deduce $\lim_{R\to\infty} \int_C \frac{dz}{z^3+1} = 0$.

Solution:



▸ We have that $f(z) = \frac{1}{z^3+1}$. We know the arc length $L = \frac{\pi}{3} R$, now we need to find the upper bound M. From the triangle inequality we know that $\left| |z_1| - |z_2| \right| \leq |z_1 + z_2|$. Let's apply this inequality using $z_1 = z^3$ and $z_2 = 1$:

$$\left| |z^3| - |1| \right| \leq |z^3 + 1|$$

We can parametrize the curve as $z(t) = R\, e^{i\,t}$ with $0 \leq t \leq \frac{\pi}{3}$.

So we have

$$\left| |R^3\, e^{3\,i\,t}| - |1| \right| \leq |z^3 + 1|$$

$$\Longrightarrow \left| |R^3| - |1| \right| \leq |z^3 + 1|$$

$$\Longrightarrow |R^3 - 1| \leq |z^3 + 1| \Longrightarrow \frac{1}{|R^3-1|} \geq \frac{1}{|z^3+1|}$$

$$\Longrightarrow \frac{1}{R^3-1} \geq \frac{1}{z^3+1} = f(z).$$

Hence our upper bound $M = \frac{1}{R^3-1}$. Then since $f(z)$ is continous on C, we know that

$$\left|\int_C \frac{dz}{z^3+1}\right| \leq ML \Longrightarrow \left|\int_C \frac{dz}{z^3+1}\right| \leq \frac{\pi}{3}\,\frac{R}{R^3-1}. \quad \checkmark$$

▸ Now we want to show that $\lim_{R\to\infty} \int_C \frac{dz}{z^3+1} = 0$:

$$\lim_{R\to\infty} \int_C \frac{dz}{z^3+1} \leq \left|\lim_{R\to\infty} \int_C \frac{dz}{z^3+1}\right| \leq \lim_{R\to\infty} \frac{\pi}{3}\,\frac{R}{R^3-1} = 0 \quad \checkmark$$

(9) Consider $I_R = \int_{C_R} \frac{e^{iz}}{z^2}\, dz$, where C_R is the semicircle with radius R in the upper half plane with endpoints $(-R,\ 0)$ and $(R,\ 0)$ (C_R is open, it does not include the x axis). Show that $\lim_{R\to\infty} I_R = 0$.

Solution:
We know that the arc length $L = \pi R$. Let's find the upper bound M.
It is true that the following inequality must hold:

$$\left|\int_{C_R} \frac{e^{iz}}{z^2}\right| \le \int_{C_R} \left|\frac{e^{iz}}{z^2}\right| = \int_{C_R} \frac{1}{z^2}$$

Hence $\frac{1}{z^2}$ is the upper bound M for $f(z)$. Since C_R can be parametrized as $z(t) = R\, e^{it}$ with $0 < t < \pi$, we have that $M = \frac{1}{R^2\, e^{2it}}$.
Then we have

$$\left|\int_{C_R} \frac{e^{iz}}{z^2}\right| \le ML \Longrightarrow \left|\int_{C_R} \frac{e^{iz}}{z^2}\right| \le \frac{\pi R}{R^2\, e^{2it}} = \frac{\pi}{R\, e^{2it}}$$

Hence we can see that the following limit is zero :

$$\lim_{R\to\infty} I_R = \lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^2}\, dz \le \left| \operatorname*{limit}_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^2}\, dz \right| \le \lim_{R\to\infty} \frac{\pi}{R\, e^{2it}} = 0.$$

Section 2.5

(1) Evaluate $\oint_C f(z)\, dz$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following :

b) e^{z^2}

Solution:
Since $f(z)$ is analytic inside the simply connected region $|z| < 1$, by Cauchy's Theorem we know that $\oint_C e^{z^2}\, dz = 0$.

d) $\frac{1}{z^2-4}$

Solution:
This time $f(z)$ is not analytic only when $z = \pm 2$, but neither one of these two points lie inside the simply connected region $|z| < 1$. Therefore, by Cauchy's Theorem we must have that

$$\oint_C \frac{1}{z^2-4}\, dz = 0 .$$

(2) Use partial fractions to evaluate the following integrals $\oint_C f(z)\, dz$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following :

a) $\frac{1}{z(z-2)}$

Solution:

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$\Longrightarrow 1 = A(z-2) + B\, z \Longrightarrow 1 = (A+B)\, z - 2\, A$$

$$\Longrightarrow A + B = 0 \;\; ; \;\; -2\, A = 1 \Longrightarrow A = -\tfrac{1}{2} \;\; ; \;\; B = \tfrac{1}{2}.$$

Hence

$$\frac{1}{z(z-2)} = \frac{-1/2}{z} + \frac{1/2}{z-2}.$$

Thus

$$\oint_C \frac{1}{z(z-2)}\, dz = -\frac{1}{2} \oint_C \frac{1}{z} + \frac{1}{2} \oint_C \frac{1}{z-2}.$$

We know that on the first integral since z is raised to the 1^{st} power, we have $-\frac{1}{2}\, 2\, \pi i = -\pi i$. On the second integral $f(z)$ is not analytic only where $z = 2$, which is outside the region $|z| < 1$. Thus we have

$$\oint_C \frac{1}{z(z-2)}\, dz = -\frac{1}{2} \oint_C \frac{1}{z} + \frac{1}{2} \oint_C \frac{1}{z-2} = -\pi i + 0 = -\pi i .$$

(3) Evaluate the integral $\oint_C \frac{e^{iz}}{z(z-\pi)}\, dz$ for each of the following four cases (all circles are centered at the origin):

a) C is the boundary of the annulus between circles of radius 1 and radius 3.

Solution:

We can notice that our integrand is analytic over the region enclosed by $1 \le |z| \le 3$. Hence, by Cauchy's theorem, we have $\oint_C \frac{e^{iz}}{z(z-\pi)}\, dz = 0$.

b) C is the boundary of the annulus between circles of radius 1 and radius 4.

Solution:
The problem this time is that the integrand is not always analytic inside the enclosed region $1 \leq |z| \leq 4$, more specifically it's not analytic when $z = \pi$.
We take a crosscut around the point $z = \pi$, that way the integrand is analytic between the boundaries of the annulus and the circle C_1 of radius ε around $z = \pi$, but it's not analytic inside the circle. By previous work we have shown that $\oint_C = \oint_{C_1}$.

First let's expand $\oint_{C_1} \frac{1}{z(z-\pi)}\, dz$ using partial fractions :

$$\frac{1}{z(z-\pi)} = \frac{A}{z} + \frac{B}{z-\pi} \Longrightarrow A(z-\pi) + B\,z = 1 \Longrightarrow (A+B)\,z - A\,\pi = 1$$
$$\Longrightarrow A = -\frac{1}{\pi} \; ; \; B = \frac{1}{\pi}\,.$$

Hence
$$\oint_{C_1} \frac{e^{iz}}{z(z-\pi)}\, dz = -\frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z}\, dz + \frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z-\pi}\, dz\,.$$

We can easily see that $-\frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z}\, dz = 0$, since the integrand is analytic in C_1.

Hence we only need to consider $\frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z-\pi}\, dz$. We can rewrite e^{iz} as $e^{iz} \cdot \frac{e^{-i\pi}}{e^{-i\pi}}$.
Thus we have

$$\frac{e^{i\pi}}{\pi} \oint_{C_1} \frac{e^{i(z-\pi)}}{z-\pi}\, dz = \frac{e^{i\pi}}{\pi} \left(\oint_{C_1} \frac{1}{z-\pi} + \frac{i(z-\pi)}{z-\pi} + \frac{(i(z-\pi))^2}{2!\,(z-\pi)} + \ldots + \frac{(i(z-\pi))^n}{n!\,(z-\pi)} \right)$$

But now we can see that only the first expanded integrand is not analytic on the enclosed region, so all the remaining terms are zero and we have
$$\frac{e^{i\pi}}{\pi} \oint_{C_1} \frac{1}{z-\pi} = \frac{e^{i\pi}}{\pi}\, 2\,\pi i = 2\,i\,e^{i\pi} = -2\,i\,.$$

Thus $\oint_C \frac{e^{iz}}{z(z-\pi)}\, dz = 0 - 2\,i = -2\,i\,.$

c) C is the circle of radius R, where $R > \pi$.

Solution:
From part b) we have that

$$\oint_{C_1} \frac{e^{iz}}{z(z-\pi)}\, dz = -\frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z}\, dz + \frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z-\pi}\, dz\,.$$

The difference this time is that the first integrand is not analytic inside C_1. Instead we have

$$-\frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z}\, dz = -\frac{1}{\pi} \left(\oint_{C_1} \frac{1}{z} + \frac{i\,z}{z} + \frac{(i\,z)^2}{2!\,z} + \ldots + \frac{(i\,z)^n}{n!\,z} \right)$$

From here we can see that only the first integrand is not analytic in the enclosed region, hence all the remaining terms are zero. Thus we have

$$-\frac{1}{\pi} \oint_{C_1} \frac{1}{z}\, dz = -\frac{1}{\pi}\, 2\,\pi i = -2\,i\,.$$

Now from part b) we saw that

$$\frac{1}{\pi} \oint_{C_1} \frac{e^{iz}}{z-\pi}\, dz = -2\,i.$$

Hence finally we have that

$$\oint_{C_1} \frac{e^{iz}}{z(z-\pi)}\, dz = -2\,i - 2\,i = -4\,i\,.$$

d) C is the circle of radius R, where $R < \pi$.

Solution:

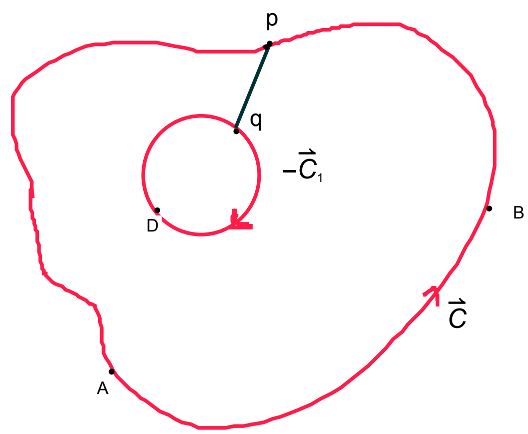
In this case our integrand is analytic everywhere except around a neighborhood of $z = 0$. Hence we only need to pay attention to the first part of the expanded integrand on c), since all the other terms are zero by Cauchy's theorem.

Thus we have that

$$\oint_{C_1} \frac{e^{iz}}{z(z-\pi)}\, dz = -2\,i.$$

(4) Discuss how to evaluate $\oint_C \frac{e^{z^2}}{z^2}\, dz$, where C is a simple closed curve enclosing the origin.

Solution:



Since C can be any simple closed curve, we want to find a way to evaluate this integral over some circle instead if possible, since circles are easy to work with. We want this circle to enclose the origin, since this is where our integrand is not analytic. In order to do that we take a cross-cut (pq on the figure above) and analyze the path of the integral as follows:

$$\oint_{\text{pABpqDqp}} \frac{e^{z^2}}{z^2}\, dz = \oint_C \frac{e^{z^2}}{z^2}\, dz + \int_{\text{pq}} \frac{e^{z^2}}{z^2}\, dz + \oint_{-C_1} \frac{e^{z^2}}{z^2}\, dz + \int_{\text{qp}} \frac{e^{z^2}}{z^2}\, dz$$

$$= \oint_C \frac{e^{z^2}}{z^2}\, dz - \oint_{C_1} \frac{e^{z^2}}{z^2}\, dz = 0$$

$$\Longrightarrow \oint_C \frac{e^{z^2}}{z^2}\, dz = \oint_{C_1} \frac{e^{z^2}}{z^2}\, dz\,.$$

So we can see that we turned an integral over some arbitrary closed curve C to an integral over a circle. Now we expand the integrand by writing e^{z^2} as a power series.

$$\oint_{C_1} \frac{e^{z^2}}{z^2}\, dz = \oint_{C_1} \frac{1}{z^2}\, dz + \oint_{C_1} \frac{z^2}{z^2}\, dz + \oint_{C_1} \frac{z^4}{2!\, z^2}\, dz + \oint_{C_1} \frac{z^6}{3!\, z^2}\, dz + \ldots + \oint_{C_1} \frac{z^{2n}}{n!\, z^2}\, dz.$$

Only the first part of the integral is not analytic in the enclosed region, so by Cauchy’s theorem all the other terms are zero, However when we evaluate $\oint_{C_1} \frac{1}{z^2}\, dz$ we can see that this integral is also zero since it is in the form $\oint_{C_1} \frac{1}{z^m}\, dz$ with $m \neq 1$. Hence $\oint_{C_1} \frac{e^{z^2}}{z^2}\, dz = 0$.