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Homework Assignment 5
General Relativity I

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Problem 1 (Exercise 5.2 (Schutz)). *Explain why a uniform external gravitational field would raise no tides on Earth.*

Solution. If a gravitational field in the vicinity of Earth is uniform, we could choose an inertial frame in a which the worldine of a particle placed in such field is described by a geodesic (i.e., the particle would be in free fall). Such particle, being in a uniform gravitational field, would not affect the Earth's own free fall (i.e., geodesic) motion. Mathematically, if we consider $\vec{X}$ to be the vector field tangent to a smooth 1-parameter family of integral curves $\{\gamma\colon [a,b] \to \mathcal{M}\}$ that are geodesics (one of which is the previously mentioned particle's geodesic, and another that of the Earth), and we let $\vec{Z}$ denote the deviation vector of the family of geodesics (so that $[\vec{X},\vec{Z}] = 0$), then, if the gravtational field is uniform,

$$\nabla_{\vec{X}}\nabla_{\vec{X}}\vec{Z} = 0.$$

That is, there is no "push" or "pull" between any of the geodesics. In general, however, if a non-uniform gravitational field is present, we have

$$\nabla_{\vec{X}}\nabla_{\vec{X}}\vec{Z} = R(\vec{X},\vec{Z})\vec{X},$$

where $\mathbf{R}$ is the Riemann tensor. In coordinates,

$$X^b\nabla_b(X^c\nabla_c Z^a) = R^a{}_{bcd}X^bX^cZ^d. \qquad \square$$

Problem 2 (Exercise 5.7 (Schutz)). *Calculate all elements of the transformation matrices $\Lambda^{\alpha'}{}_{\beta}$ and $\Lambda^{\mu}{}_{\nu'}$ for the transformation from Cartesian (x,y) –the unprimed indices– to polar (r,θ) –the primed indices.–*

Solution.

$$\Lambda^{\alpha'}{}_{\beta} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial(\sqrt{x^2+y^2})}{\partial x} & \frac{\partial(\sqrt{x^2+y^2})}{\partial y} \\ \frac{\partial\left(\arctan\frac{y}{x}\right)}{\partial x} & \frac{\partial\left(\arctan\frac{y}{x}\right)}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \frac{r\cos\theta}{r} & \frac{r\sin\theta}{r} \\ -\frac{r\sin\theta}{r^2} & \frac{r\cos\theta}{r^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}.$$

$$\Lambda^{\mu}{}_{\nu'} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial \theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Indeed, they are inverse of each other:

$$\begin{aligned} \Lambda^{\alpha'}{}_{\beta}\Lambda^{\beta}{}_{\nu'} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -r\cos\theta\sin\theta + r\cos\theta\sin\theta \\ -\frac{1}{r}\sin\theta\cos\theta + \frac{1}{r}\sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^{\alpha'}{}_{\nu'}. \qquad \checkmark \end{aligned} \qquad \square$$

Problem 3 (Exercise 5.8 (Schutz)). *Using results from Exercise 5.7,*

(a) *Let $f = x^2 + y^2 + 2xy$, and in Cartesian coordinates $\vec{V} \to (x^2+3y, y^2+3x)$, $\vec{W} \to (1,1)$. Compute f as a function of r and θ, and find the components of $\vec{V}$ and $\vec{W}$ on the polar basis, expressing them as functions of r and θ.*

(b) *Find the components of $\tilde{\mathrm{d}}f$ in Cartesian coordinates and obtain them in polars (i) by direct calculation in polars, and (ii) by transforming components from Cartesian.*

(c) *(i) Use the metric tensor in polar coordinates to find the polar components of the one-forms $\widetilde{V}$ and $\widetilde{W}$ associated with $\vec{V}$ and $\vec{W}$. (ii) Obtain the polar components of $\widetilde{V}$ and $\widetilde{W}$ by transformation of their Cartesian components.*

Solution to (a). For f, we have

$$f(r,\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 + 2r\cos\theta\, r\sin\theta = r^2(1 + 2\cos\theta\ \sin\theta) = r^2[1+\sin(2\theta)].$$

Now, we need to express both $\vec{V}$ and $\vec{W}$ in terms of the polar basis, so our first step is to transform the Cartesian basis to the polar one:

$$\begin{aligned}
\vec{e}_x &= \Lambda_x{}^r\vec{e}_r + \Lambda_x{}^\theta\vec{e}_\theta \\
&= \frac{\partial r}{\partial x}\vec{e}_r + \frac{\partial\theta}{\partial x}\vec{e}_\theta \\
&= \cos\theta\,\vec{e}_r - \frac{\sin\theta}{r}\vec{e}_\theta; \\
\vec{e}_y &= \Lambda_y{}^r\vec{e}_r + \Lambda_y{}^\theta\vec{e}_\theta \\
&= \frac{\partial r}{\partial y}\vec{e}_r + \frac{\partial\theta}{\partial y}\vec{e}_\theta \\
&= \sin\theta\,\vec{e}_r + \frac{\cos\theta}{r}\vec{e}_\theta.
\end{aligned}$$

In terms of polar coordinates, but still in the Cartesian basis, the vectors $\vec{V}$ and $\vec{W}$ are expressed as

$$\begin{aligned}
&\vec{V} \xrightarrow{\text{Cartesian}} (r^2\cos^2\theta + 3r\sin\theta, r^2\sin^2\theta + 3r\cos\theta) = (r^2\cos^2\theta + 3r\sin\theta)\,\vec{e}_x + (r^2\sin^2\theta + 3r\cos\theta)\,\vec{e}_y; \\
&\vec{W} \xrightarrow{\text{Cartesian}} \left(\sqrt{1^2+1^2}, \arctan\frac{1}{1}\right) = \left(\sqrt{2}, \frac{\pi}{4}\right) = \sqrt{2}\,\vec{e}_x + \frac{\pi}{4}\,\vec{e}_y.
\end{aligned}$$

We need to translate this to the polar basis using the transformations derived above:

$$\begin{aligned}
\vec{V}_{\text{polar}} &= \left(r^2\cos^2\theta + 3r\sin\theta\right)\left(\cos\theta\,\vec{e}_r - \frac{\sin\theta}{r}\vec{e}_\theta\right) + \left(r^2\sin^2\theta + 3r\cos\theta\right)\left(\sin\theta\,\vec{e}_r + \frac{\cos\theta}{r}\vec{e}_\theta\right) \\
&= \left\{r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta)\right\}\,\vec{e}_r + \left\{3(\cos^2\theta - \sin^2\theta) + r\cos\theta\sin\theta(\sin\theta - \cos\theta)\right\}\,\vec{e}_\theta \qquad (1) \\
\vec{W}_{\text{polar}} &= \sqrt{2}\left(\cos\theta\,\vec{e}_r - \frac{\sin\theta}{r}\vec{e}_\theta\right) + \frac{\pi}{4}\left(\sin\theta\,\vec{e}_r + \frac{\cos\theta}{r}\vec{e}_\theta\right) \\
&= \left\{\sqrt{2}\cos\theta + \frac{\pi}{4}\,\sin\theta\right\}\,\vec{e}_r + \left\{\frac{1}{r}\left(\frac{\pi}{4}\cos\theta - \sqrt{2}\,\sin\theta\right)\right\}\,\vec{e}_\theta. \qquad (2)
\end{aligned}$$

□

Solution to (b). For (i),

$$\begin{aligned}
\tilde{\mathrm{d}}f &= \tilde{\mathrm{d}}\left(r^2[1+\sin(2\theta)]\right) \\
&= \partial_r\left(r^2[1+\sin(2\theta)]\right)\tilde{\mathrm{d}}r + \partial_\theta\left(r^2[1+\sin(2\theta)]\right)\tilde{\mathrm{d}}\theta \\
&= 2r[1+\sin(2\theta)]\,\tilde{\mathrm{d}}r + 2r^2\cos(2\theta)\,\tilde{\mathrm{d}}\theta \\
&= \left(2r[1+\sin(2\theta)],\ 2r^2\cos(2\theta)\right).
\end{aligned}$$

For (ii), denote $\tilde{\mathrm{d}}f$ in Cartesian coordinates as f_μ and in polars as $f_{\mu'}$. Then,

$$f_\mu = (2x+2y,\ 2y+2x)$$

and

$$f_{\mu'} = \Lambda_{\mu'}{}^{\mu} f_\mu. \qquad (3)$$

Hence,

$$
\begin{aligned}
f_r &= \Lambda_r{}^x f_x + \Lambda_r{}^y f_y \\
&= \frac{\partial x}{\partial r}[2r(\cos\theta + \sin\theta)] + \frac{\partial y}{\partial r}[2r(\cos\theta + \sin\theta)] \\
&= \cos\theta[2r(\cos\theta + \sin\theta)] + \sin\theta[2r(\cos\theta + \sin\theta)] \\
&= 2r[1 + \sin(2\theta)]; \qquad \surd
\end{aligned}
$$

$$
\begin{aligned}
f_\theta &= \Lambda_\theta{}^x f_x + \Lambda_\theta{}^y f_y \\
&= \frac{\partial x}{\partial \theta}[2r(\cos\theta + \sin\theta)] + \frac{\partial y}{\partial \theta}[2r(\cos\theta + \sin\theta)] \\
&= -r\sin\theta\,[2r(\cos\theta + \sin\theta)] + r\cos\theta\,[2r(\cos\theta + \sin\theta)] \\
&= 2r^2\cos(2\theta). \qquad \surd
\end{aligned}
$$

□

Solution to (c). For (i), the metric tensor in polar coordinates is given by

$$
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{4}
$$

Then, using (1) and (2),

$$
\begin{aligned}
\widetilde{V} = V_\mu &= g_{\mu\nu}V^\nu \\
&= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta) \\ 3(\cos^2\theta - \sin^2\theta) + r\cos\theta\sin\theta(\sin\theta - \cos\theta) \end{pmatrix} \\
&= \begin{pmatrix} r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta) \\ 3r^2(\cos^2\theta - \sin^2\theta) + r^3\cos\theta\sin\theta(\sin\theta - \cos\theta) \end{pmatrix};
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{W} = W_\mu &= g_{\mu\nu}W^\nu \\
&= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} \sqrt{2}\cos\theta + \frac{\pi}{4}\sin\theta \\ \frac{1}{r}\left(\frac{\pi}{4}\cos\theta - \sqrt{2}\sin\theta\right) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{2}\cos\theta + \frac{\pi}{4}\sin\theta \\ r\left(\frac{\pi}{4}\cos\theta - \sqrt{2}\sin\theta\right) \end{pmatrix}.
\end{aligned}
$$

For (ii), denote $\widetilde{V}$ and $\widetilde{W}$ in Cartesian coordinates as V_μ and W_μ, respectively, and their polar counterparts as $V_{\mu'}$ and $W_{\mu'}$. Also, note that in Cartesian coordinates the flat metric has components $\delta^\mu{}_\nu$, so the one forms $\widetilde{V}$ and $\widetilde{W}$ have the same Cartesian components as their vector counterparts, namely, $\widetilde{V} = (x^2 + 3y, y^2 + 3x)$ and $\widetilde{W} = (1, 1)$. Now, using the transformation (3), we get

$$
\begin{aligned}
V_r &= \Lambda_r{}^x V_x + \Lambda_r{}^y V_y \\
&= \frac{\partial x}{\partial r}[r^2\cos^2\theta + 3r\sin\theta] + \frac{\partial y}{\partial r}[r^2\sin^2\theta + 3r\cos\theta] \\
&= \cos\theta[r^2\cos^2\theta + 3r\sin\theta] + \sin\theta[r^2\sin^2\theta + 3r\cos\theta] \\
&= r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta). \qquad \surd
\end{aligned}
$$

$$
\begin{aligned}
V_\theta &= \Lambda_\theta{}^x V_x + \Lambda_\theta{}^y V_y \\
&= \frac{\partial x}{\partial \theta}[r^2\cos^2\theta + 3r\sin\theta] + \frac{\partial y}{\partial \theta}[r^2\sin^2\theta + 3r\cos\theta] \\
&= -r\sin\theta[r^2\cos^2\theta + 3r\sin\theta] + r\cos\theta[r^2\sin^2\theta + 3r\cos\theta] \\
&= 3r^2(\cos^2\theta - \sin^2\theta) + r^3\cos\theta\sin\theta(\sin\theta - \cos\theta). \qquad \surd
\end{aligned}
$$

So we found the same components that we found in (i). Similarly for $\widetilde{W}$:

$$\begin{aligned}
W_r &= \Lambda_r{}^x W_x + \Lambda_r{}^y W_y \\
&= \frac{\partial x}{\partial r}\sqrt{2} + \frac{\partial y}{\partial r}\frac{\pi}{4} \\
&= \sqrt{2}\cos\theta + \frac{\pi}{4}\sin\theta. \qquad \checkmark
\end{aligned}$$

$$\begin{aligned}
W_\theta &= \Lambda_\theta{}^x W_x + \Lambda_\theta{}^y W_y \\
&= \frac{\partial x}{\partial \theta}\sqrt{2} + \frac{\partial y}{\partial \theta}\frac{\pi}{4} \\
&= -r\sqrt{2}\sin\theta + r\frac{\pi}{4}\cos\theta. \qquad \checkmark
\end{aligned}$$

□

Problem 4 (Exercise 5.12 (Schutz)). *For the one-form field $\tilde{p}$ whose Cartesian components are (x^2+3y, y^2+3x), compute:*

(a) *$p_{\alpha,\beta}$ in Cartesian coordinates.*

(b) *The transformation $\Lambda_{\mu'}{}^\alpha \Lambda_{\nu'}{}^\beta p_{\alpha,\beta}$ to polars.*

(c) *The components $p_{\mu';\nu'}$ directly in polars inserting the Christoffel symbols*

$$\begin{aligned}
\partial_r \vec{e}_r &= 0 &&\implies && \Gamma^\mu{}_{rr} = 0 \;\; \forall \mu; \\
\partial_\theta \vec{e}_r &= \frac{1}{r}\vec{e}_\theta &&\implies && \Gamma^r{}_{r\theta} = 0 \;\&\; \Gamma^\theta{}_{r\theta} = \frac{1}{r}; \\
\partial_r \vec{e}_\theta &= \frac{1}{r}\vec{e}_\theta &&\implies && \Gamma^r{}_{\theta r} = 0 \;\&\; \Gamma^\theta{}_{\theta r} = \frac{1}{r}; \\
\partial_\theta \vec{e}_\theta &= -r\vec{e}_r &&\implies && \Gamma^r{}_{\theta\theta} = -r \;\&\; \Gamma^\theta{}_{\theta\theta} = 0
\end{aligned} \tag{5}$$

into the expression

$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_\mu \Gamma^\mu{}_{\alpha\beta}. \tag{6}$$

Solution to (a). For $\beta = x$,

$$p_{\alpha,x} = (2x, 3),$$

and for $\beta = y$,

$$p_{\alpha,y} = (3, 2y).$$

Or, in matrix form,

$$p_{\alpha,\beta} = \begin{pmatrix} \frac{\partial p_x}{\partial x} & \frac{\partial p_x}{\partial y} \\ \frac{\partial p_y}{\partial x} & \frac{\partial p_y}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix}.$$

□

Solution to (b). We need to compute $p_{\mu',\nu'} = \Lambda_{\mu'}{}^\alpha \Lambda_{\nu'}{}^\beta p_{\alpha,\beta}$, where the primed indices are in the polar basis. We tackle one component only, since the remaining ones require essentially the same calculation, so it's just tedious work ...

$$\begin{aligned}
p_{r,r} &= \Lambda_r{}^\alpha \Lambda_r{}^\beta p_{\alpha,\beta} \\
&= \Lambda_r{}^x \Lambda_r{}^x p_{x,x} + \Lambda_r{}^x \Lambda_r{}^y p_{x,y} + \Lambda_r{}^y \Lambda_r{}^x p_{y,x} + \Lambda_r{}^y \Lambda_r{}^y p_{y,y} \\
&= \cos^2\theta(2r\cos\theta) + 3\cos\theta\sin\theta + 3\sin\theta\cos\theta + \sin^2\theta(2r\sin\theta) \\
&= 2r\left(\cos^3\theta + \sin^3\theta\right) + 3\sin(2\theta).
\end{aligned}$$

The remaining three components, $\{p_{r,\theta},\ p_{\theta,r},\ p_{\theta,\theta}\}$, are furnished by an identical calculation. □

Solution to (c). We will calculate the (r, r) component (again, the rest just follows trivially). Before starting the calculation, however, we need to express the one-form $\tilde{p}$ in the polar basis:

$$\begin{aligned}
p_r &= \Lambda_r{}^x\, p_x + \Lambda_r{}^y\, p_y \\
&= \cos\theta\left(r^2\cos^2\theta + 3r\sin\theta\right) + \sin\theta\left(r^2\sin^2\theta + 3\cos\theta\right) \\
&= r^2(\cos^3\theta + \sin^3\theta) + 3\sin\theta\cos\theta(r+1).
\end{aligned}$$

$$\begin{aligned}
p_\theta &= \Lambda_\theta{}^x\, p_x + \Lambda_\theta{}^y\, p_y \\
&= -r\sin\theta\left(r^2\cos^2\theta + 3r\sin\theta\right) + r\cos\theta\left(r^2\sin^2\theta + 3\cos\theta\right) \\
&= 3r(\cos^2\theta - r\sin^2\theta) + r^3(\sin^2\theta\cos\theta - \cos^2\theta\sin\theta).
\end{aligned}$$

Now,

$$\begin{aligned}
p_{r;r} &= p_{r,r} - p_\mu \Gamma^\mu{}_{rr} \\
&= p_{r,r} - p_r\Gamma^r{}_{rr} - p_\theta\Gamma^\theta{}_{rr} \\
&= 2r\left(\cos^3\theta + \sin^3\theta\right) + 3\sin(2\theta) - \left(r^2(\cos^3\theta + \sin^3\theta) + 3\sin\theta\cos\theta(r+1)\right)\cdot 0 \\
&\quad - \left(3r(\cos^2\theta - r\sin^2\theta) + r^3(\sin^2\theta\cos\theta - \cos^2\theta\sin\theta\right)\cdot 0 \\
&= 2r\left(\cos^3\theta + \sin^3\theta\right) + 3\sin(2\theta) \\
&= p_{r,r}.
\end{aligned}$$

The remaining calculations are identical (and even more tedious, with the non-vanishing Christoffel symbols!). □

Problem 5 (Exercise 5.14 (Schutz)). *For the tensor whose polar components are*

$$(A^{rr} = r^2,\ A^{r\theta} = r\sin\theta, A^{\theta r} = r\cos\theta,\ A^{\theta\theta} = \tan\theta),$$

compute

$$\nabla_\beta A^{\mu\nu} = A^{\mu\nu}{}_{,\beta} + A^{\alpha\nu}\Gamma^\mu{}_{\alpha\beta} + A^{\mu\alpha}\Gamma^\nu{}_{\alpha\beta}$$

in polars for all possible indices.

Solution. I'll do one calculation for $\beta = r$ and one for $\beta = \theta$, since, again, the remaining calculations follow trivially …For $\beta = r$,

$$\begin{aligned}
\nabla_r A^{rr} &= A^{rr}{}_{,r} + A^{\alpha r}\Gamma^r{}_{\alpha r} + A^{r\alpha}\Gamma^r{}_{\alpha r} \\
&= A^{rr}{}_{,r} + A^{rr}\Gamma^r{}_{rr} + A^{\theta r}\Gamma^r{}_{\theta r} + A^{rr}\Gamma^r{}_{rr} + A^{r\theta}\Gamma^r{}_{\theta r} \\
&= 2r + r^2\cdot 0 + r\cos\theta\cdot 0 + r^2\cdot 0 + r\sin\theta\cdot 0 \\
&= 2r.
\end{aligned}$$

And similarly for $\nabla_r A^{r\theta}$, $\nabla_r A^{\theta r}$, and $\nabla_r A^{\theta\theta}$.

Now for $\beta = \theta$,

$$\begin{aligned}
\nabla_\theta A^{r\theta} &= A^{r\theta}{}_{,\theta} + A^{\alpha\theta}\Gamma^r{}_{\alpha\theta} + A^{r\alpha}\Gamma^\theta{}_{\alpha\theta} \\
&= A^{r\theta}{}_{,\theta} + A^{r\theta}\Gamma^r{}_{r\theta} + A^{\theta\theta}\Gamma^r{}_{\theta\theta} + A^{rr}\Gamma^\theta{}_{r\theta} + A^{r\theta}\Gamma^\theta{}_{\theta\theta} \\
&= r\cos\theta + r\sin\theta\cdot 0 + \tan\theta\cdot(-r) + r^2\cdot\frac{1}{r} + r\sin\theta\cdot 0 \\
&= r(\cos\theta - \tan\theta + 1).
\end{aligned}$$

And similarly for $\nabla_\theta A^{rr}$, $\nabla_\theta A^{\theta r}$, and $\nabla_\theta A^{\theta\theta}$. □

Problem 6 (Exercise 5.22 (Schutz)). *Show that if* $U^\alpha \nabla_\alpha V^\beta = W^\beta$*, then* $U^\alpha \nabla_\alpha V_\beta = W_\beta$.

Beweis. Consider $U^\alpha \nabla_\alpha V^\beta = W^\beta$, and multiply both sides of the equation by the metric tensor $g_{\alpha\beta}$:

$$\begin{aligned} g_{\gamma\beta} U^\alpha \nabla_\alpha V^\beta &= \overbrace{g_{\gamma\beta} W^\beta}^{=W_\gamma} \\ U^\alpha \nabla_\alpha (\underbrace{g_{\gamma\beta} V^\beta}_{=V_\gamma}) - V^\beta U^\alpha \underbrace{\nabla_\alpha g_{\gamma\beta}}_{=0} &= W_\gamma \\ U^\alpha \nabla_\alpha V_\gamma &= W_\gamma \\ U^\alpha \nabla_\alpha V_\beta &= W_\beta. && \text{(relabeling } \gamma \leftrightarrow \beta) \end{aligned}$$

On the second equality the quantity $\nabla_\alpha g_{\gamma\beta}$ vanishes because the Levi-Civita connection ∇ is, by definition, compatible with the metric tensor. □