Geometry of General Relativity
Workshop 2 Hand-In

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Problem (**WS2 Problem 6**). *Consider a $(2,2)$ tensor T over V, with $\lambda, \mu \in V^*$ and $X, Y \in V$. Let $\{e_a\}$ be a basis of V and $\{f^a\}$ its dual basis.*

- ***a)*** *Write down $T(\lambda, \mu, X, Y)$ in terms of the components of T, λ, μ, X, Y. Hence prove that $\{e_a \otimes e_b \otimes f^c \otimes f^d\}$ is a basis for type $(2,2)$ tensors.*
- ***b)*** *Derive the transformation law for the components of T under a change of basis $f'^a = A^a{}_b f^b$ and $e'_a = (A^{-1})^b{}_a e_b$.*
- ***c)*** *Define a $(2,2)$ tensor by $T(\lambda, \mu, X, Y) = \lambda(X)\mu(Y) - \lambda(Y)\mu(X)$. Find all the contractions of T and express them in terms of the Kronecker delta tensor δ.*

*Solution to **a)***. We have

$$\begin{aligned} T(\lambda, \mu, X, Y) &= T(\lambda_a f^a, \mu_b f^b, X^c e_c, Y^d e_d) \\ &= \lambda_a \mu_b X^c Y^d \, T(f^a, f^b, e_c, e_d) \qquad \text{(By multilinearity of } T\text{)} \\ &= \lambda_a \mu_b X^c Y^d \, T^{ab}{}_{cd}\,. \end{aligned}$$

Now,

$$\begin{aligned} (e_a \otimes e_b \otimes f^c \otimes f^d)(\lambda, \mu, X, Y) &= (e_a \otimes e_b \otimes f^c \otimes f^d)(\lambda_r f^r, \mu_s f^s, X^t e_t, Y^u e_u) \\ &= \lambda_r \mu_s X^t Y^u \, (e_a \otimes e_b \otimes f^c \otimes f^d)(f^r, f^s, e_t, e_u) \\ &= \lambda_r \mu_s X^t Y^u \, f^r(e_a) f^s(e_b) f^c(e_t) f^d(e_u) \\ &= \lambda_r \mu_s X^t Y^u \, \delta^r{}_a \delta^s{}_b \delta^c{}_t \delta^d{}_u \\ &= \lambda_a \mu_b X^c Y^d. \end{aligned}$$

Combining these results we deduce that

$$T(\lambda, \mu, X, Y) = T^{ab}{}_{cd} \, (e_a \otimes e_b \otimes f^c \otimes f^d)(\lambda, \mu, X, Y).$$

In other words, any $(2,2)$ tensor T can be expressed as

$$T = T^{ab}{}_{cd} \, e_a \otimes e_b \otimes f^c \otimes f^d,$$

which tells us exactly that $\{e_a \otimes e_b \otimes f^c \otimes f^d\}$ spans the space of all $(2,2)$ tensors. Linear independence follows immediately, since setting $T = 0$ necessarily implies that the components $T^{ab}{}_{cd}$ vanish. Hence we have shown that $\{e_a \otimes e_b \otimes f^c \otimes f^d\}$ is indeed a basis for $T^2_2(V)$. Victoria!

Solution to **b)**. The transformation law for the components of T under a change of basis is given by

$$\begin{aligned} T'^{ab}{}_{cd} &= T(f'^a, f'^b, e'_c, e'_d) \\ &= T(A^a{}_r f^r, A^b{}_s f^s, (A^{-1})^t{}_c e_t, (A^{-1})^u{}_d e_u) \\ &= A^a{}_r A^b{}_s (A^{-1})^t{}_c (A^{-1})^u{}_d \, T(f^r, f^s, e_t, e_u) && \text{(By multilinearity of } T\text{)} \\ &= A^a{}_r A^b{}_s (A^{-1})^t{}_c (A^{-1})^u{}_d \, T^{rs}{}_{tu} \,. && \boxed{\text{Victoria!}} \end{aligned}$$

Solution to **c)**. We let $C^i_j T$ denote the contraction on T of the i^{th} contravariant slot with the j^{th} covariant slot (e.g. $(C^1_2 T)(\mu, X) = T(f^a, \mu, X, e_a)$). So,

$$\begin{aligned} (C^1_1 T)(\mu, Y) &= T(f^a, \mu, e_a, Y) \\ &= f^a(e_a)\mu(Y) - f^a(Y)\mu(e_a) \\ &= \delta^a{}_a \mu_b f^b(Y^c e_c) - Y^a \mu_a \\ &= n\, Y^c \mu_b f^b(e_c) - Y^a \mu_a && \text{(where } n = \dim V\text{)} \\ &= n\, Y^c \mu_b\, \delta^b{}_c - Y^a \mu_a \\ &= n\, Y^c \mu_c - Y^a \mu_a \\ &= n\, Y^a \mu_a - Y^a \mu_a && \text{(Relabelling the dummy indices)} \\ &= (n-1)\, Y^a \mu_a . \end{aligned}$$

★ Question to be discussed on next workshop! ★

The calculation above was done using the definition of a contraction that we discussed in lecture and that appears on our course notes. However, I get a slightly different result if I proceed as follows (we can discuss on the next workshop what I'm doing wrong):

Firstly,

$$\begin{aligned} T(\lambda, \mu, X, Y) &= T(\lambda_a f^a, \mu_b f^b, X^c e_c, Y^d e_d) \\ &= \lambda_a \mu_b X^c Y^d \, T^{ab}{}_{cd} \\ &= \lambda_a \mu_b X^c Y^d \, T(f^a, f^b, e_c, e_d) \\ &= \lambda_a \mu_b X^c Y^d \left(f^a(e_c) f^b(e_d) - f^a(e_d) f^b(e_c) \right) \\ &= \lambda_a \mu_b X^c Y^d \left(\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c \right) \end{aligned}$$

Then contracting on, say, the first and third components,

$$\begin{aligned} (C^1_1 T)(\mu, Y) &= \lambda_a X^a \mu_b Y^d \, T^{ab}{}_{ad} \\ &= \lambda_a X^a \mu_b Y^d \left(\delta^a{}_a \delta^b{}_d - \delta^a{}_d \delta^b{}_a \right) \\ &= \lambda_a X^a \mu_b Y^d \left(n \delta^b{}_d - \delta^b{}_d \right) \\ &= \lambda_a X^a \mu_b Y^d \delta^b{}_d (n-1) \\ &= \lambda_a X^a \mu_b Y^b (n-1). \end{aligned}$$

As you can see, there is an extra $\lambda_a X^a$ term in this result. I can tell that it's because, unlike in the lecture notes, on this procedure I didn't set the components of the slots being contracted to be equal to 1 ... but I don't know which method yields the right contraction!

Similarly (using the procedure taught in lecture),

$$\begin{aligned}
(C^1_2 T)(\mu, X) &= T(f^a, \mu, X, e_a) \\
&= f^a(X)\mu(e_a) - f^a(e_a)\mu(X) \\
&= X^a \mu_a - \delta^a{}_a X^d \mu_c \delta^c{}_d \\
&= X^a \mu_a - n X^c \mu_c \\
&= X^a \mu_a (1 - n); \\
(C^2_1 T)(\lambda, Y) &= T(\lambda, f^a, e_a, Y) \\
&= \lambda(e_a) f^a(Y) - \lambda(Y) f^a(e_a) \\
&= \lambda_a Y^a - \lambda_c Y^d \delta^c{}_d \delta^a{}_a \\
&= \lambda_a Y^a - \lambda_c Y^c n \\
&= \lambda_a Y^a (1 - n); \\
(C^2_2 T)(\lambda, X) &= T(\lambda, f^a, X, e_a) \\
&= \lambda(X) f^a(e_a) - \lambda(e_a) f^a(X) \\
&= \lambda_c X^d \delta^c{}_d \delta^a{}_a - \lambda_a X^a \\
&= \lambda_c X^c n - \lambda_a X^a \\
&= \lambda_a X^a (n - 1).
\end{aligned}$$

$\boxed{\text{Victoria!}}$