Numerical Relativity

A Brief Overview

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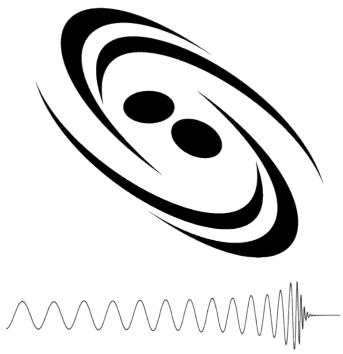

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Abstract

The Einstein Field Equations (EFE's) are nonlinear, coupled, partial differential equations that describe the relation between the geometry of a region of spacetime and its matter/energy content. A severe complication is that, with the exception of a few idealized cases characterized by high degrees of symmetry, the EFE's simply cannot be obtained analytically; we need a computer to get the job done. That being said, computers (for better or worse) lack a sense of humor; if you feed them nonsense, they will calculate nonsense. Therefore, in order to find solutions to realistic (asymmetric) spacetimes, we need to be able to somehow prescribe the right numerical recipe to the machine. This recipe comes in several different flavors (formalisms), of which we shall present two of the most widely known variants: the ADM (aka ADMY) formalism and the BSSN (aka BSSNOK) formalism. We then close out this overview by briefly mentioning some further considerations (which are huge topics on their own right!) such as the initial data problem, gauge choice, and potential applications to problems in relativistic cosmology.

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ADM Formalism of Numerical Relativity

In *General Relativity* (GR), the evolution of the gravitational field can be posed as an *initial value problem* (or *Cauchy problem*) with constraints (see the original breakthrough paper [Fourès-Bruhat (1952)]. In this formalism, the *Einstein Field Equations* (EFE's) are (roughly) determined in two steps:

i) specify g_{ab} and $\partial_t g_{ab}$ (actually, it will be related quantities) for some initial 3D spacelike hypersurface Σ_0 that has a fixed time coordinate $x^0 = t =$ constant;

ii) provided that we can obtain expressions for second-time derivatives of the 4-metric g_{ab} at all points on the hypersurface from the EFE's, we then integrate forward in time the metric quantities from step i).

However, even though this seems like a straightforward proposal, we immediately face the problem that in GR, unlike in other standard dynamical systems, space and time are two sides of the same coin; these two entities are treated on equal footing. This makes the space-time split that we are so accostumed to seeing in non-relativistic Cauchy problems much more complicated. A further complication is the *constraints* of the system; while the EFE's consists of a total of ten differential equations $^{(4)}G^{ab} = 8\pi T^{ab}$,[1] [2] [3] not all of them are evolution equations:

[1]We adopt the usual convention in which 4D objects are distinguished from their 3D counterparts by using a superscript (4). There are exceptions to this rule, where 3D objects are denoted by different symbols than their 4D cousins (examples are T_{ab} and g_{ab}, whose 3D counterparts are denoted S_{ab} and γ_{ab}, respectively; these exceptions are merely a matter of convention, of course, but they are widely used in the numerical relativity literature).

[2]We will use geometric units ($G = c = 1$) throughout.

[3]We also use standard index convention, where the "early" ($a - h$) and "late" ($o - z$) Latin letters are used for 4D spacetime indices that run from 0 to 3, whereas the letters $i - n$ are reserved for 3D spatial indices that run from 1 to 3. Lowercase Greek letters are reserved for components in a chosen basis (see Wald (1984) for reference).

By the (contracted) *Bianchi identities*,

$$\begin{aligned} 0 &= \nabla_b {}^{(4)}G^{ab} \\ &= \partial_0 {}^{(4)}G^{a0} + \partial_i {}^{(4)}G^{ai} + {}^{(4)}G^{bc}\, {}^{(4)}\Gamma^a{}_{bc} + {}^{(4)}G^{ab}\, {}^{(4)}\Gamma^c{}_{bc}, \end{aligned}$$

we get

$$\partial_t {}^{(4)}G^{a0} = -\partial_i {}^{(4)}G^{ai} - {}^{(4)}G^{bc}\, {}^{(4)}\Gamma^a{}_{bc} - {}^{(4)}G^{ab}\, {}^{(4)}\Gamma^c{}_{bc}. \tag{1.1}$$

Because there is no third-time derivatives (or higher) on the RHS of (1.1), this implies that there are no second-time derivatives contained in ${}^{(4)}G^{a0}$, and thus the four equations

$${}^{(4)}G^{a0} = 8\pi T^{a0} \tag{1.2}$$

do not yield any information whatsoever on how the fields evolve in time. Instead, they function as four *constraints* that must be satisfied from the onset on the initial hypersurface at $x^0 = t$ (and remain satisfied throughout the entire evolution!) if we are to have a physically-meaningful system. Thus, we can see that the only true dynamical (*evolution*) equations are encoded in the remaining six field equations

$${}^{(4)}G^{ij} = 8\pi T^{ij}. \tag{1.3}$$

We will see later on that certain projections of (1.2) and (1.3) onto the hypersurfaces will indeed yield the desired constraint and evolution equations of the system.

Whence, according to our discussion above, our first order of business is to somehow find a way to define the role played by space and time, as (somewhat) separate entities. Of course, by this we do not mean "forget about GR and go back to Newtonian/Galilean gravity!" It turns out that there is a special class of spacetimes, known as *globally hyperbolic* spacetimes, that will allow us this sought-after time/space split. First recall that a *Cauchy surface* is a spacelike hypersurface Σ embedded in an ambient manifold $\mathcal{M}$ such that each causal curve without endpoint in $\mathcal{M}$ intersects Σ exactly once. An equivalent way of saying this is that a Cauchy surface for a spacetime $\mathcal{M}$ is an *achronal* subspace $\Sigma \subset M$ (i.e., a subspace Σ in which no two points are timelike-related) which is transversed by every inextendible causal curve in $\mathcal{M}$. Now we properly define the concept of global hyperbolicity:

Definition 1. *A spacetime $\mathcal{M}$ is said to be* ***globally hyperbolic*** *if it admits a Cauchy surface. Equivalently, $\mathcal{M}$ is globally hyperbolic if it satisfies the strongly causal condition (i.e., if every $p \in \mathcal{M}$ has arbitrarily small neighborhoods U in which every every causal curve with endpoints in U is entirely contained in U) and if the "causal diamonds" $J^+(p) \cap J^-(q)$ are compact for all $p, q \in \mathcal{M}$.* [4]

[4] Here we used standard notation, where $J^+(p) = \{q \in \mathcal{M} \mid p \leq q\}$ and $J^-(p) = \{q \in \mathcal{M} \mid q \leq p\}$ are the *causal future* and *causal past*, respectively, of $p \in \mathcal{M}$.

The notion of global hyperbolicity is a crucial feature in Lorentzian geometry that ensures the existence of maximal causal geodesic segments. Physically, this condition is closely connected to the issue of classical determinism and the strong cosmic censorship conjecture. [Ringström (2009)] Even though this is by no means a condition satisfied a priori by all spacetimes, the 3+1 formalism assumes that all physically reasonable spacetimes are of this type. This assumption is justified by the desire to have "nice" chronological/causal features in our spacetime (i.e., no *grandfather paradox* or any similar pathological behavior). Moreover, the use of global hyperbolicity allows us to foliate our full 4D spacetime $\mathcal{M}$ in such a way that we can stack 3D spacelike Cauchy slices along a universal time axis, by virtue of $\mathcal{M}$ having topology $\Sigma \times \mathbb{R}$. This is certainly not the only way to foliate $\mathcal{M}$, but it is the most suitable one for the 3+1 formalism.

1.1 | ADM Variables & Adapted Coordinates

Given the foliation granted by the globally hyperbolic condition described above, we can now determine the geometry between two adjacent hypersurfaces Σ_t and $\Sigma_{t+\mathrm{d}t}$ from just three basic ingredients:

i) The *3D metric* γ_{ij} (metric induced on Σ: $\gamma_{ab} \equiv \Phi^* g_{ab}$, where $\Phi\colon \Sigma \hookrightarrow \mathcal{M}$ is the embedding of Σ into $\mathcal{M}$) that measures proper distances within the hypersurface itself:

$$dl^2 = \gamma_{ij}\, \mathrm{d}x^i \mathrm{d}x^j .$$

The hypersurface is then said to be

- $\underbrace{\textit{spacelike} \iff \gamma_{ab} \text{ is positive definite; i.e., it has signature } (+,+,+);}_{\text{our case}}$
- *timelike* $\iff$ γ_{ab} is Lorentzian; i.e., it has signature $(-,+,+)$;
- *null* $\iff$ γ_{ab} is degenerate; i.e., it has signature $(0,+,+)$.

(We will shortly justify why we express the spatial metric both as 3D object (γ_{ij}) and a 4D object (γ_{ab}).)

ii) The *lapse of proper time*, α, between the hypersurfaces, as measured by *Eulerian* (i.e, *normal*) observers: [5]

$$\mathrm{d}\tau = \alpha(t, x^i)\mathrm{d}t$$

Note that α, which is known as the *lapse function*, is sometimes denoted as N by other references, e.g., [Gourgoulhon (2012)], [Misner et al. (1973)].

[5] Such observers have worldlines with tangent vectors that are orthogonal/normal to the spatial hypersurfaces.

iii) The relative velocity β^i between the Eulerian observers:

$$x^i_{t+\mathrm{d}t} = x^i_t - \beta^i(t, x^i)\mathrm{d}t.$$

This 3-vector β^i measures how much the coordinates are shifted as we move from one slice to the next, and it is therefore conventionally named as the *shift vector*. (It is also denoted N^i in the literature.)

Note that, as alluded to earlier, the foliation of $\mathcal{M}$ is not unique, and neither is the coordinates shift; α determines "how much slicing" is to be done, while β^i dictates how the spatial coordinates change from slice to slice. In fact, the latitude to choose a lapse function and shift vector demonstrates the gauge freedom that is inherent to the formulation of GR, a *covariant* theory.

From the *universal time function* t (given by the foliation), we can define the future-pointing timelike *unit normal* n^a to the slice Σ to be [6]

$$n^a \equiv -\alpha \nabla^a t. \tag{1.4}$$

(We think of n^a as the 4-velocity of an Eulerian observer.) With this defined, we can see that the three scalar quantities that yield the spatial components of the shift vector, β^i, are given by

$$\beta^i = -\alpha \left(\vec{n} \cdot \vec{\nabla} x^i\right). \tag{1.5}$$

These three scalar quantities can then be used to form a full 4-vector β^a (orthogonal to n^a, by construction) which, in the adapted 3+1 coordinates we are about to introduce, will have components $\beta^\mu = (0, \beta^i)$. Equipped with the unit normal and the shift vector, we can also define a *time vector* t^a given by

$$t^a \equiv \alpha n^a + \beta^a, \tag{1.6}$$

which is nothing but the vector tangent to the *time curves*, i.e., the congruence of lines of constant spatial coordinates x^i. The importance of this vector lies in the ability to *Lie-drag* the hypersurfaces along it: the spatial basis vectors $e^a_{(i)}$ which are tangent to a particular slice Σ_t (i.e., that satisfy $\nabla_a t \, e^a_{(i)} = 0$) are Lie-dragged along t^a,

$$\mathcal{L}_{\vec{t}} e^a_{(i)} = 0.$$

Remark 1. *The* normal evolution vector $m^a \equiv \alpha n^a$ *can also be used to Lie drag the hypersurfaces (see, e.g., [Gourgoulhon (2012)])*

Now, since t^a is aligned with the basis vector $e^a_{(0)}$ while all remaining (spatial) coordinates remain constant along t^a, we get the basis components

$$t^\mu = e^\mu_{(0)} = \delta^\mu{}_0 = (1, 0, 0, 0). \tag{1.7}$$

[6] The minus sign is chosen to ensure that n^a is always future-pointing.

Thus Lie derivatives along the time curves t^a will reduce to ordinary partial derivatives with respect to t; i.e., $\mathcal{L}_{\vec{t}} = \partial_t$ (we will use this later!). Similarly, a straightforward derivation shows that in these adapted coordinates we have

$$\beta^\mu = (0, \beta^i) \tag{1.8a}$$
$$n^\mu = (\alpha^{-1}, -\alpha^{-1}\beta^i) \tag{1.8b}$$
$$n_\mu = (-\alpha, 0, 0, 0). \tag{1.8c}$$

These quantities will appear in the 3+1 coordinates expression of the spatial metric γ_{ab}. To show this, we first need to introduce the *spatial projection operator*[7]

$$P^a_{\ b} \equiv \delta^a_{\ b} + n^a n_b, \tag{1.9}$$

which we then apply (twice; once per index) to the spacetime metric g_{ab} to get

$$P_a^{\ c} P_b^{\ d} g_{cd} = (\delta^a_{\ b} + n^a n_b)(\delta^a_{\ b} + n^a n_b)\, g_{cd} = g_{ab} + n_a n_b.$$

Since the induced metric is merely a projection of the spacetime metric onto the hypersurface, we have found an expression for our *spatial metric*:

$$\gamma_{ab} = g_{ab} + n_a n_b, \tag{1.10}$$

and similarly the *inverse spatial metric*

$$\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b. \tag{1.11}$$

Hence, γ_{ab} is a projection tensor that discards components of 4D geometric objects that lie along n^a; we use it to calculate distances between points that belong to the same spatial hypersurface. We can think of γ_{ab} as first computing four-dimensional distance (with g_{ab}), and then eliminating (with $n_a n_b$) the timelike contribution to the 4D distance calculation.

Remark 2. *From (1.10) we see that, if we raise only one index of the spatial metric γ_{ab},*

$$\gamma^a_{\ b} = g^a_{\ b} + n^a n_b = \delta^a_{\ b} + n^a n_b,$$

we find out that our projection operator is merely the spatial metric with one raised index

$$P^a_{\ b} = \gamma^a_{\ b}.$$

Therefore, from now on we will exclusively use $\gamma^a_{\ b}$ to denote the spatial projection operator.

[7] This operator projects a 4D tensor onto a spatial slice. For instance, if we take an arbitrary 4-vector v^a, and hit it with the projection operator,

$$\underbrace{v^a}_{\text{arbitrary, 4D}} \xmapsto{P^a_{\ b}} \underbrace{P^a_{\ b} v^b}_{\text{purely spatial}}$$

we get a purely spatial object that lies entirely on a hypersurface.

Also from (1.10), and from (1.8c), we can see that

$$\gamma_{ij} = g_{ij} + \underbrace{n_i n_j}_{=0} = g_{ij}, \tag{1.12}$$

so that the spatial metric on Σ is nothing but the spatial part of the spacetime 4-metric g_{ab}. Note also that, even though the covariant components do not necessarily vanish ($\gamma_{0\mu} = g_{0\mu} + n_0 n_\mu = g_{0\mu} + n_0 n_0 = g_{0\mu} + \alpha^2 \neq 0$, in general), any contribution to the timelike direction can be safely ignored since $n^a \gamma_{ab} = 0$. On the other hand, timelike components of spatial contravariant tensors do vanish, so we must have $\gamma^{a0} = 0$. Therefore, from (1.11), we get the components of the inverse spacetime metric in these adapted coordinates:

$$\begin{aligned}
&g^{ab} = \gamma^{ab} - n^a n^b \\
&g^{0a} = -n^0 n^a \implies g^{00} = -\alpha^{-2} \quad \& \quad g^{0i} = \alpha^{-2}\beta^i \\
&g^{ij} = \gamma^{ij} - n^i n^j = \gamma^{ij} - (-\alpha^{-1}\beta^i)(-\alpha^{-1}\beta^j) = \gamma^{ij} - \alpha^{-2}\beta^i\beta^j.
\end{aligned}$$

In matrix form,

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}. \tag{1.13}$$

Now, by the condition $g^{ab} g_{bc} = \delta^a{}_c$, we can invert (1.13) to write the spacetime metric in 3+1 coordinates:

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}. \tag{1.14}$$

The covariant components β_i shown above come from lowering with the spatial metric, i.e., $\beta_i = \gamma_{ik}\beta^k$. We will always use the spatial metric to raise/lower indices of spatial objects, because γ_{ij} and γ^{ij} are inverses of each other in the adapted coordinates:

$$\begin{aligned}
\gamma^{ik}\gamma_{kj} &= (g^{ik} + n^i n^k)(g_{kj} + n_k n_j) \\
&= g^{ik} g_{kj} + g^{ik} n_k n_j + n^i n^k g_{kj} + n^i n^k n_k n_j \\
&= \delta^i{}_j + n^i \underbrace{n_j}_{=0} + n^i \underbrace{n_j}_{=0} - n^i \underbrace{n_j}_{=0} = \delta^i{}_j.
\end{aligned}$$

Equation (1.14) shows that the line element of the full spacetime metric in 3+1 coordinates is given by

$$ds^2 = \left(-\alpha^2 + \beta_i\beta^i\right) \mathrm{d}t^2 + 2\beta_i \, \mathrm{d}t\mathrm{d}x^i + \gamma_{ij} \, \mathrm{d}x^i \mathrm{d}x^j. \tag{1.15}$$

1.2 | ADM Evolution & Constraints

Using the projection operator (1.9) (which we now know is just $\gamma^a{}_b$), we can define the *extrinsic curvature tensor* to be [8]

$$K_{ab} = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d. \tag{1.16}$$

This quantity measures how much n^a varies as we move from point to point on a particular slice Σ, and in doing so it describes how Σ is embedded in $\mathcal{M}$. Expanding (1.16) we get

$$K_{ab} = -\nabla_a n_b - n_a n^c \nabla_c n_b \tag{1.17}$$

and, moreover, a straightforward calculation shows that

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ab}. \tag{1.18}$$

From the latter and from the time vector (1.6) we get a natural time derivative of the metric:

$$\begin{aligned} K_{ab} = -\frac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ab} &= -\frac{1}{2}\mathcal{L}_{\frac{\vec{t}-\vec{\beta}}{\alpha}}\gamma_{ab} \\ &= -\frac{1}{2\alpha}\left(\mathcal{L}_{\vec{t}}\gamma_{ab} - \mathcal{L}_{\vec{\beta}}\gamma_{ab}\right) \\ &= -\frac{1}{2\alpha}\left(\partial_t\gamma_{ab} - \mathcal{L}_{\vec{\beta}}\gamma_{ab}\right), \end{aligned} \tag{1.19}$$

where on the last line we used the fact that, in the adapted coordinates, $\mathcal{L}_{\vec{t}}$ reduces to ∂_t. Thus, expanding the Lie derivative on (1.19) (and dropping timelike components), we have an *evolution equation of the spatial metric*:

$$\partial_t\gamma_{ij} = 2D_{(i}\beta_{j)} - 2\alpha K_{ij}. \tag{1.20}$$

Here D is the affine connection compatible with the 3D metric (i.e., $D_c\gamma_{ab} = 0$), which is furnished by projecting *all* indices present in a 4D covariant derivative ∇ onto Σ; that is, for a $\binom{b}{c}$ tensor $\boldsymbol{T}$,

$$D_a T^{i_1 \ldots i_b}{}_{j_1 \ldots j_c} = \gamma_a{}^d \gamma^{i_1}{}_{k_1} \ldots \gamma^{i_b}{}_{k_b} \gamma_{j_1}{}^{\ell_1} \ldots \gamma_{j_c}{}^{\ell_c} \nabla_d T^{k_1 \ldots k_b}{}_{\ell_1 \ldots \ell_c}. \tag{1.21}$$

The evolution equation for the metric was not so difficult to derive; however the remaining evolution equation (of K_{ij}) and the constraint equations need a bit more work. Given the brevity of this presentation it would not be possible to show derivations in great detail, but nevertheless it should (in theory at least) entice the interested reader to derive all the results from scratch (this is the only true way to learn anyhow!).

[8] The minus sign is merely a convention in the NR community. In the cosmology community the sign is usually positive!

We need to find a way to formulate the EFE's in 3+1 form, which is a task we can accomplish by considering the following projections:

$$n^a n^b ({}^{(4)}G_{ab} - 8\pi T_{ab}) = 0; \tag{1.22a}$$
$$\gamma_c{}^a \gamma_d{}^b ({}^{(4)}G_{ab} - 8\pi T_{ab}) = 0; \tag{1.22b}$$
$$\gamma_c{}^b \left[n^a ({}^{(4)}G_{ab} - 8\pi T_{ab}) \right] = 0. \tag{1.22c}$$

(Note that all other projections vanish identically thanks to the symmetries of the Riemann tensor.) These equations come about from projections of the 4D Riemann tensor,

$$\gamma_a{}^e \gamma_b{}^f \gamma_c{}^g \gamma_d{}^h \, {}^{(4)}R_{efgh} = R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{cb}; \tag{1.23a}$$
$$\gamma_a{}^e \gamma_b{}^f \gamma_c{}^g n^h \, {}^{(4)}R_{efgh} = D_b K_{ac} - D_a K_{bc}; \tag{1.23b}$$
$$\gamma_a{}^q \gamma_b{}^r n^c n^d \, {}^{(4)}R_{qcrd} = \mathcal{L}_{\vec{n}} K_{ab} + \frac{1}{\alpha} D_a D_b \alpha + K_b{}^c K_{ac}. \tag{1.23c}$$

These are the so called *Gauss-Codazzi*, *Codazzi-Mainardi*, and *Ricci* equations, respectively. Note how Eqs. (1.23a) and (1.23b) depend exclusively on the 3D metric and its extrinsic curvature, as well as their spatial derivatives; they will give rise to the constraint equations. On the other hand, the first term on the RHS of (1.23c) hints that this expression will yield an evolution equation for K_{ab}.

In fact, expanding (1.23a) and doing some algebra we end up with

$$R + K^2 - K^{ab}K_{ab} = 16\pi\rho, \tag{1.24}$$

where ρ is the *total energy density as measured by a Eulerian observer* n^a,

$$\rho \equiv n^a n^b T_{ab}. \tag{1.25}$$

Dropping timelike components, we have the *Hamiltonian constraint*, which must be satisfied on each slice of the foliation,

$$R + K^2 - K^{ij}K_{ij} = 16\pi\rho. \tag{1.26}$$

Similarly, from (1.23b) we get

$$D_b K_a{}^b - D_a K = 8\pi S_a, \tag{1.27}$$

where we used the *momentum density* S_a *as measured by a Eulerian observer* n^a,

$$S_a \equiv -\gamma_a{}^b n^c T_{bc}. \tag{1.28}$$

Dropping timelike components and raising indices, we write (1.27) in its final form

$$D_j \left(K^{ij} - \gamma^{ij} K \right) = 8\pi S^i, \tag{1.29}$$

which is with the so-called *momentum constraints* that must also be satisfied on each hypersurface. Lastly, from (1.23c), some very messy algebra yields

$$\begin{aligned}\partial_t K_{ab} = \alpha(R_{ab} + KK_{ab} - 2K_{ac}K^c{}_b) - 8\pi\alpha\left(S_{ab} - \frac{1}{2}\gamma_{ab}(S-\rho)\right)\\ - D_aD_b\alpha + \mathcal{L}_{\vec{\beta}}K_{ab},\end{aligned} \tag{1.30}$$

where S_{ab} is the *spatial stress*, given from a projection of the 4D energy-momentum tensor T_{ab},

$$S_{ab} \equiv \gamma_a{}^c\gamma_b{}^d T_{cd}, \tag{1.31}$$

and S is its *trace*,

$$S \equiv \gamma^{ab}S_{ab} = S^a{}_a. \tag{1.32}$$

Then, since the entire content of spatial tensors is available from their spatial components, we can write our results as

$$\begin{aligned}\partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right)\\ - D_iD_j\alpha + \beta^k D_k K_{ij} + 2K_{k(j}D_{i)}\beta^k.\end{aligned} \tag{1.33}$$

And thus we arrived at the evolution equation for the extrinsic curvature, our last piece of the puzzle.

2

BSSN Formalism of Numerical Relativity

The 3+1 ADM (à la York) decomposition of the EFE's presented in Chapter 1 poses a very straightforward, elegant formulation. Unfortunately, however, the EFE's as presented in this form are not quite suitable for numerical implementations. Alas, we need to put in more work before we take a crack at computing anything, since in practice one finds that this form of the 3+1 decomposition results in large instabilities that develop during a computer simulation. This issue is known to be mainly due to the fact that the EFE's in this form are *weakly hyperbolic*, so that they are not *well-posed*.[1] To get around this problem, in an effort to make the EFE's more *strongly hyperbolic* (i.e., more "wave-like"), many modern NR codes utilize the so-called *BSSN* (aka *BSSNOK*) *formalism* which, together with suitable gauge conditions that we will briefly discuss on Chapter 3, does admit a more robust computational formulation of the EFE's.

In place of the ADM data $\{\gamma_{ij}, K_{ij}\}$, the BSSN formalism splits γ_{ij} into a conformal factor χ and a conformally-related metric $\bar{\gamma}_{ij}$, and it also splits K_{ij} into its trace K and a traceless part A_{ij}. Moreover, three coefficients $\bar{\Gamma}^i$ of the conformal metric are introduced as well. Then, it is these variables that are evolved instead of the original ADM physical quantities … Long story short, the dynamical variables that we consider in the BSSN formalism are

$$\{\chi, \bar{\gamma}_{ij}, \bar{A}_{ij}, K, \bar{\Gamma}^i\}. \tag{2.1}$$

We will present each of these quantities and derive their evolution equations in this section. Let us start by by considering a conformal rescaling of the spatial metric of the form

$$\gamma_{ij} = \chi^{-1}\bar{\gamma}_{ij}, \tag{2.2}$$

where χ is some positive scaling factor called the *conformal factor*, and the background auxiliary metric $\bar{\gamma}_{ij}$ is known as the *conformally-related metric* (or simply *conformal metric*). It may seem unclear why

[1] For more on the concept of hyperbolicity (in the numerical sense), see the detailed analysis on [Sarbach et al. (2002)].

we scaled the spatial metric in this way, but let it suffice to say that this "trick" will actually yield a convenient and tractable system for the EFE's. Besides the mathematical convenience that such a conformal rescaling brings about, there is also the fact that conformally-related manifolds form an equivalence class, in which objects share certain geometric features. For example, it can be shown that two strongly-causal Lorentzian metrics $g^{(1)}_{ab}$ and $g^{(2)}_{ab}$ for some manifold $\mathcal{M}$ determine the same future and past sets at all points (events) if and only if the two metrics are globally conformal, i.e., if $g^{(1)}_{ab} = \Psi g^{(2)}_{ab}$, for some smooth function $\Psi \in C^{\infty}(\mathcal{M})$. In this case, both spacetimes $(\mathcal{M}, g^{(1)}_{ab})$ and $(\mathcal{M}, g^{(2)}_{ab})$ belong to the same conformal class and share the same causal structure.

A natural choice for a representative object in a conformal equivalence class is a metric $\bar{\gamma}_{ij}$ whose determinant is the same as the determinant of a flat metric f_{ij}, in any general chart. Thus, if we adopt a Cartesian coordinate system, we can always enforce that our conformal representative must have unit determinant, i.e., $\bar{\gamma} = 1$. [2] Plugging this back into (2.2), we get

$$1 = \det \bar{\gamma}_{ij} = \det(\chi\gamma_{ij}) = \chi^3 \det \gamma_{ij} = \chi^3\gamma.$$

This would correspond to the choice $\chi = \gamma^{-1/3}$, so that $\gamma_{ij} = \gamma^{1/3}\bar{\gamma}_{ij}$. Any spatial metric γ_{ij} in this conformal class yields the same value of $\bar{\gamma}_{ij}$. However note that, since the determinant γ is coordinate-dependent, the conformal factor $\chi = \gamma^{-1/3}$ *is not* a scalar field. In fact, $\bar{\gamma}_{ij}$ is not a tensor field, but rather a *tensor density* of *weight* $-2/3$. To get around this issue of tensor densities, we could introduce a *background flat metric* f_{ij} of Riemannian signature $(+,+,+)$, and set $\chi \equiv (\gamma/f)^{-1/3}$, so that χ becomes a scalar field in this manner, and we could then use non-Cartesian coordinates. However, for our purposes of implementing the standard BSSN formalism, it is convenient to stick to Cartesian coordinates; to see the implementation of this extended BSSN formalism (where non-Cartesian coordinates are used), the reader is referred to [Gourgoulhon (2012)].

The conformal factor is one of the dynamical variables in the BSSN approach, and as such we need an evolution equation for it. A straightforward calculation shows that

$$\partial_t\chi = \frac{2}{3}\chi(\alpha K - \partial_i\beta^i) + \beta^i\partial_i\chi, \tag{2.3}$$

where K is the *trace of the extrinsic curvature*,

$$K \equiv g^{ab}K_{ab} = \gamma^{ij}K_{ij} = K^i{}_i. \tag{2.4}$$

(Note that the second equality holds because K_{ab} is a purely spatial object.) Now, before showing the evolution of the conformal metric (another BSSN dynamic variable; c.f. (2.1)), we need to briefly

[2] Here we are using the standard notation $g \equiv \det g_{ab}$, for any metric g_{ab}. (Not to be confused with boldface $\boldsymbol{g}$, which typically denotes the metric tensor g_{ab} itself when using non-index notation, which is more common in the mathematics literature.)

discuss the split of the extrinsic curvature K_{ij} into its trace K and its traceless part A_{ij},

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K. \tag{2.5}$$

Just as we rescaled the spatial metric, in the BSSN formalism we shall also rescale the traceless curvature A_{ij} as [3]

$$\bar{A}_{ij} = \chi A_{ij}, \tag{2.6}$$

which yields

$$\bar{A}_{ij} = \chi K_{ij} - \frac{1}{3}\bar{\gamma}_{ij}K. \tag{2.7}$$

In terms of these rescaled variables, it is straightforward to show that the *evolution of the conformal metric* is given by

$$\partial_t\bar{\gamma}_{ij} = -2\alpha\bar{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k, \tag{2.8}$$

where we had to use the fact that $\bar{\gamma}_{ij}$ is a tensor density of weight $-2/3$ when expanding the Lie derivative $\mathcal{L}_{\vec{\beta}}\bar{\gamma}_{ij}$. (The Lie derivative of some tensor density $\boldsymbol{\tau}$ of weight ω is given by

$$\mathcal{L}_{\vec{x}}\boldsymbol{\tau} = [\mathcal{L}_{\vec{x}}\boldsymbol{\tau}]_{\omega=0} + \omega\boldsymbol{\tau}\,\partial_i x^i,$$

where the first term is the usual Lie derivative we would compute if $\boldsymbol{\tau}$ had zero weight (i.e., if $\boldsymbol{\tau}$ was a *tensor field* rather than a *tensor density*).) Furthermore, a somewhat involved calculation shows that the *evolution of the trace of the extrinsic curvature* is given by

$$\partial_t K = \alpha\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(\rho + S) - D^2\alpha + \beta^i\partial_i K, \tag{2.9}$$

and another (yet even longer) computation yields the *evolution of the conformal traceless curvature*, [4]

$$\begin{aligned}\partial_t\bar{A}_{ij} = {} & [\chi(\alpha R_{ij} - 8\pi\alpha S_{ij} - D_iD_j\alpha)]^{\mathrm{TF}} - \alpha(2\bar{A}_{ik}\bar{A}^k{}_j + \bar{A}_{ij}K) \\ & + \beta^k\partial_k\bar{A}_{ij} + \bar{A}_{ik}\partial_j\beta^k + \bar{A}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{A}_{ij}\partial_k\beta^k.\end{aligned} \tag{2.10}$$

You may notice, however, that there is something off-putting on both (2.9) and (2.10); we have covariant derivatives D of the lapse with respect to the physical metric γ_{ij} as opposed to the BSSN conformal metric $\bar{\gamma}_{ij}$ (moreover, (2.10) also has the 3D Ricci tensor R_{ij} of γ_{ij} appearing in the expression). We correct these problems by introducing the *conformal connection* $\bar{D}$ of $\bar{\gamma}_{ij}$ and writing the Christoffel symbols of D in terms of those of $\bar{D}$,

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} - \frac{1}{2}\chi^{-1}\left(\delta^i{}_k\,\partial_j\chi + \delta^i{}_j\,\partial_k\chi - \bar{\gamma}_{jk}\bar{\gamma}^{i\ell}\,\partial_\ell\chi\right). \tag{2.11}$$

[3] Note, however, that a different scaling for A_{ij} is used when dealing with the initial data problem.

[4] Here we use the notation $[\cdots]^{\mathrm{TF}}$ to denote the trace-free part of whatever object lies inside the brackets (e.g., $[K_{ij}]^{\mathrm{TF}} = A_{ij}$). In general, for a tensor $\boldsymbol{T}$ in a D-dimensional metric $\boldsymbol{g}$, we have $[\boldsymbol{T}]^{\mathrm{TF}} = \boldsymbol{T} - \boldsymbol{g}/D\,\mathrm{Tr}(\boldsymbol{T})$.

Using this relation, a quick calculation shows that

$$D_i D_j \alpha = \bar{D}_i \bar{D}_j \alpha + \frac{1}{2\chi}\left(2\bar{D}_{(i}\chi \bar{D}_{j)}\alpha - \bar{\gamma}_{ij}\,\bar{D}_k\chi\,\bar{D}^k\alpha\right), \tag{2.12}$$

which is to be inserted into both (2.9) and (2.10). Furthermore, we may show (after a very long calculation!) that the Ricci tensor can be split as

$$R_{ij} = \bar{R}_{ij} + R^{\chi}_{ij}, \tag{2.13}$$

where

$$\begin{aligned}\bar{R}_{ij} &= -\frac{1}{2}\bar{\gamma}^{k\ell}\partial_k\partial_\ell\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\,\bar{\Gamma}^k + \bar{\Gamma}^k\,\bar{\Gamma}_{(ij)k} \\ &\quad + \bar{\gamma}^{k\ell}\left(2\,\bar{\Gamma}^m_{k(i}\bar{\Gamma}_{j)m\ell} + \bar{\Gamma}^m_{ik}\,\bar{\Gamma}_{j\ell m}\right) \qquad (2.14a)\\ R^{\chi}_{ij} &= \frac{1}{2}\left(\bar{D}_i\bar{D}_j(\log\chi) + \bar{\gamma}_{ij}\,\bar{D}_k\bar{D}^k(\log\chi)\right) \\ &\quad + \frac{1}{4}\left(\bar{D}_i(\log\chi)\bar{D}_j(\log\chi) - \bar{\gamma}_{ij}\,\bar{D}_k(\log\chi)\bar{D}^k(\log\chi)\right). \qquad (2.14b)\end{aligned}$$

Remark 3. *On (2.14a) we used the conformal coefficients $\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\,\bar{\Gamma}^i_{jk}$. The reason why we want to write $\bar{R}_{ij}$ in this form is because, with the exception of the Laplacian term $\bar{\gamma}^{k\ell}\partial_k\partial_\ell\bar{\gamma}_{ij}$, every other second derivative of the metric $\bar{\gamma}_{ij}$ is being absorbed into first derivatives of $\bar{\Gamma}^i$. This in turns makes the BSSN equations more* hyperbolic *(see, e.g., Sarbach et al. (2002)).*

Speaking of $\bar{\Gamma}^i$, this is the only remaining BSSN dynamic variable for which we need an evolution equation (c.f. (2.1)). Here we simply present it (this is another long derivation; try it!):

$$\begin{aligned}\partial_t\bar{\Gamma}^i &= -2\alpha\left(\frac{3}{2\chi}\bar{A}^{ij}\,\bar{D}_j\chi + \frac{2}{3}\bar{D}^iK + 8\pi\bar{S}^i - \bar{\Gamma}^i_{jk}\bar{A}^{jk}\right) - 2\bar{A}^{ij}\bar{D}_j\alpha \qquad (2.15)\\ &\quad + \beta^j\partial_j\bar{\Gamma}^i + \bar{\gamma}^{jk}\partial_j\partial_k\beta^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{ij}\partial_j\partial_k\beta^k.\end{aligned}$$

Lastly, we close out this section by writing the constraints in BSSN variables:

$$\bar{R} + 2\,\bar{D}^2(\log\chi) - \frac{1}{2}\bar{D}_k(\log\chi)\bar{D}^k(\log\chi) + \frac{4}{3\chi}K^2 - \frac{1}{\chi}\bar{A}_{ij}\bar{A}^{ij} = 16\pi\bar{\rho} \tag{2.16}$$

$$\bar{D}_j\bar{A}^{ij} - \frac{3}{2\chi}\bar{A}^{ij}\,\bar{D}_j\chi - \frac{2}{3}\bar{D}^iK = 8\pi\bar{S}^i, \tag{2.17}$$

where we used the rescaling $\bar{\rho}^i \equiv \chi^{-1}\rho$ and $\bar{S}^i \equiv \chi^{-1}S^i$. Moreover, we used the transformation law for the spatial Ricci scalar R,

$$R = \chi\bar{R} + 2\,\chi\bar{D}^2(\log\chi) - \frac{1}{2}\,\chi\bar{D}_k(\log\chi)\bar{D}^k(\log\chi). \tag{2.18}$$

Equations (2.16) and (2.17) are the Hamiltonian and momentum constraints, respectively, in BSSN variables.

That was a very compact presentation of the BSSN formalism of numerical general relativity. Admittedly, this formalism is not nearly as intuitive and straightforward as the ADM alternative that we presented on Chapter 1, but it is nevertheless a much more robust formulation (on the numerics side of things). This is a running theme in physics (and science in general): analytical and numerical implementations rarely play fair ball with each other. The ADM formalism is important for historical (and pedagogical) reasons, but it is nearly useless for practical purposes. We remark, however, that BSSN is not by any means the only modern successful approach to numerical relativistic studies; other flourishing alternatives such as the *Generalized Harmonic Coordinates with Constraint Damping* (GHCD) (Gundlach et al. (2005), Pretorius (2005a), Pretorius (2005b)) and $Z4$-like formalisms (Alic et al. (2012), Bernuzzi and Hilditch (2010), Bona et al. (2003), Bona and Palenzuela (2004), Sanchis-Gual et al. (2014)) are just as good as (and in some cases even superior to) BSSN.

3

Further Considerations

Having presented two of the main formalisms of numerical relativity, we now turn to a brief discussion of some further considerations that must be taken into account: the initial data problem and gauge choice. Yet another topic that we have made no mention of (nor will we get into, since it is way beyond our scope!) is the actual numerical methods employed in the field of numerical relativity; suffice it to say that the two most widely used methods used by the NR community are the good old fashioned *finite difference methods* (cho (2006)) and *spectral methods* (Grandclément and Novak (2009)). The reader is encouraged to study those references to get up to speed on the numerical side of things. Last but not least, we close out this chapter by making brief mention of some potential applications of NR outside of the usual realm of black holes/neutron stars collisions.

3.1 Initial Data

As we alluded to earlier, we are not free to impose whatever data we like on our initial time slice; the initial data has to be chosen in such a way that the Hamiltonian and momentum constraints are satisfied from the onset. [1] That being said, the constraints are just four equations that remove four degrees of freedom from the total twelve degrees of freedom of the system $\{\gamma_{ij}, K_{ij}\}$. Moreover, there is no a priori preference for which eight of the total data to use as free parameters and which remaining four to use in solving the constraint equations. This *initial data problem* is a difficult subject with a vast literature dedicated to it (see Cook (2000) or any of the standard textbooks, e.g., Chapter 9 of Gourgoulhon (2012)); the two most popular approaches to tackle this problem are known as the *conformal transverse-traceless* (CTT) decomposition and the *conformal thin-sandwich* (CTS) decomposition. Both

[1] In addition, for non-vacuum spacetimes the matter distribution (ρ, S^i) may have constraints of its own.

of these methods offer some hints as to which values to choose as free parameters and which to use as constrained data, although in highly asymmetric settings (which is typical of NR!) the choices can end up being quite arbitrary.

3.2 | Gauge Choice

Even though, in theory (i.e., analytically) all gauge choices should yield the same physical result, as it is often the case numerical simulations do not always play nice. Therefore, in order to achieve a long-term stable simulation, we need to specify the right gauge (choice for α and β^i) and determine how these quantities will evolve in coordinate time. Choosing static (i.e., time-independent) gauges is not a very good idea, since we have no *a priori* knowledge of which functions will serve us better; the best approach is to choose the lapse and shift dynamically as functions of the evolving geometry.

One may naively think, for instance, that setting $\alpha = 1$ would be an ideal choice (certainly, our calculations would simplify quite a bit!) [2]. Unfortunately however, this turns out to be a terrible pick: the acceleration of a normal observer is given in terms of the lapse function as $n^b \nabla_b n_a = D_a \log \alpha$; thus setting $\alpha = 1$ yields a vanishing acceleration of normal observers (hence the choice $\alpha = 1$ is usually referred to as *geodesic slicing*, since Eulerian observers are in free fall). A detailed examination then shows that this almost always leads to a singularity; thus singularity-avoiding techniques such as *maximal slicing* (computationally expensive) or $1 + \log$ *slicing* (lower computational cost) must be employed. The latter is the one that has been adopted by most modern NR codes; it is a generalized hyperbolic slicing condition of Bona-Massó type (Bona et al. (1995)), whose idea is to reduce the lapse in areas where the curvature is particularly strong. In general, the so-called *alpha-driver* condition is given by

$$\partial_t \alpha = -\zeta_1 \, \alpha^{\zeta_2} K + \zeta_3 \beta^i \partial_i \alpha, \tag{3.1}$$

with ζ_i being some positive scalar functions. From this equation we get the $1 + \log$ slicing by fixing $\zeta_1 = 2$ and $\zeta_2 = \zeta_3 = 1$. Similarly, we may also choose a vanishing shift vector ($\beta^i = 0$), so that the coordinates are not shifted as we move from slice to slice. This would also certainly simplify matters, and it is in fact a common gauge choice that works well in certain applications. However, in black-hole spacetime simulations if we use a vanishing β^i the event horizon grows rapidly in coordinate space, due to the normal observers falling in, which causes the computational domain to end up eventually trapped inside the black hole (see [Alcubierre (2008)]). Moreover, in order to counter the large field gradients caused by the strong gravitational field in the vicinity of black holes, a nonvanishing β^i is required. [Alcubierre and Brügmann (2001)] To deal with this slice-stretching issue, gauge conditions

[2] This corresponds to an evenly spaced slicing, so that coordinate time coincides with proper time of Eulerian observers (recall that $\mathrm{d}\tau = \alpha \mathrm{d}t$).

were designed so that second-time derivatives of the shift are proportional to first-time derivatives of the coefficients $\bar{\Gamma}^i$ (i.e., $\partial_t^2 \beta^i \sim \partial_t \bar{\Gamma}^i$). In particular, a hyperbolic shift condition

$$\partial_t^2 \beta^i = \eta \partial_t \bar{\Gamma}^i - \xi \partial_t \beta^i, \tag{3.2}$$

where η and ξ are positive scalar fields, was introduced. This is the so-called *Gamma-driver* shift condition. We may then use an auxiliary vector field $\mathcal{B}^i$ to perform the usual trick of rewriting a second order derivative in first order,

$$\partial_t \mathcal{B}^i = \vartheta_1 \, \alpha^{\vartheta_2} \, \partial_t \bar{\Gamma}^i - \varrho_1 \, \mathcal{B}^i \tag{3.3a}$$

$$\partial_t \beta^i = \varrho_2 \, \mathcal{B}^i, \tag{3.3b}$$

where we also rewrote the scalar fields η and ξ in terms of four damping parameters $\vartheta_{1,2}$ and $\varrho_{1,2}$ that fine-tune the growth of the shift.

3.3 | Scalar Fields in Cosmology

Thus far we have only discussed purely geometric aspects of the 3+1 formulation of GR, without discussing constraints or evolution of any potential matter field that might be coupled to the EFE's. We now briefly touch upon this topic (for a thorough treatment the reader may consult, e.g., Chapter 6 of Gourgoulhon (2012)). Let us focus our succinct discussion in a scalar matter field ϕ minimally coupled to the EFE's. For such scalar field, the *Lagrangian* is given by

$$L_M = \frac{1}{2} \nabla_a \phi \nabla^a \phi + V(\phi), \tag{3.4}$$

where $V(\phi)$ is the *scalar potential*, which may be decomposed as

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + V_{\text{int}}(\phi), \tag{3.5}$$

m being the *mass* of the field and V_{int} the *interaction potential* (since we are interested in the minimally-coupled case, the field is noninteracting; i.e., $V_{\text{int}} = 0$). If we then add the associated *scalar matter Lagrangian density* $\mathcal{L}_M = \sqrt{-g} L_M$ to the *gravitational Lagrangian density*, namely $\mathcal{L}_G = \sqrt{-g} R$, then minimization of the (modified) *Einstein-Hilbert action*

$$S = \frac{1}{16\pi} \int \left(\mathcal{L}_G + \mathcal{L}_M \right) \mathrm{d}^4 x \tag{3.6}$$

leads to the EFE's with stress-energy tensor defined by

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \left(\nabla_c \phi \nabla^c \phi + 2V(\phi) \right). \tag{3.7}$$

The addition of this scalar field to our 3+1 formulation of GR opens the doors to some interesting areas of study; for instance, *inhomogeneous cosmological inflation* (Clough (2017), Clough et al. (2017), Laguna et al. (1991)) and *critical gravitational collapse* (Choptuik (1993), Clough (2017), Gundlach and Martin-Garcia (2007)).

Moreover, from (3.4) the *Euler-Lagrange* equations yields our *equation of motion*, which coincides with the *Klein-Gordon* equation in curved spacetime:

$$\nabla^2\phi = \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = m^2\phi, \tag{3.8}$$

where $\nabla^2 = g^{ab}\nabla_a\nabla_b$. Since this equation of motion is of second order, it would be useful to cast it into first order for integration purposes; we accomplish this with the aid of new auxiliary variables Π and $\varkappa_i$ given by [3]

$$\Pi \equiv \frac{1}{\alpha}\left(\partial_t\phi - \beta^i\partial_i\phi\right) \tag{3.9}$$

$$\varkappa_i \equiv \partial_i\phi. \tag{3.10}$$

Using these variables, (3.8) splits as

$$\partial_t\phi = \alpha\Pi + \beta^i\varkappa_i \tag{3.11a}$$

$$\partial_t\varkappa_i = \beta^j\partial_j\varkappa_i + \varkappa_j\partial_i\beta^j + \alpha\partial_i\Pi + \Pi\partial_i\alpha \tag{3.11b}$$

$$\partial_t\Pi = \beta^i\partial_i\Pi + g^{ij}\left(\alpha\partial_j\varkappa_i + \varkappa_j\partial_i\alpha - \Gamma^k_{ij}\varkappa_k\right) + \alpha\left(K\Pi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}\right). \tag{3.11c}$$

These equations must be solved in conjunction with the gravitational field's 3+1 equations in order to determine the complete evolution of a spacetime containing a scalar matter field. Note that (3.11a) is just the definition of Π (i.e., it is simply a rewriting of (3.9)) and (3.11b) follows directly from (3.10) and (3.11a) by commuting partial derivatives; the true equation of motion is in fact determined by (3.11c). The constraint given by (3.10), namely,

$$\mathcal{R}_i \equiv \varkappa_i - \partial_i\phi = 0, \tag{3.12}$$

must be preserved by the system (3.11). Of course, if solved *exactly*, (3.11) does guarantee the preservation of (3.12) throughout the evolution. The problem is at the *numerical* level, where truncation errors can give the residual $\mathcal{R}_i$ nonzero values; therefore it is important to keep a close eye out on $\mathcal{R}_i$ (in addition to the Hamiltonian and momentum constraints!) during the evolution to make sure that we are working with an accurate simulation.

[3] Some references (e.g., Baumgarte and Shapiro (2010)) define Π as the negative of ours; here we follow the convention on Clough (2017).

Had we instead assumed homogeneity of the scalar field (i.e., $\nabla_i \phi = \partial_i \phi = 0$), then the equation of motion (3.8) for a Friedmann-Lemaître-Robertson-Walker (FLRW) metric would yield a relatively simple second-order ODE for the evolution of the "inflaton" scalar field $\phi(t)$,

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0, \tag{3.13}$$

where, per usual notation, $H(t) = \dot{a}/a$ represents *Hubble's constant*, and $a(t)$ is the *expansion parameter* that appears in the FLRW metric. The much more complicated problem of dealing with an inhomogeneous inflaton requires the full power of numerical relativity, and it is currently a very active research area. For more on this topic the reader is referred to references such as Clough (2017),Clough et al. (2017), East et al. (2016), Laguna et al. (1991).

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