Fast Approximation of Betweenness Centrality through Sampling

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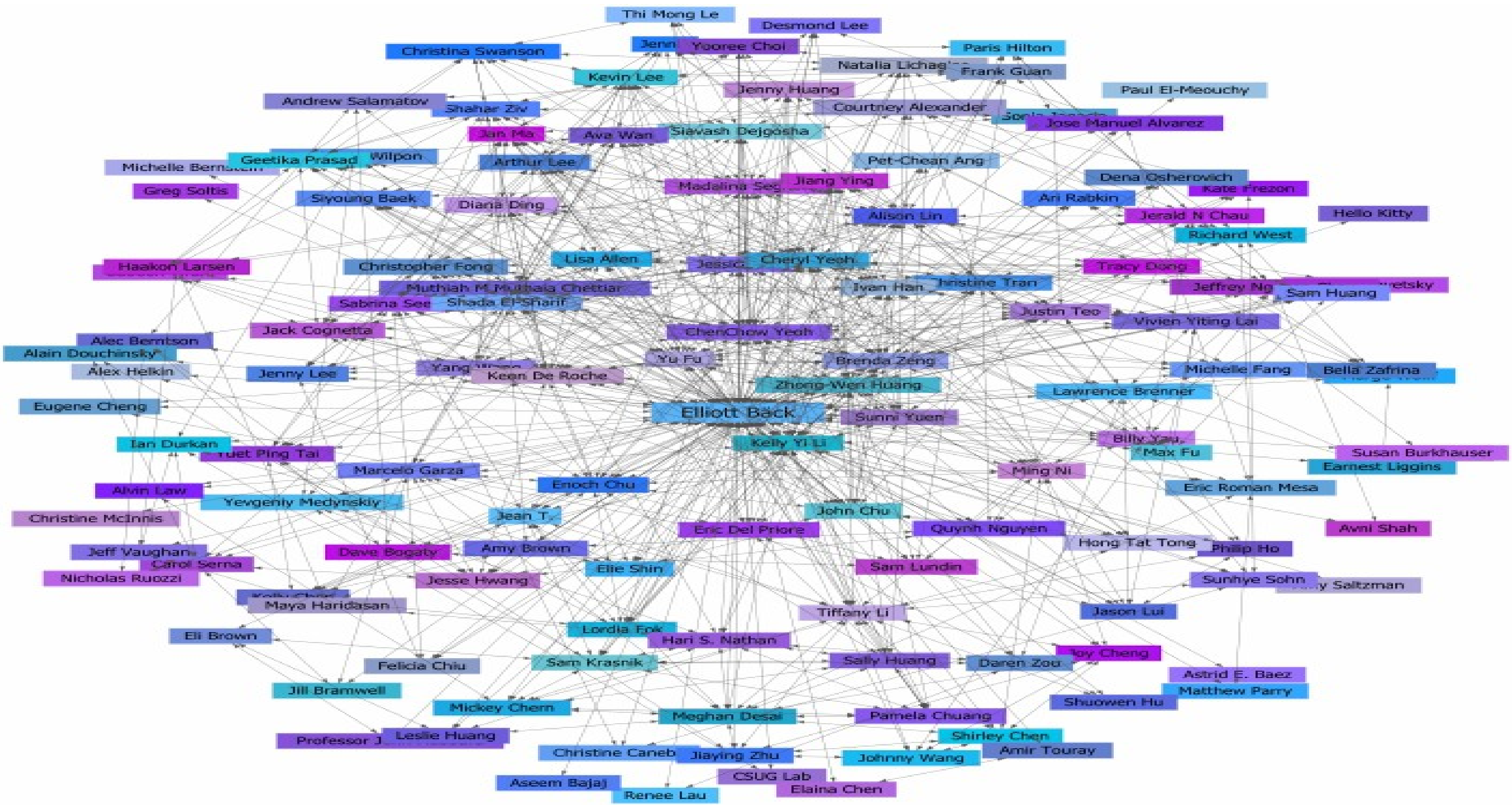
Motivation

Why is it interesting?

Online Social Networks

Relationship Graph



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Not well defined.
What is "important"?

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- Different flavours of importance
  - Many definitions of centrality indices

- Most are based on shortest paths
  - Intuition: information "spreads" along shortest paths (probably not true)

- Can be extended to include routing info

Computing Centrality

- Today's networks have 10^8 vertices

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- Too expensive, despite tricks to speed up the aggregation [Brandes01]

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- Sampling can help:
  trade-off accuracy for speed

Outline

- Motivation
- $\boxed{\text{Definition and settings}}$
- Exact and simple sampling algorithms
- Our result: a fast sampling algorithm
- Better approximation for the top-K vertices
- Experiments
- Conclusions

Graph

- Graph $G = (V, E)$
  - $|V| = n$, $|E| = m$
  - Can be weighted: $w_e \geq 0, e \in E$
  - Can be directed
  - No self-loops
  - No multiple edges between a pair of vertices

(Shortest) Paths

- Path $p = (v_1, \ldots, v_{|p|})$ ordered tuple of vertices
- End points of p: $\{v_1, v_{|p|}\}$
- Internal vertices: $\mathsf{Int}(p) = p \setminus \{v_1, v_{|p|}\}$
- $\mathcal{S}_{uv}$: Set of shortest paths from u to v
- $\sigma_{uv} = |\mathcal{S}_{uv}|$: no. of shortest paths from u to v
- $\mathbb{S}_G = \bigcup_{(u,v) \in V \times V, u \neq v} \mathcal{S}_{uv}$: all shortest paths in G

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- Let $\mathcal{T}_v = \{p \in \mathbb{S}_G \ : \ v \in \mathsf{Int}(p)\}$

- Betweenness centrality of v

$$\boxed{\mathsf{b}(v) = \frac{1}{n(n-1)} \sum_{p_{uw} \in \mathbb{S}_G} \frac{1_{\mathcal{T}_v}(p_{uw})}{\sigma_{uw}} = \frac{1}{n(n-1)} \sum_{p_{uw} \in \mathcal{T}_v} \frac{1}{\sigma_{uw}}}$$

- k-betweenness: local variant, consider only shortest paths of length up to k

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Exact Algorithm

- Naïve exact algorithm for betweenness:
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This is still too much for large networks

Sampling to the Rescue

- Solution: use sampling!
- Trade off accuracy for speed
- Can guarantee high quality approximations with high confidence

...especially when the analysis is tight!

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Key questions:

- What should we sample?
- How much should we sample?

Guarantees

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- An (ε, δ)-approximation for the betweenness is a set of estimations $\tilde{\mathsf{b}}(v)$ for all $v \in V$ such that

$$\boxed{\Pr\left(\exists v \in V \ : \ |\tilde{\mathsf{b}}(v) - \mathsf{b}(v)| > \varepsilon\right) < \delta}$$

- ε controls the accuracy

- δ controls the confidence

- Trade-off: Higher accuracy and/or confidence requires more samples!

Simple Sampling Algorithm

- [BrandesPich07] inspired by[EppsteinWang01]
- The algorithm:

For r times do
  Sample random vertex v

  Compute SSSP from v

  Perform partial computation for $\tilde{\mathrm{b}}(w), w \in V$

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Sample size
How to choose it?

Trade off between
accuracy/confidence
and speed

Like exact algorithm
[Brandes01]

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Sample Size Derivation

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- Use Hoeffding bound (method of bounded differences) to bound deviation of estimated betweenness from exact value for a single vertex:

$$\Pr(|\tilde{\mathrm{b}}(v) - \mathrm{b}(v)| > \varepsilon) < 2e^{-2r\varepsilon^2}$$

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- Use union bound over all n vertices

- Sample size r for (ε, δ)-approximation:

$$r \geq \frac{1}{2\varepsilon^2}\left(\ln n + \ln 2 + \ln \frac{1}{\delta}\right)$$

Drawbacks

- The sample size depends on $\ln n$
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- The sample size depends on $\ln n$
  - Obtained from the union bound: loose!
  - Is this the right quantity?
    - We believe not
    - It should be a characteristic quantity of the graph
- At each sample, “heavy” computation (SSSP)
  - Touch a lot of edges, has low “locality”

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Fast Sampling Algorithm

- Our algorithm solves the drawbacks:
  - Does not use the union bound
    - use VC-dimension
  - Its sample size depends on a characteristic quantity of the graph (the vertex-diameter)
    - Not on number of vertices
  - Performs a single s-t shortest path computation per sample
    - Fewer edges touched, better locality

Fast Sampling Algorithm

- The algorithm:

For r times do
    Sample a random pair of vertices (u, v)
    Compute $\mathcal{S}_{uv}$, all shortest paths from u to v
    Select a path p uniformly at random from $\mathcal{S}_{uv}$
    For each $w \in \mathsf{Int}(p)$
        $\tilde{\mathsf{b}}(w) \leftarrow \tilde{\mathsf{b}}(w) + 1/r$

Vertex Diameter

- The vertex diameter $\mathsf{VD}(G)$ of G is the maximum size of a shortest path between a pair of vertices of G:

$$\boxed{\mathsf{VD}(G) = \max\{|p| \ : \ p \in \mathbb{S}_G\}}$$

- If the graph is not weighted: $\mathsf{VD}(G) = \Delta_G + 1$
- No relationship in general if G is weighted

Analysis

- **Theorem**:

> Given $\varepsilon, \delta \in (0,1)$, if
> $$r \geq \frac{1}{\varepsilon^2}\left(\lfloor \log_2(\mathsf{VD}(G) - 2)\rfloor + 1 + \ln\frac{1}{\delta}\right)$$
> then $\Pr\left(\exists v \in V \ : \ |\tilde{\mathsf{b}}(v) - \mathsf{b}(v)| > \varepsilon\right) < \delta$

- We have an (ε, δ)-approximation

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Always less than n

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Always less than n

- We have an (ε, δ)-approximation
- For the proof, we use VC-Dimension

VC-Dimension

- [VapnikChervonenkis71]
- Combinatorial property of a collection of subsets from a domain
- Measures the “richness”, “expressivity” of the subsets

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- Combinatorial property of a collection of subsets from a domain
- Measures the “richness”, “expressivity” of the subsets
- Given a probability distribution on the domain, if we know the VC-dim of a collection of subsets, we can compute the sample size sufficient to approximate the probability mass of each subset using a sample and the empirical average

Range Sets

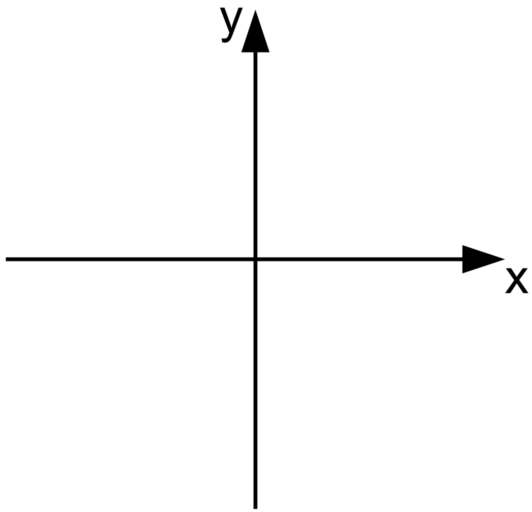
- VC-Dimension is defined on range sets
- B: Domain
- $\mathcal{R}$: collection of subsets from B (ranges)
- No restrictions:
  - B can be infinite
  - $\mathcal{R}$ can be infinite
  - $\mathcal{R}$ can contain infinitely-large subsets of B

VC-Dimension

- Range set $\mathcal{R}$ on domain B
- For any $C \subseteq B$, define $P_C = \{C \cap F \ : \ F \in \mathcal{R}\}$
- C is shattered if $P_C = 2^C$
- The VC-Dimension of $\mathcal{R}$ is the size of the largest shattered subset of B

Example of VC-Dimension

- $B = \mathbb{R}^2$



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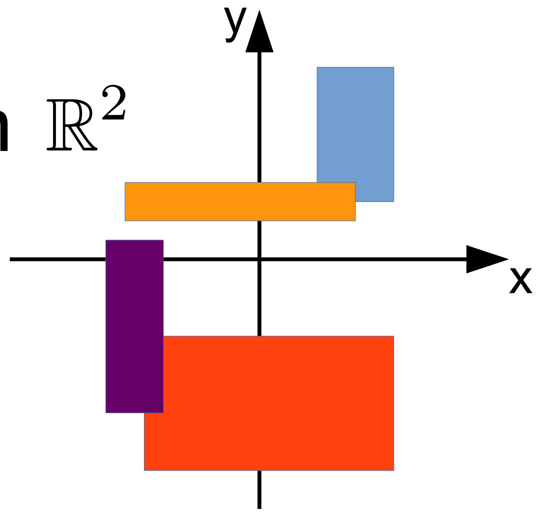

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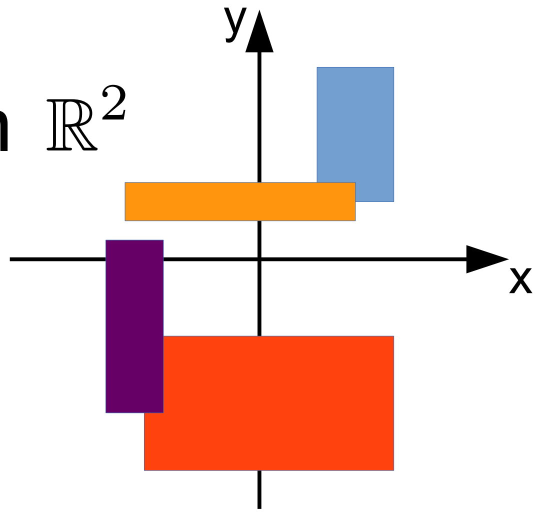
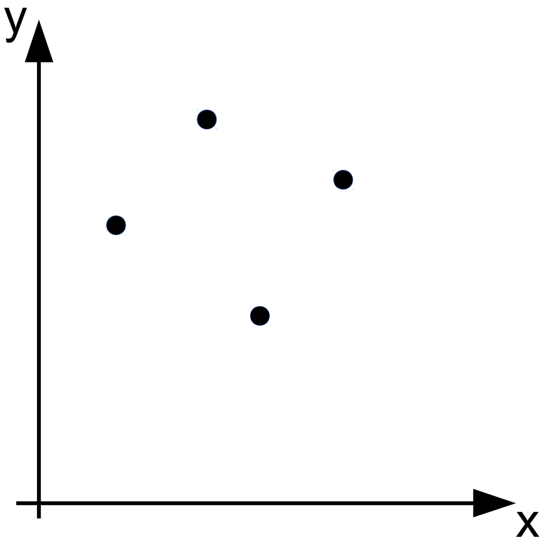


- Shattering 4 points: Easy
  - Take any 4 points s.t. no 3 of them are aligned



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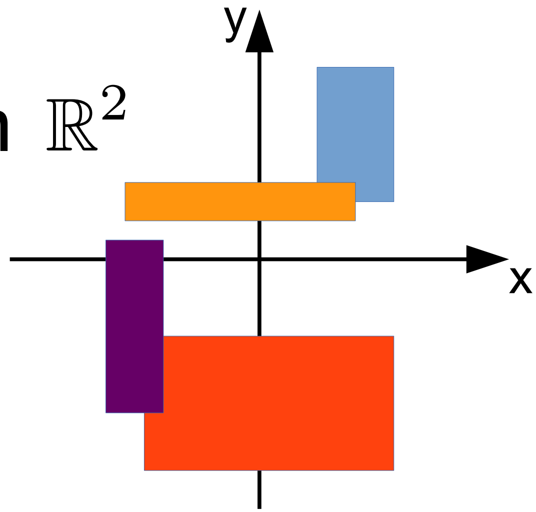

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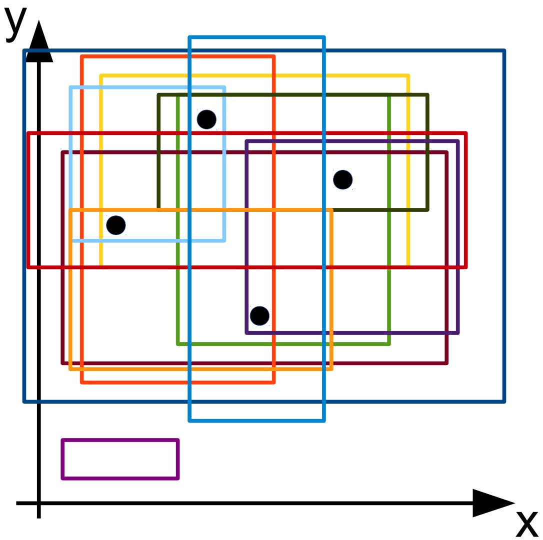

Need 16 rectangles to shatter them

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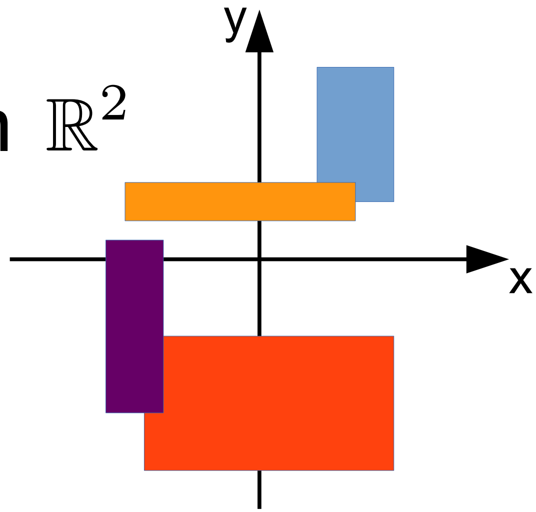

- Shattering 5 points?

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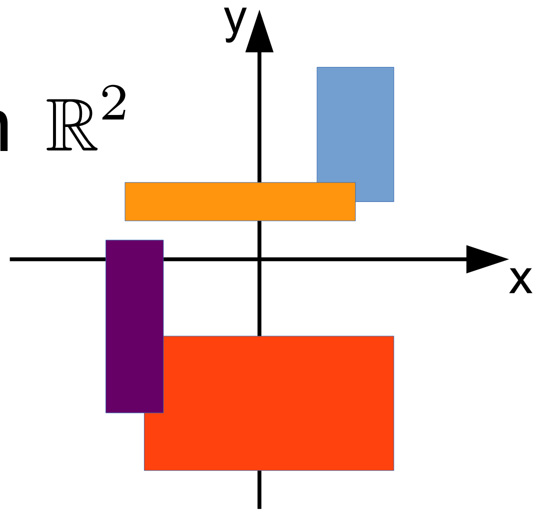

- Shattering 5 points: impossible

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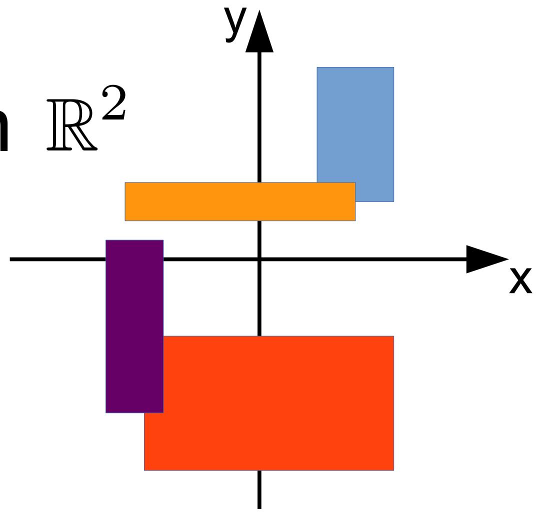
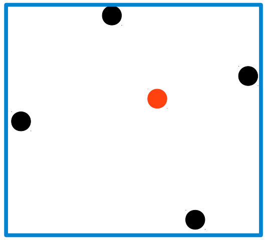


- Shattering 5 points: impossible
  - Take any 5 points
  - One of them that is contained in all rectangles containing the other four
  - Impossible to find a rectangle containing only the other four



- $\mathsf{VC}(\mathcal{R}) = 4$

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- [VapnikChervonenkis71]
- Combinatorial property of a collection of subsets from a domain
- Measures the "richness", "expressivity" of the subsets
- Given a probability distribution on the domain, if we know the VC-dim of a collection of subsets, we can compute the sample size sufficient to approximate the probability mass of each subsets using a sample and the empirical average

Sample Theorem

- Let $\mathcal{R}$ have VC$(\mathcal{R}) \leq d$
- Let π be a probability distribution on B
- Let $\pi(A)$ be the probability mass of $A \subseteq B$
- Given $\varepsilon, \delta \in [0,1]$,let S be a collection of samples from π
- If

$$\boxed{|S| \geq \frac{1}{\varepsilon^2}\left(d + \ln\frac{1}{\delta}\right)}$$

then,

$$\boxed{\Pr\left(\exists A \in \mathcal{R} \ : \ \left|\pi(A) - \frac{1}{|S|}\sum_{s \in S} 1_A(s)\right| > \varepsilon\right) < \delta}$$

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If d does not depend on |B|,
then neither does |S| !!!

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$$\Pr\left(\exists A \in \mathcal{R} \;:\; \left|\pi(A) - \frac{1}{|S|}\sum_{s \in S} 1_A(s)\right| > \varepsilon\right) < \delta$$

Roadmap

- We are going to build a range set for the problem and show an upper bound to its VC-dimension
- We define a probability distribution and use it to sample elements of the domain
- The sample theorem gives us the amount of samples we need to draw to obtain a good approximation of the betweenness of all vertices

Shortest Path Range Set

- Range set $\mathcal{R}_G$ associated to shortest paths
- Domain: $\mathbb{S}_G$, all shortest paths in G
- $\mathcal{R}_G = \{\mathcal{T}_v, v \in V\}$
- Contains one range per vertex

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- Contains one range per vertex
- Sampling probability distribution on $\mathbb{S}_G$

$$\boxed{\pi_G(p_{uv}) = \frac{1}{n(n-1)}\frac{1}{\sigma_{uv}}}$$

- The algorithm samples paths according to π_G

Bounding the VC-Dimension

- **Theorem**: $\mathsf{VC}(\mathcal{R}_G) \leq \lfloor \log_2 \mathsf{VD}(G) - 2 \rfloor + 1$

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  - p appears in $|\mathsf{Int}(p)| \leq \mathsf{VD}(G) - 2$ ranges $\mathcal{T}_v$
  - For A to be shattered, must be $2^{d-1} \leq \mathsf{VD}(G) - 2$
  - Implies the thesis

Back to the algorithm

- Recall the algorithm:

For r times do
  Sample a random pair of vertices (u, v)

  Compute $\mathcal{S}_{uv}$, all shortest paths from u to v

  Select a path p uniformly at random from $\mathcal{S}_{uv}$

  For each $w \in \mathsf{Int}(p)$
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Correctness

- From the sample theorem we get that if

$$r \geq \frac{1}{\varepsilon^2}\left(\lfloor \log_2(\mathsf{VD}(G)-2)\rfloor + 1 + \ln\frac{1}{\delta}\right)$$

then the returned collection of $\tilde{\mathrm{b}}(v)$ is an (ε, δ)-approximation:

$$\Pr\left(\exists v \in V \;:\; |\tilde{\mathsf{b}}(v) - \mathsf{b}(v)| > \varepsilon\right) < \delta$$

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(ε, δ)-approximation:

$$\Pr\left(\exists v \in V \ : \ |\tilde{\mathsf{b}}(v)\right.$$

In real world social networks,
the diameter is usually very small
(small world phenomenon)

Definitively smaller than n

Corollaries

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  - This "collapse" of the VC-dimension to a constant is somewhat surprising
  - Suggests that there may be other characteristic quantities of the graph that control the VC-Dimension
- k-betweenness: if we only consider shortest paths of size up to k, then

$$\mathsf{VC}(\mathcal{R}_G) \leq \lfloor \log_2 k - 1 \rfloor + 1$$

Diameter Approximation

- Computing $\mathrm{VD}(G)$ exactly would require APSP
  - Defeat the purpose of sampling

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- Undirected && unweighted:2-approx using BFS
- All other cases: ???
  - Return the size of the largest WCC (<= n)

Top-K Betweenness

- Often interest is top-K players in the network, and in their ranking

- Let $\mathsf{b}^{(K)}$ be the K^{th} highest betweenness (ties broken arbitrarily)

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Similar to definition of top-K Frequent Itemsets

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Multiplicative factor
(rather than addictive)

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- We proved the upper bound

$$\mathsf{VC}(\mathcal{R}_G) \leq \lfloor \log_2(\mathsf{VD}(G) - 2) \rfloor + 1$$

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  - $\mathsf{VC}(\mathcal{R}_G) = 3$

Outline

- Motivation
- Definition and settings
- Exact and simple sampling algorithms
- Our result: a fast sampling algorithm
- Better approximation for the top-K vertices
- Experiments
- Conclusions

Experimental Evaluation

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- Datasets:
  - Real networks (social, road, citation, …) from SNAP (snap.stanford.edu)

Accuracy Results

- Always within ε from real value
- Accuracy even better than guaranteed

Speedup Results

- Almost 1 order of magnitude faster than simple sampling algorithm

Scalability Results

- Scales much better than simple sampling algorithm

Conclusions

- We presented two sampling-based randomized algorithms to approximate the betweenness of (top-K) vertices in huge graphs

- The algorithms offer probabilistic guarantees on the accuracy of the approximations, using much fewer samples and performing fewer computations than previous available algorithms

- Experimental evaluation shows that the algorithm outperforms previous works in terms of execution time and accuracy

The End

- Questions or comments?
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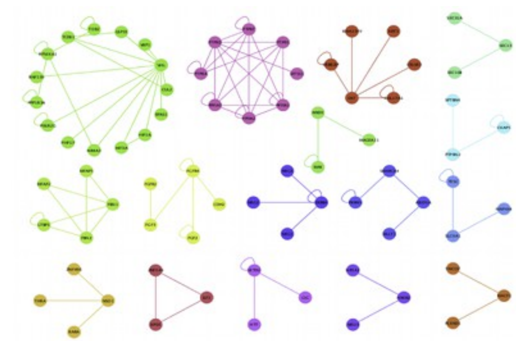
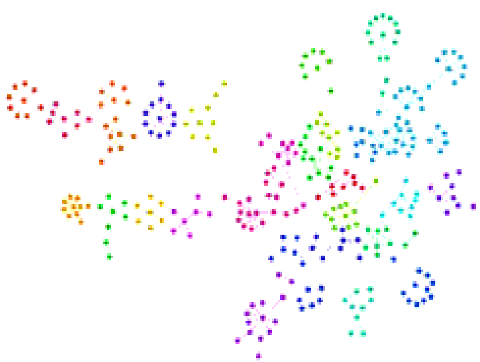
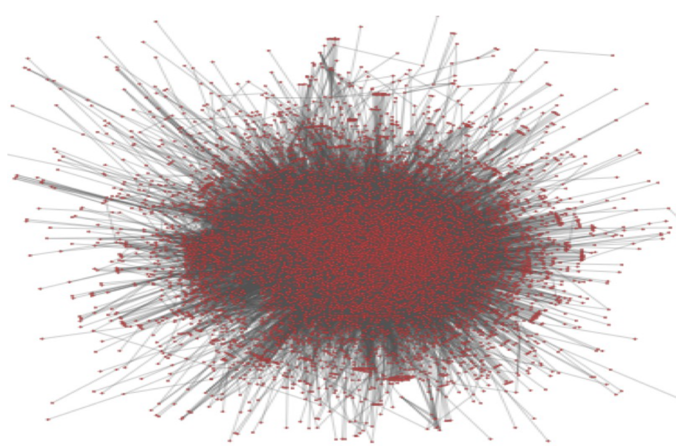
- http://bigdata.cs.brown.edu
- http://database.cs.brown.edu







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