Math Refresher

Basic Linear Algebra

Vectors

A vector is a **list** of numbers. We can think of a vector as a point in space, or as an arrow pointing from the origin to that point. For example, the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is a vector in $\mathbb{R}^3$ (three-dimensional space) that points from the origin to the point $(1, 2, 3)$.

We can add vectors together by adding their corresponding elements. For example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

We can also multiply a vector by a scalar (a single number) by multiplying each element of the vector by that number. For example,

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Matrices

A matrix is a two-dimensional array of numbers. We can think of a matrix as a list of vectors. For example, the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a matrix that concatenates 3 $\mathbb{R}^2$ vectors together.

Matrices can have different dimensions. For example, the matrix:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is a **square** matrix that concatenates 3 $\mathbb{R}^3$ vectors together.

Matrix Dimensions

Matrices are often denoted by their dimensions.

- For example, the matrix A above is a 2×3 matrix, because it has 2 rows and 3 columns.
- The matrix B above is a 3×3 matrix, because it has 3 rows and 3 columns.

In general, we can denote a matrix M with r rows and c columns as an $r \times c$ matrix.

- For Notation, I will usually refer to this like $M_{r\times c}$. In this case we have $A_{2\times 3}$ and $B_{3\times 3}$.
- We can denote the element in the ith row and jth column of M as M_{ij}. For example, the element in the 2nd row and 3rd column of B is $B_{23} = 6$.

`Stata` and Matrices

`Stata` has a powerful matrix algebra language called `mata`. We can define matrices in `mata` using the following syntax:

```
mata
//  column vector
 vv1  =  (1\2\3)
//  row vector
 vv2 =  (1,2,3)
// matrix
 a = (1,2,3 \ 4,5,6);a
 a = (1\4),(2\5),(3\6);a
 a[1,3]
end
```

<IPython.core.display.HTML object>

```
. mata
------------------------------------------------- mata (type end to exit) -----
:  vv1 = (1\2\3)

:  vv2 = (1,2,3)

:  a = (1,2,3 \ 4,5,6);a
       1   2   3
    +-------------+
  1 |  1   2   3  |
  2 |  4   5   6  |
    +-------------+

:  a = (1\4),(2\5),(3\6);a
       1   2   3
    +-------------+
  1 |  1   2   3  |
  2 |  4   5   6  |
    +-------------+

:  a[1,3]
  3

: end
--------------------------------------------------------------------------------

.
```

```
mata
    (1\2\3)+(4\5\6)
    2*(1\2\3)
end
```

```
. mata
------------------------------------------------- mata (type end to exit) -----
:     (1\2\3)+(4\5\6)
       1
    +-----+
  1 |  5  |
  2 |  7  |
  3 |  9  |
    +-----+

:     2*(1\2\3)
       1
    +-----+
  1 |  2  |
  2 |  4  |
  3 |  6  |
    +-----+

: end
--------------------------------------------------------------------------------

.
```

Special Matrices

- There are a few special matrices that we will use often. The **zero matrix** is a matrix where all of the elements are 0. For example, the zero matrix with 2 rows and 3 columns is:

$$Zero = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

`mata: J(2,3,0)`

A **square matrix** is a matrix where the number of rows is equal to the number of columns. For example, B is a square matrix.

`mata: J(3,3,0)`

Special Matrices

The **identity matrix** is a square matrix where all of the elements are 0, except for the elements along the diagonal, which are 1. For example, the identity matrix with 3 rows and 3 columns is:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

`mata: I(3)`

For simplicitly, we will use the subscript to denote the size of the identity matrix. For example, I_3 is a 3x3 identity matrix, and I_5 is a 5x5 identity matrix.

Special Matrices

A $1 \times c$ matrix is called a **row vector**. Wheras a $r \times 1$ matrix is called a **column vector**.

A **diagonal matrix** is a square matrix where all of the elements off the diagonal are 0. For example, the following matrix is a diagonal matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The identify matrix is a special case of a diagonal matrix.

`mata: diag( (1,2,3) )` or `mata: diag( (1\2\3) )`

Matrix Operations

We can add matrices together by adding their corresponding elements. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

However, both matrices must have the same dimensions. For example, we cannot add the following matrices together:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} + \begin{bmatrix} 7 & 8 \\ 10 & 11 \end{bmatrix}_{2\times 2}$$

`mata: a + b` Will work if `a` and `b` have the same dimensions.

`mata` will throw an error if you try to add matrices of different dimensions.

Matrix Scalar Multiplication

We can multiply a matrix by a scalar by multiplying each element of the matrix by that scalar. For example,

$$a \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1a & 2a & 3a \\ 4a & 5a & 6a \end{bmatrix}$$

```
mata:
2 * (1,2,3 \ 4,5,6)
end
```

```
. mata:
------------------------------------------------- mata (type end to exit) -----
: 2 * (1,2,3 \ 4,5,6)
        1    2    3
    +----------------+
  1 |   2    4    6  |
  2 |   8   10   12  |
    +----------------+

: end
-------------------------------------------------------------------------------

.
```

Matrix Multiplication

We can multiple two matrices together by taking the dot product of each row of the first matrix with each column of the second matrix. For example:

$$
\begin{aligned}
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \begin{bmatrix} 7 & 8 \\ 10 & 11 \\ 13 & 14 \end{bmatrix}_{3\times 2} &= \begin{bmatrix} 1*7+2*10+3*13 & 1*8+2*11+3*14 \\ 4*7+5*10+6*13 & 4*8+5*11+6*14 \end{bmatrix}_{2\times 2} \\
&= \begin{bmatrix} 66 & 72 \\ 156 & 171 \end{bmatrix}
\end{aligned}
$$

A good way of remembering this is to follow the flow: $\rightarrow \times \downarrow$

`mata: a = (1,2,3 \ 4,5,6) ; b = (7,8 \ 10,11 \ 13,14); a*b`

```
mata: a = (1,2,3 \ 4,5,6) ; b = (7,8 \ 10,11 \ 13,14); a*b
```

```
          1      2
    +-------------+
  1 |    66     72  |
  2 |   156    171  |
    +-------------+
```

Matrix Multiplication

Note that the number of **columns** in the first matrix must be equal to the number of **rows** in the second matrix.

$$M_{a\times b} \times N_{b\times c} = P_{a\times c}$$

For example, we cannot multiply the following matrices together:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3} \begin{pmatrix} 7 & 8 \\ 10 & 11 \end{pmatrix}_{2\times 2}$$

Some properties of matrix multiplication:

- Matrix multiplication is **not commutative**. That is, $AB \neq BA$ in general.

```
** A*B
mata:
a*b
b*a
end
```

```
. mata:
------------------------------------------------- mata (type end to exit) -----
: a*b
         1      2
    +-------------+
  1 |   66     72  |
  2 |  156    171  |
    +-------------+

: b*a
[symmetric]
         1      2      3
    +--------------------+
  1 |   39               |
  2 |   54     75        |
  3 |   69     96    123 |
    +--------------------+

: end
-------------------------------------------------------------------------------

.
```

- Matrix multiplication is **associative**. That is, $A(BC) = (AB)C$.

```
mata: c=(4,1\2,4)
```

```
** A*(B*C)
mata:
 c=(4,1\2,4)
 a*(b*c) ;  (a*b)*c
end
```

```
. mata:
------------------------------------------------- mata (type end to exit) -----
:  c=(4,1\2,4)

:  a*(b*c) ; (a*b)*c
         1     2
    +-------------+
  1 |   408   354  |
  2 |   966   840  |
    +-------------+
         1     2
    +-------------+
  1 |   408   354  |
  2 |   966   840  |
    +-------------+

: end
--------------------------------------------------------------------------------

.
```

- Any matrix multiplied by I is equal to itself. That is, $AI = IA = A$.

```
** A*(B*C)
mata:
a*I(3)
b*I(2)
end
```

```
. mata:
------------------------------------------------- mata (type end to exit) -----
```

```
: a*I(3)
       1   2   3
    +-------------+
  1 |  1   2   3  |
  2 |  4   5   6  |
    +-------------+

: b*I(2)
        1    2
    +-----------+
  1 |   7    8  |
  2 |  10   11  |
  3 |  13   14  |
    +-----------+

: end
--------------------------------------------------------------------------------
```

.

Transpose

The transpose of a matrix is a matrix where the rows and columns are swapped. For example, if the matrix A is defined as:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

then the transpose of A, denoted A^T, is:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Note that if $A_{a \times b}$, then $A^T_{b \times a}$.

```
mata: a_t = a'
```

Some properties of the transpose:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A + B)^T = A^T + B^T$
- $(aA)^T = aA^T$
- $(A^T)^{-1} = (A^{-1})^T$

Inverse

The inverse of a square matrix is a matrix that, when multiplied by the original matrix ($AA^{-1} = I$), results in the identity matrix. For example:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} -3 & 1 \\ 2 & -.5 \end{bmatrix}$$

```
mata
a = (1,2 \ 4,6)
a_inv = luinv(a); a_inv
a*a_inv
end
```

```
. mata
------------------------------------------------- mata (type end to exit) -----
: a = (1,2 \ 4,6)

: a_inv = luinv(a); a_inv
          1     2
    +-------------+
  1 |   -3     1  |
  2 |    2   -.5  |
    +-------------+

: a*a_inv
[symmetric]
        1   2
    +---------+
  1 |   1     |
  2 |   0   1 |
```

```
    +---------+

: end
--------------------------------------------------------------------------------
```

.

For a 2×2 matrix, the inverse is defined as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus, if a matrix has determinant 0, then it is not invertible.

Determinant

The determinant of a **square** matrix is a scalar value that is a function of the elements of the matrix. The determinant of a 2×2 matrix is defined as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant of a 3×3 matrix is defined as:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gbf - ceg - fha - ibd$$

```
mata: det(a)
```

Rank and linear independence

- The rank of a matrix is the number of linearly independent rows or columns in the matrix.
  - In a rectangular matrix, the rank cannot be larger than the smaller of the rows or columns.
- If we consider each column, or rows, of a matrix as a vector, then the rank of the matrix is the number of **linearly independent** vectors in the matrix.
- If a set of vectors are not linearly independent, then one of the vectors can be expressed as a linear combination of the other vectors. For example, the following vectors are not linearly independent:

$$a_1\vec{x}_1 + a_2\vec{x}_2 + a_3\vec{x}_3 = 0$$

```
mata: rank(a)
```

System of linear equations

A system of linear equations is a set of equations that can be expressed in the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

where a_{ij} and b_i are constants, and x_i are variables.

This system of equations can be written in matrix form as:

$$A_{n\times n}X_{n\times 1} = b_{n\times 1}$$

if the system has a unique solution, then the matrix A is invertible, and the solution is given by:

$$X = A^{-1}b$$

Thus if there is no solution, then A is not invertible. If the determinant of A is 0, then A is not invertible.