Math Refresher: Basic Calculus

Introduction

This is a refresher on basic calculus. It is not meant to be a substitute for a full course on calculus, but rather a quick review of the basic concepts and techniques that will be used in this semester.

Limits

The limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x gets closer and closer to a. We write this as:

$$\lim_{x \to a} f(x) = L$$

In this case, the limit of the function $f(x)$ as x approaches a is L. For example, consider the function $f(x) = x^2$. The limit of $f(x)$ as x approaches 2 is 4:

Limits to Derivatives

Limits can also be used to estimate derivatives. The derivative of a function $f(x)$ is the slope of the function at a given point. The derivative of $f(x)$ at $x = a$ is written as $f'(a)$. The derivative of $f(x)$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In other words, the deriviative is the slope of a function at a particular point a. This can be proxied using derivatives, by choosing a very small value for h.

For example, consider the function $f(x) = x^2$. The derivative of $f(x)$ at $x = a$ is:

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{(a+h)^2-(a)^2}{h} \\
&= \lim_{h\to 0} \frac{a^2+2ah+h^2-a^2}{h} \\
&= \lim_{h\to 0} 2a+h = 2a
\end{aligned}
$$

If anything else fails, one can always rely on numerical differentiation.

Derivative of common functions

For most common functions, the derivative can be calculated using the following rules:

- The derivative of a constant is zero
- The derivative of x^n is nx^{n-1}
- The derivative of $ln(x)$ is $\frac{1}{x}$
- The derivative of e^x is e^x
- The derivative of a^x is $a^x \ln a$

There are other rules for derivatives, but these are the ones that will be used most often.

Derivative of composite functions

The derivative of a composite function $f(g(x))$ is given by the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

For example, consider the function $f(x) = \ln(x^2)$. The derivative of $f(x)$ is:

$$
\begin{aligned}
\frac{d}{dx} \ln(x^2) &= \frac{1}{x^2}\frac{d}{dx}x^2 \\
&= \frac{1}{x^2}2x \\
&= \frac{2}{x}
\end{aligned}
$$

Derivative of sums and products

The derivative of a sum of functions is the sum of the derivatives of the functions.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The derivative of a product of functions is given by the product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

The derivative of a quotient of functions is given by the quotient rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Which is a special case of th product rule.

Maximization and Minimization

Derivatives can be used to identify the maximum and minimum values of a function. Consider a function $f(x)$.

To find the maximum (or minimum) value of $f(x)$, we can take the derivative of $f(x)$ and set it equal to zero. This is called the first order condition. The idea is that at the maximum (or minimum) value of $f(x)$ shouldnt change anymore (should be flat). Thus the derivative of $f(x)$ should be zero.

For example, consider the function $f(x) = 5x^2 - 4x + 2$. The derivative of $f(x)$ is:

$$\begin{gathered} f'(x) = 10x - 4 = 0 \\ x = \frac{4}{10} = 0.4 \end{gathered}$$

So when x is equal to 0.4, the function $f(x)$ does not change anymore. This, however, is insufficient to determine if the function is at a maximum or a minimum. To determine this, we can take the second derivative of $f(x)$, or second order condition:

$$f''(x) = 10 > 0$$

Because the second derivative is positive, we know that $f(x)$ is at a minimum when $x = 0.4$. If the second derivative was negative, we would know that $f(x)$ is at a maximum when $x = 0.4$.

why is this the case

- $f'(x)$ measures the changes in $f(x)$ along x. when $f'(x) = 0$, $f(x)$ is not changing anymore.
- $f''(x)$ measures the changes in $f'(x)$ (changes in those changes). Because its positive, we know that $f'(x)$ is increasing. This means that at $x = 0.4$ the changes if $f(x)$ were going from negative to positive. Thus indicating a minimum.

Optimization with multiple variables

When considering multiple variables, we also need to rely on the first and second order conditions to find minimum and maximum values. Consider a function $f(x, y)$. The first order conditions are:

$$\frac{\partial}{\partial x} f(x, y) = 0$$
$$\frac{\partial}{\partial y} f(x, y) = 0$$

This conditions now say that, in the direction of x and y, the function $f(x, y)$ is not changing anymore. Thus we have a potential maximum or minimum. Now, to identify a minimum, we need second order conditions to be:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

If $Det(H) > 0$ and $f_{xx} > 0$ then we have a minimum. If $Det(H) > 0$ and $f_{xx} < 0$ then we have a maximum. If $Det(H) < 0$ then we have a saddle point. And if $Det(H) = 0$ then we have an inconclusive result.

Optimization with constraints

When optimizing a function with constraints, we can use the method of Lagrange multipliers. Consider a function $f(x, y)$ subject to the constraint $g(x, y) = z$. The Lagrangian is:

$$L(x, y, \lambda) = f(x, y) + \lambda(z - g(x, y))$$

Notice that the Lagrangian is the function $f(x, y)$ plus the constraint $g(x, y)$ multiplied by a constant λ. The constant λ is called the Lagrange multiplier. The constrain is written as the difference between the constant z and the function $g(x, y)$. The Lagrangian is then optimized with respect to x, y, and λ. This are the equivalent of the first order conditions:

$$
\begin{aligned}
&\frac{\partial}{\partial x} L(x, y, \lambda) = 0 \\
&\frac{\partial}{\partial y} L(x, y, \lambda) = 0 \\
&\frac{\partial}{\partial \lambda} L(x, y, \lambda) = z - g(x, y) = 0
\end{aligned}
$$

The last condition is the constraint, and it implies that the constraint must be satisfied. The second order conditions are the same as before.