Math Refresher

Basic Statistics and Probability

Random Variables

- A random variable is a variable whose value is determined by the outcome of a random experiment.
  - For example, if we **toss a coin**, the outcome is random, but the possible values of X are 0 and 1.
  - If we roll a die, the outcome is random with possible values 1, 2, 3, 4, 5, and 6.
  - **Exact** temperature in a room

There are two kinds of random variables:

- **Discrete random variables** can only take on a finite number of values. For example, the number of heads in 10 coin tosses is a discrete random variable.
  - The probability of observing a particular value is not always zero
- **Continuous random variables** can take on any value in a range. For example, the height of a randomly selected person is a continuous random variable.

- If X is discrete random variable, then $P(X = c)$ is the probability that X takes on the value c. It can be any value between 0 and 1. ()
- By definition, the sum of all probabilities for all feasible values of X is 1. That is, $\sum_c P(X = c) = 1$.
- If X is continuous random variable, then $P(X = c) = 0$ for any value c.
  - The probability to observe a particular number is zero.
  - Instead, when using continuous data, we focus on the probability of observing a value in a range. For example, $P(1.7 \leq X \leq 1.8)$ is the probability that X is between 1.7 and 1.8, which can be any value between 0 and 1.

`Stata` and Random Variables

- Computers **CANNOT** generate random numbers. They can only generate pseudo-random numbers.
  - Random numbers cannot be reproduced.
  - Pseudo-random numbers can be reproduced, if we know initial conditions. (seed)
    * For most purposes, pseudo-random numbers are good enough.
- `Stata` has many built-in function to generate random numbers.
  - `help random` for more information.

Probability Distributions

- A probability distribution is a function that assigns probabilities to the values of a random variable.
  - For discrete random variables, we can use a table to describe the probability distribution. For example, the probability distribution of the number of heads in 5 coin tosses is:

Number of heads	Probability
0	0.03125
1	0.15625
2	0.3125
3	0.3125
4	0.15625
5	0.03125

In this case, the sum of all probabilities is 1.

Probability Density Functions

- For continuous random variables, we can use a function to describe the probability distribution.
  - For example, we can say that the probability distribution of the height of a randomly selected person is:

$$f(x)$$

This function has important properties:

- $f(x) \geq 0$ for all x.
- $\int_{-\infty}^{\infty} f(x)dx = 1$.
- $P(a \leq X \leq b) = \int_a^b f(x)dx$.
- $P(X \leq a) + P(X > a) = 1$.
- $P(a \leq X \leq b) = P(X < b) - P(X < a)$.

`Stata` and Empirical Distributions

Theory

- Given a dataset, you can use different tools to estimate the probability distribution or the probability density function of a random variable.
  - For example, you can use histograms, or frequency tables, to estimate the probability distribution of a discrete random variable.
  - You can use kernel density plots to estimate the probability density function of a continuous random variable.

Dicreet

```
sysuse nlsw88.dta, clear
replace grade  = 11 if grade <11
fre grade
```

```
<IPython.core.display.HTML object>
```

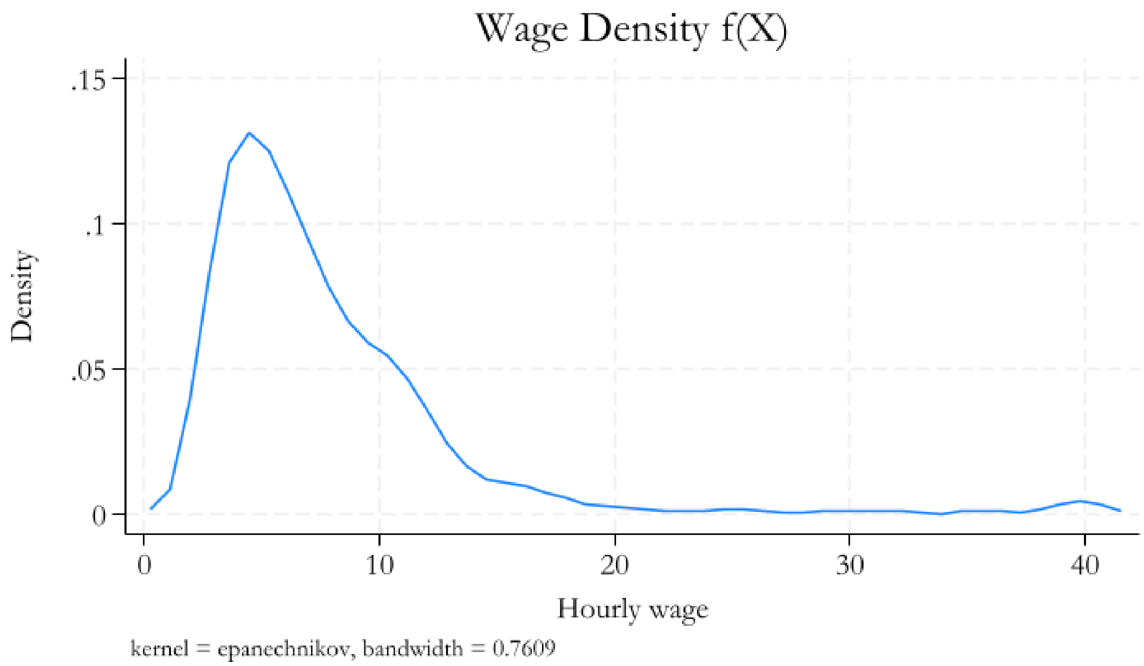
```
(NLSW, 1988 extract)
(211 real changes made)

grade -- Current grade completed
-----------------------------------------------------------
              |      Freq.    Percent      Valid       Cum.
--------------+--------------------------------------------
Valid   11    |        334      14.87      14.88      14.88
        12    |        943      41.99      42.02      56.91
        13    |        176       7.84       7.84      64.75
```

```
        14     |        187        8.33        8.33       73.08
        15     |         92        4.10        4.10       77.18
        16     |        252       11.22       11.23       88.41
        17     |        106        4.72        4.72       93.14
        18     |        154        6.86        6.86      100.00
        Total  |       2244       99.91      100.00
Missing .      |          2        0.09
Total          |       2246      100.00
-----------------------------------------------------------
```

Continuous

```
kdensity wage, scale(1.25) title("Wage Density f(X)")
```



Joint Probability Distributions

- The joint probability distribution of X and Y is a function that assigns probabilities to the values of X and Y.

- For discrete random variables, we can use a table to describe the joint probability distribution.

```
tab race married, cell nofreq
```

```
           |        Married
      Race |    Single    Married |     Total
-----------+----------------------+----------
     White |     21.68      51.20 |     72.89
     Black |     13.76      12.20 |     25.96
     Other |      0.36       0.80 |      1.16
-----------+----------------------+----------
     Total |     35.80      64.20 |    100.00
```

- It must be the case that the sum of all probabilities is 1.

- For continuous variables, estimation and graphical representation is tricky
- it must be the case that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

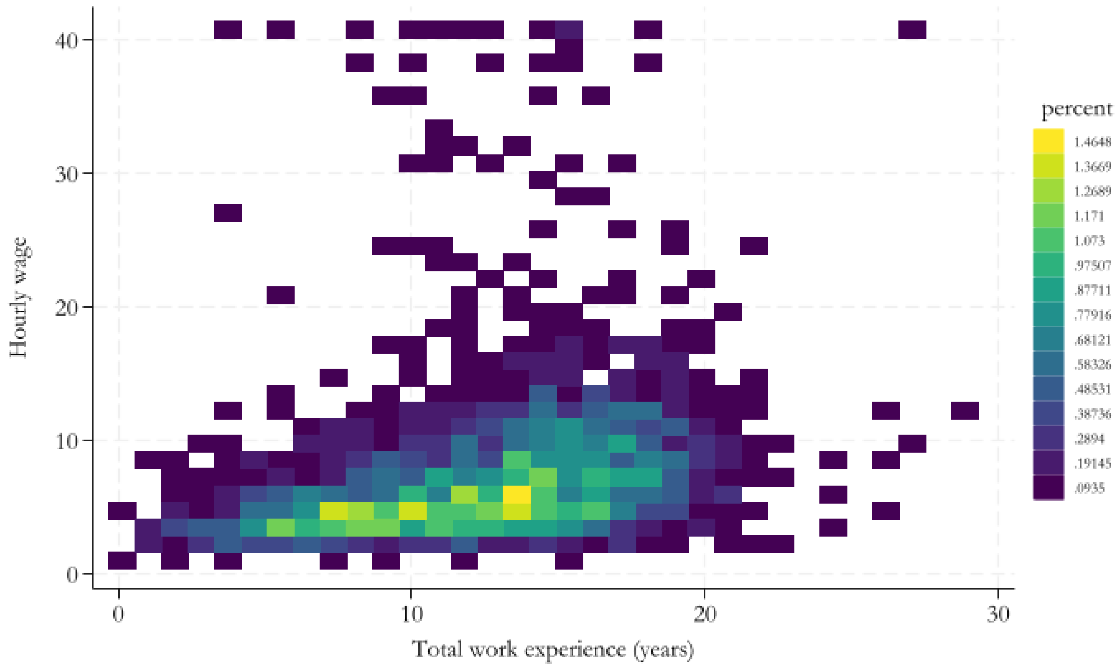
- You may be able to use scatter plots, or contour plots, to represent the joint probability distribution of two continuous random variables.

Scatter Plot

```
scatter wage ttl_exp , msize(2) mcolor(%10)
```

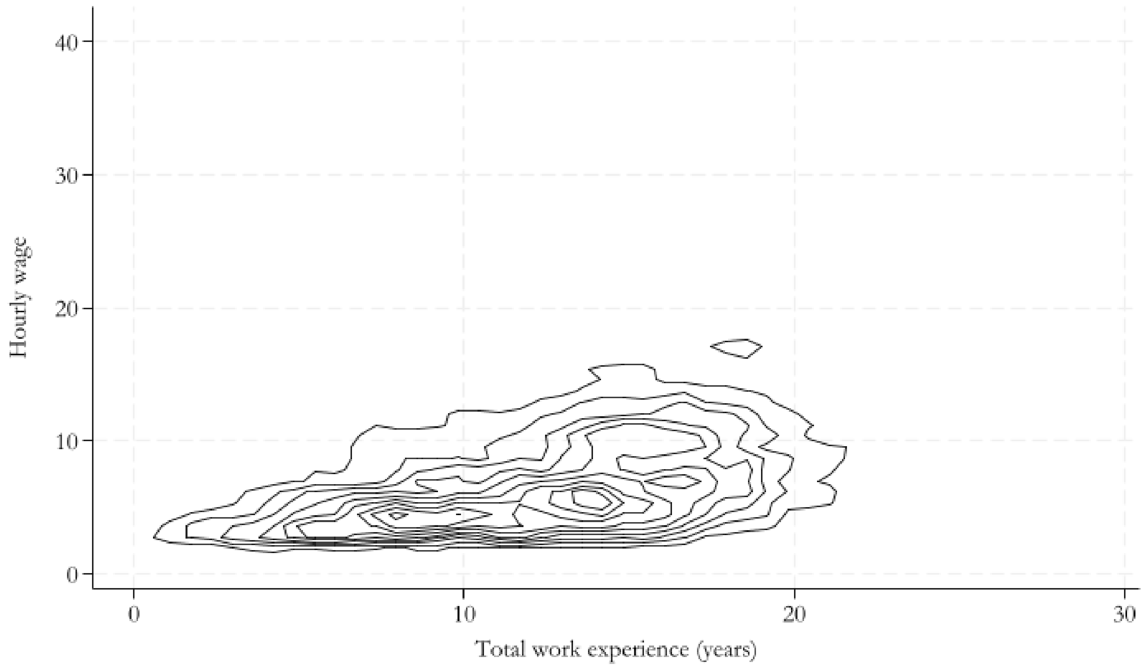
Heatplot

```
qui:ssc install heatplot
heatplot wage ttl_exp ,
```



BiDensity

```
qui:ssc install bidensity
bidensity wage ttl_exp, levels(10)
```



Conditional Probability

The conditional probability of X given Y is:

$$P(x|y) = \frac{P(x,y)}{P(y)}$$

or, the conditional probabilty density function:

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

And if X and Y are independent, then:

$P(x|y) = P(x)$ or $f(x|y) = f(x)$.

Marginal Probability Distributions

The marginal probability distribution of X is the probability distribution of X ignoring/regardless the values of Y. This can be expressed as:

$$P(x) = \sum_{z=-\infty}^{\infty} P(X = x, y = z) \text{ or } f_x(x) = \int_{z=-\infty}^{\infty} f(x, z) dz$$

This is also refer to "integrating out" the variable Y or averaging over Y.

$$P(x) = \sum_{z=-\infty}^{\infty} P(X = x|y = z) P_y(z) \text{ or } f_x(x) = \int_{z=-\infty}^{\infty} f(x|z) f_y(z) dz$$

Independence

Two random variables X and Y are independent if and only if:

$$P(x, y) = P(x)P(y) \text{ or } f(x, y) = f(x) * f(y)$$

That means the conditional probability of X given Y is the same as the marginal probability of X.

$P(x|y) = P(x)$ or $f(x|y) = f(x)$.

Summary Statistics

Given a random variable X, there are several summary statistics that can be used to describe the distribution of X, without describing the entire distribution

Central Tendency

- Mean: average value of X.

$$\bar{x} = E(X) = \sum_x xP(X = x) \text{ or } E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

- Median: middle value of X.
- Percentile: values that identify the boundaries of the distribuion. Median is the 50th percentile.

$$Q_y(p) = E(Y \leq Q_y) = p$$

- Mode: most frequent value of X.

`sum var,d` in `Stata` will give you the mean, median, and selected quantiles.

`mode` can be estimated using egen, or based on empirical distribution.

Dispersion

- Variance: Average squared deviation from the mean.

$$Var(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 P(X=x) \text{ or } Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

- Standard deviation: square root of the variance. Easier to interpret.
- Range: difference between the maximum and minimum values of X.
- Interquartile range: difference between the 75th and 25th percentiles of X.

`sum var,d` and `tabstat` can provide you with most of this information.

Some useful distributions

Discrete distributions

- **Bernoulli distribution**: $X \sim Bernoulli(p)$, where $p \in [0,1]$.
  - $E(X) = p$ and variance $Var(X) = p(1-p)$.
  - Flip a coin with probability p of getting heads.
  - `rbinomial(1, p)`
- **Binomial distribution**: $X \sim Binomial(n,p)$, where $p \in [0,1]$ and $n > 0$
  - $E(x) = np$ and $Var(X) = np(1-p)$.
  - Distribution of the number of successes in n independent Bernoulli trials.
  - `rbinomial(n, p)`
- **Poisson distribution**: $X \sim Poisson(\lambda)$, where $\lambda > 0$
  - $E(X) = Var(x) = \lambda$, Typically used for counts.
  - For example, the number of customers arriving at a store in a given hour.
  - `rpoisson(lambda)`

Continuous distributions

- **Uniform distribution**: $X \sim Uniform(a, b)$
  - $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, and $f(x) = 0$ otherwise.
  - $E(X) = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$.
  - `runiform(a, b)`
- **Normal distribution**: $X \sim Normal(\mu, \sigma^2)$
  - $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.
  - $E(X) = \mu$ and $Var(X) = \sigma^2$.
  - `rnormal(mu, sigma)`

Other useful distributions include:

- **t-distribution**, **Chi-squared distribution**, **F-distribution**
- `help density_functions` `help random_number_functions`