Conditional Quantile Regressions

Because no-one is average

Fernando Rios-Avila

Introduction

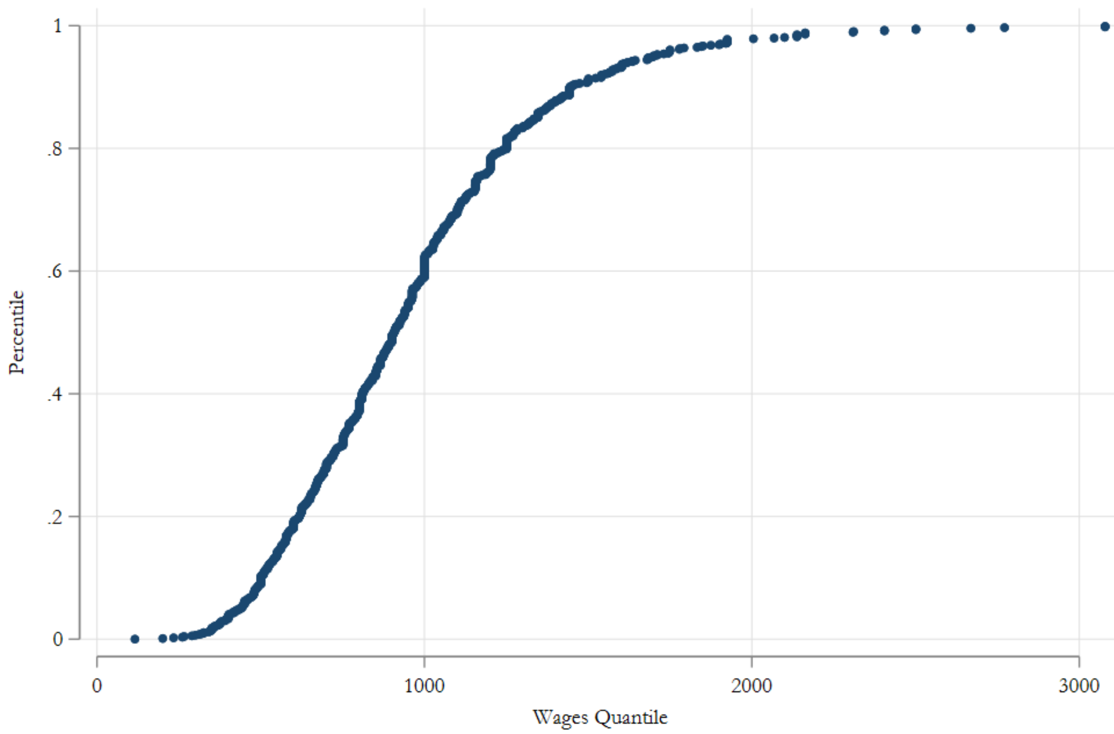
Question: What are quantiles? and why do we care??

- Quantiles are statistics that have the purpose of providing a better characterization of distributions.
  - This is possible, because it provides you with more information than standard summary statistics (means and variance)
- How so? In general, there are 3 ways you can use to know "everything" about a distribution.
  - You either have access to every single y_i
  - Or you know the distribution function $f(y)$ (or probability density function pdf)
  - Or you know the cumulative distribution function $F(y) = \int_{\infty}^{y} f(t)dt = P(Y \leq y)$
- However, there is an additional way. Quantile:

$$Q(\theta) = F^{-1}(p)$$

- Which in principle, is nothing but the inverse cumulative density function.

$Q(\theta) = F^{-1}(\theta)$



Percentile
0
.2
.4
.6
.8
1
0
1000
2000
3000
Wages Quantile

Other advantages? Yes!

- Quantiles are far more stable in the presence of outliers. Because of this, they are particularly useful as measures of central tendency (perhaps superior to the mean) (?)
  - Simple "test". In the small town of Troy-NY one of the residents wins the 2B$ lottery. How much has welfare increase for the average resident?
- Scaled IQR can be used as an alternative measure of dispersion.

$$se2 = \frac{Q_{75} - Q_{25}}{1.34898}$$

- They are also "function-transformation" resistant: $exp(Q_{log(y)}(.10)) = Q_y(.10)$
- And are also very easy to estimate:
  1. Sort data by y → Obtain weighted ranks → choose the lowest value so that θ % of the data is less of equal to that number

$$F^{-1}(tau) = inf(x : F(x) \geq t)$$

This "just" requires obtaining an approximation for $F(\theta)$, which can be approximated using nonparametric methods!

$$\hat{F}(x) = \frac{1}{N} \sum (K_F(x, x_i, h)) = \frac{1}{N} \sum 1(x_i < x)$$

then we simply "invert" the function for whichever quantile we are interested in.

Statistical Inference

As with the mean, sampling quantiles are measured with sampling error. Thus its important to recognize its sampling distribution.

However, because of the nature of how quantiles are defined, their standard errors are not as intuitive to obtain, although they can be derived using the delta Method. We start from:

$$Q_y(\tau) = F_y^{-1}(\tau) \rightarrow F_y(Q_y(\tau)) = \tau$$

$$1 = f_y(Q_y(\tau)) \frac{dQ}{d\tau} \rightarrow \frac{dQ}{d\tau} = \frac{1}{f(Q_y(\tau))}$$

So we have:

$$\hat{Q}_y(\tau) - Q_y(\tau) \simeq \frac{1}{f(Q_y(\tau)}(\hat{\tau} - \tau)$$
$$Var(\hat{Q}_y(\tau)) = \frac{Var(\hat{\tau} - \tau)}{f^2(Q_y(\tau))} = \frac{N^{-1}\tau(1-\tau)}{f^2(Q_y(\tau))}$$

Lets understand this elements

Quantile SE

$$Var(\hat{Q}_y(\tau)) = \frac{Var(\hat{\tau} - \tau)}{f^2(Q_y(\tau))} = \frac{N^{-1}\tau(1-\tau)}{f^2(Q_y(\tau))}$$

- the variance of a quantile depends on the distribution of τ which is nothing else that the distribution of a Bernoulli experiment: Is $y \geq Q_y$ or $y < Q_y$.
  - This is the largest near the center of the distribution (50%-50%) but smaller (more precise) near the tails of the distribution (more certainty that something will be larger or smaller.
- But also depends on the density of the distribution.
  - More precise estimates when the density is high (center), but less precise near tails of the distribution.
- And as usual, it depends on the sample size (N) (for more precision one needs more data)

  A minor problem. This depends on $f()$. Unless this is known, is another source of variation! (that we usually ignore

- Of course, you also have the alternative method. Bootstrap!

Example

```
frause wage2, clear
bootstrap q10=r(r1) q25=r(r2) q50=r(r3) q75=r(r4) q90=r(r5), reps(1000): _pctile wage  , p

Bootstrap results                                      Number of obs =   935
                                                       Replications  = 1,000

      Command: _pctile wage, p(10 25 50 75 90)
```

```
------------------------------------------------------------------------------
             |   Observed   Bootstrap                         Normal-based
             |  coefficient  std. err.      z    P>|z|     [95% conf. interval]
-------------+----------------------------------------------------------------
         q10 |         500   8.720396    57.34   0.000     482.9083    517.0917
         q25 |         668   14.49296    46.09   0.000     639.5943    696.4057
         q50 |         905   14.61303    61.93   0.000      876.359     933.641
         q75 |        1160   20.18665    57.46   0.000     1120.435    1199.565
         q90 |        1444   33.10919    43.61   0.000     1379.107    1508.893
------------------------------------------------------------------------------

* Analytical
sort wage
gen w1 = _n
gen w0 = _n-1
by wage:gen p=0.5*(w1[_N]+w0[1])/935
kdensity wage, at(wage) gen(fwage)
replace se = sqrt(p*(1-p)/935)/fwage
tabstat wage se if inlist(wage,500,668,905,1160,1444), by(wage)


    wage |      wage        se
---------+--------------------
     500 |       500  13.27634
     668 |       668  14.78419
     905 |       905  14.69217
    1160 |      1160   19.3574
    1444 |      1444  29.32711
---------+--------------------
   Total |  730.7619  15.88035
------------------------------
```

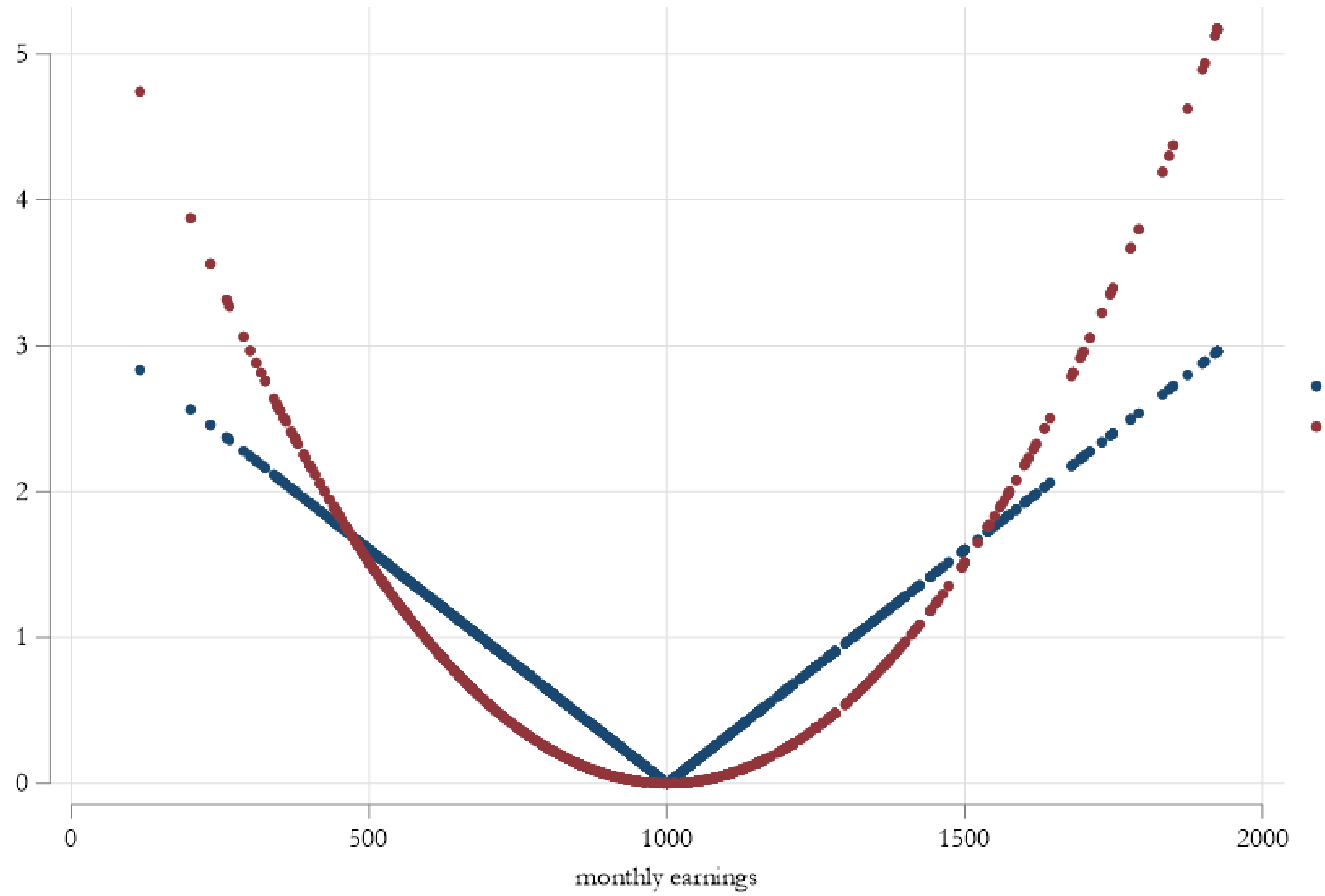
From Q_Y to $Q_{Y|X}$

The previous approaches used to identify a particular quantile are not the only ones.

Just like we can use OLS to estimate Means (Can you prove it?), we could also use a similar method to estimate the median. We only need to change the loss function $L()$ from an L^2 to a $|L|$.

Consider this:

$$median(Y) = min_\mu \frac{1}{N}\sum |y-\mu| = \frac{2}{N}\sum (y-u)(0.5 - I([y-u] < 0)$$

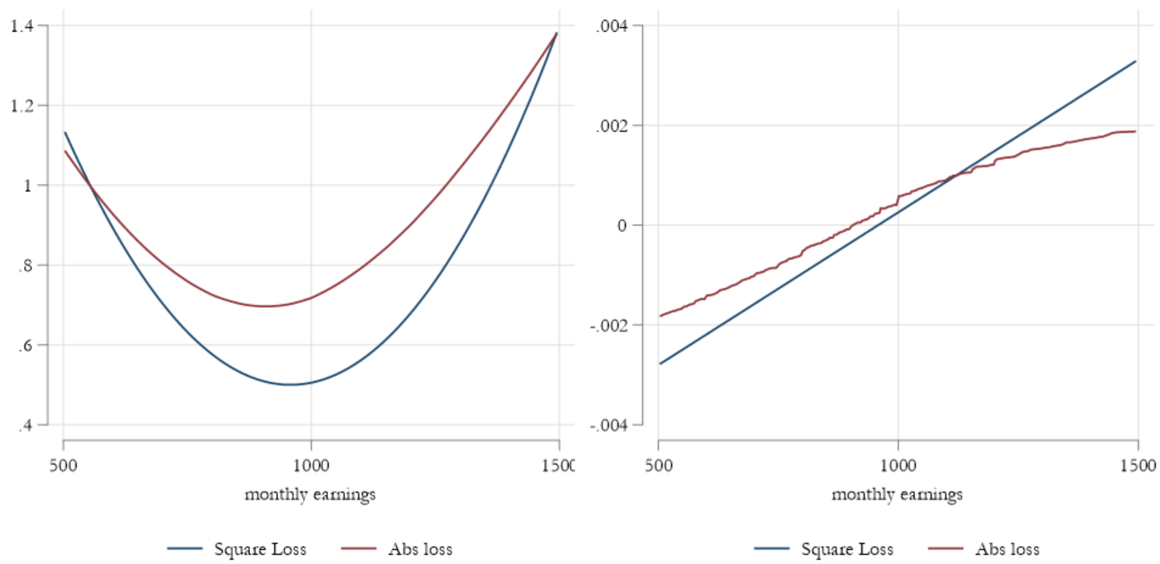


Q and Loss functions

Why does it matter?

- The loss function for Quantiles does not penalize "errors" as much as L^2 does. This is why its more robust to outliers (almost not affected by them). (the R2 will also need to be changed)

- However, the loss function is no longer differentiable (is discontinuous). So requires other methods to find the solution. Even tho it may not look like that:



From B to XB

Koenker and Bassett (1978) extended this last approximation in two ways:

- Allows for Covariates ($X's$) variation
- Allows to identify other quantiles in the distribution:

$$\beta(\tau) = \min_b N^{-1} \sum \rho_\tau(y_i - X_i'b) \rho_\tau(u) = u(\tau - I(u < 0))$$

- This implicitly states that you want to find a combination of $X's$ such that τ proportion of y_i are lower than the $X_i'\beta(\tau)$.
  - But because we are using controls, we also need that, conditional on x^c, $\tau\%$ is lower than $x'^c\beta$

$$Q_{Y|X}(\tau) = \beta_0(\tau) + \beta_1(\tau)x_1 + ... + \beta_k(\tau)x_k$$

Interpretation: Why is it so different from OLS?

- In Rios-Avila and Maroto(2022?) we stress that OLS can be interpreted at different "levels". So consider the following:

$$y_i = b_0 + b_1 x_1 + b_2 x_2 + e$$

If there errors are exogenous, and there is no heteroskedasticty, you can "obtain" marginal effects at many levels:

$$\begin{aligned}
Ind &: \frac{dy_i}{dx_{1i}} = b_1 \\
Cond &: E(y_i|X=x) = b_0 + b_1 x_1 + b_2 x_2 \\
&: \frac{dE(y|x)}{dx_1} = b_1 \\
Ucond &: E(y_i) = b_0 + b_1 E(x_1) + b_2 E(x_2) \\
&: \frac{dE(y)}{dE(x_1)} = b_1
\end{aligned}$$

- So in OLS, assuming a linear model in parameters, **Nothing** changes. The effect is the same! (although magnitude of the "experiment" changes)

But *C*Qreg?

For quantile regressions, things are not that simple.

1. There is no "individual" level quantile effect, because we do not observe individual ranks τ . If we could observe them, and we assume they are fixed, then one can obtain individual level effects.
2. Because τ is unobserved, all Qregression coefficients, should be interpreted as effects on Conditional Distributions (thus the name **C**QREG).
   - In other words, effects are just expected changes in some points in the distribution.
3. You cannot use it for unconditional effects either (not easily), because

$$E(Q_{Y|X}(\tau)) \neq Q_Y(\tau)$$

and you cannot "simply" average the CQREG effects to get unconditional.

But what does it mean? This means that CQREG interpretation are percentile τ and covariate X specific. But that is not all:

- **Fixed rank.** If you happen to be on the top of the distribution (and stay there), the quantile effect is given by the $\beta(\tau)$
- **Changes in Conditional distributions:** What we see are how distribution changes along characteristics

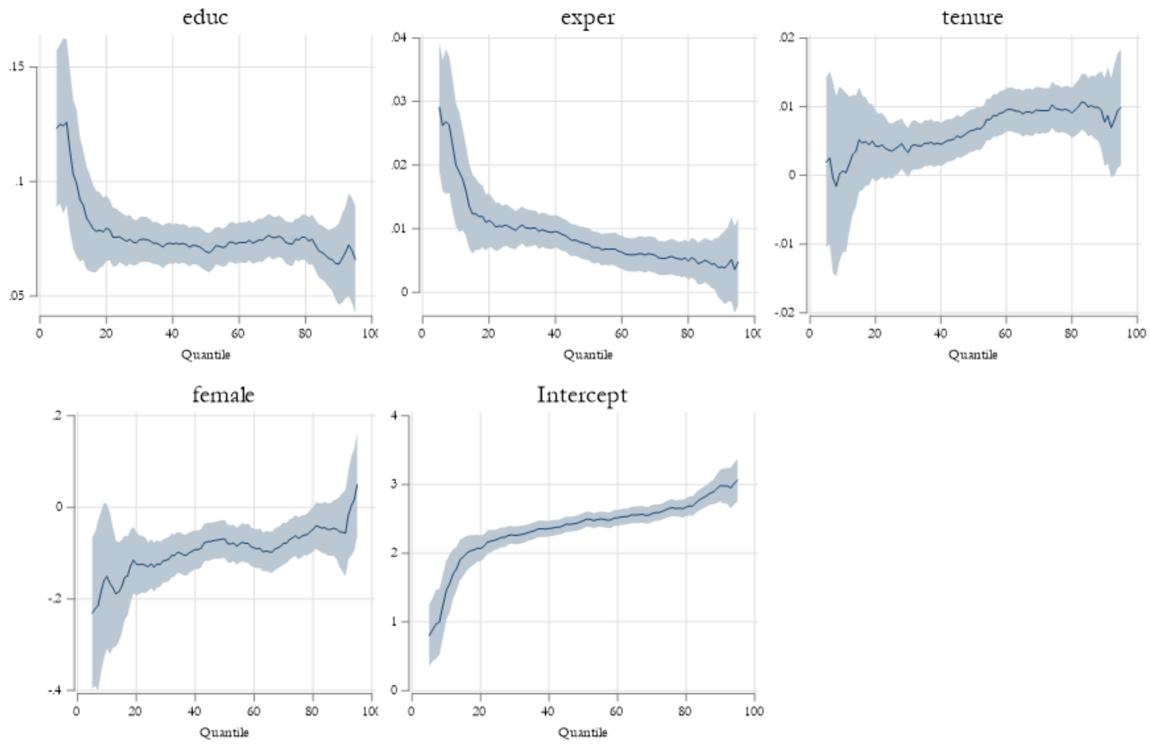
So this must be kept in mind, whenever one interpret results

Example: Wages...

```
frause oaxaca, clear
qreg  lnwage educ exper tenure female, nolog q(10)
est sto m1
qreg  lnwage educ exper tenure female, nolog q(50)
est sto m2
qreg  lnwage educ exper tenure female, nolog q(90)
est sto m3
* ssc install estout
esttab m1 m2 m3, nogaps mtitle(q10 q50 q90)
------------------------------------------------------------
                      (1)             (2)             (3)
                      q10             q50             q90
------------------------------------------------------------
educ                0.103***       0.0694***       0.0639***
                   (6.21)         (16.03)          (7.09)
exper              0.0200***      0.00758***       0.00402
                   (4.06)          (5.91)          (1.50)
tenure           0.000669         0.00657***       0.00774*
                   (0.11)          (4.19)          (2.37)
female             -0.151        -0.0689**        -0.0543
                  (-1.87)         (-3.29)         (-1.24)
_cons               1.462***        2.474***        2.984***
                   (6.67)         (43.36)         (25.10)
------------------------------------------------------------
N                    1434            1434            1434
------------------------------------------------------------
t statistics in parentheses
* p<0.05, ** p<0.01, *** p<0.001
** Also
```

```
** ssc install qregplot
qregplot educ exper tenure female, cons q(5/95)
```

Example: Wages...



Other Interpretations of Qreg

Random coefficents

One approach to both understanding, and simulating QREG is by also understanding the intuition behind the data generating process.

$$
\begin{aligned}
y = b_0(\tau) + b_1(\tau)x_1 + +b_2(\tau)x_2 + ... + b_k(\tau)x_k \\
\tau \sim runiform(0,1)
\end{aligned}
$$

where all coefficients are some function (preferably monotonically increasing or decreasing) of τ .

We want them to be monotonically increasing or decreasing because we want that

$$XB(\tau_1) \geq XB(\tau_2) \; if \; \tau_1 > \tau_2$$

This specification suggest that the unobserved component τ is a random indicator a kind to luck. If you are lucky and get a high τ then you will have better outcomes than anyone of your peers.

Also notice that this setup assumes that τ is the only random factor, and should be uncorrelated with X (you do not make your luck!)

Can you create data with these characteristics?

SVC model with a latent running variable

Another way you can think of Qreg is to align it to the -semiparametric- method we introduced ealier. SVC model.

The difference here is that the running variable is unknown. Given the outcome, and characteristics, however, we can identify something akin to the presence of a "latent" component. (but not really estimating it).

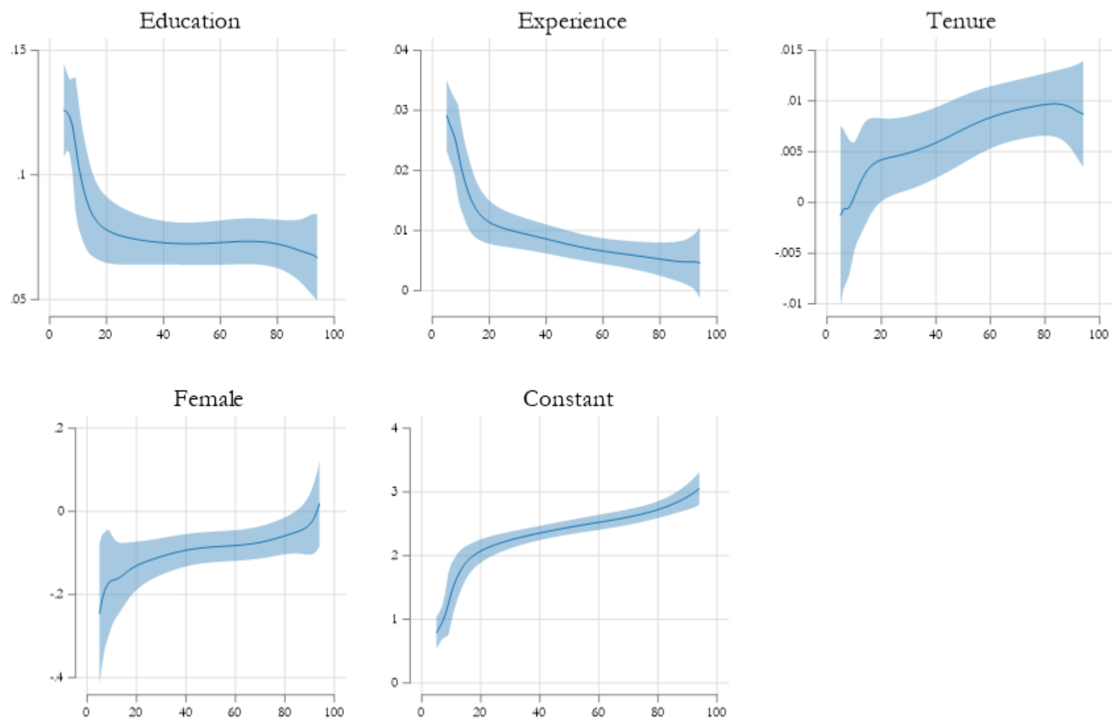
There are a few (recent) papers that focus on estimation and identification of these models. The general intuition is that the qreg model is identified by the following moment condition:

$$E\Big(1[x\beta(\tau) - y > 0] - \tau\Big) = 0$$

but substitute the indicator function with a smooth function. CDF

$$E\Big(F(x\beta(\tau) - y) - \tau\Big) = 0$$

Being differentiable, this problem is relatively easier to solve (given good initial values)


Education
.15
.1
.05
0
20
40
60
80
100
Experience
.04
.03
.02
.01
0
Tenure
.015
.01
.005
0
-.005
-.01
Female
.2
0
-.2
-.4
Constant
4
3
2
1
0

Example

Scale and Location Model

Another approach that can be used to understand Quantile regressions (and elaborate the interpretation) is to assume that the coefficients are in fact capturing two components:

$$y = Xb + Xg(\tau)$$

- **Location:** which indicates what is the average relationship between X and Y. b
- **Scale:** which indicates how far one could be from the average effect, given a relative to its position $g(\tau)$

The model Still assumes random coefficients

Estimation of this model is not standard. But can be manually implemented:

1. Estimate OLS and get residuals
2. Estimate QREG using those residuals

Requires additional care for the estimation of SE, or just bootstrap

Scale and Location 2: Heteroskedasticity

A second approach that is useful to understand and interpret CQreg is to consider a parametric version of the LS model:

$$y = Xb + \gamma(X) * e$$

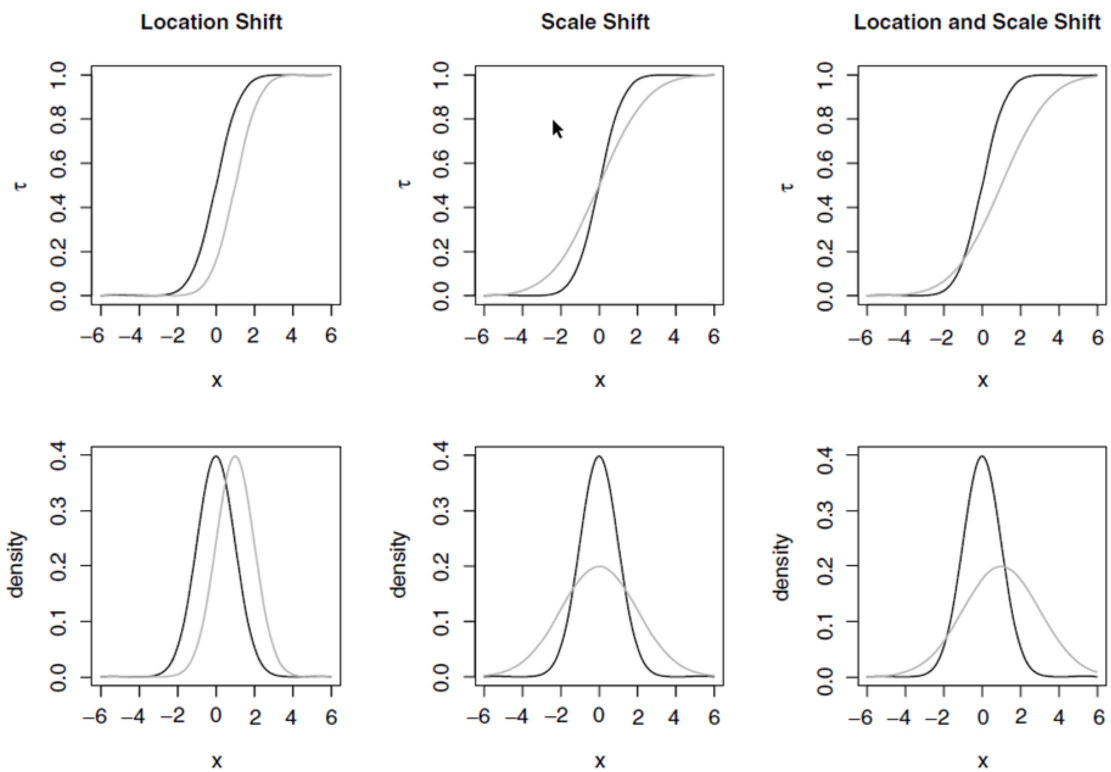
Where we assume $\gamma(X) > 0$. which directly shows the relationship between a quantile regressions and heteroskedasticity in the error term. (typically model as $X\gamma$)

Because Heteroskedasticity is parametric, it constrains the relationship across all quantile coefficients:

$$y = X(b + \gamma F^{-1}(\tau)) \rightarrow b(\tau) = b + \gamma(\tau)$$

Making it more efficient, albeit imposing constrains of the relationship.

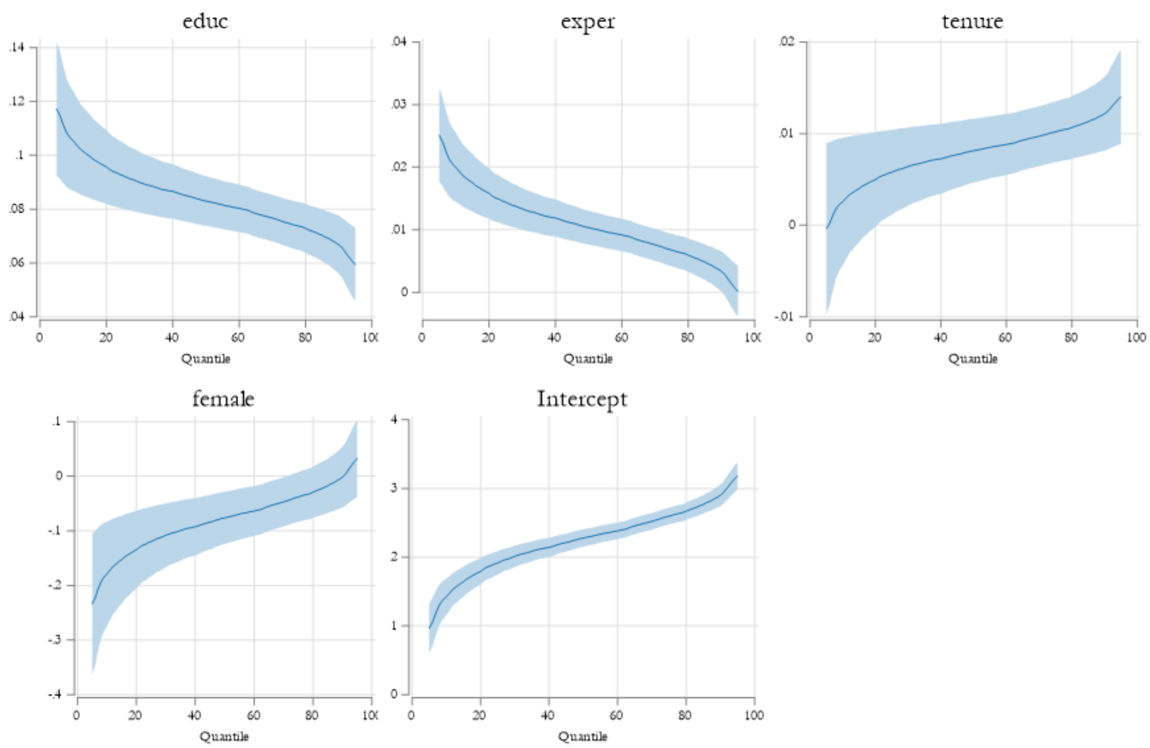
It shows more clearly the nature of the overall inequality increasing or decreasing effect.



Location Shift
Scale Shift
Location and Scale Shift
τ
density
x

Visual Loc vs Scale

Visual Loc vs Scale



Estimation and Statistical Inference

As hinted previously, there are many approaches that can be used for the estimation of Conditional Quantile regressions.

- Stata: `qreg`, `sqreg`, `bsqreg`, `qreg2`, `qrprocess`, `mmqreg`, `smqreg`, `sivqr`

Each one with its own assumptions. For Standard errors, however, there are 3 options. Under the assumption of iid error. Non iid error (robust), and assuming clustered standard errors.

$$
\begin{aligned}
iid : \Sigma_\beta &= \frac{\tau(1-\tau)}{f_y^2(F^{-1}(\tau))}(X'X)^{-1} \\
niid : \Sigma_\beta &= \tau(1-\tau)(X'f(0|x)X)^{-1}\ (X'X)\ (X'f(0|x)X)^{-1} \\
alt : \Sigma_\beta &= (IF_\beta\ 'IF_\beta)N^{-2}
\end{aligned}
$$

Or simply Bootstrap

Problems and Considerations

1. Unless otherwise specified, quantile regressions are linear in variables (and parameters?)
2. With few exceptions, quantile regressions are quantile specific. Comparisons across quantiles require joint estimation (to construct VCV matrix)
3. Because they are "local" estimators, there is risk of crossing quantiles. (Violation of Monotonicity)
4. Non-linear effects will be present if either the location or scale components are nonlinear.
5. Quantile regressions are very sensitive to measurement errors in both dependent and independent variables
6. They can be difficult to interpret (see references)

Next topic…Unconditional Quantiles

Unconditional Quantile Regressions and RIF-Regressions