

Math Refresher

Basic Linear Algebra

Vectors

A vector is a **list** of numbers. We can think of a vector as a point in space, or as an arrow pointing from the origin to that point. For example, the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is a vector in \mathbb{R}^3 (three-dimensional space) that points from the origin to the point $(1, 2, 3)$.

We can add vectors together by adding their corresponding elements. For example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

We can also multiply a vector by a scalar (a single number) by multiplying each element of the vector by that number. For example,

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Matrices

A matrix is a two-dimensional array of numbers. We can think of a matrix as a list of vectors. For example, the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a matrix that concatenates 3 \mathbb{R}^2 vectors together.

Matrices can have different dimensions. For example, the matrix:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is a **square** matrix that concatenates 3 \mathbb{R}^3 vectors together.

Matrix Dimensions

Matrices are often denoted by their dimensions.

- For example, the matrix A above is a 2×3 matrix, because it has 2 rows and 3 columns.
- The matrix B above is a 3×3 matrix, because it has 3 rows and 3 columns.

In general, we can denote a matrix M with r rows and c columns as an $r \times c$ matrix.

- For Notation, I will usually refer to this like $M_{r \times c}$. In this case we have $A_{2 \times 3}$ and $B_{3 \times 3}$.
- We can denote the element in the i th row and j th column of M as M_{ij} . For example, the element in the 2nd row and 3rd column of B is $B_{23} = 6$.

Stata and Matrices

Stata has a powerful matrix algebra language called **mata**. We can define matrices in **mata** using the following syntax:

```

mata
// column vector
vv1 = (1\2\3)
// row vector
vv2 = (1,2,3)
// matrix
a = (1,2,3 \ 4,5,6);a
a = (1\4),(2\5),(3\6);a
a[1,3]
end

```

<IPython.core.display.HTML object>

```

. mata
----- mata (type end to exit) -----
: vv1 = (1\2\3)

: vv2 = (1,2,3)

: a = (1,2,3 \ 4,5,6);a
      1   2   3
      +-----+
1 | 1   2   3 |
2 | 4   5   6 |
      +-----+

: a = (1\4),(2\5),(3\6);a
      1   2   3
      +-----+
1 | 1   2   3 |
2 | 4   5   6 |
      +-----+

: a[1,3]
3

: end
-----

.

```

```

mata
    (1\2\3)+(4\5\6)
    2*(1\2\3)
end

```

```

. mata
----- mata (type end to exit) -----
:      (1\2\3)+(4\5\6)
      1
      +-----+
      1 |  5 |
      2 |  7 |
      3 |  9 |
      +-----+

:      2*(1\2\3)
      1
      +-----+
      1 |  2 |
      2 |  4 |
      3 |  6 |
      +-----+

: end
-----

.

```

Special Matrices

- There are a few special matrices that we will use often. The **zero matrix** is a matrix where all of the elements are 0. For example, the zero matrix with 2 rows and 3 columns is:

$$Zero = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

`mata: J(2,3,0)`

A **square matrix** is a matrix where the number of rows is equal to the number of columns. For example, B is a square matrix.

`mata: J(3,3,0)`

Special Matrices

The **identity matrix** is a square matrix where all of the elements are 0, except for the elements along the diagonal, which are 1. For example, the identity matrix with 3 rows and 3 columns is:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

`mata: I(3)`

For simplicity, we will use the subscript to denote the size of the identity matrix. For example, I_3 is a 3x3 identity matrix, and I_5 is a 5x5 identity matrix.

Special Matrices

A $1 \times c$ matrix is called a **row vector**. Whereas a $r \times 1$ matrix is called a **column vector**.

A **diagonal matrix** is a square matrix where all of the elements off the diagonal are 0. For example, the following matrix is a diagonal matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The identity matrix is a special case of a diagonal matrix.

`mata: diag((1,2,3))` or `mata: diag((1\2\3))`

Matrix Operations

We can add matrices together by adding their corresponding elements. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

However, both matrices must have the same dimensions. For example, we cannot add the following matrices together:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} + \begin{bmatrix} 7 & 8 \\ 10 & 11 \end{bmatrix}_{2 \times 2}$$

`mata: a + b` Will work if `a` and `b` have the same dimensions.

`mata` will throw an error if you try to add matrices of different dimensions.

Matrix Scalar Multiplication

We can multiply a matrix by a scalar by multiplying each element of the matrix by that scalar. For example,

$$a \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1a & 2a & 3a \\ 4a & 5a & 6a \end{bmatrix}$$

```
mata:
2 * (1,2,3 \ 4,5,6)
end
```

```
. mata:
----- mata (type end to exit) -----
: 2 * (1,2,3 \ 4,5,6)
      1      2      3
    +-----+
1 |   2   4   6 |
2 |   8  10  12 |
    +-----+

: end
-----

.
```

Matrix Multiplication

We can multiply two matrices together by taking the dot product of each row of the first matrix with each column of the second matrix. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 7 & 8 \\ 10 & 11 \\ 13 & 14 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 1*7 + 2*10 + 3*13 & 1*8 + 2*11 + 3*14 \\ 4*7 + 5*10 + 6*13 & 4*8 + 5*11 + 6*14 \end{bmatrix}_{2 \times 2}$$

$$= \begin{bmatrix} 66 & 72 \\ 156 & 171 \end{bmatrix}$$

A good way of remembering this is to follow the flow: $\rightarrow \times \downarrow$

mata: a = (1,2,3 \ 4,5,6) ; b = (7,8 \ 10,11 \ 13,14); a*b

```
mata: a = (1,2,3 \ 4,5,6) ; b = (7,8 \ 10,11 \ 13,14); a*b
```

	1	2
1	66	72
2	156	171

Matrix Multiplication

Note that the number of **columns** in the first matrix must be equal to the number of **rows** in the second matrix.

$$M_{a \times b} \times N_{b \times c} = P_{a \times c}$$

For example, we cannot multiply the following matrices together:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2 \times 3} \begin{pmatrix} 7 & 8 \\ 10 & 11 \end{pmatrix}_{2 \times 2}$$

Some properties of matrix multiplication:

- Matrix multiplication is **not commutative**. That is, $AB \neq BA$ in general.

```
** A*B
mata:
a*b
b*a
end
```

```
. mata:
----- mata (type end to exit) -----
: a*b
      1      2
+-----+
1 |   66   72 |
2 |  156  171 |
+-----+

: b*a
[symmetric]
      1      2      3
+-----+
1 |   39           |
2 |   54   75       |
3 |   69   96  123   |
+-----+

: end
-----

.
```

- Matrix multiplication is **associative**. That is, $A(BC) = (AB)C$.

```
mata: c=(4,1\2,4)
```



```

** A*(B*C)
mata:
  c=(4,1\2,4)
  a*(b*c) ; (a*b)*c
end

```

```

. mata:
----- mata (type end to exit) -----
:  c=(4,1\2,4)

:  a*(b*c) ; (a*b)*c
      1      2
      +-----+
1 |  408   354 |
2 |  966   840 |
      +-----+
      1      2
      +-----+
1 |  408   354 |
2 |  966   840 |
      +-----+

: end
-----

.

```

- Any matrix multiplied by I is equal to itself. That is, $AI = IA = A$.

```

** A*(B*C)
mata:
a*I(3)
b*I(2)
end

```

```

. mata:
----- mata (type end to exit) -----

```

```

: a*I(3)
      1    2    3
    +-----+
  1 |  1    2    3 |
  2 |  4    5    6 |
    +-----+

```

```

: b*I(2)
      1    2
    +-----+
  1 |   7    8 |
  2 |  10   11 |
  3 |  13   14 |
    +-----+

```

```

: end

```

.

Transpose

The transpose of a matrix is a matrix where the rows and columns are swapped. For example, if the matrix A is defined as:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

then the transpose of A , denoted A^T , is:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Note that if $A_{a \times b}$, then $A_{b \times a}^T$.

```

mata: a_t = a'

```

Some properties of the transpose:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A + B)^T = A^T + B^T$
- $(aA)^T = aA^T$
- $(A^T)^{-1} = (A^{-1})^T$

Inverse

The inverse of a square matrix is a matrix that, when multiplied by the original matrix ($AA^{-1} = I$), results in the identity matrix. For example:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} -3 & 1 \\ 2 & -.5 \end{bmatrix}$$

```
mata
a = (1,2 \ 4,6)
a_inv = luinv(a); a_inv
a*a_inv
end
```

```
. mata
----- mata (type end to exit) -----
: a = (1,2 \ 4,6)

: a_inv = luinv(a); a_inv
      1      2
+-----+
1 |  -3      1 |
2 |   2    -.5 |
+-----+

: a*a_inv
[symmetric]
      1      2
+-----+
1 |   1      |
2 |   0      1 |
```

```

+-----+
: end
-----
.

```

For a 2×2 matrix, the inverse is defined as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus, if a matrix has determinant 0, then it is not invertible.

Determinant

The determinant of a **square** matrix is a scalar value that is a function of the elements of the matrix. The determinant of a 2×2 matrix is defined as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant of a 3×3 matrix is defined as:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gb f - ceg - fha - ibd$$

```
mata: det(a)
```

Rank and linear independence

- The rank of a matrix is the number of linearly independent rows or columns in the matrix.
 - In a rectangular matrix, the rank cannot be larger than the smaller of the rows or columns.
- If we consider each column, or rows, of a matrix as a vector, then the rank of the matrix is the number of **linearly independent** vectors in the matrix.
- If a set of vectors are not linearly independent, then one of the vectors can be expressed as a linear combination of the other vectors. For example, the following vectors are not linearly independent:

$$a_1\vec{x}_1 + a_2\vec{x}_2 + a_3\vec{x}_3 = 0$$

`mata: rank(a)`

System of linear equations

A system of linear equations is a set of equations that can be expressed in the form:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n\end{aligned}$$

where a_{ij} and b_i are constants, and x_i are variables.

This system of equations can be written in matrix form as:

$$A_{n \times n} X_{n \times 1} = b_{n \times 1}$$

if the system has a unique solution, then the matrix A is invertible, and the solution is given by:

$$X = A^{-1}b$$

Thus if there is no solution, then A is not invertible. If the determinant of A is 0, then A is not invertible.