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STRUCTURAL MATRICES OF NATIONAL ECONOMIES

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This paper is presented in the form of a general report on a new method of quantitative analysis of the national economy, a method which passed more or less certain preliminary laboratory tests and even what we call in the United States the stage of a pilot-plant production. I will make an attempt, in this half an hour, to present to you the basic procedure without trying at the same time to anticipate possible comments and criticisms. These can be brought out much better in the course of the discussion.

The first, simplest approach to the study of the quantitative relationship within a national economy is obviously a partial one: an approach in which we center our attention on some small sector—it might be the wheat market or the production function of the steel industry or the shipbuilding cycle—and using the hypothesis of the partial-equilibrium theory try to fill in the necessary numerical constants on the basis of certain more or less sophisticated statistical procedures. However, it is quite clear that most of the most interesting and significant phenomena which we as econometricians want to study are related not to partial equilibria but to the general, over-all equilibrium, which involves more or less simultaneously all parts of an economic system, the type of the relationship which traditionally the general-equilibrium theory deals with.

Up to a relatively recent time it was considered to be too difficult, and certainly not very promising, to indulge in a real empirical general-equilibrium analysis. What was used instead, particularly since the advent of Keynesian theory, was the shortcut device of aggregative analysis. It is an attempt to deal with a general-equilibrium problem involving implicitly all parts of the economic system, but at the same time to keep down the number of the variables by using extremely broad averages, *i.e.*, by dealing with such composite variables as the "total level of production," or the "general price level," "all exports," "total employment," or "average productivity," all of which are obviously very broad index numbers.

Anybody who was concerned with the practical application of econometric analysis, I think, is conscious of the fact that in a large number of instances, these aggregative measures are not very useful. Particularly in connection with many problems of policy-making and of economic planning of any kind, aggregative concepts are very limited in their application, because in this type of question we have to deal with concrete, separate industries, with individual prices, or at least outputs and prices of small commodity groups.

Sooner or later, the econometricians will have to devise some method of dealing with the national economy in rather differentiated terms comparable to those which he uses in partial analysis of a wheat or a steel market; but at the same time taking account of the general interrelationships between the separate parts of the national economy. The method that I shall now associate with the concept of a structural matrix of the national economy enables us to approach, very hesitantly and imperfectly, the solution of the problem of combining the general-equilibrium analysis with preservation of a differentiated classification of all individual aspects of the economic phenomena.

Let me first acquaint you with the basic factual material, which by its form of presentation suggests the type of theoretical handling that I shall tell you about subsequently. Table 1 is not a fictitious table; it contains actual statistics pertaining to the United States in the year 1939. The interpretation of these figures is very simple. The first row, for example, shows the amounts of the products of Agriculture and Fishing absorbed by various industries, totalling $12,575 million in the last column. Similarly, the first column shows the amounts of the products of various industries absorbed by Agriculture and Fishing. These figures represent the basic factual data on which all the subsequent analysis is based. Actually, we often use much more detailed figures; we have data for nearly one hundred groupings of American industries. There is no limit but the practical one — no conceptual limit to the amount of detail that can be put into this presentation.

Obviously these figures are not independent of each other. A planner could not possibly assign arbitrary values to all these figures. There is a necessary relationship between magnitudes in certain parts of this table and magnitudes in certain other parts. There is, for example, a clear-cut relationship between the total output of as given industry and the total input that it absorbs of commodities and services from other industries. This is the relationship that Walras describes in terms of his production function, his coefficients of production, each coefficient describing the amount of any particular input necessary to produce one unit of the final output.

The figures are given in dollars, but may be considered as representing physical quantities if the physical unit is taken as one dollar's worth of each commodity in the given year. This is a very convenient way of using the data.

Now we can pass to the theoretical manipulation of these figures; and to do that I will present a few very simple, very elementary equations:

$$
\text{(I)} \qquad
\begin{array}{ll}
X_1 - x_{21} - x_{31} - \dots - x_{m1} & = x_{n1}, \\
-x_{12} + X_2 - x_{32} - \dots - x_{m2} & = x_{n2}, \\
-x_{13} - x_{23} + X_3 - \dots - x_{m3} & = x_{n3}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\
-x_{1m} - x_{2m} - x_{3m} - \dots + X_m & = x_{nm}, \\
-x_{1n} - x_{2n} - x_{3n} \quad \dots - x_{mn} + X_n & = 0.
\end{array}
$$

The system (I) of equations represents the equilibrium conditions of production and consumption (use) of each commodity. Each of the x_{i1} in the first equation represents the amount of the commodity of industry 1 used in industry $i = 2, 3, \dots$. The X_1 shows the total output of the industry 1 and hence the combined use. The total production equals the total use, *i.e.*, all uses combined equal output. But it is one particular use x_{n_1}, namely the so-called final use in households or government, that I transfer to the right-hand side of the equation, since subsequently it is treated as an independent variable. In a similar way we can set up a separate equation to describe the balance between the production and consumption of the output of each other industry. The last of these equations describes the balance between the separate labor inputs absorbed by various industries and the total combined labor input X_n.

Now, using the coefficients of production we pass to the system (II) of equations which shows how to define certain constants a_{ik}:

$$
\text{(II)} \qquad x_{ik} = a_{ik} X_i, \; i = 1, 2, \dots, m; \; k = 1, 2, \dots, m, n; \; i \neq k.
$$

You can interpret these constants as representing the ratio between the input of a particular good in a given industry divided by the total output of that industry; in other words, it is input per unit of output.

TABLE

Allocation of Goods and Services by

[*All figures in*

INDUSTRY

INDUSTRY PRODUCING	(1) Agriculture and fishing	(2) Food tobacco, and kindred products	(3) Ferrous metals	(4) Motor vehicles, industrial and heating equipment	(5) Metal fabricating	(6) Nonferrous metals and their products	(7) Nonmetallic minerals and their products	(8) Fuel and power	(9) Chemicals
1. Agriculture and fishing. . . .	950	4,998	....	....	....	....	....	....	176
2. Food, tobacco and kindred products	645	1,530	....	....	....	....	....	....	47
3. Ferrous metals. .	24	....	1,188	479	861	43	....	....	....
4. Motor vehicles, industrial and heating equipment. .	188	72	4	1,645	7	9	19	109	7
5. Metal fabricating.	433	306	37	611	717	12	5	137	40
6. Nonferrous metals and their products	5	23	109	117	221	1,325	4	51	89
7. Nonmetallic minerals and their products.	14	137	29	70	64	6	280	6	127
8. Fuel and power. .	474	168	318	102	164	65	185	2,452	197
9. Chemicals	357	133	36	34	108	3	17	13	828
10. Lumber, paper, and their products, printing and publishing. .	94	260	1	35	63	6	46	4	69
11. Textiles and leather.	66	43	....	105	8	1	2	....	13
12. Rubber.	54	3	....	195	22	....	....	1	4
13. All other manufacturing.	2	....	....	13	23	....	....	....	1
14. Construction. . .	342	70	41	24	42	8	18	821	18
15. Transportation. .	793	392	266	108	135	75	295	2,200	222
16. Trade.	1,446	4,052	78	1,260	1,254	25	394	1,892	800
17. Foreign countries (imports from). .	337	824	22	10	17	331	63	81	161
18. Business and consumer services. .	550	376	13	85	77	4	12	39	183
19. Households and Government. . .	5,624	3,584	1,043	2,362	3,078	721	779	4,683	1,126
20. Unallocated and stocks.	1,347	1,952	536	724	2,241	599	622	1,383	819
Total gross outlays.	13,745	18,923	3,721	7,979	9,102	3,233	2,741	13,872	4,927

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Industry of Origin and Destination, 1939

millions of dollars]

PURCHASING

(10) Lumber, paper, and their products printing, and publishing	(11) Textiles and leather	(12) Rubber	(13) All other manufacturing	(14) Construction	(15) Transportation	(16) Trade	(17) Foreign countries (exports to)	(18) Business and consumer services	(19) Households and Government	(20) Unallocated and stocks	Total gross output
185	583	....	7	167	....	....	453	....	4,495	461	12,475
7	156	....	2	....	....	....	269	1	15,751	391	18,799
64	....	....	....	592	69	....	171	1	12	383	3,887
96	102	16	....	423	77	76	409	3	2,819	1,591	7,672
63	16	....	12	1,301	380	108	544	195	1,657	2,118	8,692
5	1	....	50	144	....	....	159	6	58	589	2,956
51	2	5	8	1,401	13	2	65	2	282	170	2,734
202	138	33	44	127	768	712	519	111	4,990	1,823	13,592
120	326	31	42	426	16	....	203	38	1,508	672	4,911
2,152	56	5	29	910	9	270	142	1,406	1,961	1,375	8,893
84	3,122	58	15	2	....	8	168	29	7,879	429	12,032
1	30	20	2	4	32	20	41	33	348	360	1,170
10	117	....	180	16	30	55	71	373	1,075	394	2,360
42	22	4	11	....	828	189	....	251	7,358	....	10,089
387	52	31	8	....	138	4	103	....	1,919	493	7,621
995	2,928	270	550	....	....	....	....	....	....	2,618	18,562
259	381	196	61	....	286	....	....	....	....	92	3,121
63	83	41	16	4	73	686	2	404	15,709	2,151	20,571
2,366	2,821	339	812	3,940	5,467	11,023	....	14,516	17,285	7,407	88,976
1,777	1,634	164	564	632	....	5,409	....	3,442	....	....	23,845
8,929	12,570	1,213	2,413	10,089	8,186	18,562	3,319	20,811	85,106	23,517	

We have, of course, as many of these equations of type (II)—defining the Walrasian technical coefficients—as there are separate kinds of inputs in all the different industries.

Substituting each of these equations from system (II) into system (I), we can eliminate the x_{ik} and represent each separate input in one industry as a function of the total output X_1 of that particular industry. System (III) is the resulting set of equations:

$$
\text{(III)}\quad
\begin{array}{l}
X_1 - a_{21}\,X_2 - a_{31}\,X_3 - \ldots - a_{m1}\,X_m = x_{n1},\\
-a_{12}\,X_1 + X_2 - a_{32}\,X_3 - \ldots - a_{m2}\,X_m = x_{n2},\\
-a_{13}\,X_1 - a_{23}\,X_2 + X_3 - \ldots - a_{m3}\,X_m = a_{n3},\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
-a_{1m}\,X_1 - a_{2m}\,X_2 - a_{3m}\,X_3 - \ldots + X_m = x_{nm},\\
-a_{1n}\,X_1 - a_{2n}\,X_2 - a_{3n}\,X_3 - \ldots - a_{mn}\,X_m + X_n = 0.
\end{array}
$$

It contains $m+1$ equations, based on the system of $(m+1)n$ coefficients, the a_{ik}. The variables in this case are the X_i which represent total outputs; with m industries you have m outputs plus X_n representing the total labor input.

The x_{ni} on the right-hand side—representing final demand—remain on the outside as independent variables. Thus we can solve this system of equations, expressing all the X's as functions of the final demands. In other words, if we know the magnitudes of all the a's, and if our general theoretical assumptions are correct, we can predict or compute the outputs of all individual industries, one by one, as a function of a final demand. In the same way, of course, we can compute the employment (if we know the employment coefficients) industry by industry as a function of the final demand.

The actual solution is written out in a general form in equation (IV):

$$\text{(IV)}\quad X_i = A_{i1}x_{n1} + A_{i2}x_{n2} + \ldots + A_{im}x_{nm}, \qquad i = 1, 2, 3, \ldots, m, n.$$

Here the total output of an industry, X_i, is expressed as a function of the demand for the separate commodities, the A being fixed coefficients which are computed from the determinant of the matrix coefficients of all the technical coefficients. This is what I referred to as the structural matrix of the economy. (For those of you who want more detail, it can be said that each A can be written out in determinantal form: $A_{ik} = |M_{ik}|/|M|$. M is the determinant of the matrix $|\,a_{ik}\,|$ and $|\,M_{ik}\,|$ is a complement of element a_{ik} in this matrix.)

If we want to compute the dependence of each type of output upon each kind of final demand, we simply have to find the inverse of the structural matrix. Now, of course, if this matrix is three by three, three rows by three columns, it is easy to perform such inversion with a pencil in hand; if it is ten by ten, we need a computing machine; and if it is 40 by 40, the computation becomes pretty complicated. Fortunately, the modern computing machines, for example the one we have at Harvard, do it without difficulty. So there are no serious computational difficulties even with systems as large as 90 by 90.

Let us consider now another problem, that of price relationships. The same basic method of analysis can be applied here as that used in the study of the physical structure of the economic system. Total value, *i.e.*, quantity times selling price, of a finished commodity, can be equated, by definition, to the quantity of all cost factors purchased from other industries, multiplied by their respective prices, plus the quantity of labor hired multiplied by the wage rate, plus profits π earned per unit of output.

As an economic theorist, I certainly would like to have a complete theory of profits that would enable me to explain the profits earned in an industry as a function of some other magnitudes. But we do not have such a theory yet, and in the following analyses the profits earned per unit of output in an industry are considered as a parameter, as an independent variable. In the course of future work we—let us hope—will be able to expand our theoretical system, add additional equations, and cease to consider this to be a parameter, but rather explain it as one of the dependent variables.

We have m prices, $P_1, P_2, \ldots, P_m$, one for each particular kind of output. We have the wage rate, let us call it P_n (I reserve the subscript n for price of labor) and m profit ratios π, one for each industry: in system (V)

$$\text{(V)}\quad \begin{array}{l} P_1 - a_{12}\,P_2 - a_{13}\,P_3 - \ldots - a_{1m}\,P_m - a_{1n}\,P_n - \pi_1 = 0, \\ -a_{21}\,P_1 + P_2 - a_{23}\,P_3 - \ldots - a_{2m}\,P_m - a_{2n}\,P_n - \pi_2 = 0, \\ -a_{31}\,P_1 - a_{32}\,P_2 + P_3 - \ldots - a_{3m}\,P_m - a_{3n}\,P_n - \pi_3 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -a_{m1}\,P_1 - a_{m2}\,P_2 - a_{m3}\,P_3 - \ldots + P_m - a_{mn}\,P_n - \pi_m = 0. \end{array}$$

There are m equations, and $2m + 1$ variables, obviously fewer equations than unknowns, but what we can do is to determine certain limits within which the system must lie. For example, if some central planning board were to fix all the prices, and enforce certain wage levels, there would be only one profit system—one column of profit rates—which could be earned in different industries and be consistent with the given prices and wages. Or, putting it in a different way, even the most powerful central planning board, if it were to fix the wage rate and the profit rate in all industries, could prescribe only one price system that would be consistent with these previous decisions related to the wages and the profit rate. So from any two sets of these three sets of variables, prices, wages, and profits, the remaining third can be computed.

On what basis? Again on the basis of the same matrix of technical coefficients. As a matter of fact, the system of equations corresponding to the value relationship has the same matrix as that of the quantity relationship, only turned around in the sense that its rows and columns are interchanged.

There are some quite interesting applications, if this has an empirical validity, for the explanation of the relation between the prices, wages, and profits within our system. This type of approach, for example, would enable us (and that has been done) to answer the following question: Imagine we keep the profit rate constant. We decide to increase the wage rate by ten percent. By how much will the prices increase in all the different industries? The effects, as the computation (and of course the experience without any computation) shows, are very differentiated; and when we compute them through, we get a result that is even for practical purposes certainly much more significant than some index of an "average" price increase for the system as a whole.

Now let me make some remarks on the further work in the same direction. Our system as described above is a static system, because it deals entirely with flows of commodities, or rather with the rates of flow, and as such has a validity only for relatively short-run analysis in which the stocks of commodities, at least a dependent element in the system, can be neglected. The analysis for the long run must take in account the fact that in order to produce say, steel or cotton cloth, you need not only flows of certain cost factors, but also stocks, inventories. The stock of machinery is the difference between the rate of purchase and wearing-out of machinery. The same thing applies to buildings. Theoretically or mathematically, this means that we have to introduce technical coefficients of stocks which show the relation between the integral of the difference between the inflows and the rate of use of separate factors in a given industry on the one hand and its rate of output on the other. This transforms our system of ordinary linear equations into a system of linear differential equations. Theoretically, of course, it is not difficult to handle; there is only the problem of irreversibility. When output is increased, capital has to be accumulated; when output is reduced, the full capacity of the previously accumulated plant can not be used. Fortunately, the modern computing machines can perform the difficult operation of integrating differently upwards and downwards. So this difficulty can also be overcome. The second problem is that of getting sufficient empirical data in order to replace the algebraic letters in our formulae with actual figures. The work in this direction has progressed considerably, and for many industries we have already compiled rather detailed sets of investment coefficients.

Résumé

L'article donne un aperçu général d'une méthode nouvelle d'analyse quantitative de l'économie nationale. Un secteur étroit de l'économie nationale peut être étudié sur la base d'équilibres partiels. Il est évident, toutefois, que la plupart des phénomènes les plus intéressants et significatifs sont en rapport, non avec des équilibres partiels, mais avec l'équilibre général, compréhensif, qui englobe plus ou moins simultanément toutes les parties d'un système économique. En raison des difficultés que cela entraine, les économistes ont employé le procédé simplifié d'une analyse d'ensemble, mais ceci n'est pas suffisamment détailé pour être utile, spécialement dans les problèmes de politique économique et de planification économique.

La méthode exposée ici permet de résoudre le problème de combiner l'analyse d'équilibre général avec le maintien d'une classification différentiée de tous les aspects individuels des phénomènes économiques.

Le tableau donne des statistiques concernant les Etats-Unis pour l'année 1939. Chaque ligne montre comment les produits d'une industrie sont distribués et chaque colonne montre la valeur en dollars des produits fournis à l'industrie. Il y a 20 groupes d'industries dans le tableau; toutefois, l'on dispose de chiffres plus dètaillés se rapportant à une centaine d'industries.

La manipulation théorique de ces chiffres est facilitée par quelques formules simples. Dans le système (I) chacune des x_{ik} indique le montant du produit de l'industrie k utilisé dans l'industrie i, tandis que x_k est la production totale de l'industrie k. Dans le système (II) les constantes a_{ik} indiquent le rapport entre le montant du produit k utilisé dans l'industrie i et la production totale de l'industrie i. Le système (III) est déduit en éliminant les x_{ik} des systèmes (I) et (II). Au côté droit, il reste les x_{ik} représentant les montants consommés finalement par les ménages et l'administration publique. Alors, si l'on connait la valeur de tous les a_{ik} et si nos hypothèses théoriques générales sont exactes, nous sommes à même de prédire ou de calculer la production de chaque industrie individuelle, en fonction de la demande finale, suivant le système (IV). Les A sont des coefficients fixes calculés sur la base du déterminant de la matrice de tous les coefficients techniques. Cette dernière est appelée la matrice structurelle de l'économie, et la matrice des A est son inverse.

Si nous désirons calculer la relation de chaque type de demande finale, il faut simplement trouver l'inverse de la matrice structurelle, ce qui est facile à faire à l'aide des machines à calculer modernes telles que le calculateur "Aiken Relay" de l'Université Harvard. Dans le système

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V les prix, P_1——P_m, de chaque type de marchandise produite, P_n, le taux des salaires, et le pourcentage des bénéfices, π_1——π_m pour chaque industrie, sont introduits. Il y a n équations, évidemment moins nombreuses que le nombre d'inconnues, mais il est possible de déterminer, pour ces trois groupes d'inconnues, la dépendance d'un groupe quelconque par rapport aux deux autres. C'est-à-dire, à exprimer chaque prix en fonction des salaires et des bénéfice, ou par exemple, déterminer les bénéfices dans une certaine industrie en fonction des salaires et prix donnés.

L'analyse exposée ci-dessus a un caractère statique, parce qu'elle traite de courants de marchandises ou de leurs taux de circulation, et néglige les stocks. L'analyse à longue échéance doit tenir compte des stocks, ce qui transforme notre système d'équations linéaires ordinaires en un système d'équations différentielles linéaires. Les problèmes de calcul numérique y relatifs peuvent être résolus à l'aide de machines modernes; le travail pour obtenir les données statistiques nécessaires fait des progrès satisfaisants.

Mr. Leontief's paper was discussed by Messrs. Tjalling C. Koopmans, François Divisia, Michal Kalecki, Leonid Hurwicz, Donald C. MacGregor, and the speaker.