COMMUTATIVE ALGEBRA

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Contents

1. Rings and ideals

1.1. Prime avoidance lemma.

Theorem 1.1. *Let $P_1, \cdots, P_n \in \operatorname{spec} R$ then*

$$I \subseteq \bigcup_{i=1}^{n} P_i \Rightarrow I \subseteq P_i \textit{ for some } i.$$

Proof. We have to prove that if $I \nsubseteq P_i,\ \forall\, 1 \leq i \leq n$ then $I \nsubseteq \bigcup_{i=1}^{n} P_i$. We proceed by induction on n. If $n = 1$ then we are done. Suppose, the statement is true for $n-1$ ideals. We consider $P_2, \cdots, P_n$ and we have $I \nsubseteq P_i,\ 2 \leq i \leq n$ then by induction hypothesis $I \nsubseteq \bigcup_{i=2}^{n} P_i$ then $\exists x_i \in I$ such that $x \notin \bigcup_{i=2}^{n} P_i$ i.e., $x \notin P_i,\ 2 \leq i \leq n$. If $x_1 \notin P_1$ then $x_1 \notin \bigcup_{i=1}^{n} P_i$ and hence $I \nsubseteq \bigcup_{i=1}^{n} P_i$ and we are done. So we may assume $x_1 \in P_1$ and $x_1 \notin P_i,\ 2 \leq i \leq n$. Now we consider $\{P_1, P_2, \cdots, P_n\} \setminus \{P_2\}$ and by similar approach we get $x_2 \in I$ with $x_2 \in P_2$ and $x_2 \notin P_i,\ \{1, \ldots, n\} \setminus \{2\}$ and lastly we get $x_n \in I$ with $x_n \in P_n$ and $x_n \notin P_1,\ 1 \leq i \leq n-1$. We consider

$$x = x_2 \cdots x_n + x_1 x_3 \cdots x_n + x_1 x_2 x_4 \cdots x_n + \cdots + x_1 \cdots x_{n-1}$$

then $x \in I$. We claim that $x \notin \bigcup_{i=1}^{n} P_i$ i.e., $x \notin P_1,\ 1 \leq i \leq n$. If $x \in P_i$ for some i. Let

$$y_i = x_1 \cdots \widehat{x_i} \cdots x_n$$

then $x_i | y_j$ for $i \neq j \Rightarrow y_j \in P_i$ $[x_i \in P_i] \Rightarrow \sum_{\substack{j=1 \\ j \neq i}} y_j \in P_i \Rightarrow x - \sum_{\substack{j=1 \\ j \neq i}}^{n} y_j \in P_i$ $[\because x \in P_i] \Rightarrow y_i \in P_i \Rightarrow$ $x_1 \cdots \widehat{x_i} \cdots x_n \in P_i$ but $x_j \notin P_i,\ j \neq i$ [since P_i is a prime ideal] $\Rightarrow x \notin \bigcup_{i=1}^{n} P_i$. Hence, $I \nsubseteq$. ☐

Remark 1.2. *Prime avoidance lemma is not true for infinite number of prime ideals.*

Example 1.3. *Let $R = K[x_1, \cdots, x_n, \cdots]$ (infinitely many variables). Let $I = (x_1, \cdots, x_n, \cdots), P_i = (x_1, \cdots, x_i), i \in \mathbb{N}$ then $R/P_i \cong K[x_{i+1}, x_{i+2}, \cdots,]$ (integral domain) then $P_i \in$ spec R. But $I \subseteq \bigcup_{i \in \mathbb{N}} P_i$ and $I \subseteq P_i$ $\forall i \in \mathbb{N}$.*

Theorem 1.4 (Prime avoidance lemma)**.** *Let R be a commutative ring with 1, I be an ideal of R and $f \in R$. Suppose $P_1, \cdots, P_r \in \operatorname{spec} R$ such that $f + I = \bigcup_{i=1}^{r} P_k$ then $\langle f, I \rangle \subseteq P_i$ for some $i \in \{1, \cdots, r\}$.*

Proof. Let $\sum$ be the collection of all $s \in \mathbb{N}$ such that there exist $t \in R$ and an ideal J of R such that $t + J \subseteq \bigcup_{i=1}^{s} P_i$ but $\langle t, J \rangle \nsubseteq P_i, 1 \leq i \leq s$. If $\sum \neq \emptyset$ then by well ordering principle of Natural numbers, $\sum$ has a least element say $l \in \sum$. So there exist $g \in R$ and $\mathfrak{A} \subseteq R$ such that

$g+\mathfrak{A} \subseteq \bigcup_{i=1}^{l} P_i$ but $\langle g, \mathfrak{A}\rangle \nsubseteq P_i, 1 \leq i \leq l$. We note that $l \geq 2$ and $P_i \nsubseteq P_j$. We claim that $g \in \bigcap_{i=1}^{l} P_i$. If not, $g \notin P_{i_0}$ for some $i_0 \in \{1, \cdots, l\}$, then $(g+P_{i_0}\mathfrak{A}) \cap P_{i_0} = \emptyset$ hence $g+P_{i_0}\mathfrak{A} \subseteq \bigcup_{\substack{j=1 \\ j\neq i_0}}^{l} P_j$. Since l is the minimal element of $\sum$, we have $\langle g, P_{i_0}\mathfrak{A}\rangle \subseteq P_{j_0}$ for some $j_0 \in \{1, \cdots, l\}$ but $j_0 \neq i_0$. Then $P_{i_0}\mathfrak{A} \subseteq P_{j_0} \Rightarrow P_{i_0} \subseteq P_{j_0}$ which is a contradiction (since if $\mathfrak{A} \subseteq P_{j_0}$ then $\langle g, P_{j_0}\mathfrak{A}\rangle \subseteq P_{j_0}$ implies $g \in P_{j_0} \Rightarrow \langle g, \mathfrak{A}\rangle \subseteq P_{j_0}$ but $\langle g, \mathfrak{A}\rangle \nsubseteq P_i$ for all $1 \leq i \leq l$ so, $\mathfrak{A} \nsubseteq P_{j_0}$). Therefore, $g \in \bigcap_{i=1}^{l} P_i \Rightarrow \mathfrak{A} \subseteq \bigcup_{i=1}^{l} P_i \Rightarrow \mathfrak{A} \subseteq P_s$ for some $1 \leq s \leq l$. Then by our assumption $\langle g, \mathfrak{A}\rangle \subseteq P_s$ but $\sum \neq \emptyset$ which is a contradiction. Hence our assumption is not true that is $\sum = \emptyset$. □

Proposition 1.5. *Let $I_1, \cdots, I_r$ be ideals of R and $P \in spec\,R$. If $\bigcap_{k=1}^{r} I_k \subseteq P$ then $I_k \subseteq P$ for some $k \in \{1, \cdots, r\}$.*

Proof. Since $\prod_{k=1}^{r} I_k \subseteq \bigcap_{k=1}^{r} I_k \subseteq P$, by definition of prime ideal $I_k \subseteq P$ for some $1 \leq k \leq r$. □

Theorem 1.6 (Module theoretic version)**.** *Let R be a commutative ring with 1 and $P_1, \cdots, P_m \in spec\,R$, M be an $R-$module and $x_1, \cdots, x_n \in M$. Consider the submodule $N = \langle x_1, \cdots, x_n\rangle$ of M. If $N_{P_j} \nsubseteq P_j M_{P_j}, j = 1, \cdots, m$ then there exist $a_2, \cdots, a_n \in R$ such that $x_1 + \sum_{i=1}^{n} a_i x_i \notin P_j M_{P_j}$.*

Proof.

2. Module

2.1. Tensor Product.

Definition 2.1. *Let $M_1, \cdots, M_k, N$ be $R-$modules. A map $f : M_1 \times \cdots \times M_k \to N$ is said to be linear in ith variable if, given fixed $m_j, j \neq i$, the map*

$$T : M_i \to N$$

defined by $T(m) = f(m_1, \cdots, m_{i-1}, m, m_{i+1}, m_k)$ is linear. The map f is said to be multilinear if it is linear in each variable.

Let M, N be two $R-$modules. Consider the free module F generated by the $M \times N$ over R, then the elements of F are of the form $\sum\limits_{\substack{\text{finite}\\ \text{sum}}} r_i x_i$ where $r_i \in R$ and $x_i \in M \times N$. Let D be the submodule of F generated by the elements of the form

$$\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n)\\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2)\\
&(rm, n) - r(m, n)\\
&(m, rn) - r(m, n)
\end{aligned}$$

where $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$. Let $T = F/D$. We denote $T = M \otimes_R N$ and T is said to be Tensor product of M and N. We denote $(m, n) + D \in F/D$ by $m \otimes n$ and we have a map

$$\begin{aligned}
M \times N &\xrightarrow{\pi} T\\
(m, n) &\mapsto m \otimes n
\end{aligned}$$

We will show that π is bilinear map. $\pi((m_1 + m_2, n)) = (m_1 + m_2, n) + D$. Since

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n) \in D$$

$\pi((m_1 + m_2, n)) = (m_1 + m_2, n) + D = (m_1, n) + D + (m_2, n) + D = \pi(m_1, n) + \pi(m_2, n)$ for all $m_1, m_2 \in M$ and for all $n \in N$. Similarly we can show that θ satisfies the property of bilinear map.

Theorem 2.2 (Universal Property)**.** *For every bilinear map $\beta : M \times N \to P$ where P is an $R-$module, there exists an unique $R-$linear map $\tilde{\beta} : M \otimes_R N \to P$ such that the diagram commutes.*

$$\begin{tikzcd}
M \times N \arrow[r, "\beta"] \arrow[d, "\pi"'] & P \\
M \otimes_R N \arrow[ur, "\tilde{\beta}"'] &
\end{tikzcd}$$

More over, if (T', θ') be another pair with such property then there exists a module isomorphism $M \otimes_R N \to T'$.

Proof. Define $\tilde{\beta} : M \otimes_R N \to P$ by $\tilde{\beta}(m \otimes n) = \beta(m, n)$ and extend it linearly. Let $m_1 \otimes n_1 = m_2 \otimes n_2 \Rightarrow (m_1, n_1) - (m_2, n_2) \in D$. Since

By our construction $\tilde{\beta}$ is bilinear. Suppose $\gamma : M \otimes_R N \to P$ be another $R-$linear map such that the diagram commutes. Then $\gamma(m \otimes n) = \beta(m, n) = \tilde{\beta}\pi(m, n) = \tilde{\beta}(m \otimes n)$. Hence $\gamma = \tilde{\beta}$.

Now we assume that there exists another pair (T', θ') with same property, then

$$\begin{tikzcd}
M \times N \arrow[r, "\pi"] \arrow[d, "\theta'"'] & M \otimes_R N \\
T' \arrow[ur, "\tilde{\pi}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times N \arrow[r, "\theta'"] \arrow[d, "\pi"'] & T' \\
M \otimes_R N \arrow[ur, "\tilde{\theta}'"'] &
\end{tikzcd}$$

where $\tilde{\pi}$ and $\tilde{\theta}'$ are $R-$ linear map. Since the diagrams commutes, we have $\tilde{\pi} \circ \theta' = \pi$ (from first diagram) and $\tilde{\theta}' \circ \pi = \theta'$ (from second diagram). Hence $(\tilde{\theta}' \circ \tilde{\pi}) \circ \theta' = \theta'$ and $(\tilde{\pi} \circ \tilde{\theta}') \circ \pi = \pi$. Again we consider the following diagrams

$$\begin{tikzcd}
M \times N \arrow[r, "\pi"] \arrow[d, "\pi"'] & M \otimes_R N \\
M \otimes_R N \arrow[ur, "\mathrm{id}_{M \otimes_R N}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times N \arrow[r, "\theta'"] \arrow[d, "\theta'"'] & T' \\
T' \arrow[ur, "\mathrm{id}_{T'}"'] &
\end{tikzcd}$$

By Universal property, we have $\tilde{\theta}' \circ \tilde{\pi} = id_{T'}$ and $\tilde{\pi} \circ \tilde{\theta}' = \mathrm{id}_{M \otimes_R N}$. $\square$

Tensor product of algebras. Let A and B be $R-$algebra, We consider the module $C = A \otimes_R B$. Let us define a mapping $\beta : A \times B \times A \times B \to C$ by $\beta(a, b, a', b') = aa' \otimes bb'$. Since β is multilinear, β induce a mapping $\tilde{\beta} : C \otimes_R C \to C$. This $\tilde{\beta}$ corresponds a bilinear mapping $\gamma : C \times C \to C$ given by $\gamma(a \otimes b, a' \otimes b') = aa' \otimes bb'$. Since γ is well define, it defines a multiplication on C and therefore C becomes a commutative ring with unity, $1 \otimes 1$ being the multiplicative identity. Since A and B are $R-$algebra, there exists $f : R \to A$ and $g : R \to B$ two ring morphisms. Now we define $\psi : R \to A \otimes_R B$ by $\psi(r) = f(r) \otimes g(r)$. Let $r_1, r_2 \in R$ then $\psi(r_1 + r_2) = f(r_1 + r_2) \otimes g(r_1 + r_2) = f(r_1) \otimes g(r_1 + r_2) + f(r_2) \otimes g(r_1 + r_2)$.

We note that C is both A and B algebra as $\mu_A : A \to A \otimes_R B$ is defined by $\mu_A(a) = a \otimes 1_B$ and $\mu_B : B \to A \otimes_R B$ is defined by $\mu_B(b) = 1_A \otimes b$. It is easy to check that both μ_A and μ_B is a ring homomorphism.

Theorem 2.3 (Properties of Tensor product)**.** *Let M, N, P and $\{M_i\}_{i \in \Lambda}$ be $R-$modules, $I \subseteq R$ be a ideal of R, S be a multiplicatively closed set in R then we have*

(1) $M \otimes_R N \cong N \otimes_R M$.
(2) $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R) P$.
(3) $M \otimes_R R \cong M$.
(4) $M \otimes_R R/I \cong M/IM$.
(5) $M \otimes_R S^{-1}R \cong S^{-1}M$.

(6) $\left(\bigoplus_{i\in\Lambda} M_i\right) \otimes_R N \cong \bigoplus_{i\in\Lambda}(M_i \otimes_R N)$.

Proof. (1) Consider the diagram

$$\begin{tikzcd} M \times N \arrow[r, shift left, "\alpha"] \arrow[d, "\pi"'] & N \times M \arrow[l, shift left, "\beta"] \arrow[d, "\tilde{\pi}"] \\ M \otimes_R N \arrow[r, shift left, "\alpha'"] & N \otimes_R M \arrow[l, shift left, "\beta'"] \end{tikzcd}$$

where $\alpha((m,n)) = (n,m)$ and $\beta((n,m)) = (m,n)$. We claim that $\tilde{\pi} \circ \alpha$ is bilinear. Let $(m_1 + m_2, n) \in M \times N$, $\tilde{\pi}\alpha((m_1 + m_2, n)) = \tilde{\pi}(n, m_1 + m_2) = n \otimes (m_1 + m_2) = n\otimes m_1 + n\otimes m_2 = \tilde{\pi}\alpha((m_1, n)) + \tilde{\pi}\alpha((m_2, n))$ for all $m_1, m_2 \in M$ and for all $n \in N$. Similarly other properties can be shown. By Universal property, we have a module morphism $\alpha' : M \otimes_R N \to N \otimes_R M$. Similarly the map $\beta \circ \pi$ is also bilinear so we have a $R-$ linear map $\beta' : N \otimes_R M \to M \otimes_R M$. We just need to show that $\alpha' \circ \beta = \mathrm{id}_{N\otimes_R M}$ and $\beta' \circ \alpha' = \mathrm{id}_{M\otimes_R N}$ which is easy, $\alpha' \circ \beta'(n \otimes m) = \alpha'(m \otimes n) = n \otimes n$ and $\beta' \circ \alpha'(m \otimes n) = \beta'(n \otimes m) = m \otimes n$.

(2)

(3) Let $f : M \times R \to M$ be the map where $f(m,r) = rm$. Since M is an $R-$module, f is bilinear, hence f induce a map $\tilde{f} : M \otimes_R R \to R$ such that the diagram commutes

$$\begin{tikzcd} M \times R \arrow[r, "f"] \arrow[d, "\pi"'] & M \\ M \otimes_R R \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

where $\tilde{f} \circ \pi = f \Rightarrow f(m,r) = \tilde{f}\pi(m,r) \Rightarrow rm = \tilde{f}(m \otimes r)$ and $\tilde{f}$ is $R-$linear. Let $g : M \to M \otimes_R R$ defined as $g(m) = m \otimes 1$. It is easy to show that g is $R-$linear and $\tilde{f} \circ g = \mathrm{id}_M$ and $g \circ \tilde{f} = \mathrm{id}_{M\otimes_R R}$.

(4) Let $f : M \times R/I \to M/IM$ be the bilinear map defined by $f(m, r + I) = rm + IM$. By Universal property there exists a well define module morphism $\tilde{f} : M \otimes_R R/I \to M/IM$ such that the diagram commutes,

$$\begin{tikzcd} M \times R/I \arrow[r, "f"] \arrow[d, "\pi"'] & M/IM \\ M \otimes_R R/I \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

where $\tilde{f}(m \otimes (r + I)) = rm + IM$. Let $g : M/IM \to M \otimes_R R/I$ be the map $g(m + IM) = m \otimes (1 + I)$. Then g is an $R-$linear map and $g \circ \tilde{f} = \mathrm{id}_{M\otimes_R R/I}$ and $\tilde{f} \circ g = \mathrm{id}_{M/IM}$.

(5) Consider

$$\begin{tikzcd}
M \times S^{-1}R \arrow[r, "f"] \arrow[d, "\pi"'] & S^{-1}M \\
M \otimes_R S^{-1}R \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

where $f\left(m, \dfrac{r}{s}\right) = rm/s$. First we need to check $f$is well defined. Let $\dfrac{r_1}{s_1} = \dfrac{r_2}{s_2}$ then there exists some $s \in S$ such that $s(r_1s_2 - s_1r_2) = 0 \Rightarrow s(r_1s_2 - s_1r_2)m = 0 \Rightarrow s(r_1s_2m - s_1r_2m) = 0 \Rightarrow \dfrac{r_1m}{s_1} = \dfrac{r_2m}{s_2}$. It is obvious that f is bilinear. Then there exits a unique module morphism $\tilde{f} : M \otimes_R S^{-1}R \to S^{-1}M$ where $\tilde{f}\left(m \otimes \dfrac{r}{s}\right) = \dfrac{rm}{s}$. Define $g : S^{-1}M \to M \times S^{-1}R$ by $g\left(\dfrac{m}{s}\right) = m \times \dfrac{1}{s}$. g is well defined module morphism and $g = \tilde{f}^{-1}$.

(6) Let $\theta_i : M_i \to \bigoplus_{i\in\Lambda} M_i$ be the inclusion map. Define

$$f : \left(\bigoplus_{i\in\Lambda} M_i\right) \times N \to \bigoplus_{i\in\Lambda}(M_i \otimes_R N)$$
$$((m_i)_{i\in\Lambda}, n) \mapsto (m_i \otimes n)_{i\in\Lambda}.$$

We will show that f is bilinear. $f((m_i)_{i\in\Lambda} + (m_i')_{i\in\Lambda}, n) = f((m_i + m_i')_{i\in\Lambda}, n) = ((m_i + m_i') \otimes n)_{i\in\Lambda} = (m_i \otimes n)_{i\in\Lambda} + (m_i' \otimes n)_{i\times\Lambda} = f((m_i)_{i\in\Lambda}, n) + f((m_i')_{i\in\Lambda}, n)$. Similarly other properties can be shown. Hence we have a map $\tilde{f} : \left(\bigoplus_{i\in\Lambda} M_i\right) \otimes_R N \to \bigoplus_{i\in\Lambda}(M_i \otimes_R N)$ defined by $\tilde{f}((m_i)_{i\in\Lambda} \otimes n) = (m_i \otimes n)_{i\in\Lambda}$. Define $g : \bigoplus_{i\in\Lambda}(M_i \otimes_R N) \to \left(\bigoplus_{i\in\Lambda} M_i\right) \otimes_R N$ by $g((m_i \otimes n_i)_{i\in\Lambda}) = \sum_{i\in\Lambda}(\theta_i(m_i) \otimes n_i)$. Note that g is $R-$linear. Now, $g \circ \tilde{f}((m_i)_{i\in\Lambda} \otimes n) = g((m_i \otimes n)_{i\in\Lambda}) = \sum_{i\in\Lambda}(\theta_i(m_i) \otimes n) = \left(\sum_{i\in\Lambda}\theta_i(m_i)\right) \otimes n = (m_i)_{i\in\Lambda} \otimes n \Rightarrow g \circ \tilde{f} = \mathrm{id}_{\left(\bigoplus_{i\in\Lambda} M_i\right)\otimes_R N}$.

Let $(m_i \otimes n_i)_{i\in\Lambda} \in \bigoplus_{i\in\Lambda}(M_i \otimes_R N)$, then $\tilde{f} \circ g((m_i \otimes n_i)_{i\in\Lambda}) = \tilde{f}\left(\sum_{i\in\Lambda}(\theta_i(m_i) \otimes n_i)\right) = \sum_{i\in\Lambda}\tilde{f}(\theta_i(m_i) \otimes n_i) = \sum_{i\in\Lambda} = \theta_i(m_i) \otimes n_i = (m_i \otimes n_i)_{i\in\Lambda} \Rightarrow \tilde{f} \circ g = \mathrm{id}_{\bigoplus_{i\in\Lambda}(M_i \otimes_R N)}$.

□

Remark 2.4. *Let $f : A \to B$ be a ring homomorphism. Suppose M is an $A-$module and N is an $B-$module. Then $M \otimes_A N$ has both A and B module structure,*

$$B \times M \otimes_A N \to M \otimes_A N$$
$$(n, m \otimes n) \mapsto m \otimes bm$$

Theorem 2.5. *Let B be an A algebra, M be an $A-$module and N, P be $B-$modules. Then*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

Proof. It is suffices to establish the isomorphism as $B-$module. $\square$

Theorem 2.6 (Hom-Tensor adjunction)**.** *Let M, N, P be $R-$modules. Then*

$$Hom_R\left(M \otimes_R N, P\right) \cong Hom_R\left(M, Hom_R\left(N, P\right)\right).$$

Proof. Define

$$\psi : \mathrm{Hom}_R\left(M \otimes_R N, P\right) \to \mathrm{Hom}_R\left(M, \mathrm{Hom}_R\left(N, P\right)\right)$$
$$f \mapsto \psi(f)$$

where $\psi(f)(m)(n) = f(m \otimes n)$ and

$$\phi : \mathrm{Hom}_R\left(M, \mathrm{Hom}_R\left(N, P\right)\right) \to \mathrm{Hom}_R\left(M \otimes_R N, P\right)$$
$$g \mapsto \phi(g)$$

where $\phi(g)(m \otimes n) = g(m)(n)$. We shall now show that $\phi(g)$ is well defined. Consider the diagram

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\pi"'] & P \\ M \otimes_R N \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

where $f(m, n) = g(m)(n)$. We claim that f is bilinear. $f(m_1 + m_2, n) = g(m_1 + m_2)(n) = g(m_1)(n) + g(m_2)(n) = f(m_1, n) + f(m_2, n)$ for all $m_1, m_2 \in M$ and for all $n \in N$. Now $f(m, n_1 + n_2) = g(m)(n_1 + n_2) = g(m)(n_1) + g(m)(n_2) = f(m, n_1) + f(m, n_2)$ for all $m \in M$ and for all $n_1, n_2 \in N$. Pick $r \in R, m \in M$ and $n \in N$, $f(rm, n) = g(rm)(n) = rg(m)(n) = rf(m, n)$ and $f(m, rn) = g(m)(rn) = rg(m)(n) = rf(m, n)$. By Universal property $\tilde{f}$ is well defined map such that $\tilde{f} \circ \pi = f$ and $\tilde{f} = \phi(g)$. Now it is easy to show that $\phi \circ \psi = \mathrm{id}_{\mathrm{Hom}_R(M \otimes_R N, P)}$ and $\psi \circ \phi = \mathrm{id}_{\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))}$. $\square$

Theorem 2.7. *Let B be an A algebra, M be an $A-$module and N, P be B modules. Then*

$$Hom_B\left(M \otimes_A N, P\right) \cong Hom_A\left(M, Hom_B\left(N, P\right)\right).$$

Proof. Note that $\mathrm{Hom}_A\left(M, \mathrm{Hom}_B\left(N, P\right)\right)$ is an $B-$module,

$$B \times \mathrm{Hom}_A\left(M, \mathrm{Hom}_B\left(N, P\right)\right) \to \mathrm{Hom}_A\left(M, \mathrm{Hom}_B\left(N, P\right)\right)$$
$$(b, f) \mapsto (bf)$$

where $(bf) : M \to \mathrm{Hom}_B\left(N, P\right)$ is defined by $(bf)(m) := b \cdot (f(m))$.

Now, we define

$$\theta : \mathrm{Hom}_A\left(M, \mathrm{Hom}_B\left(N, P\right)\right) \to \mathrm{Hom}_B\left(M \otimes_A N, P\right)$$

where $\theta(f)(m \otimes n) = f(m)(n)$. We will show that $\theta(f)$ is well defined.

$$\begin{tikzcd}
M \times N \arrow[r, "\alpha"] \arrow[d, "\pi"'] & P \\
M \otimes_R N \arrow[ur, "\tilde{\alpha}"'] &
\end{tikzcd}$$

Where $\alpha(m,n) = f(m)(n)$. We claim that α is $A-$linear in first component and $B-$linear in second component. Let $m_1, m_2 \in M$ and $m \in N$, $\alpha(m_1+m_2, n) = f(m_1+m_2)(n) = f(m_1)(n) + f(m_2)(n) = \alpha(m_1, n) + \alpha(m_2, n)$. Let $m \in M, n_1, n_2 \in N$ then $\alpha(m, n_1+n_2) = f(m)(n_1+n_2) = f(m)(n_1) + f(m)(n_2) = \alpha(m, n_1) + \alpha(m, n_2)$. Now, for all $a \in A, m \in M, n \in N$, $\alpha(am, n) = f(am, n) = af(m)(n) = a\alpha(m, n)$ and for all $b \in B, m \in M, n \in N$, $\alpha(m, bn) = f(m)(bn) = bf(m)(n) = b\alpha(m, n)$. Hence α is $A-$linear in first component and $B-$linear in second component. Hence $\theta(f)$ is a well defined $B-$linear map. Let

$$\psi : \mathrm{Hom}_B\left(M \otimes_A N, P\right) \to \mathrm{Hom}_A\left(M, \mathrm{Hom}_B\left(N, P\right)\right)$$
$$g \mapsto \psi(g)$$

where $\psi(g) : M \to \mathrm{Hom}_B\left(N, P\right)$ is the map $\psi(g)(m)(n) = g(m \otimes n)$. It is easy to show that ψ is a $B-$linear map and $\psi \circ \theta = \mathrm{id}_{\mathrm{Hom}_A(M, \mathrm{Hom}_B(N,P))}$ and $\theta \circ \psi = \mathrm{id}_{\mathrm{Hom}_B(M \otimes_A N, P)}$. ☐

Corollary 2.8. *Let*

(1) $$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of $R-$modules. Let N be another $R-$module then the sequence

(2) $$M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is exact.

Proof. Let P be any $R-$module. Since (11) is exact, the sequence

(3) $$\mathrm{Hom}_R\left(M'', \mathrm{Hom}_R\left(N, P\right)\right) \to \mathrm{Hom}_R\left(M, \mathrm{Hom}_R\left(N, P\right)\right) \to \mathrm{Hom}_R\left(M', \mathrm{Hom}_R\left(N, P\right)\right) \to 0$$

is exact and by Theorem 14.53 we have

$$\mathrm{Hom}_R\left(M'' \otimes_R N, P\right) \to \mathrm{Hom}_R\left(M \otimes_R N, P\right) \to \mathrm{Hom}_R\left(M' \otimes_R N, P\right) \to 0$$

is exact. Hence we have (12). ☐

2.1.1. *Flat module.*

Definition 2.9. *A module N is said to be flat $R-$module if for every short exact sequence of $R-$modules*

$$0 \to M' \to M \to M'' \to 0$$

we have the following short exact sequence

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0.$$

Remark 2.10. (1) *An R-mdoule N is said to be flat if and only if for every short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of $R-$modules, we have the following exact sequence

$$0 \to M' \otimes_R N \to M \otimes_R N.$$

(2) *An $R-$module N is said to be flat if for every exact sequence*

$$\sum \equiv \cdots \to M_i \to M_{i+1} \to M_{i+2} \to \cdots$$

of $R-$modules, we have the following exact sequence

$$\sum \otimes_R N \equiv \cdots \to M_i \otimes_R N \to M_{i+1} \otimes_R N \to M_{i+2} \otimes_R N \to \cdots .$$

Definition 2.11. *An $R-$module N is said to be faithfully flat module if it is a flat module and any sequence of*

$$\sum \equiv \cdots \to M_i \to M_{i+1} \to M_{i+2} \to \cdots$$

of $R-$modules, $\sum \otimes_R N$ is exact implies $\sum$ is an exact sequence.

Definition 2.12. *Let S be an $R-$algebra. S is said to be flat over R if S is a flat $R-$module.*

Example 2.13. *Let S be a multiplicatively closed set of a ring R then $S^{-1}R$ is a flat $R-$module.*

Question 2.14. *Let I be an ideal of R. Is R/I flat?*

Lemma 2.15. *Let M, N be flat $R-$modules then $M \otimes_R N$ and $M \oplus N$ is also flat.*

Proof. (1) Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of $R-$modules, since M is flat, the following sequence

$$0 \to M_1 \otimes_R M \to M_2 \otimes_R M \to M_3 \otimes_R M \to 0$$

is exact and so the sequence

$$0 \to (M_1 \otimes_R M) \otimes_R N \to (M_2 \otimes_R M) \otimes_R N \to (M_3 \otimes_R M) \otimes_R N \to 0.$$

Hence

$$0 \to M_1 \otimes_R (M \otimes_R N) \to M_2 \otimes_R (M \otimes_R N) \to M_3 \otimes_R (M \otimes_R N) \to 0$$

is exact. Therefore, $M \otimes_R N$ is flat.

(2) Since M and N are flat the sequences

$$0 \to M_1 \otimes_R M \xrightarrow{\alpha_M} M_2 \otimes_R M \xrightarrow{\beta_M} M_3 \otimes_R M \to 0$$

and

$$0 \to M_1 \otimes_R N \xrightarrow{\alpha_N} M_2 \otimes_R N \xrightarrow{\beta_N} M_3 \otimes_R N \to 0$$

are exact. Therefore the sequence

$$0 \to M_1 \otimes_R M \oplus M_1 \otimes N \xrightarrow{(\alpha_M, \alpha_N)} M_2 \otimes_R M \oplus M_2 \otimes N \xrightarrow{(\beta_M, \beta_N)} M_3 \otimes_R M \oplus M_3 \otimes N \to 0$$

is exact. So we have the following exact sequence,

$$0 \to M_1 \otimes_R (M \oplus N) \to M_2 \otimes_R (M \oplus N) \to M_3 \otimes_R (M \oplus N) \to 0.$$

Hence $M \oplus N$ is a flat $R-$module.

$\square$

Remark 2.16. *Let S be a flat $R-$ algebra and N be a flat $S-$module. Then N is a flat $R-$module.*

Proof. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of $R-$modules. Since S is flat $R-$module,

$$0 \to M_1 \otimes_R S \to M_2 \otimes_R S \to M_3 \otimes_R S \to 0$$

is an exact sequence of $R-$module. Since S is an $R-$algebra, each $M_i \otimes_R S, 1 \leq i \leq 3$ also has $S-$module structure. So the above sequence is an exact sequence of $S-$module. Since N is flat $S-$module,

$$0 \to (M_1 \otimes_R S) \otimes_S N \to (M_2 \otimes_R S) \otimes_S N \to (M_3 \otimes_R S) \otimes_S N \to 0$$

is exact and so the sequences are

$$\begin{tikzcd}
0 \arrow[r] & M_1 \otimes_R (S \otimes_S N) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & M_2 \otimes_R (S \otimes_S N) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & M_3 \otimes_R (S \otimes_S N) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & M_1 \otimes_R N \arrow[r] & M_2 \otimes_R N \arrow[r] & M_3 \otimes_R N \arrow[r] & 0
\end{tikzcd}$$

therefore, N is a flat $R-$module. $\square$

Theorem 2.17. *Let M and N be two $S^{-1}R$ modules, then M, N are also $R-$modules via $\psi : R \to S^{-1}R$. Then $M \otimes_{S^{-1}R} N \cong M \otimes_R N$.*

Proof. We note that $M \otimes_R N$ is an $S^{-1}R-$module. We will show that $M \otimes_R N$ and $M \otimes_{S^{-1}R} N$ is same as $S^{-1}R-$module, hence they are same as $R-$module also. In $M \otimes_R N$,

$$\frac{a}{s}(m \otimes n) = \frac{am}{s} \otimes n = \frac{am}{s} \otimes \frac{ns}{s} = \frac{sm}{s} \otimes \frac{an}{s} = m \otimes \frac{an}{s}.$$

Thus $\frac{a}{s} m \otimes n = m \otimes \frac{an}{s}$ in $M \otimes_R N$. So they are same as $S^{-1}R-$module. $\square$

Theorem 2.18. *Let S be an $R-$algebra and M be an $S-$module. A necessary and sufficient condition for M to be flat over R is that for every $p \in spec\, S$, M_p is flat R_q-module where $q = p \cap R$.*

Proof. First we note that M_p is an R_q module. As S is an $R-$algebra, there exists $f : R \to S$ and $f(p) \subseteq q$ then by Universal property of localization there exists an unique morphism $f_p : R_q \to S_p$ to make S_p an R_q-algebra. Now $S_p \otimes_S M \cong M_p$. Thus M_p is an S_p-module hence M_p is an A_q-module. Suppose M is flat. Consider the exact sequence of R_q-modules (also as $R-$modules)

$$0 \to M_1 \to M_2 \to M_3 \to 0 \tag{4}$$

By previous theorem,

$$M_p \otimes_{R_q} M_i \cong M_p \otimes_R M_i, 1 \leq i \leq 3. \tag{5}$$

Now From (14)

$$0 \to M_1 \otimes_R M \to M_2 \otimes_R M \to M_3 \otimes_R M \to 0$$

is an exact sequence of $S-$mdoule (since M is an $S-$module). As S_p is flat over S we have the following exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (M_1 \otimes_R M) \otimes_S S_p & \longrightarrow & (M_2 \otimes_R M) \otimes_S S_p & \longrightarrow & (M_3 \otimes_R M) \otimes_S S_p & \longrightarrow & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
0 & \longrightarrow & M_1 \otimes_R (M \otimes_S S_p) & \longrightarrow & M_2 \otimes_R (M \otimes_S S_p) & \longrightarrow & M_3 \otimes_R (M \otimes_S S_p) & \longrightarrow & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
0 & \longrightarrow & M_1 \otimes_R M_p & \longrightarrow & M_2 \otimes_R M_p & \longrightarrow & M_3 \otimes_R M_p & \longrightarrow & 0
\end{array}$$

From (15) we have the following exact sequence

$$0 \to M_1 \otimes_{R_q} M_p \to M_2 \otimes_{R_q} M_p \to M_3 \otimes_{R_q} M_p \to 0.$$

Thus M_p is a flat R_q module.

Conversely, let M_p be flat over R_q for all $p \in \operatorname{spec} S$ and $q = p \cap R$. Consider the exact sequence of $R-$modules $0 \to N' \xrightarrow{\phi} N$ then

$$0 \to \operatorname{Ker}(\phi \otimes 1) \xrightarrow{i} N' \otimes_R M \xrightarrow{\phi \otimes 1} N \otimes_R M$$

where $\operatorname{Ker}(\phi \otimes 1), N' \otimes_R M$ and $N \otimes_R M$ are $S-$modules and S_p is flat over S. Thus we have the exact sequence

$$\begin{array}{ccccccl}
0 & \longrightarrow & (\operatorname{Ker}(\phi \otimes 1)) \otimes_S S_p & \longrightarrow & (N' \otimes_R M) \otimes_S S_p & \longrightarrow & (N \otimes_R M) \otimes_S S_p \text{ is exact} \\
 & & \wr\| & & \wr\| & & \wr\| \\
0 & \longrightarrow & (\operatorname{Ker}(\phi \otimes 1))_p & \longrightarrow & N' \otimes_R (M \otimes_S S_p) & \longrightarrow & N \otimes_R (M \otimes_S S_p) \text{ is exact} \\
 & & \wr\| & & \wr\| & & \wr\| \\
0 & \longrightarrow & (\operatorname{Ker}(\phi \otimes 1))_p & \longrightarrow & N' \otimes_R M_p & \longrightarrow & N \otimes_R M_p \text{ is exact} \\
 & & \wr\| & & \wr\| & & \wr\| \\
0 & \longrightarrow & (\operatorname{Ker}(\phi \otimes 1))_p & \longrightarrow & N' \otimes_R (R_q \otimes_{R_q} M_p) & \longrightarrow & N \otimes_R (R_q \otimes_{R_q} M_p) \text{ is exact} \\
 & & \wr\| & & \wr\| & & \wr\| \\
0 & \longrightarrow & (\operatorname{Ker}(\phi \otimes 1))_p & \longrightarrow & (N' \otimes_R R_q) \otimes_{R_q} M_p & \longrightarrow & (N \otimes_R R_q) \otimes_{R_q} M_p \text{ is exact} \\
 & & \wr\| & & \wr\| & & \wr\| \\
0 & \longrightarrow & (\operatorname{Ker}(\phi \otimes 1))_p & \longrightarrow & N'_q \otimes_{R_q} M_p & \longrightarrow & N_q \otimes_{R_q} M_p \text{ is exact}
\end{array}$$

Again we have the exact sequence $0 \to N_q \to N_q$, since R_q is flat over R. As M_p is flat over R_q, the following sequence

$$0 \to N'_q \otimes_{R_q} M_p \to N_q \otimes_{R_q} M_p$$

is exact. Therefore, $(\operatorname{Ker}(\phi \otimes 1))_p = 0$ for all $p \in \operatorname{spec} S$. By Local-global property, $\operatorname{Ker}(\phi \otimes 1) = 0$. So the sequence $0 \to N' \otimes_R M \to N \otimes_R M$ is exact. $\square$

Lemma 2.19. *Let M be an $R-$module. For $p \in maxspec\, R$, we have the map $\theta_p : M \to M_p$ given by $m \mapsto \dfrac{m}{1}$. Let $x \in M$ such that $\theta_p(x) = 0$ for all $p \in maxspec\, R$ then $x = 0$.*

Proof. Let $x \neq 0$ then $\mathrm{Ann}_R(x) \neq R$ so there exists $m \in \mathrm{maxspec}\, R$ such that $\mathrm{Ann}_R(x) \subseteq m$. Consider the map $\theta_m : M \to M_m$. Since $\theta_m(x) = 0 \Rightarrow \dfrac{x}{1} = \dfrac{0}{1} \Rightarrow u(x \cdot 1 - 0 \cdot 1) = 0 \Rightarrow ux = 0 \Rightarrow u \in \mathrm{Ann}_R(x)$ which is a contradiction. Hence $x = 0$. □

Theorem 2.20 (Local-global property)**.** *Let M be an $R-$module. Then the followings are equivalent:*

(1) $M = 0$.
(2) $M_p = 0$ *for all* $p \in spec\, R$.
(3) $M_m = 0$ *for all* $m \in maxspec\, R$.

Proof. (3) $\Rightarrow$ (1) □

Lemma 2.21. *Let $N \subseteq M$ be an $R-$module and P be a flat $R-$module. Then $\dfrac{M \otimes_R P}{N \otimes_R P} \cong M/N \otimes_R P$.*

Proof. Consider the exact sequence $0 \to N \to M \to M/N \to 0$. Since P is flat, the resulting sequence

$$0 \to N \otimes_R P \to M \otimes_R P \to M/N \otimes_R P \to 0$$

is exact. □

Corollary 2.22. *Let M, N be $R-$modules and $f \in Hom_R(M, N)$. Then the followings are equivalent.*

(1) f *is injective (surjective).*
(2) f_p *is injective (surjective) for all* $p \in spec\, R$.
(3) f_m *is injective (surjective) for all* $m \in maxspec\, R$.

Proof. □

2.2. **Projective module.**

Theorem 2.23. *Let P be an $R-$module. Then the followings are equivalent:*

(1) $Hom_R(P, -)$ *is an exact functor that is given any exact sequence of $R-$modules,*

$$0 \to M' \to M \to M'' \to 0$$

the sequence

$$0 \to Hom_R\left(P, M'\right) \to Hom_R(P, M) \to Hom_R\left(P, M''\right) \to 0 \tag{6}$$

is exact.

(2) *Given*

$$\begin{tikzcd} & P \arrow[d, "\psi"] & \\ M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

we have $\phi : P \to M$ such that the diagram commutes that is $g \circ \phi = \psi$.

$$\begin{tikzcd} & P \arrow[dl, "\phi"'] \arrow[d, "\psi"] & \\ M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

(3) *There exist an $R-$module Q such that $P \oplus Q$ is free.*

(4) *For any epimorphism $f : M \to P$, there exists $s : P \to M$ such that $f \circ s = id_P$.*

Proof. (1) $\Rightarrow$ (2) Since (16) is exact $g_*(\alpha) = \beta \Rightarrow g \circ \alpha = \beta$. Take $\alpha = \phi$ and $\beta = \psi$.

(2) $\Rightarrow$ (1) We just need to show that g_* is surjective. Let $\gamma \in \mathrm{Hom}_R(P, M'')$. By (2) there exists $\phi \in \mathrm{Hom}_R(P, M)$ such that $g \circ \phi = \gamma \Rightarrow g_*(\phi) = \gamma$.

(2) $\Rightarrow$ (3) Given P, there exists a free module F and a surjective map $f : F \to P$.

$$\begin{tikzcd} & & & P \arrow[dl, "g"'] \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & \mathrm{Ker}\, f \arrow[r] & F \arrow[r, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

Since $f \circ g = \mathrm{id}_P$ the above sequence is split exact. Hence $F = P \oplus \mathrm{Ker}\, f$. So $Q = \mathrm{Ker}\, f$ is the desired module.

(3) $\Rightarrow$ (2) Consider the diagram

$$\begin{tikzcd} & F \arrow[d, "\pi"] \arrow[ddl, "\tilde{\alpha}"'] & \\ & P \arrow[d, "\psi"] & \\ M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

Let $S \subseteq F$ be a basis, define $\alpha : S \to M$ given by $\alpha(x) = \tau_x$ where $\tau_x \in g^{-1}(\psi \circ (x))$ is a fixed element. Then there exists $\tilde{\alpha} : F \to M$ such that $\tilde{\alpha} \circ g = \psi \circ \pi$. Then $\tilde{\alpha}|_P : P \to M$ is the required map.

(2) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (3) Given P, there exists a free module F and $f : F \to P$ is a surjection. Then there is also a map $s : P \to F$ such that $f \circ s = \mathrm{id}_P$. Since the following sequence

$$0 \to \mathrm{Ker}\, f \to F \to P \to 0$$

is split exact, $F \cong P \oplus \mathrm{Ker}\, f$. $\square$

Definition 2.24. *Any $R-$module P which satisfies any one of the above condition is called projective module.*

Remark 2.25. *Any free module F is projective since $F = F \oplus 0$. But converse is not true. Let $R = \mathbb{Z}/6\mathbb{Z}$ and $P = \mathbb{Z}/3\mathbb{Z}$. Note that P is an $R-$module, take $Q = \mathbb{Z}/2\mathbb{Z}$. Then $P \oplus Q = R$ hence P is a projective module over R but P is not free. If P is free R module then $\mathbb{Z}/3\mathbb{Z} \cong (\mathbb{Z}/6\mathbb{Z})^{|S|}$ where S is a basis of P. Therefore $3 = |\mathbb{Z}/3\mathbb{Z}| = |S||\mathbb{Z}/6\mathbb{Z}| = 6|S|$ which is impossible.*

Note 2.26. *Therefore we have the following implication*

$$\textit{Free} \Longrightarrow \textit{Projective} \Longrightarrow \textit{Flat}$$

but the reverse implications are not true. Let F be a free module, then $F \cong \bigoplus_{i\in\Lambda} R_i$ where $R_i = R$ for all $i \in \Lambda$ and

$$0 \to M' \to M \to M'' \to 0 \tag{7}$$

be an exact sequence of $R-$modules. Then we have

$$0 \to M' \otimes_R R_i \to M \otimes_R R_i \to M'' \otimes_R R_i \to 0$$

is an exact sequence of $R-$modules for all $i \in \Lambda$. Hence

$$0 \to \bigoplus_{i\in\Lambda}(M' \otimes_R R_i) \to \bigoplus_{i\in\Lambda}(M \otimes_R R_i) \to \bigoplus_{i\in\Lambda}(M'' \otimes_R R_i) \to 0$$

is exact. Therefore

$$0 \to M' \otimes_R F \to M \otimes_R F \to M'' \otimes_R F \to 0$$

is exact that is F is a flat module. Now let P be a projective module then there exist an $R-$module Q such that $P \oplus Q$ is free. By previous result we have

$$0 \to (M' \otimes_R P) \oplus (M' \otimes_R Q) \to (M \otimes_R P) \oplus (M \otimes_R Q) \to (M'' \otimes_R P) \oplus (M'' \otimes_R Q) \to 0$$

is exact. Therefore

$$0 \to M' \otimes_R P \to M \otimes_R P \to M'' \otimes_R P \to 0$$

is exact and P is flat. Note that $\mathbb{Q}$ is flat $\mathbb{Z}$ module since $\mathbb{Q} = S^{-1}\mathbb{Z}$ where $S = \mathbb{Z} \setminus \{0\}$ but $\mathbb{Q}$ is not projective. Suppose $\mathbb{Q}$ is projective $\mathbb{Z}-$module then $\mathbb{Q}$ is a free $\mathbb{Z}-$module which is a contradiction.

Definition 2.27. *Let $R-$ be a ring.n A projective module is said to be stably free if there exists a free module Q such that $P \oplus Q$ is free.*

Example 2.28. (1) *Any free module.*

Question 2.29. *Give an example of a module M and a free module F such that $F \oplus M \cong M$.*

Ans. Let $F = R^n$, $M = \bigoplus_{i\in\mathbb{N}} R_i$ where $R_i = F^n$ for all $i \in \mathbb{N}$.

Theorem 2.30. *Let (R, m) be a local ring. Then any finitely generated projective $R-$module P is free over R.*

Proof. Let $S \subseteq P$ be a minimal generating set. Let $S = \{x_1, \cdots, x_n\}$ then $\overline{S} = \{x_1 + mP, \cdots, x_n + mP\}$ is the basis of P/mP over R/m. Since $P = \langle S \rangle$ there exists a surjective map $\phi : R^n \to P$. Consider the exact sequence

$$0 \to \operatorname{Ker}\phi \xrightarrow{i} R^n \xrightarrow{\phi} P \to 0. \tag{8}$$

Then we have

$$\begin{array}{ccccccc}
\operatorname{Ker}\phi \otimes_R R/m & \xrightarrow{\tilde{i}} & R^n \otimes_R R/m & \xrightarrow{\tilde{\phi}} & P \otimes_R R/m & \longrightarrow & 0 \\
\wr\| & & \wr\| & & \wr\| & & \\
\dfrac{\operatorname{Ker}\phi}{m \operatorname{Ker}\phi} & \xrightarrow{\tilde{i}} & (R/m)^n & \xrightarrow{\tilde{\phi}} & P/mP & \longrightarrow & 0
\end{array}$$

Since $\dim(R/m)^n = n = \dim P/mP$, $\tilde{\phi}$ is an isomorphism $\dfrac{\operatorname{Ker}\phi}{m \operatorname{Ker}\phi} = 0$. Since P is projective (17) is split exact. Therefore $R^n \cong \operatorname{Ker}\phi \oplus P$ and hence $\operatorname{Ker}\phi$ is finitely generated. By NAK, $\operatorname{Ker}\phi = 0$. Hence P is free. □

Proposition 2.31. *Let R be a commutative ring with 1 and $\phi : R^k \to R^n$ be an endomorphism. Then $n \leq k$.*

Proof. Let $m \in \operatorname{maxspec} R$. Consider the exact sequence

$$0 \to \operatorname{Ker}\phi \xrightarrow{i} R^k \xrightarrow{\phi} R^n \to 0. \tag{9}$$

of $R-$ modules. We have

$$\begin{array}{ccccccc}
\operatorname{Ker}\phi \otimes_R R/m & \xrightarrow{\tilde{i}} & R^k \otimes_R R/m & \xrightarrow{\tilde{\phi}} & R^n \otimes_R R/m & \longrightarrow & 0 \\
\wr\| & & \wr\| & & \wr\| & & \\
\operatorname{Ker}\phi \otimes_R R/m & \xrightarrow{\tilde{i}} & (R/m)^k & \xrightarrow{\tilde{\phi}} & (R/m)^n & \longrightarrow & 0
\end{array}$$

Since $(R/m)^k$ is vector space over R/m and the map $\tilde{\phi}$ is onto, by Rank-Nullity theorem $n \leq k$. □

Theorem 2.32. *Let R be a commutative ring with 1 such that $R^m \cong R^n$ then $m = n$.*

Proof. Let $\psi : R^m \to R^n$ be the isomorphism then there exists $\phi : R^n \to R^m$ such that $\phi \circ \psi = \mathrm{id}_{R^m}$ and $\psi \circ \phi = \mathrm{id}_{R^n}$. Since ψ is onto, $n \leq m$ and ϕ is onto implies $m \leq n$. Hence $m = n$. □

For a commutative ring R with 1, we define rank $R^n = n$. For a finitely generated free module F, there exists $n \in \mathbb{R}$ such that $F \cong R^n$. So we define rank $F = n$. Let P be a finitely generated projective module over R. Define rank : $\operatorname{spec} R \to P$ given by $p \mapsto$ rank (P_p).

Note 2.33. *Let P be a projective module, then there exists Q such that $P \oplus Q \cong F$ where F is a free module. Let $p \in$ spec R. then $(P \oplus Q) \otimes_R R_p \cong F \otimes_R R_p \Rightarrow P_p \otimes_R Q_p \cong F_p$. Since P_p is a finitely generated over a local ring in R_p, and F_p is free R_p module, therefore P_p is projective R_p module and hence P_p is free over R_p. So rank (P_p) is well defined. Note that if R is local then the rank function is constant.*

Theorem 2.34. *Let R be a semi local ring and P be a finitely generated projective module over R of constant rank then P is free.*

Proof. Let $\operatorname{maxspec} R = \{m_1, \cdots, m_r\}$ and $J = \displaystyle\bigcap_{i=1}^{r} m_i$ be the Jacobson radical. By Chinese Remainder theorem $P/JP \cong P/m_1P \times \cdots \times P/m_rP$ and $R/J \cong R/m_1 \times \cdots \times R/m_r$ and P/JP

is R/J module. Let $S = \{s_1, \cdots, s_k\}$ be a minimal generating set of P over R. We claim that $\overline{S} = \{s_1 + JP, \cdots, s_k + JP\}$ be the minimal generating set of P/JP over R/J. If not, we assume that P/JP is generated by $\{s_1 + JP, \cdots, s_{k-1} + JP\}$. Let $N = \langle s_1, \cdots, s_{k-1}\rangle$. Pick $x \in P$ then $x + JP = \sum_{i=1}^{k-1}(r_i + J)(s_i + JP) \Rightarrow x - \sum_{i=1}^{k-1} r_i s_i \in JP \Rightarrow x \in N + JP \Rightarrow P = N + JP$. By NAK, $P = N$ which is a contradiction. So our claim is proved. Thus P/JP is free R/J module. Now we consider the exact sequence

$$0 \to \operatorname{Ker} f \to R^k \to P \to 0. \tag{10}$$

Since P is projective, this above sequence is split exact and therefore $\operatorname{Ker} f$ is finitely generated. From (19)

$$\begin{array}{ccccccc}
\operatorname{Ker} f \otimes_R R/J & \xrightarrow{i\otimes 1} & R^k \otimes_R R/J & \xrightarrow{f\otimes 1} & P \otimes_R R/J & \longrightarrow & 0 \\
\wr\Vert & & \wr\Vert & & \wr\Vert & & \\
\dfrac{\operatorname{Ker} f}{J \operatorname{Ker} f} & \xrightarrow{\quad i\otimes 1 \quad} & (R/J)^k & \xrightarrow{\quad f\otimes 1 \quad} & P/JP & \longrightarrow & 0
\end{array}$$

We claim that $\{s_1 + JP, \cdots, s_k + JP\}$ is a R/J basis of P/JP. If we prove the claim then $f \otimes 1$ is an isomorphism and $\operatorname{Ker} f/J \operatorname{Ker} f = 0 \Rightarrow \operatorname{Ker} f = 0$ by NAK and $P \cong R^k$ hence P is free.

Proof of the claim.

Note 2.35. *Let F_i be free R_i module of same rank for all $1 \le i \le k$, then $F = F_1 \times \cdots \times F_k$ is free $R_1 \times \cdots \times R_k$ module. That is $F_i \cong (R_i)^l$ for some $l \in \mathbb{N}, 1 \le i \le n$. Then $F = F_1 \times \cdots \times F_k \cong (R_1)^l \times \cdots \times (R_k)^l \cong (R_1 \times \cdots \times R_k)^l$. We will prove this by induction on k. Let $\theta : R_1^l \times R_2^l \to (R_1 \times R_2)^l$ defined by $((x_1, \cdots, x_l), (x'_1, \cdots, x'_l)) \mapsto ((x_1, x'_1), \cdots, (x_l, x'_l))$ be the required isomorphism.*

Note 2.36. *Since P is projective of constant rank, let $P_m \cong (R_m)^l$ for all $m \in mspecR$ and for some $l \in \mathbb{N}$. Let $P/mP \cong (R/m)^s$ for some $s \in \mathbb{N}$. Then $P/mP \otimes_R R_m \cong (R/m)^s \otimes_R R_m \Rightarrow \dfrac{P_m}{mP_m} \cong \left(\dfrac{R_m}{mR_m}\right)^s \cong \left(\dfrac{R_m}{mR_m}\right)^l \Rightarrow l = s$. Hence for any $m \in maxspec\, R$, $P/mP \cong (R/m)^l$. Therefore $P/JP \cong \prod_{i=1}^{r} P/m_iP \cong \prod_{i=1}^{r}(R/m_i)^l \cong \left(\prod_{i=1}^{r} R/m_i\right)^l \cong (R/J)^l$.*

Question 2.37. *Let R be a semi local ring and F be a finitely generated free module over R. Is any minimal generating set of F an $R-$basis of F?.*

Definition 2.38. *Let M be an $R-$module. M is said to be finitely presented if there exists finitely generated free modules F_1 and F_2 such that the following sequence is exact*

$$F_1 \to F_2 \to M \to 0.$$

Note 2.39. *Suppose M is a finitely generated module over R. If $Ker f$ is finitely generated then we have the following sequence*

$$\begin{tikzcd} R^k \arrow[rr, "i\circ\phi"] \arrow[dr, "\phi"'] & & R^n \arrow[r, "f"] & M \arrow[r] & 0 \\ & Ker\, f \arrow[ur, "i"'] \arrow[dr] & & & \\ & & 0 & & \end{tikzcd}$$

is exact because $Ker\,\phi = Im\,\phi = Im(i\circ\phi)$. Thus a finitely generated module may not be finitely presented. If R is Noetherian then it is true. Conversely any finitely presented module is finitely generated.

Theorem 2.40. *Let R be a ring and M, N be $R-$modules and S be a flat $R-$algebra. Suppose M is of finite presentation then we have*

$$Hom_R\,(M,N)\otimes_R S \cong Hom_S\,(M\otimes_R S, N\otimes_R S)\,.$$

Proof. Since M is of finite presentation, there exists two finitely generated free module R^p and R^q such that

$$R^p \to R^q \to M \to 0 \tag{11}$$

is exact. Then for any $R-$module N the following sequence

$$0 \to \mathrm{Hom}_R\,(M,N) \to \mathrm{Hom}_R\,(R^q,N) \to \mathrm{Hom}_R\,(R^p,N) \tag{12}$$

is exact. As S is flat,

$$0 \to \mathrm{Hom}_R\,(M,N)\otimes_R S \to \mathrm{Hom}_R\,(R^q,N)\otimes_R S \to \mathrm{Hom}_R\,(R^p,N)\otimes_R S$$

is exact. Now consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R\,(M,N)\otimes_R S \arrow[r] \arrow[d, "\lambda_M"] & \mathrm{Hom}_R\,(R^q,N)\otimes_R S \arrow[r] \arrow[d, "\lambda_{R^q}"] & \mathrm{Hom}_R\,(R^p,N)\otimes_R S \arrow[d, "\lambda_{R^p}"] \\ 0 \arrow[r] & \mathrm{Hom}_S\,(M\otimes_R S, N\otimes_R S) \arrow[r] & \mathrm{Hom}_S\,(R^q\otimes_R S, N\otimes_R S) \arrow[r] & \mathrm{Hom}_S\,(R^p\otimes_R S, N\otimes_R S) \end{tikzcd}$$

where $\lambda_M : \mathrm{Hom}_R\,(M,N)\otimes_R S \to \mathrm{Hom}_S\,(M\otimes_R S, N\otimes_R S)$ is defined by $\lambda_M(f\otimes s) = \tilde{f}$ and $\tilde{f} : M\otimes_R S \to N\otimes_R S$ is defined by $\tilde{f}(m\otimes s) = f(m)\otimes s$. By Universal property $\tilde{f}$ is well defined. Since $\mathrm{Hom}_R\,(R^q,N)\otimes_R S \cong (\mathrm{Hom}_R\,(R,N))^q\otimes S \cong N^q\otimes S = (N\otimes_R S)^q$ and $\mathrm{Hom}_S\,(R^q\otimes_R S, N\otimes_R S) \cong \mathrm{Hom}_S\,(S^q, N\otimes_R S) \cong (N\otimes_R S)^q$. Thus the mappings λ_{R^q} and λ_{R^p} are isomorphism. Since the bottom sequence of the above diagram is exact and the diagram ia commutative, λ_M is also an isomorphism. $\square$

Corollary 2.41. *Let M and N be $R-$modules with M be of finite presentation. Then for each $p\in spec\,R$,*

$$(Hom_R\,(M,N))_p \cong Hom_{R^p}(M_p,N_p).$$

Proof. Take $S = R_p$. $\square$

Theorem 2.42. *Let R be any ring and M be a finitely presented. Then the followings are equivalent:*

(1) *The map* $\theta : M \otimes_R M^* \to R$ *defined by* $\theta(m, f) = f(m)$ *is an isomorphism.*
(2) *There exists an* $R-$*module* N *such that* $M \otimes_R N \cong R$.
(3) $M_m \cong R_m$ *for all* $m \in maxspec\, R$.
(4) $M_p \cong R_p$ *for all* $p \in spec\, R$.
(5) M *is projective of rank 1.*

Proof. (1) $\Rightarrow$ (2) Take $N = M^*$.
(2) $\Rightarrow$ (3) $M \otimes_R N \cong R \Rightarrow M_m \otimes_R N_m \cong R_m \Rightarrow M_m \otimes_{R_m} N_m \cong R_m \Rightarrow (M_m \otimes_{R_m} N_m) \otimes_{R_m} \dfrac{R_m}{mR_m} \cong \dfrac{R_m}{mR_m} \Rightarrow M_m \otimes_{R_m} \dfrac{N_m}{mN_m} \cong \dfrac{R_m}{mR_m} \Rightarrow \dfrac{M_m}{mR_m} \otimes_{R_m} \dfrac{N_m}{mN_m} \cong \dfrac{R_m}{mR_m}$[1]. Therefore, $\dfrac{M_m}{mM_m} \cong \dfrac{R_m}{mR_m}$. By NAK $M_m = \langle x \rangle, x \in M_m \Rightarrow M_m \cong \dfrac{R_m}{Ann_{R_m}(x)} \Rightarrow Ann_{R_m}(x)(M_m \otimes_{R_m} N_m) = 0 \Rightarrow Ann_{R_m}(x) R_m = 0 \Rightarrow Ann_{R_m}(x) = 0 \Rightarrow M_m = R_m$.

(3) $\Rightarrow$ (4) Further localization.

(4) $\Rightarrow$ (5) By definition.

(5) $\Rightarrow$ (1) Since M is of finite presentation, $(\mathrm{Hom}_R(M, R))_m \cong \mathrm{Hom}_{R_m}(M_m, R_m)$ for all $m \in \mathrm{maxspec}\, R$, that is $(M^*)_m \cong (M_m)^*$. Now M is projective of rank 1 so $M_m \cong R_m$. So we have $M_m \otimes_{R_m} (M_m)^* \cong R_m \otimes_{R_m} (R_m)^* \cong R_m$. Again from the above equation,

$$\begin{aligned}
M_m \otimes_{R_m} (M_m)^* &\cong M_m \otimes_{R_m} (M^*)_m \\
&\cong M_m \otimes_R (M^*)_m \\
&\cong M_m \otimes_R (M^* \otimes_R R_m) \\
&\cong (M \otimes_R R_m) \otimes_R (M^* \otimes_R R_m) \\
&\cong (M \otimes_R M^*) \otimes_R (R_m \otimes_R R_m) \\
&\cong (M \otimes_R M^*) \otimes_R R_m \\
&\cong (M \otimes_R M^*)_m
\end{aligned}$$

Hence $(M \otimes_R M^*)_m \cong R_m$ for all $m \in \mathrm{maxspec}\, R$. By Local-global property $M \otimes_R M^* \cong R$.

Note 2.43. *Let* I *and* J *be two ideals of* R *then* $R/I \otimes_R R/J \cong \dfrac{R/I}{J(R/I)} \cong \dfrac{R/I}{(J+I)/I} \cong \dfrac{R}{I+J}$. *(Check this isomorphism as ring.)*

Picard group. Let $\sum$ be the isomorphism classes of projective $R-$modules of rank 1. Define

$$\begin{aligned}
\cdot : \sum \times \sum &\to \sum \\
([P], [Q]) &\mapsto [P \otimes_R Q]
\end{aligned}$$

[1]As $K(m) := \dfrac{R_m}{mR_m}$ and $K(m)^l \otimes_{K(m)} K(m)^s \cong K(m)^{ls}$.

We need to show that $(\sum, \cdot)$ is a group with inverse of $[P]$ is $[P^*]$. This group is called Picard group of R and it is denoted by $Pic\ R$. Let P, Q be two projective module of rank 1 then

$$(P \otimes_R Q) \otimes_R R_m \cong P_m \otimes_R Q_m \cong P_m \otimes_{R_m} Q_m \cong R_m \otimes_{R_m} R_m \cong R_m.$$

Thus $P \otimes_R Q$ is also a projective module of rank 1. By Corollary 14.88 $(M^*)_p \cong (M_p)^* \cong (R_p)^* \cong R_p$ for all $p \in \operatorname{spec} R$. Therefore M is projective of rank 1 implies M^* is also projective of rank 1.

Free, Projective and Flat resolution.

Definition 2.44. *Let M be an $R-$module. A free (or projective or flat) resolution of M over R is an exact sequence of $R-$modules*

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

where each P_i is a free (or projective or flat respt.) $R-$module.

Lemma 2.45. *Let M be an $R-$module. Then projective resolution of M over R exists.*

Proof. For any module M, there exists a free module F and a surjective map $F_0 \xrightarrow{f_0} M \to 0$. Consider the $\operatorname{Ker} f_0$, then there exists a free module F_1 with the diagram

$$\begin{tikzcd}
F_1 \arrow[rr, "f_1=i\circ\pi"] \arrow[dr, "\pi_1"] & & F_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
& \operatorname{Ker} f_0 \arrow[ur, hook, "i"] \arrow[dr] & & & \\
& & 0 & &
\end{tikzcd}$$

The above diagram is exact since $\operatorname{Ker} f_0 = \operatorname{Im} \pi_1 = \operatorname{Im} i \circ \pi_1 = \operatorname{Im} f_1$ since i is the inclusion map and π_1 is onto. Next we consider $\operatorname{Ker} f_1$, then there exists F_2 such that

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[rr, "f_2"] \arrow[dr, "\pi_2"] & & F_1 \arrow[rr, "f_1=i\circ\pi"] \arrow[dr, "\pi_1"] & & F_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
& & \operatorname{Ker} f_1 \arrow[ur, hook, "i"] \arrow[dr] & & \operatorname{Ker} f_0 \arrow[ur, hook, "i"] \arrow[dr] & & & \\
& & & 0 & & 0 & &
\end{tikzcd}$$

Inductively we can construct a free resolution of M. Since every free module is projective and therefore flat, we have a projective (or flat) resolution. $\square$

Tor and Ext.

Definition 2.46. *Let M be an $R-$module. We consider a projective resolution of M that is*

$$\mathcal{C} \equiv \cdots \to P_2 \xrightarrow{f_2'} P_1 \xrightarrow{f_1'} P_0 \xrightarrow{f_0'} M \to 0.$$

Let N be another $R-$module. We consider,

(1)

$$\mathcal{C} \otimes_R N \equiv \cdots \to P_2 \otimes_R N \xrightarrow{f_2} P_1 \otimes_R N \xrightarrow{f_1} P_0 \otimes_R N \xrightarrow{f_0} M \otimes_R N \to 0$$

where $f_i = f_i' \otimes 1$ *for all* $i \in \mathbb{N}$. *Then we define* $Tor_i^R(M,N) := H_i(\mathcal{C} \otimes_R N) = \dfrac{Ker\, f_i}{Im\, f_{i+1}}$.

(2)

$$Hom_R\,(\mathcal{C}, N) \equiv \cdots \xleftarrow{f_2^*} Hom_R\,(P_1, N) \xleftarrow{f_1^*} Hom_R\,(P_0, N) \xleftarrow{f_0^*} Hom_R\,(M, N) \leftarrow 0.$$

we define $Ext_R^i(M,N) := H^i(Hom_R\,(\mathcal{C}, \mathcal{N})) = \dfrac{Ker\, f_{i+1}^*}{Im\, f_i^*}$.

Remark 2.47. *These definition doesn't depend on the choice of resolution of* M.

3. Integral Dependence and Valuation

Definition 3.1. *Let* B *be a ring and* $A \subseteq B$ *be a subring. An element* $x \in B$ *is said to be integral over* A *if* x *is a root of a monic polynomial in* $A[T]$.

Proposition 3.2. *Let* $A \subseteq B$ *where* A *and* B *are commutative ring with 1. Then the followings are equivalent:*

(1) $x \in B$ *is integral over* A.
(2) $A[x]$ *is a finitely generated* $A-$ *module.*

Proof. (1) $\Rightarrow$ (2) We note that $A[x] = \text{span}\,\{1, x, x^2, \cdots\}$ over A. As $x \in B$ is integral over A, there exist $f(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in A[T]$ such that $f(x) = 0$. Let $g(T) \in A[T]$ then by division algorithm,

$$g(T) = q(T)f(T) + r(T), r(T) = 0 \text{ or } \deg r(T) < \deg f(T) = n.$$

Therefore, $g(x) = r(x) \in \text{span}\,\{1, x, \cdots, x^{n-1}\}$. Hence $A[x]$ is a finitely generated $A-$module and $A[x] = \langle 1, x, \cdots, x^{n-1}\rangle$.

(2) $\Rightarrow$ (1) Suppose $A[x]$ is a finitely generated $A-$module. Let $\{f_1, \cdots, f_r\}$ be a finite generating set of $A[x]$ over A. Let $d > \deg f_i(T), 1 \leq i \leq r$. Since $x^d \in A[x]$,

$$x^d = c_1 f_1 + \cdots + c_r f_r, c_i \in A[x]; 1 \leq i \leq r.$$

Therefore x satisfies a polynomial equation $T^d - \sum_{i=1}^{r} c_i f_i(T) \in A[T]$. So, x is integral over A. $\square$

Theorem 3.3 (Going up theorem)**.** *Let* B *be a ring and* A *be a subring of* B *such that* B *is integral over* A. *Then* A *is field if and only if* B *is field.*

Proof. Suppose A is field. Pick $u \in B \setminus A$, since u is integral over $A, A[u] = A(u) \subseteq B$. Therefore $u^{-1} \in B$.

Conversely, Suppose B is a field. Let $a \in A \subseteq B \Rightarrow a^{-1} \in B$. Since B is integral over A, a^{-1} satisfies a monic polynomial in A that is $(a^{-1})^n + \cdots + a_1(a^{-1}) + a_0 = 0$. Clearing the denominator,

$$a^{-1} = -(a_{n-1} + \cdots + a_0 a^{n-1}) \in A.$$

$\square$

Lemma 3.4. *Let D be an integral domain and $f \in D[X_1, \cdots, X_n]$ and $N \geq 1$ be an integer such that $N >$ total degree of f. Suppose $\phi \in Aut_D\, D[X_1, \cdots, X_n]$ such that $\phi(X_i) = X_i + X_n^{N^i}, 1 \leq i \leq n-1$ and $\phi(X_n) = X_n$. Then the highest degree term of $\phi(f)$ involving X_n is of the form cX_n^p where $c \in D$.*

Proof. We consider any non zero term of f which is of the form $c_\alpha X_1^{a_1} \cdots X_n^{a_n}$ where $\alpha = (a_1, \cdots, a_n)$ and $c_\alpha \neq 0$. Then

$$\phi(c_\alpha X_1^{a_1} \cdots X_n^{a_n}) = c_\alpha \left(X_1 + X_n^N\right)^{a_1} \left(X_2 + X_n^{N^2}\right)^{a_2} \cdots \left(X_{n-1} + X_n^{N^{n-1}}\right)^{a_{n-1}} X_n^{a_n}.$$

After expanding we get the highest degree term is $c_\alpha X_n^{a_n + a_1 N + \cdots + a_{n-1}N^{n-1}}$. If there exist $\beta = (b_1, \cdots, b_n)$ such that $c_\beta X_1^{b_1} \cdots X_n^{b_n}$ is a term of f and the highest degree power of $\phi(c_\beta X_1^{b_1} \cdots X_n^{b_n}) = c_\beta X_n^{b_n + b_1 N + \cdots + b_{n-1}N^{n-1}}$ cancels $c_\alpha X_n^{a_n + a_1 N + \cdots + a_{n-1}N^{n-1}}$ then $c_\beta = -c_\alpha$ and $b_N + b_1 N + \cdots + b_{n-1}N^{n-1} = a_n + a_1 N + \cdots + a_{n-1}N^{n-1} \Rightarrow (b_1, \cdots, b_n) = (a_1, \cdots, a_n)$ (by division algorithm) hence $\alpha = \beta$ and which implies $c_\alpha X^\alpha = -c_\beta X^\beta$ which is a contradiction as both are monomials of f. ☐

Definition 3.5. *Let K be a field. The elements $y_1, \cdots, y_q$ in some $K-$algebra are called algebraically independent if there is no polynomial $p \in K[X_1, \cdots, X_q]$ such that $p(y_1, \cdots, y_q) = 0$.*

Observation 3.6. *Suppose $y_1, \cdots, y_q$ are algebraically independent over K. Then the map $\theta : K[X_1, \cdots, X_q] \to K[y_1, \cdots, y_q]$ defined by $X_i \mapsto y_i, 1 \leq i \leq q$ is an isomorphism. Conversely, suppose $K[X_1, \cdots, X_q] \cong K[y_1, \cdots, y_q]$ and $\phi : K[X_1, \cdots, X_n] \to K[y_1, \cdots, y_q]$ be an isomorphism. Let $\alpha : K[X_1, \cdots, X_q] \to K[X_1, \cdots, X_q]$ be a map where $\alpha(X_i) = p_i$ and $p_i = \phi^{-1}(y_i), 1 \leq i \leq q$. We note that $Im\,\phi^{-1} = Im\,\alpha = K[p_1, \cdots, p_q]$. Because ϕ^{-1} is an isomorphism, we have $K[p_1, \cdots, p_q] = K[X_1, \cdots, X_q]$, hence α is surjective and thus α is an isomorphism. Now $\phi \circ \alpha(X_i) = y_i, 1 \leq i \leq q$ and $\phi \circ \alpha$ is an isomophism. Suppose $y_1, \cdots, y_q$ are algebraically dependent so there exist $0 \neq f(X_1, \cdots, X_q) \in K[X_1, \cdots, X_q]$ such that $f(y_1, \cdots, y_q) = 0 \Rightarrow \phi \circ \alpha(f) = 0 \Rightarrow f = 0$ which is a contradiction.*

Lemma 3.7 (Vasconcelous)**.** *Let R be a ring and M be a finitely generated $R-$module. $\phi : M \to M$ is a surjective $R-$linear map then ϕ is an isomorphism.*

Proof. We consider M as an $R[X]$ module via ϕ, i.e., the scalar multiplication map $\cdot : R[X] \times M \to M$ is $(f, m) \mapsto f(\phi)m$. Since ϕ is surjective, $\phi(M) = M \Rightarrow X \cdot M = M$. Take $I = \langle X \rangle$, so by NAK there exist $f(X) \in I$ such that $(1 + f(X))M = 0$. Let $m \in \operatorname{Ker} \phi \Rightarrow \phi(m) = X \cdot m = 0$. So $(1 + f(X)) \cdot m = m + 0 = m$ (as $f(x) \in I$). Therefore $m = 0$ as $(1 + f(X))M = 0$. ☐

Lemma 3.8. *Let R be an Noetherian ring. If $\phi : R \to R$ is an epimorphism then ϕ is an isomorphism.*

Proof. Note that we have the following ascending chain of ideals of R,

$$\operatorname{Ker} \phi \subseteq \operatorname{Ker} \phi^2 \subseteq \cdots.$$

Since R is Noetherian, $\operatorname{Ker} \phi^{n_0} = \operatorname{Ker} \phi^{n_0 + k}$ for some $n_0 \in \mathbb{N}$ and for all $k \in \mathbb{N}$. Let $x \in \operatorname{Ker} \phi$, as ϕ is surjective, ϕ^n is also surjective for all $n \in \mathbb{N}$, hence there is $y \in R$ such that $\phi^{n_0}(y) = x \Rightarrow \phi^{n_0+1}(y) = \phi(x) = 0 \Rightarrow y \in \operatorname{Ker} \phi^{n_0+1} = \operatorname{Ker} \phi^{n_0} \Rightarrow \phi^{n_0}(y) = 0 \Rightarrow x = 0$. ☐

Corollary 3.9. *Let M be an Noetherian $R-$module and $\phi : M \to M$ be a surjective $R-$linear map. Then ϕ is an isomorphism.*

Note 3.10. *Note that the statement is not true if surjectivity is replaced by injectivity. For example let $R = \mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{Z}$ be the map where $\phi(x) = 2x$. Here ϕ is injective but not surjective.*

Theorem 3.11 (Noether Normalization lemma)**.** *Let K be a field and suppose $A = K[r_1, \cdots, r_m]$ is a finitely generated $K-$algebra. Then for some $q, 0 \leq q \leq m$, there are algebraically independent elements $y_1, \cdots, y_q \in A$ such that A is integral over $K[y_1, \cdots, y_q]$.*

Proof.

Example 3.12. *Let $A = K[X,Y,Z]/\langle Y - X^2, Y^2 - XZ\rangle = K[r_1, r_2, r_3]$ where $r_1 = X + \langle Y - X^2, Y^2 - XZ\rangle, r_2 = Y + \langle Y - X^2, Y^2 - XZ\rangle$ and $r_3 = Z + \langle Y - X^2, Y^2 - XZ\rangle$. Let $\phi : K[X,Y,Z] \to K[T]$ be a map defined by $X \mapsto T, Y \mapsto T^2$ and $Z \mapsto T^3$. Then ϕ is a ring morphism and $Ker\,\phi = \langle Y - X^2, Y^2 - XZ\rangle$. By first isomorphism theorem $K[T] \cong A$.*

$$\begin{tikzcd} K[r_1, r_2, r_3] \arrow[d, dash, "\textit{integral}"] \\ K[r_1] \arrow[d, dash, "\textit{transcendental}"] \\ K \end{tikzcd}$$

Theorem 3.13 (Weak Nullstellensatz)**.** *Let K be an algebraically closed field. Then $\mathfrak{m}$ is a maximal ideal in a polynomial ring $K[X_1, \cdots, X_n]$ if and only if $\mathfrak{m} = \langle X_1 - a_1, \cdots, X_n - a_n\rangle$ for some $a_1, \cdots, a_n \in K$. Equivalently, there is a one to one correspondence between points in K^n and maximal ideals in $K[X_1, \cdots, X_n]$.*

Proof. It is easy to check that $\langle X_1 - a_1, \cdots, X_n - a_n\rangle \in \operatorname{maxspec} K[X_1, \cdots, X_n]$. Conversely, suppose $\mathfrak{m} \in \operatorname{maxspec} K[X_1, \cdots, X_n]$ and denote $x_i = X_i + \mathfrak{m} \in A/\mathfrak{m}, 1 \leq i \leq n$. Then $A/\mathfrak{m}$ is a finitely generated $K-$algebra. By Noether normalization lemma, there exist $y_1, \cdots, y_q \in A/\mathfrak{m}; 0 \leq q \leq n$, algebraically independent elements over K such that $A/\mathfrak{m}$ is integral over $K[y_1, \cdots, y_q]$. Since $A/\mathfrak{m}$ is field and $A/\mathfrak{m}|K[y_1, \cdots, y_q]$ is an integral extension, $K[y_1, \cdots, y_q]$ is also field. But $K[y_1, \cdots, y_q] \cong K[T_1, \cdots, T_q]$, therefore $q = 0$ and $A/\mathfrak{m}|K$ is an algebraic extension. As K is algebraically closed, $A/\mathfrak{m} = K$ and therefore $x_i \in K$. Let $X_i + \mathfrak{m} = a_i + \mathfrak{m} \Rightarrow X_i - a_i \in \mathfrak{m}, 1 \leq i \leq m \Rightarrow \langle X_1 - a_1, \cdots, X_m - a_m\rangle \subseteq \mathfrak{m}$. Since both are maximal ideals of $K[X_1, \cdots, X_n]$. We have $\mathfrak{m} = \langle X_1 - a_1, \cdots, X_n - a_n\rangle$. $\square$

Remark 3.14. *The result is not true if K is not algebraically closed. For example take $K = \mathbb{R}$ and $m = \langle X^2 + 1\rangle$.*

Theorem 3.15 (Hilbert's Nullstellensatz-Zariski form)**.** *Let K be a field and E be a finitely generated $K-$algebra. If E is field then $E|K$ is a finite algebraic extension.*

Proof. Let $E = K[r_1, \cdots, r_n]$. Since E is finitely generated $K-$algebra, by Noether normalization lemma, there exist $y_1, \cdots, y_q \in E; 0 \leq q \leq n$ algebraically independent over K such that E is integral over $K[y_1, \cdots, y_q]$. But E is field and $E|K[y_1, \cdots, y_q]$ is integral, this implies $K[y_1, \cdots, y_q]$ is also a field and therefore $q = 0$. Hence $E|K$ is algebraic. Since E is finitely generated, extension is also finite. □

Definition 3.16. *Let K be a field. An affine space over K of dimension n is just the set $K^n := \{(a_1, \cdots, a_n) : a_i \in K, 1 \leq i \leq n\}$.*

Notation. An affine space over K of dimension n will be denoted by $\mathbb{A}^n_K$.

Definition 3.17. (1) *Let $S \subseteq \mathbb{A}^n_K$. Define*

$$I(S) := \{f \in K[X_1, \cdots, X_n] : f(a_1, \cdots, a_n) = 0 \text{ for all } (a_1, \cdots, a_n) \in S\}.$$

(2) *Let $T \subseteq K[X_1, \cdots, X_n]$. Then we define*

$$Z(T) = \{(a_1, \cdots, a_n) \in \mathbb{A}^n_K : f(a_1, \cdots, a_n) = 0 \text{ for all } f \in T\}.$$

Note 3.18. (1) *The set $I(S)$ in the Definition 1.17. (1) is an ideal of $K[X_1, \cdots, X_n]$ and*
(2) *The set in the Definition 1.17. (2) is called 'Algebraic set'.*

Observation 3.19 (Zariski topology on $\mathbb{A}^n_K$)**.** *We define a topology on $\mathbb{A}^n_K$ whose closed sets are algebraic sets. Check that this is a topology on $\mathbb{A}^n_K$.*

Theorem 3.20 (Nullstellensatz)**.** *Let K be an algebraically closed field and $I \subseteq K[X_1, \cdots, X_n]$ be an ideal. Then $Z(I) = \emptyset$ if and only if $1 \in I$.*

Proof. If $1 \in I$ then it is clear that $Z(I) = \emptyset$. Conversely, suppose $Z(I) = \emptyset$ but $1 \notin I$, then there exist a maximal ideal $\mathfrak{m}$ of $K[X_1, \cdots, X_n]$ such that $I \subseteq \mathfrak{m}$. Since K is algebraically closed, $\mathfrak{m} = \langle X_1 - a_1, \cdots, X_n - a_n\rangle$ for some $a_1, \cdots, a_n \in K$. But $(a_1, \cdots, a_n) \in Z(\mathfrak{m}) \subseteq Z(I)$ which is a contradiction. Hence $1 \in I$. □

Remark 3.21. *It is not true if K is not algebraically closed. For example lets take $K = \mathbb{R}$ and $I = \langle X^2 + 1\rangle$. Then I is a proper ideal of $\mathbb{R}[X]$ but $Z(I) = \emptyset$.*

Theorem 3.22 (Strong Nullstellensatz)**.** *Let K be an algebraically closed field and $J \subseteq K[X_1, \cdots, X_n]$ be an ideal. Then $I(Z(J)) = \sqrt{J}$.*

Proof. Let $(a_1, \cdots, a_n) \in Z(J)$ and $g \in \sqrt{J} \Rightarrow g^N \in J$ for some $N \in \mathbb{N}$. Then $g^N(a_1, \cdots, a_n) = 0 \Rightarrow g(a_1, \cdots, a_n) = 0 \Rightarrow g \in I(Z(J)) \Rightarrow \sqrt{J} \subseteq I(Z(J))$. Since $K[X_1, \cdots, X_n]$ is Noetherian, J is finitely generated. Let $J = \langle f_1, \cdots, f_r\rangle$ and $g \in I(Z(J))$. Introduce an extra variable Z and consider $f_1, \cdots, f_r, 1 - Zg \in K[X_1, \cdots, X_n, Z]$. Let $\mathfrak{A} = \langle f_1, \cdots, f_r, 1 - Zg\rangle$. We claim that $Z(\mathfrak{A}) = \emptyset$. If $(a_1, \cdots, a_n, b) \in Z(\mathfrak{A}) \Rightarrow (a_1, \cdots, a_n) \in Z(J)$. Since $g \in I(Z(J)) \Rightarrow g(a_1, \cdots, a_n) = 0 \Rightarrow 1 - bg(a_1, \cdots, a_n) = 0$ leads to a contradiction. Hence our claim is proved and by Hilbert's Nullstellensatz, $1 \in \mathfrak{A}$. Let

$$1 = h_1 f_1 + \cdots + h_r f_r + h(1 - Zg) \tag{13}$$

where $h_i \in K[X_1, \cdots, X_n, Z], 1 \leq i \leq r, h \in K[X_1, \cdots, X_n, Z]$. We consider the ring morphism $\theta : K[X_1, \cdots, X_n, Z] \to K(X_1, \cdots, X_n)$ defined by $X_i \mapsto X_i, 1 \leq i \leq n$ and $Z \mapsto 1/g$. We apply

θ on (1) and we have $1 = \sum_{i=1}^{r} \theta(h_i)\theta(f_i) \Rightarrow \sum_{i=1}^{r} f_i \frac{\tilde{h_i}}{g^{n_i}}, \tilde{h_i} \in K[X_1, \cdots, X_n], 1 \le i \le n$. Clearing the denominator, $g^P = \sum_{i=1}^{r} f_i \alpha_i \Rightarrow g \in \sqrt{J}$. Therefore $I(Z(J)) = \sqrt{J}$. □

Note 3.23. *The above method is known as Rabinowitch's trick.*

Theorem 3.24 (Artin-Tate)**.** *Let $A \subseteq B \subseteq C$ be rings. Suppose that A is Noetherian and C is finitely generated as an $A-$algebra and that C is either*

(1) *finitely generated as a $B-$module or*
(2) *integral over B*

then B is finitely generated as an $A-$algebra.

Proof. Since (1) and (2) are equivalent, we assume (1). Let $C = A[x_1, \cdots, x_n]$ ($x_1, \cdots, x_n$ generates C as an $A-$algebra) and $y_1, \cdots, y_m$ generates C as a $B-$module. As $x_i \in C, 1 \le i \le n$ we have

$$(*) \quad x_i = \sum_{j=1}^{m} b_{ij} y_j, b_{ij} \in B, 1 \le i \le n \text{ and } (**) \quad y_i y_j = \sum_{k=1}^{m} b_{ijk} y_k, 1 \le i \le m, \le j \le m.$$

Let B_0 be the algebra over A generated by b_{ij} and b_{ijk}. By Hilbert basis theorem, B_0 is Noetherian (since A is Noetherian). Let $f \in C = A[x_1, \cdots, x_n]$. Substituting $(*)$ and $(**)$ repeatedly we can write $f = \sum_{i=1}^{m} h_i y_i, h_i \in B_0$. Therefore C is finitely generated as B_0-module. Hence C is Noetherian B_0-module. As B is a submodule of C, so B is finitely generated B_0-module and B_0 is finitely generated $A-$algebra. Hence B is finitely generated $A-$algebra. □

Theorem 3.25. *Let $F|K$ be an algebraic extension and $S = \{\alpha_1, \cdots, \alpha_n\} \subseteq F$, then $K[\alpha_1, \cdots, \alpha_n] = K(\alpha_1, \cdots, \alpha_n)$. Consider the map $\theta : K[X_1, \cdots, X_n] \to K(\alpha_1, \cdots, \alpha_n)$ is defined by $X_i \mapsto \alpha_i, 1 \le i \le n$. Then θ is a $K-$algebra homomprhism and $Ker\, \theta = \langle f_1(X_1), f_2(X_1, X_2), \cdots, f_n(X_1, \cdots, X_n)\rangle$.*

Proof. Since $K(\alpha_1)|K$ is an algebraic extension, we consider the minimal polynomial $f_1(X_1) \in K[X_1]$ of α_1 over K. Again α_2 is algebraic over K so over $K(\alpha_1)$. Let $f_2(X_1, X_2) \in K[X_1, X_2]$ such that $f_2(\alpha_1, X_2) \in K(\alpha_1)[X_2]$ is the minimal polynomial of α_2 over $K(\alpha_1)$. Here we note that $K[\alpha_1] = K(\alpha_1)$. Therefore we can consider the coefficient of the minimal polynomial of α_2 over $K(\alpha_1)$ are in $K[\alpha_1]$. Therefore we have $K[X_1]/f_1(X_1) \cong K(\alpha_1)$ and $K(\alpha_1)[X_2]/f_2(\alpha_1, X_2) \cong K(\alpha_1, \alpha_2)$. Inductively we can consider $f_i(X_1, \cdots, X_i)$ such that

$$K(\alpha_1, \cdots, \alpha_{i-1})[X_i]/f_i(X_1, \cdots, X_i) \cong K(\alpha_1, \cdots, \alpha_i).$$

We observe that each $f_i(X_1, \cdots, X_i) \in K[X_1, \cdots, X_i]$ is monic in X_i. We claim that $\mathrm{Ker}\, \theta = \langle f_1(X_1), \cdots, f_n(X_1, \cdots, X_n)\rangle$. By construction of $f_i(X_1, \cdots, X_i)$, we have $f_i(X_1, \cdots, X_i) \in \mathrm{Ker}\, \theta$ for all $1 \le i \le n$. We assume that degree of X_i in $f_i(X_1, \cdots, X_i)$ is $d_i, 1 \le i \le n$. Now pick $g(X_1, \cdots, X_n) \in \mathrm{Ker}\, \theta$. By division algorithm

$$g(X_1, \cdots, X_n) = q(X_1, \cdots, X_n) f_n(X_1, \cdots, X_n) + r_0^{(n)}(X_1, \cdots, X_{n-1}) + \cdots + r_{d_n-1}^{(n)}(X_1, \cdots, X_{n-1}) X_n^{d_n-1}. \tag{14}$$

Again dividing $r_i^{(n)}(X_1,\cdots,X_{i-1})$ by $f_{n-1}(X_1,\cdots,X_{n-1}), 1 \le i \le d_n - 1$ we get

$$r_i^{(n)}(X_1,\cdots,X_{n-1}) = q_i(x_1,\cdots,X_{n-1})f_{n-1}(X_1,\cdots,X_{n-1}) + r_0^{(n-1)}(X_1,\cdots,X_{n-2}) \\ + \cdots + r_{d_{n-1}-1}^{(n-1)}(X_1,\cdots,X_{n-2})X_{n-1}^{d_{n-1}-1}$$

for all $1 \le i \le d_n - 1$. Repeated application of division algorithm shows that

$$r_i^{(2)}(X_1) = q_i(X_1)f_1(X_1) + r_0^{(1)} + r_1^{(1)}X_1 + \cdots + r_{d_1-1}^{(1)}X_1^{d_1-1}$$

for all $1 \le i \le d_2 - 1$. Putting all these together in (14) and applying θ both sides, we get $g(\alpha_1,\cdots,\alpha_n) = 0$ that is $g \in \langle f_1(X_1),\cdots,f_n(X_1,\cdots,X_n)\rangle$. Therefore our claim is proved. $\square$

4. Primary decomposition

Definition 4.1. (1) *Let A be a ring and M be an $A-$module. A prime ideal p is called associated prime ideal of M if there exists $x \in M$ such that $p = Ann_A(x)$. We define*

$$Ass_A(M) = \{p \in spec\, A : p \text{ is an associated prime of } M\}.$$

(2) *For an ideal $I \subseteq A$, the associated primes of the $A-$modules A/I are referred to as the prime divisors of I.*

Observation 4.2. *Let A be a Noetherian ring and M be a non zero $A-$module. We consider*

$$\sum = \{Ann_A(x) : x \in M \setminus \{0\}\}.$$

Since A is Noetherian, every chain of ideals has an upper bound. By Zorn's lemma, $\sum$ has a maximal element. We claim that maximal elements of $\sum \subseteq Ass_A(M)$. In particular $Ass_A(M) \neq \emptyset$ if $M \neq 0$. Let $Ann_A(y)$ is a maximal element of $\sum$ for some $y \in M \setminus \{0\}$. Let $ab \in Ann_A(y) \Rightarrow (ab)y = 0 \Rightarrow a(by) = 0$. If $by \neq 0$ then $a \in Ann_A(by)$. Since $Ann_A(y) \subseteq Ann_A(by)$ and $Ann_A(y)$ is a maximal element in $\sum$, we have $Ann_A(y) = Ann_A(by) \Rightarrow a \in Ann_A(y)$ that is $Ann_A(y) \in Ass_A(M)$.

Corollary 4.3. *The set of all zero divisors of $M, Z(M) = \bigcup_{p \in Ass_A(M)} p$.*

Proof. Let $a \in Z(M)$ then there is $x_0 \in M \setminus \{0\}$ such that $ax_0 = 0 \Rightarrow a \in \mathrm{Ann}_A(x_0)$. Consider a maximal element of $\sum$ containing $\mathrm{Ann}_A(x_0)$. Since maximal elements of $\sum$ are associated primes we have $Z(M) \subseteq \bigcup_{p \in \mathrm{Ass}_A(M)} p$. Now pick $b \in \bigcup_{p \in \mathrm{Ass}_A(M)} p \Rightarrow b \in p$ for some $p \in \mathrm{Ass}_A(M)$ that is $bx = 0$ for some non zero $x \in M \Rightarrow b \in Z(M)$. This completes the proof. □

Observation 4.4. *Let A be a ring and M be an $A-$module, $p \in spec\, A$. $p \in Ass_A(M)$ if and only if there is an exact sequence $0 \to A/p \to M$.*

Proof. Let $p \in \mathrm{Ass}_A(M)$ then p is of the form $\mathrm{Ann}_A(x)$ for some $x \in M$. Define $\theta_x : A \to M$ by $\theta_x(a) = ax$. Then $\mathrm{Ker}\, \theta_x = p$ and by first isomorphism theorem $A/p \hookrightarrow M$.

Conversely, there is an exact sequence $0 \to A/p \xrightarrow{f} M$. Pick $a + p \in A/p$ such that $a \notin p$ and consider the element $f(a + p) = m$. We shall show that $\mathrm{Ann}_A(m) = p$. Let $s \in \mathrm{Ann}_A(m) \Rightarrow sm = 0 \Rightarrow sf(a + p) = 0 \Rightarrow f(sa + p) = 0 \Rightarrow sa \in p \Rightarrow s \in p$ (since $a \notin p$). Similarly take $b \in p$. Now

$$sx = sf(a + p) = f(sa + p) = f(0 + p) = 0.$$

Therefore, $s \in \mathrm{Ann}_A(x)$. □

Observation 4.5. *Let A be a ring, M be an $A-$module and S be a multiplicative set in A. Then*

$$Ass_{S^{-1}A}(S^{-1}M) \supseteq \{S^{-1}p : p \in Ass_A(M) \text{ and } p \cap S = \emptyset\}.$$

Equality occurs if A is Noetherian.

Proof. Let $p \in \mathrm{Ass}_A(M)$ and $p \cap S = \emptyset$. We have an exact sequence $0 \to A/p \to M$ of $A-$module. Since $S^{-1}A$ is flat,

$$\begin{array}{ccccc} 0 \longrightarrow & A/p \otimes_A S^{-1}A & \longrightarrow & M \otimes_A S^{-1}A & \text{is exact} \\ & \wr\| & & \wr\| & \\ 0 \longrightarrow & \dfrac{S^{-1}A}{S^{-1}p} & \longrightarrow & S^{-1}M & \text{is exact.} \end{array}$$

Therefore by previous observation $S^{-1}p \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$.

Suppose A is Noetherian. Let $S^{-1}p \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$. Then $S^{-1}p$ is of the form $\mathrm{Ann}_{S^{-1}A}(x/s)$ and $p \cap S = \emptyset$. We observe that $\mathrm{Ann}_{S^{-1}A}(x/s) = \mathrm{Ann}_{S^{-1}A} = (x/1)$ as $\dfrac{x}{s} = \dfrac{1}{s} \cdot \dfrac{x}{1}$ and $\dfrac{1}{s}$ is unit in $S^{-1}A$. Consider the set

$$G = \{\mathrm{Ann}_A(ux) : u \in S \text{ and } ux \neq 0\}.$$

Corollary 4.6. *Let A be a Noetherian ring and M be an $A-$module. Then $p \in Ass_A(M)$ if and only if $pA_p \in Ass_{A_p}(M_p)$.*

Theorem 4.7. *Let A be a ring and*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of $A-$module. Then $Ass_A(M) \subseteq Ass_A(M') \cup Ass_A(M'')$.

Proof. Let $p \in \mathrm{Ass}_A(M)$. Then there is an exact sequence $0 \to A/p \to M$. Let $\theta(A/p) = N$ be a submodule of M.

Case 1. If $N \cap f(M') \neq \{0\}$. Let $\theta(a+p) \in N \cap f(M')$. We know that $\mathrm{Ann}_A(\theta(a+p)) = p$. Let $\theta(a+p) = f(x')$ for some $x' \in M'$. Since f is injective, $\mathrm{Ann}_A(f(x')) = \mathrm{Ann}_A(x') \Rightarrow p \in \mathrm{Ass}_A(M')$.

Case 2. If $N \cap f(M') = \{0\} \Rightarrow N \cap \mathrm{Ker}\, g = \{0\}$. Let $\theta(a+p) \in N$ where $a \notin p$. We claim that $\mathrm{Ann}_A(g \circ \theta(a+p)) = \mathrm{Ann}_A(\theta(a+p))$. It is quite obvious that $\mathrm{Ann}_A(\theta(a+p)) \subseteq \mathrm{Ann}_A(g \circ \theta(a+p))$. For the reverse inclusion, Let $\alpha \in \mathrm{Ann}_A(g \circ \theta(a+p)) \Rightarrow \alpha g(\theta(a+p)) = 0 \Rightarrow g(\alpha\theta(a+p)) = 0 \Rightarrow \alpha\theta(a+p) \in \mathrm{Ker}\, g \cap N = \{0\} \Rightarrow \mathrm{Ann}_A(\theta(a+p))$. ☐

Corollary 4.8. *Let A be a ring and M be an $A-$module, N a submodule of M. Then,*

(1) $Ass_A(M) \subseteq Ass_A(N) \cup Ass_A(M/N)$.
(2) $Ass_A(N) \subseteq Ass_A(M)$.

When does the equality holds in (1)?

Let A be a Noetherian ring, M a finitely generated $A-$module. Then we know that $\mathrm{Ass}_A(M) \neq \emptyset$. Consider $p_1 \in \mathrm{Ass}_A(M)$, then there exists an exact sequence $0 \to A/p_1 \xrightarrow{\theta_1} M$. Let $\theta_1(A/p_1) = M_1 \subseteq M$. If $M/M_1 = 0$ then we stop. If not, then pick $p_2 \in \mathrm{Ass}_A(M/M_1)$ and we have an exact sequence $0 \to A/p_2 \xrightarrow{\theta_2} M/M_1$. Let $\theta_2(A/p_2) = M_2/M_1$ where $M_1 \subseteq M_2 \subseteq M$. If $\dfrac{M/M_1}{M_2/M_1} \cong M/M_2 = 0$ then we stop, otherwise pick $p_3 \in \mathrm{Ass}_A(M/M_2)$ and continue this process. Then we get a chain of submodules

$$(*) \qquad 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

of M where $M_i/M_{i-1} \cong A/p_i$ and $p_i \in \mathrm{Ass}_A(M/M_{i-1})$. Since M is Noetherian, the chain $(*)$ becomes stationary after some finite steps. So there exists $k \in \mathbb{N}$ such that $M_k = M$. For each

$1 \le i \le k$ we have an exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

and by the previous corollary we have $\mathrm{Ass}_A(M_i) \subseteq \mathrm{Ass}_A(M_{i-1}) \cup \mathrm{Ass}_A(M_i/M_{i-1})$. If we put $i = k$ then

$$\begin{aligned}
\mathrm{Ass}_A(M) &\subseteq \mathrm{Ass}_A(M_{k-1}) \cup \mathrm{Ass}_A(M/M_{k-1}) \\
&\subseteq \mathrm{Ass}_A(M_{k-2}) \cup \mathrm{Ass}_A(M_{k-1}/M_{k-2}) \cup \mathrm{Ass}_A(M/M_{k-1}) \\
&\;\vdots \\
&\subseteq \mathrm{Ass}_A(M_0) \cup \left(\bigcup_{i=1}^{k} \mathrm{Ass}(M_i/M_{i-1}) \right).
\end{aligned}$$

Note that $\mathrm{Ass}_A(M_0) = \emptyset$ and since $M_i/M_{i-1} \cong A/p_i$, so $\mathrm{Ass}_A(M_i/M_{i-1}) = \{p_i\}$. Therefore $\mathrm{Ass}_A(M) \subseteq \{p_1, \cdots, p_n\}$.

Corollary 4.9. *If A is Noetherian and M is a finitely generated $A-$module then $Ass_A(M)$ is finite.*

Theorem 4.10. *Let A be a Noetherian ring and M be an $A-$module. Then $Ass_A(M) \subseteq Supp\ (M)$.*

Proof. Let $p \in \mathrm{Ass}_A(M)$ then there is an exact sequence $0 \to A/p \to M$. Since A_p is flat,

$$\begin{array}{ccccc}
0 \longrightarrow & A/p \otimes_A A_p & \longrightarrow & M \otimes_A A_p & \quad \text{is exact} \\
& \wr\| & & \wr\| & \\
0 \longrightarrow & \dfrac{A_p}{pA_p} & \longrightarrow & M_p & \quad \text{is exact.}
\end{array}$$

Since pA_p is maximal in A_p, $A_p/pA_p \neq 0$ and therefore $M_p \neq 0 \Rightarrow p \in \mathrm{Supp}\ (M)$. $\square$

Theorem 4.11. *Let A be a Noetherian ring and M be an $A-$module. Then $\min Ass_A(M) = \min Supp\ (M)$ where $\min Ass_A(M)$ and $\min Supp\ (M)$ are the collection of minimal primes of $Ass_A(M)$ and $Supp\ (M)$ respectively.*

Proof. Let $p \in \min \mathrm{Ass}_A(M) \Rightarrow p \in \mathrm{Supp}\ (M)$. Suppose $p \notin \min \mathrm{Supp}\ (M)$ then there is a $q \in \mathrm{Supp}\ (M)$ such that $q \subsetneq p$. Since $q \in \mathrm{Supp}\ (M) \Rightarrow M_q \neq 0$ so there exists $p_1 \in \mathrm{spec}\, A$ such that $p_1 A_q \in \mathrm{Ass}_{A_q}(M_q) \Rightarrow p_1 \in \mathrm{Ass}_A(M)$ but $p_1 \subseteq q \subsetneq p$ which is a contradiction as p is a minimal prime in $\mathrm{Ass}_A(M)$. Therefore $\min \mathrm{Ass}_A(M) \subseteq \min \mathrm{Supp}\ (M)$.

Conversely, let $p \in \min \mathrm{Supp}\ (M) \Rightarrow M_p \neq 0 \Rightarrow \mathrm{Ass}_{A_p}(M_p) \neq 0$.

Claim. $\mathrm{Ass}_{A_p}(M_p) = \{pA_p\}$. If $qA_p \in \mathrm{spec}\, A_p$ with $q \subsetneq p$ such that $qA_p \in \mathrm{Ass}_{A_p}(M_p)$ then $q \in \mathrm{Ass}_A(M)$ so there exists an exact sequence $0 \to A/q \to M$. Flatness of A_q gives

$$\begin{array}{ccccc}
0 \longrightarrow & A/q \otimes_A A_q & \longrightarrow & M \otimes_A A_q & \quad \text{is exact} \\
& \wr\| & & \wr\| & \\
0 \longrightarrow & \dfrac{A_q}{qA_q} & \longrightarrow & M_q & \quad \text{is exact.}
\end{array}$$

but $M_q = 0$ gives us a contradiction as $q \subsetneq p$ and $p \in \min \text{Supp }(M)$. Hence $\text{Ass}_{A_p}(M_p) = \{pA_p\} \Rightarrow p \in \text{Ass}_A(M)$. Since $p \in \min \text{Supp }(M)$ and $\text{Ass}_A(M) \subsetneq \text{Supp }(M) \Rightarrow p \in \min \text{Ass}_A(M)$. $\square$

Observation 4.12. *Let A be a Noetherian ring and M a finitely generated $A-$module. Let $p \in$ Supp (M) and $p \subseteq q$ then we observe that $q \in$ Supp (M). If not then $M_q = 0$ so there exists $u \in A \setminus q$ such that $uM = \{0\}$ but $A \setminus q \subseteq A \setminus p \Rightarrow M_p = 0$ which is a contradiction. Suppose* $\min$ *Supp $(M) = \{p_1, \cdots, p_r\}$ then Supp $(M) = \bigcup_{i=1}^{r} V(p_i)$ and $V(p_i)$'s are the irreducible component of the closed set Supp (M) in spec A.*

Definition 4.13. *The prime ideals $\{p_1, \cdots, p_r\} = \min Supp\ (M) = \min Ass_A(M)$ are called isolated primes of M and the remaining assocoated primes are called embedded primes.*

Definition 4.14. *Let A be a ring and M be an $A-$module. A submodule N of M is said to be primary submodule of M if the following condition holds for all $a \in A$ and $m \in M, m \notin N$ and $am \in N \Rightarrow a^k M \subseteq N$ for some $k > 0$. Equivalently, if a is a zero divisor for M/N then $a \in \sqrt{Ann_A(M/N)}$.*

If we take $M = A$ and $N = I$ and ideal of A then I is said to be primary ideal if $ab \in I$ with $b \notin I \Rightarrow a \in \sqrt{I}$ for all $a, b \in A$.

Example 4.15. *Let A be a ring and $m \in$ maxspec A. Then m^k is a primary ideal. Let $ab \in m^k$ with $b \notin m^k$. We need to show that $a \in \sqrt{m^k} = m$. As $ab \in m^k \subseteq m$, Since $b + m^k \neq 0 + m^k, b + m^k \in m/m^k$ (notice that it is not an unit in A/m^k). Then there is an element $\alpha + m^k \in m/m^k$ such that*