Complex Analysis

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1 Problem 1

Starting from the definition, we derive the identity.

$$\begin{aligned}
\sinh z &= \frac{e^z - e^{-z}}{2} \\
&= \frac{1}{2}(e^x e^{iy} - e^{-x}e^{-iy}) \\
&= \frac{1}{2}\Big(e^x(\cos y + i\sin y) - e^{-x}(\cos(-y) + i\sin(-y))\Big) \\
&= \frac{1}{2}\Big(e^x(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)\Big) \\
&= (\frac{e^x - e^{-x}}{2})\cos(y) + i\Big((\frac{e^x + e^{-x}}{2})\sin(y)\Big) \\
&= \sinh x\cos(y) + i\cosh x\sin(y)
\end{aligned}$$

Thus, we have proven the identity.

2 Problem 2

First, we begin with the definition and derive the derivative.

$$\begin{aligned}
\frac{d}{dz}\tan^{-1} &= \frac{d}{dz}\frac{i}{2}\log(\frac{i+z}{i-z}) \\
&= (\frac{i}{2})(\frac{i-z}{i+z})\frac{d}{dz}(\frac{i+z}{i-z})
\end{aligned}$$

For compact notation, I will use ' to denote the derivative with respect to z, as we will be doing the quotient rule (which expands out quite large)

$$
\begin{aligned}
&(\frac{i}{2})(\frac{i-z}{i+z})(\frac{i+z}{i-z})' \\
&= (\frac{i}{2})(\frac{i-z}{i+z})\Big(\frac{(i+z)'(i-z)-(i-z)'(i+z)}{(i-z)^2}\Big) \\
&= (\frac{i}{2})(\frac{i-z}{i+z})\Big(\frac{(1)(i-z)-(-1)(i+z)}{(i-z)^2}\Big) \\
&= (\frac{i}{2})(\frac{i-z}{i+z})\Big(\frac{(i-z)+(i+z)}{(i-z)^2}\Big) \\
&= (\frac{i}{2})(\frac{i-z}{i+z})\Big(\frac{2i}{(i-z)^2}\Big) \\
&= i(\frac{i-z}{i+z})\Big(\frac{i}{(i-z)^2}\Big) \\
&= (\frac{i-z}{i+z})\Big(\frac{-1}{(i-z)^2}\Big) \\
&= (\frac{1}{i+z})\Big(\frac{-1}{(i-z)}\Big) \\
&= \Big(\frac{-1}{(i+z)(i-z)}\Big) \\
&= \frac{-1}{-1+zi-zi-z^2} \\
&= \frac{-1}{-1-z^2} \\
&= \frac{1}{1+z^2}
\end{aligned}
$$

Thus, we have derived the identity:

$$\frac{d}{dz}\tan^{-1} = \frac{1}{1+z^2}$$

3 Problem 3

(a)

First, we start with the definition of $\mathrm{Log}(z)$

$$\mathrm{Log}(z) = \ln \mid z \mid + i\mathrm{Arg}(z)$$

Thus, we have:

$$\mathrm{Log}(-ei) = \ln \mid -ei \mid + i\mathrm{Arg}(-ei)$$

Which we can rewrite as:

$$
\begin{aligned}
\mathrm{Log}(-ei) &= \ln \mid e \mid + i\mathrm{Arg}(-ei) \\
&= 1 + i\mathrm{Arg}(-ei)
\end{aligned}
$$

Since e is just a scalar in this instance, the Arg of $-ei$ is $-\frac{\pi}{2}$ since it is just pointing directly down in the complex plane. Thus, we can finally rewrite the expression to:

$$\text{Log}(-ei) = 1 - \frac{\pi}{2}i$$

(b)

This is a *natural* extension from (a), except with the new definition of log. We have:

$$\log(z) = \ln \mid z \mid + i\arg(z)$$

And we know the definition of arg is: $\text{Arg(z)} + 2\pi n$, where n is any integer. Thus, rewriting this, we have:

$$\begin{aligned}\log(-ei) &= \ln \mid -ei \mid + i\arg(-ei) \\ &= 1 + i(-\frac{\pi}{2} + 2\pi n)\end{aligned}$$

4 Problem 4

(a)

Let $f(z) = \sin(\bar{z})$. Using a useful identity that was given to us by two excellent students, we can rewrite $\sin(\bar{z})$ as:

$$\sin(\bar{z}) = \sin(x)\cosh(-y) + i\cos(x)\sinh(-y)$$

Since cosh is even and sin is odd, we rewrite it to be:

$$\sin(\bar{z}) = \sin(x)\cosh(y) - i\cos(x)\sinh(y)$$

Which is exactly $\overline{\sin(z)}$ obtained by using the same identity above.

(b)

To be analytic, at a point z_0, the function must be differentiable everywhere in some ϵ neighbordhood around z_0. To confirm whether or not the function is differentiable, we will utilize C.R derivatives. Recall, if the function is differentiable, it satisfies the following pair of equations:

$$\begin{aligned}u_x &= v_y \\ u_y &= -v_x\end{aligned}$$

Where $f(z) = u(x, y) + iv(x, y)$. From part a, we easily define u, v as follows:

$$\begin{aligned}u(x, y) &= \sin(x)\cosh(y) \\ v(x, y) &= -\cos(x)\sinh(y)\end{aligned}$$

We get the following derivatives:

$$\begin{aligned} u_x &= \cos(x)\cosh(y) \\ u_y &= \sin(x)\sinh(y) \\ v_x &= \sin(x)\sinh(y) \\ v_y &= -\cos(x)\cosh(y) \end{aligned}$$

Thus, putting these into the form of the aforementioned pair of criterion, we have:

$$\begin{aligned} \cos(x)\cosh(y) &= -\cos(x)\cosh(y) \\ \sin(x)\sinh(y) &= -\sin(x)\sinh(y) \end{aligned}$$

Because we have essentially the same thing on either side in both pairs, just the negative on one side, this is only ever satisfied when both sides are 0 in both pairs. This only occurs at a collection of disjoint points, where $y = 0, x = \pi/2 + n\pi$, (where n is any integer) at the point $z = (\pi + n\pi) + 0i$. However, since there is no continuous region of space where the derivative exists, there exists no ϵ neighborhood where the derivative exists everywhere. Thus, the function is analytic nowhere.

5 Problem 5

To find the solutions to the equation $\sin(z) = i$, we use the inverse sine to rewrite it to solve for z:

$$\arcsin z = -i\log(iz + (1 - z^2)^{1/2})$$

So, we take the arcsin of both sides, and we have:

$$\begin{aligned} z = \arcsin i &= -i\log(i^2 + (1 - i^2)^{1/2}) \\ &= -i\log(-1 \pm (2)^{1/2}) \\ &= -i\log(\pm\sqrt{2} - 1) \end{aligned}$$

We then split this into the two scenarios based on the $\pm\sqrt{2}$. In either scenario, the number in question is purely real, so if it is positive, the principle argument is 0, if it is negative, it is $-\pi$. For the positive, we have:

$$\begin{aligned} z &= -i\log(\sqrt{2} - 1) \\ &= -i(\ln \mid \sqrt{2} - 1 \mid + 0 + i2\pi n) \\ &= -i\ln \mid \sqrt{2} - 1 \mid + 2\pi n \end{aligned}$$

For the negative we have:

$$\begin{aligned} z &= -i\log(-\sqrt{2} - 1) \\ &= -i(\ln \mid -\sqrt{2} - 1 \mid + i(-\pi + 2\pi n)) \\ &= -i\ln \mid -\sqrt{2} - 1 \mid + (-\pi + 2\pi n) \end{aligned}$$

We then simplify this into two instances to consider, one for odd, and one for even n. This gives us the following two solutions, first, for even valued n:

$$z = -i \ln \mid \sqrt{2} - 1 \mid + 2\pi n$$

For odd:

$$z = -i \ln \mid -\sqrt{2} - 1 \mid + \pi(2n - 1)$$

6 Problem 6

This is the contour integral over the unit semi circle. Recall we can express $z(t) = x(t)+iy(t)$, in this case, since we are along the unit circle, we split this into two lines. C_1 is the straight line along the real axis from -1 to 1, parameterized by $(y(t) = 0, x(t) = t)$. C_2 is the semi-circular arc from -1 to 1, which can be expressed by $z(t) = e^{it}$. Thus we have the following statement:

$$\int_S f(z)dz = \int_{C_1} tdt + \int_{C_2} e^{-it} * ie^{it} dt$$

Which can be further rewritten as such:

$$\int_S f(z)dz = \int_{-1}^{1} tdt + \int_0^{\pi} idt$$

We then evaluate, which gives us:

$$\int_S f(z)dz = \frac{t^2}{2} \mid_{-1}^{1} + it \mid_0^{\pi}$$

Plugging in, we have:

$$\int_S f(z)dz = 0 + i\pi = i\pi$$

7 Problem 7

We have $f(z) = \frac{\bar{z}}{z+i}$. For a maximum, M, we have the following statement.

$$\mid \int_C f(z(t))z'(t)dt \mid \leq M* \mid \int_C z'(t)dt \mid$$

Which can be rewritten:

$$\int_C f(z(t))z'(t)dt \leq M * L$$

Where L is the arclength (length of the contour path). To find M, we consider the worst case of both the numerator and the denominator, and then construct a worst worst case from these instances, giving us the largest possible upper bound. For the numerator, $\mid \bar{z} \mid = \mid z \mid$, which has a max at 1 or i, which is 1, since everything in between is less than 1. Thus the max value of the top is 1. For the denominator, we want to find the smallest it could be.

Given the function $z+i$, we want to minimize the magnitude, which is how far the point is away from the point $-i$. The closest point along the path to $-i$ is z $= 1$. Thus, the smallest the denominator can be is $\sqrt{2}$. So, our worst worst case scenario is a maximum of $\frac{1}{\sqrt{2}} = M$. The length of the curve is just given by the hypotenuse of the triangle with sides of 1, which gives us $\sqrt{2}$. Thus, $M * L = \frac{\sqrt{2}}{\sqrt{2}} = 1$, thus our upper bound is 1.

$$\int_C f(z(t))z'(t)dt \leq 1$$