Problem Set 1

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February 2020

Problem 1

a

Assume:

$$0 <| z - i |< \delta$$

We have:

$$| 4z + i - 5i |$$

Rearranging this we have:

$$\begin{aligned} | 4z - 4i | \\ | 4(z - i) | \\ 4\delta < \epsilon \\ \delta < \frac{\epsilon}{4} \end{aligned}$$

b

Assume:

$$0 <| z - 2 - i |< \delta \tag{1}$$

We have:

$$| \frac{z + \bar{z}}{2} - 2 |$$

Rearranging, we get:

$$
\begin{aligned}
&\mid \frac{z+\bar{z}}{2} - 2 \mid \\
&\frac{1}{2} \mid z+\bar{z}-4 \mid \\
&\frac{1}{2} \mid (z-(2+i)) + (2+i) + \bar{z} - 4 \mid \\
&\frac{1}{2}(\delta + \mid (2+i) + \bar{z} - 4 \mid \\
&\textbf{Now, using the triangle ineq.} \\
&\leq \frac{1}{2}(\delta + \mid \overline{(2+i) + \bar{z} - 4} \mid) \\
&\frac{1}{2}(\delta + \mid (2-i) + z - 4 \mid) \\
&\frac{1}{2}(\delta + \mid z - i - 2 \mid) \\
&\frac{1}{2}(\delta+\delta) < \epsilon \\
&\delta < \epsilon
\end{aligned}
$$

c

Assume:

$$0 <\mid z+i \mid< \delta$$

We have:

$$\mid z^2+1 \mid$$

Rearranging, we get:

$$
\begin{aligned}
&\mid z^2+1 \mid \\
&\mid (z+i)(z-i) \mid \\
&\delta \mid (z-i) \mid \\
&\delta \mid z-i+2i-2i \mid \\
&\delta \mid (z+i)-2i \mid \\
&\delta \mid \delta - 2i \mid \\
&\delta^2 + 2\delta < \epsilon \\
&\textbf{Now, if we assume } \delta < 1\textbf{, this becomes:} \\
&\delta + 2\delta < \epsilon \\
&\delta < \min\{1, \frac{\epsilon}{3}\}
\end{aligned}
$$

d

Assume:

$$0 <\mid z-i \mid< \delta$$

We invert the function to $(\frac{1}{f(z)})$ to show the corresponding finite limit, giving us:

$$\mid \frac{z-i}{2} \mid$$

Rearranging:

$$\begin{aligned} \mid \frac{z-i}{2} \mid \\ \mid \frac{\delta}{2} \mid < \epsilon \\ \delta < 2\epsilon \end{aligned}$$

e

First we invert the input of the function $(f(\frac{1}{z}))$ to show the limit. Now, we assume:

$$0 < \mid z \mid < \delta$$

Next, we have:

$$\mid \frac{1+\frac{1}{z}}{3\frac{1}{z}} - \frac{1}{3} \mid$$

Rearranging, we have:

$$\begin{aligned} \mid \frac{1+\frac{1}{z}}{3\frac{1}{z}} - \frac{1}{3} \mid \\ \frac{1}{3} \mid \frac{1+\frac{1}{z}}{\frac{1}{z}} - 1 \mid \\ \frac{1}{3} \mid \frac{z}{z}\frac{1+\frac{1}{z}}{\frac{1}{z}} - 1 \mid \\ \frac{1}{3} \mid z+1-1 \mid \\ \frac{1}{3} \mid z \mid \\ \frac{1}{3}\delta < \epsilon \\ \delta < 3\epsilon \end{aligned}$$

Problem 2

Given the function:

$$(\frac{\bar{z}}{z})^2$$

We rewrite this into polar form, and simplify:

$$(\frac{re^{-i\theta}}{re^{i\theta}})^2$$
$$(e^{-2i\theta})^2$$
$$e^{-4i\theta}$$

Thus, this function gives different answers depending on the angles of the path taken towards zero, showing the limit does not exist. The limit as z goes to zero implies to the limit as r goes to zero, since we are fixing theta. If the limit exists, regardless of the path, it will evaluate to the same thing. Let theta be $\frac{\pi}{3}$ or $\frac{\pi}{5}$, which evaluate to $e^{-\frac{4\pi}{3}i}$ and $e^{-\frac{4\pi}{5}i\theta}$ respectively, which are not the same. Thus the limit does not exist.

Problem 3

a

Let $c = 0$, then we have:

$$T(z) = \frac{az+b}{d}$$

Using the theorem, we turn this function into $\frac{1}{T(\frac{1}{z})}$, giving us:

$$\frac{d}{a\frac{1}{z}+b}$$

Rearranging, we have:

$$\frac{d}{a\frac{1}{z}+b}$$
$$\frac{dz}{a+bz}$$

We can evaluate the limit as z goes to 0, and this expression becomes 0, thus via the theorem, the limit is infinity.

b

Let $c \neq 0$, we use the theorem to prove this first part by inverting the input, $f(\frac{1}{z})$ and look at the limit as we approach 0 to be $\frac{a}{c}$. Thus, we have:

$$T(\frac{1}{z}) = \frac{a\frac{1}{z}+b}{c\frac{1}{z}+d}$$

We then rearrange the expression:

$$\frac{a\frac{1}{z}+b}{c\frac{1}{z}+d}$$
$$\frac{a\frac{1}{z}+b}{c\frac{1}{z}+d}\left(\frac{z}{z}\right)$$
$$\frac{a+bz}{c+dz}$$

We can evaluate the limit as z goes to 0, and this becomes $\frac{a}{c}$, thus via the theorem, the limit as T(z) goes to infinity is $\frac{a}{c}$. For the next part we invert the function itself $(\frac{1}{f(z)})$, and see if it goes to 0, to prove it goes to infinity at $-\frac{d}{c}$. Doing so gives us:

$$\frac{1}{T(z)}=\frac{cz+d}{az+b}$$

We then look at this as we approach $-\frac{d}{c}$. Evaluating this limit as z goes to 0 in the numerator, which makes the expression 0. Thus, via the theorem, the limit at this point is infinity.

Problem 4

Using the definition of the derivative gives us:

$$\lim_{\Delta z\to 0}\frac{\overline{z_0+\Delta z}-\bar{z_0}}{\Delta z}$$

Rearranging gives us:

$$\lim_{\Delta z\to 0}\frac{\overline{z_0+\Delta z}-\bar{z_0}}{\Delta z}$$
$$\lim_{\Delta z\to 0}\frac{\bar{z_0}+\overline{\Delta z}-\bar{z_0}}{\Delta z}$$
$$\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$$

Which will give you different values if approach it from different paths, showing the limit does not exist, regardless of the point chosen.