Topology Problem Set 4

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Problem 1

Let $K = \{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z}_+\}$. Assume $\mathbb{R}$ has the standard topology.

a

Show that $\mathcal{T}_K$ is the discrete topology

Proof. To show that $\mathcal{T}_K$ is the discrete topology, we will show that every singleton set is open in $\mathcal{T}_K$. Suppose we have an arbitrary element of K, which is $\frac{1}{n}$. We create an open interval $(\frac{1}{n+1}, \frac{1}{n-1})$ around $\frac{1}{n}$. The only element this open interval intersects with is $\frac{1}{n}$, thus we have shown the arbitrary element's singleton set $\{\frac{1}{n}\}$ to be open. Moreover, because this element was arbitrary, we have now shown every element of K to have a open singleton set in $\mathcal{T}_K$. Hence, because every singleton set of K is open, we know this is the discrete topology. $\square$

b

Let $K^* = K \cup \{0\}$. Is $\mathcal{T}_{K^*}$ also the discrete topology?

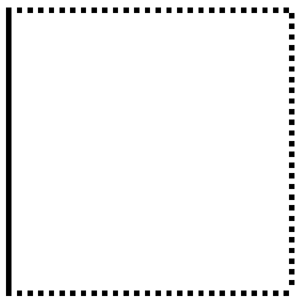
Proof. To show that $\mathcal{T}_{K^*}$ is not the discrete topology, we will show that not every singleton set is open, specifically, the set $\{0\}$ is not open. Suppose we create an open interval around 0, of the form $(\frac{1}{n}, \frac{1}{n})$, where n is very large. As K is defined, there will always exist an element, $\frac{1}{n+1}$ such that $\frac{1}{n+1} < \frac{1}{n}$. Moreover, due to this, we cannot create an open interval including 0, without including more than just 0. Hence, $\{0\}$ is not open, so $\mathcal{T}_{K^*}$ is not the discrete topology.

$\square$

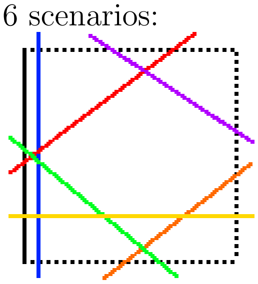
Problem 2

Let L be a straight line in the plane. Then we have two possibilities: $L = \{(a, b) \in \mathbb{R}^2 \mid a \text{ a fixed constant}\}$ (a vertical line); or $L = \{(x, y) \in \mathbb{R}^2 \mid y = mx+b, m \text{ and } b \text{ constants}\}$ (a non-vertical line).
Determine the topology on L as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$.

Proof. ($\mathbb{R}_l \times \mathbb{R}$)
We will show the topology is the lower limit topology, because the open sets are of the form $[a, b)$ and (a, b). A standard open rectangle in $\mathbb{R}_l\mathbb{R}$ looks like:



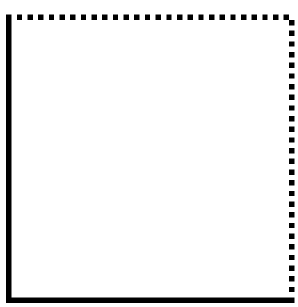
We can draw all possible types intersections with all possible types of lines, and we get these 6 scenarios:



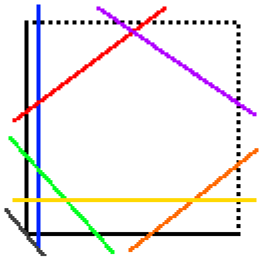
Red, a positive sloped line intersecting a closed bound and an open bound, gives us the open set of $[a, b)$. Orange, a positive slope intersecting two open bounds, gives us the open set of (a, b). Yellow, a horizontal slope, intersecting a closed bound and an open bound, gives us the open set $[a, b)$. Green, a negative slope intersecting a closed bound and an open bound, gives us the open set $[a, b)$. Blue, a vertical line intersecting two open bounds, gives us the open set of (a, b). And finally, purple, a negative slope intersecting two open bounds, gives us the open set of (a, b). Given all these possible combinations of intersections, the open sets we can create are $[a, b)$ and (a, b). Thus, this is the lower limit topology.

□

Proof. ($\mathbb{R}_l \times \mathbb{R}_l$)
We will show the topology is the discrete topology, because every singleton set is open. A standard open rectangle in $\mathbb{R}_l\mathbb{R}$ looks like:



We can draw all possible types intersections with all possible types of lines, and we get these 6 scenarios:



Red, a positive sloped line intersecting a closed bound and an open bound, gives us the open set of $[a, b)$. Orange, a positive slope intersecting a closed bound and an open bound, gives us the open set of $[a, b)$. Yellow, a horizontal slope, intersecting a closed bound and an open bound, gives us the open set $[a, b)$. Green, a negative slope intersecting two closed bounds, gives us the open set $[a, b]$. Blue, a vertical line intersecting a closed bound and an open bound, gives us the open set of $[a, b)$. Purple, a negative slope intersecting two open bounds, gives us the open set of (a, b). Finally, grey, a line intersecting the point $\{a\}$, gives us the open singleton set $\{a\}$. Given all these possible combinations of intersections, the open sets we can create are $[a, b)$ and (a, b), $[a, b]$, $(a, b]$, and $\{a\}$. Thus, this is the discrete topology, since every singleton set for every point is open.

□

Problem 3

Let $\mathbb{R}$ have the standard topology and define $p : \mathbb{R} \to \mathbb{Z}$ by $p(x) = x$ if x is an integer, and $p(x) = n$, where $x \in (n-1, n+1)$ and n is an odd integer. For example, $p(\pi) = 3$. What topology is the quotient topology on $\mathbb{Z}$ induced by p?

Proof. To show the quotient topology induced by p is the digital line topology, we will show that each topology is finer than the other.
$\rightarrow$ (The quotient topology is finer than the digital line topology)

We will show each element open in the digital line is also open in the quotient topology by showing the basis elements $\{n\}$ (if n is odd) and $\{n-1, n, n+1\}$ (if n is even) are open. The pre-image $p^{-1}(\{n\})$ is the interval $(n-1, n+1)$ which is open. The pre-image $p^{-1}(\{n-1, n, n+1\})$ is $(n-2, n) \cup \{n\} \cup (n, n+2)$ which is exactly the interval $(n-2, n+2)$ which is open. Thus, since the basis elements of the digital line topology are open, the quotient topology induced by p is finer than the digital line topology.

$\leftarrow$ (The digital line topology is finer than the quotient topology)

To show the digital line topology is finer than the quotient topology, we will use the contrapositive: If we have a non-open element in the digital line, we have a non-open in the quotient. Suppose we have a set U, that is a non-open element in the digital line. Then U is a set that contains an even integer n without $n-1$ and $n+1$. We claim this same element U is not open in the quotient topology induced by p. If the set U was the singleton $\{n\}$, (n is even) its pre-image would be a single, discrete point which is not open in the standard topology. If U containted $\{n\}$ along with other elements, but not both $n-1$ and $n+1$, then

WLOG, the pre-image of U would be a half closed interval of the form $[n, b)$ or $(a, n]$. If there were more than one such n, the interval could be fully closed, $[n_1, n_2]$, where n_1 and n_2 are both even n without their respective $n+1$'s and $n-1$'s. Thus, if U is not open in the digital line, it is not open in the quotient topology. Thus the digital line topology is finer than the quotient topology. □

Problem 4

a

Show $f(x) = x + 2$ is continuous.

Proof. Let $\epsilon > 0$. Suppose we have δ such that $\delta = \epsilon$ and $\delta > |x - a|$. Then $\delta > |(x + 2) - (a + 2)|$, which can be rewritten as $\delta > |f(x) - f(a)|$. We now have $\epsilon > |f(x) - f(a)|$. Thus by 183, $f(x)$ is continuous. □

b

Show $f(x) = 2x$ is continuous.

Proof. Let $\epsilon > 0$. Suppose we have δ such that $\delta = \frac{\epsilon}{2}$ and $\delta > |x - a|$. Then $2\delta > |2x - 2a|$, which can be rewritten as $2\delta > |f(x) - f(a)|$. Then we have $\epsilon > |f(x) - f(a)|$. Thus by 183, $f(x)$ is continuous. □

c

Show $h(x) = x^2$ is continuous.

Proof. Let $\epsilon > 0$. Our goal is to find $\delta > 0$ such that if $|x - a| < \delta$, then $|x^2 - a^2| = |x - a||x + a| < \epsilon$. Since we only care about x close to a, we suppose that $|x - a| < 1$. Given this assumption, we can show that $|x + a| \leq M$, where $M = \max\{2a - 1, 2a + 1\}$. Since $|x - a| < 1$, then $x \in [a - 1, a + 1]$. Thus, substituting what x can be into $|x + a|$, we have a bound of $[2a - 1, 2a + 1]$, and the max possible distance will be the max of the two bounds, $M = \max\{2a - 1, 2a + 1\}$, thus $|x + a| \leq \max\{2a - 1, 2a + 1\}$. Next, we will show if $\delta = \min\{1, \frac{\epsilon}{M}\}$ and if $|x - a| < \delta$, then $|x^2 - a^2| < \epsilon$. If $\delta = \frac{\epsilon}{M}$ then $\frac{\epsilon}{M} > |x - a|$ and $M|x - a| < \epsilon$. We know $|x^2 - a^2| = |x - a||x + a| \leq |x - a|M < \epsilon$. Thus, if $\delta = \frac{\epsilon}{M}$, $|x^2 - a^2| < \epsilon$. Since it is the min of the two options, if $1 < \frac{\epsilon}{M}$, it quickly follows this also holds. □

Problem 5

Let X and Y be topological spaces and $X \times Y$ have the product topology. Suppose $A \subseteq X$ and $B \subseteq Y$ Show that $\bar{A} \times \bar{B} = \overline{A \times B}$,where we view the closure of these sets inside $X \times Y$

Proof. $\rightarrow$
Let $(x, y) \in \bar{A} \times \bar{B}$ and $C = \bigcup_i (U_i \times V_i)$ such that U_i is open in A and V_i is open in B. Then for every element $x \in U_i$, $U_i \cap A \neq \varnothing$, and for every element $y \in V_i$, $V_i \cap B \neq \varnothing$. So, if $(x, y) \in U_i \times V_i$, the $U_i \times V_i \cap A \times B \neq \varnothing$. Since $U_i \times V_i \in C$, then $C \cap A \times B \neq \varnothing$ thus, $(x, y) \in \overline{A \times B}$. $\square$

Proof. $\leftarrow$
Let $(x, y) \in \overline{A \times B}$. Then, for every open set $U \times V$ containing the element (x, y), $(U \times V) \cap (A \times B) \neq \varnothing$. The intersection $(U \times V) \cap (A \times B)$ can be rewritten (via class notes on March 24, page 7) as $(U \cap A) \times (V \cap B) \neq \varnothing$. Because this intersection is not equal to the empty set, $U \cap A \neq \varnothing$ and $V \cap B \neq \varnothing$. Thus, $x \in \bar{A}$ and $y \in \bar{B}$, thus $(x, y) \in \bar{A} \times \bar{B}$.

$\square$

Meme