Problem Set 5

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Problem 1

Let A be the matrix:

$$\begin{bmatrix} 2 & -2 & 4 \\ 4 & -3 & 11 \\ -8 & 9 & -14 \end{bmatrix}$$

We then use the three subsequent matrices, E_1, E_2, E_3 to cancel the first and second columns below the diagonal.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

We then have:

$$E_3E_2E_1A = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = U$$

We will call this resulting matrix U. We then have:

$$E_1^{-1}E_2^{-1}E_3^{-1}U = A$$

Thus, our lower triangular matrix is given by $E_1^{-1}E_2^{-1}E_3^{-1}$, which gives us the matrix:

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}$$

We then have:

$$Lc = b$$
$$Ux = c$$

First we solve for c, and we get:

$$c = L^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ -13 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$$

Next, we use this result below:

$$x = U^{-1}c = \begin{bmatrix} 1/2 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 2

The converse is not true. Consider this counter example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The determinant is non-zero, thus it is invertible, yet it is not s.d.d.

Problem 3

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$. Using this, we solve for x, y in terms of u, v. We know,

$$A \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus we have:

$$A^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

Solving this out, we get two equations:

$$u = 3/5x - 1/5y$$
$$v = 2/5y - 1/5x$$

Thus the unit ball becomes:

$$u^2 + v^2 = 1$$
$$(3/5x - 1/5y)^2 + (2/5y - 1/5x)^2 = 1$$
Doing some distribution and multiplying by 25 we get:
$$5y^2 - 10xy + 10x^2 = 25$$
Divide by 5:
$$y^2 - 2xy + 2x^2 = 5$$

Thus, the unit ball in the coordinates of x, y becomes the ellipse above.
The eigenvalues of A are $\lambda_1 = 3.61, \lambda_2 = 1.38$, which are the major and minor axes of the ellipse respectively.

Problem 4

In this instance, the exact soultion is $(1, 1, 1, 1)^T$, starting with $(0, 0, 0, 0)$ we have an initial error of 4 in the 1 norm.

(a)

2 Jacobi iterations give us:

$$x_1^{(1)} = 3/4$$
$$x_2^{(1)} = 2/5$$
$$x_3^{(1)} = 1/2$$
$$x_4^{(1)} = 3/4$$
Err measure in 1 norm: 1.6
$$x_1^{(2)} = 0.85$$
$$x_2^{(2)} = 0.8$$
$$x_3^{(2)} = 0.75833$$
$$x_4^{(2)} = 0.875$$
Err measure in 1 norm: 0.7166

(b)

2 Gauss Seidel iterations give us:

$$x_1^{(1)} = 0.75$$
$$x_2^{(1)} = 0.7$$
$$x_3^{(1)} = 0.7333$$
$$x_4^{(1)} = 0.93333$$

Err measure in 1 norm:0.883337

$$x_1^{(2)} = 0.925$$
$$x_2^{(2)} = 0.91666$$
$$x_3^{(2)} = 0.9611$$
$$x_4^{(2)} = 0.9902$$

Err measure in 1 norm:0.20704

(c)

2 SOR iterations give us:

$$x_1^{(1)} = 0.9$$
$$x_2^{(1)} = 0.912$$
$$x_3^{(1)} = 0.9648$$
$$x_4^{(1)} = 1.189$$

Err measure in 1 norm: 0.4122

$$x_1^{(2)} = 0.9936$$
$$x_2^{(2)} = 1.006$$
$$x_3^{(2)} = 1.047$$
$$x_4^{(2)} = 0.9763$$

Err measure in 1 norm: 0.08310

Problem 5

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Dia = {{4, 0, 0, 0}, {0, 5, 0, 0}, {0, 0, 6, 0}, {0, 0, 0, 4}};

Lower = {{0, 0, 0, 0}, {2, 0, 0, 0}, {0, 2, 0, 0}, {0, 0, 1, 0}};

Upper = {{0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}};

IDia = {{1/4, 0, 0, 0}, {0, 1/5, 0, 0}, {0, 0, 1/6, 0}, {0, 0, 0, 1/4}};

Tj = IDia.(Lower + Upper)
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$$\left\{\left\{0, \frac{1}{4}, 0, 0\right\}, \left\{\frac{2}{5}, 0, \frac{1}{5}, 0\right\}, \left\{0, \frac{1}{3}, 0, \frac{1}{6}\right\}, \left\{0, 0, \frac{1}{4}, 0\right\}\right\}$$

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Eigenvalues[Tj] // N

{-0.431187, 0.431187, -0.149702, 0.149702}

TGS = Inverse[Dia - Lower].Upper
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$$\left\{\left\{0, \frac{1}{4}, 0, 0\right\}, \left\{0, \frac{1}{10}, \frac{1}{5}, 0\right\}, \left\{0, \frac{1}{30}, \frac{1}{15}, \frac{1}{6}\right\}, \left\{0, \frac{1}{120}, \frac{1}{60}, \frac{1}{24}\right\}\right\}$$

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Eigenvalues[TGS] // N

{0.185923, 0.0224108, 0., 0.}
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Thus, the spectral radius for $\rho(T_J) = 0.431187$ and $\rho(T_{GS}) = 0.224108$

Problem 6

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \omega a_{ii}^{-1}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

We can rewrite this as:

$$x^{(k+1)} = x_i^{(k)} - \omega x^{(k)} + \omega D^{-1}\Big(b + Lx^{(k+1)} + Ux^{(k)}\Big)$$

We then will get it in to a more, familar form:

$$\begin{aligned}
x^{(k+1)} &= x^{(k)} - \omega x^{(k)} + \omega D^{-1}\Big(b + Lx^{(k+1)} + Ux^{(k)}\Big) \\
Dx^{(k+1)} &= Dx^{(k)} - \omega Dx^{(k)} + \omega\Big(b + Lx^{(k+1)} + Ux^{(k)}\Big) \\
\frac{1}{w}Dx^{(k+1)} &= \frac{1}{w}Dx^{(k)} - Dx^{(k)} + \Big(b + Lx^{(k+1)} + Ux^{(k)}\Big) \\
\frac{1}{w}Dx^{(k+1)} - Lx^{(k+1)} &= \frac{1}{w}Dx^{(k)} - Dx^{(k)} + b + Ux^{(k)} \\
\Big(\frac{1}{w}D - L\Big)x^{(k+1)} &= \Big(\frac{1}{w}D - D + U\Big)x^{(k)} + b
\end{aligned}$$

Thus, we have $M = \frac{1}{w}D - L$ and $N = \frac{1}{w}D - D + U$