Combinatorics PS 2

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Problem 1

If f is a function and $f(i) = i$, then we call i a fixed point of f.

(a)

How many functions $[5] \to [5]$ have atleast 1 fixed point?

There are a total of 5^5 functions from $[5] \to [5]$. There are a total of 4^5 functions with no fixed points (4 options per point, since we remove the fixed possibility). Thus, there are a total of $5^5 - 4^5 = 2101$ functions with at least one fixed point.

(b)

If the function has no fixed point, we just remove itself as an option. Thus, we have $n-1$ options per point, and n points to map. Thus, there are $(n-1)^n$ total functions with no fixed points.

(c)

We can count the derangements here. The number of derangements of a n element set is $C_n = (n-1)(C_{n-1} + C_{n-2})$, where $C_0 = 1, C_1 = 1$. So, $C_4 = 3(C_3 + C_2) = 3(2+1) = 9$.

Problem 2

Determine the number of nonnegative solutions to each of the following equations, assume all z_i are nonnegative integers unless otherwise stated.

(a)

$z_1 + z_2 + z_3 + z_4 = 1$

For this instance, we have 1 item to distribute into 4 distinct bins. We are allowed 0 in any of the bins. Thus, we have 4 different ways to put 1 item into these 4 bins, the rest being 0's.

(b)

$z_1 + z_2 + 10z_3 = 8$
For this instance, we note that we can only put 0 into bin 3 (z_3), since any non-zero integer would put us over 8. Thus, we have 8 things to distribute into 2 distinct bins, which gives us $\left(\binom{2}{8}\right)$.

(c)

$z_1 + z_2 + \cdots + z_20 = 401$, and $z_i \geq 1$. Since we must guarantee at least 1 per bin, this becomes 401 objects into 20 distinct bins, or $\left(\binom{20}{381}\right)$

(d)

$z_1 + z_2 + z_3 + z_4 = 12$, where $z_1, z_2 \geq 1$ and $z_3, z_4 \geq 2$. Similar logic as (c), but we subtract off 1+1+2+2, so we get $\left(\binom{4}{6}\right)$.

1 Problem 3

Prove

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}$$

If we note that we can split 3^n into $(1+2)^n$, we see we can apply the binomial theorem. Thus we have

$$3^n = (2+1)^n = \binom{n}{0} 2^n 1^0 + \binom{n}{1} 2^{n-1} 1^1 + \cdots + 2^0 1^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}$$

This becomes exactly the right side.

2 Problem 4

First, we find all the ways to make the first block of 5. Note that the order does not matter here, so we do not want to overcount. There are $\binom{10}{5}$ ways to pick the elements of the first block. Then the remaining 5 elements will be partitioned into 2,3,4, or 5 blocks. For each case, we have the following possible blocks: 1-4 or 2-3, 1-1-3 or 1-2-2, 1-1-1-2, or 1-1-1-1-1. For each of these instances, 1-4 gives 5 possibilities, 2-3 gives $\binom{5}{2}$ possibilities, 1-1-3 gives $\binom{5}{3}$ possibilities, 1-2-2 gives $\binom{5}{2} * \binom{3}{2}/2$, 1-1-1-2 gives $\binom{5}{2}$ possibilities, and 1-1-1-1-1 is one possibility. Thus, we have

$$\binom{10}{5} * \left(5 + 10 + 10 + 15 + 10 + 1\right) = 12852$$

Problem 5

Using the recurrence relation on page 80, we have

$$P(n,k) = P(n-1,k-1) + P(n-k,k)$$

Subbing in n-2 for k we have

$$P(n,n-2) = P(n-1,n-3) + P(2,n-2)$$

(We have already proven this recurrence to be true)
Base case, we have P(3,1) = 1. Next base case we have P(4,2) = 2 We can note that when $n \geq 5$, P(2,n-2) becomes 0 for every subsequent n. Thus we need only consider P(n-1,n-3), which becomes 2 for every iteration afterwards, since P(5,3) = P(4,2) = 2, P(6,4) = P(5,3) = P(4,2) = 2, and so on. Thus, P(n,n-2) = 2 for $n > 3$ and 1 for $n = 3$.

Problem 6

Using inclusion-exclusion and the floor operation trick we learned, we find there are 25 elements ($\frac{100}{4}$) that are divisible by 4, 16 divisible by 6, and 14 divisible by 7. In order to avoid over-counting, we need to remove the elements that these 3 groups share. For 6 and 4, we divide by 24 and find they share 4, for 6 and 7, they share 2, for 4 and 7, they share 3 elements, and all 3 share no elements. Thus, if we take 100 - 25 - 16 - 14, we have 45 elements. But this is too much subtracted. We then add back the overcounts, and get 45 + 4 + 2 + 3, or 54 elements that are not divisible by 4,6, or 7.

Challenge