PDE {Problem}

Rippy

September 20, 2019

1 Problem 1

Determine the Fourier Series for $f(x) = 1 - 2x + x^2$ for $x \in [-1, 1]$

Proof. The Fourier Series is given by $f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \Big)$, where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \tag{1}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L}) dx \tag{2}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L}) dx \tag{3}$$

Solving for a_0

$$a_0 = \frac{1}{2} \int_{-1}^{1} 1 - 2x + x^2 dx = \frac{4}{3} \tag{4}$$

$$\tag{5}$$

Solving for a_n

$$a_n = \int_{-1}^{1} (1 - 2x + x^2) \cos(n\pi x) \tag{6}$$

$$= \int_{-1}^{1} \cos(n\pi x) - 2x \cos(n\pi x) + x^2 \cos(n\pi x) \tag{7}$$

$$= \int_{-1}^{1} x^2 \cos(n\pi x) \tag{8}$$

$$= \frac{4 \cos(n\pi x)}{(n\pi)^2} \tag{9}$$

Solving for b_n

$$\int_{-1}^{1} (1 - 2x + x^2)\sin(n\pi x) \tag{10}$$

$$= \int_{-1}^{1} \sin(n\pi x) - 2x\sin(n\pi x) + x^2\sin(n\pi x) \tag{11}$$

$$= \int_{-1}^{1} -2x\sin(n\pi x) \tag{12}$$

$$= \frac{4\cos(n\pi x)}{n\pi} \tag{13}$$

Thus, the Fourier Series for $f(x)$ for $x \in [-1, 1]$ is:

$$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \left(\Big(\frac{4\cos(n\pi x)}{(n\pi)^2} \Big) \cos(\frac{n\pi x}{L}) + \Big(\frac{4\cos(n\pi x)}{n\pi} \Big) \sin(\frac{n\pi x}{L}) \right) \tag{14}$$

□

2 Problem 2

Determine the Fourier Series for $f(x) = 5\cos(2\pi x)$ for $x \in [-1, 1]$

Proof. The Fourier Series is given by $f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \Big)$

Solving for a_0, a_n, b_n

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx = \int_{-1}^{1} 5\cos(2\pi x) = 0 \tag{15}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx = \int_{-1}^{1} 5\cos(2\pi x)\sin(n\pi x) = 0 \tag{16}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx = \int_{-1}^{1} 5\cos(2\pi x)\cos(n\pi x) = 0 \Big(\forall\ n \neq 2 \Big) \tag{17}$$

$$\Big(n = 2 \Big)\ a_2 = \int_{-1}^{1} 5\cos^2(2\pi x) = 5 \tag{18}$$

Thus the Fourier Series for $f(x)$ for $x \in [-1, 1]$ is:

$$5\cos(2\pi x) \tag{19}$$

□

3 Problem 3

Consider the function $f(x) = 1 - x$ on $[0, 2]$.
(a) Find $F_s(x)$, the Fourier Sine series for $f(x)$.
(b) Find $F_c(x)$, the Fourier Cosine series for $f(x)$.
(c) For what values in $[0, 2]$ is $F_s(x) = f(x)$? For what values in $[0, 2]$ is $F_c(x) = f(x)$? Explain.

Proof. (a) The Fourier Sine Series for $f(x)$ is given by $F_s(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx \tag{20}$$

Solving we get:

$$b_n = \int_0^2 \sin(\frac{n\pi x}{2}) - x\sin(\frac{n\pi x}{2})dx \tag{21}$$

$$= \frac{2}{n\pi}\Big(1 + 2cos(n\pi)\Big) \tag{22}$$

Thus, solving we get:

$$F_s(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\Big(1 + 2cos(n\pi)\Big)\sin\left(\frac{n\pi x}{2}\right) \tag{23}$$

□

Proof. (b) The Fourier Cosine Series for $f(x)$ is given by $F_c(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ where

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \tag{24}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L}) \tag{25}$$

Solving we get:

$$a_0 = \frac{1}{2}\int_0^2 1 - x = 0 \tag{26}$$

$$a_n = \int_0^2 (1-x)\cos(\frac{n\pi x}{2}) = \frac{-1}{(n\pi)^2}(4\cos(n\pi) + 4) \tag{27}$$

Thus, solving we get:

$$F_c(x) = \sum_{n=1}^{\infty}\left(\frac{-1}{(n\pi)^2}(4\cos(n\pi) + 4)\right)\cos\left(\frac{n\pi x}{2}\right) \tag{28}$$

□

4 Problem 4

Prove Parseval's Formula

Proof. Given $f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \Big)$

Then,

$$[f(x)]^2 = \Big[a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \Big) \Big]^2 \tag{29}$$

$$= a_0^2 + 2a_0 \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \tag{30}$$

$$+ \Big[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \cos(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) + b_n a_m \sin(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) \tag{31}$$

$$+ a_n b_m \cos(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) + b_n b_m \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) \Big] \tag{32}$$

Integrating both sides from -L to L, and breaking down each part, we get $\int_{-L}^{L} [f(x)]^2 dx =$

$$\int_{-L}^{L} a_0^2 dx = 2La_0^2 \tag{33}$$

$$\int_{-L}^{L} 2a_0 \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) dx = 0 \text{ because of Periodic Bounds} \tag{34}$$

$$\tag{35}$$

Because we are looking at the case where n = m:

$$\int_{-L}^{L} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\textbf{Stuff}) dx = \int_{-L}^{L} \sum_{n=1}^{\infty} (\textbf{Stuff}) dx \tag{36}$$

Then we can rewrite the expression and cancel the sin*cos functions because they are odd functions, giving us:

$$\sum_{n=1}^{\infty} \int_{-L}^{L} a_n^2 \cos^2(\frac{n\pi x}{L}) dx + \int_{-L}^{L} b_n^2 \sin^2(\frac{n\pi x}{L}) dx \tag{37}$$

Using Trig identities to rewrite further, we get:

$$\sum_{n=1}^{\infty} \frac{a_n^2}{2} \int_{-L}^{L} 1 + \cos(\frac{2n\pi x}{L}) dx + \frac{b_n^2}{2} \int_{-L}^{L} 1 - \cos(\frac{2n\pi x}{L}) dx \tag{38}$$

Integrating, both cosines go to 0, and we get 2L for each integral, giving us:

$$\sum_{n=1}^{\infty} a_n^2 L + b_n^2 L \tag{39}$$

Bringing down the $2a_0L$ we get:

$$2a_0 L + \sum_{n=1}^{\infty} a_n^2 L + b_n^2 L = \int_{-L}^{L} [f(x)]^2 dx \tag{40}$$

Thus, by dividing out L, we finally get:

$$2a_0 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{L} \int_{-L}^{L} [f(x)]^2 dx \tag{41}$$

□

5 Problem 5

Suppose a laterally insulated $3m$ aluminum rod is initially at $100°C$ throughout. At $t = 0$ the ends are cooled to $0°$ instantly and held at that temperature. Approximately, how long will it take until the center of the rod has a temperature of less than $50°$? (The thermal diffusivity of Al is $k = 8.41810^{-5} m^2/s$.)

Proof. Assume $u(x,t) = \phi(x)G(t)$. We know $u(x,t)$ must satisfy $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ Thus:

$$\phi(x)G'(t) = k\phi'' G(t) \tag{42}$$

Rewriting, and using separation of variables:

$$\frac{G'}{kG} = \frac{\phi''}{\phi} = -\lambda \tag{43}$$

Thus, solving for ϕ, G and λ we get:

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \tag{44}$$

$$\text{Where } b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L}) \tag{45}$$

$$G(t) = e^{-kt\lambda} \tag{46}$$

$$\lambda = \frac{L^2}{(n\pi)^2} \tag{47}$$

Finally, using these solved functions we get $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(1 - cos(n\pi))\sin(\frac{n\pi x}{3})e^{-kt(\frac{L^2}{(n\pi)^2})} \tag{48}$$

Solving for t, we get that the middle of the rod will be at $50°$ at $t = 6932$ seconds □

6 Problem 6

Find the same generic solution with imperfect lateral insulation this time

Proof. Identical work to problem 5, except after separation of variables, we end with:

$$\frac{G'}{kG} = \frac{\phi''}{\phi} - \frac{\gamma}{k} = -\lambda \tag{49}$$

Again, same process for solving for ϕ and G, only thing that changes is the λ, which comes out to be:

$$\lambda = \frac{L^2}{(n\pi)^2} + \frac{\gamma}{k} \tag{50}$$

Now, the final function $u(x,t)$ is now:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})e^{-kt(\frac{L^2}{(n\pi)^2}+\frac{\gamma}{k})} \tag{51}$$

□

7 Problem 7

Recalculate the t where the middle of the rod reaches $50°$ from 5, this time with imperfect insulation

Proof. This time assuming T is nonzero, we must do a substitution. Starting with:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \gamma(u - T) \tag{52}$$

Let $v = u - T$, then:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \gamma(v) \tag{53}$$

Following the EXACT same process from 6, and using the fact $u(x,t) = v(x,t) + T$ we get:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(1 - cos(n\pi))\sin(\frac{n\pi x}{3})e^{-kt(\frac{L^2}{(n\pi)^2}+\frac{\gamma}{k})} + T \tag{54}$$

Using this approximation, we get the center of the rod will reach $50°$ at $t = 1104$ seconds □