Combinatorics PS 1

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1 Problem 1

(a)

How many subset of [20] have smallest element 4 and largest element 15?
For this we fix two elements, 4 and 15, then remove choices 1-3 and 16-20, then form the subsets from there. This leaves 5-14 as options to add to the subset. The smallest subset is $\{4, 15\}$. Next for set of size 3, there are 10 elements left as choices if we move up, and each time we add an element, we remove that as an option since we don't have repeats. We also note that order doesn't matter here, since the subset $\{4, 15\} = \{15, 4\}$. Thus, we conclude this is a summation of 10 choose k per each size of the subset until we are at our maximum of 10 elements.

$$\sum_{k=0}^{10} \binom{10}{k}$$

This is equivalent to the subsets of any size of a set of cardinality 10, which is $2^{10} = 1024$. Thus this comes out to be 1024 possible subsets.

(b)

How many subset of [20] contain no even numbers?
We use a similar argument to earlier, except this time we remove even options. Thus, our options are 1,3,5, $\cdots$, 19. For subsets of size one, we have 10 options, for size two, since we don't allow repeats and order doesn't matter, we have 10 choose 2 possibilities, and so on. Thus we have the total number to be:

$$\sum_{k=1}^{10} \frac{10!}{k!(10-k)!}$$

We then also include the instance where nothing exists in the set, as this contains no even number. Which comes out to be 1024 possible subsets. Similarly, this comes out to the subsets of a cardinality 10 set once more, which is $2^{10} = 1024$. (since we consider the set to be the odds)

(c)

How many subset of [20] have no elements larger than 17?
Again, similar reasoning, except this time we remove options 18,19, and 20. 17 choose 10.

$$\frac{17!}{10!(7)!}$$

Problem 2

We have a group of 12 independents and 8 fraternity members.

(a)

How many ways can we form a committee of 5 people?
We consider each person to be distinct. To form a committee of size 5, this becomes 20 options to choose from, and we choose 5 of them. The order doesn't matter, since we only care if members are in the group, not the order they are picked in. Thus we have the total number of options to be:

$$\binom{20}{5} = \frac{20!}{5!(15!)}$$

Thus the total number of 5 person committees we can choose is 15504.

(b)

How many ways can we form a committee of 5 people, with 3 being fraternity members and 2 being independent?
We apply similar reasoning as before, but here we split the 5 group into two subset of size 2 and size 3 for independents and fraternity members respectively. So, it becomes 12 choose 2 (for independents) and 8 choose 3 (for fraternity members). We then multiply these together to get the total number of combinations, since we want all the possible parings of these two sets of subsets.

$$\binom{12}{2} * \binom{8}{3} = \frac{12!}{2!10!} * \frac{8!}{3!5!}$$

Thus the total number of such possible committees is: 3696.

(c)

How many ways can we form a committee of 5 people, with at least 3 fraternity members?
We apply a similar approach to part (b). We split the committee into two sets, one of independents and one of fraternity members. For one of the size 3 set, we fix this to be all possible 3-sets of fraternity members, so 8 choose 3. We assume the other 3 are independents. We then multiply these outcomes to find all possible pairings of sets, and we then repeat this for the different instances where we have 4,5, and 6 fraternity members, the others being

2,1, and 0 independents. We then add all these scenarios, and thus we have the number of possible combinations to be:

$$\sum_{k=3}^{6} \binom{8}{k} * \binom{12}{6-k}$$

Problem 3

If n is odd, then yes, this mapping is a Bijection.

Proof. Let $\{f(E) : \mathcal{E} \to \mathcal{O} \mid f(A) = A^c\}$ (The complement of $A \in \mathcal{E}$)

First, the function f is well defined. Because n is odd, if the set A is of even cardinality, the complement's cardinality $n - |A|$ will be an odd number minus an even number, which always returns an odd value. Similarly, the inverse function is well defined, as an odd cardinality set B's complement will be of cardinality $n - |B|$ which is an odd number minus and odd number, which always returns an even value.

First we prove one-to-one.

Let $E_1, E_2 \in \mathcal{E}$ and let $E_1 \neq E_2$. Assume $E_1^c = E_2^c$. The complement of a complement is the original set, so $(E_1^c)^c = E_1$ and $(E_2^c)^c = E_2$, thus $E_2 = E_1$, which is a contradiction. Thus, we now know a subset and the subset's complement form a unique partitioning of the entire set. Thus, for every $E_1, E_2 \in \mathcal{E}$, if $E_1 \neq E_2$, then $E_1^c \neq E_2^c$, thus f is one-to-one.

Next we prove onto.

Let $B \in \mathcal{O}$. By definition, $|B|$ is odd, and because n is odd, $B^c = [n] \setminus B$ has even cardinality, and exists within $\mathcal{E}$. Thus, for every $B \in \mathcal{O}$, there exists some element $B^c \in \mathcal{E}$ such that $f(B^c) = (B^c)^c = B$. Thus f is onto.

Thus f is a bijection. □

Problem 4

How many ways are there to seat 20 professors around a table with the alternating Div-1, Div-2 style? By phrasing of the question, there are 10 Div-1 professors, and 10 Div-2 professors. The order in which we seat the professors matters. If we consider this to be a circular table, then we must also take into account that an ordering of say, $\{1, 2, 3, 4\}$, is equivalent to an ordering $\{2, 3, 4, 1\}$, thus, there exists 20 equivalent cycles per permutation of professor seating, so based on equivalence classes, we must divide by 20. The first choice could be any of the 20, and subsequent choices would be 10,9,9,8,8...

$$20 * 10 * (9!)^2/20 = 10 * (9!)^2$$

Problem 5

If S is a subset of $[2n]$ of size $n + 1$, then S can contain values 1 through 2n. Partition [2n] into n distinct parings $\{1, 2n\}, \{2, 2n - 1\} \cdots \{n, n + 1\}$. Note that each of these pairings sum to exactly $2n + 1$, they partition the entire set, and there are exactly n distinct parings.

S is of size $n+1$, so we can choose up to n unpaired elements of $[2n]$, but the $+1$ element, by the pigeon hole principle, *must* pair with an element already chosen, and thus some subset of size two of these chosen elements must sum to $2n+1$.

Challenge

Let $n \in \mathbb{N}$. Consider the mapping f of a sequence such that f maps any positive integer a_i into a summation of a_i 1's and preserves commas. For example, the sequence $(3, 2, 2)$ would map to $1+1+1, 1+1, 1+1$. This sequence (composition) in question adds up to $n = 7$. Now consider each + and each , to be a choice in between each 1. The sequence composition of n becomes n 1's separated by $n-1$,'s and +'s.

First we prove f to be one-to-one. Let c_1 and c_2 be compositions of n. Assume $f(c_1) = f(c_2)$. The inverse mapping of f would be equivalent to adding the 1's together and again, preserving commas. There are an equal amount of 1's in both $f(c_1)$ and $f(c_2)$ with commas and +'s in the same places, we add the 1's separated by commas together, to form the sequences $c_1 = (a_1, a_2, \cdots, a_k)$ and $c_2 = (a_1, a_2, \cdots, a_k)$. Note because $f(c_1), f(c_2)$ were equal, they shared the positions of 1's, commas, and +'s, thus these compositions are then equivalent, meaning $c_1 = c_2$. WLOG, this is the same argument the other direction. Thus, f is one-to-one.

Next we prove onto. Let b be an element that is comprised of a n 1's and $n-1$,'s and +'s, and is a summation of a_1 1's followed by a comma, followed by a summation of a_2 1's followed by a comma, and so on until some final summation of a_k 1's, where $a_1 + a_2 + \cdots + a_k = n$. We use the inverse mapping of adding the a_i ones together, and preserving the commas. This element $f^{-1}(b)$ then becomes the sequence $(a_1, a_2, \cdots, a_k)$, which is a composition of n. Thus, for every element b, there exists some composition c_i such that $f(c_i) = b$.

Thus f is a bijection between these two spaces.

Now consider a list of 1's, separated by either a + or a comma. If we have n 1's, we will have $n-1$ instances to choose either a + or a comma to separate the list of 1's. This then becomes $\binom{2}{1}$ a total of $n-1$ times, which becomes 2^{n-1} possible compositions that sum to n.