PDE {Problem}

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Problem 1

Given the initial conditions:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \text{ in } (0,1)^2 \\ u(x,0) = f_1(x) &= \frac{3}{2}|x - \frac{1}{3}| \\ u(1,y) = f_2(y) &= (1-y)^2 \\ u(x,1) = f_3(x) &= sin(\pi x) \\ u(0,y) = f_4(y) &= \frac{1}{2}(1-y) \end{aligned}$$

We must break the PDE into 4 parts to then superimpose on top of each other, where in each part, everything is set to zero except that boundary condition.

BC 1

We assume $u(x,y) = \phi(x)G(y)$ and use sep of vars. From our boundary conditions and sep of vars, we have:

$$\begin{aligned} \phi(1) &= 0 \\ \phi(0) &= 0 \\ \phi'' &= -\lambda\phi \\ \lambda &= (n\pi)^2 \end{aligned}$$

This leads us to determining ϕ to be:

$$\phi_n = \sin(n\pi x)$$

We can assume G is of the form:

$$G(y) = c_1 \sinh(n\pi y) + c_2 \cosh(n\pi y)$$

Using the initial conditions, we can simplify:

$$\begin{aligned} G(1) = c_1 \sinh(n\pi) + c_2 \cosh(n\pi) = 0 \\ \Rightarrow -c_2 \cosh(n\pi) = c_1 \sinh(n\pi) \\ \Rightarrow c_1 = -c_2 \coth(n\pi) \end{aligned}$$

Using the final initial condition, we get:

$$u(x,0) = \sum_{n=1}^{\infty} \sin(n\pi x)\Big(c_1 \sinh(0) + c_2 \cosh(0)\Big) = \sum_{n=1}^{\infty} \sin(n\pi x)\Big(c_2\Big) = f_1(x)$$

Which is exactly a Fourier sine series. Solving for c_2 we get:

$$c_2 = 2\int_0^1 \frac{3}{2}|x - \frac{1}{3}| \sin(n\pi x) dx$$

THUS, plugging it all in, we get:

$$u(x,y) = \sum_{n=1}^{\infty} c_2 \sin(n\pi x)\Big(\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y)\Big)$$

BC 2

We assume $u(x,y) = \phi(x)G(y)$ and use sep of vars. From our boundary conditions and sep of vars, we have:

$$\begin{aligned} G(0) &= 0 \\ G(1) &= 0 \\ G'' &= -\lambda G \\ \lambda &= (n\pi)^2 \end{aligned}$$

This leads us to determine that $G = \sin(n\pi y)$. Now we must find ϕ. We know that it must be of the form:

$$\phi(x) = c_1 \sinh(n\pi x) + c_2 \cosh(n\pi x)$$

Given the initial conditions, we can simplify this, since:

$$\phi(0) = c_1 \sinh(0) + c_2 \cosh(0) = 0$$

Which indicates:

$$\phi(x) = c_1 \sinh(n\pi x)$$

Thus we now have:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi x) \sin(n\pi y)$$

Using our last boundary condition, we have:

$$u(1,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi) \sin(n\pi y)$$

Which is just a Fourier sine series. If we solve for b_n, we get:

$$b_n = \frac{2}{\sinh(n\pi)} \int_0^1 (1-y)^2 \sin(n\pi y) dy$$

THUS, We have:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi x) \sin(n\pi y)$$

BC 3

We assume $u(x,y) = \phi(x)G(y)$ and use sep of vars. From our boundary conditions and sep of vars, we have:

$$\begin{aligned} \phi(0) &= 0 \\ \phi(1) &= 0 \\ \phi'' &= -\lambda\phi \\ \lambda &= (n\pi)^2 \end{aligned}$$

This leads us to determine that $\phi_n(x) = \sin(n\pi x)$. Now we must determine $G(y)$. From sep of vars and initial conditions, we have:

$$\begin{aligned} G(0) &= 0 \\ G'' &= (n\pi)^2 G \end{aligned}$$

From these conditions, we can deduce that G must be of the form:

$$G(y) = c_1 \sinh(n\pi y) + c_2 \cosh(n\pi y)$$

Using the initial conditions, we know:

$$G(0) = 0 = c_1 \sinh(0) + c_2 \cosh(0) = c_2 \cosh(0)$$

Thus, c_2 must be 0. Thus we have:

$$G(y) = c_1 \sinh(0)$$

Reassembling our $u(x,y)$ we have:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \sinh(n\pi y)$$

We have the final boundary condition, $\phi(x)G(1) = \sin(\pi x)$, which we use to deduce that:

$$b_n = \begin{cases} \frac{1}{\sinh(\pi)} & n = 1 \\ 0 & n \neq 1 \end{cases}$$

THUS, We have:

$$u(x,y) = \frac{\sin(\pi x)\sinh(\pi y)}{\sin(\pi)}$$

BC 4

We assume $u(x,y) = \phi(x)G(y)$ and use sep of vars. From our boundary conditions and sep of vars, we have:

$$\begin{aligned} G(1) &= 0 \\ G(0) &= 0 \\ G'' &= -\lambda G \\ \lambda &= (n\pi)^2 \end{aligned}$$

This leads us to determining G to be:

$$G_n = \sin(n\pi y)$$

We can assume ϕ_n is of the form:

$$\phi(x) = c_1 \sinh(n\pi x) + c_2 \cosh(n\pi x)$$

Using the initial conditions, we can simplify:

$$\begin{aligned} \phi(1) = c_1 \sinh(n\pi) + c_2 \cosh(n\pi) &= 0 \\ \Rightarrow -c_2 \cosh(n\pi) &= c_1 \sinh(n\pi) \\ \Rightarrow c_1 &= -c_2 \coth(n\pi) \end{aligned}$$

Using the final initial condition, we get:

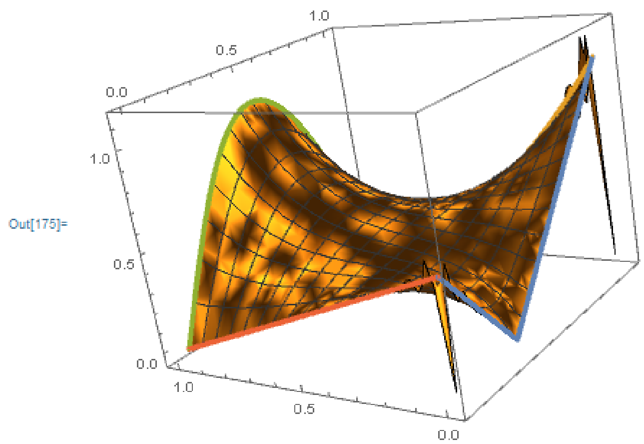
$$u(0,y) = \sum_{n=1}^{\infty} \sin(n\pi y)\Big(c_1 \sinh(0) + c_2 \cosh(0)\Big) = \sum_{n=1}^{\infty} \sin(n\pi y)\Big(c_2\Big) = f_4(x)$$

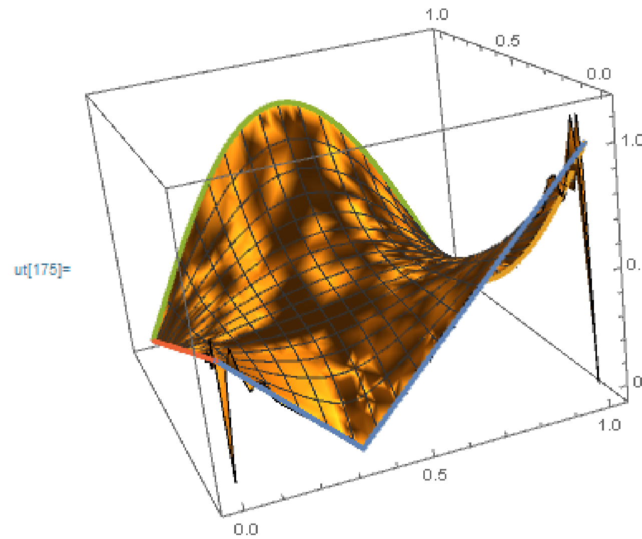
Which is exactly a Fourier sine series. Solving for c_2 we get:

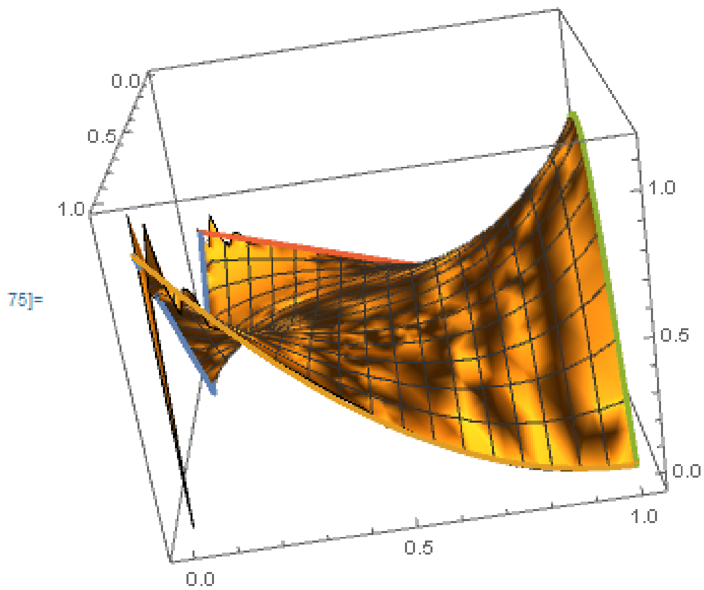
$$c_2 = 2\int_0^1 \frac{1}{2}(1-y)\sin(n\pi y)dy$$

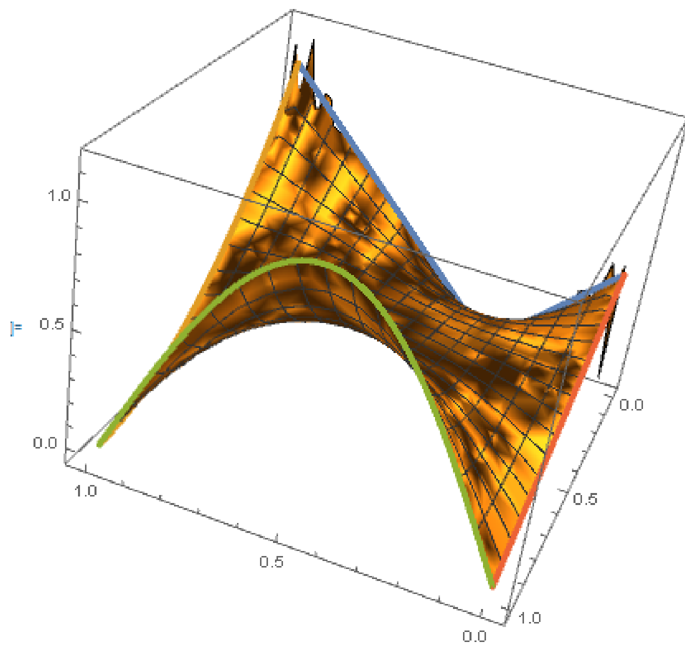
THUS, plugging it all in, we get:

$$u(x,y) = \sum_{n=1}^{\infty} c_2 \sin(n\pi y)\Big(\cosh(n\pi x) - \coth(n\pi)\sinh(n\pi x)\Big)$$









Problem 2

Show:

$$u(r,\theta) = (r^{\beta} - r^{-\beta})sin(\beta\theta)$$

Satisfies Laplace's Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solving for the derivatives, we get:

$$u_{\theta\theta} = -\beta^2(r^{\beta} - r^{-\beta})sin(\beta\theta)$$
$$u_r = (\beta r^{\beta-1} + \beta r^{-\beta-1})\sin(\beta\theta)$$

Plugging in we get:

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}(r(\beta r^{\beta-1} + \beta r^{-\beta-1})\sin(\beta\theta)) + \frac{1}{r^2}(-\beta^2(r^{\beta} - r^{-\beta})sin(\beta\theta)) \\
&\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}((\beta r^{\beta} + \beta r^{-\beta})\sin(\beta\theta)) + (-\beta^2(r^{\beta-2} - r^{-\beta-2})sin(\beta\theta)) \\
&\Rightarrow \frac{1}{r}((\beta^2 r^{\beta-1} - \beta^2 r^{-\beta-1})\sin(\beta\theta)) + (-\beta^2(r^{\beta-2} - r^{-\beta-2})sin(\beta\theta)) \\
&\Rightarrow \beta^2\sin(\beta\theta)\Big((r^{\beta-2} - r^{-\beta-2}) + (-r^{\beta-2} + r^{-\beta-2})\Big) \\
&\Rightarrow \beta^2\sin(\beta\theta)\Big(r^{\beta-2} - r^{\beta-2} - r^{-\beta-2} + r^{-\beta-2})\Big) \\
&\Rightarrow \beta^2\sin(\beta\theta)\Big(0\Big) = 0
\end{aligned}
$$

Problem 3

The mean value property states that if we draw a circle centered at a point, a, the average value of the function along the circle will equal the value of point a. Given the initial conditions, specifically $u(\alpha,\theta) = f(\theta)$, this is a circle, centered around $r = 0$. Given that:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$$

When $r = 0$, $u(0,\theta) = A_0$. Thus the point at the center of the circle with radius α has a value of A_0. Conveniently, A_0 is given by:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

Which is *exactly* the average value of $f(\theta)$ over the bounds of a circle with radius α. Thus this problem/ solution illustrates the mean value property!