Topology Final Exam

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May 2020

Problem 1

Let X and Y be topological spaces, and give $X \times Y$ the product topology. Let $(x_1, y_1), (x_2, y_2)$... be a sequence in $X \times Y$. Show that the sequence x_n, y_n converges to the point (x, y) if and only if (x_n) converges to x in X and (y_n) in Y.

Proof. $\rightarrow$
If x_n converges to x, for all open neighborhoods U of x, for some $N \in \mathbb{N}$, $x_n \in U$ for all $n \geq N$. If y_n converges to y, for all open neighborhoods V of y, for some $M \in \mathbb{N}$, $y_n \in V$ for all $n \geq M$. By definition of the product topology, all open neighborhoods of (x, y) are of the form $U \times V$, (where U, V are open neighborhoods of x, y respectively). (Since by definition, an open neighborhood of (x, y) must be made of open neighborhoods containing both x, y) Let a neighborhood of (x, y) be denoted as $U \times V$. WLOG, let $N \geq M$. Then for some N, for all open neighborhoods $U \times V$ of (x, y), $(x_n, y_n) \in U \times V$ for all $n \geq N$. Thus, if x_n converges to x, and y_n converges to y, (x_n, y_n) converges to (x, y). □

Proof. $\leftarrow$

Let (x_n, y_n) converge to (x, y) in $X \times Y$. Let π_1, π_2 be the projection maps $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$ respectively. By 193,160, and 161 these functions are continuous and open. By 198, if (x_n, y_n) converges, $\pi_1(x_n, y_n) = x_n$ and $\pi_2(x_n, y_n) = y_n$ converge. Thus, if (x_n, y_n) converges to (x, y) in $X \times Y$, x_n converges to x and y_n converges to y. Essentially, every open neighborhood of (x, y) is an open neighborhood of x and an open neighborhood of y, so if (x_n, y_n) converges to (x, y), x_n converges to x and y_n converges to y. □

Problem 2

Suppose X is a connected topological space and Y is a space with the discrete topology. Let $f : X \rightarrow Y$ be a continuous map. Describe f.

Proof. If X is a connected space, then the only clopen sets are $\varnothing$ and X (Theorem 255). Since Y is a space with the discrete topology, Y is disconnected and since every set is clopen.

Since f is a continuous function and X is connected, by Theorem 263, the image of f must be connected. Thus, in order to preserve connectedness, as well as maintain continuity, f must map to either Y, V or $\varnothing$, where $V \neq Y, \varnothing$ is a open set in Y. More plainly, if U is a open set in X,

$$f(U) = \begin{cases} Y & U = X \\ V & U \neq X, \varnothing \\ \varnothing & U = \varnothing \end{cases}$$

□

Problem 3

It is not true in general that a subspace of a normal space is normal, although counterexamples are non-trivial. However, it does work for closed subspaces: Let X be a normal space, and A be a closed subspace of X. Show that A is normal with respect to the subspace topology.

Proof. Let A be a closed subspace of X. Let B, C be disjoint closed sets in A. By Theorem 147, if B, C are closed sets in A, we can write $B = F_1 \cap A$ and $C = F_2 \cap A$, where F_1, F_2 are closed sets in X. Since the intersection of closed sets is closed, and A is closed, then B, C are closed sets in X. Thus, because X is normal, there exists a pair of disjoint U, V such that $B \subseteq U$ and $C \subseteq V$ for every pair B, C. Thus, we have the disjoint open sets on the subspace topology $U \cap A$ and $V \cap A$ for every disjoint closed set B, C in the closed subspace of A with the subspace topology. Thus if A is a closed subspace with the subspace topology of the normal space X, A is normal.

□

Problem 4

Let X be a topological space. If $f : X \to Y$ is continuous and X is path-connected, then $f(X)$ is path connected.

Proof. Because X is a path connected space, for every $x_1, x_2 \in X$, x_1 and x_2 are path connected via a continuous function. We know a composite function of two continuous functions is also continuous. Let α be an arbitrary path from x_1 to x_2. Then the function, $g = f \circ \alpha$ is also continuous. Now, we have $\alpha(0) = x_1$ and $\alpha(1) = x_2$, which implies $f(\alpha(0)) = f(x_1)$ and $f(\alpha(1)) = f(x_2)$ or more clearly, $g(0) = f(x_1)$ and $g(1) = f(x_2)$, which is a path from $f(x_1)$ to $f(x_2)$. Thus, we now have a continuous function, $g : [0, 1] \to Y$ that defines a path for every $f(x_1), f(x_2) \in f(X)$. Thus, $f(X)$ is path connected. □

Problem 5

Let $f, g : X \to Y$ be two continuous maps, and suppose Y is Hausdorff. Define $A = \{x \in X \mid f(x) = g(x)\}$. Show that A is closed in X.

Proof. Assume $x \notin A$, then $f(x) \neq g(x)$. Because Y is Hausdorff, we know there exists some U, V such that $f(x) \in U$ and $g(x) \in V$, and $U \cap V = \varnothing$. Because both f, g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open in X. Define $W = f^{-1}(U) \cap g^{-1}(V)$. W is an open set since $f^{-1}(U) \cap g^{-1}(V)$ are both open, and this is a finite intersection. To prove it is disjoint from A, let $x \in W, A$. If $x \in W$, then $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$, which implies $f(x) \in U$ and $g(x) \in V$. However, since $x \in A$, $f(x) = g(x)$, which would mean $U \cap V \neq \varnothing$, which is a contradiction. Thus, if $x \in W$ then $x \notin A$, implying for every $x \notin A$, there exists an open set W such that $x \in W$, and $W \cap A = \varnothing$. We can then write A^c as the arbitrary union of all open W_x containing $x \notin A$, as such $A^c = \bigcup\limits_{x \notin A} W_x$. Thus, since A^c is open, A is closed. □

Problem 6

Define an equivalence relation on $\mathbb{R}^2$ by $(a, b) \sim (x, y)$ if and only if (a, b) and (x, y) lie on the same circle centered at the origin.

(a)

The space $\mathbb{R}^2/\sim$ is homemorphic to $\mathbb{R}_+$.

(b)

Proof. We have $(a, b) \sim (x, y)$ if and only if they lie on the same circle centered at the origin so, $(a, b) \sim (x, y) \Leftrightarrow \sqrt{a^2 + b^2} = \sqrt{x^2 + y^2}$
Define the map $f : \mathbb{R}/\sim \to \mathbb{R}_+$ as $f([(x, y)]) = \sqrt{x^2 + y^2}$, where $[(x, y)]$ is an equivalence class element in $\mathbb{R}^2/\sim$. Thus we have:

$$\begin{aligned} f([(x, y)]) &= f([(a, b)]) \\ \Leftrightarrow \sqrt{x^2 + y^2} &= \sqrt{a^2 + b^2} \\ \Leftrightarrow (x, y) &\sim (a, b) \end{aligned}$$

Bottom to top proves well defined. Top to bottom proves 1-1. An open element U, in $\mathbb{R}^2/\sim$ is $\{[(0, y)] \mid a < y < b\}$ which maps to (a, b) in $\mathbb{R}_+$ which is also open since it is guaranteed to be positive (by calculus). This is surjective since x, y can be anything in $\mathbb{R}$, by calculus the domain is continuous and $[0, \infty)$, which is exactly $\mathbb{R}_+$.

Likewise, given $r \in \mathbb{R}_+$ it can be see the inverse $f^{-1}(r) = \{[(a, b)] \mid a^2 + b^2 = r^2\}$ is also true, as an open interval $(a, b) \in \mathbb{R}_+$ implies the open interval of the equivalence classes $\{[(0, y)] \mid a < y < b\}$ which is an open set in $\mathbb{R}/\sim$. This is surjective, since any open interval of equivalence classes can be made, and it covers the whole space.

□

Problem 7

Let X be a topological space. Then prove that X is T_3 (regular + T_1) if and only if for every point $x \in X$ and open neighborhood U of x, there exists a neighborhood V of x such that $\bar{V} \subseteq U$.

Proof. $\rightarrow$

Assume X is T_3. Thus, X is regular and T_1. Because X is T_3 by 245, X is Hausdorff. So for every pair of points there exists disjoint open sets U, V such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$. This implies that there exists disjoint closed sets, U^c, V^c, such that $x \in V^c$, $y \notin V^c$, and $y \in U^c$, $x \notin U^c$ and $U \subseteq V^c$ and $V \subseteq U^c$ for every x, y. Because X is regular, (WLOG) since V^c is closed and $y \notin V^c$, there exists disjoint open sets A, B such that $y \in A$, $V^c \subseteq B$. The important takeaway is there exists an open B such that $V^c \subseteq B$. Because $x \in U \subseteq V^c$, this implies V^c is an closed neighborhood of x, and B is an open neighborhood of x. Because V^c is closed, it is equivalent to the closure $\bar{V^c}$. Let this specific $V^c = G$. Thus, for every point $x \in X$ and open neighborhood B of x, there exists a neighborhood G of x such that $\bar{G} \subseteq B$

Proof. $\leftarrow$ Assume X is T_1. Let F be a closed set in X, such that $x \notin X$. Then F^c is an open neighborhood of x. Because F^c is an open neighborhood of x, there exists a neighborhood of V of x such that $\overline{V} \subseteq F^c$. Thus, the set $\overline{V}^c$ is an open set such that $F \subseteq \overline{V}^c$. By definition of closure, $V \subseteq \overline{V}$. Because V is a neighborhood, there exists an open set U with $x \in U$ and $U \subseteq V$. We know that $\overline{V} \cap \overline{V}^c = \varnothing$, and because $U \subseteq V \subseteq \overline{V}$, $U \cap \overline{V}^c = \varnothing$. Thus, for every $x \in X$ and closed set F in X such that $x \notin F$, there exists disjoint open sets $\overline{V}^c$ and U such that $F \subseteq \overline{V}^c$ and $x \in U$. Thus X is regular, and thus X is T_1.

□

□

Problem 8

(a)

Consider one mathematical idea from this course that you found beautiful, and explain why it is beautiful to you.

I find the idea of being able to map spaces to one another by warping and squishing really cool and amazing. It makes for some really amazing mind-melting ideas, like the klein-bottle and the torus. It gives a way to create understandable notions for spaces that would otherwise be very difficult to understand. For example, the torus I find beautiful because of how it takes a seemingly 2D square, and makes it a 3D surface, through an equivalence relation. Building upon that further, I find the creativity that topology requires quite intriguing, as it forces you, quite literally, to warp your mind space to try and understand the spaces that get created. I still absolutely am in love with configuration spaces. I think it is so cool to explain these crazy spaces in understandable ways, and have so much practical use, in physics and engineering alike!

(b)

What is the most important thing you have learned from this course? Why? How will this benefit you moving forward?

Concept wise? In general I feel that I've gotten much better with my proofs. I feel like I can formulate an argument and execute it relatively well. However, probably the most important thing I've learned is how difficult a class can be. I learned that I am capable of getting large quantities of intense, difficult work done. Topology forced me to consistently finish all of my other work efficiently so that I would have time to devote to the class. I learned some things just can't be forced, like understanding difficult concepts like this. You have to sit on it, and let it sink in. If you work at something long enough, eventually it will come to you. I'm used to things coming to me quickly, and topology (pure math) is definitely an exception. I have to put my very soul into the work I do in order to get it to the quality I want. Often I found myself during a hard workout, or a hard assignment in another class saying that "this is nothing compared to topology". I consider this class one of the most difficult trials I've had to endure thus far class wise, and I'm better for it. I can confidently say, moving forward, I will fear no class. If I make it through this, I sure as heck can make it through any other class. Moving forward, I'll be prepared for tougher things to come, because I learned how to cope with difficulty through this class. I'm excited for the future, and can't wait to take it head on. I'm ready.

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