PDE {Problem}

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1 Problem 1

Theorem 1. Let $u(x,t)$ be a continuous function on $(x,t) = [0,L] \times [0,T] = \Omega$ that satisfies the conditions below in the interior of Ω. Then $u(x,t)$ attains its maximum and minimum on either $x = 0, x = L,$ or $t = 0$.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } (x,t) \in (0,L) \times (0,T) \tag{1}$$
$$u(x,0) = f(x) \tag{2}$$
$$u(0,t) = g(t) \tag{3}$$
$$u(L,t) = h(t) \tag{4}$$

Proof. Assume that u does not have its max on the set $B = \Big\{(x,t)|\ x = 0, x = L, t = 0\Big\}$.
Then the max of u must occur at some point (x_0, t_0) where $0 < x_0 < L$ and $0 < t_0 \leq T$
$\Rightarrow u(x,t) \leq M \quad \forall (x,t) \in \Omega$.
Now consider u constrained to B.
On B, $u(x,t) \leq M - \epsilon$ for some $\epsilon > 0$
$\Rightarrow$ Max of $u(x,t)$ on B is equal to $(M - \epsilon)$
Define:

$$\mu(x,t) = u(x,t) + \frac{\epsilon}{2L}(x, x_0)^2 \tag{5}$$
$$\frac{\partial \mu}{\partial t} = \frac{\partial u}{\partial t} \tag{6}$$
$$\frac{\partial^2 \mu}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\epsilon}{L^2} \tag{7}$$

$$\text{On B: } \mu(x,t) = u(x,t) + \frac{\epsilon}{2L^2}(x - x_0)^2 \leq M - \epsilon + \frac{\epsilon}{2} \tag{8}$$

Rewriting this, we get:

$$\mu(x,t) = u(x,t) + \frac{\epsilon}{2L^2}(x - x_0)^2 \leq M - \frac{\epsilon}{2} \tag{9}$$

And if we plug in x_0, t_0 we get:

$$\mu(x_0, t_0) = u(x_0, t_0) = M \tag{10}$$

Thus the maximum of μ also does not occur on B.
Given $\forall (x, t) \in \Omega$

$$\frac{\partial \mu}{\partial t} - k\frac{\partial^2 \mu}{\partial x^2} \tag{11}$$

Can be rewritten by substituting $\dfrac{\partial \mu}{\partial t}$ and $\dfrac{\partial^2 \mu}{\partial x^2}$ with equations (6) and (7) respectively.
Subbing in gives us:

$$\frac{\partial u}{\partial t} - k\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\epsilon}{L^2}\Big) \tag{12}$$

Distributing, we get:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} - \frac{k\epsilon}{L^2} \tag{13}$$

We are given that:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{14}$$

Thus:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \tag{15}$$

Substituting this back in to equation (13), we get:

$$-\frac{k\epsilon}{L^2} \tag{16}$$

Since k, ϵ, and L are all strictly positive quantities that cannot equal zero, we can write:

$$-\frac{k\epsilon}{L^2} < 0 \tag{17}$$

Let (x_1, t_1) be the point **NOT** on B where the Max of μ occurs.
Now, plugging in (x_1, t_1)

$$\frac{\partial u}{\partial t}(x_1, t_1) - k\frac{\partial^2 u}{\partial x^2}(x_1, t_1) \tag{18}$$

Because $\dfrac{\partial u}{\partial t}(x_1, t_1) = 0$ (if $t_1 < T$) or positive (if $t_1 = T$) because (x_1, t_1) is the maximum and $k\dfrac{\partial^2 u}{\partial x^2}(x_1, t_1)$ will always be negative since (x_1, t_1) is the maximum, we can write:

$$\frac{\partial u}{\partial t}(x_1, t_1) - k\frac{\partial^2 u}{\partial x^2}(x_1, t_1) \geq 0 \tag{19}$$

But we proved in equation (17) that this same expression must be strictly less than 0. Thus, we have a contradiction, meaning the maximum **MUST** occur on B! $\square$

2 Problem 2

Prove that if the boundary value problem has a continuous solution, then it must be unique.

Proof. Assume that $u_1(x,t)$ and $u_2(x,t)$ are solutions to this problem.
Let $u_3(x,t) = u_1(x,t) - u_2(x,t)$, then:

$$u_3(x,0) = 0 \tag{20}$$

$$u_3(0,t) = 0 \tag{21}$$

$$u_3(L,t) = 0 \tag{22}$$

$$\frac{\partial u_3}{\partial t} = \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} \tag{23}$$

$$\text{Which can be rewritten as } \frac{\partial u_3}{\partial t} = k\Big(\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2}\Big) \tag{24}$$

$$\text{Which can be further rewritten as } \frac{\partial u_3}{\partial t} = k\Big(\frac{\partial^2 u_3}{\partial x^2}\Big) \tag{25}$$

The Max/Min theorem states that the maximum and the minimum must exist in B. Since the entirety of the region $B = 0$, then everything must be 0 since both the maximum and the minimum equal 0, thus $u_3(x,t) = 0$. Thus, $u_1 - u_2 = 0$, therefore $u_1 = u_2$.

$\square$

3 Problem 3

Prove the corollary to the max/min theorem.

Proof. Let $u_3(x,t) = u_1(x,t) - u_2(x,t)$, then:

$$u_3(x,0) = f_1(x) - f_2(x) \tag{26}$$
$$u_3(0,t) = g_1(x) - g_2(x) \tag{27}$$
$$u_3(L,t) = h_1(x) - h_2(x) \tag{28}$$
$$\frac{\partial u_3}{\partial t} = \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} \tag{29}$$
$$\text{Which can be rewritten as } \frac{\partial u_3}{\partial t} = k\Big(\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2}\Big) \tag{30}$$
$$\text{Which can be further rewritten as } \frac{\partial u_3}{\partial t} = k\Big(\frac{\partial^2 u_3}{\partial x^2}\Big) \tag{31}$$

The Max/Min Theorem states that the maximum and minimum exist on B. If the maximum and minimum exist on B, given:

$$\text{Max value of } |f_1(x) - f_2(x)| \leq \epsilon \tag{32}$$
$$\text{Max value of } |g_1(x) - g_2(x)| \leq \epsilon \tag{33}$$
$$\text{Max value of } |h_1(x) - h_2(x)| \leq \epsilon \tag{34}$$

Thus, since the max/min of u_3 exists on B, and given the above conditions, we can write $|u_3| \leq \epsilon$ which can be rewritten as $|u_1 - u_2| \leq \epsilon$
This corollary indicates that the heat equation has continuous dependence of the solution on the initial data. What this means is that if the initial starting conditions change by a small amount, the solution won't change by more than that small amount. $\square$