Rubik's Cube Group

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1 Introduction

1.1 History

The Rubik's Cube is a puzzle invented in 1974 by Hungarian sculptor and architecture professor Ernő Rubik. Originally the Rubik's Cube was referred to as the Magic Cube, and the puzzle was licensed by Rubik to be sold by Ideal Toy Corp in 1980, and won the German 'Game of the Year' award for Best Puzzle that same year. As of January 2009, it the world's top-selling puzzle game, with over 350 million cubes sold worldwide. Each of the six faces Rubik's Cube are covered by nine stickers, each of one of six solid colors: white, red, blue, orange, green, and yellow. In current models, white is opposite yellow, blue is opposite green, and orange is opposite red, and the red, white and blue are arranged in that order in a clockwise arrangement. An internal pivot mechanism enables each face to turn independently, allowing one to scramble the cube. For the puzzle to be solved, each face must be returned to have only one solid color. Similar puzzles have now been made with various numbers of sides, dimensions, and stickers. The Rubik's Cube reached its height of mainstream popularity in the 1980s, however it is still widely known and used. Many 'speedcubers' continue to practice the cube and similar puzzles by competing for the fastest times in various categories. Since 2003, the World Cube Association, (the Rubik's Cube's international governing body) has organized competitions worldwide and recognizes world records.

1.2 Mechanics

A standard Rubik's Cube measures $5.7 \times 5.7 \times 5.7$ centimeters. The cube consists of twenty-six unique miniature cubes, referred to as "cubies" or "cubelets". Each of these includes a concealed inward extension that interlocks with the other cubes while permitting them to move to different locations. However, the center cube of each of the six faces is merely a single square facade; all six centers do not move as they are directly affixed to the core mechanism. These provide structure for the other pieces to fit into and rotate around. So there are technically twenty-one pieces: a single core piece consisting of three intersecting axes holding the six centre squares in place but letting them rotate, and twenty smaller plastic pieces which fit into it to form the assembled puzzle. Each of the six center pieces pivots on a screw held by the single center piece, the '3D cross'. A spring between each screw head and its corresponding piece tensions the piece inward, so that collectively, the whole assembly remains compact, but still remains easily manipulated. The screw can be tightened or loosened to change the "feel" of the Cube. In newer cubes, they are built with rivets instead of screws and cannot be adjusted. The Cube can be taken apart without much difficulty, typically by rotating the top layer by 45 degrees and then prying one of its edge cubes away from the other two layers. *Side note, if you ever get so infuriated with the puzzle, you can *solve* the Cube by taking it apart and reassembling it 'solved'.

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2.1 The Cube

There are six central pieces which show one coloured face, twelve edge pieces which show two coloured faces, and eight corner pieces which show three colored faces. Each piece shows a unique color combination, but not all combinations are present (for example, if red and orange are on opposite sides of the solved Cube, there is no edge piece with both red and orange sides). The location of these cubes relative to one another can be altered by twisting an outer third of the Cube 90, 180 or 270 degrees, but the location of the colored sides relative to one another in the completed state of the puzzle cannot be altered: it is fixed by the relative positions of the center squares.

2.2 The Permutations

There are $8! = (40{,}320)$ ways to arrange the corner cubes. Each corner has three possible orientations, however only seven can be oriented independently, as the orientation of the eighth corner depends on the other seven, giving 3^7 (2,187) possibilities. There are $\frac{12!}{2}$ (239,500,800) ways to arrange the edges, half of 12! because edges must be in an even permutation exactly when the corners are. The first eleven edges can be independently flipped, but similar to the corners, the twelfth's orientation depends on the preceding eleven, giving 2^{11} (2,048) possibilities. Thus, the total permutations of the Rubik's cube is $8! * 3^7 * \frac{12!}{2} * 2^{11} = 43,252,003,274,489,856,000$. <u>That's a lot of permutations.</u>

2.3 The Rubik's Cube Group

Let us finally define the generators of the group to be $\{F, B, U, D, L, R\}$ Where

F	turns the front clockwise	F'	turns the front counterclockwise
B	turns the back clockwise	B'	turns the back counterclockwise
U	turns the top clockwise	U'	turns the top counterclockwise
D	turns the bottom clockwise	D'	turns the bottom counterclockwise
L	turns the left face clockwise	L'	turns the left face counterclockwise
R	turns the right face clockwise	R'	turns the right face counterclockwise

This notation of the generators is referred to as Singmaster notation, named after the mathematician David Singmaster.

The cube itself consists of 9 faces, or 54 facets. Since 6 of these facets are the center pieces, which cannot move, there are essentially 48 facets that can be permuted, since we fixed the orientation of the entire cube in order to define the generators. Thus, numbering the facets 1 through 48, the Rubik's Cube group, G, is (By Caley's Theorem) then defined to be the subgroup of S_{48} generated by the 6 face rotations, $\{F, B, U, D, L, R\}$. G is not abelian, since $FR \neq RF$.

2.4 Subgroups of The Rubik's Cube Group

Let's consider two types of subgroups of G: First the subgroup C_o of cube orientations, which are the moves that leave the *position* of every block fixed, but can change the *orientations* of blocks. This a normal subgroup of G. It can be represented as the normal closure of moves that flip edges or twist corners.

The other subgroup C_p is cube permutations, the moves which change the *positions* of the blocks, but leave the *orientation* fixed. This subgroup is created by two disjoint normal subgroups, the subgroup of even permutations on the corners (A_8) (these swap two corners) and the subgroup of even permutations on the edges (A_{12}) (these swap two edges).

Together, these subgroups' semi-direct product form the entire group, thus $G = C_o \rtimes C_p$.

2.5 Algorithms

Many algorithms used to solve the cube only transform a small part of the cube without touching other parts that have already been solved, so that they can be applied repeatedly to different parts of the cube until it is solved. For example, there are well-known algorithms that cycle three corners without changing the rest of the puzzle.

Some algorithms do have the desired effect on the cube (like swapping two corners or edges) but may also have the side-effect of changing other parts of the cube (such as permuting some edges). These algorithms are often far simpler than the ones without side-effects and are used early on when most of the puzzle has yet to been solved and the side-effects are not important.

David Singmaster first published his solution in the book Notes on Rubik's "Magic Cube" in 1981. Singmaster's solution involved solving the Cube layer by layer, in which one layer (denoted the top) is solved first, then the middle layer, and then the bottom layer.

There are many more commonly known algorithms, such as Thistlethwaite's, Kociemba's, and Korf's algorithms

2.6 God's Number

In July 2010, a team of researchers gave the final computer-assisted proof that showed the so-called 'God's number', or minimax value, to be 20. This is the optimal solution, since there exist some starting positions which require a minimum of 20 moves to solve. It has been shown that an $n \times n \times n$ Rubik's Cube can be solved optimally in $\Theta(\frac{n^2}{log(n)})$ moves.

3 References

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https://en.wikipedia.org/wiki/Rubik%27s_Cube_group