PDE {Problem}

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Problem 1

a

Using the change of variables where $u = ve^{\alpha x - \beta t}$ we have:

$$\begin{aligned} u_t &= v_t e^{\alpha x - \beta t} - \beta v e^{\alpha x - \beta t} \\ u_x &= v_x e^{\alpha x - \beta t} + \alpha v e^{\alpha x - \beta t} \\ u_{xx} &= v_{xx} e^{\alpha x - \beta t} + \alpha v_x e^{\alpha x - \beta t} + \alpha v_x e^{\alpha x - \beta t} + \alpha^2 v e^{\alpha x - \beta t} \end{aligned}$$

Plugging these into the equation :

$$\frac{\partial u}{\partial t} = 0.05 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 2u$$

We get (canceling $e^{\alpha x - \beta t}$ in every term) :

$$\begin{aligned} v_t - \beta v &= 0.05(v_{xx} + 2\alpha v_x + \alpha^2 v) + (v_x + \alpha) + 2v \\ \Rightarrow v_t &= 0.05 v_{xx} + (.1\alpha + 1) v_x + (\beta + 0.05\alpha^2 + \alpha + 2) v \end{aligned}$$

If we just want $v_t = 0.05 v_{xx}$ then:

$$\begin{aligned} \alpha &= -10 \\ \beta &= 3 \end{aligned}$$

b

Using the initial condtions, we have:

$$\begin{aligned} u(x,0) &= v(x,0)e^{\alpha x} = f(x) \\ u(0,t) &= v(0,t)e^{-\beta t} = 0 \Rightarrow v(0,t) = 0 \\ u(1,t) &= v(1,t)e^{-\beta t} = 0 \Rightarrow v(1,t) = 0 \end{aligned}$$

Using the same old separation of variables, we get the x and t-components of the $v(x,t)$ function to be:

$$\sin(n\pi x)$$
$$e^{-k(n\pi)^2 t}$$

Thus we have:

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-k(n\pi)^2 t}$$

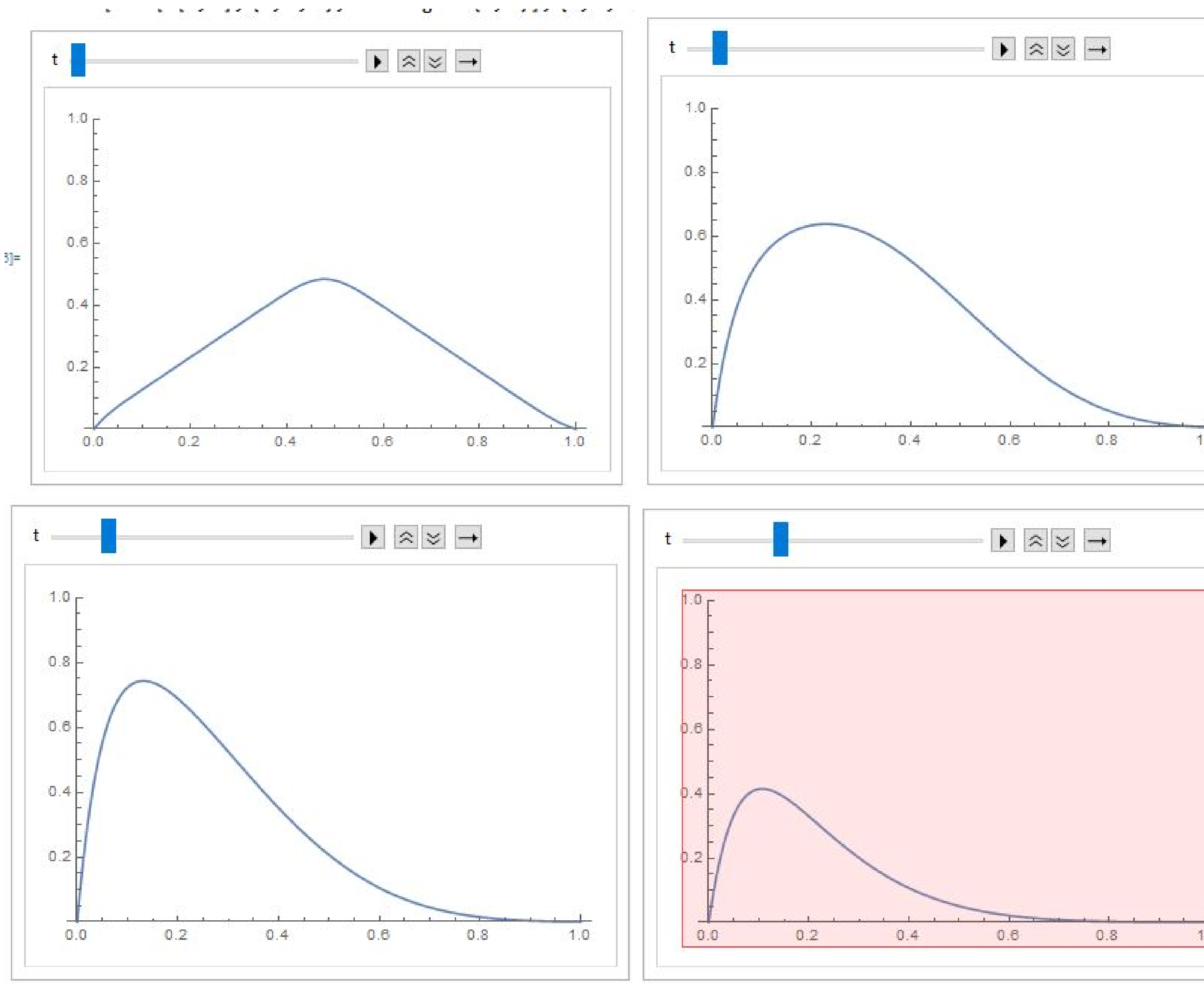
Where:

$$b_n = 2\int_0^1 f(x)e^{-\alpha x} sin(n\pi x)dx$$

Thus, we have

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-k(n\pi)^2 t} e^{\alpha x - \beta t}$$

c



Problem 2

a

We have:

$$\nabla \cdot (A \nabla u) = 0$$

Solving this out, we get:

$$\nabla \cdot \left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \right)$$

Which, when we simplify, we get:

$$au_{xx} + 2bu_{xy} + cu_{yy}$$

We know the PDE is ellpitic if:

$$4b^2 - 4ac < 0$$

Solving the eigenvalue matrix and setting it's determinant to 0, we get:

$$\lambda^2 - (c+a)\lambda - (b^2 - ac) = 0$$

Using the quadradic formula, we get:

$$\lambda = \frac{(c+a) \pm \sqrt{(c+a)^2 - 4(b^2 - ac)}}{2}$$

Using the fun fact that:

$$\Pi\lambda_i = det(A)$$

If the eigen values are both positive or both negative, the determinant will be positive. That means that

$$ac - b^2 > 0$$

Thus we then would know that

$$ac > b^2$$

Making the equation:

$$4b^2 - 4ac$$

Always less than 0, thus always ellptic.

b

Laplaces equation states that:

$$u_{xx} - u_{yy} = 0$$

Using:

$$au_{xx} + 2bu_{xy} + cu_{yy}$$

We can conclude that a matrix that would satisfy Laplace's equation would be:

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

c

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

Would satisfy anisotropic diffusion.

d

$$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$$

Would satisfy the basic wave equation.

1 Problem 3

See attached paper