PDE {Problem}

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1 Problem 1

The solution to the first order PDE is

$$f(x - ct)e^{kt}$$

The graph has convection to the right with a speed of c, and a growing/decaying reaction at a rate of k.

2 Problem 2

(a)

If

$$\frac{dx}{dt} = 1$$

Then

$$\frac{du}{dt} = xu$$

Solving for u, we get

$$\frac{du}{u} = xdt$$

Giving us:

$$u = f(a)e^{xt}$$

Solving for f(a) we have:

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ 1dx &= 1dt \\ x &= t + C \\ x(0) &= a \\ x &= t + a \\ f(a) &= f(x - t) \end{aligned}$$

Thus,

$$u(x, t) = f(x - t)e^{xt}$$

(b)

If

$$\frac{dx}{dt} = \sin(t), \text{ Then } \frac{du}{dt} = 0$$

Solving for $x(t)$ we get:

$$x = -\cos(t) + a$$

Given u is constant with time, $u(a, 0) = u(x, t)$ Thus we have

$$u(x, t) = f(a) = f(x + cos(t))$$

(c)

If

$$\frac{dx}{dt} = \frac{x}{t+1}, \text{ Then } \frac{du}{dt} = 0$$

Since u does not change with time, u(a, 0) = u (a, t). Solving for $x(t)$

$$\begin{aligned}
\frac{1}{x}dx &= \frac{1}{t+1}dt \\
lnx &= ln(t+1) + C \\
x &= e^c(t+1) \\
x(0) &= a \\
x &= a(t+1)
\end{aligned}$$

Thus, we have:

$$u(x, t) = f(\frac{x}{1+t})$$

(d)

If

$$\frac{dx}{dt} = -x, \text{ Then } \frac{du}{dt} = u^2$$

Solving for x

$$\begin{aligned}
\frac{1}{x}dx &= -1dt \\
ln(x) &= -t + C \\
x &= e^{-t}e^c \\
x &= ae^{-t} \\
a &= \frac{x}{e^{-t}}
\end{aligned}$$

Solving for u

$$\begin{aligned}
\frac{du}{dt} &= u^2 \\
\frac{1}{u^2}du &= 1dt \\
-\frac{1}{u} &= t + C \\
u &= \frac{1}{-t + C} \\
u(a,0) = \frac{1}{C} &= f(a)
\end{aligned}$$

Given that $f(a) = \frac{\sin(a)}{a}$. We can solve for C:

$$C = \frac{x}{e^{-t}\sin(\frac{x}{e^{-t}})}$$

Thus we have

$$u(x,t) = \frac{1}{-t + \frac{x}{e^{-t}\sin(\frac{x}{e^{-t}})}}$$

Characteristic Curves

(a) Convecting to the right with a speed 1, and the reaction is growing with x
(b) The curve is convecting left to right on a period (wiggling). There is no reaction. (constant)
(c) Initially convects to the left or right based on negative or positive x. It has no reaction. (constant)
(d) Convecting to the middle with a speed proportional to x. Grows or decays based on u. When u is positive, the curve will be growing, when u is negative, the curve is decaying towards 0. When u is 0, it is constant.

3 Problem 3

If

$$\frac{d\rho}{dx} = (1 - 2\rho), \text{ Then } \frac{d\rho}{dt} = 0$$

Solving for x(t), given $x(0) = a$, we get

$$\begin{aligned}
x &= (1 - 2\rho)t + a \\
a &= x - (1 - 2\rho)t
\end{aligned}$$

Since ρ is constant with time, we know $\rho(x,t) = \rho(a,0)$. So, given this, we get

$$\rho = e^{-a^2}$$

Substituting in, and given that a is a constant:

$$x = a + (1 - 2e^{-a^2})t$$