Topology Problem Set

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Problem 1

Prove $f(f^{-1}(V)) \subseteq V$

Proof. Let $y \in f(f^{-1}(V))$. If this is true, then there must exist some x, such that $f(x) = y$, so then by definition, $x \in f^-1(V)$, thus $f(x) \in V$, so $y \in V$. Thus, $f(f^{-1}(V)) \subseteq V$. □

Prove $f(f^{-1}(V)) = V$ if and only if f is surjective.

Proof. Let $y \in V$. Assume f is surjective. Since f is surjective, there exists some $f(x)$ such that $f(x) = y$. Thus $x \in f^{-1}(y) = f^{-1}(V)$. We know that $f(x) = y$, so $y \in f(x) \to y \in f(f^{-1}(y))$. Thus, $V \subseteq f(f^{-1}(V))$. The other direction is proved above. Thus, if f is surjective, then $f(f^{-1}(V)) = V$.

Assume $f(f^{-1}(V)) = V$. Assume f is not surjective. Then, there does not exist, for every $v \in V$, some $f(x)$ such that $f(x) = v$. Then, there exists some $y \in V$ such that for every $x\ inX$, $f(x) \neq y$. Then, $f^{-1}(y) = \varnothing$, and $f(\varnothing) = \varnothing$. Thus, $f(f^{-1}(y)) = \varnothing$, which is a direct contradiction $f(f^{-1}(v)) = V$. Thus, f must be surjective. So, if $f(f^{-1}(V)) = V$, then f is surjective. □

Problem 2

$1 \to 2$

Proof. Assume X is countable. If X is countably infinite, then there exists a bijection $\mathbb{N} \to X$, which proves by definition there exists a surjection from $\mathbb{N} \to X$. If X is finite of cardinality n, define f as $\mathbb{N} \to \{1, 2, ..., n\}$ which maps $\{1 \to 1, 2 \to 2, ..., x \to n \mid x \in \mathbb{N}, x \geq n\}$ then $\{1, 2, ..., n\} \to X$ which maps $\{1 \to x_1, 2 \to x_2, ..., n \to x_n\}$. Since, there exists some $n \in \mathbb{N}$ such that for every $x \in X$, such that $f(n) = x$, f is surjective. Thus, if X is countable, there exists a surjection $\mathbb{N} \to X$.

□

$2 \to 3$

Proof. Assume there exists a surjective function $f\colon \mathbb{N} \to X$. Then, for every $x \in X$, there exists some $n \in \mathbb{N}$ such that $f(n) = x$. Let $x \in X$, define $g(x) = \min\{i \mid f(i) = x_i\}$. Then, if $g(x) = g(y) = i$ then $f(i) = x = y$. Thus, $g(x)$ is a injection from $X \to \mathbb{N}$, which proves, if there exists a surjective function $\mathbb{N} \to X$, there exists an injective function $X \to \mathbb{N}$. $\square$

$3 \to 1$

Proof. Assume there exists a injective function g: $X \to \mathbb{N}$. If X is finite, by definition it is countable. If X is infinite, define x_1 as the smallest element, recursively define all subsequent x_i as the smallest element in $\{X \backslash \{x_1, ..., x_{i-1}\}\}$. Now, define $f\ X \to \mathbb{N}$ such that $\{x_1, x_2, ..., x_i\}$ maps $x_1 \to 1, x_2 \to 2, ..., x_i \to i$. This is a surjection into $\mathbb{N}$, and we already have an injection (which is both ways) thus, we now have a bijection from $x \to \mathbb{N}$, which by definition proves X is countable. Thus, if there a injective function, then X is countable. $\square$

Problem 3

Prove A countable union of countable sets is countable.

Proof. Let X_i be countable. Let $i, j \in \mathbb{N}$. Define the mapping $f : \bigcup_{i \in \mathbb{N}} X_i \to \mathbb{N} \times \mathbb{N} \mid (i, j)$ where i is the ith set, and j is the jth component of the ith set. Define X_{i1} as the smallest element of X_i, and recursively define every subsequent element X_{ij} as the smallest element of $X_i \backslash \{X_{i1}, ..., X_{i(j-1)}\}$ Since both $i, j \in \mathbb{N}$, this mapping now puts the countable union of countable sets into terms of $\mathbb{N} \times \mathbb{N}$, and we know via exercise 44, that this is countable. Thus, the countable union of countable sets is countable. $\square$

Problem 4

Prove $\left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{j \in J} B_j\right) = \bigcup_{i,j \in I,J} A_i \times B_j$

Proof. Let $a \in A_i$ for some i and let $b \in B_j$ for some j, then $(a, b) \in \left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{j \in J} B_j\right)$ for some i, j. (a, b) is also an arbitrary element of $\bigcup_{i,j \in I,J} A_i \times B_j$ for some i, j. Thus, $\bigcup_{i \in I} A_i \times \bigcup_{j \in J} B_j \subseteq \bigcup_{i,j \in I,J} A_i \times B_j$.

Let $(a, b) \in \bigcup_{i,j \in I,J} A_i \times B_j$ for some i, j. $a \in A_i$ for some i and $b \in B_j$ for some j. Thus, $(a, b) \in \left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{j \in J} B_j\right)$. Thus $\bigcup_{i,j \in I,J} A_i \times B_j \subseteq \left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{j \in J} B_j\right)$. $\square$

Problem 5

Prove $|A| < |\mathcal{P}(A)|$

Proof. Let $f\ A \to \mathcal{P}(A) \mid a \in A, f(a) = \{a\}$. Thus, there exists for every $a \in A$, an element $\{a\} \in \mathcal{P}(A)$. Thus, $|A| \leq |\mathcal{P}(A)|$. Assume there exists a bijection from $A \to \mathcal{P}(A)$. By definition, the set $A \in \mathcal{P}(A)$, thus A must exist in the image of g. However, because there exists a set element $\{a\} \in \mathcal{P}(A)$ for every $a \in A$, in order for the bijection to exist, there must exist a unique mapping for all $\{a\}$ and A. However, this creates a contradiction as A cannot exist in the image if all $\{a\}$ exist. Thus, $|A| < \mathcal{P}(A)$. $\square$

Problem 6

a

We are given that for any positive numbers a, b, there exists some $n \in \mathbb{N}$ such that $na > b$ Assume that $0 \leq x < y$. Let $b = 1$, and $a = y - x$. Substituting this into our equation, $b < an$ Thus, we have $1 < n(y - x)$ which we can rewrite as $\frac{1}{n} < y - x$.