Class Notes

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Hyperbolic Complex Functions

Derivation of the derivatives

Given the real counterparts, we expect the derivatives to mirror in the complex, however we will still confirm. Let us start with the definitions of sinh and cosh.

$$\sinh z = \frac{e^z - e^{-z}}{2} \tag{1}$$
$$\cosh z = \frac{e^z + e^{-z}}{2} \tag{2}$$

Starting from the definition, we have the derivative of sinh:

$$\frac{d}{dz}\sinh z \tag{3}$$
$$= \frac{d}{dz}\Big(\frac{e^z - e^{-z}}{2}\Big) \tag{4}$$
$$= \frac{e^z - (-e^{-z})}{2} \tag{5}$$
$$= \frac{e^z + e^{-z}}{2} \tag{6}$$
$$= \cosh z \tag{7}$$

Following the same path of logic, we have for the derivative of cosh:

$$\frac{d}{dz}\cosh z \tag{8}$$
$$= \frac{d}{dz}\Big(\frac{e^z + e^{-z}}{2}\Big) \tag{9}$$
$$= \frac{e^z + (-e^{-z})}{2} \tag{10}$$
$$= \frac{e^z - e^{-z}}{2} \tag{11}$$
$$= \sinh z \tag{12}$$

Derivation of an Identity

We will derive the identity $\cosh^2 z - \sinh^2 z = 1$ We begin from the definitions and work from there.

$$\begin{aligned}
&\cosh^2 z - \sinh^2 z \\
&= \Big(\frac{e^z + e^{-z}}{2}\Big)^2 - \Big(\frac{e^z - e^{-z}}{2}\Big)^2 \\
&= \frac{1}{4}\Big(e^{2z} + 2e^0 + e^{-2z}\Big) - \frac{1}{4}\Big(e^{2z} - 2e^0 + e^{-2z}\Big) \\
&= \frac{1}{4}\Big(e^{2z} - e^{2z} + e^{-2z} - e^{-2z} + 2e^0 + 2e^0\Big) \\
&= \frac{1}{4}\Big(4e^0\Big) \\
&= 1
\end{aligned}$$

Derivation of the Inverse Functions

To derive the inverses, $\sinh^{-1}$ and $\cosh^{-1}$, we begin with the following:

$$w = \sinh^{-1} z \tag{13}$$

$$z = \sinh w \tag{14}$$

Here, $'w'$ acts sort of as an intermediary to make our lives easier. Now, let's use the definition of sinh and go from there.

$$z = \frac{e^w - e^{-w}}{2} \tag{15}$$

$$2z = e^w - e^{-w} \tag{16}$$

$$0 = e^w - 2z - e^{-w} \tag{17}$$

$$\tag{18}$$

From here, we multiply by e^w to get it into a quadratic form. From there, we solve it using the complex quadratic equation.

$$0 = e^w\Big(e^w - 2z - e^{-w}\Big) \tag{19}$$

$$0 = \Big(e^w\Big)^2 - \Big(e^w\Big)2z - 1 \tag{20}$$

Here, the coefficients of our quadratic with respect to e^w are $a = 1, b = -2z$, and $c = -1$. Recall, the solution to the quadratic is given by:

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}$$

In this case, we plug in e^w, since that is what we are solving for, giving us:

$$e^w = \frac{2z + (4z^2 + 4)^{1/2}}{2} \tag{21}$$

$$e^w = \frac{2z + (4(z^2 + 1))^{1/2}}{2} \tag{22}$$

$$e^w = \frac{2z + 2(z^2 + 1)^{1/2}}{2} \tag{23}$$

$$e^w = z + (z^2 + 1)^{1/2} \tag{24}$$

From here, we just take the log, and solve for w in terms of z, which gives us the inverse:

$$w = \log[z + (z^2 + 1)^{1/2}] = \sinh^{-1} z \tag{25}$$

The derivation for $\cosh^{-1} z$ is very similar, barring more P.D.A, we have:

$$\cosh^{-1} z = \log[z + (z^2 - 1)^{1/2}] \tag{26}$$