Abstract Problem Set 1

Will Lillis

January 2019

1 Problem Set

1.1 Problem 1

Exercise 1. Given $(a*b)^2 = a^2 * b^2$ must the operation be commutative and associative?

Proof. Let $a, b \in A$

*	a	b
a	a	a
b	b	a

Note That $a * b = a \neq b = b * a$ ie. $*$ is not commutative

Given $(a*b)^2 = a^2 * b^2$
$(a*b)^2 = (a*b)*(a*b) = a*a = a$
$a^2 * b^2 = a*a*b*b = a*b*b = a*b = a$

The given equation $(a*b)^2 = a^2 * b^2$ is proven true with an operation that is not commutative, therefore, * need not be commutative and associative given $(a*b)^2 = a^2 * b^2$ to be true

$\square$

1.2 Problem 2

Exercise 2. Let $R_{\geq 0}$ be the set of all non-negative real numbers, and define * on $R_{\geq 0}$ by $a * b = |a - b|$
(a) Prove that $*$ is a binary operation on $R_{\geq 0}$.
(b) Is * commutative? Prove your answer.
(c) Is ($R_{\geq 0}$, *) a group? Prove your answer

Proof. (a)

Given $(a * b) = a * b = |a - b|$
Say $a, b, c \in \ R_{\geq 0}$

Case 1, $a < b$
$a * b = |a - b| = |-c| = c, \ c \in \ R_{\geq 0}$

Case 2, $a = b$
$a * b = |a - b| = |0| = 0, \ 0 \in \ R_{\geq 0}$

Case 3, $a > b$
$a * b = |a - b| = |c| = c, \ c \in \ R_{\geq 0}$

Thus $*$ is a binary operation on $R_{\geq 0}$ □

Proof. (b)

Say $a, b, c \in \ R_{\geq 0}$

Case 1, $a > b$
$a * b = |a - b| = |c| = c, \ c \in \ R_{\geq 0}$

$b * a = |b - a| = |-c| = c, \ c \in \ R_{\geq 0}$

Case 2, $a < b$
$a * b = |a - b| = |-c| = c, \ c \in \ R_{\geq 0}$

$b * a = |b - a| = |c| = c, \ c \in \ R_{\geq 0}$

Case 3, $a = b$
$a * b = |a - b| = |0| = 0, \ 0 \in \ R_{\geq 0}$

$b * a = |b - a| = |0| = 0, \ 0 \in \ R_{\geq 0}$

Thus $*$ is Commutative □

Proof. (c)

Given $(a * b) = a * b = |a - b|$

$a, b, c \in\ R_{\geq 0}$ To be associative $(a * b) * c = ||a - b| - c| = a * (b * c) = |a - |b - c||$
However, this is disproven, when $a = 9, b = 11, c = 15$, $|||9-11|-15| = 13 \neq 5 = |9-|11-15||$
The identity would be 0, as $0 * a = a * 0 = |a| = a$
Thus $(R_{\geq 0}, *)$ is not a group □

1.3 Problem 3

Exercise 3. Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}^+$ denote the sets of integers, rational numbers, and positive rational numbers, respectively. Define $*$ by $a * b = \frac{a}{b}$ For each of these three sets, determine whether or not * is a binary operation and justify your answer.

Proof. Given $a\ *\ b = \frac{a}{b}$ and $b = 0 \in \mathbb{Z}$ then $a * b$ is undefined, so $*$ is not a binary operation on $\mathbb{Z}$

Given $a * b = \frac{a}{b}$ and $b = 0 \in \mathbb{Q}$ then $a * b$ is undefined, so $*$ is not a binary operation on $\mathbb{Q}$

Given $a * b = \frac{a}{b}$ and any $\frac{a}{b}$ is considered rational $\frac{a}{b} \in \mathbb{Q}^+$ and $0 \notin \mathbb{Q}^+$ then $a * b$ is defined for all $a, b \in \mathbb{Q}^+$, so * is a binary operation on $\mathbb{Q}^+$ □

1.4 Problem 4

Exercise 4. (a) Subtraction is a binary operation on $\mathbb{Z}$. We know that ($\mathbb{Z}$, +) is a group. Is ($\mathbb{Z}$, -) also a group? Prove your answer.
(b) Define $*$ on $\mathbb{R}^+$, the set of positive real numbers, by a * b $=\sqrt{ab}$. Is ($\mathbb{R}^+$, *) a group? Prove your answer.

Proof. (a)

There exists no identity such that $a * a^{-1} = a^{-1} * a = identity$
Say the identity was e, then $a * e = a - e = a$, however, $e * a = e - a = -a \neq a$
Therefore there is no identity, and thus ($\mathbb{Z}$, -) is not a group.

□

Proof. (b)

$a, b, c \in \mathbb{R}^+$
$(a * b) * c = \sqrt{c\sqrt{ab}}$
$a * (b * c) = \sqrt{a\sqrt{bc}}$
$\sqrt{a\sqrt{bc}} \neq \sqrt{c\sqrt{ab}}$
$ex.\ a = 1, b = 2, c = 3,\ 6^{\frac{1}{4}} \neq 2^{\frac{1}{4}}3^{\frac{1}{2}}$
Thus ($\mathbb{R}^+$, *) is not a group

□

1.5 Problem 5

Exercise 5. Let A be a set and let $\mathcal{P}(A)$ denote the power set of A. We define the symmetric difference Δ on $\mathcal{P}(A)$ by $X\Delta Y = (X \cup Y)\backslash(X \cap Y) = (X\backslash Y) \cup (Y\backslash X)$, for any subsets X, Y of A. (You can use whichever definition feels more natural to you. The symmetric difference is all elements in X or Y but not in both.) Prove that $(\mathcal{P}(A), \Delta)$ is a group.

Proof. $\{a\}, \{b, c\} \in \mathcal{P}(A)$
$\{a\}\Delta\{b, c\} = \{a, b, c\} \in \mathcal{P}(A)$
Closed

$\{a\}, \{b, c\}, \{a, b, c\} \in \mathcal{P}(A)$
$\{a\}\Delta(\{b, c\}\Delta\{a, b, c\}) = \{a\}\Delta\{a\} = \{\}$
$(\{a\}\Delta\{b, c\})\Delta\{a, b, c\} = \{a, b, c\}\Delta\{a, b, c\} = \{\}$
Associativity Proven

Identity e is the empty set $\{\varnothing\} \in \mathcal{P}(A)$
as $e\Delta\{a\} = \{a\} = \{a\}\Delta e = \{a\}$

Inverse is itself, as $\{a\}\Delta\{a\} = \{\varnothing\}$
Thus $\mathcal{P}(A)$ is a group

□