Abstract Problem Set 3

Rippy

February 2019

1 Problems

1.1 Problem 1

If G is a group such that $H, K \leq G$, then $H \cap K \leq G$. Moreover, $H \cap K$ is the largest subgroup contained in both H and K.

Proof. In order for $H \cap K \leq G$, $H \cap K$ (i) must be nonempty, (ii) contain an inverse $a^{-1} \in H \cap K$ for every $a \in H \cap K$ and (iii) be closed under the binary operation of G.

If $H, K \leq G$ they must both contain the identity element e, so if nothing else exists in $H \cap K$, the trivial subgroup e is guaranteed to be within $H \cap K$. (Therefore the subgroup is nonempty, and contains the identity) Because $H, K \leq G$, for any element $a \in H, K$, there also exists some $a^{-1} \in H, K$, meaning that for every element $a \in H \cap K$ there exists some $a^{-1} \in H \cap K$. Because $H, K \leq G$, they are both closed under the binary operation of G, therefore, anything within $H \cap K$ will also be closed under the binary operation of G since anything that can be made with $a, b \in H, K$ will also be contained within $H \cap K$.
Because the subgroup $H \cap K$ contains every possible element of overlap between H, K, it must be the largest subgroup contained in both H and K, since any subgroup in both H and K must be within $H \cap K$ by definition. $\square$

1.2 Problem 2

(a) Provide an example of a group G and subgroup H and K such that $H \cup K$ is not a subgroup of G.

(b) Provide an example of a group G and proper, nontrivial subgroups H and K such that $H \cup K$ is a subgroup of G. Justify your answer.

Proof. (a)
Let V_4 be the group G, H be the subgroup consisting of the elements $\{e, h\}$ and K be the subgroup consisting of the elements $\{e, v\}$. Then $H \cup K$ would contain the elements $\{e, h, v\}$ however, since $hv \notin H \cup K$, $H \cup K$ is not closed, thus $H \cup K \not\leq G$

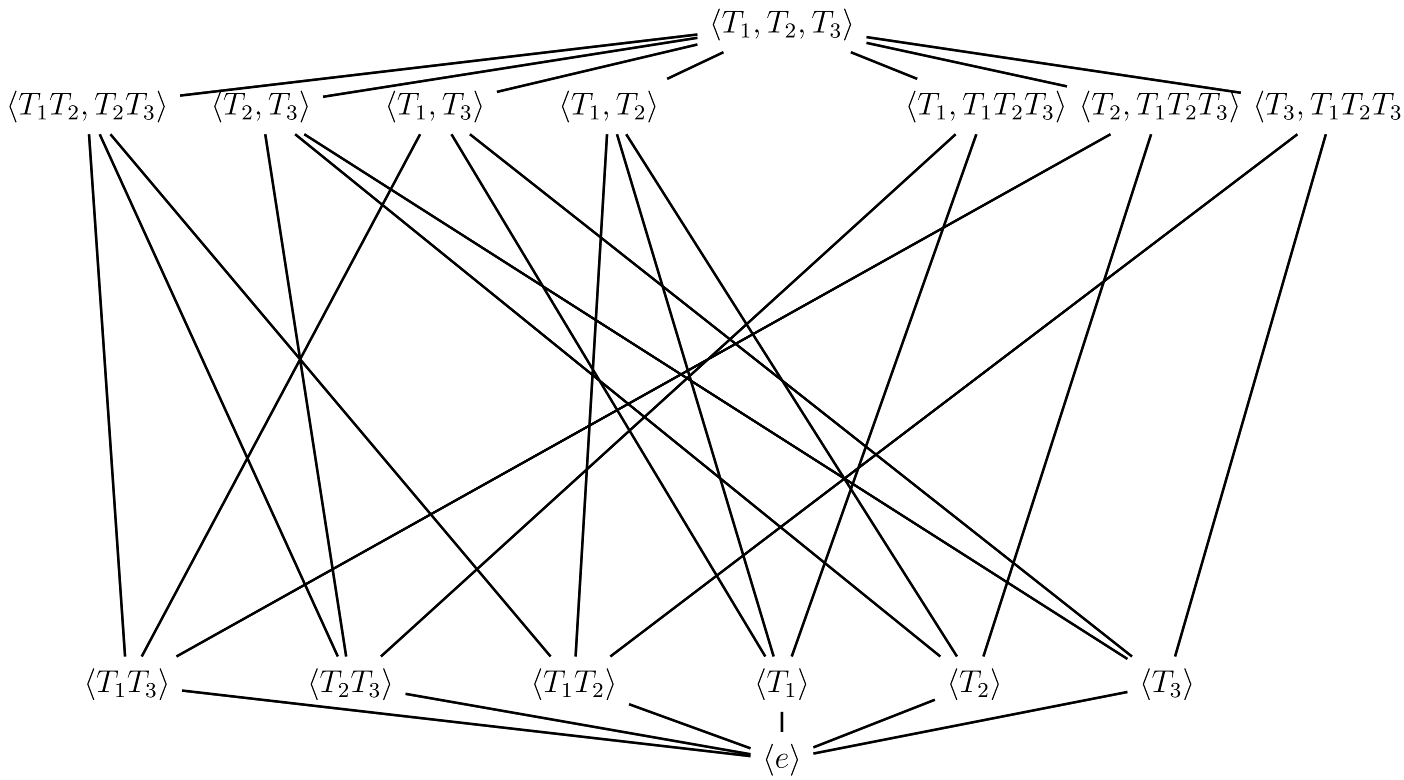
$\square$

Proof. (b) Let D_4 be the group G, H be the subgroup consisting of the elements $\{e, r, r^2, r^3\}$ and K be the subgroup consisting of the elements $\{e, r^2\}$ $H \cup K$ would consist of the elements $\{e, r, r^2, r^3\}$, becoming the subgroup H, and we know $H \leq G$

$\square$

1.3 Problem 3

Draw the subgroup lattices of the groups L_2 and L_3. Justify why this is the entire lattice for each.



Lattice for L_3

The 16 unique subgroups of L_3 are

$\{e, T_1, T_2, T_3, T_1T_2, T_2T_3, T_1T_3, T_1T_2T_3\} = \langle T_1, T_2, T_3\rangle$
$\{e, T_1, T_2, T_1T_2\} = \langle T_1, T_1T_2\rangle = \langle T_2, T_2T_1\rangle = \langle T_1, T_2\rangle$
$\{e, T_1, T_3, T_1T_3\} = \langle T_1, T_1T_3\rangle = \langle T_3, T_1T_3\rangle = \langle T_1, T_3\rangle$
$\{e, T_2, T_3, T_2T_3\} = \langle T_2, T_2T_3\rangle = \langle T_3, T_2T_3\rangle = \langle T_2, T_3\rangle$
$\{e, T_1, T_2T_3, T_1T_2T_3\} = \langle T_2T_3, T_1T_2T_3\rangle = \langle T_1, T_2T_3\rangle = \langle T_1, T_1T_2T_3\rangle$
$\{e, T_2, T_1T_3, T_1T_2T_3\} = \langle T_1T_3, T_1T_2T_3\rangle = \langle T_2, T_1T_3\rangle = \langle T_2, T_1T_2T_3\rangle$
$\{e, T_3, T_1T_2, T_1T_2T_3\} = \langle T_1T_3, T_1T_2T_3\rangle = \langle T_3, T_1T_2\rangle = \langle T_3, T_1T_2T_3\rangle$
$\{e, T_1T_2, T_2T_3, T_1T_3\} = \langle T_1T_3, T_1T_2\rangle = \langle T_1T_3, T_2T_3\rangle = \langle T_1T_2, T_2T_3\rangle$
$\{e, T_1\} = \langle T_1\rangle$
$\{e, T_2\} = \langle T_2\rangle$
$\{e, T_3\} = \langle T_3\rangle$
$\{e, T_1T_2\} = \langle T_1T_2\rangle$
$\{e, T_2T_3\} = \langle T_2T_3\rangle$
$\{e, T_1T_3\} = \langle T_1T_3\rangle$
$\{e, T_1T_2T_3\} = \langle T_1T_2T_3\rangle$
$\{e\} = \langle e\rangle$

All unique subgroups are represented in this Lattice for L_3, therefore this is the entire lattice

$$\begin{tikzcd}[column sep=small]
 & \langle T_1, T_2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle T_1 \rangle \arrow[dr, dash] & \langle T_1 T_2 \rangle \arrow[d, dash] & \langle T_2 \rangle \arrow[dl, dash] \\
 & \langle e \rangle &
\end{tikzcd}$$

Lattice for L_2

The 4 unique subgroups of L_2 are
$\{e, T_1, T_2, T_1T_2\} = \langle T_2, T_1T_2 \rangle = \langle T_1, T_1T_2 \rangle = \langle T_1, T_2 \rangle$
$\{e, T_1\} = \langle T_1 \rangle$
$\{e, T_2\} = \langle T_2 \rangle$
$\{e, T_1T_2\} = \langle T_1T_2 \rangle$
$\{e\} = \langle e \rangle$
All unique subgroups are represented in this Lattice for L_2, therefore this is the entire lattice

1.4 Problem 4

Suppose G is a cyclic group with only one generator. That is, there exists only one element $g \in G$ such that $G = \langle g \rangle$. Prove that G has at most two elements.

Proof. Let G be a cyclic group with only one element $g \in G$ such that $\langle g \rangle = G$. Because we know it is a cyclic group with only one generator, any arbitrary element $a \in G$ can only be expressed in the form $g^n = a$. By the definition of a generating set, we know that $g^{-1} \in G$. Because we know a can also be expressed as $(g^{-1})^{-n} = a$, this indicates g^{-1} is a generator for the group G. Because every element $a \in G$ can only be expressed of the form g^n, we know $g = g^{-1}$. We then know $gg^{-1} = g^2 = e$, therefore, the only two possible elements in G are e, g, in one case e, and g are different elements, and the other case, where $g = e$. $\square$

1.5 Problem 5

Suppose that H is a nonempty subset of a group G such that for any $a, b \in H$, the element $a^{-1}b^{-1} \in H$. Is this enough to ensure that H is a subgroup of G? Prove your answer.

Proof. In order for $H \leq G$, H (i) must be nonempty, (ii) contain an inverse $a^{-1} \in H$ for every $a \in H$ and (iii) be closed under the binary operation of G.
Let $(H \subseteq G) \neq \varnothing$ and let $a, b, (a^{-1}b^{-1}) \in H$, then on the binary operation of G, in order for H to be closed, $ab \in H$. However, there is no guarantee $ab \in H$ since $(H = \{a, b, (a^{-1}b^{-1})\}) \subseteq G$, without the necessary element for closure, so we cannot guarantee $H \leq G$. For example, the subset H of $R_3 = \{r\}$, say $a = r, b = r$, then $(a^{-1}b^{-1} = r^3 = r$. Given this information, you can only ensure $r \in H$, and in order to be a subgroup, you need to be able to ensure $r^2 \in H$, but cannot. Therefore, you cannot ensure H is closed.

□

1.6 Problem 6

(a) Define $\sim$ on $\mathbb{R}$ by $a \sim b$ if and only if $a \leq b$ for $a, b \in \mathbb{R}$. Determine whether or not $\sim$ is an equivalence relation on $\mathbb{R}$.
(b) Define $\sim$ on $\mathbb{R}$ by $a \sim b$ if and only $ab \geq 0$ for $a, b \in \mathbb{R}$. Determine whether or not $\sim$ is an equivalence relation on $\mathbb{R}$.

In order for $\sim$ to be an equivalence relation, $\sim$ must be reflexive for all a $(a \sim a)$, symmetric for all a,b $(a \sim b \textit{ and } b \sim a)$ and must be transitive for all a,b,c $(a \sim b, b \sim c, \textit{ and } a \sim c)$

Proof. (a)
Breaks with symmetric. Let $a, b \in \mathbb{R}$
$(a \leq b) \neq (b \leq a)$ ex. Let $a, b \in \mathbb{R}$ so that $a = 1, b = 2$. It is true that $1 \leq 2$, the opposite is not true, that is, $2 \leq 1$ is not true. Thus $\sim$ is not an equivalence relation

□

Proof. (b) Let $a \in \mathbb{R}$
Reflexive ✓ $(-a)(-a) = (a)(a) = a^2$ which is always ≥ 0
Symmetric ✓ $ab = ba$ (multiplication is communtative) so if $a \sim b$ then $b \sim a$ since if $ab \geq 0, ba \geq 0$
Transitive ✓ If $ab \geq 0$ either a, b are both negative, or both positive. If $bc \geq 0$, c would have to be the same sign as b in order for this to be true. Therefore, a, c are the same sign, and $ac \geq 0$ must be true.

□