Topology Problem Set 5

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Problem 1

Show that $f(x) = |x|$ is continuous from $\mathbb{R}$ to $\mathbb{R}$ using the open set definition of continuity.

Proof. Let (a, b) be an arbitrary open set in $\mathbb{R}$. We will show that $f^{-1}(a, b)$ is open, thus proving continuity via the open set definition. So, what does the pre-image of (a, b) look like? We have three cases to consider.

Case 1: $a < 0, b \leq 0$
This would give us an interval of the form $(-a, -b)$ or $(-a, 0)$. The pre-image of these intervals would be $\varnothing$, since nothing can map to negative values from $f(x)$. The empty set is open, thus this case's pre-image is open.

Case 2: $a, b > 0$
This would give us an interval of the form (a, b). Given $a < b$, the pre-image of this interval would be $(-b, -a) \cup (a, b)$, which is an open set in $\mathbb{R}$. Thus this case's pre-image is open.

Case 3: $a < 0$**,** $b > 0$
This would give us an interval of the form (a, b). Given $a < b$, the pre-image of this interval would be $(-b, 0] \cup [0, b)$, which can be rewritten as $(-b, b)$, which is an open set in $\mathbb{R}$. Thus, this case's pre-image is open.

In all three cases, the preimage, $f^{-1}(x)$ is open, thus via the open set definition, $f(x)$ is continuous. $\square$

Problem 2

Show that $\mathbb{R}^2$ is homeomorphic to the upper half-plane $H = \{(x, y) \mid y > 0\}$. Find an explicit map.

Proof. Let $f(x, y) : \mathbb{R} \to H$ be defined as $f(x, y) = (x, e^y)$. We can break the product function $f(x, y)$ into two coordinate functions, $g(x) = x$ and $h(y) = e^y$. $g(x)$ is the identity map, which we know to be continuous in both directions. $h(y)$ and it's inverse, $h^{-1}(y) = ln(y)$ are continuous on their domains by calculus. Thus, since both coordinate functions are continuous in both directions, the product functions, $f(x,y)$ and $f^{-1}(x,y)$ are both continuous. Thus f is a homeomorphism, and $\mathbb{R}^2$ is homeomorphic to H. $\square$

Problem 3

Let X be a topological space and A a set.
Suppose we have a quotient map $p : X \to A$ and endow A with the quotient topology induced by p. Is p continuous?

Proof. p is continuous by the definition of the quotient topology. $\square$

Show that the quotient topology on A induced by p is the finest topology on A such that the function p is continuous. That is, if $\mathcal{T}$ is another topology on A such that p is continuous, then $\mathcal{T} \subseteq \mathcal{T}_p$, where $\mathcal{T}_p$ is the quotient topology on A induced by p.

Proof. Let $\mathcal{T}_p$ be the quotient topology on A, and $\mathcal{T}$ be another topology on A such that p is continuous. Because p is continuous, for all open sets $U \in \mathcal{T}$, $p^-1(U)$ is an open set in X. By the definition of a quotient topology, $U \in \mathcal{T}_p$, thus $\mathcal{T} \subseteq \mathcal{T}_p$. Thus, the quotient topology on A induced by p is the finest topology on A such that the function p is continuous.

$\square$

Problem 4

(a)

Suppose $f : X \to Y$ is a continuous function between topological spaces X and Y. If a sequence $(x_n) = (x_1, x_2, ...)$ in X converges to $x \in X$, show that the sequence $f(x_n) = (f(x_1), f(x_2), ...)$ in Y converges to $f(x) \in Y$.

Proof. Given x_n converges, then for some $N \in \mathbb{N}$, x_n exists within every open neighborhood of x for all $n \geq N$. Because f is continuous, for all open sets $V \in Y$, $f^{-1}(V)$ are also open in X. Thus, all open sets V containing $f(x)$ are also open in sets in X containing x, moreover, every open neighborhood of $f(x)$ is an open neighborhood of x. If x_n converges to x, every open neighborhood of x contains x_n for sufficiently large N. Since every open neighborhood of $f(x)$ is an open neighborhood of x, every open neighborhood of $f(x)$ would also contain $f(x_n)$ for sufficiently large N. Thus, $f(x_n)$ converges to $f(x)$. $\square$

(b)

Let X be $\mathbb{R}$ with the countable complement topology. Show that a sequence (x_n) in X converges to $x \in X$ if and only if it is eventually constant, i.e. there exists some $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

Proof. $\to$
Assume x_n is eventually a constant value, x, after some N. Then for every open neighborhood containing x, the sequence eventually will remain within all open neighborhoods of x after

sufficiently large N, since the sequence will be equal to x. Thus, the sequence is eventually constant x, it converges to $x \in X$. □

Proof. $\leftarrow$
Let U be an open neighborhood of x, such that $\mathbb{R} - \{x_n\}$ for all $x_n \neq x$ This neighborhood is an open neighborhood of x, but never contains x_n unless it is equal to x. Thus, if x_n converges to x, then x_n must be within every open neighborhood of x, including U. However this is only possible for U if x_n eventually equals x after large enough N. Thus, if x_n converges within $\mathbb{R}_{cc}$, for some $N \in \mathbb{N}$, it must eventually be constant $x_n = x$ for $n \geq N$.

□

(c)

With $X = \mathbb{R}_{cc}$, prove 199 by finding a space Y and a map f such that f is not continuous but f still preserves limits of sequences.

Proof. Let Y be $_s$ with the standard topology, and let $f : \mathbb{R}_{cc} \to \mathbb{R}$ be the identity map, that is, $f(\mathbb{R} - \{a, b\}) = (-\infty, a) \cup (a, b) \cup (b, \infty)$. If x_n converges in R_{cc}, then by part (b), the sequence must be eventually constant for sufficiently large N. Thus, because f is the identity mapping, if x is constant, $f(x)$ is constant. If x_n is eventually constant, then $f(x_n)$ is eventually constant, so by part (b), $f(x_n)$ also converges. Thus, f still preserves the limits of sequences in R_{cc}. However, the set $f^{-1}(a, b)$ is not open with R_{cc} since the complement of $R - (a, b)$ is not constant. Thus, f is not continuous, but does preserve limits of sequences. □

Problem 5

(a)

Use the fact that

$$(0,1) = \bigcup_{n \in \mathbb{N}} \left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right] \tag{1}$$

and

$$(0,1] = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right] \tag{2}$$

to describe a bijection between (0,1) and (0,1].

Proof. Since we can break the two intervals up into n parts, we can map each individual n-part using $f(a, b) = (1 - b, 1 - a)$. Each n-part becomes of (0,1):

$$f\left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right] = \left(1 - 1 + \frac{1}{n+1}, 1 - 1 + \frac{1}{n}\right] = \left(\frac{1}{n+1}, \frac{1}{n}\right] \tag{3}$$

Since the union of maps is the same as the map of a union, we can union up all the mappings for each n-part and we now have a bijection for each from (0,1) to (0,1]. □

(b)

Show that there exists no continuous bijection $f : (0,1) \to (0,1]$ by contradiction. That is, assume there is continuous bijection, and find a contradiction using the Intermediate Value Theorem, and the fact that something must map to 1.

Proof. Assume there exists a continuous function $f : (0,1) \to (0,1]$. Then, there exists some $x \in (0,1)$ such that $f^{-1}(1) = x$. However, because $(0,1)$ is an open interval, there exists values a, b such that $a < x < b$. Because no value can be larger than 1 on the interval $(0,1]$, we have that $f(a), f(b) < 1$. However, because the function is continuous, by the intermediate value theorem, there must exist a value $f(c)$ such that $f(b) < f(c) < 1$ and $f(a) < f(c) < 1$. However, this violates 1-1 since this same value would then be mapped twice. Thus, there exists no continuous bijection. □

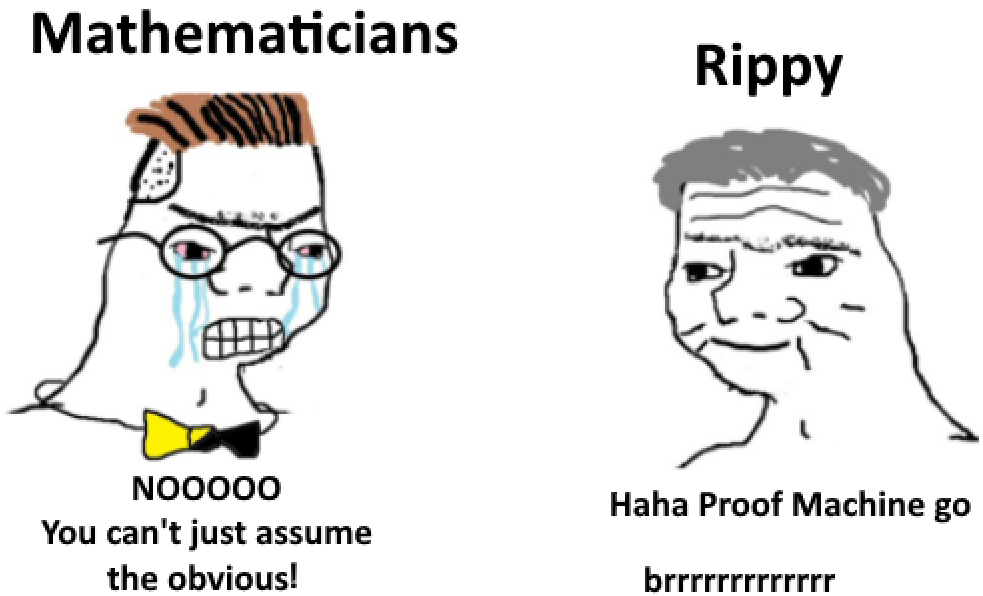
Problem 6

Show that the quotient space X^* from 178 is homeomorphic to S_1.

Proof. The quotient space X^* is obtained from an equivalence relation on $\mathbb{R}$. Thus, using 218, we only need show that there exists a quotient map $g : \mathbb{R} \to S_1$. Consider the map $g : \mathbb{R} \to S_1$ where $g(x) = (1, 2\pi x)$, where 1 is the radius, and $2\pi x$ is in terms of the angle θ (Polar Coordinates). Every interval (a,b) such that $|b-a| = 1$ maps to one full "turn" around the unit circle, S_1, and intervals large than this continue to "wrap" around the circle. The map $g(x)$ is onto, since for every $(1,\theta) \in S_1$, there exists atleast one $\frac{\theta}{2\pi} \in \mathbb{R}$ such that $g(\frac{\theta}{2\pi}) = (1,\theta)$.
Let (a,b) be an open interval in $\mathbb{R}$. $g(a,b) = \Big(1, (2\pi a, 2\pi b)\Big)$ which is an open arc on S_1. Let $\Big((1,\theta_1),(1,\theta_2)\Big)$ be an open arc on S_1, so $g^{-1}\Big((1,\theta_1),(1,\theta_2)\Big) = \bigcup_{n\in\mathbb{Z}} (\frac{\theta_1}{2\pi} + n, \frac{\theta_2}{2\pi} + n)$, which is an open set on $\mathbb{R}$. Thus, g is a quotient map from $X \to S_1$, so by Theorem 218, X^* is homeomorphic to S_1.

□

Meme


Mathematicians
NOOOOO
You can't just assume
the obvious!
Rippy
Haha Proof Machine go
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