PDE {Problem}

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1 Problem 1

Given these conditions:

$$\begin{aligned}
L &= 5 \\
\rho &= .08 \\
\tau &= 200 \\
u(x,0) &= 0 \\
u(0,t) &= 0 \\
u(5,t) &= 0 \\
u_t(x,0) &= -0.05\sin(\pi x) \\
\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Let's say that $u(x,t) = \phi(x)G(t)$, and for simplicity, $\phi(x) = \phi$ and $G(t) = G$
Using the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

We can sub in the derivatives:

$$G''\phi = c^2\phi''G$$

Using separation of variables we get:

$$\frac{G''}{c^2G} = \frac{\phi''}{\phi} = -\lambda$$

Using the initial conditions

$$\begin{aligned}
u(0,t) &= 0 \\
u(5,t) &= 0
\end{aligned}$$

We can determine that

$$\phi_n(x) = \sin(\frac{n\pi x}{5})$$
$$\text{and } \lambda = (\frac{n\pi}{5})^2$$

Now, solving for G, we get

$$G'' = -\lambda c^2 G$$

Given this result, that $u_t(x,0)$ is a sin function, and that $\phi_n(x) = \sin(\frac{n\pi x}{5})$ our G' must be 1 at time $t = 0$, implying:

$$G_n = b_n \sin(\frac{n\pi ct}{5})$$

We can rewrite $u(x,t)$ as:

$$\sum_{n=1}^{\infty} \sin(\frac{n\pi x}{5}) b_n \sin(\frac{n\pi ct}{5}))$$

To solve for b_n, we solve for u_t (velocity):

$$\sum_{n=1}^{\infty} (b_n \frac{n\pi c}{5}) \sin(\frac{n\pi x}{5}) \cos(\frac{n\pi ct}{5}))$$

Using the given initial condition at $t = 0$:

$$-\frac{1}{20}\sin(\pi x) = \sum_{n=1}^{\infty} (b_n \frac{n\pi c}{5}) \sin(\frac{n\pi x}{5})$$

This indicates that,

$$b_n = \begin{cases} -\dfrac{1}{4\pi c} & \text{when } n = 1 \\ 0 & \text{when } n \neq 0 \end{cases}$$

Now, we can rewrite $u(x,t)$

$$u(x,t) = \left(\frac{1}{4\pi c}\right) \sin(\pi x) \sin(\frac{\pi ct}{5})$$

We know the natural frequency in cycles/s is $\dfrac{10}{c}$, thus the final natural frequency is $10\sqrt{\dfrac{\rho}{\tau}}$, which is 0.2 cycles/second.

2 Problem 2

Given:

$$\rho \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial u}{\partial x} - \alpha \frac{\partial u}{\partial t} - \beta u$$

2.1 (a)

In order for a PDE to be hyperbolic, $b^2 - 4ac > 0$. In this instance,

$$b = 0$$
$$a = \rho$$
$$c = -\tau$$

Thus, in order for the PDE to be hyperbolic, $4\rho\tau > 0$. Thus to be positive, ρ and τ need to be the same sign.

2.2 (b)

Assume $u(x,t) = \phi(x)G(t)$ and for simplicity, $\phi(x) = \phi$ and $G(t) = G$. First we rewrite the premise in terms of ϕ and G, giving us:

$$\rho\phi G'' = \tau\phi'' G + \gamma\phi' G - \alpha\phi G - \beta\phi G$$

Then using separation of variables we get:

$$\frac{G''}{G} + \alpha\frac{G'}{G} = \tau\frac{\phi''}{\phi} + \gamma\frac{\phi'}{\phi} - \beta = -\lambda$$

2.3 (c)

Using the result from (b), we can rewrite the ϕ portion to be:

$$\tau\frac{\phi''}{\phi} = -\lambda$$

Taking ϕ_n to be $\sin(\dfrac{n\pi x}{L})$ we can solve for λ

$$\tau(\frac{n\pi}{L})^2$$

Using the result from (b), we can rewrite the G portion to be:

$$G'' + \alpha G' + \lambda G = 0$$

Assuming G is some exponential, e^{rt} we get the quadratic equation:

$$r^2 + \alpha r + \lambda = 0$$

Solving using the quadratic equation, we get:

$$\frac{-\alpha \pm \sqrt{\alpha^2 - 4\lambda}}{2}$$

A over-damped scenario would mean a solely real solution to the quadratic. This occurs when $\alpha^2 > 4\tau(\frac{n\pi}{L})^2$. A under-damped scenario would mean a complex solution to the quadratic. This occurs when $\alpha^2 < 4\tau(\frac{n\pi}{L})^2$

3 Problem 3

3.1 (a)

Physically, the string is of uniform density, it is clamped at both ends, there is no restorative force, there is uniform tension in the string, it is not moving initially, the initial position is given by $f(x)$, and it has light damping.

3.2 (b)

Assume that $u(x,t) = \phi(x)G(t)$ and for simplicity, $\phi(x) = \phi$ and $G(t) = G$

Rewriting this and subbing in the derivatives, we get:

$$G''\phi = 5\phi''G - 0.25G'\phi$$

Using separation of variables, we get:

$$\frac{G''}{G} + \frac{0.25G'}{G} = 5\frac{\phi''}{\phi} = -\lambda$$

Using the initial conditions

$$u(0,t) = 0$$
$$u(2,t) = 0$$

We can determine that

$$\phi_n(x) = \sin(\frac{n\pi x}{2})$$
$$\text{and } \lambda = 5(\frac{n\pi}{2})^2$$

Solving for G, we get:

$$G'' + 0.25G' + \lambda G = 0$$

We get this quadratic, where $r = -a \pm ib$:

$$r^2 + 0.25r + \lambda$$

Using the quadratic formula to solve for r, we get:

$$\frac{-0.25 \pm i\sqrt{20\left(\frac{(n\pi)^2}{4}\right) - (0.25)^2}}{2}$$

This then gives us a and b:

$$a = \frac{1}{8}$$
$$b = \sqrt{\frac{5}{4}n^2\pi^2 - \frac{1}{16}}$$

Using the form of $G(t) = c_1e^{-at}\cos(bt) + c_2e^{-at}\sin(bt)$

We then take the derivative of this to determine G from the initial conditions of G'.

$$\text{Given: } u_t(x,0) = 0$$
$$G' = -ae^{at}\Big(c_1\sin(bt) + c_2\cos(bt)\Big) + e^{at}\Big(bc_1\cos(bt) - bc_2\sin(bt)\Big)$$
$$0 = -ac_2 + bc_1$$
$$ac_2 = bc_1$$
$$c_1 = c_2\frac{a}{b}$$
$$G(t) = c_2\frac{a}{b}e^{-at}\cos(bt) + c_2e^{-at}\sin(bt)$$

Pluggin this all back in, we get:

$$u(x,t) = \sum_{n=1}^{\infty} c_2 \sin(\frac{n\pi x}{2}) e^{-at}\Big(\frac{a}{b}\sin(bt) + \cos(bt)\Big)$$

Now, using the initial condition for position, we get:

$$f(x) = \sum_{n=1}^{\infty} c_2 \sin(\frac{n\pi x}{2})$$

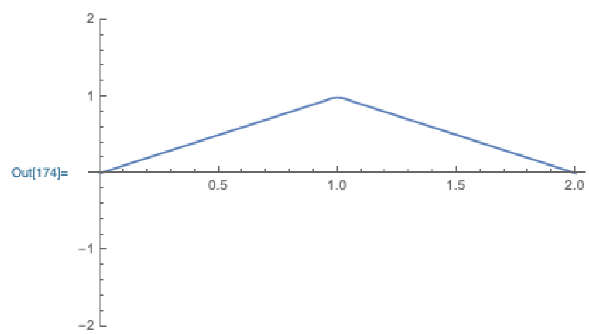
Which is exactly a Fourier Sine series. Solving for the constant, we get:

$$c_2 = \int_0^2 f(x)\sin(\frac{n\pi x}{2})dx$$

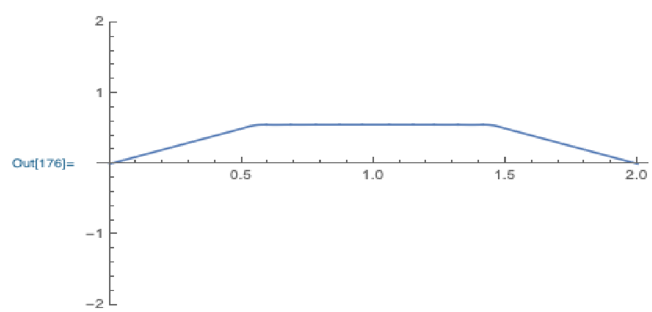
3.3 (c)

Plugging the equations above into Mathematica, we get several snapshots of the motion:

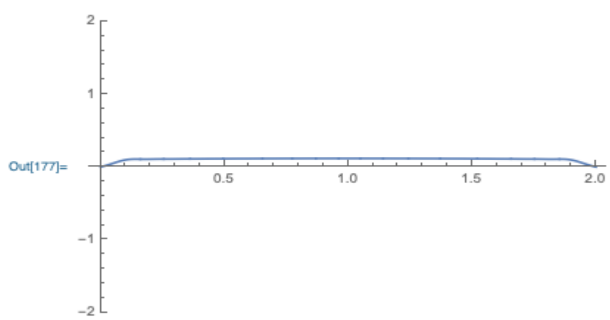
```
In[174]:= Plot[u[x, 0], {x, 0, 2}, PlotRange → {-2, 2}]
```



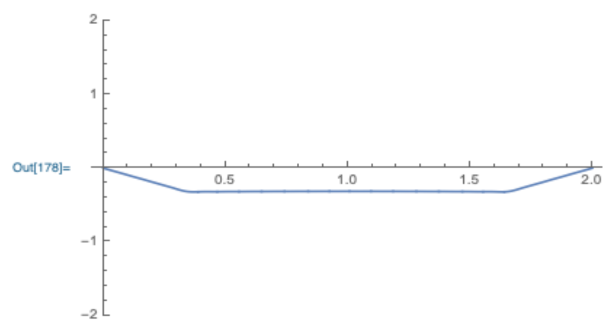

```
In[176]:= Plot[u[x, .1], {x, 0, 2}, PlotRange → {-2, 2}]
```




```
In[177]:= Plot[u[x, .2], {x, 0, 2}, PlotRange → {-2, 2}]
```




```
In[178]:= Plot[u[x, .3], {x, 0, 2}, PlotRange → {-2, 2}]
```



4 Problem 4

The solution would be of the form:

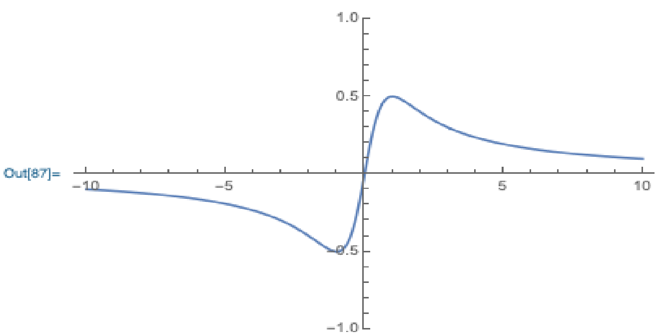
$$\frac{1}{2}\Big(f(x+\sqrt{3}t)+f(x-\sqrt{3}t)\Big)$$

Given the initial conditions, we get:

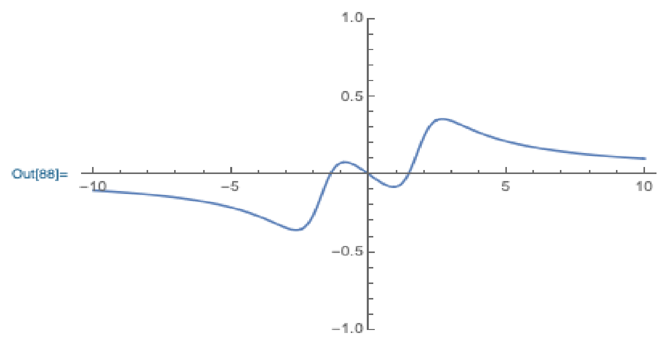
$$u(x,t)=\frac{1}{2}\left(\frac{x+\sqrt{3}t}{(x+\sqrt{3}t)^2+1}+\frac{x-\sqrt{3}t}{(x-\sqrt{3}t)^2+1}\right)$$

Plugging into Mathematica, we get several snapshots of the motion:

In[87]:= `Plot[v[x, 0], {x, -10, 10}, PlotRange → {-1, 1}]`



In[88]:= `Plot[v[x, 1], {x, -10, 10}, PlotRange → {-1, 1}]`



In[89]:= `Plot[v[x, 2], {x, -10, 10}, PlotRange → {-1, 1}]`