# Combinatorics PS 5

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## Problem 1

### (a)

The generating function for this would be :

$$(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4+x^5)$$

Where each section of parenthesis represents the purple, blue, and yellow balls respectively

### (b)

The generating function for this would be:

$$(x+x^2+x^3+x^4+x^5)(x+x^2+x^3)(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8)$$

Where each section of parenthesis represents the Milky Ways, Reeses, and Snickers respectively

### (c)

Here we take the short hand for infinite generating functions:

$$\frac{1}{(1-x)}\frac{1}{(1-x)^5}\frac{1}{(1-x)^10}\frac{1}{(1-x)^25}$$

For pennies, nickles, dimes, quarters

### (d)

$$\frac{x^2}{(1-x)^4}\frac{1}{(1-x)^5}$$

For the first two types being odd, and the last 5 being even

# Problem 2

## (a)

Using our formulas, we can simplify this into :

$$[\![\frac{1}{(1-x)^4}]\!]_{10} - 3 * [\![\frac{1}{(1-x)^4}]\!]_{11}$$

Which becomes:

$$\left(\binom{4}{10}\right) - 3 * \left(\binom{4}{11}\right)$$

## (b)

Using our formula from class, we find the coefficient to be:

$$[\![(1-x)^8]\!]_{12} = (-1)^{12}\binom{8}{12}$$

## (c)

Using our formula, we have the coefficient to be:

$$[\![\frac{1}{(1+x)^8}]\!]_{12} = (-1)^{12}\left(\binom{8}{12}\right)$$

## (d)

Using our formula, we have the coefficient to be:

$$[\![\frac{1}{(1-4x)^5}]\!]_{12} = (4)^{12}\left(\binom{5}{12}\right)$$

# Problem 3

For this, we limit each integer's participation to at most 3 times. This gives us the generating function for $r$ to be:

$$= \Pi_{i=1}^{\infty}(1 + x^i + x^{2i} + x^{3i})$$

Where each set of parenthesis represents the numbers 1 to $r$.

# Problem 4

## (a)

For this instance, we use the form of a second order linear with constant coefficients. We can immediately identify $a_0 = a_1 = 1$ and $\alpha = 3$ and $\beta = 4$. Thus, we can use our formula to solve the recurrence. We have:

$$\begin{aligned}
&a_n - \alpha a_{n-1} - \beta a_{n-2} = 0 \\
&1 - 3x - 4x^2 = 0 \\
&(1 + x)(1 - 4x) = 0 \\
&a_n = c_1(-1)^n + c_2(4)^n
\end{aligned}$$

Solving for $c_1, c_2$:

$$\begin{aligned}
1 &= c_1 + c_2 \\
1 &= -c_1 + 4c_2 \\
c_1 &= \frac{3}{5} \\
c_2 &= \frac{2}{5}
\end{aligned}$$

Thus we have:

$$a_n = \frac{3}{5}(4)^n + \frac{2}{5}(-1)^n$$

## (b)

Again, second order linear with constant coefficients. We simply identify $a_0 = a_1 = 2$, $\alpha = 2$, $\beta = 1$ Then we just plug these in and get:

$$\begin{aligned}
&a_n - \alpha a_{n-1} - \beta a_{n-2} = 0 \\
&1 - 2x - x^2 = 0 \\
&(1 + (\frac{1}{1 + \sqrt{2}})x)(1 - (1 + \sqrt{2})x) = 0 \\
&a_n = c_1(\frac{-1}{1 + \sqrt{2}})^n + c_2(1 + \sqrt{2})^n
\end{aligned}$$

Solving for $c_1, c_2$:

$$\begin{aligned}
2 &= c_1 + c_2 \\
2 &= (\frac{-1}{1 + \sqrt{2}})c_1 + c_2(1 + \sqrt{2}) \\
c_1 &= 1 \\
c_2 &= 1
\end{aligned}$$

Thus we have:

$$a_n = (\frac{-1}{1 + \sqrt{2}})^n + (1 + \sqrt{2})^n$$

## Challenge

The generating for at most two parts is given by:

$$\Pi_i^\infty (1 + x^i + x^{2i})$$

We multiply by by the $(1 - x^{3i})$ to prevent divisibility by 3, and constrict the function to only allowing parts to show up at most twice.

$$\begin{aligned}
&\Pi_{i=1}^\infty \frac{(1 - x^{3i})}{(1 - x^i)} \\
&= \frac{(1 - x^3)(1 - x^6)(1 - x^9)(1 - x^{12})(1 - x^{15})(1 - x^{16}) \cdots}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6) \cdots} \\
&= \frac{1}{(1 - x)(1 - x^2)(1 - x^4)(1 - x^5) \cdots}
\end{aligned}$$

1,2,3,5,7,8 etc. are all parts not divisible by 3. If we have any of these more than twice, they become a multiple of 3, thus this shows the number of partitions where each part appears at most twice.