Combinatorics PS 6

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Problem 1

(a)

For this problem, we merely permute the various letters. However, the I's are indistinguishable, so we don't want to over count. Thus, without the I's, there are 7 remaining letters, each with their own separate bin to distribute into. This leaves us with a multinomial coefficient of :

$$\binom{12}{5,1,1,1,1,1,1,1}$$

(b)

Optional, yay.

1 Problem 2

To find the x^n coefficient in $\sqrt{1-8x}$, we use the general form then fit the equation to find the proper coefficients to plug in. Thus we have:

$$(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k \tag{1}$$

$$\sqrt{1-8x} = \sum_{k\geq 0} \binom{\frac{1}{2}}{k} (-8x)^k \tag{2}$$

We simply plug in the alpha and adjust x to reflect the equation. Thus, the x^n coefficient is given by

$$\sum_{k\geq 0} \binom{\frac{1}{2}}{k} (-8x)^k$$

Which is:

$$-8^n * \binom{\frac{1}{2}}{n}$$

2 Problem 3

Prove for all $n \geq 1$, $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$ Since this deals with multiple levels of previous terms, we need to use strong induction rather than weak.

Proof. Base Case 1:

$$F_1^2 - F_2F_0 = 1 - 2 = -1 = (-1)^1$$

Base Case 2:

$$F_2^2 - F_3F_1 = 4 - 3 = 1 = (-1)^2$$

With both of our base cases proved, we assume the kth case for all k at or below n. Now we manipulate our statement into a more useful form, and utilize the definition of $F_{n+1} = F_n + F_{n-1}$ to rewrite $F_{n-1} = F_{n+1} - F_n$.

$$\begin{aligned}
F_n^2 - F_{n+1}F_{n-1} &= (-1)^n \\
F_n^2 - (F_{n+1})(F_{n+1} - F_n) &= (-1)^n \\
F_n^2 - (F_{n+1}^2) + F_{n+1}F_n &= (-1)^n \\
F_n(F_n + F_{n+1}) - F_{n+1}^2 &= (-1)^n \\
F_nF_{n+2} - F_{n+1}^2 &= (-1)^n \\
F_{n+1}^2 - F_{n+2}F_n &= (-1)^{n+1}
\end{aligned}$$

Thus, we have proved via strong induction that:

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^n$$

□

3 Problem 4

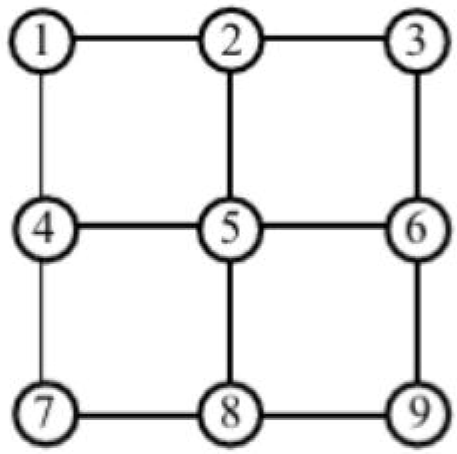
For exactly two cycles, we either can decompose them into a 1-4 or a 2-3 cycles. For 1-4, we have 5 ways to chose the first one cycle, and then $(4!/4)$ ways to choose the second 4 cycle. We divide by 4, since we can shift the cycle from the permutation and get the same cycle. For the 2-3 cycle, we have $(5 * 4)/2$ ways to choose the first two cycle, then $(3!/3)$ ways to choose the 3 cycle. Thus, there are

$$30 + 20 = 50$$

different 2 cycles.

For exactly 3 cycles, there are multiple cases. Since we do not care about the order, we need only consider the different ways to partition and choose the numbers in the blocks. For this instance, the possibilities are 1-1-3,1-2-2 cycles. For the first case, there are $5*4*(3!/3) = 40$ ways to choose this instance. For the next instance, there are $5 * (4!/2) * (1) = 60$ Thus there are 100 different permutations with exactly 3 cycles.

4 Problem 5



Element	Cycle Notation
I	(1)(2)(3)(4)(5)(6)(7)(8)(9)
R_1	(1 3 9 7)(2 6 8 4)(5)
R_2	(1 9)(2 8)(3 7)(4 6)(5)
R_3	(1 7 9 3)(2 4 8 6)(5)
$F_{1,9}$	(1)(5)(9)(2 4)(3 7)(6 8)
$F_{2,8}$	(2)(5)(8) (1 3)(4 6)(7 9)
$F_{3,7}$	(3)(5)(7)(1 9)(2 6)(4 8)
$F_{4,6}$	(4)(5)(6)(1 7)(2 8) (3 9)

Thus, the subgroup is $\{I, R_1, R_2, R_3, F_{1,9}, F_{2,8}, F_{3,7}, F_{4,6}\}$

Challenge

I would like to meet with you to get a possible few hints on this one, as I continue to iron out the other problem set revisions.