# PDE {Problem}

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## Problem 1

In class, we derived the Green's function with these boundary conditions to be:

$$G(x;x_0) = \begin{cases} \frac{x(L-x_0)}{L} & x_0 > x \\ \frac{x_0(L-x)}{L} & x_0 < x \end{cases}$$

So, we know that:

$$u(x) = \int_0^L G(x;x_0) f(x_0) dx_o$$

In this instance, we are given:

$$f(x_0) = x$$

Thus, we have:

$$\begin{aligned} u(x) &= \int_0^x x_0 \frac{x_0(L-x)}{L} + \int_x^L x_0 \frac{x(L-x_0)}{L} \\ &= \frac{L-x}{L} \int_0^x {x_0}^2 dx_0 + \frac{x}{L} \int_x^L Lx_0 - {x_0}^2 dx_0 \\ &= 1 - x \int_0^x {x_0}^2 dx_0 + x \int_x^1 x_0 - {x_0}^2 dx_0 \\ &= (1-x)\frac{x^3}{3} + (x)(\frac{1}{2} - \frac{1}{3} - \frac{x^2}{2} + \frac{x^3}{3}) \\ &= \frac{x^3}{3} - \frac{x^4}{3} + \frac{x}{2} - \frac{x}{3} - \frac{x^3}{2} + \frac{x^4}{3} \\ u(x) &= -\frac{x}{6} - \frac{x^3}{6} \end{aligned}$$

# Problem 2

## (a)

Given the initial conditions:

$$u'' + u = f(x)$$
$$u(0) = u(\frac{\pi}{2}) = 0$$

And that:

$$u_1'' + u_1 = 0$$
$$u_1(0) = 0$$
$$u_2'' + u_2 = 0$$
$$u_2(\frac{\pi}{2}) = 0$$

We can determine that:

$$u_1 = \sin(x)$$
$$u_2 = \cos(x)$$

## (b)

Letting $u = u_1v_1 + u_2v_2$ we can choose $v_1$ and $v_2$ such that:

$$u_1v_1' + u_2v_2' = 0$$
$$u_1'v_1' + u_2'v_2' = f$$

Using that

$$u_1 = \sin(x)$$
$$u_2 = \cos(x)$$

$$u' = u_1'v_1 + u_1v_1' + u_2'v_2 + u_2v_2' = u_1'v_1 + u_2'v_2 = \cos(x)v_1 - \sin(x)v_2$$
$$u'' = u_1''v_1 + u_1'v_1' + u_2''v_2 + u_2'v_2' = -\sin(x)v_1 + \cos(x)v_1' - \cos(x)v_2 - \sin(x)v_2'$$
$$f = -\sin(x)v_1 + \cos(x)v_1' - \cos(x)v_2 - \sin(x)v_2' + \sin(x)v_1 + \cos(x)v_2$$
$$\Rightarrow -\sin(x)v_1 + \sin(x)v_1 - \cos(x)v_2 + \cos(x)v_2 = 0$$
$$\Rightarrow u_1v_1' + u_2v_2' = 0$$
$$\Rightarrow \cos(x)v_1' - \sin(x)v_2' = f$$
$$\Rightarrow u_1'v_1' + u_2'v_2' = f$$

We now have that

$$\begin{aligned}\sin(x)v_1' + \cos(x)v_2' &= 0\\ \cos(x)v_1' - \sin(x)v_2' &= f\end{aligned}$$

We can put these into a system of equations:

$$\begin{bmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Solving this out we get:

$$\begin{aligned}v_1' &= \cos(x)f(x)\\ v_2' &= -\sin(x)f(x)\end{aligned}$$

Integrating to solve for $v_1$ and $v_2$ we get:

$$\begin{aligned}v_1 &= \int_0^x \cos(x_0)f(x_0)dx_0\\ v_2 &= -\int_0^x \sin(x_0)f(x_0)dx_0\end{aligned}$$

Using the initial conditions, we have:

$$\begin{aligned}u(x) = u_1v_1 + u_2v_2 &= \sin(x)v_1 + \cos(x)v_2\\ u(0) = v_2(0) &= 0\\ u(\frac{\pi}{2}) = v_1(\frac{\pi}{2}) &= 0\end{aligned}$$

Using this, we can plug in:

$$\begin{aligned}v_1(\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \cos(x_0)f(x_0)dx_0 + c_1 &= 0\\ \Rightarrow c_1 = -\int_0^{\frac{\pi}{2}} \cos(x_0)f(x_0)dx_0\\ \Rightarrow v_1(x) = \int_0^x \cos(x_0)f(x_0)dx_0 - \int_0^{\frac{\pi}{2}} \cos(x_0)f(x_0)dx_0\\ \Rightarrow v_1(x) = \int_{\frac{\pi}{2}}^x \cos(x_0)f(x_0)dx_0\end{aligned}$$

And we can plug $v_2$, which just gives us the integral from 0 to 0, which is 0, implying $c_2$ is 0, giving us:

$$v_2 = -\int_0^x \sin(x_0)f(x_0)dx_0$$

## (c)

Plugging back in to the original equation for $u(x)$ we get:

$$u(x) = \sin(x)\int_{\frac{\pi}{2}}^{x}\cos(x_0)f(x_0)dx_0 - \cos(x)\int_0^x \sin(x_0)f(x_0)dx_0$$

Rearranging this to make bounds easier, we get:

$$u(x) = -\int_x^{\frac{\pi}{2}}\sin(x)\cos(x_0)f(x_0)dx_0 - \int_0^x \cos(x)\sin(x_0)f(x_0)dx_0$$

We can then rewrite this in terms of a green's equation

$$u(x) = \int_0^{\frac{\pi}{2}} G(x;x_0)f(x_0)$$

Where

$$G(x;x_0) = \begin{cases} -\sin(x)\cos(x_0) & x_0 > x \\ -\cos(x)\sin(x_0) & x_0 < x \end{cases} \tag{1}$$

# Problem 3

Using the setup from above, and using we are given:

$$u'' + u = -1 = f$$

Plugging in the parts of the Green's function, we get:

$$\begin{aligned} u(x) = \sin(x)\int_x^{\frac{\pi}{2}}\cos(x_0)dx_0 + \cos(x)\int_0^x \sin(x_0)dx_0 \\ \Rightarrow u(x) = \sin(x)(1-\sin(x)) + \cos(x)(1-\cos(x)) \\ \Rightarrow u(x) = \sin(x) - \sin^2(x) + \cos(x) - \cos^2(x) \\ \Rightarrow u(x) = \sin(x) + \cos(x) - 1 \end{aligned}$$

# Problem 4

Let

$$u_1 = x$$
$$u_2 = 1$$

We let:

$$u(x) = u_1v_1 + u_2v_2$$

Plugging in our $u_1, u_2$, we have:

$$u = xv_1 + v_2$$

Then we have

$$v_1 + xv_1' + v_2'$$

Since we know from the initial conditions that:

$$-u'' = f$$

And if we set $xv_1' + v_2' = 0$ we have:

$$u'' = v_1' = -f$$

We can then use substitution to solve for $v_2'$ since we know $v_1$. Thus we have:

$$\begin{gathered} v_1' = -f \\ v_2' = xf \end{gathered}$$

Further, we have now that, using the initial conditions:

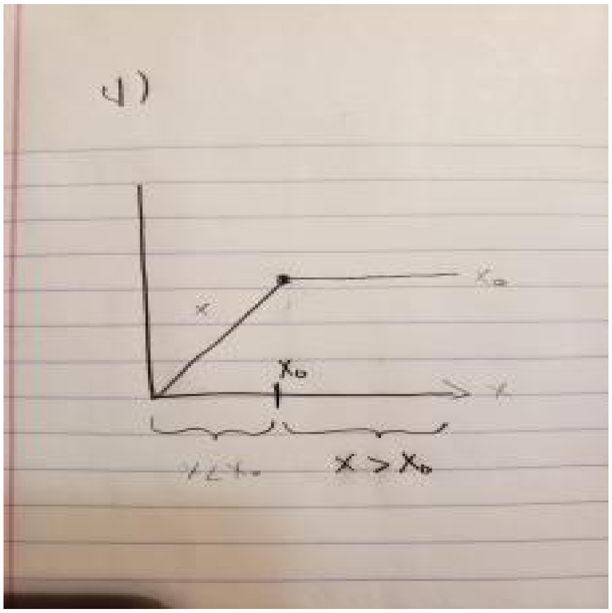
$$\begin{aligned} u(0) = (0)v_1(0) + v_2(0) = v_2(0) = \int_0^0 v_2' dx_0 + c_2 = 0 \\ \Rightarrow c_2 = 0 \\ \Rightarrow v_2 = \int_0^x x_0 f(x_0) dx_0 \\ u'(L) = v_1(L) = -\int_0^L f(x_0) dx_0 + c_1 = 0 \\ \Rightarrow c_1 = \int_0^L f(x_0) dx_0 \\ \Rightarrow v_1 = -\int_0^x f(x_0) dx_0 + \int_0^L f(x_0) dx_0 \\ \Rightarrow v_1 = \int_x^L f(x_0) dx_0 \end{aligned}$$

Now, subbing in for $u(x)$ we get:

$$u(x) = x \int_x^L f(x_0) dx_0 + \int_0^x x_0 f(x_0) dx_0$$

Which then gives the Green's function of:

$$G(x; x_0) = \begin{cases} x & x < x_0 \\ x_0 & x > x_0 \end{cases}$$




## Problem 5

Using what we derived in class, we know that a happy goldfish named delta lives in this integral:

$$u(x) = \int_0^L G(x; x_0)\delta(x_0 - x_s)dx_0 = G(x; x_s)$$

So then this just becomes a simple plug and chug.

$$\begin{aligned} u(.5) &= 3G(x; .2) + 2G(x; .6) \\ &= 3(.2(.5)) + 2(.5(.4)) = .7 \end{aligned}$$

Thus, $u(.5) = .7$