Topology Super Fun Exercises

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Problem 1

Let $A_1, A_2, ...$ be a collection of subsets of a set X. Define

$$\tilde{A}_n = A_n - \bigcup_{i=1}^{n-1} A_i$$

Show that if $i \neq j$,

$$\tilde{A}_i \bigcap \tilde{A}_j = \varnothing$$

and

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \tilde{A}_n$$

Proof. If $i > j$, then for all $a \in \tilde{A}_i, a \notin \bigcup_{k=1}^{i-1} A_k$, by definition of $\tilde{A}_i$. Thus, $a \notin A_k$ for all $k < i$. Then, if $i \neq j$, WLOG, either $i < j$ or $j < i$, so if $a \in \tilde{A}_i$, and $i \neq j$, then $\tilde{A}_i \bigcap \tilde{A}_j = \varnothing$.
Let $x \in \bigcup_{n \in \mathbb{N}} \tilde{A}_n$. Then there exists an A_n such that $x \in A_n$, thus $x \in \bigcup_{n \in \mathbb{N}} A_n$.
Let $x \in \bigcup_{n \in \mathbb{N}} A_n$, then for some *minimal* i, $x \in A_i$, and by definition of $\tilde{A}$, then $x \in \tilde{A}_i$, thus $x \in \bigcup_{n \in \mathbb{N}} \tilde{A}_n$ □

Problem 2

Given that X is a non-empty set with the Particular Point Topology.

Sets A that contain the point x

Int$(A) = A$.
Since for every A, $x \in A$, thus A is open. A is the largest open set contained within A, so the union will be A.
Cl$(A) =$ X.
All A contain x, so $x \notin A^c$ for all A^c. Thus no A^c is open. The only closed set that still contains A is X, since the empty set is open in the topology, and $\varnothing^c = X$.

Sets A that do not contain the point x

$\text{Int}(A) = \varnothing$
All A^c contain x, thus $x \notin A$ for all A. If $U \subset A$, and U was open, then $x \in U$, however this presents a contradiction, as $x \notin A$, so there are no open subsets of A.
$\text{Cl}(A) = A$
For all A, $x \notin A$, so then, for all A^c, $x \in A^c$, so all A^c are open, thus all A are closed. Thus, via 73.5, $\text{Cl}(A) = A$

Problem 3

Show the collection $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

1

$$\varnothing \in \mathcal{T}(\text{Empty Union})$$

$X \in \mathcal{T}$, since for every $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$, so by theorem 12, we have $\bigcup_{x \in X} B_x$

2

Since for every open set, $U_i \in \mathcal{T}$ can be written as the union of basis elements B_i, thus the arbitrary union of all U_i can also be written as the union of basis elements within $\mathcal{B}$.

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} \Big(\bigcup_{j \in J} B_j \Big)_i$$

3

Let $x \in U_1, U_2$, and let $U_1 = \bigcup A_i$, and $U_2 = \bigcup B_j$. We have:

$$x \in U_1 \bigcap U_2$$
$$x \in \Big(\bigcup A_i \bigcap \bigcup B_j \Big)$$
$$x \in A_i \bigcap B_j \text{ for some } i, j$$
$$x \in \bigcup_{i,j} \Big(A_i \bigcap B_j \Big),$$
$$\text{thus, there exists some } B \in \mathcal{B} \text{ such that } x \in B \subseteq A_i \bigcap B_j \text{ since}$$
$$A_i \bigcap B_j = \bigcup_{A_i \bigcap B_j} B_{x_{ij}}$$
$$\text{Since } A_i \bigcap B_j \text{ is open, the union of open sets is open.}$$

Now, we take this one step further, to prove for an arbitrary amount of intersections.

$$\bigcap_{i=1}^{k} U_i = \bigcap_{i=1}^{k-1} U_i \bigcap U_k$$

By our hypothesis, we assume $\bigcap_{i=1}^{k-1} U_i$ to be open, and we know that U_k is open, and we know the intersection of open sets is open, thus this has become the case we have already proven.

Problem 4

Is $\mathcal{T}_\infty$ is a Topology?

Proof. No. By definition of this set, U must be finite to be open. However, the arbitrary union of finite sets can be infinite. Let $U_n = \{n\}$, for all $n \in \mathbb{N}$. $X - U_n$ is infinite, since $\mathbb{N} - \{n\}$ is infinite. However, $\bigcup_{n\in\mathbb{N}} U_n = \mathbb{N}$, which is infinite, which cannot exist since $\mathbb{N} - \mathbb{N} = \varnothing$, which is finite. $\square$

Problem 5

a

1

For every $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$ (by definition of $\mathcal{B}$ in this problem)

2

$\mathcal{B}$ is a collection of *open* sets. Thus, since B_1 and B_2 are open, their intersection $B_1 \bigcap B_2$ will also be open, thus their intersection, $B \in \mathcal{B}$. We have now, $x \in B \subseteq B_1 \bigcap B_2$. Thus, $\mathcal{B}$ is a basis.

b

1

Let $B \in \mathcal{B}$ be the interval $(x-1, x+1)$. If x is rational, then $x-1, x+1, \in \mathbb{Q}$. If X is irrational, then we know via 6 PS 1, that between any two real numbers, there must exist a rational number. Thus, we know there must exist a rational number between $x-1$ and x, and x and $x+1$. Thus, there must exist an interval $x \in B \in \mathcal{B}$.

Prove standard topology equivalence

Let $\mathcal{T}_{\mathcal{B}'}$ be the standard topology. $B \in \mathcal{B}$. B is an open set in $\mathcal{T}_{\mathcal{B}'}$. $x \in B = \bigcup_{i \in I}(a_i, b_i)$, thus for some B', $x \in B' \subset B$, where B' is a basis element of the topology. Thus via exercise 92, $\mathcal{T}_{\mathcal{B}'}$ is finer than $\mathcal{T}_{\mathcal{B}}$. Let U be in $\mathcal{B}$ and $x \in U$. Since all basis elements in this topology are open, then there exists a basis element, B, such that $x \in B \subseteq U$ Thus via 92, $\mathcal{T}_{\mathcal{B}}$ is finer than $\mathcal{T}_{\mathcal{B}'}$ Thus, $\mathcal{B}$ is the standard topology.

Problem 6

a

Via exercise 50, we know that a topology is the discreet topology if and only if $\{x\} \in \mathcal{T}$ for all $x \in X$. Since B is a basis, we know if $x \in B_1 \bigcap B_2$, there must exist a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \bigcap B_2$. Let $B = (x - 1.x + 1)$, this is equivalent to the singleton set for x. Thus, for every $x \in X$, there exists some interval $(x - 1, x + 1) \in \mathcal{T}$, thus via exercise 50, this is the discreet topology.

b

All singleton sets of X are open, except for (2,1). Every singleton set except this one can be written as the interval of (a_{i-1}, a_{i+1}), $[a_0, a_1)$, (b_{i-1}, b_{i+1}), or (b_{0-1}, b_0) EXCEPT $(2, 1)$ since we cannot pick a maximal a_{max} element since it exists in $\mathbb{N}$, which has no 'max'. Thus, we cannot include *only* (2,1) in an interval, thus it cannot be written as union of basis elements, thus it is not open within this topology.