Problem Set 2

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Problem 1

a

Using the theory for fixed point iterations, we first have $g(x) = \frac{f(x)}{f'(x)}$. In this case, $f(x) = \sin(x)$, so $g(x) = x - \tan(x)$. We then have $g'(x) = 1 - \sec^2(x)$, $g''(x) = -2\sec^2(x)\tan(x)$, and $g'''(x) = -4\sec^2(x)\tan(x) - 2\sec^4(x)$. Since this has a root at 0, $p = 0$, so we evaluate these at $x = 0$. $g(0), g'(0)$, and $g''(0)$ are all 0. $g'''(0) = -2$, which is a non-zero constant. Thus, by using the theory for fixed point, we find for this case that the order of convergence, α is 3.

b

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Newtons[f, 1.]
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n	Values	Ratio
1	-0.557408	-0.557408
2	0.0659365	-0.380721
3	-0.0000957219	-0.333914
4	2.92357×10^{-13}	-0.333333
5	0.	0.

This is not initially consistent with cubic convergence, as it goes from 10^{-1}, to 10^{-2}, to 10^{-5}, to 10^{-13}, which is not cubed each time. However, if you look at the ratio of the errors, $\frac{p_n+1}{(p_n)^3}$, you see that it is approaching a constant, namely $-\frac{1}{3}$. So, it ultimately does become cubic convergence. Any other power other than cubic causes it to blow up.

Problem 2

a

The Lagrange form of the polynomial is:

$$\begin{aligned} P_3(x) = 1 * \frac{(x-2)}{(1-2)}\frac{(x-5)}{(1-5)}\frac{(x-6)}{(1-6)} + 2.4 * \frac{(x-1)}{(2-1)}\frac{(x-5)}{(2-5)}\frac{(x-6)}{(2-6)} \\ +3 * \frac{(x-1)}{(5-1)}\frac{(x-2)}{(5-2)}\frac{(x-6)}{(5-6)} + 1.2 * \frac{(x-1)}{(6-1)}\frac{(x-2)}{(6-2)}\frac{(x-5)}{(6-5)} \end{aligned}$$

It's approximation of f(4) is 3.64.

b

The Newton form of the polynomial is:

$$P_3(x) = 1 + 1.4(x-1) - 0.3(x-1)(x-2) + 0.04(x-1)(x-2)(x-5)$$

It's approximation of f(4) is 3.64.

Problem 3

a

Using the newton form, we calculate $P_4(x)$ to be:
$P_4(x) = 1 - \frac{2}{3}x + \frac{1}{3}(x)(x-0.5) - \frac{2}{15}(x)(x-0.5)(x-1) + \frac{2}{45}(x)(x-0.5)(x-1)(x-1.5)$

b

This is a plot of the actual error from 0 to 2

$$h[x\_] := 1 - \frac{2}{3}x + \frac{1}{3}x * (x - 0.5) - \frac{2}{15}(x)(x - 0.5)(x - 1) + \frac{2}{45}(x)(x - 0.5)(x - 1)(x - 1.5);$$

$$k[x\_] := \frac{1}{x + 1};$$

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Plot[{Abs[k[x] - h[x]]}, {x, 0, 2}]
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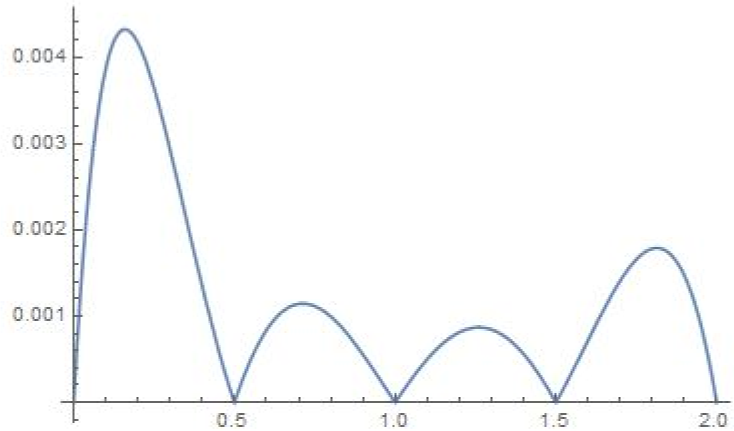

c

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g[x_] := Abs[(x) (x - 0.5) (x - 1) (x - 1.5) (x - 2)];
Plot[g[x], {x, 0, 2}]
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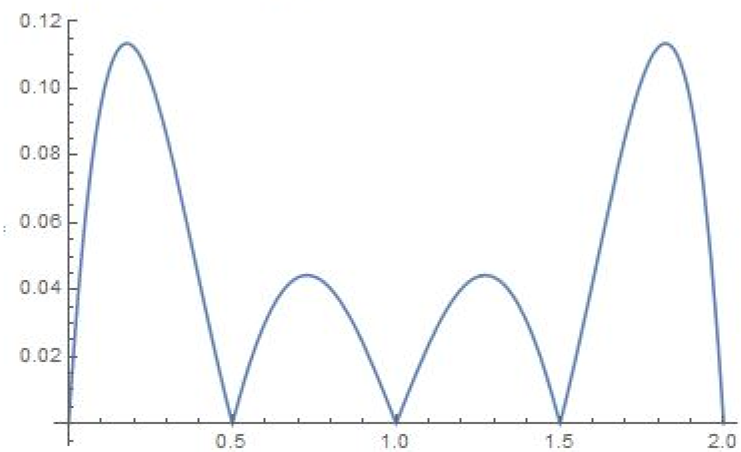

The relationship between the actual error and the estimated error is they have approximately the same shape, and the estimated error acts like an upper bound for the actual error.

Problem 4

The Hermite polynomial for the data is calculated to be:

$0.6(x-1)-0.095(x-1)^2+0.22(x-1)^2(x-3)-0.0878(x-1)^2(x-3)^2+0.055(x-1)^2(x-3)^2(x-5)$

It estimates f(1.5) to be 0.035 and f'(1.5) to be -0.160

Problem 5

The error of the Hermite polynomial is given by:

$$\frac{(x-x_0)^2...(x-x_n)^2}{(2n+2)!}f^{(2n+2)}(\xi(x))$$

We know that $f^{(2n+2)}(\xi(x))$ will be a maximum of 1, so we can simplify to this out solve for the worst case scenario to guarantee an error threshold. This now leaves the error to be:

$$\frac{(x-x_0)^2...(x-x_n)^2}{(2n+2)!}$$

By looking at how the numerator behaves as we increase n (featured in the graph below), we see the maximum value of the numerator of the error strictly decreases. The largest this will ever be is at the minimum number of nodes $n = 1$, which gives us a maximum numerator

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y1[x_] := (x^2 * (x - Pi)^2);
y2[x_] := (x^2 * (x - Pi)^2 * (x - Pi/2)^2);
y3[x_] := (x^2 * (x - Pi)^2 * (x - Pi/3)^2 * (x - (2*Pi)/3)^2);

Plot[{y1[x], y2[x], y3[x]}, {x, 0, Pi}]
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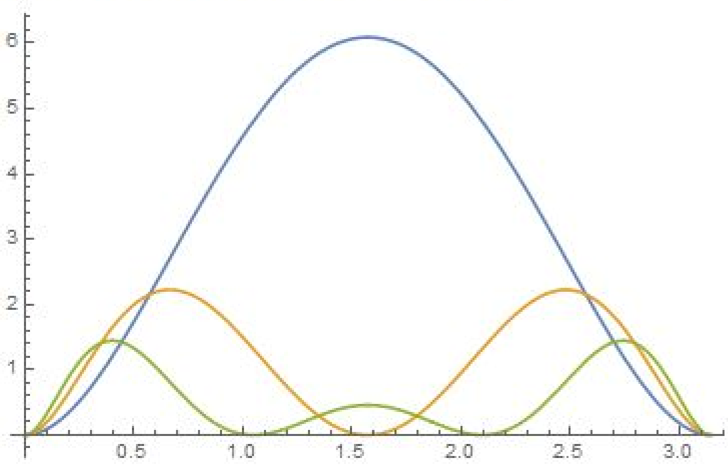

value of 6.08807.

This then allows to simplify the maximum error further, giving us:

$$\frac{(6.08807)}{(2n+2)!}$$

Solving for the n that guarantees the error is less than 10^{-6}, gives us a value of 5.