Abstract Problem Set

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1 Problems

1.1 Problem 1

Let $H \leq G$ and define $X := \{aH \mid a \in G\}$ and $Y := \{Ha \mid a \in G\}$.

(a) Show that the map $\psi : X \to Y$ given by $\psi(aH) = Ha$ is not well-defined in general (i.e. is not a function) by showing that there exists $a, b \in S_3$ such that $aH = bH$ but $Ha \neq Hb$ where $H = \langle(1,2)\rangle$.

(b) Prove that the map $\phi : X \to Y$ given by $\phi(aH) = Ha^{-1}$ is a 1-1 and onto function. Consequently, there are always the same number of left and right cosets of H in G.

Proof. (a)
Let $a = (1,2,3)$, $b = (1,3)$, and $H = \langle(1,2)\rangle = \{(1)(2),(1,2)\}$.
$aH = (1,2,3)\{(1)(2),(1,2)\} = \{(1,2,3),(1,3)\}$
$bH = (1,3)\{(1)(2),(1,2)\} = \{(1,3),(1,2,3)\}$
Thus, $aH = bH$ and $a \neq b$.
$Ha = \{(1)(2),(1,2)\}(1,2,3) = \{(1,2,3),(1)(2,3)\}$
$Hb = \{(1)(2),(1,2)\}(1,3) = \{(1,3),(1,3,2)\}$
Thus $Ha \neq Hb$. □

Proof. (b)
Onto: For every $Ha \in Y$ there exists some $a^{-1}H \in X$ such that $\phi(a^{-1}H) = H(a^{-1})^{-1} = Ha$
1-1: $\phi(aH) = \phi(bH)$
$\Rightarrow Ha^{-1} = Hb^{-1}$
$\Rightarrow H = Hb^{-1}a$
$\Rightarrow b^{-1}a \in H$
Then there exists some $h \in H$ such that $h = b^{-1}a$.
$\Rightarrow bh \in bH$
$\Rightarrow b(b^{-1}a)H = bH$
$\Rightarrow aH = bH$
Since ϕ is 1-1, for every left coset $\in X$, there is a unique right coset $\in Y$ that ϕ maps to. Since ϕ is onto, every right coset $\in Y$ has some left coset $\in X$ that maps to it. Thus, the number of left cosets exactly matches the number of right cosets.

□

1.2 Problem 2

(a) Consider the arbitrary k-cycle $(a_1, a_2, ..., a_k)$ from S_n(with $k \leq n$). When will this cycle be odd versus even?

(b) The set A_n forms a group under composition of permutations and has order $\frac{n!}{2}$.

Proof. (a)
Let the k-cycle $(a_1, a_2, ..., a_k) \in S_n$ (with $n \leq k$).
Induction:
Base Case: n = 2, (a_1, a_2) thus, k-cycle can be written as $n - 1$ cycles.
Assume kth case: $(a_1, a_2, ..., a_k)$ can be written as $k - 1$ cycles.
Prove $kth + 1$ case: $(a_1, a_2, ..., a_k, a_{k+1}) = (a_1, a_2, ..., a_k)(a_k, a_{k+1})$ which is $(k - 1) + 1 = k$ transposition cycles. The cycle $(a_1, a_2, ..., a_k)$ can be written as a composition of $k-1$ transpositions, $(a_1, a_2)(a_2, a_3) \cdots (a_{k-1}, a_k)$. Thus, if k is odd, the cycle will be even, and if k is even, the cycle will be odd.

□

Proof. (b)
A_4 contains all cycles of even length.
Closure: Let $a, b \in A_n$ such that $a = (a_1, a_2 \cdots a_t)$ and $b = (b_1, b_2, \cdots b_k)$ where $ab = (a_1, a_2 \cdots a_t)(b_1, b_2, \cdots b_k)$ which still can be written as an even number of transpositions, since a, b are even.
Associative: We know composition of permutations is associative.
Inverse and Identity: Any element is a product of transpositions and can be flipped, while still remaining the same number of transpositions, thus the inverse cycle remains even, so the inverse cycles exists within A_n. Since the inverse cycle exists in A_n, then the original cycle composed the inverse gives the trivial cycle, the identity, which is even.
Let X be even permutations$\in S_n$, and Y be odd permutations $\in S_n$. Consider the map from X to Y, as $\phi(\alpha) \to (1, 2)(\alpha)$.
1-1: Let $\alpha, \beta \in X$
$\phi(\alpha) = \phi(\beta)$
$\Rightarrow (1, 2)(\alpha) = (1, 2)(\beta)$
$\Rightarrow (1, 2)(1, 2)(\alpha) = (1, 2)(1, 2)(\beta)$
$\Rightarrow \alpha = \beta$
Onto: Let $\gamma \in Y$.
For every $\gamma \in Y$, there exists some $(1, 2)(\gamma) \in X$ such that $\phi((1, 2)\gamma) \to (1, 2)(1, 2)(\gamma) = \gamma$
Thus, by this bijection, there is exactly as many even permutations as odd. Since $|S_n| = n!$, and S_n is composed of odd and even permutations, there are $\frac{n!}{2}$ even permutations. Thus $|A_4| = \frac{n!}{2}$

□

1.3 Problem 3

(a) Find all abelian groups up to isomorphism of order 48. Which, if any, are cyclic?

(b) Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$.

Proof. (a)
48 factors down into 3,2,2,2,2, or $3, 2^4$. Thus, the 5 abelian groups of isomorphism of 48 are:
$\mathbb{Z}_3 \times \mathbb{Z}_{16}$ (only cyclic group)
$\mathbb{Z}_3 \times \mathbb{Z}_2 \times Z_8$
$\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$
$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$
$\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

□

Proof. (b) Take the least common multiple of the orders of the groups. Thus the lcm(4,18,15) is 180. Maximum possible order for an element is 180.

□

1.4 Problem 4

Prove that the converse to Lagrange's Theorem is false by showing that A_4 has no subgroup of order 6. (Hint: Prove this by contradiction. Suppose there is a subgroup H of A_4. Think about its index, and consider elements of A_4 of order 3.)

Proof. Let there be a subgroup, H, of A_4 such that $|H| = 6$. Since we know $|A_4| = 12$, the index is $12/6 = 2$, indicating H is a normal subgroup. Thus, $aH = Ha$ for all $a \in G$. Because the index is 2, there are two distinct cosets, one of them being H since $eH = He = H$. Let b be an element of order $3 \in A_4$.
$\Rightarrow$ Assume that $b \notin H$. Since b is order 3, $b \neq b^2 \neq e$. Thus, since $b, b^2 \notin H$, $bH, b^2H \neq H$. Thus since there is only one other distinct coset, bH must be the other coset. However, since $b^2H \neq H$, it must be equal to bH. Thus, $bH = b^2H$, multiplying by the inverse, we get $H = bH$, which is not true, indicating by contradiction that b, b^2 must exist within H. However, since there are 8 elements of order $3 \in A_4$ and $8 > 6$, this creates a contradiction, thus there can be no subgroup of order 6, disproving the converse. □

1.5 Problem 5

Let G be a group of order pq, where p and q are prime numbers. Show that every proper subgroup of G is cyclic.

Proof. Let $H \leq G$. Since G is of finite order pq, by Lagrange's Theorem, the proper subgroups of G will be of order p, q or 1 since these are the only numerical divisors of pq. The subgroup of order 1 is the trivial subgroup and is always cyclic. Let $H \leq G$, and let $|H| = p$ (WLOG, since p is an arbitrary prime, it could just as easily be q). By Lagrange's Theorem, the subgroups of H must be of order p or 1. Let $h \in H \neq e$, then $|\langle h \rangle| = p$. This proves h is a generator of H since the subgroup generated by h is an improper subgroup, thus H is cyclic. Thus, every proper subgroup of G is cyclic. $\square$