Topology Super Fun Exercises

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Problem 1

a

Show that K is closed in $\mathbb{R}_K$

Proof. Let K exist within $\mathbb{R}_K$. To show K is closed in $\mathbb{R}_K$, we want to show that K^c is open. Hence, we will show that $\mathbb{R} - \{K\}$ is open in $\mathbb{R}_K$. In order for a set to be open, we must be able to write the set as a union of basis elements. So, we will do just that. We can exactly write $\mathbb{R} - \{K\}$ as a union of the baisic open intervals $\bigcup_{i \in I} \Big((a, b) - \{K\} \Big)_i$. Thus, K^c is open, and we prove that K is closed within $\mathbb{R}_K$ □

b

Show K is not closed in $\mathbb{R}$ with the standard topology by showing $0 \in \bar{K}$

Proof. To show K is not closed in $\mathbb{R}$ with the standard topology, we will use a contradiction by showing 0 exists within $\bar{K}$. In order to contain 0 within an open interval, we must include a negative and a positive bound. Let this interval be $(-\epsilon, \epsilon)$. Using the Archimedian property, we conclude there exists a $\frac{1}{n} < \epsilon$, thus every open interval containing 0 will intersect K. Via 79, we can then conclude that 0 exists within $\bar{K}$. It is here that the contradiction arises. If K is closed, $\bar{K} = K$, however this obviously cannot be the case as we cannot write 0 as $\frac{1}{n}$. Thus, we can conclude that K is not closed in $\mathbb{R}$ with the standard topology. □

c

Show that the K-topology is strictly finer than the standard topology on $\mathbb{R}$

Proof. To show that the K-topology is strictly finer than the standard topology, we will show that it contains something the standard topology does not. Following directly from (a) and (b) we know that K is closed in the K-topology and not in the standard topology, so K^c is open in the K-topology, and K^c is *not* open in the standard topology. Since the K-topology also contains all of $\mathbb{R}$ by definition, *and* it contains K^c, the K-topology is strictly finer than the standard topology. □

d

Show that the K-topology is not comparable to the lower-limit topology.

Proof. $\rightarrow$
We will show the K-topology is not a subset of the lower limit topology. In order to contain 0 within an lower limit interval, we must include 0 and a positive bound (the other case already accounted for in b). Let this interval be $[0, \epsilon)$. Using the Archimedian property, we conclude there exists a $\frac{1}{n} < \epsilon$, thus every interval containing 0 will intersect K. Via 79, we can then conclude that 0 exists within $\bar{K}$. It is here that the contradiction arises. If K is closed, $\bar{K} = K$, however this obviously cannot be the case as we cannot write 0 as $\frac{1}{n}$. Thus, we can conclude that K is not closed in $\mathbb{R}_{ll}$ with the standard topology, so K^c is not open in the lower limit topology, but is open in the K-topology, so the K-topology is not a subset of the lower limit topology. □

Proof. $\leftarrow$
We will show the lower limit topology is not a subset of the K-topology. The element [0,a) is open in the lower limit topology, but not the K-topology. The element $[0, a)$ cannot be created with open intervals by definition of open intervals, as an open interval cannot include only 0, it must include the points around 0. (proven in b). The element $[0, a)$ also cannot be created from the open intervals $(a, b) - \{K\}$ either, for the same reason. Thus, the lower limit topology is not a subset of the K-topology.
Thus, the K-topology is not comparable to the lower limit topology. □

Problem 2

Prove that $\bar{B}_r(x)$ is a closed set in $\mathbb{R}^2$

Proof. To show that $\bar{B}_r(x)$ is a closed set in R^2, we will show that $\bar{B}_r(x)^c$ is open. Let a be an arbitrary point outside of $\bar{B}_r(x)$. Let l_a be the minimum distance from the point a to the ball $\bar{B}_r(x)$. Let $B_l(a)$ be an open ball of radius l centered at a. Since a is not an element of $\bar{B}_r(x)$, we know $l > 0$, thus $B_l(a)$ is a open ball, for any arbitrary a. Then, we can write $\bar{B}_r(x)^c$ as the union all elements, a, as $\bigcup_{a \in \bar{B}_r(x)^c} B_l(a)$, showing $\bar{B}_r(x)$ is closed. □

Problem 3

a

Let X be a topological space, and $A \subseteq X$. Then A is closed if and only if $A' \subseteq A$.

Proof. $\rightarrow$
Let A be closed, and let x' be an element of A'. Via the definition of A', for every open set U, $U \cap A - x' \neq \varnothing$. Via 79, if every open set containing x' intersects A, then x' is an element of $\bar{A}$. Hence, because A is closed, $\bar{A} = A$, thus x' is an element of A, thus $A' \subseteq A$. □

Proof. $\leftarrow$
Let $A' \subseteq A$. If x is an element of A', then x is an element of A. Suppose that x is not an element of A, then x is not an element of A'. So, for all open sets U that contain x, $U \cap (A - \{x\}) = U \cap A = \varnothing$. Thus, via 79, x is not an element of $\bar{A}$, so via the contrapositive, x must be an element of A. Hence, $\bar{A} \subseteq A$ and clearly $A \subseteq \bar{A}$. Thus, $A = \bar{A}$, so A is closed. □

b

Let $X = \{1, 2\} \times \mathbb{N}$. Define $A = \{a_i \mid i \in \mathbb{N}\}$ and $B = \{b_i \mid i \in \mathbb{N}\}$. Find $A'\bar{A}, B'$ and $\bar{B}$.

Proof. $A' = b_1$. In order to include a_i, the interval must always intersect with b_1 since there is no defined maximum of a_i. □

Proof. $\bar{A} = A \cup b_1$, since $\bar{A} = A' \cup A$ □

Proof. $B' = \varnothing$. Since we cannot find a maximum value b_i, we cannot find a limit point of b_i. □

Proof. $\bar{B} = B$, since $\bar{B} = B' \cup B$ □

Problem 4

Show it is not enough to show that any element of one basis contains some element from the other basis and vice versa.

Give an example of two bases $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbb{R}$ such that for all $B \in \mathcal{B}$, there exists some $B' \in \mathcal{B}'$ such that $B' \subseteq B$ and for all $B' \in \mathcal{B}'$ there exists some $B \in \mathcal{B}$ such that $B \subseteq B'$ but $\mathcal{T}_{\mathcal{B}} \neq \mathcal{T}_{\mathcal{B}'}$

Proof. To show this, we will use the lower limit topology, and the upper limit topology, two distinct topologies on $\mathbb{R}$ (ie, $\mathcal{T}_{ll} \neq \mathcal{T}_{ul}$. We denote the collection basis elements of the upper and lower limit topologies as $\mathcal{B}_{ul}$ and $\mathcal{B}_{ll}$ respectively. Let $B_i \in \mathcal{B}_{ll}$ and $B'_i \in \mathcal{B}_{ul}$. Any B_i will be of the form $[a, b)$, where $a < b$, and any B'_i will be of the form $(a, b]$, where $a < b$. Let $B_i = [a, b)$. Via the Archemedian property, there exists a real number, let's call ϵ, such that $a < \epsilon < b$. Hence, we can create an element, B'_i of the form $(a, \epsilon]$ that exists within the interval $[a, b)$. Thus, for every $B_i \in \mathcal{B}_{ll}$, there exists a $B'_i \in \mathcal{B}_{ul}$ such that $B'_i \subseteq B_i$ for all i. Without loss of generality, we can apply this same argument the other way. For the element $B'_j = (a, b]$ there exists an ϵ such that $a < \epsilon < b$, thus we can create an element $B_j = [\epsilon, b)$ that exists within the element B'_j, showing for all $B'_j \in \mathcal{B}_{ul}$ there exists a $B_j \in \mathcal{B}_{ll}$ such that $B_j \subseteq B'_j$. □

Problem 5

Let $\mathcal{B}$ be a basis for a topology on a set X. Prove that $\mathcal{T}_{\mathcal{B}}$ is the intersection of all topologies on X containing $\mathcal{B}$. Conclude that $\mathcal{T}_{\mathcal{B}}$ is the smallest topology on X. (With respect to inclusion) containing $\mathcal{B}$.

Proof. To show that $\mathcal{T}_{\mathcal{B}}$ is the intersection of all topologies on X containing $\mathcal{B}$, we will show subset equality on both sides. Let U be an element of $\mathcal{T}_{\mathcal{B}}$. We want to show that for every i, U is an element of $\mathcal{T}_i$. Let $\mathcal{B} = \{B_\alpha\}_{\alpha \in I}$. Let $U = \bigcup_{\alpha \in I} B_\alpha$ (since it is an element of $\mathcal{T}_{\mathcal{B}}$) and $B_\alpha \in \mathcal{T}_{\mathcal{B}}$ for all α. By definition, B_α is an element of $\mathcal{T}_i$ for all α, i. Because $\mathcal{T}_i$ is a topology, and $B_\alpha \in \mathcal{T}_i$ for all α, i, we can conclude that $\bigcup_{\alpha \in I} B_\alpha \in \mathcal{T}_i$. Thus, U is an element of $\mathcal{T}_i$ for all i, showing $\mathcal{T}_{\mathcal{B}} \subseteq \bigcap_{i \in I} \mathcal{T}_i$. Showing $\bigcap_{i \in I} \mathcal{T}_i \subseteq \mathcal{T}_{\mathcal{B}}$ is trivial, as it is true by definition since $\mathcal{T}_{\mathcal{B}}$ exists within the set of topologies $\mathcal{T}_i$. Thus, $\mathcal{T}_{\mathcal{B}} = \bigcap_{i \in I} \mathcal{T}_i$. It then follows quickly since the intersection is the equal to $T_{\mathcal{B}}$, it must be the smallest topology on X containing $\mathcal{B}$. $\square$

Problem 6

Let X be a topological space and A, B be subsets of X.

a

If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$

Proof. Let $A \subseteq B$. If x is an element of $\bar{A}$, then via 79, every open set containing x intersects A. Hence, since A exists within B, every set containing x also intersects B. Thus, via 79, if every set containing x intersects B, then x is an element of $\bar{B}$. $\square$

b

$\overline{A \cup B} = \bar{A} \cup \bar{B}$

Proof. $\rightarrow$
We know the finite union of closed sets is closed. Hence, $\bar{A} \cup \bar{B}$ is closed. Thus, because it is closed, and $A \cup B$ is an element of $\bar{A} \cup \bar{B}$, $\overline{A \cup B}$ is an element of $\bar{A} \cup \bar{B}$. Thus, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. $\square$

Proof. $\leftarrow$
Let x be an element of $\bar{A} \cup \bar{B}$. Either x is an element of $\bar{A}$ or an element of $\bar{B}$. Hence, via 79, either every open set containing x intersects A or B. Thus, every open set containing x intersects $A \cup B$, thus x is an element of $\overline{A \cup B}$ $\square$

c

$\bigcup_{\alpha \in I} \bar{A}_\alpha \subset \overline{\bigcup_{\alpha \in I} A_\alpha}$

Proof. To show this is a subset, we will show every element of the set exists within the other. Let x be an element of $\bigcup_{\alpha \in I} \bar{A}_\alpha$. Then for some α, x is an element of $\bar{A}_\alpha$. Hence, via 79, for some α, all open sets containing x intersect As_α. Hence, all open sets containing x will intersect $\bigcup_{\alpha \in I} A_\alpha$. Hence, via 79, x is an element of $\overline{\bigcup_{\alpha \in I} A_\alpha}$. $\square$

d

Show that the equality does not hold in general for c.

Proof. To show the equality does not hold in general, we will show a counterexample. Let us look at $\mathbb{R}$ with the standard topology on $\mathbb{R}$. Let $\bigcup_{\alpha\in I} A_\alpha = \mathbb{Q}$, where each A_α is the singleton set $\{a\}$, where a is a rational number. The closure of a singleton set $\{a\}$ is the element a itself. Hence, $\bigcup_{\alpha\in I} \bar{A}_\alpha = \bigcup_{\alpha\in I}\{a\}_\alpha = \mathbb{Q}$. However, via 97, we know the closure of $\mathbb{Q}$ is equal to $\mathbb{R}$, and herein lies the counterexample! We now have $\overline{\bigcup_{\alpha\in I} A_\alpha} \subseteq \bigcup_{\alpha\in I} \bar{A}_\alpha$, which we can rewrite as $\mathbb{R} \subseteq \mathbb{Q}$, which is clearly incorrect. Thus, the equality does not hold in general. □

Meme