Combinatorics PS 3

Rippy

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Problem 1

Discover and prove a formula for the sum $\sum_{j=1}^{n}(-1)^j j^2$.

We can find a recursive formula for this. Writing out the first 10 iterations, we can find a pattern.

j	1	2	3	4	5	6	7	8	9	10
Sum	-1	3	-6	10	-15	21	-28	36	-45	55

We can see a pattern of a flipping sign and adding the numbers 1 to n.

Thus, we can make a conjecture that the formula is

$$(-1)^j\frac{j(j+1)}{2}$$

Base Case:

a_1, $\frac{1(2)}{2} = 1$ Inductive Step: Assume

$$a_k = (-1)^k\frac{k(k+1)}{2}$$

Prove the $k+1$

$$\begin{aligned}
a_{k+1} &= a_k + -1^{k+1}(k+1)^2 \\
&= (-1^k)\frac{k(k+1)}{2} + -1^{k+1}(k+1)^2 \\
&= -1^k\Big(\frac{k^2+k}{2} - k^2 - 2k - 1\Big) \\
&= -1^k\Big(\frac{k^2+k}{2} - \Big(\frac{2k^2+4k+2}{2}\Big)\Big) \\
&= -1^k\Big(- (\frac{k^2+3k+2}{2})\Big) \\
&= -1^{k+1}\Big(\frac{k^2+3k+2}{2}\Big) \\
&= -1^{k+1}\Big(\frac{(k+1)(k+2)}{2}\Big)
\end{aligned}$$

Which proves our formula.

Problem 2

How many unlabeled graphs are on 4 vertices? There are 11 unlabeled graphs on 4 verities using direct counting. (If I need to show more work here, I can also send in my sketches of each graph)

Problem 3

Find the number of:

(a)

5-5 walks of length 8 in the Graph G_1 of Fig. 6.1.
We must start and end at 5, so we have the walk $\{5, u_1, u_2, \cdots, u_n, 5\}$. We can count this using an adjacency matrix.

In[2]:= $B = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix};$

In[6]:= `B.B.B.B.B.B.B.B // MatrixForm`

Out[6]//MatrixForm=

$$\begin{pmatrix} 688 & 1073 & 0 & 899 & 920 & 675 \\ 1073 & 1736 & 0 & 1446 & 1446 & 1073 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 899 & 1446 & 0 & 1235 & 1201 & 920 \\ 920 & 1446 & 0 & 1201 & 1235 & 899 \\ 675 & 1073 & 0 & 920 & 899 & 688 \end{pmatrix}$$

Figure 1: The Adjacency Matrix and The 8-walk product

Thus we have 1235 possible 5-5 walks of length 8.

(b)

The number of walks of length 8 in the cube graph Q_3 can be found using an adjacency matrix.

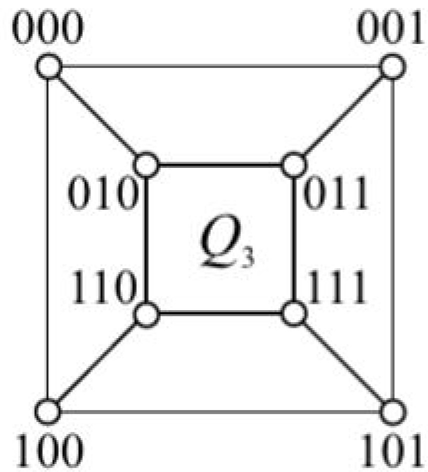

Figure 2: Q_3

In[1]:= $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix};$

In[3]:= `A.A.A.A.A.A.A.A // MatrixForm`

Out[3]//MatrixForm=

$$\begin{pmatrix} 1641 & 0 & 0 & 1640 & 0 & 1640 & 1640 & 0 \\ 0 & 1641 & 1640 & 0 & 1640 & 0 & 0 & 1640 \\ 0 & 1640 & 1641 & 0 & 1640 & 0 & 0 & 1640 \\ 1640 & 0 & 0 & 1641 & 0 & 1640 & 1640 & 0 \\ 0 & 1640 & 1640 & 0 & 1641 & 0 & 0 & 1640 \\ 1640 & 0 & 0 & 1640 & 0 & 1641 & 1640 & 0 \\ 1640 & 0 & 0 & 1640 & 0 & 1640 & 1641 & 0 \\ 0 & 1640 & 1640 & 0 & 1640 & 0 & 0 & 1641 \end{pmatrix}$$

Figure 3: The Adjacency Matrix

Thus we can see there are 0 walks of length 8 between points 000 and 001.

(c)

We can again use an adjacency matrix to do this. WLOG, I will arbitrarily choose the labels for the entries (since we can just rotate the cycle otherwise)

In[7]:= $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix};$

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In[9]:= A.A.A.A.A.A.A.A // MatrixForm
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Out[9]//MatrixForm=

$$\begin{pmatrix} 70 & 36 & 57 & 57 & 36 \\ 36 & 70 & 36 & 57 & 57 \\ 57 & 36 & 70 & 36 & 57 \\ 57 & 57 & 36 & 70 & 36 \\ 36 & 57 & 57 & 36 & 70 \end{pmatrix}$$

Figure 4: The adjacency matrix for a generic C_5 graph

Thus, we see, the number of walks of length 8 between any two adjacent vertices is 36.

Problem 4

Prove a forest with n vertices and k components has $n - k$ edges. We know from class that any tree with n vertices has $n - 1$ edges. Thus, for each tree component in the forest, it will have 1 less edge than the total number of vertices in that tree component. So, for each component in the forest, we subtract 1 possible edge from the entire edge set, so if we have k tree components, we have k less edges. Thus, the number of edges in a forest of n vertices and k components is $n - k$.

Challenge

If G is isomorphic to $\bar{G}$, then we know the number of edges in both are equal. Since $\bar{G}$ is the complement of G relative to the K_n graph, where n is the number of vertices in G, we know that for the number of edges to be equal, the total number of edges in K_n must be an even. The total number of edges in a K_n is given by $\dfrac{n(n-1)}{2}$. If we express n as multiple of four plus a remainder of $0, 1, 2, 3$, we can rewrite n as $4k + b$. We can then rewrite the number of edges in a K_n graph as

$$\begin{aligned}
&\frac{n(n-1)}{2} \\
&= \frac{(4k+b)(4k+b-1)}{2} \\
&= \left(8k^2 + 4bk - 2k\right) + \frac{1}{2}\left(b^2 - b\right)
\end{aligned}$$

We can note that the term $\left(8k^2 + 4bk - 2k\right)$ will *always* be an even number regardless of what k or b happen to be, since each term is multiplied by a multiple of 2. For the $\frac{1}{2}\left(b^2 - b\right)$ term, this comes out to be:

b	0	1	2	3
$\frac{1}{2}\left(b^2 - b\right)$	0	0	1	3

The b value directly corresponds to the mod value, ie the remainder after dividing by 4. Thus, if the modulo remainder is 2 or 3, then the $\frac{1}{2}\left(b^2 - b\right)$ term is 1,3 respectively. Since we are adding an even number to this term, this results in an odd number of edges. Thus, if G is isomorphic to $\bar{G}$, then $n(G) \bmod 4 = 0, 1$.