Complex Problem Set 5

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Problem 1

To complete this problem, we utilize the following formula:

$$\text{Res}(f,\infty) = -\text{Res}\Big(\frac{1}{z^2}f(\frac{1}{z}),0\Big)$$

Plugging in f in this form, we get:

$$\frac{2z^{-1}-1}{z^2z^{-2}(z^{-1}+1)(z^{-1}-i)}$$

Simplifying:

$$\frac{2z^{-1}-1}{(z^{-1}+1)(z^{-1}-i)}$$

Multiply by $\frac{z^3}{z^3}$ and we get:

$$\frac{2z^2-z^3}{(z)(1+z)(1-iz)} \tag{1}$$

And now we evaluate at 0 this to find the residue at infinity:

$$\frac{0-0}{(1)(1)}$$

And, we have the residue is 0! This is consistent with the previous problem set's number 3 result.

Problem 2

From the theorem in class (Omitting much of the PDA involved in showing why this works), we can utilize a semi-circular path in the positive Imaginary plane and along the Real axis from -R to R, with the limit as R goes to infinity. As a result of this setup, we need only the residues in the upper plane to find the answer, and can ignore the others. We have the following from the theorem:

$$\lim_{R\to\infty} f(x)dx = 2\pi i\sum_{k=1}^{n}\text{Res}_{z=z_k}f(z) - \int_{C_R} f(z)dz$$

First we take care of the C_R arc. Since we are looking at R as it goes to ∞, R is subsequently large enough we can bound $\mid z \mid^2 = R^2$. Through triangle ineq we can bound $\mid z^2+1 \mid\mid z^2+4 \mid \leq (\mid z \mid^2 -1)(\mid z \mid^2 -4)$. We have then our $M_R = \frac{R^2}{(R^2-1)(R^2-4)}$. Thus, we can bound the following:

$$\mid f(z) \mid = \frac{\mid z \mid^2}{(\mid z \mid^2 -1)(\mid z \mid^2 -4)} \leq M_R$$

Then we can bound the integral using the arclength to be:

$$\mid \int_{C_R} f(z)dz \mid \leq M_R \pi R$$

Our bound now becomes:

$$\frac{\pi R^3}{(R^2-1)(R^2-4)}$$

This effectively becomes $\frac{R^3}{R^4}$, which since we are taking the limit as R goes to infinity, effectively becomes 0.

Now for the residues. Next, we factor the function in question into:

$$\frac{x^2}{(x+i)(x-i)(x+2i)(x-2i)}$$

From here, we calculate the two residues at i and 2i. At i we have:

$$\frac{-1}{(2i)(-i)(3i)} = \frac{i}{6}$$

At 2i we have:

$$\frac{-4}{(i)(3i)(4i)} = -\frac{i}{3}$$

Thus, as an answer we have for the total integral (from $-\infty$ to ∞) we have:

$$(2\pi i)(-\frac{i}{6}) = \frac{\pi}{3}$$

Now, since we only want the integral from 0 to ∞, and since this is an even function, we can use this symmetry to divide this answer in two to get the final answer:

$$\frac{\pi}{6}$$

Problem 3

Proof. Let $f(z) = \frac{az+b}{cz+d}$, and set $f(z_1) = f(z_2)$ and assume $ad \neq bc$. We now have:

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$$
$$(az_1+b)(cz_2+d) = (cz_1+d)(az_2+b)$$

Do some expanding...

$$acz_1z_2 + cbz_2 + adz_1 + bd = acz_1z_2 + cbz_1 + adz_2 + bd$$

Getting rid of the same terms on either side...

$$cbz_2 + adz_1 = cbz_1 + adz_2$$

Further simplifying...

$$cb(z_2 - z_1) = ad(z_2 - z_1)$$

Further simplifying...

$$(z_2 - z_1)(bc - ad) = 0$$

Since $bc \neq ad$, we divide this out...

$$z_2 = z_1$$

Thus, the function is 1 to 1 provided that $bc \neq ad$. $\square$

Problem 4

Let $f(z) = ze^z$. We consider $z_0 = i$.

a

$f(i) = ie^i$. To prove it is conformal, we show the derivative at this point is not zero. We have $f'(z) = e^z + ze^z$, which at i is $e^i + ie^i$, which is non-zero. Thus, the mapping is conformal at i.

b

To find the scale factor, we take the modulus of the derivative at i. We have:

$$\mid e^i + ie^i \mid = \mid e^i(1+i) \mid = \sqrt{2}$$

Next we find the argument of $f'(z)$ (to find the angle of rotation):

$$\text{Arg}(i + ie^i) = \text{Arg}(e^i(1+i)) = \text{Arg}(\sqrt{2}e^{1+\frac{\pi}{4}}) = \frac{\pi}{4} + 1$$

**Important sidenote, your notes say Arg($f(z)$) not Arg($f'(z)$), but I'm pretty sure it is supposed to be Arg($f'(z)$).

c

Via the theorem in class, we know the inverse's scale factor is the invert of $f(z)$ and the angle of rotation is negative that of $f(z)$. Thus...

$$\text{Scale factor} = \frac{1}{\sqrt{2}}$$

$$\text{Angle of Rotation} = -(\frac{\pi}{4} + 1)$$

Problem 5

We have:

$$\Gamma = f(C) = f(z(t))$$

We now prove $\Gamma' \neq 0$ to show it is smooth (since continuity follows from the continuous function mapping a continuous arc).

$$\Gamma' = f'(z(t))(z'(t))$$

We are given C is smooth, thus the parameterization's derivative, $z'(t)$, is non-zero. We are also given $f(z)$ is conformal, thus $f'(z) \neq 0$. Thus, neither of these terms are non-zero, showing Γ is smooth!

Problem 6

Using the theorem we know we can express the following:

$$H(x, y) = h[u(x, y), v(x, y)]$$

We are given:

$$h(u, v) = \text{Re}(w^2)$$

We now do some algrbra....

$$\begin{aligned} &\text{Re}(e^{2z}) \\ &= \text{Re}(e^{2x+2iy}) \\ &= \text{Re}(e^{2x}(\cos(2y) + i\sin(2y)) \\ &= e^{2x}\cos(2y) \end{aligned}$$

Which is consistent with the theorem's form, proving it is harmonic. Thus, $H(x, y) = e^{2x}\cos(2y)$

Problem 7

The following mapping meets the requirements:

$$f(z) = e^{i\pi}\frac{(z-1)}{(z-i)}$$

Where $\alpha = \pi, z_0 = 1, z_1 = i$. To find these, we found values z_0, z_1 whose modulus equalled one another along the split line (Im(z) = Re(z)).