Problem Set 3

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September 2020

Problem 1

(a)

We have

$$
\begin{aligned}
P_3(x) &= f_0L_0 + f_1L_1 + f_2L_2 + f_3L_3 \\
P_3'(x) &= f_0L_0' + f_1L_1' + f_2L_2' + f_3L_3' \\
x_{j+1} - x_j &= h
\end{aligned}
$$

Where

$$
\begin{aligned}
L_0 &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-x_1)(x-x_2)(x-x_3)}{-6h^3} \\
L_1 &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-x_0)(x-x_2)(x-x_3)}{2h^3} \\
L_2 &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-x_0)(x-x_1)(x-x_3)}{-2h^3} \\
L_3 &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-x_0)(x-x_1)(x-x_2)}{6h^3}
\end{aligned}
$$

And

$$
\begin{aligned}
L_0' &= \frac{1}{-6h^3}\Big((x-x_2)(x-x_3) + (x-x_1)(x-x_3) + (x-x_1)(x-x_2)\Big) \\
L_1' &= \frac{1}{2h^3}\Big((x-x_2)(x-x_3) + (x-x_0)(x-x_3) + (x-x_0)(x-x_2)\Big) \\
L_2' &= \frac{1}{-2h^3}\Big((x-x_1)(x-x_3) + (x-x_0)(x-x_3) + (x-x_0)(x-x_1)\Big) \\
L_3' &= \frac{1}{6h^3}\Big((x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)\Big)
\end{aligned}
$$

To then find the stencil, we have the formula:

$$
P_3'(x) = f_0L_0'(x) + f_1L_1'(x) + f_2L_2'(x) + f_3L_3'(x) \tag{1}
$$

For the forward stencil, we plug in x_0, then simplify in terms of h. This gives us:

$$P'(x_0) = \frac{1}{h}(f_0(-\frac{11}{6}) + f_1(3) + f_2(-\frac{3}{2}) + f_3(\frac{1}{3}))$$

Further simplifying, we get the forward stencil $\frac{1}{h}[-\frac{11}{6}, 3, -\frac{3}{2}, \frac{1}{3}]$, confirming the result.
For the two center stencils, we evaluate Eq. 1 at x_1 and x_2, and for the backward stencil, we evaluate Eq. 1 at x_3. This gives us:

$$P_3'(x_1) = \frac{1}{h}(f_0(-\frac{1}{3}) + f_1(-\frac{1}{2}) + f_2(1) + f_3(-\frac{1}{6}))$$
$$P_3'(x_2) = \frac{1}{h}(f_0(\frac{1}{6}) + f_1(-1) + f_2(\frac{1}{2}) + f_3(\frac{1}{3}))$$
$$P_3'(x_3) = \frac{1}{h}(f_0(-\frac{1}{3}) + f_1(\frac{3}{2}) + f_2(-3) + f_3(\frac{11}{6}))$$

(b)

the error for $P_3'(x_j)$ is given by

$$\frac{f^{(4)}(\xi)}{(4!)}\prod_{k=0,k\neq j}^{3}(x_j - x_k)$$

So, for each of the stencils, the product evaluates to 3 multiples of h times some constants, (which we ultimately don't care about, since we are only interested in Big O). For example, the error on the forward stencil, at x_0, becomes the constant stuff out front, $\frac{f^{(4)}(\xi)}{(4!)}$ times $(x_0-x_1)(x_0-x_2)(x_0-x_3)$, which, when put in terms of h becomes $(-h)(-2h)(-3h) = -6h^3$, which makes the error $\frac{f^{(4)}(\xi)}{(4!)} * (-6h^3)$ which is $O(h^3)$. This is easily repeatable for the other stencils, as the product always creates a multiple of h^3, times some constant, making each stencil's error $O(h^3)$

Problem 2

(a)

x_j	1.0	1.5	2	2.5	3
$f(x_j)$	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$
$f'(x_j) \approx$	$-\frac{9}{10}$	$-\frac{7}{15}$	$-\frac{7}{30}$	$-\frac{7}{45}$	$-\frac{11}{90}$
exact $f'(x_j)$	-1	$-\frac{4}{9}$	$-\frac{1}{4}$	$-\frac{4}{25}$	$-\frac{1}{9}$
$\mid$ error $\mid$	$\frac{1}{10}$	$\frac{1}{45}$	$\frac{1}{60}$	$\frac{1}{225}$	$\frac{1}{90}$

(b)

Using the stencil with $h = 0.25$, we approximate the derivative at 1.5 to be $-\frac{47}{105}$. The exact derivative at this point is $-\frac{4}{9}$. The stencil with $h = 0.5$ approximates it to be $-\frac{7}{15}$. The error for $h = 0.25$ is 0.00317, and the error when $h = 0.5$ is $\frac{1}{45}$. The error when $h = 0.25$ is 7 times less than when $h = 0.5$. This is consistent with $O(h^3)$, which is what we expected when we derived the error for this stencil.

Problem 3

(a)

Using the composite Trapezoid Rule $\frac{h}{2}[1, 2...2, 1]$ we find the approximation to be: 3.9625

(b)

Using the composite Simpson's Rule $\frac{h}{3}[1, 4, 2, 4, 2...4, 1]$ we find the approximation to be: 3.9917

Problem 4

(a)

Exact value $= -0.1972245$

(b)

The roots of the second order polynomial are $\pm\frac{\sqrt{3}}{3}$. Using these along with the coefficients, 1 and 1, we find the integral to be equal to -0.181818

(c)

The roots of the third order polynomial are $0, \pm\sqrt{\frac{3}{5}}$. Using these along with the coefficients, $\frac{5}{9}, \frac{8}{9}, \frac{5}{9}$ respectively, we find the integral to be equal to -0.196078

Problem 5

Using the strategy outlined in class, we create the following system of equations:

$$c_1 + c_2 = \int_0^1 1 = 1$$
$$c_1\sqrt{x_1} + c_2\sqrt{x_2} = \int_0^1 \sqrt{x} = \frac{2}{3}$$

Using this system of equations, we solve for c_1 and c_2 to be:

$$c_1 = 0.62650$$
$$c_2 = 0.37349$$

To confirm this, we will use $\int_0^1 3 - 2\sqrt{x}$. Using calculus, the answer we'd expect is $\frac{5}{3}$. Using Guassian quadrature, we get:

$$0.625650 * (3 - 2\sqrt{\frac{1}{3}}) + 0.37349 * (3 - 2\sqrt{\frac{2}{3}}) = 1.667 = \frac{5}{3}$$

Thus confirming our answer.