Combinatorics PS 4

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Problem 1

Give a combinatorial proof that:

$$\tau(G) = \tau(G - e) + \tau(G \cdot e)$$

A tree has a total of $n - 1$ edges, where n is the number of nodes in G. As we traverse through the graph, we can choose how to add our node based on the edges that connect that node in the graph. In the case without e, $\tau(G - e)$, we calculate all the possible spanning trees where we have no e. In the case for $\tau(G \cdot e)$, since e is just contracted in the case, the two nodes that were pushed together still retain all their other edges. If we calculate the number of spanning trees in this case, it's as if we are baking in the assumption we are always including e since we are utilizing the merged node in every case, which contains e, so this equivalently becomes the case always with e. Since there are only two possible cases, either we include e or don't, these two are complements of one another, and together form the total number of spanning trees for the graph G.

Problem 2

Some useful info:
Total degree of a tree is $2n - 2$. A planar graph should have at most $3n - 6$ edges.

(a)

Total degree is 15, this is odd and not a simple graph.

(b)

- 10 vertices, Ore's Thrm, every vertex does not have a degree of at least 5, thus we cannot tell if this is Hamiltonian or not.
- All degrees are even, thus this is Eulerian.
- 10 verticies, total degree should be 18 to be a tree, in this case it is 36, thus this is not a tree.

- Via Handshake Lemma, 18 edges, a planar graph should have at most 24 edges, so this could be planar.

(c)

- 7 vertices, Ore's Thrm, every vertex does not have a degree of at least 4, thus we cannot tell for certain if this graph is Hamiltonian or not.
- More than 2 degrees are odd, not Eulerian.
- 7 verticies, total degree should be 12 to be a tree, in this case it is 22, thus this is not a tree.
- Via Handshake Lemma, 11 edges, a planar graph should have at most 15 edges, so this could be planar.

(d)

- 10 vertices, Ore's Thrm, every vertex does not have a degree of at least 5, thus we cannot tell for certain if this graph is Hamiltonian or not.
- More than 2 degrees are odd, not Eulerian.
- 10 verticies, total degree should be 18 to be a tree, in this case it is 50, thus this is not a tree.
- Via Handshake Lemma, 25 edges, a planar graph should have at most 24 edges, so this is definitely not planar.

Problem 3

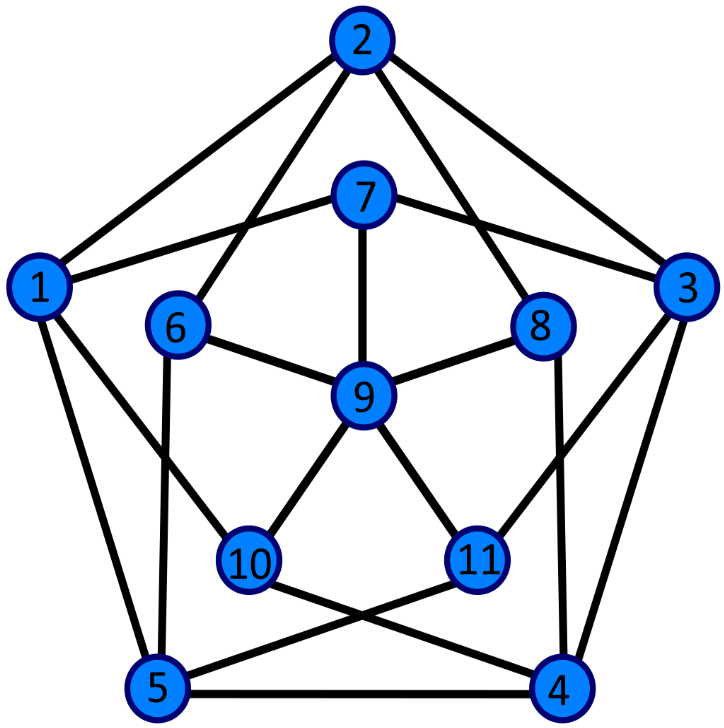

Figure 1: The Grötzsch Graph

This graph is Hamiltonian, a path to prove this is 2,6,5,11,3,7,1,10,4,8,9

Problem 4

Since the center vertex connects to all others in the cycle surrounding it, this adds one more color required to that of the cycle surrounding it. Given n is odd, then the cycle around it must be even, we know for even cycles, the minimum coloring is 3, however if n is even, then the cycle around it must be odd, and thus only requires 2 colors to color it. Thus, if n is odd, the coloring is 4, if n is even, the coloring is 3.

Challenge

Proof. Assume $a, b \geq 3$. Define $n = R(a-1, b) + R(a, b-1) - 1$. Consider an arbitrary red-blue coloring of K_n. n is odd because even + even - 1 = odd. Consider the subgraph of K_n with only red edges. Since n is odd, there exists at least one vertex with even degree, so vertex 1 is incident to an odd number of red edges.

Case 1: There are at least R(a-1,b) -1 red edges incident to vertex 1. Since vertex 1 is incident to an even number of red edges, it is incident to at least R(a-1,b) red edges. Look at R(a-1,b) subgraph of vertices adjacent to vertex 1 by red edges. Since there are R(a-1,b) red vertices, this subgraph either has a complete red K_{a-1} or complete blue K_b. If it has a blue K_b, done. If it has a red K_{a-1} add vertex 1 to get a red K_a.

Case 2: There are at least R(a,b-1) blue edges incident to vertex 1. Since vertex 1 is incident to an even number of blue edges, it is incident to atleast R(a,b-1) blue edges. Since there are R(a,b-1) blue vertices this subgraph either has a complete blue K_{b-1} or complete red K_a. If it has a red K_a, done. If it has a blue K_{b-1}, add vertex 1 to get a blue K_b.

$\square$