Abstract Problem Set

Rippy

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1 Problems

1.1 Problem 1

Let $f : A \to B$ and $g : B \to C$ be functions

(a) If $g \circ f$ is 1-1, then f is 1-1.
(b) If $g \circ f$ is onto, then g is onto.
(c) Are the converses of the statements in (a) and (b) true? If so, prove your answer. If not, provide a counterexample.

Let $x, y \in A$

Proof. (a)

If $g \circ f$ is 1-1, this implies if $g \circ f(x) = g \circ f(y)$ then $x = y$. Assume $f(x) = f(y)$ then $g \circ f(x) = g \circ f(y)$ which implies $x = y$, so if $f(x) = f(y)$ and $x = y$, then f is 1-1.

□

Proof. (b)

If $g \circ f$ is onto, we know for every $c \in C$, there exists some $a \in A$ such that $g \circ f(a) = c$. Let $f(a) = b \in B$. Then rewriting the previous statement, for every $c \in C$ there exists some $b \in B$ such that $g(b) = c$, thus g is onto.

□

Proof. (c)

(a) If f is 1-1, we know $f(x) = f(y)$ implies that $x = y$. Let $x \neq y$ and let $g : B \to C$ be the trivial map, that is, $g \circ a = e$ such that $a \in B$ and $e \in C$. Then $g \circ f(x) = g \circ f(y) = e$, however $x \neq y$, therefore, if f is 1-1, g need not be 1-1.

(b) If g is onto, we know the image of $g : B \to C$ is C on the given domain. However, f need not be onto. Let f be the trivial transformation, Let C consist of elements a, b. The image of of f would be e, and thus the image of $g \circ f$ would not be onto C. Thus, $g \circ f$ is not necessarily onto if g is onto.

□

1.2 Problem 2

Determine whether the following functions satisfy the homomorphic property. If a function does satisfy the homomorphic property, then prove whether or not it is an isomorphism. **The homomorphic property is $\phi(xy) = \phi(x)\phi(y)$

(a) det : $(M_{n\times n}(\mathbb{R}), +) \to (\mathbb{R}, +)$, where det is the determinant map

Proof. (a)
Let A, B be diagonal matrices $\in M_{n\times n}(\mathbb{R})$ with diagonal entries $a_{11}, a_{22}, \cdots, a_{nn}$ and $b_{11}, b_{22}, \cdots, b_{nn}$ respectively. $[det(A+B) = (a_{11}+b_{11})(a_{22}+b_{22})\cdots(a_{nn}+b_{nn})] \neq [(a_{11})(a_{22})\cdots(a_{nn}) + (b_{11})(b_{22})\cdots(b_{nn}) = detA + detB]$ Thus ϕ does not satisfy the homomorphic property for $n > 1$, as we know $det(A+B) \neq detA + detB$.

Let $n = 1$, and let A, B be matrices $\in M_{1\times 1}(\mathbb{R})$ with entries a_{11} and b_{11} respectively. $det(A+B) = a_{11} + b_{11} = (a_{11}) + (b_{11}) = detA + detB$, thus for $n = 1$, ϕ satisfies the homomorphic property. We know $detA = a_{11}$ and $detB = b_{11}$, so if $detA = detB$, this implies $a_{11} = b_{11}$, which implies $A = B$, thus ϕ is 1-1 for $n = 1$. For all $c \in \mathbb{R}$, there exists some matrix $A \in M_{1\times 1}(\mathbb{R})$ with entry $a_{11} = c \in \mathbb{R}$ such that $detA = c$, therefore ϕ is onto for $n = 1$. Thus, ϕ is an isomorphism for $n = 1$.

□

(b) det : $(GL_n(\mathbb{R}), +) \to (\mathbb{R}\backslash\{0\}, \cdot)$

Proof. (b)
Let $A, B \in GL_n(\mathbb{R})$. ϕ does satisfy the homomorphic property, as $det(AB) = (detA)\cdot(detB)$ because this is a property of determinants.

However ϕ is not 1-1 for $n > 1$.

Let $A =$ be a $n \times n$ Matrix such that $A = \begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$

Let $B =$ be a $n \times n$ Matrix such that $B = \begin{bmatrix} 2 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$

We know $A \neq B$ however, $detA = detB$, thus ϕ is not 1-1 for $n > 1$, and thus not isomorphic for $n > 1$.

Let $n = 1$, and let $A, B \in GL_1(\mathbb{R})$ with entries $a_{11} \in \mathbb{R}\backslash\{0\}$ and $b_{11} \in \mathbb{R}\backslash\{0\}$ $detA = a_{11}$

and $detB = b_{11}$, so if $detA = detB$, this implies $a_{11} = b_{11}$ which implies $A = B$, thus ϕ is 1-1. Let $A \in GL_1(\mathbb{R})$ with entry $a_{11} \in \mathbb{R}\backslash\{0\}$. For all $c \in \mathbb{R}\backslash\{0\}$, there exists some matrix A with entry $a_{11} = c \in \mathbb{R}\backslash\{0\}$ such that $detA = c$. Therefore ϕ is onto. And thus, ϕ is an isomorphism for $n = 1$.

□

(c) tr : $(M_{n\times n}(\mathbb{R}), +) \to (\mathbb{R}, +)$, where tr is the trace map, the sum of the diagonal entries of a matrix

Proof. (c)
Let $A, B \in M_{n\times n}(\mathbb{R})$. Let the diagonal entries of A, B be denoted as $a_{11}, a_{22}, \cdots, a_{nn}$ and $b_{11}, b_{22}, \cdots, b_{nn}$ respectively. ϕ does satisfy the homomorphic property, as $tr(A+B) = a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = (a_{11} + a_{22} + \cdots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn}) = tr(A) + tr(B)$.

However, ϕ is not 1-1 for $n > 1$

Let $A =$ be a $n \times n$ Matrix such that $A = \begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$

Let $B =$ be a $n \times n$ Matrix such that $B = \begin{bmatrix} 2 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$

$trA = 2 = trB$. Since $A \neq B$ and $trA = trB$, ϕ is not 1-1, and thus not an isomorphism for $n > 1$.

Let n = 1. Let, $A, B \in M_{1\times 1}(\mathbb{R})$ with entries $a_{11} \in \mathbb{R}$ and $b_{11} \in \mathbb{R}$ respectively. $trA = a_{11}$ and $trB = b_{11}$, thus if $trA = trB$, this implies $a_{11} = b_{11}$, which implies $A = B$, thus ϕ is 1-1 Let $c \in \mathbb{R}$, for all $c \in \mathbb{R}$, there exists some matrix $A \in M_{1\times 1}(\mathbb{R})$ with entry $a_{11} = c \in \mathbb{R}$ such that $trA = c$. Therefore, ϕ is onto. Thus, ϕ is an isomorphism for $n = 1$.

□

(d) $\lfloor\cdot\rfloor : (\mathbb{R}, +) \to (\mathbb{Z}, +)$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Proof. (d)
ϕ does not satisfy the homomorphic property. Let $a, b \in \mathbb{R}$ such that $a = 4.5$ and $b = 5.5$. $(\lfloor(4.5 + 5.5)\rfloor = 10) \neq (\lfloor 4.5 \rfloor + \lfloor 5.5 \rfloor = 9)$

□

(e) $\phi : (C(\mathbb{R}), +) \to (\mathbb{R}, +)$, where $C(\mathbb{R})$ is the set of all continuous functions $\mathbb{R} \to \mathbb{R}$ and $\phi(f) = \int_0^1 f(x)dx$

Proof. Let $f(x), g(x) \in C(\mathbb{R})$
ϕ does satisfy the homomorphic property as $\phi(f(x)+g(x)) = \int_0^1 f(x)+g(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = \phi(f(x)) + \phi(g(x))$

However, for ϕ is not 1-1
Let $f(x) = 2x$ and $g(x) = 1$, $f(x) \neq g(x)$
$\phi(f(x)) = \int_0^1 f(x)dx = 1$
$\phi(g(x)) = \int_0^1 g(x)dx = 1$
Thus ϕ is not 1-1

□

1.3 Problem 3

Prove there is no isomorphism from $(\mathbb{Q}, +)$ to $(\mathbb{Q}\backslash\{0\}, \cdot)$.

Proof. Let an isomorphism ϕ : $(\mathbb{Q}, +) \to (\mathbb{Q}\backslash\{0\}, \cdot)$ exist. Let $a \in \mathbb{Q}$ such that $\phi(a) = b \in \mathbb{Q}\backslash\{0\}$. $\mathbb{Q}$ is defined as all numbers that can be expressed as a fraction, thus if a exists in $\mathbb{Q}$, $\frac{a}{2}$ must also exist in $\mathbb{Q}$. We know $a = \frac{a}{2} + \frac{a}{2}$, and since ϕ is an isomorphism, $\phi(a) = \phi(\frac{a}{2}+\frac{a}{2}) = \phi(\frac{a}{2}) \cdot \phi(\frac{a}{2}) = \phi(\frac{a}{2})^2 = b$. $\phi(\frac{a}{2})^2 = b$ can simplify to $\phi(\frac{a}{2}) = \sqrt{b}$. The square root operation yields numbers not in $\mathbb{Q}$, $\sqrt{b} \notin \mathbb{Q}$ or $\mathbb{Q}\backslash\{0\}$, thus ϕ is not an isomorphism. For example, let $a = 6$, then $\frac{6}{2} = 3 \in \mathbb{Q}$, then $\phi(3)^2 = b$ and then $\sqrt{b} = \sqrt{3} \notin \mathbb{Q}$ or $\mathbb{Q}\backslash\{0\}$, thus proving ϕ is not an isomorphism. □

1.4 Problem 4

If G is cyclic and innite, then G $\cong (\mathbb{Z}, +)$.

Proof. We know $\mathbb{Z}$ is cyclic and infinite, let $\langle 1 \rangle$ be the generator of the $\mathbb{Z}$. Suppose G is also cyclic and infinite, let $\langle g \rangle$ be the generator. Let $\phi : \mathbb{Z} \to G$ be defined such that $k \to g^k$.

It would satisfy the homomorphic property, as $\phi(k)\phi(l) = g^k g^l = g^{k+l} = \phi(k+l)$

Assume ϕ is not 1-1
Let $\phi(k) = \phi(b)$, $k \neq b$, then $\phi(k) = g^k = g^b = \phi(b)$ However, since $G, \mathbb{Z}$ are cyclic and infinite, by Theorem 4.17, if $\langle g \rangle$ is infinite, there does not exist an finite n such that $g^n = e$. Therefore if $g^k = g^b$, then without loss of generality, $b > k$, $g^{-k}g^b = e = g^{b-k}$ Since $b - k$ is a finite number, this is a contradiction. Therefore ϕ must be 1-1

Given ϕ is 1-1, we know every $k \in \mathbb{Z}$ maps to a unique element $g^k \in G$. This implies if $g^m = g^n$, $m = n$. Because $|G| = |\mathbb{Z}| = \infty$, for any element $g^k \in G$ there exists an element $k \in Z$ such that $\phi(k) = g^k$. Therefore, ϕ is onto.

Thus G $\cong (\mathbb{Z}, +)$

$\square$