Topology Super Fun Take Home Exam 1

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Problem 1

a

Consider $\mathbb{R}$ with the standard topology, and let $A = (a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Prove $\text{Int}(A) = (a, b)$ and $\bar{A} = [a, b]$. Is A open, closed, neither, or both?

Proof. Let U be an open set in $\mathbb{R}$ with the standard topology. Then, U is a union of open intervals, which is itself an open interval. A is $(a, b]$. The union of all open sets within A is (a, b). The largest open interval within A is (a, b) since we cannot include b without including something outside the bound (by definition of an open bound), so b is the maximum upper bound of an open interval within A, and a itself is an open bound. All other open intervals within A are contained within (a, b). Thus, $\text{Int}(A) = (a, b)$. The closure is the intersection of all closed sets containing A. Closed sets required that their complement in open. All closed sets contain A at a minimum (by definition), and the minimum closed set that still contains A is $[a, b] = \Big((-\infty, a) \cup (b, \infty)\Big)^c$. This is the minimum closed set, since any bound larger than a on the open interval would no longer include all elements of A in its complement, and there can be no smaller bound than b, since any smaller would also no longer include all elements of A. Thus, the intersection becomes the smallest closed set containing A, which is $[a, b]$.
A is neither open or closed. Because $\text{Int}(A) \neq A$, A is not open. $A^c = (-\infty a] \cup (b, \infty)$, the interval $(-\infty, a])$ cannot be written as a union of open intervals, thus A^c is not open, and A is not closed. □

b

Show any interval of the form $(a, b]$ is both open and closed in $\mathbb{R}_u$ (Upper Limit Topology)

Proof. The interval $(a, b]$ is exactly a open basis element of R_u, and thus is open. The complement of the interval $(a, b]$ is $(-\infty, a] \cup (b, \infty)$. We can express both of these intervals as a union of open elements $(c, d]$ in $\mathbb{R}_u$. For $(-\infty, a]$, since we are in the upper limit topology, we can include the upper bound a in the element $c, a]$, and extend the union on to negative infinity, giving us the interval. Similarly, for (b, ∞), we have the open bound b in an open element of the topology, $(b, d]$ and we can extend the union on towards infinity. This gives us $(b, \infty) = \bigcup_{r_i \in \mathbb{R}} (b, r_i] \mid r_{i+1} < r_i$ and $(-\infty, a]) = \bigcup_{r_i \mathbb{R}} (-r_i, a] \mid r_{i+1} < r_i$. Thus since we can write the complement as a combination of basis elements, the complement is open, and thus the interval is closed. □

c

Show that R_{cc} (Countable Complement Topology) has no nontrivial clopen sets.

Proof. All nontrivial open elements that exist within $\mathbb{R}_{cc}$ must be of the form $\mathbb{R}-$ {countable set}, since $\mathbb{R} - \{\mathbb{R} -$ countable set$\}$ = {countable set}. If the element is open, then it's complement is open, so the complement of the complement is the countable set. However, $\mathbb{R}-$ {countable set} $\neq$ {countable set}, since $\mathbb{R}$ is uncountable, and an uncountable - countable is uncountable, so the complement is not open. Thus, there are no nontrivial clopen sets. (WLOG, this is the same the other way). If a closed element, B, existed in $\mathbb{R}$, it's complement is open, thus B is countable. If B is countable, it's complement is uncountable, since $\mathbb{R} - B$ is not coutable. (Same idea, uncountable - countable = uncountable). Thus, a nontrivial closed set cannot be open. Thus, there are no nontrivial clopen sets in R_{cc}. □

Problem 2

a

Prove that $\mathcal{T}$ is a topology by showing that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis.

Proof. We will show the two criteria to be a basis:
(1) Since we can intersect with nothing, every element of $\mathcal{S}$ is contained within $\mathcal{B}$. Since all elements of $\mathcal{S}$ union to create X, there exists an element for every $x \in X$ in $\mathcal{S}$ such that $x \in S_x$. Thus, for each $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$.
(2) Let B_1, B_2 be elements of $\mathcal{B}$, and let $x \in B_1 \cap B_2$. Since $\mathcal{B}$ is the collection of all finite intersections of elements of $\mathcal{S}$, all elements $B \in \mathcal{B}$ can be written as $\bigcap S_n$ where S_n are elements in $\mathcal{S}$. Thus, $B_1 \cap B_2$ can be rewritten as $\left(\bigcap S_n\right) \cap \left(\bigcap S_m\right)$, which, is exactly a finite intersection of elements of $\mathcal{S}$, thus there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.
Thus, $\mathcal{B}$ is a basis, and consequently, $\mathcal{T}$ is a topology.

□

b

Let $\mathcal{S} = \{(-\infty, a) | a \in \mathbb{R}\} \cup \{(a, \infty) | a \in \mathbb{R}\}$. Show $\mathcal{S}$ is a subbasis, but not a basis.

Proof. Consider the elements in $\mathcal{S}$, $(-\infty, a)$ and $(a - \epsilon, \infty)$, where $\epsilon > 0$. These elements union together to span all of $\mathbb{R}$, $(-\infty, \infty)$, thus $\mathcal{S}$ is a collection of subsets whose union equals X, in this case, $\mathbb{R}$. Thus $\mathcal{S}$ is a subbasis.
However, $\mathcal{S}$ is not a basis. Consider this counterexample. Let $x = 0$. Let $B_1 = (-\infty, 1)$ and

$B_2 = (-1, \infty)$. $x \in B_1, B_2, B_1 \cap B_2$, however, $B_1 \cap B_2 = (-1, 1)$ which is not an element of $\mathcal{S}$. So, there does not exist a $B \in \mathcal{S}$ such that $x \in B \subseteq B_1 \cap B_2$.

□

Problem 3

Consider the set $\mathcal{T} = \{(-\infty, a) | a \in \mathbb{R}\} \cup \{\varnothing\} \cup \mathbb{R}$. (The left Ray Topology).

a

Prove $\mathcal{T}$ is a topology on $\mathbb{R}$.

Proof. We will prove the 3 criteria to be a topology:
(1) $\varnothing$ and $\mathbb{R}$ exist within $\mathcal{T}$, via the union.
(2) Let a^* be the maximum value of all a in $\bigcup_{a \in R}(-\infty, a)$, then the union becomes the interval $(-\infty, a^*)$, and since $a^* \in \mathbb{R}$, this is still an element of the form $(-\infty, a), a \in \mathbb{R}$, thus the union exists within the topology.
(3) Let b^* denote the minimum value of all b in $\bigcap_{b \in \mathbb{R}}(-\infty, b)$, then the intersection becomes the interval $(-\infty, b^*)$, and since $b^* \in \mathbb{R}$, this still is an element of the form $(-\infty, b), b \in \mathbb{R}$, thus the intersection exists within the topology.
Thus, $\mathcal{T}$ is a topology. □

b

Determine how $\mathcal{T}$ compares to the standard topology, and countable complement topology on $\mathbb{R}$.

Proof. The standard topology is strictly finer than the left ray topology.
For every basis element $(-\infty, a)$ of $\mathcal{T}$, there exists a basis element (b, a) in the standard topology such that $(b, a) \subseteq (-\infty, a)$. Thus, the standard topology is finer than $\mathcal{T}$. What makes it strictly finer is, for every basis element, (b, a) in the standard topology, there does not exist a basis element, B, in $\mathcal{T}$ such that $B \subseteq (b, a)$. For example, $(1, 2)$ is an open basis element of the standard topology, but there exists no open basis element in $\mathcal{T}$ of the form $(-\infty, a)$ that is a subset of this element. Thus, the standard topology is strictly finer. □

Proof. The countable complement is incomparable to $\mathcal{T}$ The open elements of $\mathcal{T}$ are not open in R_{cc} since $(-\infty, a)^c$ is not countable. For every open element in R_{cc}, we can express them as $\mathbb{R} - \{a\}$, or $(-\infty, \infty) - a$. Open elements such as $\mathbb{R} - \{-3, 5\}$ are not open in $\mathcal{T}$ since they cannot be expressed as a union of basis elements, since trying to write it as $(-\infty, -3) \cup (-\infty, 5)$ would overlap and include -3, thus making it not open in $\mathcal{T}$. Thus, they are incomparable to one another.

□

Problem 4

Consider $\mathbb{Z}$ with digital line topology.

a

Let $A = \{1, 2, 3, 6, 10, 11\}$. Find A', $\text{Int}(A)$, and $\bar{A}$

Proof. $\text{Int}(A) = \{1\} \cup \{1, 2, 3\} \cup \{11\} = \{1, 2, 3, 11\}$ (A union of the open sets within A)
$A' = \{2, 10, 12\}$ (The points that, for every open set $U, U - \{x\} \cap A \neq \varnothing$) Since the only open set containing 2 is {1,2,3}, which intersects A without 2, the only open set containing 10 is {9,10,11}, which intersects A without 10, and the only open set containing 12 is {11,12,13} which intersects A without 12.
$\bar{A} = A \cup A' = \{1, 2, 3, 6, 10, 11, 12\}$ □

b

Construct a sequence in $\mathbb{Z}$ which converges to more than one point.

Proof. A sequence x_n that converges to more than one point would be $x_n = 3$. This converges to $2, 3, 4$, since the all open neighborhoods for each of these values intersect 3. For 2, the open neighborhoods are $\{1, 2, 3\}$ which intersect 3, for 3, the open neighborhoods are $\{1, 2, 3\}, \{3\}, \{3, 4, 5\}$, which all intersect 3, and for 4, the open neighborhoods are $\{3, 4, 5\}$ which intersect 3. □

Problem 5

Consider the Sierpinski space $\mathcal{S} = \{0, 1\}$ with the topology $\mathcal{T} = \{\varnothing, \{1\}, \mathcal{S}\}$

a

Is S Hausdorff with the topology?

Proof. No. There are not unique disjoint neighborhoods for the points $0, 1$. The only open neighborhood for 0 is S, and the open neighborhoods for 1 are $\{1\}$ and S, $\{1\}$ is not disjoint from S since $\{1\} \cap S = \{1\}$ and S is the same neighborhood (in addition to it not being disjoint). Thus it is not Hausdorff. □

b

Prove that every sequence in S converges to 0.

Proof. There is only 1 open neighborhood for 0, which is S. The definition of convergence for a sequence states that a sequence converges to a point, if for every open neighborhood containing that point and the sequence after sufficient N, then the sequence converges to that point. Because the one and only open neighborhood for 0 is S, and all sequences must exist within S, every sequence in S converges to 0. □

c

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$. Find a necessary and sufficient condition for this sequence to converge to 1.

Proof. The sequence must, for sufficient N, exist within all open neighborhoods of 1, which are $S, \{1\}$. Thus, the necessary and sufficient condition is, for sufficient N, $(x_n)_{n\in\mathbb{N}}$ MUST become 1, and remain 1, in order to exist within both $S, \{1\}$, and thus converge to the point 1. $\square$

Problem 6

Let A, B be subsets of a topological space X. Determine whether to following equalities are true.

a

$\overline{A \cap B} = \bar{A} \cap \bar{B}$

Proof. $\rightarrow$
If $x \in \overline{A \cap B}$ then via 79, every open set containing x intersects $A \cap B$. Thus, every open set containing x intersects A and B. Thus, via 79, $x \in \bar{A}$ and $x \in \bar{B}$, thus $x \in \bar{A} \cap \bar{B}$. $\square$

Proof. $\leftarrow$
Let $x \in \bar{A} \cap \bar{B}$. Then, via 79, for every open U containing x, $U \cap A \neq \varnothing$ and $U \cap B \neq \varnothing$. Hence, $U \cap \Big(A \cap B\Big) \neq \varnothing$. Thus, via 79, $x \in \overline{A \cap B}$

$\square$

b

$\overline{A - B} = \bar{A} - \bar{B}$

Proof. $\leftarrow$
Let $x \in \bar{A} - \bar{B}$. Thus, $x \in \bar{A}$ and $x \notin \bar{B}$. Thus, via 79, $U \cap A \neq \varnothing$ for every open U containing x. Hence, $U \cap (A - B) \neq \varnothing$ for every open U containing x, so via 79, $x \in \overline{A - B}$.
$\square$

Proof. However, it does not go the other way.
Consider the counter-example. We are on the lower limit topology. Let $A = [0, 1), B = [\frac{1}{2}, 2)$. Then, $\bar{A} = [0, 1], \bar{B} = [\frac{1}{2}, 2]$, $A - B = [0, \frac{1}{2}), \overline{A - B} = [0, \frac{1}{2}]$. So, $\overline{A - B} \subsetneq \bar{A} - \bar{B}$ since $= [0, \frac{1}{2}] \subsetneq [0, \frac{1}{2})$.

$\square$

Meme