Configuration Spaces

Rippy

April 28th, 2020

Configuration, State, and Phase Spaces: What exactly are they?

A configuration space is almost synonymous with state spaces and phase spaces in studies within physics. In physics, these spaces are used to describe the state of a whole system as a "single point in a high-dimensional space". In mathematics specifically, these spaces are used to describe assignments of a collection of points to positions in a topological space. A state space is the set of all possible configurations of a system, with incredibly useful applications within robotics, algorithm analysis and artificial intelligence. A phase space is a space in which all possible states of a system are represented at once, with each possible state corresponding to one unique point in the phase space. For mechanical systems, the phase space usually is made of all possible values of position and momentum variables.

How do we define Configuration spaces?

For the sake of simplicity, understanding, and the length of this paper, I will be focusing on topological spaces where X is a closed, smooth manifold, that is, a locally Euclidean where each neighborhood is $\cong \mathbb{R}^n$, for some $n \in \mathbb{N}$. For all intents and purposes, X is also a connected topological space.

The configuration space of n distinct labeled points on a topological space X, denoted as $\mathcal{C}^n(X)$, is the complement of the diagonal inside the n-fold Cartesian product of X with itself:

$$\mathcal{C}^n(X) = \prod_{1}^{n} X - \Delta_X^n$$

where the fat diagonal Δ is defined as,

$$\Delta = \{(x_i)_1^n : x_i = x_j \text{ for some } i \neq j\}$$

We define the fat diagonal of X as the topological subspace of X^n, that is, comprised of n-tuples of points for which at least one pair of components coincide. $\mathcal{C}^n(X)$ is the space of ordered but otherwise unlabeled configurations of n points in X.

The space of unordered, unlabeled configurations of n points in X, denoted as $\mathcal{UC}^n(X)$ is the quotient space defined as the quotient of $C^n(X)$ by the natural action of the symmetric group S_n:

$$\mathcal{C}^n(X) := \mathcal{C}^n(X)/S_n$$

which permutes the ordered points in the topological space X. For the unordered unlabeled configuration space of *any* finite number of points, we write it as the disjoint union of these spaces over all natural numbers n. We are essentially "gluing together" the points that are mapped to each other by the action of S_n.

1 Applications

Configuration spaces in physics come in the form of phase spaces. In a phase space, every degree of freedom or parameter of the system is represented as an axis of a multidimensional space; a one-dimensional system is called a phase line, while a two-dimensional system is called a phase plane. For every possible state of the system or allowed combination of values of the system's parameters, a point is included in the multidimensional space. The system's evolving state over time traces a path, which we refer to as a "phase space trajectory for the system", through the high-dimensional space. The phase space trajectory represents the set of states compatible with starting from one particular initial condition, located in the full phase space that represents the set of states compatible with starting from any initial condition. As a whole, the phase diagram represents all that the system can be. Due to the nature of the "glued" points, it can easily reveal qualities of the system that might not have been obvious otherwise. How? Consider a flat square space, but now apply the quotient action to make it a torus. We can now visualize the space in an entirely different way, illuminating the properties we might not have otherwise seen.

A phase space can contain a great number of dimensions. For instance, a phase space for a gas requires a separate dimension for each particle's x, y and z positions and their respective momenta, which is at a minimum 6 dimensions. Phase spaces are also quite intuitive to use when analyzing the behavior of mechanical systems that are restricted to motion around and along various axes of rotation or translation, for example, in robotics, analyzing the range of motion of a robotic arm or determining the optimal path to achieve a particular position/momentum result.

In classical mechanics, generalized coordinates and momenta refer to the parameters that describe the configuration of the system relative to some reference configuration. These parameters uniquely define the configuration of the system relative to the reference configuration. Any choice of generalized coordinates q_i on the configuration space define generalized momenta p_i which together define coordinates on phase space. An example of a generalized coordinate is the angle that locates a point moving on a circle. The number of independent generalized coordinates is defined by the number of degrees of freedom of the system.

2 Examples

2.1 A simple particle

The position of a single particle moving in ordinary Euclidean 3-space is defined by the vector (x, y, z), and therefore its configuration space is $\mathcal{C}^n = \mathbb{R}^3$, since there are 3 degrees of freedom. Consider a particle that is constrained to move on a specific manifold. Let the particle be attached to a rigid leash pinned at the origin, so it is essentially constrained to lie on a sphere, thus its configuration space is the subset of coordinates in $\mathbb{R}^3$ that lie on the sphere, S^2. For n disconnected, non-interacting point particles, the configuration space is $\mathbb{R}^{3n}$ (Since each n has its own $\mathbb{R}^3$). With the addition of restrictions, the configuration space essentially becomes is the subspace of allowable positions that the point particles can take.

2.2 Phase Planes

The phase space of a two-dimensional system is called a phase plane, which occurs in classical mechanics for a single particle moving in one dimension, and where the two variables are position and velocity. In this case, a sketch of the phase portrait may give qualitative information about the dynamics of the system, such as the limit cycle of the Van der Pol oscillator shown in the diagram. Here, the horizontal axis gives the position and vertical axis the velocity. As the system evolves, its state follows one of the lines (trajectories) on the phase plot.

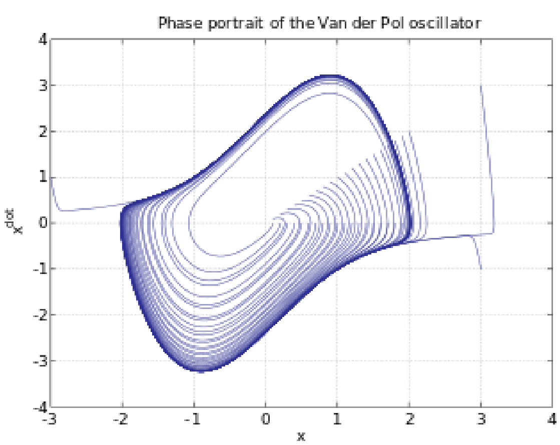

Figure 1: A phase plane of the Van der Pol oscillator

2.3 Robotic Arms (3D)

For a robotic arm consisting of some number of n rigid links, the configuration space would consist of the location of each link subject to the constraints of how the linkages are attached to each other, and their allowed range of motion. Let SO(3) represent the rotation of the specific link's frame relative to the ground frame. We have 6 total degrees of freedom between

$\mathbb{R}^3$ and $SO(3)$. Thus, for n links, the total space would be

$$\mathcal{C}^n(X) = [\mathbb{R}^3 \times \mathrm{SO}(3)]^n$$

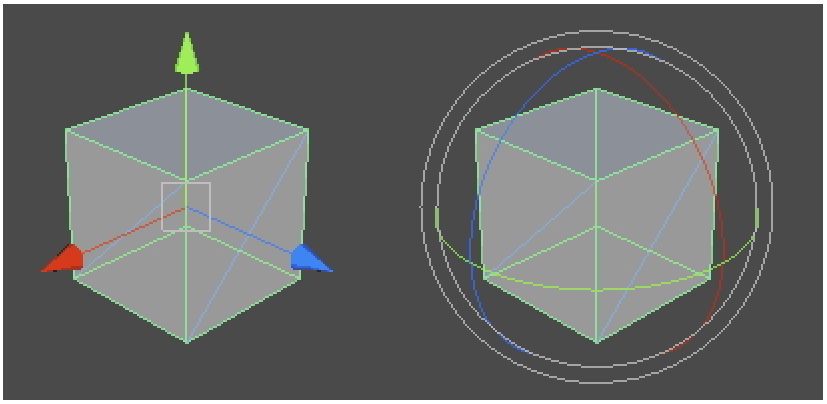

Figure 2: 3 Degrees of freedom in $\mathbb{R}^3$ (Left) and 3 Degrees of freedom in SO(3) (Right)

However, various further constraints would mean that not every point in this space is reachable, so it would be a further subspace of this. In robotics, the term configuration space refers to this further-reduced subset, the set of reachable positions by a robot's end-effector, which is the device at the end of a robotic arm, designed to interact with the environment in some way.

The resulting configuration space gives us: Where the colored regions refer to the respec-

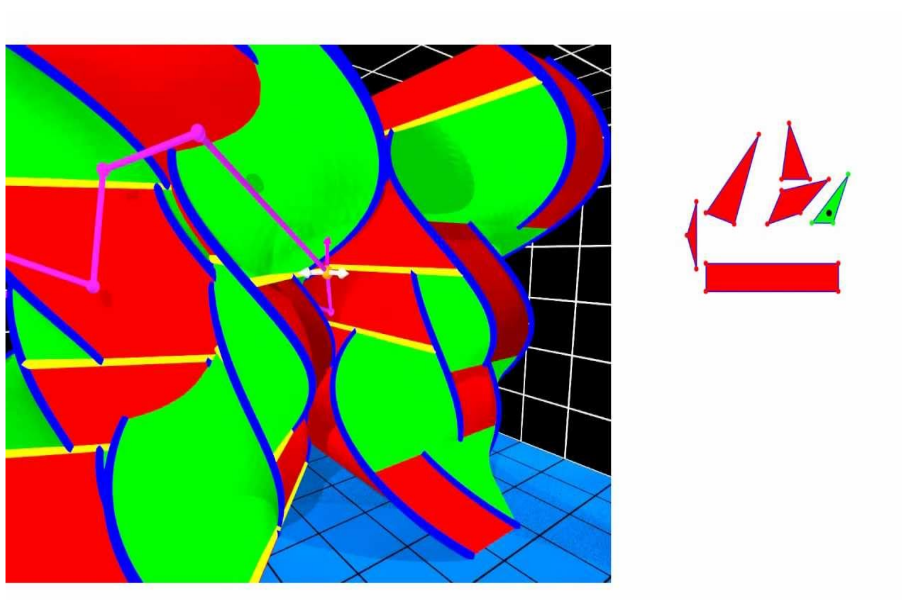

Figure 3: A configuration in 3-space, with multiple links

tive restricted regions. The green triangle represents the "shape" that must traverse through the red obstacles. White space is movable, unrestricted space.

2.4 Robotic Arms (2D)

To look at an more "easy" to visualize example, consider constricting the arm to Euclidean 2-space, with 2 links. In this instance, we are in $\mathbb{R}^2$, and SO(2), thus we have 4 degrees of freedom, with $n = 2$ links. Our configuration space would be:

$$\mathcal{C}^2(X) = [\mathbb{R}^2 \times \text{SO}(2)]^2$$

Now, we can visualize the configuration space below:

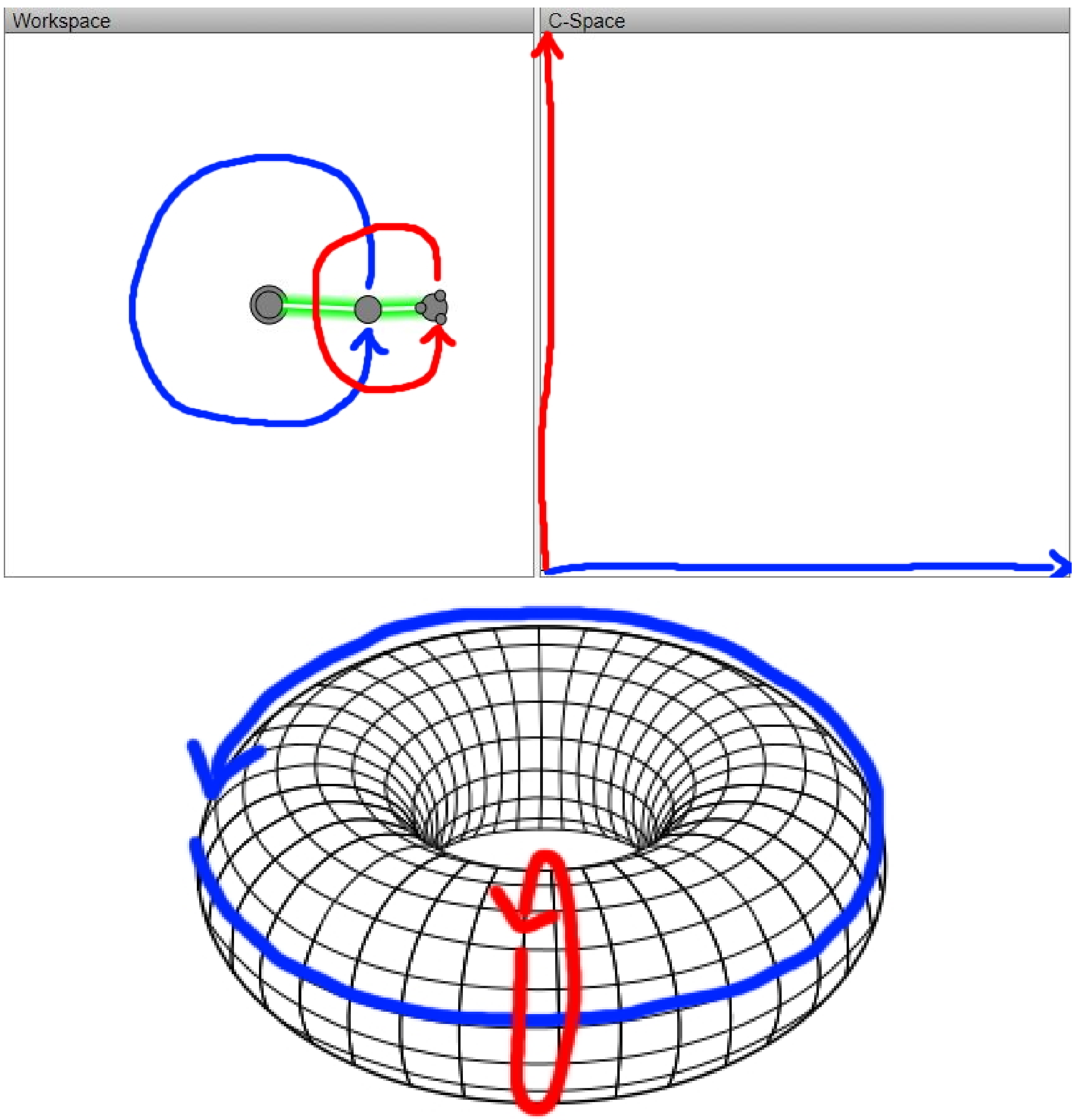

Figure 4: An Example of 2-space configuration, with 2 links

We can visualize the configuration space as a torus. Now, we can take a look at how the geometry warps in the configuration space, so lets look at another example to see exactly how it would warp.

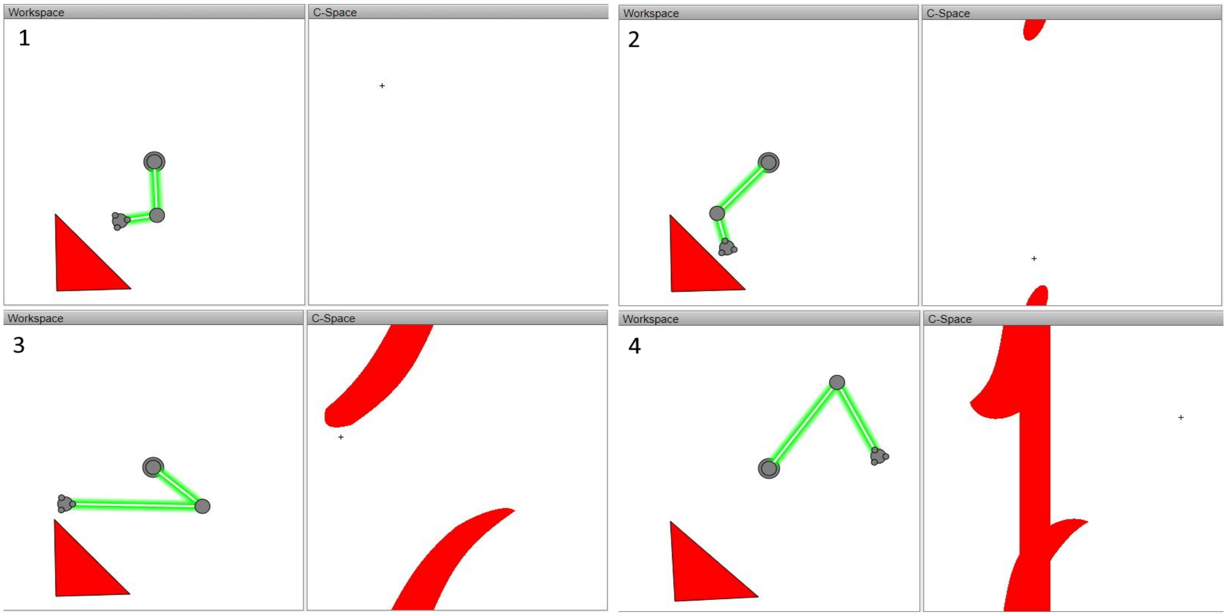

Figure 5: A look at how the geometry warps

As you can see, the geometry becomes warped as you convert to the configuration space via the quotient action of the S_2 group. (Similar to how the surface of a torus warps). The normal geometry as we see it is on the left, colored to show the distinctions. The resulting, restricted configuration space that our robotic arm can work within is the white region, and the colored regions represent their respective obstacles. The '+' is the location of the end-effector, which is the three prong circle. As the link "leashes" are extended, The space changes accordingly since the subspace changes along with the length of these link "leashes".

Finally, we look at a "boxed in" example. Try imagining how the robotic arm would move, and how the restrictions would make sense, since hitting an onstacle would be a forbidden region.

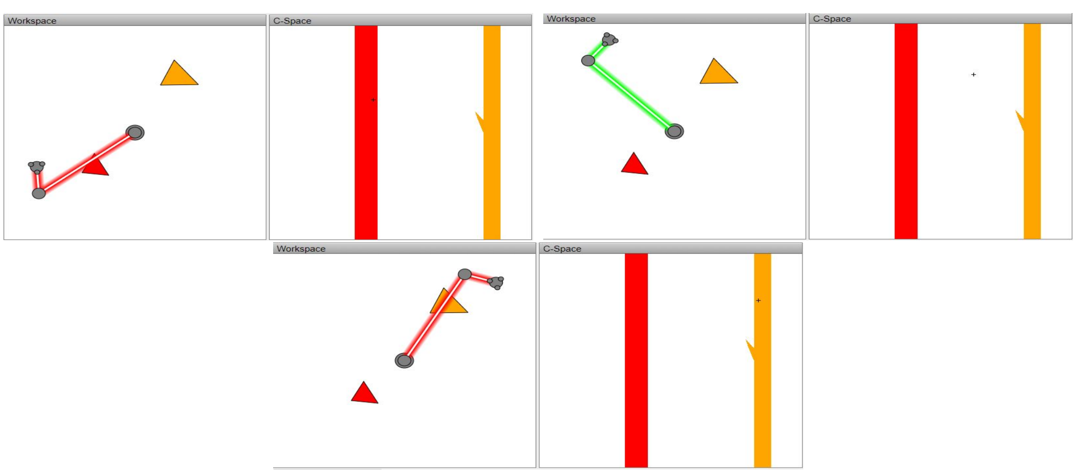

Figure 6: An example with a boxed in robotic arm

With further uses in quantum physics, thermodynamics, and motion planning (to name only a few), the possibilities for the uses of configuration spaces are endless and far-reaching. I hope you enjoyed reading this paper as much as I enjoyed explaining configuration spaces!

References

[1] Dr. Dror Atariah. *Configuration Space Visualization*, 2012

[2] Jeffrey Ichnowski. *Configuration Space Visualization of 2-D Robotic Manipulator*, 2010

[3] Howie Choset. *Robotic Motion Planning: Configuration Space*, 2010

[4] Adams and Franzosa *Introduction to Topology: Pure and Applied*, 2011

[5] The nLab. *Configuration space of points*, 2010-2019

[6] R. Ghrist. *Configuration Spaces, Braids, and Robotics*, 2007

[7] Jerry Ginsberg. *Engineering Dynamics*, 2008