Complex Problem Set 4

Rippy

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Problem 1

We can express the nth derivative of $f(z)$ at some arbitrary point z via the the form:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}ds$$

So, we express the second derivative:

$$| f''(z) | = | \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}ds |$$

Then using the triangle inequ, and using the definition that $R = | s - z |$, since this is the arbitrary circle centered around the point z, we rewrite this:

$$\begin{aligned} &\leq \frac{1}{\pi}\int_C | \frac{f(s)}{(s-z)^3} | d | s | \\ &\leq \frac{1}{\pi}\int_C | \frac{f(s)}{R^3} | d | s | \end{aligned}$$

Given our initial condition, $| f(z) | \leq c | z |$, we can further bound this previous statement, rewriting:

$$\leq \frac{c}{R^3\pi}\int_C | s | d | s |$$

Here, we know $| s | \leq | z | + R$, since there is no point on the circle further than that quantity. Thus, we can further bound the expression as follows:

$$\leq \frac{c}{R^3\pi}\int_C (| z | + R) d | s |$$

Since we can control how large R is, we can further bound this statement by making R sufficiently large enough to be larger than $| z |$, so we can rewrite it to be:

$$\begin{aligned} &\leq \frac{c}{R^3\pi}\int_C (R+R) d | s | \\ &= \frac{2c}{R^2\pi}\int_C d | s | \end{aligned}$$

Then we can evaluate the integral, which is just the arclength, which is the circumference of the circular path, with radius R. Thus we are left with:

$$= \frac{2c}{R^2\pi}2\pi R$$
$$= \frac{4c}{R}$$

Now, for any sufficiently small ϵ, we can choose our R to fit the equation:

$$\frac{4c}{R} < \epsilon$$
$$R > \frac{4c}{\epsilon}$$

Thus, we can choose R such that the expression becomes effectively 0 since we have shown $\mid f''(z) \mid < \epsilon$

Problem 2

Using the residue theorem, we can calculate the residue for problematic points in the region, then add them up to find the value of the original path integral. For (a) we have the only problematic point to be 0. For (b), the problematic points are at 0, and $\pm 2\sqrt{2}i$. We rewrite the function as an analytic function in the region, and then the non-analytic part, and then evaluate the residue. For the first region (a) , we have the function:

$$\frac{\cos z}{z(z^2+8)}$$

Which we rewrite into two parts:

$$\Big(\frac{1}{z}\Big)\Big(\frac{\cos z}{(z^2+8)}\Big)$$

Using the theorem, $\frac{1}{z}$ becomes $2\pi i$, and we plug in 0 into $\frac{\cos z}{(z^2+8)}$ to get the residue, $\frac{1}{8}$. Thus, the answer for (a) is $\frac{\pi i}{4}$, since this is the only problematic point. For (b), we can use the result from (a), but must in addition evaluate the residue at $\pm 2\sqrt{2}i$. We rewrite the function:

$$\frac{\cos z}{z(z^2+8)}$$

into the function:

$$\Big(\frac{1}{z}\Big)\Big(\frac{\cos z}{(z+2\sqrt{2}i)(z-2\sqrt{2}i)}\Big)$$

For each point, we factor out that part as the 'problematic part' and evaluate the remaining part of the function. For $2\sqrt{2}i$, we have:

$$\Big(\frac{1}{(z-2\sqrt{2}i)}\Big)\Big(\frac{\cos z}{z(z+2\sqrt{2}i)}\Big)$$

And for $-2\sqrt{2}i$, we have:

$$\Big(\frac{1}{(z+2\sqrt{2}i)}\Big)\Big(\frac{\cos z}{z(z-2\sqrt{2}i)}\Big)$$

In both instances, $\frac{1}{(z\pm 2\sqrt{2}i)}$ evaluates to $2\pi i$. Now we consider the analytic parts. For $2\sqrt{2}i$:

$$\frac{\cos{(2\sqrt{2}i)}}{2\sqrt{2}i(2\sqrt{2}i+2\sqrt{2}i)}$$
$$\frac{\cos{(2\sqrt{2}i)}}{2\sqrt{2}i(4\sqrt{2}i)}$$
$$\frac{\cos{(2\sqrt{2}i)}}{(-16)}$$

For $-2\sqrt{2}i$:

$$\frac{\cos(-2\sqrt{2}i)}{-2\sqrt{2}i(-2\sqrt{2}i-2\sqrt{2}i)}$$
$$\frac{\cos{(-2\sqrt{2}i)}}{-2\sqrt{2}i(-4\sqrt{2}i)}$$
$$\frac{\cos{(-2\sqrt{2}i)}}{(-16)}$$

Since cos is an even function, we know $cos(-z) = cos(z)$, thus, both instances evaluate to residues of:

$$\frac{\cos{(2\sqrt{2}i)}}{(-16)}$$

THUS, we have for a answer:

$$\int_C \frac{\cos z}{z(z^2+8)}dz = 2\pi i\Big(-\frac{\cos{(2\sqrt{2}i)}}{8}+\frac{1}{8}\Big)$$

Problem 3

To evaluate this, we will utilize partial fractions and residues. We begin with the expression:

$$\int_C \frac{2z-1}{z^2(z+1)(z-i)}$$

Which we then break into:

$$\frac{Az+B}{z^2}+\frac{C}{z+1}+\frac{D}{z-i}$$

We solve for A, B, C, D with a lot.... of algebra.

$$\begin{aligned}(Az+B)(z+1)(z-i)+(C)(z-i)(z^2)+D(z+1)(z^2)=2z-1\\ =A(z^3+z^2-iz^2-iz)+B(z^2+z-iz-i)+C(z^3-iz^2)+D(z^3+z^2)=2z-1\end{aligned}$$

Solving all this out, we get:

$$\begin{aligned}A&=3i-1\\ B&=-i\\ C&=\frac{3}{2}-\frac{3}{2}i\\ D&=-\frac{1}{2}-\frac{3}{2}i\end{aligned}$$

Which then gives us:

$$\frac{3i-1}{z}-\frac{i}{z^2}+\frac{3/2-3/2i}{z+1}+\frac{-1/2-3/2i}{z-i}$$

Now we use the residue theorem to calculate each of the respective parts' residues, and add them together. The z^2 term becomes zero, and the others become the numerators added together, times $2\pi i$. Giving us:

$$\begin{aligned}&(3i-1+3/2-3/2i-1/2-3/2i)\\ &=(3i-6/2i-1+3/2-1/2)\\ &=(3i-3i-1+1)\\ &=0\end{aligned}$$

Thus, this evaluates to 0.

Problem 4

(a)

$\frac{1-\cos(z-i)}{(z-i)^2}$

(b)

$\frac{1}{(z-2-i)z^3}$

(c)

$\log(z-1)$

Problem 5

We begin with the expression:

$$\int_C \frac{e^z}{z^m} dz$$

We use the taylor series expansion to rewrite e^z:

$$\int_C \sum_n \frac{z^n}{n! z^m} = \int_C \sum_n \frac{z^{n-m}}{n!}$$

We know (via class) that any instance where the power is not -1 results in 0. So, we can rewrite it to be:

$$\int_C \frac{1}{z(m-1)!}$$

Which evaluates to: $\frac{2\pi i}{(m-1)!}$