CHAPTER

7

Linear Systems in the Complex Frequency Domain: The Laplace Transform

7.1 GOALS OF THIS CHAPTER

We now have two techniques to find the response of a system to any input… well, almost any input: it has to be either periodic or aperiodic. In Chapter 5, we divided the input into short time segments and used convolution to sum up the resultant impulse responses. In Chapter 6, we used the Fourier transform to divide the input into sinusoids, found the output to each sinusoid by multiplying it with the transfer function, and then summed the output sinusoids using the inverse Fourier transform. We should always keep in mind that both of these methods involve the principle of superposition and time invariance so they apply only to linear, time invariant (LTI) systems. Even so, neither of these approaches can handle a third class of signals: waveforms that suddenly change and never return to a baseline level (recall Section 1.4.2). A classic example is the "step function," which changes from one value to another at one instant of time (often taken as $t = 0$) and remains at the new value for all eternity, Figure 7.1.

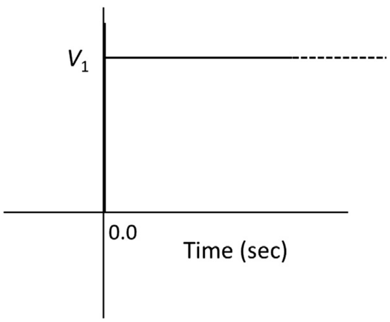

FIGURE 7.1 Time plot of a step function that changes from 0 to value V_1 at $t = 0.0$ s. It remains at this value for all time.

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One-time changes, or changes that do not return to some baseline level, are common in nature and even in everyday life. For example, this text should leave the reader with a lasting change in his or her knowledge of biosystems and biosignals, one that does not return to the baseline, precourse level. However, unlike a step function, this change is not expected to occur instantaneously. An input signal could change a number of times, either upward or downward, and might even show a partial return to its original starting point. But if the signal never returns to some original level, it cannot be either periodic or aperiodic and we do not have the tools to analyze how a system responds to these signals. As noted in Chapter 1, signals that change and never return to baseline are sometimes referred to as "transient" signals. Of course all signals vary in time and therefore could technically be called transient signals, but this term is often reserved for signals that have one-time or step-like changes. This is another linguistic issue in engineering where context can vary the meaning.

The goal of this chapter is to master the Laplace transform, a technique that allows us to analyze the response of LTI systems to transient or step-like inputs. It also enables us to analyze systems that have initial conditions. The downside is that this is a purely analytical technique and cannot be applied to discrete signals. So this rules out computer applications. Moreover, think back on the difficulty of working out the convolution integral analytically as in Example 5.3. Maybe we should avoid the Laplace transform and try working around the signal limitations of our previous methods. Could we not approximate a step function as a pulse having a very long pulse width? You could, and that is very likely what you would do in many situations. But we still cannot avoid the Laplace transform; it is just too important a concept in systems analysis.

The importance of the Laplace transform lies in its theoretical implications, although it does have some practical applications. Like the Fourier transform, it converts signals or functions from the time domain to something we call the Laplace domain. The Laplace domain is actually more general than the Fourier transform and, since it is easy to go from the Laplace domain to the frequency domain (but not vice versa), it is the preferred domain for transfer function equations. So most systems are described in the Laplace domain even if they do not use Laplace transforms for evaluation. MATLAB's powerful simulation language Simulink® described in Chapter 9 uses the Laplace notation to define system elements. If the input signals are periodic or aperiodic, we can easy slip back into the frequency domain and apply all the computer-friendly methods described in Chapters 5 and 6.

Topics in this chapter reflect many of those in Chapter 6 and include:

- The development of the Laplace transform as an extension of the Fourier transform and the introduction of the concept of complex frequency;
- How to determine the Laplace transform of a signal;
- The transfer function in the Laplace domain;
- How to find the output to any input in the Laplace domain (the output signal is found in the same way as in Chapter 6, by multiplying the signal by the transfer function, but this time in the Laplace domain);
- How to take inverse Laplace transform to get the time domain signal (this may involve additional algebraic tools such as partial fraction expansion);
- The relationships between the various methods for defining systems and describing their behavior.

7.2 THE LAPLACE TRANSFORM

The Laplace transform is used to define systems using transfer functions similar to the frequency domain transfer functions used in the last chapter.[1] But the Laplace domain can also represent signals that do not return to a baseline level. With this tool, we can extend all of the techniques developed in the last chapter to systems exposed to this wider class of signals. The Laplace domain can also be used to analyze systems that begin with nonzero initial conditions, a situation we have managed to avoid thus far, but one that does occur in the real world.

7.2.1 Definition of the Laplace Transform

Transfer functions written in Laplace notation differ only slightly from the frequency domain transfer function used in the last chapter. The reason that frequency domain transfer functions cannot be used when signals change in a step-like manner is simply that these functions cannot be decomposed into sinusoidal components using either Fourier series analysis or the Fourier transform. Consider a function similar to that shown in Figure 7.1, a function that is 0.0 for a $t \leq 0$ and 1.0 for all $t > 0$:

$$x(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases} \tag{7.1}$$

The function defined in Equation 7.1 is known as the "unit step function" since it begins at zero and jumps to 1.0 at $t = 0$, but a generic step function could begin at any level and jump to any other level. Try to find the Fourier transform of this function and you get:

$$FT(\omega) = \int_0^\infty x(t)e^{-j\omega t}dt = \int_0^\infty 1e^{-j\omega t}dt \Rightarrow \infty \tag{7.2}$$

The problem is that because $x(t)$ does not return to its baseline level (zero in this case), the limits of the integration must be infinite. Since the sinusoidal function $e^{-j\omega t}$ has nonzero values out to infinity, the integral becomes infinite. In the past, our input signals had a finite life span and the Fourier transform integral need only be taken over that finite time period. But for transient signals this integral cannot be computed.

The trick used to solve this infinite integral problem is to modify the exponential function so that it converges to zero at large values of t even if the signal, $x(t)$, does not. This can be accomplished by multiplying the sinusoidal term, $e^{-j\omega t}$, by a decaying exponential such as $e^{-\sigma t}$ where σ is some positive real variable. In this case the sinusoid term in the Fourier transform becomes a complex sinusoid, or rather a sinusoid with a complex frequency:

$$e^{-j\omega t}e^{-\sigma t} = e^{-(\sigma + j\omega)t} = e^{-st} \tag{7.3}$$

[1]In fact, some purists believe that only transfer functions written in Laplace notation are worthy of the term "transfer function."

where $s = \sigma + j\omega$ and is termed the "complex frequency" because it is a complex variable, but has the same role as frequency, ω, in the Fourier transform exponential. The complex variable, s, is also known as the "Laplace variable" since it plays a critical role in the Laplace transform (another example of multiple names for the same thing). A modified version of the Fourier transform can now be constructed using complex frequency in place of regular frequency; that is, s (which is $\sigma + j\omega$) instead of just $j\omega$. This modified transform is termed the Laplace transform:

$$X(s) = \mathscr{L}x(t) = \int_0^\infty x(t)e^{-st}dt \tag{7.4}$$

where the script $\mathscr{L}$ indicates the Laplace transformation.

So the trick is to use complex frequency, with its decaying exponential component, to cause convergence for functions that would not otherwise converge. For any general signal, $x(t)$, the product of $x(t)e^{-st} = x(t)e^{-(\sigma+j\omega)t}$ may not necessarily converge to zero as $t \to \infty$, in which case the Laplace transform does not exist.[2] Some advanced signal processing gurus have spent a lot of time worrying about such functions: which functions converge, their ranges of convergence, or how to get them to converge. Fortunately, such matters need not concern us since most common real-world signals, including the step function, do converge for some values of σ and so have a Laplace transform. The range of σ's over which a given product of $x(t)$ and $e^{-(\sigma+j\omega)t}$ converges is another occupation of signal processing theoreticians, but again is not something bioengineering signal processors need worry about. If a signal has a Laplace transform (and all the signals we use do), then the product $x(t)e^{-st}$ converges as $t \to \infty$.

The Laplace transform is a purely analytical technique so it cannot be solved on a computer and also has two other issues: it cannot be applied to functions for negative values of t, and it is difficult to evaluate using Equation 7.4 for any but the simplest of functions. The restriction against representing functions for negative time values comes from the fact that e^{-st} becomes a positive exponential and will go to infinity as t goes to large negative values. (For negative t, the real part of the exponential becomes $e^{+\sigma t}$ and does exactly the opposite of what we want it to do: it forces divergence rather than convergence.) The only way around this is to limit our analyses to $t > 0$, but this is usually not a serious restriction. The other problem, the difficulty in evaluating Equation 7.4, stems from the fact that s is complex, so although the integral in Equation 7.4 does not look so complicated, the complex integration becomes very involved for all but a few simple functions of $x(t)$. To get around this problem, we use tables that give us the Laplace transforms of frequently used functions. Such a table is given in Appendix B, and a more extensive list can easily be found on the internet. A Laplace transform table is used both to determine the Laplace transform of a signal and, using it in reverse, the inverse Laplace transform. The only difficulty in finding the inverse Laplace transform is rearranging the output Laplace function into one of the formats found in the table.

[2]For example, the function $x(t) = e^{t^2}$ will not converge as $t \to \infty$ even when multiplied by e^{-st}, so it does not have a Laplace transform.

EXAMPLE 7.1

Find the Laplace transform of the step function in Equation 7.1.

Solution: The step function is one of the few functions that can be evaluated easily using the basic defining equation of the Laplace transform, Equation 7.4.

$$X(s) = \int_0^\infty x(t)e^{-st}dt = \int_0^\infty 1e^{-st}dt = -\frac{e^{-st}}{s}\bigg|_0^\infty = 0 - \left(-\frac{1}{s}\right)$$
$$X(s) = \frac{1}{s}$$

7.2.2 Calculus Operations in the Laplace Domain

Just as in the frequency domain, the calculus operations of differentiation and integration can be reduced to algebraic operations in the Laplace domain. The Laplace transform of the derivative operation can be determined from the defining equation, Equation 7.4.

$$\mathcal{L}\frac{dx(t)}{dt} = \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt$$

Integrating by parts:

$$\mathcal{L}\frac{dx(t)}{dt} = x(t)e^{-st}\big|_0^\infty + s\int_0^\infty x(t)e^{-st}dt$$

From the definition of the Laplace transform, $\left(\int_0^\infty x(t)e^{-st}dt = X(s)\right)$, the right-most term in the summation is $sX(s)$, and the equation becomes:

$$\mathcal{L}\frac{dx(t)}{dt} = x(\infty)e^{-\infty} - x(0)e^{-0} + sX(s)$$

$$\mathcal{L}\frac{dx(t)}{dt} = sX(s) - x(0-) \tag{7.5}$$

Equation 7.5 shows that in the Laplace domain, differentiation becomes multiplication by the Laplace variable s with the additional subtraction of the value of the function at $t = 0$. The value of the function at $t = 0$ is known as the "initial condition." This value can be used, in effect, to account for all negative time history of $x(t)$. In other words, all of the behavior of $x(t)$ when t was negative can be lumped together as a single initial value at $t = 0$. This trick allows us to include some aspects of the system's behavior over negative values of t even if the Laplace transform does not itself apply to negative time. If the initial condition is zero, as is frequently the case, then differentiation in the Laplace domain is simply multiplication by s.

Multiple derivatives can also be taken in the Laplace domain, although this is not such a common operation. Multiple derivatives involve multiplication by s n-times, where n is the number of derivative operations, and taking the derivatives of the initial conditions:

$$\mathfrak{L}\frac{d^n x(t)}{dt^n} = s^n X(s) - s^{n-1}x(0-) - s^{n-2}\frac{dx(0-)}{dt}\ldots\frac{d^{n-1}x(0-)}{dt} \tag{7.6}$$

Again, if there are no initial conditions, taking n derivatives becomes just a matter of multiplying $x(t)$ by s^n.

If differentiation is multiplication by s in the Laplace domain, then integration is simply a matter of dividing by s. Again, a second term account for the initial conditions:

$$\mathfrak{L}\left[\int_0^T x(t)dt\right] = \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0} x(t)dt \tag{7.7}$$

If there are no initial conditions, then integration is accomplished by simply dividing by s. The second term of Equation 7.7 is a direct integral that accounts for the initial conditions and is again a way of accounting for the negative time history of the system.

7.2.3 Sources—Common Signals in the Laplace Domain

In the Laplace domain, both signals and systems are represented by functions of s. As mentioned earlier, the Laplace domain representation of signals is determined from a table. Although it might seem that there could be a large variety of such signals, which would require a very large table, in practice only a few signal types commonly occur in systems analysis. The signal most frequently encountered in Laplace analysis is the step function shown in Figure 7.1, or its more constrained version, the unit step function given in Equation 7.1 and repeated here:

$$x(t) = u(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases} \tag{7.8}$$

The symbol u is used frequently to represent the unit step function. The Laplace transform of the step function was found in Example 7.1 and repeated here:

$$X(x) = U(s) = \mathfrak{L}u(t) = \frac{1}{s} \tag{7.9}$$

As with the Fourier transform, it is common to use capital letters to represent the Laplace transform of a time function. Two functions closely related to the unit step function are the ramp and impulse functions, Figure 7.2.

These functions are related to the step function by differentiation and integration. The unit ramp function is a straight line with slope of 1.0.

$$r(t) = \begin{cases} t & t > 0 \\ 0 & t \le 0 \end{cases} \tag{7.10}$$

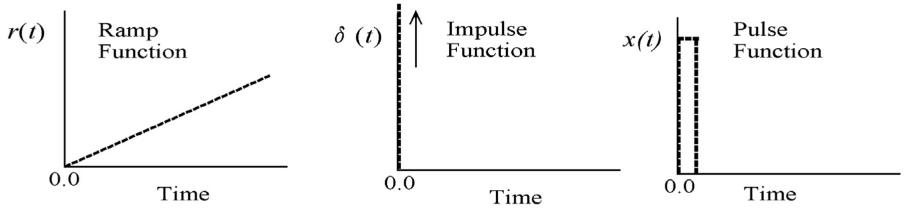


FIGURE 7.2 The ramp and impulse are two signals related to the step function and are commonly encountered in Laplace analysis. As described in Chapter 5, an ideal impulse occurs at $t = 0$ and is infinitely narrow and infinitely tall. Real-world impulse signals are approximated by short pulses (see Example 5.1).

Since the unit ramp function is the integral of the unit step function, its Laplace transform will be that of the step function divided by s:

$$R(s) = \mathcal{L}r(t) = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2} \tag{7.11}$$

The impulse function is the derivative of a unit step, which leads to one of those mathematical fantasies: a function that becomes infinitely short, but as it does its amplitude becomes infinite so the area under the function remains 1.0 as shown in Equation 7.12.

$$x(t) = \delta(t) = \lim_{a \to 0} \frac{1}{a} \quad \frac{-a}{2} \le t \le \frac{a}{2} \tag{7.12}$$

In practice, a short pulse is used as an impulse function and an approach to finding the appropriate pulse width for any given system is described in Example 5.1.

Since the impulse response is the derivative of the unit step function, its Laplace transfer function is that of a unit step multiplied by s:

$$\Delta(s) = \mathcal{L}\delta(t) = s\left(\frac{1}{s}\right) = 1 \tag{7.13}$$

Hence the Laplace transform of an impulse function is a constant, and if it is a unit impulse (the derivative of a unit step) then that constant is 1. As you might guess, this fact will be especially useful in the analysis of Laplace transfer functions. The Laplace transforms of other common signal functions are given in a table in Appendix B.

7.2.4 Converting the Laplace Transform to the Frequency Domain

Remember that s is a complex frequency and equals a real term plus the standard imaginary frequency term: $s = \alpha + j\omega$. To convert a Laplace function to the frequency domain, we simply substitute $j\omega$ for s. When we do this, we are agreeing to restrict ourselves to sinusoidal steady-state signals, or those that can be decomposed into such signals, so the real component of s with its convergence properties is no longer needed. Converting the Laplace transform function to the frequency domain allows us to determine the frequency characteristics of that function using Bode plot techniques or computer-based methods. This approach is used in the next section to evaluate the frequency characteristics of a time delay process.

TABLE 7.1 Transformations Between the Time, Frequency, and Laplace Domains.

From	To	Transformation
Time	Frequency	Fourier transform
Frequency	Time	Inverse Fourier transform
Time	Laplace	Laplace transform
Laplace	Time	Inverse Laplace transform
Laplace	Frequency	$s \to j\omega$
		Assumes sinusoidal steady state
Frequency	Laplace	$j\omega \to s$
		Assumes no initial conditions

7.2.5 The Inverse Laplace Transform

Working in the Laplace domain is pretty much the same as working in the frequency domain: the math is still algebra. The transfer function of system elements will be in Laplace notation as described later and summarized in Table 7.1. First, the input signal is converted to the Laplace domain using the table in Appendix B. Then multiplying the input signal with the transfer function provides the output response, but again as a function of s, not $j\omega$. Sometimes the Laplace representation of the solution or even just the Laplace transfer function is sufficient, but if a time domain solution is desired, then the inverse Laplace transform must be determined.

The equation for the inverse Laplace transform is given as:

$$x(t) = \mathcal{L}^{-1}X(x) = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \tag{7.14}$$

Unlike the inverse Fourier transform, this equation is quite difficult to solve even for simple functions. So to evaluate the inverse Laplace transform, we use the Laplace transform table in Appendix B in the reverse direction: find a function (on the right side of the table) that matches your Laplace output and convert it to the equivalent time domain signal. The difficulty is usually in rearranging the Laplace output function to conform to one of the formats given in the table. Methods for doing this are described next and specific examples given.

7.3 THE LAPLACE DOMAIN TRANSFER FUNCTION

The analysis of systems using the Laplace transform is no more difficult than in the frequency domain, except that there may be the added task of accounting for initial conditions. In addition, to go from the Laplace domain back to the time domain may require some algebraic manipulation of the Laplace output function.

The transfer function introduced in Chapter 6 is ideally suited to Laplace domain analysis, particularly when there are no initial conditions. In the frequency domain, the transfer function is used primarily to determine the spectrum of a system or system element, but it can also be used to determine the system's response to any input provided that input can be expressed as a sinusoidal series or is aperiodic. In the Laplace domain, the transfer function can be used to determine a system's output to a broader class of input signals. The Laplace domain transfer function is similar to its cousin in the frequency domain, except the frequency variable, ω, is replaced by the complex frequency variable, s:

$$TF(s) = \frac{Output(s)}{Input(s)} \tag{7.15}$$

Like its frequency domain cousin, the general Laplace domain transfer function consists of a series of polynomials. The general transfer function in the frequency domain was given in Equation 6.50 and repeated here:

$$TF(\omega) = \frac{Gj\omega\left(1 + j\dfrac{\omega}{\omega_1}\right)\left(1 - \left(\dfrac{\omega}{\omega_{n1}}\right)^2 + j\dfrac{2\delta_1\omega}{\omega_{n1}}\right)\ldots}{j\omega\left(1 + j\dfrac{\omega}{\omega_2}\right)\left(1 - \left(\dfrac{\omega}{\omega_{n2}}\right)^2 + j\dfrac{2\delta_2\omega}{\omega_{n2}}\right)\ldots} \tag{7.16}$$

In the Laplace domain, the general form of the transfer function differs in three ways: (1) the frequency variable $j\omega$ is replaced by the complex frequency variable s; (2) the polynomials are normalized so that the highest order of the frequency variable is normalized to 1.0 (as opposed to the constant term as in Equation 7.16); and (3) the order of the frequency terms is reversed with the highest-order term (now normalized to 1) coming first:

$$TF(s) = \frac{Gs(s + \omega_1)(s^2 + 2\delta_1\omega_{n1}s + \omega_{n1}^2)\ldots}{s(s + \omega_2)(s^2 + 2\delta_2\omega_{n2}s + \omega_{n2}^2)\ldots} \tag{7.17}$$

The constant terms, δ, ω, ω_n, etc. have the same meaning as in Equation 6.50. Even the terminology used to describe the elements is the same. For example, $\frac{1}{s+\omega_1}$, which is related to $\frac{1}{1+\frac{j\omega}{\omega_1}}$, is called a first-order element, and $\frac{1}{s^2+2\delta\omega_n s+\omega_n^2}$ is a second-order element. Note that in Equation 7.17, the constants δ and ω_n have a more orderly arrangement in the transfer function equation.

As described in Section 7.1.4, it is possible to go from the Laplace domain transfer function to the frequency representation simply by substituting $j\omega$ for s, but you should also rearrange the coefficients and frequency variable to fit the frequency domain format. As in the frequency domain, the assumption is that any higher-order polynomial (third-order or above) can be factored into the first- and second-order terms.

As with the frequency domain transfer function, we cover each of the element types separately: gain and first-order elements followed by the intriguing behavior of the second-order element. But first we introduce a new element, the "time delay" element, commonly found in physiological systems.

7.3.1 Time Delay Element: The Time Delay Theorem

Many physiological processes have a delay before they begin to respond to a stimulus. Such physiological delays are termed "reaction time," or "response latency," or simply "response delay." In these processes, there is a period of time between the onset of the stimulus and the beginning of the response. In systems involving neurological control, this delay is due to processing delays in the brain. The Laplace domain has an element to represent such delays.

The Time Delay theorem can be derived from the defining equation of the Laplace transform, Equation 7.4. Assume a signal $x(t)$ that is zero for negative time, but normally changes at $t = 0$ s. If this signal is delayed by T seconds, then the delayed function would be $x(t - T)$. From the defining equation (Equation 7.4), the Laplace transform of such a delayed function would be:

$$\mathcal{L}[x(t-T)] = \int_0^\infty x(t-T)e^{-st}dt$$

Defining a new variable: $\gamma = t - T$

$$\mathcal{L}[x(t-T)] = \int_0^\infty x(\gamma)e^{-s(T+\gamma)}d\gamma = \int_0^\infty x(\gamma)e^{-sT}e^{-s\gamma}d\gamma = e^{-sT}\int_0^\infty x(\gamma)e^{-s\gamma}d\gamma$$

But the integral in the right-hand term is the same as Laplace transform of the function, $x(t)$, it just has a different variable for time. So the right-hand integral is the Laplace transform of an unshifted $x(t)$; i.e., it is $\mathcal{L}[x(t)]$. Hence the Laplace transform of the shifted function becomes:

$$\mathcal{L}[x(t-T)] = e^{-sT}\mathcal{L}[x(t)] \tag{7.18}$$

Equation 7.18 is the Time Delay theorem, which can also be used to construct an element that represents a pure time delay, an element with a transfer function:

$$TF(s) = e^{-sT} \tag{7.19}$$

where T equals the delay, usually in seconds. So an element with a transfer function $TF(s) = e^{-sT}$ is a time delay of T seconds. This element is often found in models of neurological control where it represents neural processing delays.

EXAMPLE 7.2

Find the spectrum of a system time delay element for two values of delay: 0.5 and 2.0 s. Use MATLAB to plot the spectral characteristics of this element.

Solution: The Laplace transfer function of a pure time delay of 2.0 s would be:

$$TF(s) = e^{-2s}$$

To determine the spectrum of this element, we need to convert the Laplace transfer function to the frequency domain. This is done simply by substituting $j\omega$ for s in the transfer function equation:

$$TF(\omega) = e^{-j2\omega}$$

The magnitude of this frequency domain transfer function is:

$$|TF(\omega)| = \left|e^{-j2\omega}\right| = 1 \tag{7.20}$$

for all values of ω since changing ω (or the constant 2 for that matter) only changes the angle of the imaginary portion of the exponential, not its magnitude. The phase of this transfer function is:

$$\angle TF(\omega) = \angle e^{-j2\omega} = -2\omega \text{ rad.} \tag{7.21}$$

Since the magnitude of the transfer function is 1.0, the input and output signals have the same magnitude at all frequencies. The time delay process only changes the phase of the output signal by making it more negative with increasing frequency (i.e., adding -2ω to the phase of the input signal).

For a 0.5-s delay, $\angle TF(\omega) = \angle e^{-0.5j\omega}$, so the phase curve would be -0.5ω. A pure time delay decreases the phase component of a signal's spectrum in a linear manner, as shown in Figure 7.3. A time delay process is also explored in one of the problems.

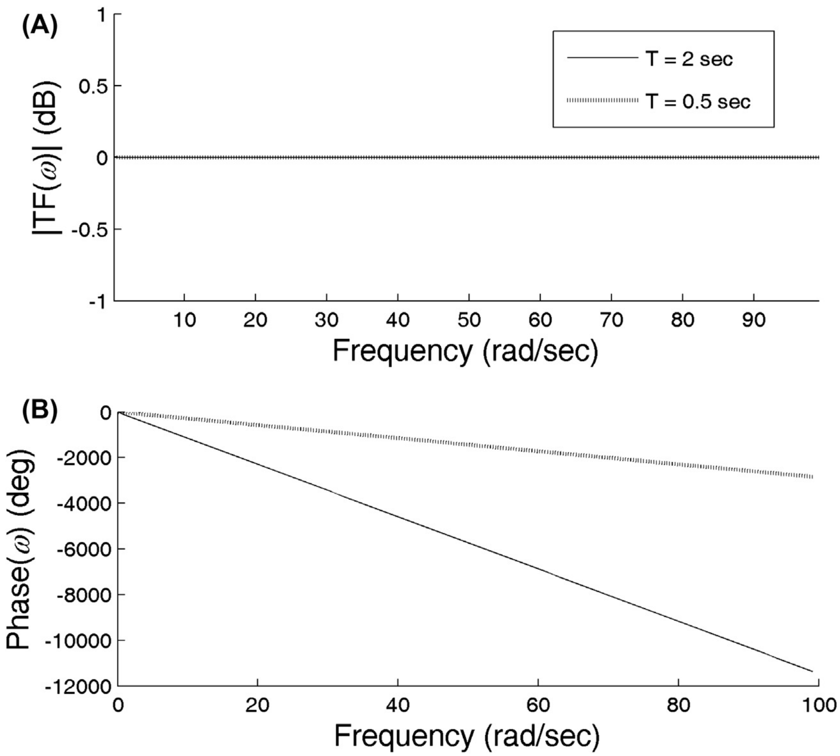


FIGURE 7.3 The magnitude and phase spectrum of a time delay element with two time delays: 2 and 0.5 s. In both cases the magnitude spectrum is a constant 0 dB (output equal input) as expected and the phase spectrum decreases linearly with frequency. Note that when $\omega = 100$ rad/s the phase angle is $-12{,}000$ degrees for the 2-s time delay as predicted by Equation 7.21 ($-2*100*360/2\pi = -12{,}000$ degrees).

The MATLAB code used to generate the magnitude and phase spectrum is a minor variation of Example 6.7. The plotting is done using linear frequency rather than log, to emphasize that the phase spectrum is a linear function of frequency.

```
% Example 7.2 Use MATLAB to plot the transfer function of a time delay
%
T = 2;                                   % Time delay in sec.
w = .1:1:100;                            % Frequency vector
TF = exp(-j*T*w);                        % Transfer function
Mag = 20*log10(abs(TF));                 % Calculate magnitude spectrum
Phase = unwrap(angle(TF))*360/(2*pi);            % Calculate phase spectrum
  ....... Repeat for T = 0.5 and plot and label......
```

Results: The time delay element increases the phase linearly with frequency as shown in Figure 7.3.

7.3.2 Constant Gain Element

The gain element is not a function of frequency, so its transfer function is the same irrespective of whether standard or complex frequency is involved:

$$TF(s) = G \tag{7.22}$$

The system representation for a gain element is the same as in the frequency domain representation, Figure 6.9.

7.3.3 Derivative Element

Equation 7.5 shows that the derivative operation in the Laplace domain is implemented simply by multiplication with the Laplace variable s. This is analogous to this operation in the frequency domain, where differentiation is accomplished by multiplying by $j\omega$. So in the absence of initial conditions, the Laplace transfer function for a derivative element is:

$$TF(s) = s \tag{7.23}$$

The Laplace system representation of this element is shown in Figure 7.4A

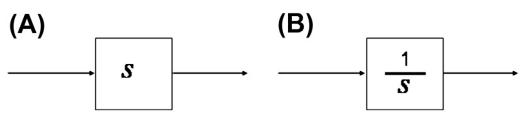


FIGURE 7.4 (A) Laplace system representation of a derivative element. (B) Laplace system representation of an integrator element.

7.3.4 Integrator Element

As shown in Equation 7.7, integration in the Laplace domain is accomplished by dividing by s. So in the absence of initial conditions, the system representation of integration in the Laplace domain has a transfer function of

$$TF(s) = 1/s \tag{7.24}$$

This parallels the representation in the frequency domain where $1/s$ becomes $1/j\omega$. The representation of this element is shown in Figure 7.4B.

EXAMPLE 7.3

Find the Laplace and frequency domain transfer function of the system in Figure 7.5. The gain term k is a constant.

Solution: The approach to finding the transfer function of a system in the Laplace domain is exactly the same as in the frequency domain used in Chapter 6. Here we are asked to determine both the Laplace and frequency domain transfer function. First we find the Laplace transfer function, then substitute $j\omega$ for s, and rearrange the format for the frequency domain transfer function.

We could solve this several different ways, but this is clearly a feedback system so the easiest solution is to use the feedback equation, Equation 6.7. All we need to do is find the equivalent feedforward gain function, $G(s)$, and the equivalent feedback function, $H(s)$. The feedback function is $H(s) = 1$ since all of the output feeds back to the input in this system.[3] The feedforward gain is the product of the two elements:

$$G(s) = k\frac{1}{s}$$

Substituting $G(s)$ and $H(s)$ into the feedback equation:

$$TF(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{k}{s}}{1 + \frac{k}{s}} = \frac{k}{s + k}$$

Result: Again, the Laplace domain transfer function has the highest coefficient of complex frequency, in this case s, normalized to 1. Substituting $s = j\omega$ and rearranging the normalization:

$$TF(\omega) = \frac{k}{j\omega + k} = \frac{1}{1 + j\frac{\omega}{k}}$$

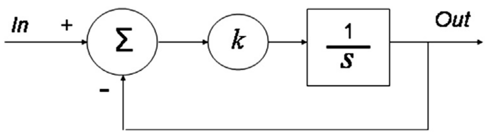


FIGURE 7.5 Laplace representation of a system used in Example 7.3.

This has the same form as a first-order transfer function given in Equation 6.31 and shown in Figure 6.12B, where the constant term ω_1 is k.

[3]Such systems are known as "unity gain feedback systems" since the feedback gain, $H(s)$, is 1.0 (unity).

7.3.5 First-Order Element

First-order processes contain an $s + \omega_1$ term in the denominator (or possibly only an s). Later we find that these systems contain a single energy storage device. The Laplace transfer function for a first-order process is:

$$TF(s) = \frac{1}{s+\omega_1} = \frac{\frac{1}{\tau}}{s+\frac{1}{\tau}} \tag{7.25}$$

In this transfer function equation, a new variable is introduced, τ, which is termed the "time constant" for reasons shown in Example 7.4. As seen in Equation 7.25, this time constant variable, τ, is simply the inverse of the frequency constant, ω_1:

$$\tau = \frac{1}{\omega_1} \tag{7.26}$$

In Chapter 6, we found that the frequency constant ω_1 is where the magnitude spectrum is -3 dB (Equation 6.34) and the phase spectrum is -45 degrees (Equation 6.37). In the next example, we show that the time constant τ has a direct relationship to the time behavior of a first-order element.

Now that the transfer function is in the Laplace domain, we can explore the response of these two similarly behaving systems using input signals other than sinusoids. Two popular signals that are used are the step function, shown in Figure 7.1, and the impulse function, shown in Figure 7.2. Typical impulse and step responses of a first-order system are found in the next example.

EXAMPLE 7.4

Find the arterial pressure response of the linearized model of the Guyton−Coleman body fluid balance system (Ridout, 1991) to a step increase of fluid intake of 0.5 mL/min. The frequency domain version of this model is shown in Figure 6.4. Also find the pressure response to fluid intake of 250 mL administered as an impulse.[4] This is equivalent to drinking approximately 1 cup of fluid quickly. After finding the analytical solution, use MATLAB to plot the outputs in the time domain.

Solution, step response: The first step is to convert to Laplace notation from the frequency notation used in Figure 6.4. The feedforward gain becomes $G(s) = 16.67\left(\frac{0.06}{s}\right) = \frac{1}{s}$ and the feedback gain becomes $H(s) = 0.05$. Again applying the feedback equation (Equation 6.7), the transfer function is:

$$TF(s) = \frac{P_A(x)}{F_{IN}(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{s}}{1+0.05\frac{1}{s}} = \frac{1}{s+0.05}$$

where P_A is arterial blood pressure in mmHg and F_{IN} is a small change in fluid intake in mL/min.

To find the step response, multiply the Laplace input signal by the transfer function:

$$P_A(s) = F_{IN}(s)TF(s) = \frac{1}{s}TF(s) \tag{7.27}$$

Substituting in for $TF(s)$, Equation 7.27 becomes:

$$P_A(s) = \frac{1}{(s+0.05)}\left(\frac{1}{s}\right) = \frac{1}{s(s+0.05)} \tag{7.28}$$

To take the inverse Laplace transform and get the time domain output, we need to rearrange the resulting Laplace output function so that it has the same form as one of the functions in the Laplace Transform Table of Appendix B. Often this can be the most difficult part of the problem. Fortunately, in this problem, we see that the right-hand term of Equation 7.28 matches the Laplace function in entry number 4 of the table.

$$\frac{1}{s(s+0.05)} \text{ has the form : } \frac{\alpha}{s(s+\alpha)} \Leftrightarrow (1-e^{-\alpha t})/\alpha$$

where $\alpha = 0.05$. Hence the step response in the time domain for this system is an exponential:

$$p_A(t) = \frac{1}{0.05}\left(1-e^{-0.05}\right) = 20\left(1-e^{-0.05}\right) \text{ mmHg}$$

Solution, impulse response: Solving for the impulse response is even easier since for the impulse function, $F_{IN}(s) = 1$. So the impulse response of a system in the Laplace domain is the transfer function itself. Therefore, the impulse response in the time domain is the inverse Laplace transform of the transfer function:

In this example the impulse input was 250 mL, so $F_{IN}(s) = 250$, and $P_A(s) = 250\ TF(s)$:

$$P_A(s) = \frac{250}{s+0.05} \text{ mmHg}$$

The Laplace output function is matched by entry 3 in the Laplace Transform Table except for the constant term. From the definition of the Laplace transform in Equation 7.4, any constant term can be removed from the integral, so the Laplace transform of a constant times a function is the constant times the Laplace transform of the function. Similarly, the inverse Laplace transform of a constant times a Laplace function is the constant times the inverse Laplace transform. Stating these two characteristics formally:

$$\mathcal{L}[kx(t)] = k\mathcal{L}x(t) \tag{7.29}$$

$$\mathcal{L}^{-1}[kX(s)] = k\mathcal{L}^{-1}[X(s)] \tag{7.30}$$

So the time domain solution for $P_A(s)$ is obtained:

$$V_{out}(s) = 250\left(\frac{1}{s+0.05}\right) \text{ which has the same form as : } k\frac{1}{s+\alpha} \Leftrightarrow ke^{-\alpha t}$$

$$v_{out}(t) = 250e^{-0.05} \text{ mmHg}$$

Plotting this result with the help of MATLAB is easy as shown.

```
% Example 7.4 First-order system impulse and step response for two time constants
%
t = 0:.1:100;                       % Define time vector: 0 to 100 min
x = 250*exp(-0.05);                 % Impulse response
plot(t,x,'k');                      % Plot impulse response
  .......labels, title, and other text.....
x = 20*(1 - exp(-0.05));            % Step response.
plot(t,x,'k');                      % Plot step response
  .......plot, labels, title, and other text.....
```

Results: The system responses are shown in Figure 7.6. Both the impulse and step responses are exponential with time constants of $1/0.05 = 20$ min. When time equals the time constant (i.e., $t = \tau$), the value of the exponential becomes: $e^{-t/\tau} = e^{-1} = 0.37$. Hence at time $t = \tau$, the exponential is within 37% of its final value. In other words, the exponential has attained 73% of its final value after one time constant.

The time constant makes a good measure of the relative speed of an exponential response. As a rule of thumb, an exponential is considered to have reached its final value when $t > 5\tau$, although in theory the final value of an exponential is reached only when $t = \infty$.

[4]Logically, the response to a step change in input is called the "step response," just as the response due to an impulse is termed the "impulse response."

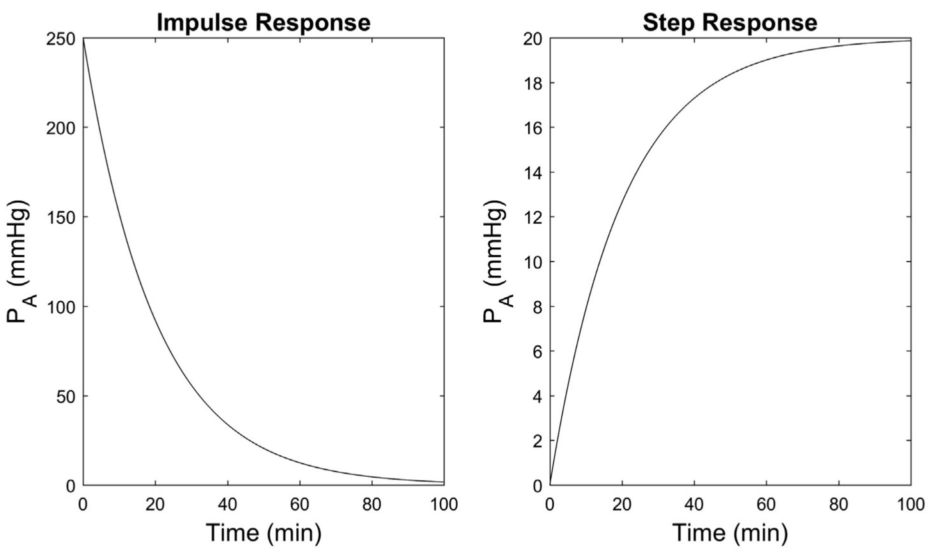


FIGURE 7.6 Response of a linearized model of the Guyton–Coleman body fluid balance system. The graph shows the change in arterial pressure to an increase of fluid intake, either taken in a step-like manner (right) or as an impulse (left).

There are many other possible input waveforms in addition to the step and impulse function. As long as the waveform has a Laplace representation, the response to any signal can be found from the transfer function.[5] However, the step and/or impulse responses usually provide the most insight into the general behavior of the system.

7.3.5.1 The Characteristic Equation

First-order transfer functions can be slightly different than that of Equation 7.25. There can be a single s or even an $(s + 1/\tau)$ in the numerator. What bonds all first-order systems is the denominator term, $(s + 1/\tau)$, or equivalently $(s + \omega_1)$. All first-order systems will contain a s in the denominator irrespective of what is in the numerator. In the next section, we find that all second-order systems contain an s^2 in the denominator of the transfer function, usually as a quadratic polynomial of s. Again the numerator can be a variety of polynomials of s.

The characteristics of the transfer function are primarily determined by the denominator, which explains why the denominator is called the "characteristic equation." The characteristic equation describes the general behavior of the system. Often, only the characteristic equation is required to determine the salient features of a response: a time domain solution is not needed. For example, the characteristic equation $s + 3$ tells us that the system's response will include an exponential having a time constant of 1/3 or 0.33 s. Second-order characteristic equations are even more informative, as the next section shows.

EXAMPLE 7.5

Find the transfer function of the system shown in Figure 7.7.

Solution: As in Example 7.3 and previous examples, we use the feedback equation. We just need to find $G(s)$ and $H(s)$. Again $H(s) = 1$: another unity gain feedback system. The feedforward gain function is just the product of the two feedforward elements:

$$G(s) = \frac{1}{s+7}\left(\frac{5}{s}\right) = \frac{5}{s^2+7s}$$

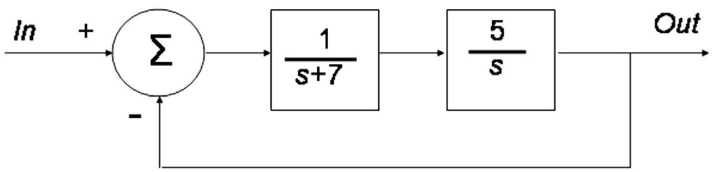


FIGURE 7.7 System used in Example 7.5. The transfer function of this system is to be determined.

[5]Even a sine wave has a Laplace transform, but it is complicated. It is much easier to use the frequency domain techniques developed in Chapter 6 for sine waves as long as they are in steady state. If the sine wave is not in steady state, but starts at some particular time, say $t = 0$, then Laplace techniques and the Laplace transform of a sine wave must be used.

Result: Substituting into the feedback equation gives the transfer function:

$$TF(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{5}{s^2+7s}}{1+\frac{5}{s^2+7s}} = \frac{5}{s^2+7s+5}$$

The s^2 in the characteristic equation indicates that this is a second-order system. These elements are covered in the next section.

7.3.6 Second-Order Element

Second-order processes have an s^2 in the denominator of the transfer function, usually as a quadratic polynomial:

$$TF(s) = \frac{1}{s^2+2\delta\omega_n s+\omega_n^2} = \frac{1}{s^2+bs+c} \tag{7.31}$$

The denominator of the right-hand term is the familiar notation for a standard quadratic equation and can be factored using Equation 7.32. As we find later, second-order systems must contain two energy storage devices.

One method for dealing with second-order terms would be to factor them into first-order terms. We could then use partial fraction expansion to expand the two factors into two terms of the form $s+\alpha$. This approach is perfectly satisfactory if the factors, the roots of the quadratic equation, are real. Examination of the classic quadratic equation demonstrates when this approach will work. Since the coefficient of the s^2 term is always normalized to 1.0, the a coefficient in the quadratic is always 1.0 and the roots of the quadratic equation become:

$$r_1, r_2 = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2-4c} \tag{7.32}$$

If $b^2 \geq 4c$ then the roots will be real and the quadratic can be factored into two first-order terms: $s-r_1$ and $s-r_2$. However, if $b^2 < 4c$, both roots will be complex and have real and imaginary parts:

$$r_1 = \frac{-b}{2} + j\frac{1}{2}\sqrt{4c-b^2} \text{ and } r_2 = \frac{-b}{2} - j\frac{1}{2}\sqrt{4c-b^2} \tag{7.33}$$

If the roots are complex, they both have the same real part ($-b/2$), whereas the imaginary parts also have the same values but with opposite signs. Complex number pairs that feature this relationship, the same real part but oppositely signed imaginary parts, are called "complex conjugates."

Whether or not the roots of a second-order characteristic equation are real or complex has important consequences in the behavior of the system. Sometimes all we need to know about a second-order system is whether the roots in the characteristic equation are real or imaginary. This saves the effort of finding the inverse Laplace transform.

The second-order transfer function variables δ and ω_n were introduced in Chapter 6. Recall that the parameter δ is called the "damping factor," whereas ω_n is called the "undamped natural frequency." As with the term "time constant," applied to first-order systems, these names relate to the step and impulse response behavior of the second-order systems. As is shown later, second-order systems with low damping factors will respond with an exponentially decaying oscillation, and the smaller the damping factor, the slower the decay. The rate of oscillation is related to ω_n. Specifically, the rate of oscillation, ω_d, of these underdamped systems is:

$$\omega_d = \omega_n\sqrt{1-\delta^2} \tag{7.34}$$

So as δ becomes smaller and smaller, the oscillation frequency ω_d approaches ω_n. When δ equals 0.0, the system is "undamped" and the oscillation continues forever at frequency ω_n. Hence the term "undamped natural frequency" for ω_n: it is the frequency at which the system oscillates if it has no damping; i.e., $\delta \rightarrow 0$.

Here we equate the variables ω_n and δ to coefficients a and b in Equation 7.31 and insert them into the solution to the quadratic equation:

$$r_1, r_2 = \frac{-2\delta\omega_n}{2} \pm \frac{\sqrt{4\delta^2\omega_n^2 - 4\omega_n^2}}{2} = -\delta\omega_n \pm \omega_n\sqrt{\delta^2-1} = -\delta\omega_n \pm \omega_d \tag{7.35}$$

From this equation, we see that the damping factor δ alone determines if the roots will be real or complex. Specifically, if $\delta > 1$ then the constant under the square root is positive and the roots will be real. Conversely, if $\delta < 1$ the square root will be a negative number and the roots will be complex. If $\delta = 1$, the two roots are also real, but are the same: both roots equal $-\omega_n$.

Again, the behavior of the system is quite different if the roots are real or imaginary, and the form of the inverse Laplace transform is also different. Accordingly, it is best to examine the behavior of a second-order system with real roots and complex roots separately: they act as two different animals.

7.3.6.1 Second-Order Elements With Real Roots

If the roots are real (i.e., $\delta > 1$), the system is said to be "overdamped" because it does not exhibit oscillatory behavior in response to a step or impulse input. Such systems have responses consisting of double exponentials.

The best way to analyze overdamped second-order systems is to factor the quadratic equation into two first-order terms each with its own time constant, τ_1 and τ_2:

$$TF(s) = \frac{1}{s^2 + 2\delta\omega_n s + \omega_n^2} = \frac{1}{\left(s+\frac{1}{\tau_1}\right)\left(s+\frac{1}{\tau_2}\right)} \tag{7.36}$$

These two time constants are the negative of the inverted roots of the quadratic equation as given by Equation 7.32. In other words, $\tau_1 = -1/r_1$ and $\tau_2 = -1/r_2$. The second-order transfer function can also have numerator terms other than 1, but the essential behavior does not change and the analysis strategy does not change. Typical overdamped impulse and step responses will be shown in the following example.

After the quadratic equation is factored, the next step is either to separate this function into two individual first-order terms $\frac{k_1}{s+1/\tau_1}+\frac{k_2}{s+1/\tau_2}$ using partial fraction expansion (see later discussion), or find a Laplace transform that generally matches the unfactored equation in Equation 7.36. If the numerator is a constant, then entry 9 in the Laplace Transform Table (Appendix B) matches the unfactored form. If the numerator is more complicated, a match may not be found and partial fraction expansion will be necessary.

Figure 7.8 shows the response of a second-order system to a unit step input. The step responses shown are for an element with four different combinations of the two time constants: τ_1, $\tau_2 = 1$, 0.2; 1, 2; 4, 0.2; and 4, 2 s. The time it takes for the output of this element to reach its final value depends on both time constants, but is primarily a factor of the slowest time constant. The next example illustrates the use of Laplace transform analysis to determine the impulse and step responses of a typical second-order overdamped system.

7.3.6.2 Partial Fraction Expansion—Manual Methods

Partial fraction expansion is the opposite of finding the common denominator: instead of trying to combine fractions into a single fraction, we are trying to break them apart. We want to determine what series of simple fractions will add up to the fraction we are trying to decompose. The technique described here is a simplified version of partial fraction expansion that deals with distinct linear factors, that is, denominator components of the form $(s - p)$. Moreover, this analysis will be concerned only with single components, not multiple components such as $(s - p)^2$.

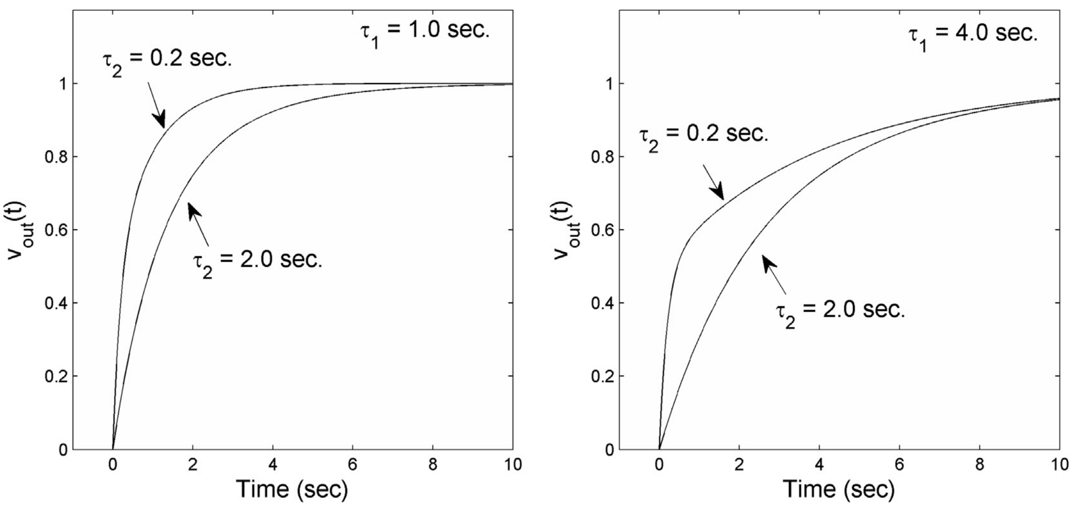


FIGURE 7.8 Typical step responses of a second-order system with real roots. These responses are termed overdamped relating to the exponential-like behavior of the response. Four different combinations of τ_1 and τ_2 are shown. The speed of the response depends on both time constants, but is dominated by the slower of the two.

Under these restrictions: the partial fraction expansion can be defined as:

$$TF(s) = \frac{N(s)}{(s-p_1)(s-p_2)(s-p_3)\ldots} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \frac{k_3}{s-p_{31}} + \ldots \tag{7.40}$$

$$k_n = (s-p_n)TF(s)\big|_{s=p_n} \tag{7.41}$$

Since the constants in the Laplace equation denominator will always be positive, the values of p will always be negative. The next example uses partial fraction expansion to find the step response of a second-order system.

EXAMPLE 7.6

Find the impulse and step responses of the following second-order transfer function. Use Laplace analysis to find the time functions and MATLAB to plot the two time responses.

$$TF(s) = \frac{25}{s^2 + 12.5s + 25} \tag{7.37}$$

Solution, General: First find the values of δ and ω_n by equating coefficients with the basic equation shown in Equation 7.31:

$$\omega_n^2 = 25; \;\; \omega_n = 5$$

$$2\delta\omega_n = 12.5; \;\; \delta = \frac{12.5}{2\omega_n} = \frac{12.5}{10} = 1.25$$

Since $\delta > 1$, the roots are real. In this case, the next step is to factor the denominator using the quadratic equation, Equation 7.32[6]:

$$r_1, r_2 = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c} = \frac{-12.5}{2} \pm \frac{1}{2}\sqrt{12.5^2 - 4(25)} = -6.25 \pm 3.75$$

$$r_1, r_2 = -10.0 \text{ and } -2.5$$

Solution, Impulse Response: For an impulse function input, the output $V_{out}(s)$ is the same as the transfer function since the Laplace transform of the input is $V_{in}(s) = 1$:

$$V_{out}(s) = \frac{25}{(s+10)(s+2.5)} = \left(\frac{25}{7.5}\right)\frac{7.5}{(s+10)(s+2.5)}$$

The right hand term of $V_{out}(s)$ has been rearranged to match entry #9 in the Laplace Transform Table:

$$\frac{\gamma - \alpha}{(s+\alpha)(s+\gamma)} \Leftrightarrow e^{-\alpha t} - e^{-\gamma t} \quad \text{where } \alpha = 2.5 \text{ and } \gamma = 10$$

So $v_{out}(t)$ becomes:

$$v_{out}(t) = \left(\frac{25}{7.5}\right)\left(e^{-2.5t} - e^{-10t}\right) = 3.33\left(e^{-2.5t} - e^{-10t}\right) \tag{7.38}$$

Solution, Step response: To find $V_{out}(s)$, multiply the transfer function by the Laplace transform of the step function, $1/s$:

$$V_{out}(s) = \left(\frac{1}{s}\right)\frac{25}{(s+2.5)(s+10)} = \frac{25}{s(s+2.5)(s+10)} \tag{7.39}$$

With the extra s added to the denominator, this function no longer matches any in the Laplace Transform Table. However, we can expand this function using partial fraction expansion into the form:

$$V_{out}(s) = \frac{k_1}{s} + \frac{k_2}{s+2.5} + \frac{k_3}{s+10}$$

Applying partial fraction expansion to the Laplace function of Equation 7.39, the values for p_1, p_2, and p_3 are, respectively: -0, -2.5, and -10, which produces the numerator terms k_1, k_2, and k_3:

$$k_1 = (s+0)\left.\frac{25}{s(s+2.5)(s+10)}\right|_{S=-0} = \frac{25}{2.5(10)} = 1.0$$

$$k_2 = (s+2.5)\left.\frac{25}{s(s+2.5)(s+10)}\right|_{S=-2.5} = \frac{25}{-2.5(-2.5+10)} = -1.33$$

$$k_3 = (s+10)\left.\frac{25}{s(s+2.5)(s+10)}\right|_{S=-10} = \frac{25}{-10(2.5-10)} = 0.33$$

This gives rise to the expanded version of Equation 7.39:

$$V_{out}(s) = \frac{25}{s(s+2.5)(s+10)} = \frac{1}{s} - \frac{1.33}{s+2.5} + \frac{0.33}{s+10} \tag{7.42}$$

Each of the terms in Equation 7.42 has an entry in the Laplace Table. Taking the inverse Laplace transform of each of these terms separately gives:

$$v_{out}(t) = 1.0 - 1.33e^{-2.5t} + 0.33e^{-10t} \tag{7.43}$$

MATLAB is used as in the Example 7.4 to plot the two resulting time domain equations, Equations 7.38 and 7.43 in Figure 7.9.

The next example reiterates the role of partial fraction expansion in the solution of the inverse Laplace transform.

[6]If you are challenged by basic arithmetic you can easily factor any polynomial using the MATLAB `roots` routine. An example of using this routine to factor a fourth-order polynomial is given in Example 7.9.

7.3.6.3 *Partial Fraction Expansion—MATLAB*

Not surprisingly, MATLAB has a routine to perform partial fraction expansion. The routine also finds the roots of the denominator. (A MATLAB routine that only calculates the roots of a polynomial is described in the next section.) To expand a ratio of polynomials, you first define the numerator and denominator polynomials using two vectors: one that specifies the numerator coefficients, the other the denominator coefficients. For example, for the transfer function:

$$TF(s) = \frac{s^2+5s+15}{s(s^2+12.5s+25)} = \frac{s^2+5s+15}{s^3+12.5s^2+25s}$$

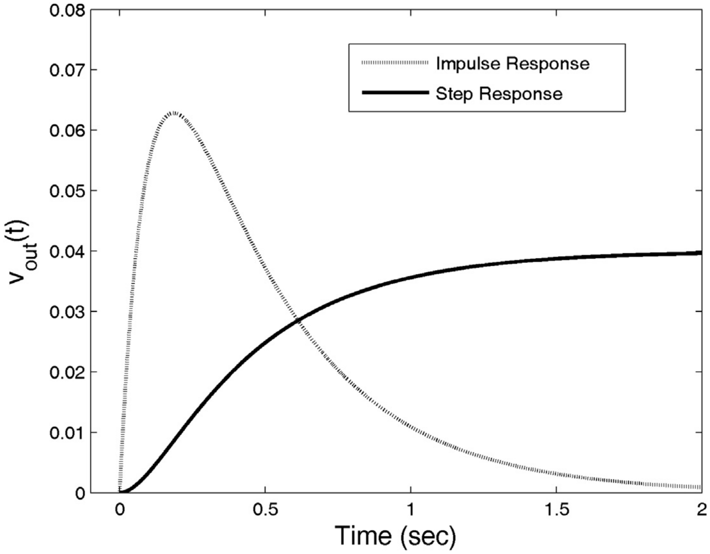


FIGURE 7.9 The impulse (*dashed curve*) and step (*solid curve*) responses of the transfer function used in Example 7.7.

The vector defining the numerator would be `b = [1, 5, 15];`, whereas the denominator would be defined as `a = [1, 12.5, 25 0];`. Note that since the constant term is missing in the denominator, a zero must be added to the vector definition. Once these vectors are defined, the partial fraction expansion terms are found using:

```
[numerators, roots, const] = residue(b,a)  % Partial fraction expansion
```

This routine gives the outputs:

```
numerators =
   0.8667
  -0.4667
   0.6000

roots =
   -10.0000
   -2.5000
     0

const =
   [ ]
```

and leads to the expanded transfer function.

$$TF(s) = \frac{0.6}{s} - \frac{0.47}{s+2.5} + \frac{0.87}{s+10}$$

The roots are the same as in Example 7.9 because the denominator equation is the same. The basic components of the time domain solution found from the inverse Laplace transform will also be the same. The numerator influences only the amplitude of these components. This again shows why the denominator polynomial is called the "characteristic equation" (Section 7.2.5.1).

EXAMPLE 7.7

Find the step response of the system having the following transfer function. Determine the solution analytically and also use MATLAB's `residue` routine.

$$TF(s) = \frac{\frac{s}{4}}{s^2 + 15s + 50}$$

Solution, Analytical: Since $In(s) = 1/s$, the output in Laplace notation becomes:

$$Out(s) = \left(\frac{1}{s}\right)\frac{\frac{s}{4}}{s^2 + 15s + 50} = \frac{.25}{s^2 + 15s + 50} \tag{7.44}$$

Next, we evaluate the value of δ by equating coefficients:

$$2\delta\omega_n = 15; \; \delta = \frac{15}{2\omega_n} = \frac{15}{2\sqrt{50}} = 1.06$$

Since $\delta = 1.07$, the roots will be real and the system will be overdamped. The next step is to factor the roots using the quadratic equation, Equation 7.30:

$$r_1, r_2 = \frac{-15}{2} \pm \frac{1}{2}\sqrt{15^2 - 4(50)} = -7.5 \pm 2.5 = -10.0, -5.0$$

$$\text{and the output becomes: } Out(s) = \frac{.25}{(s+10)(s+5)}$$

The inverse Laplace transform for this equation can be found in Appendix B (#9) where $\gamma = 10$ and $\alpha = 5$. To get the numerator constants to match, multiply top and bottom by $10 - 5 = 5$:

$$Out(s) = \left(\frac{0.25}{5}\right)\frac{5}{(s+10)(s+5)} = (0.05)\frac{5}{(s+10)(s+5)}$$

From the Laplace Transform Table we get the time function:

$$out(t) = 0.05\left(e^{-5t} - e^{-10t}\right) \tag{7.45}$$

Solution, MATLAB: For those of us who are arithmetically challenged, this problem could be solved using the MATLAB `residue` routine described in Section 7.2.6.3. First define the numerator and denominator vectors: `b = 0.25;` and `a = [1, 15, 50];`. The MATLAB command:

```
[numerators, roots, const] = residue(b,a)    % Partial fraction expansion
```

```
Gives:

Numerators =
  -0.0500
   0.0500

Roots =
  -10
  -5

Const =
  [ ]
```

This gives the expanded equation:

$$Out(s) = \frac{0.05}{s+5} - \frac{0.05}{s+10}$$

Result: Taking the inverse Laplace transform of the two first-order terms leads to the same time function found using the unfactored transfer function. Figure 7.10 shows $out(t)$ plotted using MATLAB code.

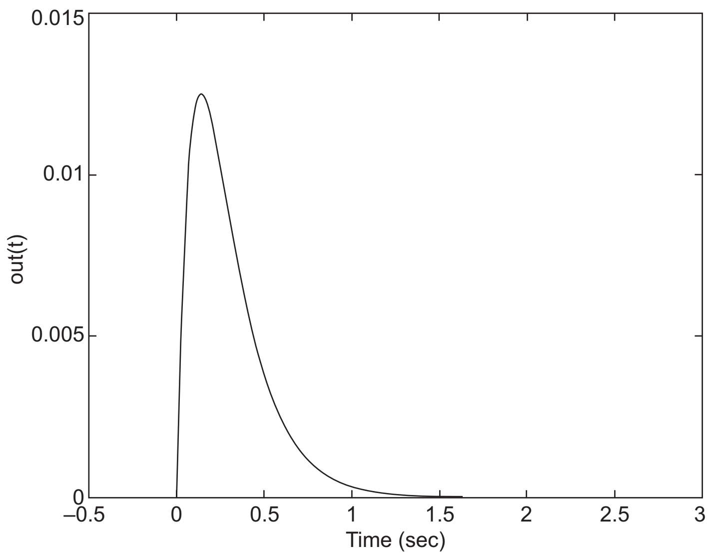


FIGURE 7.10 The impulse response of the second-order system in Example 7.7.

7.3.6.4 Second-Order Processes With Complex Roots

If the damping factor, δ, of a second-order transfer function is <1, then the roots of the characteristic (i.e., denominator) equation are complex and the step and pulse responses have the behavior of a damped sinusoid: a sinusoid that decreases in amplitude exponentially with time (i.e., of the general form $e^{-\alpha t}\sin\omega_n t$). Second-order systems that have complex roots are said to be "underdamped," a term relating to the oscillatory behavior of such systems. This terminology can be a bit confusing. Underdamped systems respond with damped sinusoidal behavior, whereas overdamped responses have double exponential responses and no oscillatory behavior.

If the roots are complex, the quadratic equation is not factored as the inverse Laplace transform can usually be determined directly from the Table. Inverse Laplace transforms for second-order underdamped responses are provided in the Table in terms of ω_n and δ and in terms of general coefficients (Transforms #13—17). Usually, the only difficulty in finding the inverse Laplace transform to these systems is in matching coefficients and scaling the transfer function to match the constants in the Table. The next example demonstrates the solution of a second-order underdamped system.

EXAMPLE 7.8

In a simple biomechanics experiment, a subject grips a rotating handle that is connected through an opaque screen to a mechanical system that can generate an impulse of torque. To create the torque impulse, a pendulum behind the screen strikes a lever arm attached to the handle. The subject is unaware of when the strike and the torque impulse it generates will occur. The resulting rotation of the wrist in degrees is measured under relaxed conditions. The rotation can be represented by the following second-order equation (Equation 7.46). As shown in Chapter 14, such an equation is typical of mechanical systems that contains a mass, elasticity, and friction or viscosity. In this example, we solve for the angular rotation, $\theta(t)$.

$$TF(s) = \frac{\theta(s)}{T(s)} = \frac{150}{s^2 + 6s + 310} \tag{7.46}$$

where $T(s)$ is the torque input and $\theta(s)$ is the rotation of the wrist in radians. The impulse function has a value of $s\ 2 \times 10^{-2}$ newton-meters.

Solutions: The values for δ and ω_n are found from the quadratic equation's coefficients.

$$\omega_n = \sqrt{310} = 17.6 \quad \text{and } \delta \text{ is}: \ 2\delta\omega_n = 6; \ \delta = \frac{6}{2\omega_n} = \frac{6}{2(17.6)} = 0.17$$

Since $\delta < 1$, the roots are complex.

The input $T(s)$ was an impulse of torque with a value of 2×10^{-2} newton-meters, so in the Laplace domain: $T(s) = 2 \times 10^{-2}$. $\theta(s)$ becomes:

$$\theta(s) = \left(2 \times 10^{-2}\right)\frac{150}{s^2 + 6s + 310} = \frac{3}{s^2 + 6s + 310} \tag{7.47}$$

Two of the transforms given in the Laplace tables will work (#13 and #15): entry #13 is used here with $b = 0$ (although #15 is more direct):

$$e^{-\alpha t}\left(\frac{c - b\alpha}{\beta}\sin(\beta t) + b\cos(\beta t)\right) \Leftrightarrow \frac{bs + c}{s^2 + 2\alpha s + \alpha^2 + \beta^2}$$

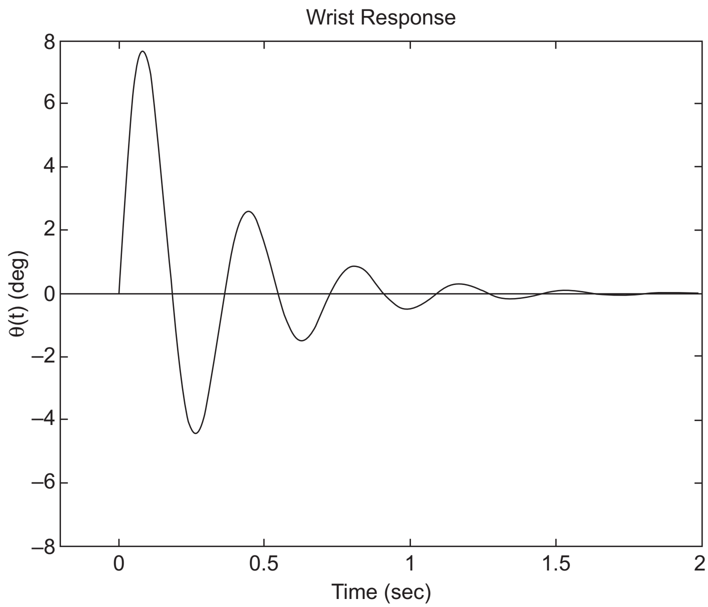


FIGURE 7.11 The response of the relaxed human wrist to an impulse of torque. This system is of second order, underdamped so the impulse response is a damped sinusoid.

Equating coefficients with Equation 7.47 we get:

$$c = 3; \ b = 0; \ \alpha = \frac{6}{2} = 3; \ \alpha^2 + \beta^2 = 310; \ \beta = \sqrt{310 - \alpha^2} = \sqrt{310 - 9} = 17.3$$

Substituting these values into the time domain equivalent on the left side of the Transform Table gives $\theta(t)$:

$$\theta(t) = e^{-3t}(0.173\sin(17.3t)) = 10e^{-3t}(\sin(17.3t))\ rad$$

Result: This response, scaled to degrees, is plotted in Figure 7.11 using MATLAB code.

7.3.7 Higher-Order Transfer Functions

Transfer functions that are higher than second-order can usually be factored into multiple second-order terms, or a combination of second- and first-order terms. The math would be tedious, so we turn to MATLAB for assistance. The MATLAB routine `roots` computes the roots of a polynomial. The polynomial equation is defined by a vector, `c`, that contains the polynomial's coefficients beginning with the highest power in the polynomial and continuing down to the constant. For example, the roots of the polynomial:

$$s^4 + 14s^3 + 25s^2 + 10s + 35$$

are found using: `roots([1, 14, 25, 10, 35]);`.

If a root is real, it is represented by a first-order term. Complex roots are given as complex conjugates and should be combined into second-order terms. This is achieved by multiplying the roots together. For example, the polynomial above has four roots: two are real and two are complex conjugates, $0.1431 \pm j1.1122$. We could combine these by multiplying $(s + 0.1431 + j1.1122)\ (s + 0.1431 - j1.1122)$. If we want to make our life easier and avoid doing the complex arithmetic, we could use the MATLAB routine `poly`, which performs the inverse of `roots`. So the command:

```
poly([0.1431 + j*1.1122, 0.1431 – j*1.1122]);
```

gives the polynomial coefficients: 1.0, −0.2862, 1.2575 corresponding to the second-order polynomial $s^2 - 0.2862s + 1.2575$.

The use of `roots` and `poly` to factor a fourth-order transfer function is illustrated in the next example. After factoring, we find the magnitude and phase spectrum. This example uses MATLAB to generate the plot, but the plotting could also have been done using the graphical Bode plot methods developed in Chapter 6.

EXAMPLE 7.9

Factor the transfer function shown below into first- and second-order terms. Write out the factored transfer function and plot the Bode plot (magnitude and phase). Use `roots` to factor the numerator and denominator and use `poly` to rearrange complex congregate roots into second-order terms. Generate the Bode plot from the factored transfer function.

$$TF(s) = \frac{s^3 + 5s^2 + 3s + 10}{s^4 + 12s^3 + 20s^2 + 14s + 10}$$

Solution: The numerator contains a third-order term and the denominator contains a fourth-order term. We use the MATLAB `roots` routine to factor these two polynomials and rearrange them into first- and second-order terms, then substitute $s = j\omega$ and plot. Of course, we could plot the transfer function directly without factoring, but the factors provide some insight into the system. The first part of the code factors the numerator and denominator.

```
% Example 7.9 Find the Bode plot of a higher order transfer function.
%
num = [1 5 3 10];          % Define numerator polynomial
den = [1 12 20 14 10];     % Define denominator polynomial
n_root = roots(num)        % Factor numerator and display
d_root = roots(den)        % Factor denominator and display
```

The results of this program are:

```
n_root = -4.8087
         -0.0957 + 1.4389i
         -0.0957 - 1.4389i
```

```
d_root = -10.1571
         -1.4283
         -0.2073 + 0.8039i
         -0.2073 - 0.8039i
```

The numerator factors into a single root and a complex pair indicating a second-order term. The denominator factors into two roots and one complex pair. To combine two complex roots into a single second-order term we use `poly`:

```
nun_sec_ord = poly([-0.0957+j*1.4389,-0.0957-j*1.4389])      % 2nd-order numerator
den_sec_order = poly([-0.2073+j*0.8039,-0.2073-j*0.8039])    % 2nd-order denominator
```

This gives the output:

```
nun_sec_ord =  1.0000  0.1914  2.0797
den_sec_order =  1.0000  0.4147  0.7893
```

Results: Combining these first- and second-order roots, the factored transfer function is written as:

$$TF(s) = \frac{(s+4.81)(s^2+0.19s+2.08)}{(s+10.2)(s+1.43)(s^2+0.41s+0.69)}$$

The Bode plot can be obtained by substituting $j\omega$ for s in this equation and evaluating over ω where now each of the terms is identifiable as one of the Bode plot elements presented in Chapter 6. As mentioned, although we could use MATLAB to plot the unfactored transfer function, factoring shows us the Bode plot primitives (see Table 6.2), providing insight into what the elements make up the system. In this example, we use MATLAB to plot this function so there is no need to rearrange this equation into the standard Bode plot format (i.e., normalized so that the constant terms are equal to 1).

```
% Plot the Bode Plot
w = 0.1:.05:100;                          % Define the frequency vector
TF = (j*w + 4.81).*(-w.^2+j*0.19*w + 2.08)...
  ./((j*w +10.2).*(j*w+1.43).*(-w.^2+j*0.41*w+0.79));   % Define TF
subplot(2,1,1);
semilogx(w,20*log10(abs(TF)));            % Plot magnitude spectrum in dB
  .......labels......
subplot(2,1,2);
  semilogx(w,angle(TF)*360/(2*pi));       % Plot the phase spectrum
  .......labels.......
```

The resulting spectrum plot is shown in Figure 7.12.

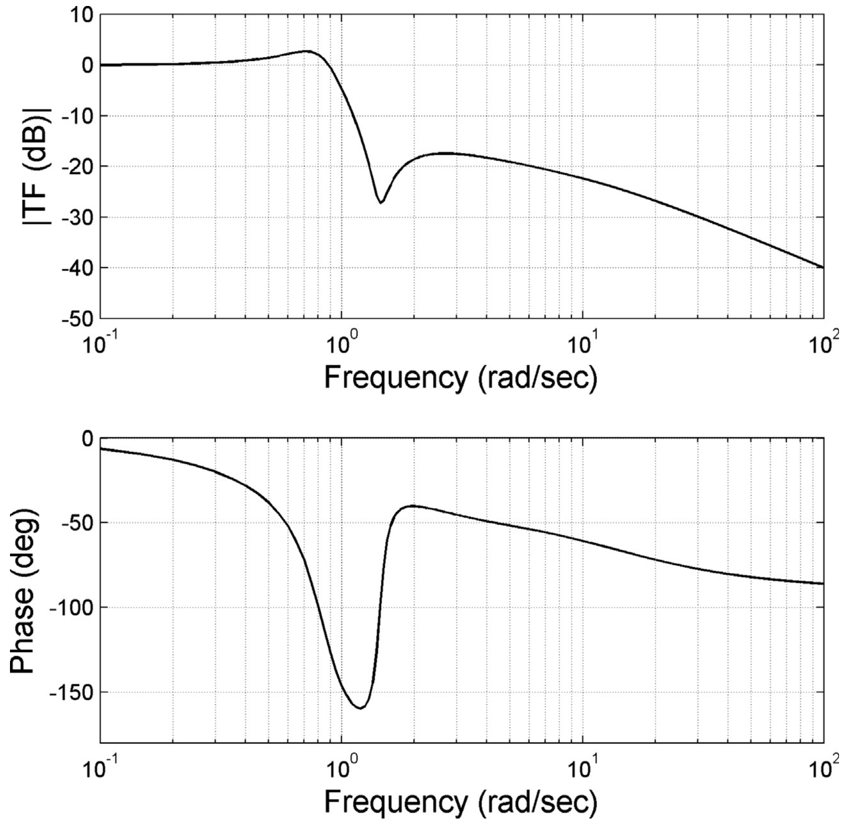


FIGURE 7.12 The magnitude and phase frequency characteristic (i.e., Bode plot) of the higher-order transfer function used in Example 7.9.

7.4 NONZERO INITIAL CONDITIONS—INITIAL AND FINAL VALUE THEOREMS

7.4.1 Nonzero Initial Conditions

Although the Laplace analysis method cannot deal with negative values of time, it can handle systems that have nonzero conditions at $t = 0$. So one way of dealing with systems that have a history for $t < 0$ is to summarize that history as an initial condition at $t = 0$. To evaluate systems with initial conditions, the full Laplace domain equations for differentiation and integration must be used. These equations, Equations 7.5 and 7.7 are repeated here.

$$\mathfrak{L}\frac{dx(t)}{dt} = sX(s) - x(0-) \tag{7.48}$$

$$\mathfrak{L}\left[\int_0^T x(t)dt\right] = \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0} x(t)dt \tag{7.49}$$

In both cases, we add a term to the standard Laplace operator. In taking the derivative, the term is a constant: the negative of the initial value of the variable; that is, $-x(0-)$. With integration, the term is a constant divided by s where the constant is the integral of the past history of the variable from minus infinity to zero. This is because the current state of an integrative process is the result of integration over all past values.

So for systems with nonzero initial conditions, it is only necessary to add the appropriate terms in Equations 7.48 and 7.49. An example is given in the solution of a one-compartment diffusion model. This model is a simplification of diffusion in biological compartments such as the cardiovascular system. The one-compartment model would apply to large molecules in the blood that cannot diffuse into tissue and are only slowly eliminated. The compartment is assumed to provide perfect mixing; that is, the mixing of blood and substance is instantaneous and complete.

EXAMPLE 7.10

Find the concentration, $c(t)$, of a large molecule solute delivered as a step input to the blood compartment. Assume an initial concentration of $c(0)$ and a diffusion coefficient, K.

Solution: The kinetics of a one-compartment system with no outflow is given by the differential equation:

$$V\frac{dc(t)}{dt} = F_{in} - Kc(t)$$

where V is the volume of the compartment, K is a diffusion coefficient, and F_{in} is the input, which is assumed to be a step change of a given amount A: $F_{in} = A\mu(\mathrm{t})$. Converting this to the Laplace domain, $F_{in}(s) = A/s$. Next we assume that the initial concentration of solute is $c(0)$. Substituting A/s for $F_{in}(s)$, converting to Laplace notation, and applying the derivative operation with the initial condition (Equation 7.48):

$$V(sC(s) - c(0)) = \frac{A}{s} - KC(s)$$

Solving for the concentration, $C(s)$:

$$C(s)(Vs + K) = Vc(0) + \frac{A}{s}$$

$$C(s) = \frac{A}{s(Vs + K)} + \frac{Vc(0)}{Vs + K} = \frac{\frac{A}{V}}{s\left(s + \frac{K}{V}\right)} + \frac{c(0)}{s + \frac{K}{V}}$$

Taking the inverse Laplace transform:

$$c(t) = \frac{\frac{A}{V}}{\frac{K}{V}}\left(1 - e^{-tK/V}\right) + c(0)e^{-tK/V} = \left(\frac{A}{K}\right)\left(1 - e^{-tK/V}\right) + c(0)e^{-tK/V}$$

Results: We can interpret the solution as consisting of two components: the exponentially increasing concentration from an initial value of $c(0)$ to A/K resulting from the step input and an

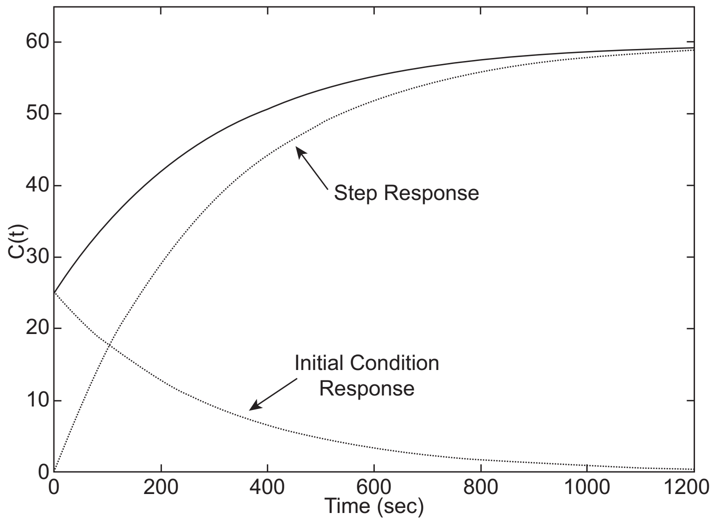


FIGURE 7.13 The diffusion of a large molecule solute delivered as a step input into the blood compartment. The molecule had an initial concentration of 25 g/mL. The solution (*dark curve*) can be viewed as having two components (*light curves*): the exponentially increasing concentration due to the step input and the exponential decay of the initial concentration. Perfect mixing in the blood compartment is assumed.

exponential decay of the initial value to zero. This second component is what we would see if the step input is not present. Using typical values from Rideout (1991) of $A = 0.7$ g, $V = 3.0$ mL, $K = 0.01$, $c(0) = 25.0$ g/mL, leads to the results for both components and their sum as plotted in Figure 7.13.

Nonzero initial conditions also occur in circuits where preexisting voltages exist or in mechanical systems that have nonzero initial velocities, so you can look forward to more examples of systems with nonzero initial conditions in Chapter 13.

7.4.2 Initial and Final Value Theorems

The time representation of a Laplace function is obtained by taking the inverse Laplace transform using tables such as that found in Appendix B. But sometimes we are only looking for the value of the function at the very beginning of the stimulus, $t = 0$, or at its very end, $t \to \infty$. Two useful theorems can supply us with this information without the need to take the inverse Laplace transform: the "Initial and Final Value Theorems." These theorems give us the initial and the final output values directly from the Laplace transform equation.

Recall that frequency and time are inversely related: $f = 1/T$. For example, as the period of a sine wave increases, its frequency decreases and vice versa. The Laplace variable, s, is complex frequency so we might expect to find the value of a Laplace function at $t = 0$ by letting $s \to \infty$. The Initial Value Theorem supports this idea, but we must first multiply $X(s)$ by s, then let $s \to \infty$:

$$x(0+) = \lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s) \tag{7.50}$$

The Final Value Theorem follows the same logic, but now since $t \rightarrow \infty$, it is s that goes to 0. The Final Value Theorem states:

$$x(\infty) = \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \tag{7.51}$$

The application of either theorem is straightforward as is shown in the following example.

EXAMPLE 7.11

Use the Final Value Theorem to find the final value of $x(t)$ to a step input for the system whose transfer function is given below. Also find the final value the hard way: by determining $x(t)$ from the inverse Laplace transform, then letting $t \rightarrow \infty$.

$$TF(s) = \frac{.28s + .23}{s^2 + 0.3s + 2} \tag{7.52}$$

Solution, Inverse Laplace Transform: First find the output Laplace function $X(s)$ by multiplying $TF(s)$ by the step input function in the Laplace domain:

$$X(s) = \frac{1}{s}TF(s) = \frac{.28s + 0.92}{s(s^2 + 0.03s + 2)} \tag{7.53}$$

Next we find the full expression for $x(t)$ from the inverse Laplace transform. We need to examine δ to determine if the roots are real or complex:

$$2\delta\omega_n = 0.3; \;\; \delta = \frac{0.3}{2\omega n} = \frac{0.3}{2\sqrt{2}} = 0.106 < 1.$$

So the system is underdamped and one of the transfer functions #13–17 in the Laplace Transform Table should be used to find the inverse. The function $X(s)$ matches entry #14 in the Laplace Transform Table, but requires some rescaling to match the numerator.

$$1 - e^{-\alpha t}\left(\frac{\alpha - b}{\beta}\right)\sin \beta t + b \cos \beta t \Leftrightarrow \frac{bs + \alpha^2 + \beta^2}{s\left(s^2 + 2\alpha s + \alpha^2 + \beta^2\right)}$$

Considering only the denominator, the sum $\alpha^2 + \beta^2$ should equal 2, but then the numerator needs to be rescaled so the numerator constant is also 2 (i.e., $\alpha^2 + \beta^2 = 2$). To make the numerator constants match we need to multiply the numerator in Equation 7.53 by: $2/0.92 = 2.17$. Multiplying top and bottom by 2.17, the rescaled Laplace function becomes:

$$X(s) = \left(\frac{1}{2.17}\right)\frac{0.61s + 2}{s(s^2 + 0.3s + 2)} = \frac{0.46(0.61s + 2)}{s(s^2 + 0.3s + 2)}$$

Now equating coefficients with entry #14:

$$b = .61; \;\; \alpha = \frac{0.3}{2} = 0.15; \;\; \alpha^2 + \beta^2 = 2; \;\; \beta^2 = 2 - 0.15^2; \;\; \beta = \sqrt{1.98} = 1.41$$

The inverse Laplace Transform becomes:

$$x(t) = 0.46\left[1 - e^{-.15t}\left(\frac{.15 - .61}{1.41}\sin(1.41t) + \cos(1.41t)\right)\right]$$

$$x(t) = 0.46\left(1 - e^{-.15t}(-0.33\sin(1.41t) + \cos(1.41t))\right)$$

Now letting $t \to \infty$, the exponential term goes to zero and the final value becomes:

$$x(t) = 0.46$$

Solution, Final Value Theorem: This approach is much easier. Substitute the output Laplace function given in Equation 7.52 into Equation 7.51:

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left[\frac{0.281s + 0.92}{s(s^2 + 0.3s + 2)}\right] = \lim_{s\to 0}\left[\frac{0.28s + 0.92}{s^2 + 0.3s + 2}\right] = \frac{0.921}{2} = 0.46$$

This is the same value that is obtained when letting $t \to \infty$ in the time solution $x(t)$.

An example of the application of the Initial Value Theorem is found in the problems.

7.5 THE LAPLACE DOMAIN, THE FREQUENCY DOMAIN, AND THE TIME DOMAIN

We know how to move between the Laplace domain and the time domain: take the Laplace transform or its inverse. To move between the frequency domain and the time domain: take the Fourier transform or its inverse. To move from the Laplace domain to the frequency domain, we note that s is complex frequency, $\sigma + j\omega$, so we just do away with the real part, σ, and substitute $j\omega$ for s. (We should also renormalize the coefficients so the constant term equals 1, particularly if we are planning to use Bode plot techniques.) To make this transformation, we must assume that we are dealing with periodic steady-state signals only. Since we usually work with systems in the Laplace domain, converting the other way, from the frequency domain to the Laplace domain ($j\omega \to s$), is not common and we would also need to assume zero initial conditions. These transformations are summarized in Table 7.1.

Some of the relationships between the Laplace transfer function and the frequency domain characteristics have already been mentioned, and these depend largely on the characteristic equation. A first-order characteristic equation gives rise to first-order frequency characteristics such as those shown in Figures 6.13 and 6.14.

Second-order frequency characteristics, like second-order time responses, are highly dependent on the value of the damping coefficient, δ. As shown in Figure 6.17, the frequency curve shows a peak for values of $\delta < 1$, and the height of that peak increases as δ decreases. This peak occurs at the undamped natural frequency, ω_n.

The peaks in the frequency domain have dramatic correlates in the time domain. As illustrated in the next example, when $\delta < 1$ the response overshoots the final value, oscillating around this value at frequency ω_d. This oscillation frequency is not quite the same as the undamped frequency, ω_n, although for small values of δ the oscillation frequency approaches ω_n (Equation 7.34). The undamped natural frequency, ω_n, is the frequency that the system would like to oscillate at if there was no damping, that is, if it could oscillate unimpeded. But the decay in the oscillation caused by the damping lowers the actual oscillation frequency to ω_d. As the damping factor, δ, decreases, its influence on oscillation frequency, ω_d, is reduced, so it approaches the undamped natural frequency, ω_n, as described in Equation 7.34.

The next example uses MATLAB to compare the frequency and time characteristics of a second-order system for various values of damping.

EXAMPLE 7.12

A system with the following transfer function has an undamped natural frequency of 10,000 rad/s and can have three different values of the damping factor: 0.05, 0.1, and 0.5.

$$TF(s) = \frac{1}{s^2 + 200\delta s + 10^4}$$

Plot the frequency spectrum (i.e., Bode plot) of the time domain transfer function and the step response of the system for the three damping factors. Use the Laplace transform to solve for the time response and MATLAB for calculation and plotting.

Solution, Time Domain: The step response can be obtained by multiplying the transfer function by $1/s$, then determining the inverse Laplace transform leaving δ as a variable:

$$V_{out}(s) = \frac{1}{s(s^2 + 2 * 10^2\delta s + 10^4)}$$

This matches entry #17 for $\delta < 1$, although a minor rescaling is required:

$$V_{out}(s) = \left(\frac{1}{10^4}\right)\frac{10^4}{s(s^2 + 200\delta s + 10^4)}$$

The time domain solution is:

$$v_{out}(t) = \left(\frac{1}{10^4}\right)\left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\sin\left(\omega_n\sqrt{1-\delta^2}\;t + \theta\right)\right); \;\; \theta = \tan^{-1}\left(\frac{\sqrt{1-\delta^2}}{\delta}\right)$$

where $\omega_n^2 = 10^4$, $\omega_n = 100$.

The equivalent time response will be programmed directly into the MATLAB code.

Solution: Frequency Domain: To find the frequency response, convert the Laplace transfer function to a frequency domain transfer function by substituting $j\omega$ for s and rearranging into frequency domain format where the constant is normalized to 1.0:

$$TF(\omega) = \frac{1}{(j\omega)^2 + 200\delta j\omega + 10^4} = \frac{1}{1 - 10^{-4}\omega^2 + 0.02\delta j\omega}$$

This equation can also be programmed directly into MATLAB. The following is the resulting program.

```
% Example 7.12 Comparison of time and frequency characteristics of a second-order
system
% with different damping factors.
%
w = 1:1:10000;                  % Frequency vector: 1 to 10,000 rad/sec
t = 0:.0001:.4;                 % Time vector: 0 to 0.4 sec.
delta = [0.5 0.1 0.05];         % Damping factors
% Calculate spectra
hold on;                        % Plot superimposed
for k = 1:length(delta)         % Repeat for each damping factor
```

```
  TF = 1./(1 - 10.^-4*w.^2 + j*delta(k)*.02*w); % Transfer function
  Mag = 20*log10(abs(TF));                      % Take magnitude in dB
  semilogx(w,Mag);                              % Plot semilog (log frequency)
end
.........labels and title........
% Calculate and plot time characteristics
figure ; hold on;                               % Plot superimposed
wn = 100;                                       % Undamped natural frequency
for k = 1:length(delta)
  c = sqrt(1-delta(k)^2);                       % Useful constant
  theta = atan(c/delta(k));                     % Calculate theta
  xt = (1 - (1/c)*(exp(-delta(k)*wn*t)).*(sin(wn*c*t + theta)))/1000; % Time
function
  plot(t,xt);
end
.......label, title, and text.......
```

Results: The results are shown in Figures 7.14 and 7.15. The correspondence between the frequency characteristics and the time responses is evident. When a system's spectrum has a peak, the system's time response has an overshoot. Note that the frequency peak associated with a δ of 0.5 is modest, but the time domain response still has some overshoot. As shown here and in one of the problems, the larger the spectral peak, the greater the overshoot. The minimum δ for no overshoot in the response is left as an exercise in one of the problems.

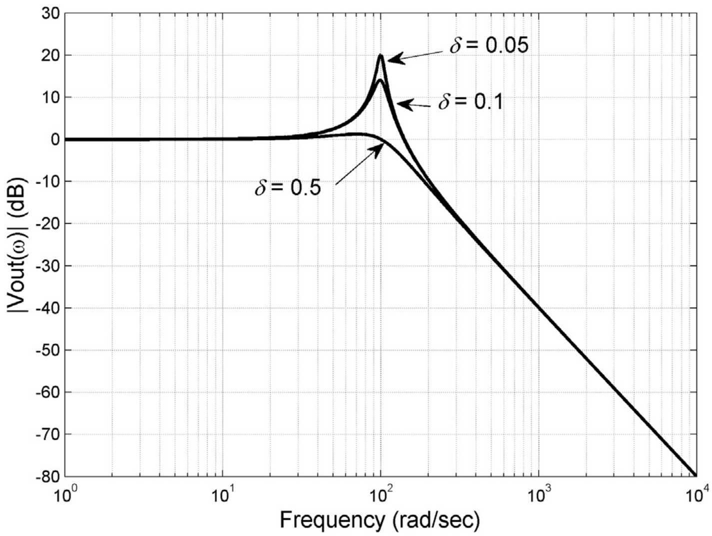


FIGURE 7.14 Comparison of magnitude spectra of the second-order system used in Example 7.12 having three different damping coefficients: $\delta = 0.05$, 0.1, and 0.05.

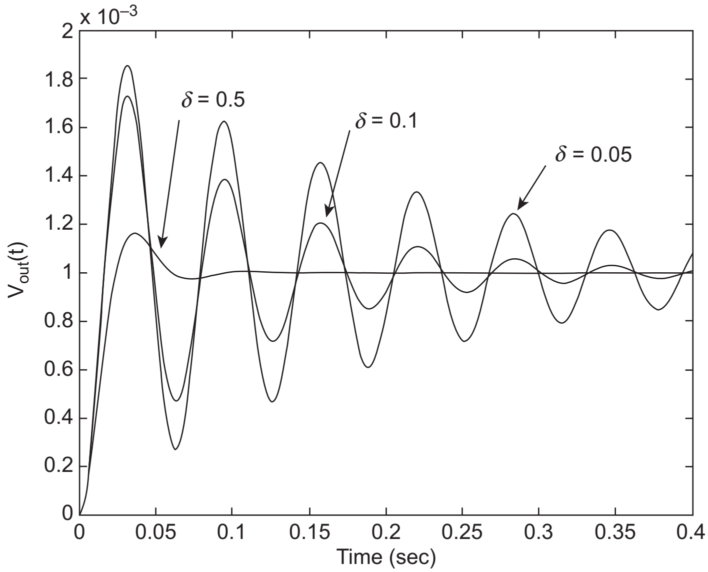


FIGURE 7.15 Comparison of time domain step responses for a second-order system with three different damping factors. The magnitude spectrum for this system is shown in Figure 7.14.

7.6 SYSTEM IDENTIFICATION

Bioengineers often face complex systems with unknown internal components. In such cases, it is usually impossible to develop equations for system behavior to construct the frequency characteristics. However, if you can control the stimulus to the system, and measure its response, you should be able to determine the system's spectrum experimentally, provided that the system is linear or can be taken as linear. Once you determine the system's spectrum, its transfer function can be estimated using Bode plot methods from Chapter 6. Then the system's response to any input can easily be computed. Finding a system's transfer function from external behavior is called "system identification."

The are several approaches to identifying a system if we can control, or at least have access to, its input. If we can generate an impulse input, we can determine its spectrum from the impulse response by taking the Fourier transform of the impulse response (see Section 5.3.2). If the inputs are sinusoidal, or are decomposed into sinusoids, then we can estimate the frequency characteristics by taking the ratio of output amplitude to input amplitude at each frequency. That is, we transform the input (if needed) and output signals to the frequency domain and divide *Output*(ω) by *Input*(ω) to get *TF*(ω), as shown in Equation 6.23. This approach works if we can measure the input signal as long as that signal contains energy over the frequency range of the system.

Another powerful method that works as long as we can measure both input and output signals is model-based simulation. In this approach, we construct a model we believe represents the system, then adjust the model parameters (or elements) until the inputs and outputs match our data. The result is not just a transfer function that matches our system, but a representation of possible internal biological elements. Simulation approaches are explored in Chapter 9.

Here we use frequency-based methods to find the spectrum of several unknown systems, then apply Bode plot primitives to estimate the transfer function. In the following three examples of system identification, the first uses Fourier decomposition, the second uses sinusoidal and impulse inputs, and the last example uses measurements of the input and output signals.

EXAMPLE 7.13

Use white noise to estimate the magnitude spectrum of a system represented by MATLAB routine y = `unknown_sys7_1(x).m`. The input argument, `x`, is taken as the input signal and the output argument, `y`, is the output signal. Plot the magnitude spectrum, then apply Bode plot primitives to estimate the transfer function.

Solution: Since we have complete control of the input signal (not often the case in dealing with biological systems), we use an input signal that contains energy over a broad range of frequencies. As discussed in Chapter 1, a random white noise signal has equal energy over all frequencies or, for digital signals, equal energy up to $f_s/2$ (see Figure 1.13). With a random signal as our input, we take the Fourier transform of both this input signal and the system's output. We divide the two frequency domain signals and plot the magnitude (in dB) as the spectrum of our transfer function. As always, we only plot the meaningful spectral points below $f_s/2$.

We initially select a sampling frequency of 1000 Hz. This gives us a spectrum that ranges between 0 and 500 Hz. As we have no knowledge of the system's transfer function, we do not know if this range is sufficient to cover the frequencies of interest. We should be prepared to adjust f_s to cover other frequencies if they are needed to define the transfer function. We use a signal with a large number of points, $N = 10{,}000$, to improve the resolution.

```
% Example 7.13 Use MATLAB to find the transfer function of an unknown
% system from its input and output signals.
%
fs = 1000;                          % Assumed sample rate
N = 10000;                          % Number of points
nf = round(N/2);                    % Number of valid spectral points
f = (0:N-1)*fs/N;                   % Frequency vector for plotting
t = (1:N)/fs;                       % Time vector for plotting
x = rand(1,N);                      % Input signal
y = unknown_sys7_1(x);              % Unknown system
X = fft(x);                         % Convert input to freq. domain
Y = fft(y);                         % Convert output to freq. domain
TF = Y./X;                          % Find TF
Mag = 20*log10(abs(TF));            % TF in dB
semilogx(f(1:nf),Mag(1:nf),'k');    % Plot magnitude, valid points only
```

Results: The spectrum produced by this code is shown in Figure 7.16. To convert these frequency characteristics into transfer functions, we need to rely on the skills developed in the last chapter. The spectrum looks like a combination of three Bode plot primitives: a second-order underdamped

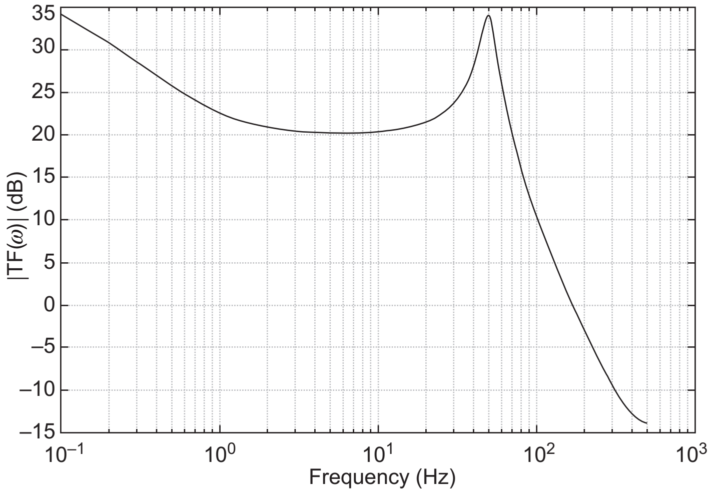


FIGURE 7.16 The spectrum of a signal from the unknown system used in Example 7.13 when the input is white noise. Since white noise has energy at all frequencies, this spectrum is the same as the system's spectrum and can be used to estimate the transfer function.

element, an integrator (i.e., $1/j\omega$), and an inverted low-pass filter (a numerator $1 + j\omega/\omega_1$). In fact, it looks a lot like the spectrum in Figure 6.25, which was derived from a transfer function having these three elements. Thus the transfer function of this system has the general form:

$$TF(\omega) = K\frac{1 + j\dfrac{\omega}{\omega_1}}{j\omega\left(1 + \left(\dfrac{\omega}{\omega_n}\right)^2 + j2\delta\dfrac{\omega}{\omega_n}\right)}$$

We can get the parameters for ω_1, ω_n, and δ from the spectral plot. The peak occurs at 50 Hz, so $\omega_n = 2\pi f = 2\pi(50) = 314$ rad sec. and the descending low frequency slope is within 3 dB of leveling off at about 2 Hz so $\omega_1 = 2\pi f = 2\pi(2) = 12.6$ rad/sec. To find the baseline gain, K, note that at $f = 0$ Hz, the integrator element has a gain of 0.0 (Figure 6.11) as do the other elements, but the spectrum has a gain of 20 dB. So K must be 10 (i.e., 20 dB). To find δ, note that the peak is about 14 dB above the baseline, so:

$$-20\log(2\delta) = 14dB; \quad -\log(2\delta) = 0.7; \quad 1/(2\delta) = 10^{-0.7}; \quad \delta = .1999/2 = 0.1$$

This gives the complete transfer function as:

$$TF(\omega) = 10\frac{1 + j.08\omega}{j\omega\left(1 - (0.0032\omega)^2 + j0.00064\omega\right)}$$

If we ignore initial conditions: we can rewrite this transfer function in Laplace notation:

$$TF(s) = \frac{0.8s\left(s + \dfrac{1}{0.08}\right)}{0.0032^2 s\left(s^2 + \dfrac{0.00064s}{0.0032^2} + \dfrac{1}{0.0032^2}\right)} = \frac{78125(s + 12.6)}{s(s^2 + 62.5s + 97656)}$$

White noise is not an easy stimulus to induce in most biological systems. Another way to determine frequency response experimentally is to take advantage of the fact that a sinusoidal stimulus into a linear system will produce a sinusoidal response at the same frequency. By stimulating the biological system with sinusoids over a range of frequencies and measuring the change in amplitude and phase of the response, we can construct a plot of the frequency characteristics by simply combining all the individual measurements. This approach is illustrated in the next example.

EXAMPLE 7.14

Find the magnitude of spectral characteristics of the process represented by `unknown_sys7_2(x).m`. Use sinusoids to identify the spectrum and Bode plot primitives to estimate the transfer function. Also determine the system spectrum from the impulse response and compare. The actual magnitude spectrum in dB can be found as the second output argument of `unknown_sys7_2(x).m`.

Solution: Generate a sinusoid with an RMS value of 1.0. This requires the amplitude to be 1.414. Input this sinusoid to the unknown process, and measure the RMS value of the output. The RMS value is usually a more accurate measurement of a signal value than the peak-to-peak amplitude as it is less susceptible to noise-induced error. Repeat this protocol for increasing frequencies until the output falls to very low levels (in this case up to 400 Hz). Plot the results in dB against log frequency. Also construct an impulse signal and input it to the system. Take the Fourier transform of the impulse response and plot in dB against log frequency. Finally, plot the actual system spectrum and compare it with the two experimentally obtained spectra.

To find the transfer function from the spectrum, use the sinusoidal responses as they are likely to be more accurate. Estimate asymptotes if need be and apply Bode techniques to determine the transfer function.

```
% Example 7.14 Identify an unknown system unknown_sys7_2.
%
N = 1000;                        % Input signal length
t = (0:N-1)/N;                   % Time vector ( 1 sec)
for k = 1:400                    % Try frequencies up to 400 Hz
  f(k) = k;                      % Freq = k = 1-400Hz
  x = 1.414*cos(2*pi*f(k)*t);    % Sinusoid with RMS = 1.0
  y = unknown_sys7_2(x);         % Input stimulus to process
  Y(k) = sqrt(mean(y.^2));       % Calculate RMS value as magnitude
end
Y = 20*log10(Y);                 % Convert to dB
% Now use the impulse response for system identification
x1 = [1.0, zeros(1,N-1)];        % Generate impulse response
[y1, spec] = unknown_sys7_2(x1);        % Get impulse response and true spectrum
Mag = abs(fft(y1));              % Get magnitude spectrum
Y1 = 20*log10(Mag(1:400));       % Convert to dB
semilogx(f,Y,'k','LineWidth',1); hold on;     % Plot magnitude as dB/log f
semilogx(f,Y1,':k','LineWidth',2);            % Plot as dB/log f
semilogx(f,spec(1:400),'--k','LineWidth',2);  % Plot actual spectrum
  .......labels.......
```

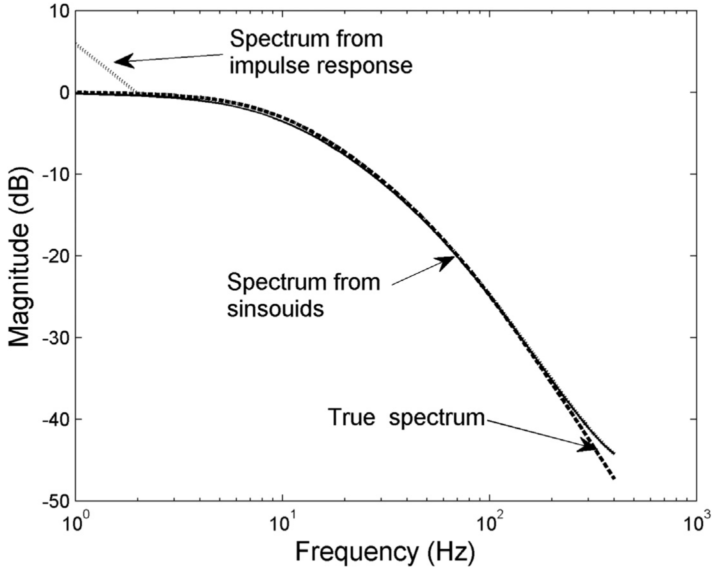


FIGURE 7.17 Three spectra generated in Example 7.14: one from sinusoidal simulation (*solid line*), another from the impulse response (*dashed line*), and a third showing the true spectrum (*dotted line*). Except at the lowest frequencies, the three spectra overlap and are hard to distinguish.

Results: The three spectra are plotted superimposed in Figure 7.17 with labels. Note that the spectrum generated from sinusoids (solid line) closely matches the true spectrum (dotted line). The spectrum determined from the impulse response (dashed line) deviates slightly at the higher frequencies. This is likely due to computational error at the low output amplitudes.

Based on the spectrum from sinusoidal stimulation, the process appears overdamped, as there are no spectral peaks (Figure 7.18). At the higher frequencies, the slope is 40 dB/decade, indicating a second-order system. The lower frequencies show a slope of 20 dB/decade. Applying Bode plot techniques, we fit the curve with lines of 20 and 40 dB/decade (Figure 7.18 dashed lines).

From the spectrum of Figure 7.18, it looks like this system consists of two first-order elements with cutoff frequencies somewhere around 10 and 80 Hz (or 63 and 503 rad/sec). The gain is around 0 dB or 1.0. So an estimate of the frequency domain transfer function of this system would be:

$$TF(\omega) = \frac{1.0}{\left(1+\dfrac{j\omega}{63}\right)\left(1+\dfrac{j\omega}{503}\right)} = \frac{1.0}{(1+j0.0159\omega)(1+j0.002j\omega)}$$

Assuming no initial conditions, the transfer function in Laplace notation becomes:

$$TF(s) = \frac{1.0}{(0.0159)(0.002)\left(s+\dfrac{1}{0.0159}\right)\left(s+\dfrac{1}{0.002}\right)} = \frac{3.14\times 10^4}{(s+63)(s+503)}$$

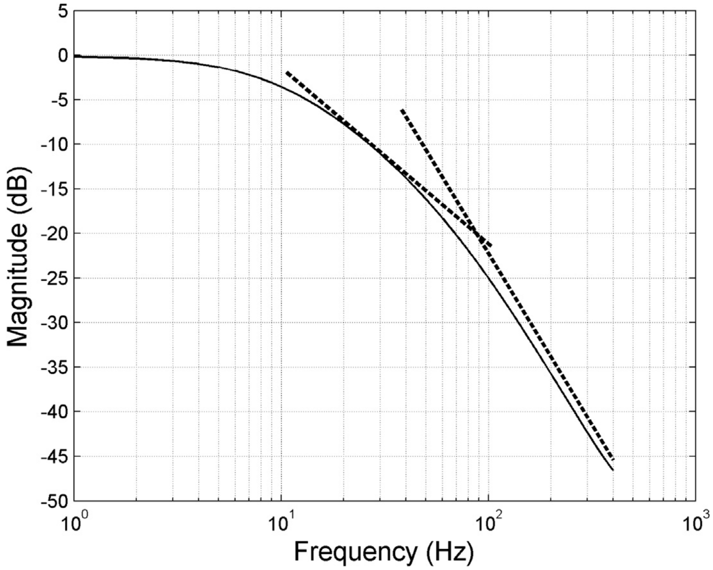


FIGURE 7.18 The magnitude spectrum of an unknown system that is represented in the routine `unknown_sys7_2.m`. The spectrum was determined by stimulating the system with sine waves ranging in frequency from 1 to 400 Hz. The stimulus sine waves all had root mean square (RMS) values of 1.0, so the RMS values of the output indicate the magnitude spectrum of the unknown system.

When it is not practical to simulate biological systems, the system's natural input can be used as long as it contains energy covering the range of the system's spectrum. To estimate the Fourier transform, you take the Fourier transform of the output signal and divide it by the Fourier transform of the input signal. The next example embodies this approach to identify a biological system.

EXAMPLE 7.15

The data file `bio_sys.mat` contains the input and output signals of a biological system in variables `x` and `y`, respectively. $f_s = 150$ Hz. This file also contains the true spectrum of the system in variable `spec`. Determine if the input signal can be used to accurately determine the system's spectrum. If so, estimate that spectrum and then use Bode plot primitives to find the related transfer function.

Solution: Use the Fourier transform to calculate both the input and output magnitude spectra, then plot. Check to see if the input spectrum has energies out to a frequency range where the output spectrum is considerably attenuated. In other words, ensure that a decrease in the output spectrum is due to the system and not insufficient energy in the input spectrum. Then divide the magnitude spectrum of the output by the input to get an estimate of the system's spectrum. Use Bode plot methods to estimate the transfer function.

```
% Example 7.15 Identify a biological system from input/output data.
%
load bio_sys;
fs = 150;                   % Sample frequency
N = length(x);              % Signal length
N_2 = round(N/2);           % Half signal length for fft
f = (1:N)*fs/N;             % Frequency vector for plotting
X = abs(fft(x));            % Fourier transform of the input signal
Y = abs(fft(y));            % Fourier transform of the output signal
.......linear plot of x and y, label, new figure.........
TF = Y./X;                  % Calculate magnitude transfer function
TF_dB = 20*log10(TF);       % in dB
.......semilog plot, label.........
```

Results: Figure 7.19 shows the input and output spectra plotted as linear functions. Although the energy in the input spectrum falls off at the higher frequencies, it does appear to have energy over the range of system output frequencies except possibly at 30 and 60 Hz.

Since the input spectrum appears to contain sufficient energy over the frequency range of interest, the ratio of output spectrum to input spectrum should give a reasonable estimate of the system's spectrum. The result of dividing the output spectrum by the input spectrum produces the system's spectrum estimate shown in Figure 7.20 (solid line). This estimated spectrum closely follows the actual spectrum (dashed line) except at the two frequency extremes. Despite the deviations at the high and low frequencies, the estimated spectrum is sufficient to determine the transfer function using Bode plot methods.

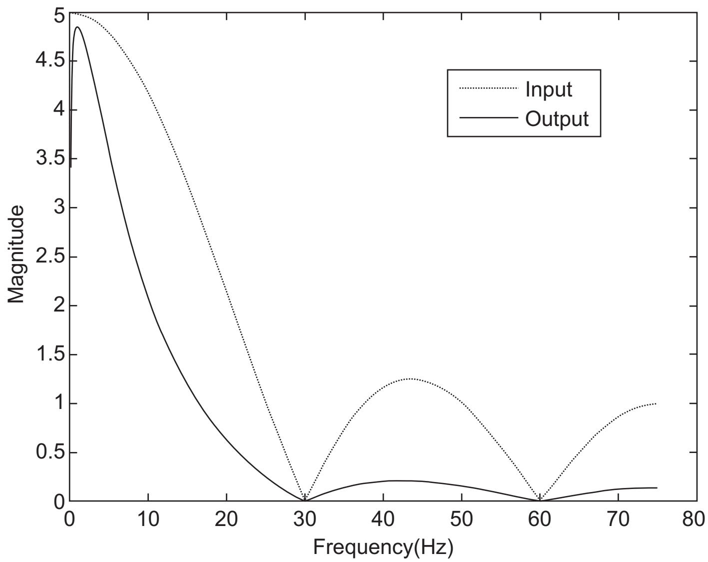


FIGURE 7.19 The magnitude spectra of input and output signals associated with an unknown biological system in Example 7.15. The input spectrum is seen to decrease with increasing frequency, but still exceeds that of the output spectrum except possibly at 30 and 60 Hz. This indicates that the attenuation seen in the output spectrum is not a result of insufficient energy in the input spectrum.

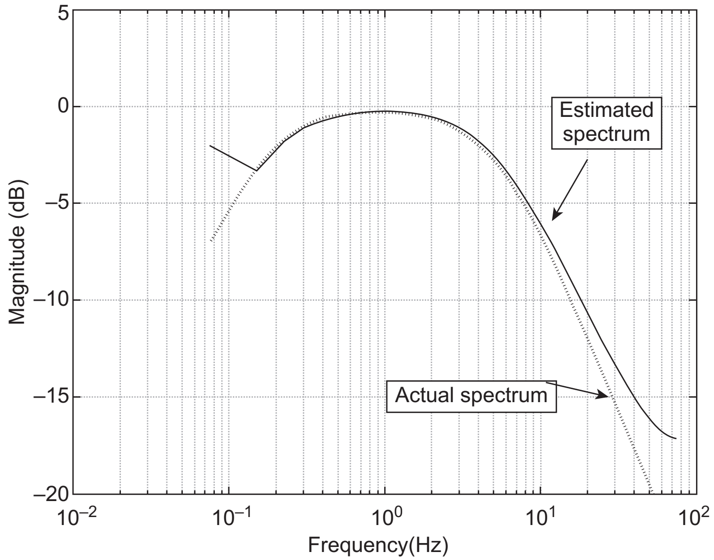


FIGURE 7.20 An estimate of system spectrum obtained by dividing the output spectrum by the input spectrum (*solid line*) and the system's actual spectrum (*dotted line*).

The system spectrum in Figure 7.20 has the shape of a band-pass filter with cutoff frequencies around 0.2 and 5 Hz corresponding to 1.26 and 31.4 rad/sec. The slope on either side appears to be 20 dB/decade and the midrange gain is near 0 dB or 1.0. Applying Bode plot techniques with this type of curve (see Example 6.11 and Equation 6.56) with these parameters leads to an estimated transfer function.

$$TF(\omega) = \frac{j\omega}{\left(1+\dfrac{j\omega}{\omega_1}\right)\left(1+\dfrac{j\omega}{\omega_2}\right)} = \frac{j\omega}{\left(1+\dfrac{j\omega}{1.26}\right)\left(1+\dfrac{j\omega}{31.4}\right)} = \frac{j\omega}{(1+j0.794\omega)(1+j0.032\omega)}$$

Again assuming no initial conditions, the Laplace transfer function is:

$$TF(s) = \frac{s}{(0.794)(0.032)(s+1.26)(s+31.4)} = \frac{39.4\,s}{(s+1.26)(s+31.4)}$$

If you can control, or at least measure, the stimulus and response of a system, the approaches used here can be very useful. Variations of both impulse and frequency response methods have been used to estimate the transfer function of the extraocular muscles, the iris, and lens muscles in the eye, and the response of chemoreceptors in the respiratory system and numerous other biosystems.

7.7 SUMMARY

With the Laplace transform, all of the analysis tools developed in Chapter 6 can be applied to systems exposed to a broader class of signals. Transfer functions written in terms of the

Laplace variable s (i.e., complex frequency) serve the same function as frequency domain transfer functions, but now include transient signals such as the step function. Here only the response to the step and impulse signals is used in examples because these are the two stimuli that are most commonly used in practice. Their popularity stems from the fact that they provide a great deal of insight into system behavior, and they are usually easy to generate in practical situations. However, responses to other signals such as ramps or exponentials, or any signal that has a Laplace transform, can be analyzed using these techniques. Laplace transform methods can also be extended to systems with nonzero initial conditions, a useful feature explored later.

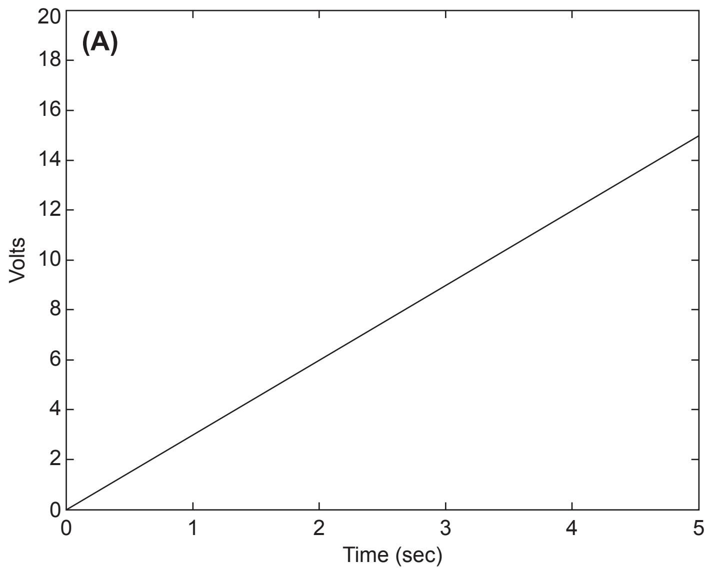
The Laplace transform can be viewed as an extension of the Fourier transform where complex frequency, s, is used instead of imaginary frequency, $j\omega$. With this in mind, it is easy to convert from the Laplace domain to the frequency domain by substituting $j\omega$ for s in the Laplace transfer functions. Bode plot techniques can be applied to these converted transforms to construct the magnitude and phase spectra. Thus the Laplace transform serves as a gateway into both the frequency domain and the time domain through the inverse Laplace transform.

Determining a system's transfer function from external behavior is a broad and expanding area of signal processing known as systems identification. If it is practical to generate a sinusoidal or impulse input to a system and measure the response, we should be able to determine the spectral characteristics of that system. Alternatively, we can use the system's own natural stimulus as long as we can measure that stimulus and it contains energy of the spectral frequencies of interest. From the system's spectrum, we can use Bode plot methods developed in Chapter 6 to reconstruct the system's transfer function.

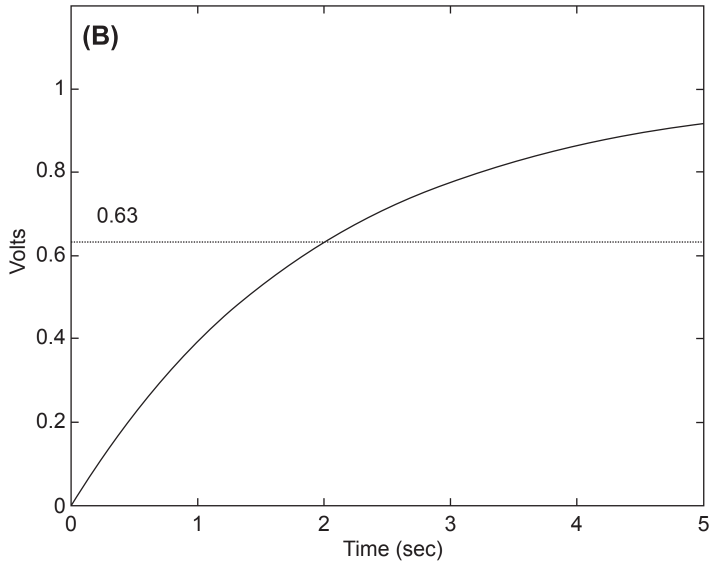
PROBLEMS

1. Find the Laplace transform of the following time functions:

a.



b.



c. $(e^{-2t} - e^{-5t})$

d. $2e^{-3t} - 4e^{-6t}$

e. $5 + 3e^{-10t}$

2. Find the inverse Laplace transform of the following Laplace functions:

a. $\frac{10}{s+5}$

b. $\frac{10}{s(s+5)}$

c. $\frac{5s+4}{s^2+5s+20}$ (Hint: Check roots.)

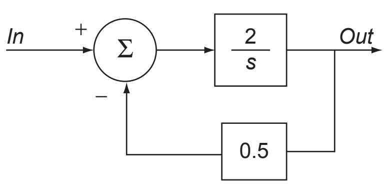
d. $\frac{5s+4}{s(s^2+5s+20)}$

3. Use partial fraction expansion to find the inverse Laplace transform of these functions:

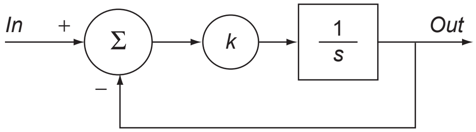
a. $\frac{s+4}{s^2+10s+10}$

b. $\frac{10}{s(s^2+4s+3)}$

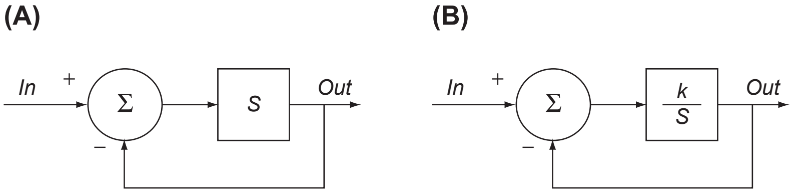
4. Find the time response of the following system if the input is a step from 0 to 5:



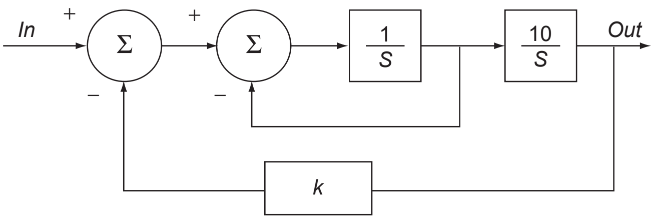
5. Find the time response of the following system to a unit step function. Use Laplace methods to solve for the time response as a function of k. Then use MATLAB to plot the time function for $k = 0.1$, 1, and 10.



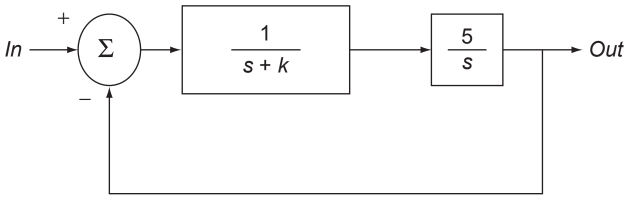
6. Find the time response of the following two systems if the input is a step from 0 to 8. Use MATLAB to plot the time responses. Plot superimposed.



7. Solve for the Laplace transfer function of the following system where $k = 1$. Find the time response to a step from 0 to 4 and an impulse having a value of 4. (Hint: You can apply the feedback equation twice.)



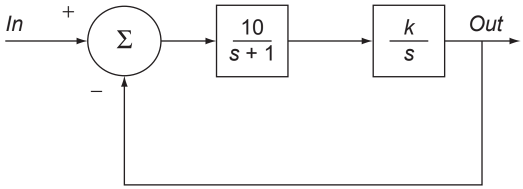
8. Find the time response of the system with the following transfer function if the input is a unit step. Also find the response to a unit impulse function. Assume $k = 5$.



9. Repeat Problem 8 assuming $k = 0.2$.

10. Find the unit impulse response to the system in Problem 7 for $k = 1$ as on Problem 7 and $k = 0.1$. Use MATLAB to plot the two impulse responses. How does decreasing k change the output behavior?

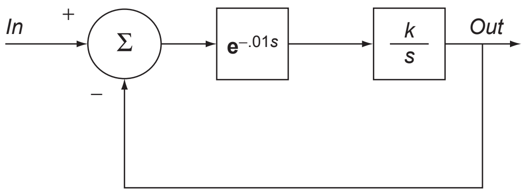
11. Find the impulse response to the following system for $k = 1$ and $k = 0.1$. Use MATLAB to plot the two impulse responses. How does decreasing k change the output behavior? Compare with the results in Problem 10 to the same values of k.



12. Use MATLAB to plot the magnitude spectra to the two systems shown in Problem 6. Plot superimposed.

13. Use MATLAB to plot the magnitude spectra of the system in Problem 10 to the two values of k. Repeat for the system in Problem 11.

14. Use Laplace analysis to find the transfer function of the following system that contains a time delay of 0.01 sec ($e^{-0.01s}$). Find the Laplace domain response of a step from 0 to 10. Then convert to the frequency domain and use MATLAB to find the magnitude and phase spectrum of the response. Note that such a time delay is typical in biological systems. Plot the system spectrum over a frequency range of 1 to 200 rad/sec. Owing to the delay, the phase curve will exceed -180 deg and will wraparound, so use the MATLAB `unwrap` routine.



15. Demonstrate the effect of a 0.2-sec delay on the frequency characteristics of a second-order system. The system should have an ω_n of 10 rad/sec and a δ of 0.7. Plot the magnitude and phase with and without the delay. (Hint: Use MATLAB to plot the spectrum of the second-order system by substituting $j\omega$ for s. Then replot adding an $e^{-0.2s}$ ($=e^{-j0.2\omega}$) to the transfer function.) Plot for a frequency range of 1 to 100 rad/sec. Again the `unwrap` routine should be used since the phase plot will exceed -180 deg.

16. Find the time function of the following higher-order Laplace function. Use `roots` to factor the denominator (and `poly` if needed). Then apply partial fraction expansion to separate out the denominator terms and find the inverse Laplace transform.

$$TF(s) = \frac{s^3 + 17s^2 + 80s + 100}{s^4 + 10s^3 + 45s^2 + 110s + 104}$$

17. Find the time function of the following higher-order Laplace function. Use `roots` to factor the denominator (and `poly` if needed). Then apply partial fraction expansion to separate out the denominator terms and find the inverse Laplace transform. Alternatively, use MATLAB's residue to find the partial fractions directly,

$$TF(s) = \frac{3(s+5)}{s^3 + 6s^2 + 11s + 6}$$

18. The impulse response of a first-order system is:

$$V_{out}(s) = \left(\frac{1}{\tau}\right)\left(\frac{1}{s + \frac{1}{\tau}}\right)$$

Use the Initial Value Theorem to find the filter output's value at $t = 0$ (i.e. $v_{out}(0)$) for the filter.

19. The transfer function of an electronic system has been determined as:

$$TF(s) = \frac{5s + 4}{s^2 + 5s + 20}$$

Use the Final Value Theorem to find the value of this system's output for $t \rightarrow \infty$ if the input is a step function that jumps from 0 to 5 at $t = 0$.

20. The MATLAB file `unknown_sys7_4(x).m` found on the associated files represents a linear system as in Examples 7.13 and 7.14. The input is `x` and the output is the output argument, i.e., `y = unknown_sys7_4(x);`. Assume a sampling frequency of 1.0 kHz and use 40,000 points to get good spectral resolution. Determine the magnitude spectrum for this unknown process using a random input as in Example 7.13. Estimate the transfer function of this system based on Bode plot primitives.

21. Find the magnitude spectrum of the unknown system as represented by `unknown_sys7_5.m`using sinusoids as in Example 7.14. Vary the range of frequencies of the sine wave between 0 and 400 Hz in 1.0 Hz intervals. Make $N = 1000$ and assume $f_s = 1$ kHz. Estimate the transfer function for this system based on Bode plot primitives.

22. Use the impulse response to find the magnitude spectrum of the system represented by `unknown_sys7_6(x).m`. Estimate the transfer function for this based on Bode plot primitives.

Use an impulse input of 1000 points and assume they are spaced 1.0 msec apart. As always, be sure to plot only valid spectral points.

23. The file `bio_sys1.mat` contains the input and output signals of a biological system sampled at 1000 Hz. Follow the approach used in Example 7.15 to find the magnitude transfer function, then use Bode plot primitives to estimate the transfer function. Ignore obvious artifacts.