Appendix G
Determinants and Cramer's Rule

The solution of simultaneous equations can be greatly facilitated by matrix algebra. When the solutions must be done by hand, the use of determinants is helpful, at least when only two or three equations are involved. A determinant is a specific single value defined for a square array of numbers. Given a 2×2 array, the determinant would be found by the application of the so-called diagonal rule where the product of the main diagonal (solid arrow) is subtracted by the product of the off-diagonal (dotted arrow):

$$\text{det} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

This gives rise to the equation:

$$\text{det} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{G.1}$$

For a 3×3 array, the determinant is found by an extension of the diagonal rule. One way to visualize this extension is to repeat the first two columns at the right side of the array. Then the diagonals can be drawn directly:

$$\text{det} = \left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right.$$

This procedure produces the equation:

$$\text{det} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}) - (a_{13}a_{22}a_{31} + a_{23}a_{32}a_{11} + a_{33}a_{21}a_{12}) \tag{G.2}$$

It is a lot easier using MATLAB where the determinate is obtained by the command: det(A), where A is the matrix.

Cramer's rule is used to solve simultaneous equations using determinants. The equations are first put in matrix format (shown here using electrical variables):

$$\begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \end{vmatrix} \tag{G.3}$$

The current i_1 is found using:

$$i_1 = \frac{\det Z_1}{\det Z} = \frac{\det \begin{vmatrix} v_1 & Z_{12} \\ v_2 & Z_{22} \end{vmatrix}}{\det \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}} = \frac{v_1 Z_{22} - v_2 Z_{12}}{Z_{11} Z_{22} - Z_{21} Z_{12}} \tag{G.4}$$

And in a similar fashion, the current i_2 is found by:

$$i_2 = \frac{\det Z_1}{\det Z} = \frac{\det \begin{vmatrix} Z_{11} & v_1 \\ Z_{21} & v_2 \end{vmatrix}}{\det \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}} = \frac{Z_{11} v_2 - Z_{21} v_1}{Z_{11} Z_{22} - Z_{21} Z_{12}} \tag{G.5}$$

Extending Cramer's rule to 3×3 matrix equation:

$$\begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \\ i_3 \end{vmatrix}$$

The three currents are obtained as:

$$i_1 = \frac{\det \begin{vmatrix} v_1 & Z_{12} & Z_{13} \\ v_2 & Z_{22} & Z_{23} \\ v_3 & Z_{32} & Z_{33} \end{vmatrix}}{\det \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix}} \quad i_2 = \frac{\det \begin{vmatrix} Z_{11} & v_1 & Z_{13} \\ Z_{21} & v_2 & Z_{23} \\ Z_{31} & v_3 & Z_{33} \end{vmatrix}}{\det \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix}} \quad r_3 = \frac{\det \begin{vmatrix} Z_{11} & Z_{12} & v_1 \\ Z_{21} & Z_{22} & v_2 \\ Z_{31} & Z_{32} & v_3 \end{vmatrix}}{\det \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix}}$$

where each determinant would be evaluated using Equation G.2 or using MATLAB.