CHAPTER

14

Circuit Reduction: Simplifications

14.1 GOALS OF THE CHAPTER

Sometimes it is desirable to simplify a circuit or system by combining elements. Many quite complex circuits can be reduced to just a few elements using an approach known as "network reduction." Such simplifications make it easier to analyze and understand the network by providing a summary-like representation of a complicated system. New insights can be had about the properties of a network after it has been simplified. Network reduction can be particularly useful when two networks or systems are to be connected together. We only need the reduced networks to figure out how connecting them together affects the passage of information between them. Finally, the principles of network reduction are useful in understanding the behavior of real sources.

Network reduction principles also help us understand the problems that arise when making medical measurements. Making a measurement on a biological system is essentially connecting the measurement system to the biological system. All measurements require drawing some energy from the system being measured. The energy required depends on the match between the biological and measurement systems. This match can be quantified in terms of a generalized concept of impedance: the difference between the impedance of the biological system and the input impedance of the measurement system.

In this chapter we will learn how to:

- Combine series and parallel elements and apply this approach to complex configurations of passive elements.
- Reduce complex networks of circuits to a single source and impedance by successively combining elements.
- Reduce complex circuits to a single source and impedance using an applied voltage source (either real or theoretical) and calculating (or measuring voltage and current).
- Use passive elements to construct circuits that have sharp resonance characteristics.
- How to represent real current and voltage sources using ideal elements.
- Determine if, when one circuit is connected to another, the influence of the second circuit on the first can be ignored.
- Determine how, when one circuit is connected to another, maximum power from the first circuit to the second is transferred.

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- Determine the equivalent impedance of a mechanical system and apply the concepts of source and load impedance to mechanical systems.
- Determine the influence, when one mechanical system is connected to another, of the loading mechanical system on the source mechanical system, particularly in measurement situations.

14.2 SYSTEM SIMPLIFICATIONS—PASSIVE NETWORK REDUCTION

Before we can reduce complex networks or systems, we must first learn to reduce simple configurations of elements such as series and parallel combinations. Network reduction is based on a few simple rules for combining series and/or parallel elements. The approach is straightforward, although implementation can become tedious for large networks (that is when we turn to MATLAB). After we introduce the reduction rules for networks consisting only of passive elements, we will expand our guidelines to include networks with sources. These reduction tools can be applied whenever two systems are interconnected.

14.2.1 Series Electrical Elements

Electrical elements are said to be in "series" when they are connected to each other and no other elements share that common connection, Figure 14.1A. Although series elements are often drawn in line with one another, they can be drawn in any configuration and still be in series as long as they follow the "no other connection" rule. The three elements in Figure 14.1B are also in series as long as no other elements are connected between the elements. A simple application of Kirchoff's voltage law (KVL) demonstrates that when

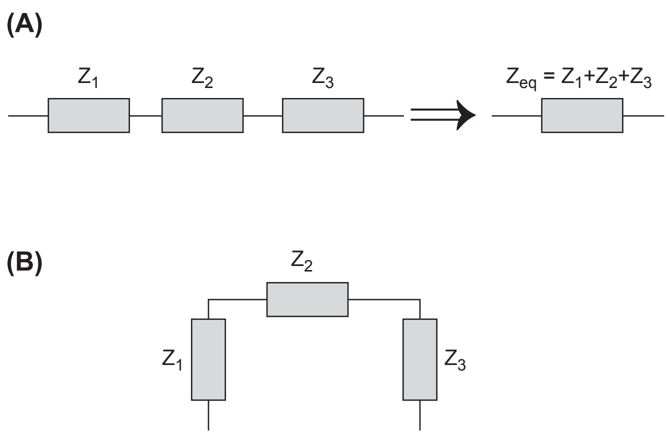


FIGURE 14.1 (A) Three elements in series, Z_1, Z_2, and Z_3, can be converted into a single equivalent element, Z_{eq}, that is the sum of the three individual elements. Elements are in series when they share one node and no other elements share this node. (B) Series elements are often drawn in *line*, but these elements are also in series as long as nothing else is connected between the elements.

elements are in series their impedances add. The voltage across three series elements in Figure 14.1B is:

$$v_{total} = v_1 + v_2 + v_3 = (Z_1 + Z_2 + Z_3)i$$

The total voltage can also be written as:

$$v_{total} = Z_{eq}\ i;$$

where $Z_{eq} = Z_1 + Z_2 + Z_3$.

So series elements can be represented by a single element that is the sum of the individual elements, Equation 14.1.

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \ldots + Z_N \tag{14.1}$$

If the series elements are all resistors or all inductors, they can be represented by a single resistor or inductor that is the sum of the individual elements:

$$R_{eq} = R_1 + R_2 + R_3 + \ldots \tag{14.2}$$

$$L_{eq} = L_1 + L_2 + L_3 + \ldots \tag{14.3}$$

If the elements are all capacitors, their reciprocals add, since the impedance of a capacitor is a function of $1/C$, i.e., $1/j\omega C$ or $1/sC$:

$$1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3 + \ldots \tag{14.4}$$

If the string of elements includes different element types, the individual impedances can be added using complex arithmetic to determine a single equivalent impedance. In general, this single impedance is complex, as shown in the following example.

EXAMPLE 14.1

Series element combination. Find the equivalent single impedance, Z_{eq}, of the series combination in Figure 14.2.

Solution: First combine the two resistors into a single 25-Ω resistor, then combine the two inductors into a single 11-h inductor, and then add the three impedances (R_{eq}, $j\omega L_{eq}$, and $1/j\omega C$). Alternatively, convert each element to its equivalent phasor impedance, and then add these impedances.

$$Zeq = 10 + j\omega 5 + \frac{1}{j\omega .01} + 15 + j\omega 6 = 25 + j\omega 11 + \frac{100}{j\omega}$$

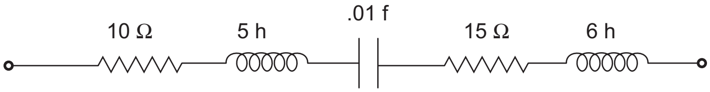


FIGURE 14.2 A combination of different series elements that can be mathematically combined into a single element. Usually the resulting element is complex (i.e., contains a real and an imaginary part).

If a specific frequency is given, for example, $\omega = 2.0$ rad/s, then Z_{eq} can be evaluated as a single complex number.

$$Z(\omega = 2) = 25 + j(11(2) - 100/2) = 25 - j28\ \Omega = 37.5\angle -48\ \Omega$$

EXAMPLE 14.2

Find the equivalent capacitance for the three capacitors in series in Figure 14.3.

Solution: Since all the elements are capacitors, they add as reciprocals:

$$1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3 = 1/0.1 + 1/0.5 + 1/0.2 = 10 + 2 + 5 = 17$$

$$C_{eq} = 1/17 = 0.059\ \mathrm{f}$$

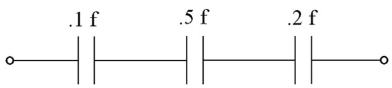


FIGURE 14.3 Three series capacitors. They can be combined into a single capacitor by adding reciprocals as shown in Example 14.2.

14.2.2 Parallel Elements

Elements are in parallel when they share both connection points as shown in Figure 14.4. For parallel electrical elements, it does not matter if other elements share these mutual connection points, as long as both ends of the elements are connected to each other.

When you are looking at electrical schematics, it is important to keep in mind the definition of parallel and series elements because series elements may not be drawn in line (Figure 14.1B) and parallel elements may not be drawn as geometrically parallel. For example, the two elements, Z_1 and Z_2, on the left side of Figure 14.5 are in parallel because they connect at both ends, even though they are not drawn in parallel geometrically. Conversely, elements Z_1 and Z_2 are drawn parallel, but they are not in parallel electrically because they are not connected.

As can be shown by KCL, parallel elements combine as the reciprocal of the sum of the reciprocals of each impedance. With application of KCL to the upper node of the three parallel elements in Figure 14.4, the total current flowing through the three impedances is:

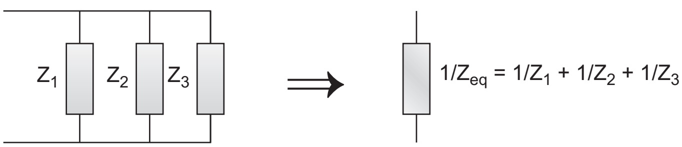


FIGURE 14.4 Parallel elements share connection points and both ends. Three elements in parallel, Z_1, Z_2, and Z_3, can be converted into a single equivalent impedance, Z_{eq}, that is the reciprocal of the sum of the reciprocals of the three individual impedances.

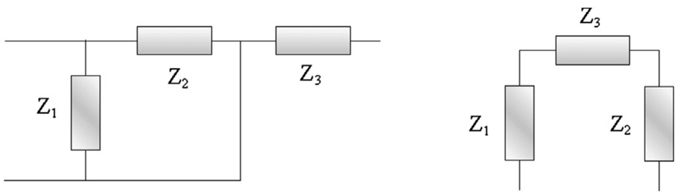


FIGURE 14.5 Left circuit: the elements Z_1 and Z_2 are connected at both ends and are therefore electrically in parallel, even though they are not drawn parallel. Right circuit: although they are drawn parallel, elements Z_1 and Z_2 are not electrically parallel because they are not connected at both ends (in fact they are not connected at either end).

$i_{total} = i_1 + i_2 + i_3$. Substituting in v/Z for the currents through the impedances, the total current is:

$$i_{total} = v/Z_1 + v/Z_2 + v/Z_3 = v(1/Z_1 + 1/Z_2 + 1/Z_3).$$

This equation, restated in terms of an equivalent impedance, becomes:

$$i_{total} = v/Z_{eq},$$

where $1/Z_{eq} = 1/Z_1 + 1/Z_2 + 1/Z_3$.

Hence:

$$1/Z_{eq} = 1/Z_1 + 1/Z_2 + 1/Z_3 + \ldots \tag{14.5}$$

Equation 14.5 also holds for the value of parallel resistors and inductors:

$$1/R_{eq} = 1/R_1 + 1/R_2 + 1/R_3 + \ldots \tag{14.6}$$

$$1/L_{eq} = 1/L_1 + 1/L_2 + 1/L_3 + \ldots \tag{14.7}$$

Parallel capacitors, however, simply add.

$$C_{eq} = C_1 + C_2 + C_3 + \ldots \tag{14.8}$$

If the three capacitors in Example 14.2 were in parallel, their equivalent capacitance would be the addition of the three capacitance values:

$$C_{eq} = 0.1 + 0.5 + 0.2 = 0.8 \text{ f}$$

EXAMPLE 14.3

Find the equivalent single impedance for the parallel resistor, inductor, capacitor (RLC) combination in Figure 14.6.

Solution: First take the reciprocals of the impedances:

$$1/R = 1/10 = 0.1\ \Omega;\ \ 1/j\omega L = 1/j5\omega\ \Omega;\ \ j\omega C = j.01\omega\ \Omega$$

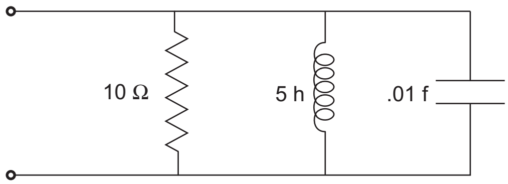


FIGURE 14.6 Three parallel elements can be combined into a single element as shown in Example 14.3.

Then add, and invert:

$$\frac{1}{Z_{eq}} = Y_{eq} = 0.1 + \frac{1}{j\omega 5} + j\omega .01 = 0.1 + j\left(.01\omega - \frac{1}{5\omega}\right)$$

$$Z_{eq} = \frac{1}{Y_{eq}} = \frac{1}{0.1 + j\left(.01\omega - \frac{1}{5\omega}\right)}\ \Omega$$

Once a value of frequency, ω, is given, this equation can be solved for a specific impedance value. Alternatively, we can solve for Z_{eq} over a range of frequencies using MATLAB. This is given as a problem at the end of this chapter.

EXAMPLE 14.4

Find the equivalent resistance of the parallel combination of three resistors: 10, 15, and 20 Ω.

Solution/Result: Calculate reciprocals, add them and invert:

$$1/R_{eq} = 1/R_1 + 1/R_2 + 1/R_3 = 1/10 + 1/15 + 1/20 = 0.1 + 0.0667 + .05 = 0.217\ \Omega$$

$$R_{eq} = 1/0.217 = 4.61\ \ \Omega$$

Note that the equivalent resistance of a parallel combination of resistors is always less than the smallest resistor in the group. The same is true of inductors, but the opposite is true of parallel capacitors—the parallel combination of capacitors is always larger than the largest capacitor in the group.

14.2.2.1 Combining Two Parallel Impedances

Combining parallel elements via Equation 14.5 is mildly irritating, with its double inversions. Most parallel element combinations involve only two elements, so it is useful to have an equation that directly states the equivalent impedance of two parallel elements without the inversion. Starting with Equation 14.5 for two elements:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{Z_2}{Z_2 Z_1} + \frac{Z_1}{Z_1 Z_2} = \frac{Z_1 + Z_2}{Z_1 Z_2}$$
$$\text{Inverting:} \quad Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} \tag{14.9}$$

Hence the equivalent impedance, Z_{eq}, of two parallel elements equals the product of the two impedances divided by their sum.

14.3 NETWORK REDUCTION—PASSIVE NETWORKS

The rules for combining series and parallel elements can be applied to networks that include a number of elements. Even very involved configurations of passive elements can usually be reduced to a single element. Obviously, it is easier to grasp the significance of a single element than a confusing combination of many elements.

14.3.1 Network Reduction—Successive Series–Parallel Combination

In the last section of this chapter, we see that it is possible to combine a number of series or parallel combinations. Even when most of the elements are not in either series or parallel configurations, it is possible to combine them into a single impedance using the techniques of network reduction. In this section, the networks being reduced consist only of passive elements, but in Section 14.4 we learn how to expand network reduction to include networks with sources.

In the network in Figure 14.7, most of the elements are neither in series nor in parallel. It is important to realize that the elements across the top of this network, inductor–resistor–inductor, are not in series because their connection points are shared by other elements, capacitors in this case. To be in series, the elements must share one connection point and must be the only elements to share that point. If we could somehow eliminate the two capacitors (we cannot), then these three elements would be in series. Nor are any of the elements in parallel, since no elements share both connection points. If they did, they would be in parallel even if other elements share these connection points. However, there are two elements in series: the 4-h inductor and the 20-Ω resistor on the right-hand side of the network. We could combine these two elements using Equation 14.1. After combining these two elements, we now find that the new element is in parallel with the 0.02-f capacitor. We can then combine these two parallel elements using the parallel rule given in Equation 14.9. Although the

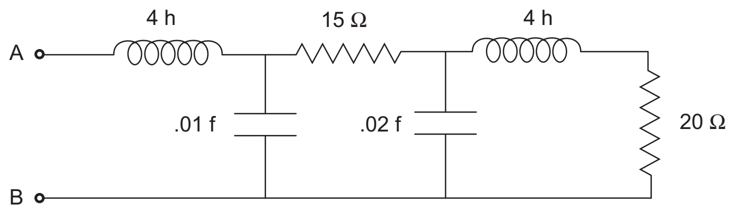


FIGURE 14.7 A network containing R, L, and C's where most of the elements are neither in series nor in parallel. Nonetheless, this network can be reduced to a single equivalent impedance (with respect to nodes A and B) as is shown in Example 14.5.

argument becomes difficult to follow at this point without actually going through it, the single element newly combined from the parallel elements is now in series with the 15-Ω resistor. Most reductions of passive networks proceed in this fashion: find a series or parallel combination to start with, combine them into a single element, and then look to see if the new combination produces a new series or parallel combination. Then just continue down the line:

The following example uses the approach based on sequential series–parallel combinations to reduce the network in Figure 14.7 to a single equivalent impedance.

EXAMPLE 14.5

Network reduction using sequential series–parallel combinations. Find the equivalent impedance between the nodes A and B in Figure 14.7. Find the impedance at only one frequency, $\omega = 5.0$ rad/s. Using network reduction, we can find the equivalent impedance leaving frequency as a variable (i.e., $Z(\omega)$), but this will make the algebra more difficult. By using a specific frequency, we are able to use complex arithmetic instead of complex algebra.

Solution: Convert all elements to their equivalent impedances at $\omega = 5.0$ rad/s (to simplify subsequent calculations). Then begin the reduction by combining the two series elements on the right-hand side. As a first step, first convert the elements to their phasor impedances (at $\omega = 5$ rad/s). Then combine the two series elements using Equation 14.1, leading to the network shown in Figure 14.8 on the right-hand side.

This combination puts two elements in parallel: the newly formed impedance and the $-j10$-Ω capacitor. These two parallel elements can be combined using Equation 14.9:

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(20 + j20)(-j10)}{20 + j20 - j10} = \frac{200 - j200}{20 + j10} = \frac{282.8\angle -45}{22.36\angle 26.6}$$

$$Z_{eq} = 12.65\angle -71.6\ \Omega = 4 - j12\ \Omega$$

This leaves a new series combination that can be combined as follows. In the third step of network reduction, the newly formed element from the parallel combination ($4 - j12\ \Omega$) is now in series with the 15-Ω resistor, and this equivalent series element will be in parallel with the 0.01-f capacitor, Figure 14.9.

Combining these two parallel elements:

$$Z_{eq} = \frac{-j20(19 - j12)}{-j20 + 19 - j12} = \frac{-240 - j380}{19 - j32} = \frac{449.4\angle 238}{37.2\angle -59}$$

$$Z_{eq} = 12.07\angle 297\ \ \Omega = 5.49 - j10.76\ \ \Omega$$

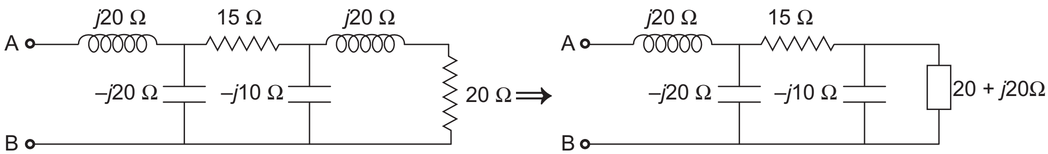


FIGURE 14.8 The network on the left is a phasor representation of the network in Figure 14.7; a partial reduction of this network is shown on the right.

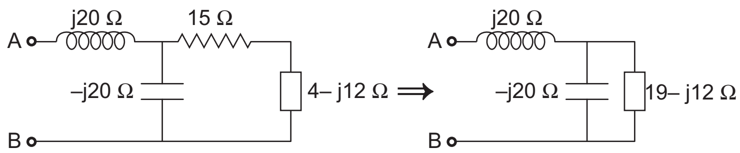


FIGURE 14.9 The next step in the reduction of the network is shown in Figure 14.7. The newly combined elements are now in series with the 15-Ω resistor, and the element formed from that combination will be in parallel with the 0.01-f capacitor.

Result: As shown in Figure 14.10, combining the parallel elements leads to the final series combination and, after combination, a single equivalent impedance with a value of:

$$Z_{eq} = 5.49 + j9.24\ \Omega = 10.75\angle 59\ \Omega$$

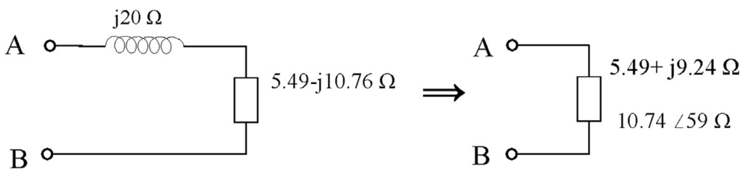


FIGURE 14.10 A single element represents the effective impedance between Nodes A and B in the network of Figure 14.7.

Network reduction is always done from the point of view of two nodes, such as nodes A and B in this example. In principle, any two nodes can be selected for analysis and the equivalent impedance can be determined between these nodes. Generally the nodes selected have some special significance, e.g., the nodes that make up the input or output of the circuit. Network reduction usually follows the format of this example: sequential combinations of series elements, then parallel elements, then series elements, and so on. In a few networks, there are no elements either in series or in parallel to start with, and an alternative method described in the next section must be used. This method works for all networks and any combination of two nodes, but it is usually more computationally intensive. On the other hand, it lends itself to a computer solution using MATLAB, which ends up being less computationally intensive.

14.3.1.1 Resonance Revisited

In Chapter 13 we observe the resonance that occurs in electrical and mechanical systems when the impedance of the two types of energy storage devices, inertial and capacitive, are equal and cancel. This leads to a minimum in total impedance: zero if there are no dissipative elements present. For example, in a series resistor, inductor, capacitor (RLC) circuit the total impedance is $R + j\omega L + 1/j\omega C$ and resonance occurs when the impedances of the inductor and capacitor are the same. The total impedance is just that of the resistor. If the two energy storage elements are in parallel, then an antiresonance occurs where the net impedance goes to infinity, since by Equation 14.9:

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{j\omega L(1/j\omega C)}{j\omega L + 1/j\omega C} \rightarrow \infty \quad \text{when } j\omega L = 1/j\omega C$$

An example of the impedance of a 5.0-μf capacitor in series with a 50-mh inductor is shown as a function of log frequency in Figure 14.11A. The sharp resonance peak at 2000 rad/s is evident.

A network can exhibit both resonance and antiresonance, at different frequencies of course. In the network shown in Figure 14.12, the parallel inductor and capacitor produce an antiresonance at 2000 rad/s, but at a higher frequency the combination becomes capacitive and that parallel combination will resonate with the series inductor. This is shown in the next example.

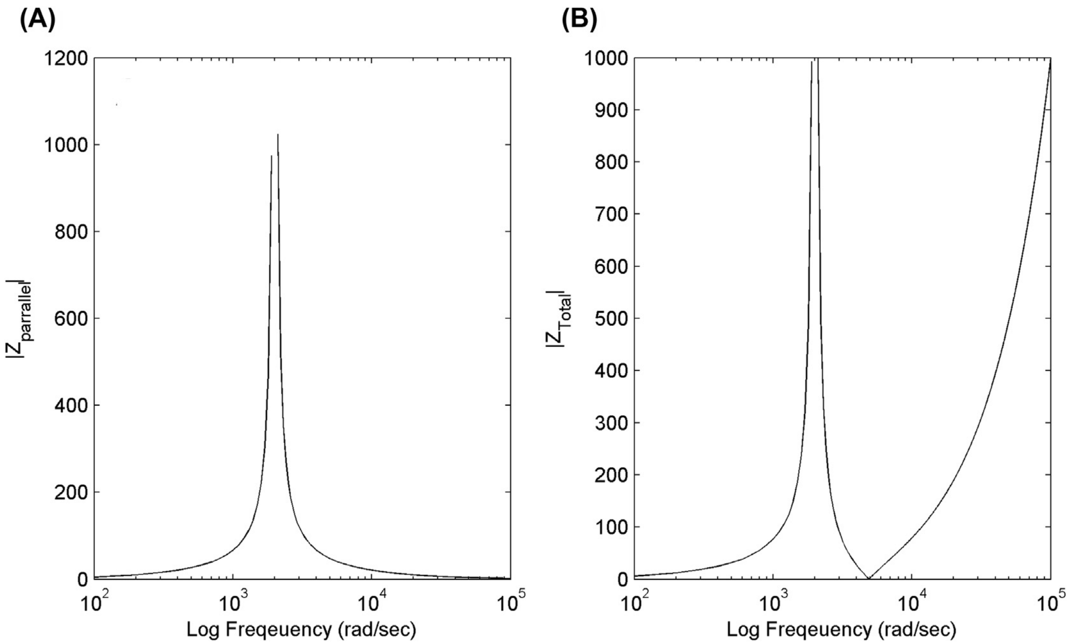


FIGURE 14.11 (A) The impedance of a parallel inductor-capacitor (LC) combination. An antiresonance peak (i.e., a very large impedance) is seen at 2000 rad/s when the impedances of the two elements are equal. (B) An example of both resonance and antiresonance phenomena produced by the network shown in Figure 14.12. This graph is constructed in Example 14.6 using MATLAB.

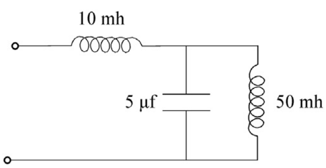


FIGURE 14.12 A circuit that exhibits both resonance and antiresonance. This circuit is analyzed in Example 14.6.

EXAMPLE 14.6

Use MATLAB to find the net impedance of the circuit in Figure 14.12 and plot the impedance as a function of log frequency. Also plot the impedance characteristic of the parallel inductor-capacitor (LC) separately.

Solution: After defining the component values, calculate the impedance of each component. Use the parallel element equation, Equation 14.9, to find the impedance of the parallel LC, and then add in the impedance of the series inductor to find the total impedance. The program is as follows.

```
% Example 14.6 Example to show resonance and antiresonance
%
w = (100:100:1000000);               % Define frequency vector
C = 05e-6;                           % Assign component values
L1 = 0.05;
L2 = .01;
ZL1 = j*w*L1;                        % Calculate component impedances
ZL2 = j*w*L2;
ZC = 1./(j*w*C);
Zp = (ZL1 .* ZC)./(ZL1 + ZC);        % Calculate parallel impedance
Z = ZL2 + Zp;                        % Calculate total impedance
  ......plot using semilogx and label.......
```

Result: The resulting plots from this program are shown in an earlier figure, Figure 14.11. Figure 14.11 B plots the total impedance and shows both the antiresonance peak, when $Z_{Total} \to \infty\ \Omega$ at 2000 rad/s, and the resonance peak, when $Z_{Total} \to 0\ \Omega$ at 5000 rad/s. Although the impedance of the network in Figure 14.12 was found for a fixed set of component values, it would be easy to rerun the program using different component values or even a range of component values.

14.3.2 Network Reduction—Voltage—Current Method

The other way to find the equivalent impedance of a network follows the approach that you would use in the real world on an existing physical system. Suppose you are asked to determine the impedance between two nodes of some physical network. This is called a "two-terminal" problem. Perhaps the actual network is inside a difficult-to-open box and all you have available are two nodes as shown in Figure 14.13.

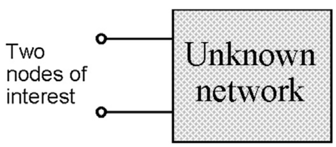


FIGURE 14.13 An unknown network that has only two terminals available for measurement, or for possible use. This is a classic "two-terminal" device.

The actual network could contain any number of nodes, but in this scenario only two are accessible for measurement. Without opening the box it is possible to determine the effective impedance between these two terminals. What you do in this situation (or, at least, what I would do) is to apply a known voltage to the two terminals, measure the resulting current, and calculate the equivalent impedance, Z_{eq}, using Ohm's law:

$$Z_{eq} = \frac{V_{known}}{I_{measured}} \tag{14.10}$$

Of course, V_{known} has to be a sinusoidal source of known amplitude, phase, and frequency unless you know, *a priori*, that the network is purely resistive, in which case a direct current (DC) source suffices. Moreover, you are limited to determining Z_{eq} at only one frequency, the frequency of the voltage source, but most laboratory generators can produce a sine wave over a wide range of frequencies, so the impedance can determined for a range of frequencies. Before you apply this method, you should first see if there is a voltage between the two terminals. If so, the unknown network is not a collection of passive elements, but rather contains voltage or current sources. In this case, a slightly different strategy developed in the next section can be used.

This same approach can be applied to a network that exists only on paper, such as the network in the last example. Using the tools that we have acquired thus far, we simply connect a source of our choosing to the network and solve for the current into the network. The source can be anything. Moreover, this approach can be applied to simplify any network, even one that does not have any series or parallel elements. In the next example, this method is applied to the network in Example 14.5 and, in a subsequent example, to an even more challenging network.

EXAMPLE 14.7

Passive network reduction using the source−current method. Find the equivalent impedance between nodes A and B in the network of Figure 14.7 for a frequency of $\omega = 5$ rad/s.

Solution: To the given network, apply a known (if theoretical) source across nodes A and B and solve for the resulting current using mesh analysis. Theoretically we can choose any sinusoidal source as long as it has a frequency of 5 rad/s, but why not choose something simple like 1 V RMS at 0.0 degree phase: $v(t) = \cos(5t) \rightarrow V(\omega) = 1\angle 0$. The desired impedance Z_{ab} will then be: $Z_{ab} = 1/I_1(\omega)$. Mesh analysis can be used to solve for $I_1(\omega)$. The three-mesh network is shown using phasor notation in Figure 14.14.

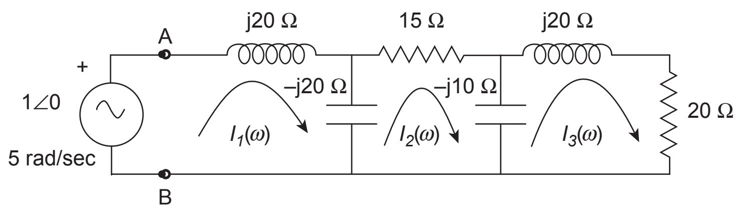


FIGURE 14.14 The network shown in Figure 14.7 with a voltage source attached to nodes A and B and represented in the phasor domain.

To find the input current, we analyze this as a straightforward three-mesh problem solving for I_1. However, several alternatives are also possible: using network reduction to convert the last three elements to a single element and solving as a two-mesh problem; converting the voltage source to an equivalent current source (as shown later in this chapter) and solving as a two-node problem; or combining these approaches and solving as a one-node problem. Here we solve the three-mesh problem directly, but use MATLAB to reduce the computational burden. Applying standard mesh analysis to the above-mentioned network, the basic matrix equation can be written as:

$$\begin{vmatrix} 1.0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} j20 - j20 & +j20 & 0 \\ +j20 & 15 - j20 - j10 & +j10 \\ 0 & +j10 & 20 + j20 - j10 \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \\ i_3 \end{vmatrix}$$

Simplifying by complex addition:

$$\begin{vmatrix} 1.0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 & j20 & 0 \\ j20 & 15 - j30 & j10 \\ 0 & j10 & 20 + j10 \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \\ i_3 \end{vmatrix}$$

Solving for I_1, then Z_{eq}, using MATLAB:

```
% Example 14.7 Find the equivalent impedance of a network by applying a source and
solving for the resultant current.
%
% First assign values for v and Z
v = [1 0 0];
Z = [0 20j 0; 20j 15–30j 10j; 0 10j 20+ 10j];
%
i = Z\v                                 % Solve for currents

Zeq = 1/i(1)                            % Solve for Zeq. Output as real and imaginary

Zeq = [abs(Zeq) angle(Zeq)*360/(2*pi)]  % Output in polar form
```

Result: The output from this program is: Zeq = 5.49 + 9.24i Ω or 10.75∠59 Ω, which is the same as that found in the previous example using network reduction. Again, this approach could be used to reduce any passive network of any complexity to an equivalent impedance between any two terminals.

The latter portion of this chapter shows how to reduce, and think about, networks that also contain sources. Before proceeding to the next section, we look at another example that shows how to reduce a passive network when the two terminals of interest are in more complicated positions (for example, when separated by more than a single element). We will also solve for the impedance over a range of frequencies.

EXAMPLE 14.8

Find the equivalent impedance of the circuit shown in Figure 14.15 between terminals A and B. Use MATLAB to find Z_{eq} over frequencies and plot Z_{eq} as a function of log frequency. Use a frequency range of 0.01–100 rad/s. (This frequency range was established by trial and error in MATLAB.)

Solution: Apply a source between terminals A and B and solve for the current flowing out of the source. In this problem, the source has a fixed amplitude of 1.0 V with a phase of 0 degree, but the frequency varies so that Z_{eq} can be determined over the specified range of frequencies. Before you apply the source, it is helpful to rearrange the network so that the meshes can be more readily identified. Sometimes this topographical reconfiguration can be the most challenging part of the problem, especially to individuals who are spatially challenged.

In this case, simply rotating the network by 90 degrees makes the mesh arrangement evident as shown in Figure 14.16.

This network cannot be reduced by a simple series–parallel combination strategy as used previously. Although more complicated transformations can be used,[1] the voltage–current method coupled with computer evaluation is far easier. Again the problem is solved using standard network analysis. Apply a voltage source and define the mesh currents as shown in Figure 14.16. Note that the total current out of the source flows into both the 15-Ω resistor and the 0.01-f capacitor. Hence the current into node B, the current we are looking for, is the sum of $i_1 + i_2$.

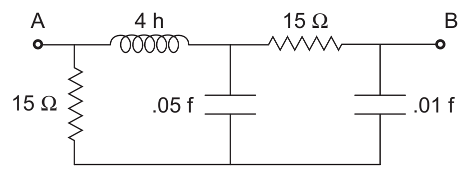


FIGURE 14.15 Network used in Example 14.8. The goal is to find the equivalent impedance between nodes A and B over a frequency range of 0.01 to 100 rad/s.

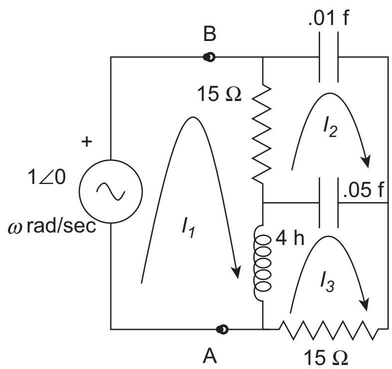


FIGURE 14.16 The network shown in Figure 14.15 after rotating by 90 degrees to make the three-mesh topology more evident.

After you convert all elements to phasor notation, write the matrix equation as:

$$\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 15 + j4\omega & -15 & -j4\omega \\ -15 & 15 - j100/\omega - j20/\omega & j20/\omega \\ -j4\omega & j20/\omega & 15 + j4\omega - j20/\omega \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \\ I_3 \end{vmatrix}$$

Note that the frequency, ω, remains a variable in this equation because we want to find the value of Z_{eq} over a range of frequencies. MATLAB is used to find the values of Z_{eq} over the desired range of frequencies and also plots these impedance values as a function of frequency.

```
% Example 14.8 Find and plot the values of an equivalent impedance
%  between .01 and 100 rad/sec
%
% Define frequency range, use .01 rad/sec increments
w = .01:.01:100;                   % Define frequency vector
v = [1; 0; 0];                     % Define voltage vector
%
% Loop over all frequencies, solving for Zeq2
for k = 1:length(w)
  % Define impedance vector (Use continuation statement)
  Z = [15+4j*w(k), -15, -4j*w(k); -15, 15-120j/w(k), 20j/w(k);
    -4j*w(k), 20j/w(k), 15+4j*w(k)-20j/w(k)];
  i = Z\v;                         % Solve for current
  Zeq(k) = 1/(i(1) + i(2));        % Solve for Zeq
end
  .......plot and label magnitude and phase.......
```

The graph produced by this program is shown in Figure 14.17. Both the magnitude and phase of Z_{eq} are functions of frequency and both reach a maximum value at between 3 and 5 rad/s.

It is easy to find the maximum and minimum of the magnitude and phase impedances using the MATLAB `max` and `min` functions. Applying these functions to the `Zeq`:

```
[max_Zeq_abs, abs_freq] = max(abs(Zeq));
[max_Zeq_phase, phase_freq] = min(angle(Zeq)*360/(2*pi));
%
% Now display the maximum and minimum values including frequency
disp([max_Zeq_abs, w(abs_freq); max_Zeq_phase, w(phase_freq)])
```

This produces the values:

$$\begin{aligned} |Z_{eq}|_{max} &= 15.43\,\Omega & f_{max} &= 1.6 \text{ rad/s} \\ \angle Z_{eq\ max} &= -21.48 \text{ degrees} & f_{max} &= 8.15 \text{ rad/s} \end{aligned}$$

The `max` and `min` functions give the maximum or minimum value of their arguments along with an index indicating where these values occur. To convert the index to the appropriate frequency, use `w(index)` (where `index` is the second output of the `max` and `min` functions) as in the earlier code.

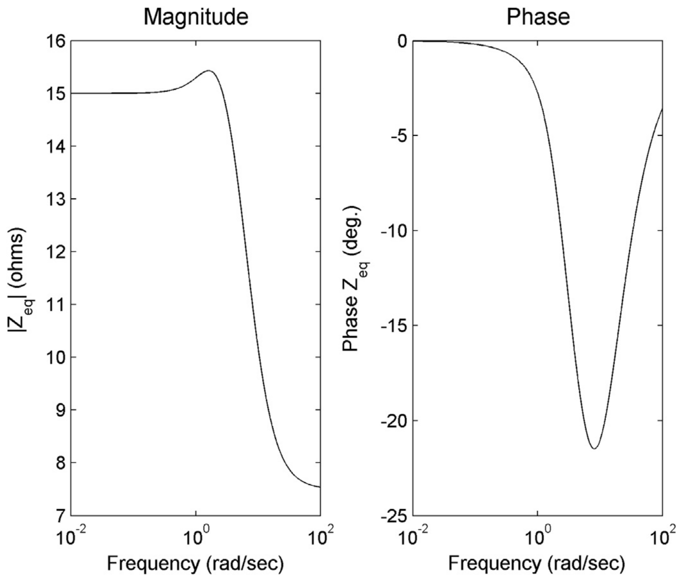


FIGURE 14.17 The value of Z_{eq} as a function of frequency for the network in Example 14.8. The impedance is plotted in terms of magnitude and phase. These plots are obtained from the MATLAB code used in Example 14.8.

[1]A slightly more complicated transformation exists that allows configurations such as this to be reduced by the reduction method, specifically a transformation known as the "Π (pi) to T" transformation. But the voltage–current approach used in this example applies to any network and lends itself well to computer analysis.
[2]You could avoid the loop and use MATLAB's vectorization capabilities. You would need to use ./ and .* operators. A loop to modify frequency is used here because it is easy and more understandable for the reader.

The remainder of this chapter examines the characteristics of sources, both real and ideal, and develops methods for reducing networks that contain sources. Many of the principles used here in passive networks are also used with these more general networks.

14.4 IDEAL AND REAL SOURCES

Before we develop methods to reduce networks that contain sources, it is helpful to revisit the properties of ideal sources described in Chapter 12 (Section 12.4.2) and examine how ideal and real sources differ. In this discussion, only constant output sources (i.e., DC sources) are considered, but the arguments presented generalize with only minor modifications to time-varying sources as shown in the next section.

14.4.1 The Voltage–Current or v–i Plot

Essentially an ideal source can supply any amount of energy that is required by whatever is connected to that source. Discussions of real and ideal sources often use plots of voltage against current (or force against velocity), which provide a visual representation of the source

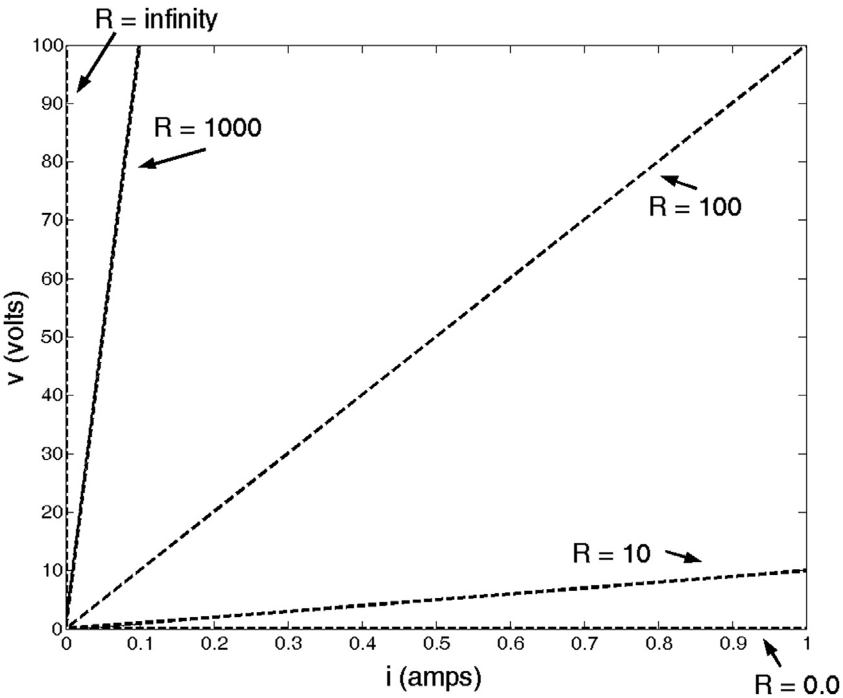


FIGURE 14.18 A v–i plot, showing voltage against current, for resistors having five different resistance values between 0.0 Ω and infinity Ω. This shows that the v–i plot of a resistor is a straight line passing through the origin and having a slope equal to the resistance.

characteristics. Such "v–i" plots are particularly effective at demonstrating the equivalent resistance of an element. Consider the v–i plot of pure resistors as shown in Figure 14.18. The plots of five different resistors are shown: 0, 10, 100, 1000, and ∞ Ω. The voltage–current relationship for a resistor is given by Ohm's law, $v = Ri$. A comparison of Ohm's law with the equation for a straight line, $y = mx + b$, where m is the slope of the line and b is the intercept, shows that the voltage–current relationship for a resistor plot is a straight line with a slope equal to the value of the resistance and an intercept of 0.0.

The reverse argument says that an element that plots as a straight line on a v–i plot is either a resistor or contains a resistance, and the slope of the line indicates the value of the resistance. The steeper the slope, the greater the resistance: a vertical line with a slope of infinity indicates the presence of an infinite resistance, whereas a horizontal line with a zero slope indicates the presence of a 0.0 Ω resistance.

The v–i plot of an ideal DC voltage source follows directly from the definition: a source of voltage that is constant irrespective of the current through it. For a time-varying source, such as a sinusoidal source, the voltage varies as a function of time, but not as a function of the current. An ideal voltage source cares naught about the current through it. The current will be whatever it has to be to maintain the specified voltage. Hence the v–i plot of an ideal DC voltage source, V_S, is a horizontal line intersecting the vertical (voltage) axis at $v = V_S$,

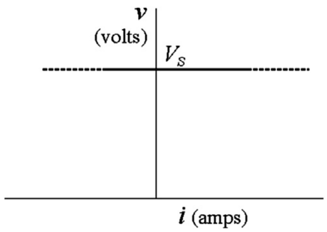


FIGURE 14.19 A $v-i$ plot of an ideal voltage source. This plot shows that the resistor-like properties of a voltage source have a zero value.

Figure 14.19. If the voltage source were time varying, the $v-i$ plot would look essentially the same except that the height of the horizontal line would vary as with time.

The $v-i$ plot of a voltage source with its horizontal line demonstrates that the resistive component of an ideal voltage source is zero. In other words, the equivalent resistance of an ideal voltage source is 0.0 Ω. With regard to the $v-i$ plot, an ideal source looks like a resistor of 0.0 Ω with an offset of V_S. It may seem strange to talk about the equivalent resistance of a voltage source, especially since it has a resistance of 0.0 Ω, but it turns out to be a very useful concept. The equivalent resistance of a source is its resistance ignoring other electrical properties. It is especially useful in describing real sources where the equivalent resistance is no longer zero. The concept of equivalent resistance or, more generally, equivalent impedance, is an important concept in network reduction and has a strong impact on biotransducer analysis and design.

Resistive elements having zero or infinite resistance have special significance and have their own terminology. Resistances of zero will produce no voltage drop regardless of the current running through them. As mentioned in Chapter 12, an element that has zero voltage for any current is termed a "short circuit" since current flows freely through such elements. Although it may seem contradictory, voltage sources are actually short-circuit devices, but with a voltage offset.

Resistors with infinite resistance have the opposite voltage—current relationship: they allow no current irrespective of the voltage (assuming that it is finite). Devices that have zero current irrespective of the voltage are termed "open circuits" since they do not provide a path for current. These elements have infinite resistance. As shown later, current sources also have nonintuitive equivalent resistance: they are open-circuit devices but with a current offset.

The $v-i$ plot of an ideal current source is evident from its definition: an element that produces a specified current irrespective of the voltage across it. This leads to the $v-i$ plot shown in Figure 14.20 of a vertical line that intersects the current axis at $i = I_S$. By the earlier arguments, the equivalent resistance of such an ideal current source is infinite.

The concepts of ideal voltage and ideal current sources are somewhat counterintuitive. An ideal voltage source has the resistive properties of a short circuit, but it also somehow maintains a nonzero voltage. The trick is to understand that an ideal voltage source is a short circuit with respect to current, but not with respect to voltage. A similar apparent contradiction applies to current sources: they are open circuits with respect to voltage, yet produce a specified current. Understanding these apparent contradictions is critical to understanding ideal sources.

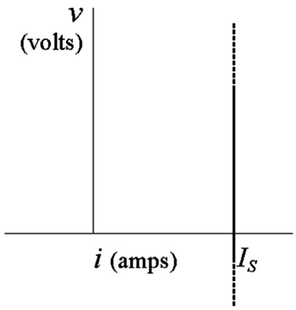


FIGURE 14.20 A v–i plot of an ideal current source. This plot shows that the resistor-like properties of a current source have an infinite value.

In this section, voltage and current sources have been described in terms of fixed values (i.e. DC sources), but the basic arguments do not change if V_S or I_S are time varying. This generalization also holds for the concepts presented next.

14.4.2 Real Voltage Sources—The Thévenin Source

Unlike ideal sources, real voltage sources are not immune to the current flowing through them, nor are real current sources immune to the voltage falling across them. In real voltage sources, the source voltage drops as more current is drawn from the source. This gives rise to a v–i plot such as shown in Figure 14.21, where the line is no longer horizontal but decreases with increasing current. The decrease indicates the presence of an internal, nonzero, resistance having a value equal to the negative slope of the line. The slope is negative because the voltage drop across the internal resistance is opposite in sign to V_s and hence subtracts from the values of V_s.

Real sources, then, are simply ideal sources with some nonzero resistance, so they can be represented as an ideal source in series with a resistor as shown in Figure 14.22. This configuration is also known as a Thévenin source, named after an engineer who developed a network reduction theory described later. Finding values for V_T and R_T given a physical (i.e., a real) source is fairly straightforward. The value of the internal ideal source, V_T, is

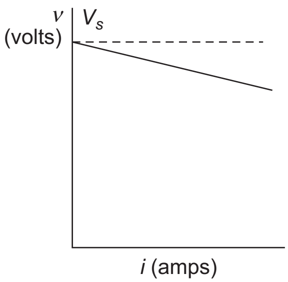


FIGURE 14.21 The v–i plot of a real voltage source (*solid line*). The nonzero slope of this line shows that the source contains an internal resistor. The slope indicates the value of this internal resistance.

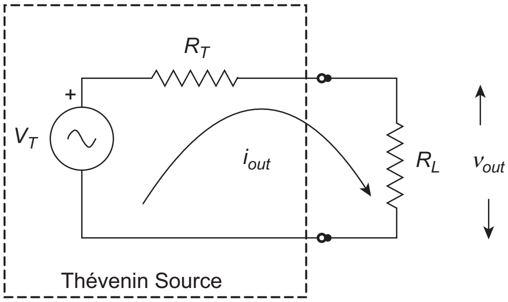


FIGURE 14.22 Representation of a real source using a Thévenin circuit. The Thévenin circuit is inside the *dashed rectangle*. The load resistor, R_L, draws current from the source. In a high-quality voltage source, the internal resistance, R_T, will be small.

the voltage that would be measured at the output, v_{out}, if no current was flowing through the circuit, that is, if the source is connected to an open circuit. For this reason, V_T is equivalent to the "open-circuit voltage," denoted v_{oc}. To find the value of the internal resistance, R_T, we need to draw current from the circuit and measure how much the output voltage decreases. A resistor placed across the output of the Thévenin source will do the trick. This resister, R_L, is often referred to as a "load resistor" or just the "load" because it makes the source do work by drawing current from the source $\left(P = v_{out}{}^2/R_L\right)$. The smaller the load resistor, the more current that will be drawn for the source, and the more power the source must supply.

Assuming a current i_{out} is being drawn from the source and the voltage measured is v_{out}, then the difference in voltage between the no-current voltage and current conditions is: $v_D = V_T - v_{out}$. Since the voltage difference v_D is due entirely to R_T, the value of R_T can be determined as:

$$R_T = \frac{v_D}{i_{out}} = \frac{(V_T - v_{out})}{i_{out}} \tag{14.11}$$

The maximum current the source is capable of producing occurs when the source is connected to a short circuit (i.e., $R_L \to 0\ \Omega$), and the current out of the source, the "short-circuit current," i_{sc}, is:

$$i_{sc} = \frac{V_T}{R_T} \tag{14.12}$$

Remembering that V_T is the voltage that would be measured under open-circuit conditions and defining the open-circuit voltage as v_{oc}, we write:

$$\begin{aligned} V_T &= v_{oc} = R_T i_{sc} \\ R_T &= \frac{v_{oc}}{i_{sc}} \end{aligned} \tag{14.13}$$

Putting Equation 14.13 in words, the internal resistance is equal to the open-circuit voltage (v_{oc}) divided by the short-circuit current (i_{sc}). Using Equation 14.13 is a viable method for determining R_T in theoretical problems, but is not practical in real situations with real sources because shorting a real source may draw excessive current and damage the source.[3] It is safer to draw only a small amount of current out of a real source by placing a large resistor across the source, not a short circuit. In this case, Equation 14.11 is used to find R_T since v_{out} and i_{out} can be measured and V_T can be determined by a measurement of open-circuit voltage. Example 14.9 takes this approach to determine the internal resistance of a voltage source.

EXAMPLE 14.9

In the laboratory, the voltage of a real source is measured using a "voltmeter" that draws negligible current from the source; hence the voltage recorded can be taken as the open-circuit voltage. (Most high-quality voltmeters require very little current to measure a voltage.) The voltage measured is 9.0 V. Figure 14.22 shows a resistor, R_L, placed across the output terminals of the source. In this example, this load resistor results in a current flow out of the source of: $i_{out} = 5$ mA (5×10^{-3} A). Assume this current is measured using an ideal current measurement device, although it could also be calculated from v_{out} if the value of R_L is known, i.e., $i_{out} = v_{out}/R_L$. Under this load condition, the output voltage, v_{out}, falls to 8.6 V. What is the internal resistance of the source, R_T? What is the load resistance, R_L, that produced this current?

Solution/Result: When there is no load resistor (i.e., $R_L = \infty\ \Omega$), then $i_{out} = 0$ and $v_{out} = V_T$. When the load resistor is attached to the output, $i_{out} = 5$ mA, and $v_{out} = 8.6$ V. Applying Equation 14.11:

$$R_T = \frac{v_D}{i_{out}} = \frac{(V_T - v_{out})}{i_{out}} = \frac{(9.0 - 8.6)}{0.005} = \frac{0.4}{0.005} = 80\ \Omega$$

To find the load resistor, R_L, use Ohm's law:

$$R_L = \frac{v_{out}}{i_{out}} = \frac{8.6}{0.005} = 1720\ \Omega$$

In summary, a real voltage source can be represented by an ideal source with a series resistance. In the examples shown here the sources are DC and the series element a pure resistor. In the more general case, the Thévenin source could generate sinusoids or other waveforms (but would still be ideal), and the series element might be a Thévenin impedance, Z_T. In the case where the source is sinusoidal, the equations mentioned previously still hold, but require phasor analysis for their solution.

14.4.3 Real Current Sources

Current sources are difficult to grasp because we intuitively think of current as an effect of voltage: voltage pushes current through a circuit. Current sources can be viewed as sources

[3] Many real sources such as laboratory "power supplies" have short-circuit protection where the output voltage drops to zero if the source is short circuited. Alternatively, some laboratory sources have current limiting where the voltage is automatically lowered to maintain a given maximum current.

that somehow adjust their voltage to produce the desired current. For an ideal current source, the larger the load resistor, the more work it has to do since it must generate a larger voltage to produce the desired current. Current sources prefer small load resistors, the opposite of voltage sources. For a real current source, as the load resistor increases the voltage requirement increases and will eventually exceed the voltage capabilities of the current source so the output current will fall off. This is reflected by the $v-i$ plot circuit in Figure 14.23, and can be represented by an internal resistor.

For current sources, the negative slope of the line in the $v-i$ plot is equal to the internal resistance just as for voltage sources. The circuit diagram of a real current source is shown in Figure 14.24 to be an ideal current source in parallel with an internal resistance. This configuration is often referred to as a "Norton equivalent" circuit. Inspection of this circuit shows how it represents the falloff in output current that occurs when higher voltages are needed to push the current though a large resistor. As the voltage needed at the output increases, more current flows though the internal resistor, R_N, and less is available for the

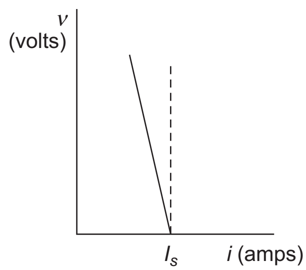


FIGURE 14.23 The $v-i$ plot of a real current source (*solid line*). The less-than-infinite slope of this *line* shows that the source cannot produce the voltage required when the load resistance increases. The source is not capable of producing the higher voltages that are needed to attain the desired current. The resultant reduction in current output with increased voltage requirements can be represented by an internal resistor, and the slope is indicative of its value.

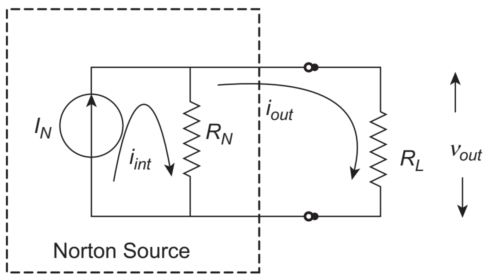


FIGURE 14.24 A circuit diagram of a real current source inside the *dashed rectangle* that is connected to a resistor load. The current source circuit is referred to as a Norton circuit. When a large external load is presented to this circuit, some of the current flows through the internal resistor, R_N. As R_L increases, a higher percentage of current flows through R_N and less through the output. In a high-quality current source R_N will be large.

output. If there were no internal resistor, all of the current would have to flow out of the source irrespective of the output voltage as expected from an ideal current source.

As shown in Figure 14.24, when the output is a short circuit (i.e., $R_L = 0.0\ \Omega$), the current that flows through R_N is zero, so all the current flows through the output. (Note that analogous to the open-circuit condition for a voltage source, the short-circuit condition produces the least work for a current source, in fact no work at all.) By KCL:

$$\begin{aligned} I_N - i_{R_N} - i_{out} &= 0; \\ i_{out} &= i_{sc} = I_N \end{aligned} \tag{14.14}$$

Hence I_N equals the short-circuit current, i_{sc}. When R_L is not a short circuit, some of the current flows through R_N and i_{out} decreases. Essentially, the internal resistor steals current from the current source when the load resister is anything other than zero. Applying KCL to the upper node of the Norton circuit (Figure 14.24), paying attention to the current directions:

$$\begin{aligned} I_N - i_{R_N} - i_{out} &= 0 \\ I_N - \frac{v_{out}}{R_N} - i_{out} &= 0 \end{aligned} \tag{14.15}$$

Solving for R_N:

$$\begin{aligned} &\frac{v_{out}}{R_N} = I_N - i_{out}; \quad R_N = \frac{v_{out}}{(I_N - i_{out})} \\ &\text{Since } I_N = isc: \quad R_N = \frac{v_{out}}{(i_{sc} - i_{out})} \end{aligned} \tag{14.16}$$

When the output of the Norton circuit is an open circuit (i.e., R_L 6 4), all the current flows through the internal resistor, R_N. Hence:

$$v_{oc} = I_N R_N \tag{14.17}$$

Combining this equation with Equation 14.14, we can solve for R_N in terms of the open-circuit voltage, v_{oc}, and the short-circuit current, i_{sc}:

$$\begin{aligned} v_{oc} &= I_N R_N = i_{sc} R_N \\ R_N &= \frac{v_{oc}}{i_{sc}} \end{aligned} \tag{14.18}$$

This relationship is the same as for the Thévenin circuit as given in Equation 14.13 if we make $R_T = R_N$.

EXAMPLE 14.10

A real current source produces a current of 500 mA under short-circuit conditions and a current of 490 mA when the short is replaced by a 20-Ω resistor. Find the internal resistance.

Solution/Result: Find v_{out} when the load is 20 Ω, then apply Equation 14.16 to find R_N.

$$v_{out} = R_L i_{out} = 20(.49) = 9.8 \text{ V}$$

$$R_N = \frac{v_{out}}{i_{sc} - i_{out}} = \frac{9.8}{0.5 - 0.49} = \frac{9.8}{0.01} = 980 \ \Omega$$

The Thévenin and Norton circuits have been presented in terms of sources, but they can also be used to represent entire networks as well as mechanical and other nonelectrical systems. These representations can be especially helpful when two systems are being connected. Imagine you are connecting two systems and you want to know how the interconnection will affect the behavior of the overall system. If you represent the source system as a Thévenin or Norton equivalent and then determine the effective input impedance of the loading system, you are able to calculate the loss of signal due to the interconnection. The same could be stated for a biological measurement where the biological system is the source and the measurement system is the load. You may not have much control over the nature of the source, but, as a biomedical engineer, you have some control over the effective impedance of the load. These concepts are explored further in a later section.

14.4.4 Thévenin and Norton Circuit Conversion

It is easy to convert between the Thévenin and Norton equivalent circuits, that is, to determine a Norton circuit that has the same voltage–current relationship as a given Thévenin circuit and vice versa. Such conversions allow you to apply KVL to systems with current sources (by converting them to an equivalent voltage source) or to use Kirchoff's current law (KCL) in systems with voltage sources (by converting them to an equivalent current). Consider the voltage–current relationship shown in the v–i plot of Figure 14.25. Since the curve is a straight line, it is uniquely determined by any two points. The horizontal and vertical intercepts, v_{oc} and i_{sc}, are particularly convenient points to use.

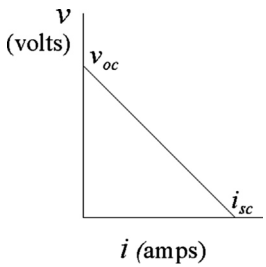


FIGURE 14.25 The v–i plot of the output of a device that can be represented as either a Thévenin or Norton circuit. The voltage–current relationship plots as a *straight line* and can be uniquely represented by two points such as v_{oc} and i_{sc}.

The equations for equivalence are easy to derive based on previous definitions:

$$\text{For a Thévenin circuit: } v_{ocT} = V_T; \quad \text{and since: } R_T = \frac{v_{ocT}}{i_{scT}}; \quad i_{scT} = \frac{v_{ocT}}{R_T}$$

$$\text{For a Norton circuit: } I_N = i_{scN}; \quad \text{and since} \quad R_N = \frac{v_{ocN}}{i_{scN}}; \quad v_{ocN} = R_N i_{scN}$$

For a Norton circuit to have the same v–i relationship as a Thévenin, $i_{scN} = i_{scT}$ and $v_{ocN} = v_{ocT}$. Equating these terms in the earlier equations:

$$\begin{aligned} I_N &= i_{scN} = i_{scT} = \frac{v_{ocT}}{R_T}; \quad \text{but } v_{ocT} = V_T \\ I_N &= \frac{V_T}{R_T} \end{aligned} \tag{14.19}$$

$$\begin{aligned} R_N &= \frac{v_{ocN}}{i_{scN}} = \frac{v_{ocT}}{i_{scT}}; \quad \text{but } \frac{v_{ocT}}{i_{scT}} = R_T \\ R_N &= R_T \end{aligned} \tag{14.20}$$

To go the other way and convert from a Norton to an equivalent Thévenin:

$$\begin{aligned} V_T &= v_{ocT} = v_{ocN}; \quad \text{but } v_{ocN} = I_N R_N \\ V_T &= I_N R_N \end{aligned} \tag{14.21}$$

$$\begin{aligned} R_T &= \frac{v_{ocT}}{i_{scT}} = \frac{v_{ocN}}{i_{scN}}; \quad \text{but } \frac{v_{ocN}}{i_{scN}} = R_N \\ R_T &= R_N \end{aligned} \tag{14.22}$$

These four equations allow for easy conversion between Thévenin and Norton circuits. Note that the internal resistance, R_N or R_T, is the same for either configuration. This is reasonable, since the internal resistance defines the slope of the v–i curve, so to achieve the same v–i relationship you need the same-sloped curve and hence the same resistor.

The ability to represent any linear v–i relationship by either a Thévenin or a Norton circuit implies that it is impossible to determine whether a real source is in fact a current or a voltage source based solely on external measurement of voltage and current. If the v–i relationship of a source is more or less a vertical line as in Figure 14.24, indicating a large internal resistance, we might guess that the source is probably a current source. In fact, one simple technique for constructing a crude current source in practice is to place a voltage source in series with a large resistor. Alternatively, if the v–i relationship is approximately horizontal as in Figure 14.21, a nonideal voltage source would be a better guess. However, if the v–i curve is neither particularly vertical nor horizontal, as in Figure 14.25, it is anyone's guess as to whether it is a current or voltage source and either would be an equally appropriate representation unless other information was available.

Conversion between Thévenin and Norton circuits can be used for nodal analysis in circuits that contain voltage sources or for mesh analysis in circuits that contain current sources. This application of Thévenin–Norton conversion is shown in the following example.

EXAMPLE 14.11

Find the voltage, V_A, in the circuit shown in Figure 14.26 using nodal analysis.

Solution: This circuit can be viewed as containing two Thévenin circuits: a 5-V source and 10-Ω resistor, and a 10-V source and 40-Ω resistor. After converting these two Thévenin circuits to equivalent Norton circuits using Equations 14.19 and 14.20, we apply standard nodal analysis. Figure 14.27 shows the circuit after Thévenin–Norton conversion.

Result: Writing the KCL equation around node A.

$$i_{S1} + i_{S2} + i_{R1} + i_{R2} + i_{R3} = 0$$

$$0.5 + 0.25 - \frac{V_A}{R_1} - \frac{V_A}{R_2} - \frac{V_A}{R_3} = 0$$

$$0.5 + 0.25 - V_A\left(\frac{1}{10} + \frac{1}{20} + \frac{1}{40}\right) = 0$$

$$V_A = \frac{0.5 + 0.25}{1/10 + 1/20 + 1/40} = \frac{.75}{0.1 + 0.05 + 0.025} = \frac{0.75}{0.175} = 4.29 \text{ V}$$

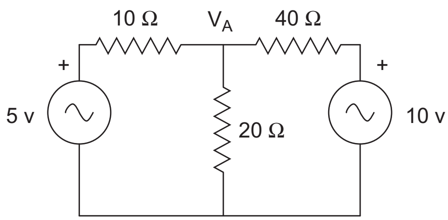


FIGURE 14.26 A two-mesh network containing voltage sources. If the two Thévenin circuits on either side are converted to their Norton equivalents, this circuit becomes a single-node circuit and can be evaluated using a single nodal equation. (The sources are shown as sinusoidal, but since the passive elements are all resistors, the equations will be algebraic.)

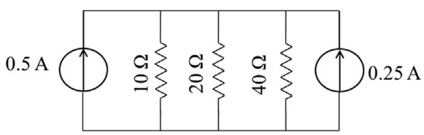


FIGURE 14.27 The network in Figure 14.26 after the two Thévenin circuits have been converted to their Norton equivalents. This is now a single-node network.

The Thévenin and Norton circuits and their interconversions are useful for network reduction of circuits that contain sources as shown in the next section. These concepts also apply to mechanical systems, with appropriate modifications, as illustrated at the end of this chapter.

14.5 THÉVENIN AND NORTON THEOREMS: NETWORK REDUCTION WITH SOURCES

The "Thévenin Theorem" states that any network of passive elements and sources can be reduced to a single voltage source and series impedance. Such a reduced network would look like a Thévenin circuit such as that shown in Figure 14.22, except that the internal resistance, R_T, would be replaced by a generalized impedance, $Z_T(\omega)$. The Norton theorem makes the same claim for Norton circuits, which is reasonable since Thévenin circuits can easily be converted to Norton circuits via Equations 14.19 and 14.20.

There are a few constraints on these theorems. The elements in the network being reduced must be linear, and if there are multiple sources in the network they must be at the same frequency. As has been done in the past, the techniques for network reduction will be developed using phasor representation and hence will be limited to networks with sinusoidal sources. However, the approach is the same in the Laplace domain.

There are two approaches to finding the Thévenin or Norton equivalent of a general network. One is based on solving for the open-circuit voltage, v_{oc}, and the short-circuit current, i_{sc}. The other method evaluates only the open-circuit voltage, v_{oc}, and then determines R_T (or R_N) through network reduction. During network reduction, a source is replaced by its equivalent resistance, that is, short circuits ($R = 0$) substitute for voltage sources, and open circuits ($R = \infty$) substitute for current sources. Both network reduction methods are straightforward, but the open-circuit voltage/short-circuit current approach can be implemented on a computer. These approaches are demonstrated in the next two examples.

EXAMPLE 14.12

Find the Thévenin equivalent of the circuit in Figure 14.28 using both the v_{oc}–i_{sc} method and the v_{oc}–network reduction technique.

Solution: v_{oc}-Reduction method: First find the open-circuit voltage, v_{oc}, using standard network analysis. Convert all network elements to their phasor representations as shown in Figure 14.29.

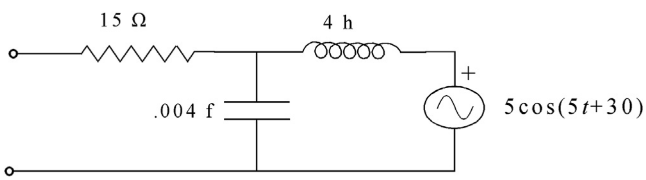


FIGURE 14.28 A network that will be reduced to a single impedance and source using two different strategies.

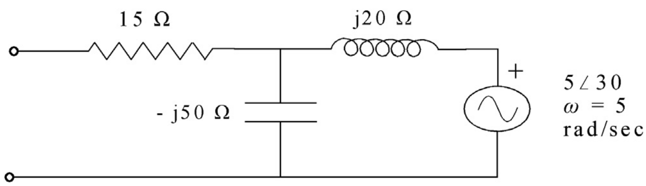


FIGURE 14.29 The network of Figure 14.28 after conversion to the phasor domain.

Since in the open-circuit case no current flows through the 15-Ω resistor, there is no voltage drop across this resistor, so the open-circuit voltage is the same as the voltage across the capacitor. The resistor is essentially not there with respect to open-circuit voltage. The resistor is not totally useless: it does play a role in determining the equivalent impedance and is involved in the calculation of short-circuit current. The open-circuit voltage, the voltage across the capacitor, can be determined by writing the mesh equation around the loop consisting of the capacitor, inductor, and source. Using the usual directional conventions, defining the mesh current as clockwise and going around the loop in a clockwise direction, note that the voltage source will be negative as there is a voltage drop going around in the clockwise direction.

$$-5\angle 30 - i(-j50 + j20) = 0$$
$$i = \frac{-5\angle 30}{-j50 + j20} = \frac{-5\angle 30}{-j30} = \frac{-5\angle 30}{30\angle -90} = -0.167\angle 120 \text{ A}$$
$$v_{oc} = iZ_C = -0.167\angle 120(-j50) = -0.167\angle 120(50\angle -90)$$
$$v_{oc} = -8.35\angle 30 \text{ V}$$

Note that the magnitude of the Thévenin equivalent voltage is actually larger than the source voltage. This is the result of a partial resonance between the inductor and capacitor.

Next, find the equivalent impedance by reduction. To reduce the network, essentially turn off the sources and apply network reduction techniques to what is left. Turning off a source does not mean you remove it from the circuit; rather, to turn off a source, you replace it by its equivalent resistance. For an ideal voltage source $R_T \rightarrow 0$ Ω, so the equivalent resistance of an ideal voltage source is 0 Ω, i.e., a short circuit. To turn off a voltage source you replace it with a short circuit. After you replace the source with a short circuit, you are left with the network shown in Figure 14.30 on the left-hand side. Series−parallel reduction techniques will work for this network.

After replacing the voltage source with a short circuit, we are left with a parallel combination of inductor and capacitor. This combines to a single impedance, Z_P:

$$Z_P = \frac{Z_C Z_L}{Z_C + Z_L} = \frac{-j50(j20)}{-j50 + j20} = \frac{1000}{-j30} = \frac{1000}{30\angle -90}$$
$$Z_P = 33.33\angle 90 \ \Omega$$

We are left with the series combination of Z_P and the 15-Ω resistor:

$$Z_T = 15 + 33.33\angle 90 = 15 + j33.33 = 36.55\angle 65.8 \ \Omega$$

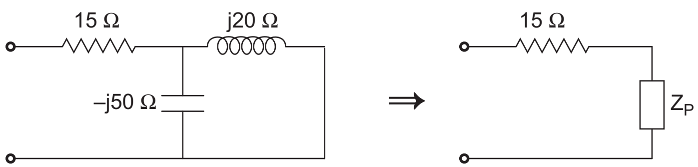


FIGURE 14.30 (Left) The network given in Figure 14.29 after effectively turning off the voltage source. The voltage source is replaced by its equivalent resistance, which is, since it is an ideal source, 0.0 Ω or a short circuit. (Right) The network after combining the two parallel impedances into a single impedance, Z_P.

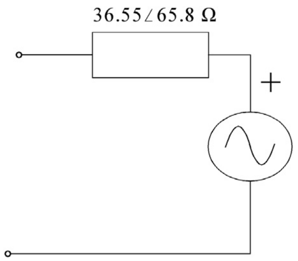


FIGURE 14.31 The Thévenin equivalent of the network given in Figure 14.28 determined by the v_{oc}-reduction method.

Results: v_{oc}-Reduction method: The original circuit can be equivalently represented by the Thévenin circuit shown in Figure 14.31.

Solution $v_{oc}-i_{sc}$: In this method, we solve for the open-circuit voltage and short-circuit current. We have already found the open-circuit voltage so it is only necessary to find the short-circuit current. After shorting out the output and converting to phasor notation, the circuit is shown in Figure 14.32.

If we solve this using mesh analysis, it is a two-mesh circuit, but if we convert the inductor—source combination to a Norton equivalent, it becomes a single-node equation. To implement the conversion, use Equations 14.19 and 14.20:

$$I_N = \frac{V_T}{R_T} = \frac{5\angle 30}{j20} = \frac{5\angle 30}{20\angle 90} = 0.25\angle -60 \text{ V}; \quad R_N = R_T = j20 \ \Omega$$

The new network is shown in Figure 14.33.

$$.25\angle -60 - v\left(\frac{1}{15} + \frac{1}{-j50} + \frac{1}{j20}\right) = 0$$

$$v = \frac{.25\angle -60}{\frac{1}{15} + \frac{1}{-j50} + \frac{1}{j20}} = \frac{.25\angle -60}{0.0667 + j0.02 - j0.05} = \frac{.25\angle -60}{0.0667 - j0.03}$$

$$v = \frac{.25\angle -60}{0.073\angle -24.2} = 3.42\angle -35.8 \text{ V}$$

$$i_{sc} = \frac{v}{15} = \frac{3.42\angle -35.8}{15} = 0.228\angle -35.8 \text{ A}$$

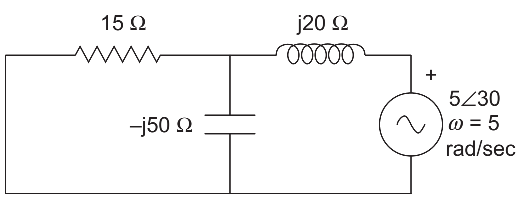


FIGURE 14.32 The circuit shown in Figure 14.28 after shorting the output terminals and converting to phasor notation. Note that the capacitor and resistor are in parallel.

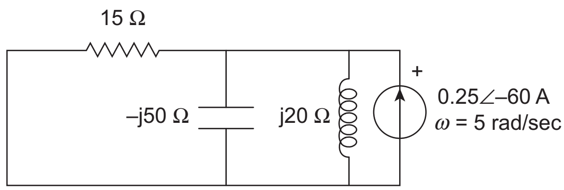


FIGURE 14.33 The circuit of Figure 14.32 after the voltage source and series impedance are converted to a Norton equivalent source. This is a one-node problem that can be solved from a single equation.

Now solve for R_T:

$$R_T = \frac{v_{oc}}{i_{sc}} = \frac{8.35\angle 30}{0.228\angle -35.8} = 36.6\angle 65.8\ \Omega$$

This is the same value for R_T found using the v_{oc}-reduction method and leads to the Thévenin equivalent shown in Figure 14.31. More complicated networks can be reduced using MATLAB as shown in the following example.

EXAMPLE 14.13

Find the Norton equivalent of the circuit of Figure 14.34 with the aid of MATLAB.

Solution: The open-circuit voltage and short-circuit current can be solved directly using mesh analysis in conjunction with MATLAB. In fact, the mesh equations in both cases (solving for v_{oc} and i_{sc}) are similar. The only difference is that when solving for the short-circuit current, the 20-Ω resistor will be short circuited and not appear in the equation.

First convert to phasor notation, and then encode the network directly into MATLAB. Since we are using MATLAB and the computational load is reduced, we keep ω as a variable in case we want to find the Norton equivalent for other frequencies.

After converting to phasor notation and assigning the mesh current, the circuit is shown in Figure 14.35. Note that the open-circuit voltage, v_{oc}, is the voltage across the 20-Ω resistor and the short-circuit current, i_{sc}, is just i_3 when the resistor is shorted.

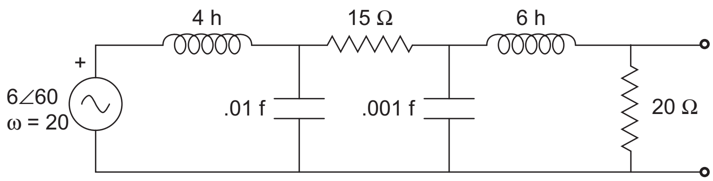


FIGURE 14.34 A complicated network that is reduced to a Norton equivalent in Example 14.13. To ease the mathematical burden, MATLAB is used to solve the equation.

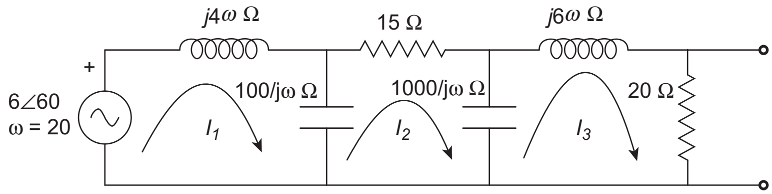


FIGURE 14.35 The circuit of Figure 14.34 after conversion to phasor notation. Since mesh analysis will be used, the mesh currents are shown.

Writing the KVL equations for the open-circuit condition:

$$\begin{vmatrix} 6\angle 60 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} j4\omega + \dfrac{100}{j\omega} & -\dfrac{100}{j\omega} & 0 \\ -\dfrac{100}{j\omega} & 15 + \dfrac{100+1000}{j\omega} & -\dfrac{1000}{j\omega} \\ 0 & -\dfrac{1000}{j\omega} & 20 + j6\omega + \dfrac{1000}{j\omega} \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \\ i_3 \end{vmatrix}$$

where $v_{oc} = 20\ i_3$.

The mesh equation for the short-circuit condition is quite similar:

$$\begin{vmatrix} 6\angle 60 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} j4\omega + \dfrac{100}{j\omega} & -\dfrac{100}{j\omega} & 0 \\ -\dfrac{100}{j\omega} & 15 + \dfrac{100+1000}{j\omega} & -\dfrac{1000}{j\omega} \\ 0 & -\dfrac{1000}{j\omega} & j6\omega + \dfrac{1000}{j\omega} \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \\ i_3 \end{vmatrix}$$

where $i_{sc} = i_3$.

The following is the program to solve these equations and find I_N and R_N:

```
% Example 14.13 Find the Norton equivalent of a three-mesh circuit
%
w = 20;                                 % Define frequency
theta = 60*2*pi/360;
VS = 6*cos(theta) + 6*sin(theta)*j;     % Define Vs as rectangular
v = [VS 0 0]';                          % Define voltage vector
%
% Define open-circuit impedance matrix
Zoc = [4j*w+100/(j*w), -100/(j*w), 0; -100/(j*w), 15+1100/(j*w),...
      -1000/(j*w); 0, -1000/(j*w), 20+6j*w+1000/(j*w)];
ioc = Zoc\v;                            % Solve for currents
voc = 20*ioc(3);                        % and open-circuit voltage
%
% Define short-circuit impedance matrix and solve for short-circuit current
```

```
Zsc = [4j*w+100/(j*w), −100/(j*w), 0; −100/(j*w), 15+1100/(j*w),...
      −1000/(j*w); 0, −1000/(1j*w), 6j*w+1000/(j*w)];
i = Zsc\v                               % Solve for currents
isc = i(3);                             % Find isc
Zn = voc/isc;                           % Solve for Z_N (Equation 14.18)
%
% Output magnitude and phase of IN and RN
IN_mag = abs(isc)
IN_phase = angle(isc)*360/(2*pi)
ZN_mag = abs(Zn)
ZN_phase = angle(Zn)*360/(2*pi)
```

This program produces the following outputs:

$$I_N = 0.0031\angle 20.6 \text{ A}$$
$$Z_N = 19.36\angle 9.8 \ \Omega$$

It is easy to modify this program to find, and plot, the Norton element values over a range of frequencies. This exercise is given as one of the problems at the end of the chapter.

14.6 MEASUREMENT LOADING

We now have the tools to analyze the situation when two systems are connected together. For biomedical engineers not involved in electronic design, this situation most frequently occurs when making a measurement, so we analyze the problem in that context. However, the approach followed here applies to any situation when two systems are connected together.

14.6.1 Ideal and Real Measurement Devices

One of the important tasks of biomedical engineers is to make measurements, usually on a living system. Any measurement requires withdrawing some energy from the system of interest and that, in turn, alters the state of the system and the value of the measurement. This alteration is referred to here as "measurement loading." The word "load" is applied to any device that is attached to a system of interest, and "loading" is the effect the attached device has on the system. Measurement loading is a well-known phenomenon that extends down to the smallest systems and has a significant impact on fundamental principles in particle physics. The concepts developed earlier can be used to analyze the effect a measurement device has on the system being measured. In fact, an analysis of measurement loading is one of the major applications of Thévenin and Norton circuit models.

Just as there are ideal and real sources, there are ideal and real measurement devices or, equivalently, ideal and real loads. As mentioned previously, an ideal voltage source supplies a given voltage at any current and an ideal current source supplies a given current at any voltage. Since power is the product of voltage times current ($P = vi$), an ideal source can supply any amount of energy or power required: infinite if need be. Ideal measurement devices (or ideal loads) have the opposite characteristics: they can make a measurement without drawing any energy or power from the system being measured. Of course, we know from basic physics that such an idealization is impossible, but some measurement devices can provide nearly ideal measurements, at least for all practical purposes.

The goal in practical situations is to be able to make a measurement without significantly altering the system being measured. The ability to attain this goal depends on the characteristics of the source as well as the load; a given device might have little effect on one system, thus providing a reliable measurement, yet significantly alter another system, giving a measurement that does not reflect the underlying conditions. It is not just a matter of how much energy a measurement device requires (i.e., load impedance), but how much energy the system being measured can supply without a significant change in the quantity being measured (i.e., the source impedance).

Just as ideal voltage and current sources have quite different properties, ideal measurement devices for voltage and current differ significantly. A device that measures voltage is termed a "voltmeter." An ideal voltmeter would draw no power from the circuit being measured. Since $P = vi$ and v cannot be zero (that is what is being measured), an ideal voltmeter must draw no current while making the measurement. The current will be zero for any voltage only if the equivalent resistance of the voltmeter is infinite. An ideal voltmeter is effectively an open circuit, and the $v-i$ plot is a vertical line. Practical voltmeters do not have infinite resistances, but they do have very large impedances, of the order of 100s of megohms (1 megohm = 1 MΩ = 10^6 Ω), and this can be considered ideal for all but the most challenging conditions.

Current measuring devices are termed "ammeters." An ideal ammeter also needs no power from the circuit to make its measurement. Again, $P = vi$ and i cannot be zero in an ammeter, so now it is voltage that must be zero to draw no energy from the system being measured. This means that an ideal ammeter is effectively a short circuit having an equivalent resistance of 0.0 Ω. The $v-i$ plot of an ideal ammeter would be a horizontal line. Practical ammeters are generally not very ideal, having resistances approaching a tenth of an ohm or more. However, current measurements are rarely made in practice because that involves breaking a circuit connection to make the measurement unless special "clip-on" ammeters are used. The characteristics of ideal sources and loads are summarized in Table 14.1.

An illustration of the error caused by a less-than-ideal ammeter is given in the following example.

TABLE 14.1 Electrical Characteristics of Ideal Sources and Loads

	Sources		Measurement Devices (Loads)	
Characteristics	**Voltage**	**Current**	**Voltage**	**Current**
Impedance	0.0 Ω	∞ Ω	∞ Ω	0.0 Ω
Voltage	V_S	Up to ∞ A	$V_{measured}$	0.0 A
Current	Up to ∞ V	I_S	0.0 A	$I_{measured}$

EXAMPLE 14.14

A practical ammeter having an internal resistance of 2.0 Ω is used to measure the short-circuit current of the three-mesh network used in Example 14.13. How large is the error, that is, how much does the measurement differ from the true short-circuit current?

Solution: As with all issues of measurement loading, it is easiest to use the Thévenin or Norton representation of the system being loaded. The Norton equivalent of the three-mesh circuit is determined in Example 14.13 and is shown in Figure 14.36 loaded by the ammeter. From the Norton circuit, we know the true short-circuit current is: $i_{sc} = I_N = 0.0031\pi20.6$ A. The measured short-circuit current can be determined by applying nodal analysis to the circuit.

Applying KCL:

$$.0031\angle 20.6 - v\left(\frac{1}{19.36\angle 9.8} + \frac{1}{2}\right) = 0$$

$$v = \frac{.0031\angle 20.6}{\dfrac{1}{19.36\angle 9.8} + \dfrac{1}{2}} = \frac{.0031\angle 20.6}{.052\angle -9.8 + 0.5} = \frac{.0031\angle 20.6}{.051 - j.009 + 0.5} = \frac{.0031\angle 20.6}{0.55\angle -0.9}$$

$$v = 0.0056\angle 21.5 \text{ V}$$

$$i_{sc\_measured} = \frac{v}{R} = \frac{.0056\angle 21.5}{2} = .0028\angle 21.5 \text{ A}$$

The measured short-circuit current is slightly less than the actual short-circuit current, which is equal to I_N. The error is:

$$\text{Error} = \frac{(.0031 - .0028)}{.0031}100 = 9.7\%$$

The difference in the measured current and the actual current is due to current flowing through the internal resistor. Since the external load is much less (an order of magnitude) than the internal resistor, most of the current still flows through the ammeter, so the error is small. The smaller the ammeter's impedance with respect to the system's internal impedance, the closer the measurement loading is to ideal. In this case, the load resistor is approximately one-tenth the value of the internal resistor, leading to an error of approximately 10%. For some measurements, this may be sufficiently accurate. Usually a ratio of 1:100 (i.e. two orders of magnitude or 1% error) is adequate for the type of accuracy required in biomedical engineering measurements.

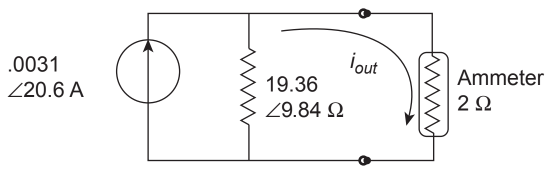


FIGURE 14.36 The output current of the Norton equivalent of the circuit shown in Figure 14.34. A less-than-ideal ammeter with an internal resistance of 2.0 Ω is being used to measure the short-circuit current. The effect of the nonzero resistance of the ammeter on the resultant current measurement is determined in Example 14.14.

The same rules regarding source and load resistance apply to voltage measurements, except now the load resistance should be much greater than the internal resistance (or impedance). In voltage measurements, if the load resistor is 100 or more times the internal resistance, the loading can usually be considered negligible and the measurement sufficiently accurate.

These general rules also apply whenever one network is attached to another. If voltages carry the signal (usually the case), the influence of the second network on the first can be ignored if the effective input impedance of the second network is much greater than the effective output impedance of the first network, Figure 14.37.

The relationship between source and load impedances has major consequences on system analysis and specifically the transfer function. Recall the two basic assumptions used in constructing a system's transfer function: (1) the input is an ideal source (i.e., $Z_1 \to 0$) and (2) the output is an ideal load ($Z_2 \to \infty$). Referring to Figure 14.37, if $Z_2 >> Z_1$, the transfer functions derived for each individual network independently will still work when the two are interconnected. If $Z_2 >> Z_1$ (say, 100 times), these assumptions are reasonably met and the original transfer functions can be taken as valid. The transfer function of the combined network is $TF_1(\omega)\ TF_2(\omega)$. Of course the input to the source network and the next load on loading network could still present problems that depend on the relative input and output impedances connected to the combined network. An analysis of voltage loading is found in the problems.

If the signal is carried as a current, then the opposite would be true. The output impedance of Network 1 should be much greater than the input impedance of Network 2, i.e., $Z_2 << Z_1$. In this situation, the current loading of Network 2 can be considered negligible with respect to Network 1. Signals are rarely viewed in terms of currents except for the output of certain transducers, particularly those that respond to light, and in these cases the signal is converted to a voltage by a special amplifier circuit (see Section 15.8.4).

What if these conditions are not met? For example, what if Z_2 is approximately equal to Z_1? In this case, you have two choices: calculate the transfer function of the two-network combination, or estimate the error that will occur owing to the interconnection as in the last example. Sometimes you actually want to increase the load on a system, purposely making Z_2 close to the value of Z_1. The motivation for such a strategy is explained in the next section.

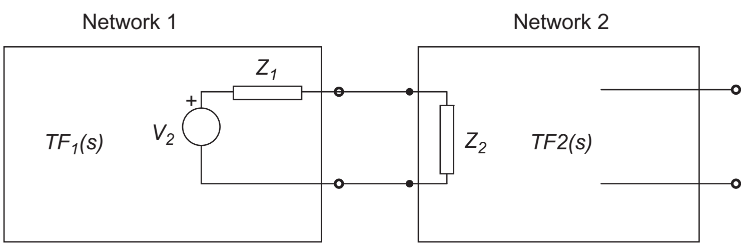


FIGURE 14.37 The input of Network 2 is connected to the output of Network 1. If the equivalent input impedance of Network 2, Z_L, is much greater than the output impedance of Network 1, Z_T, the output of Network 1 will not be altered by the connection to Network 2. The transfer functions derived for each of these networks separately are still valid.

14.6.2 Maximum Power Transfer

The goal in most measurement applications is to extract minimum energy from the system. This is also the usual goal when one system is connected to another because that way the original transfer function remains unchanged. Assuming voltage signals, the load impedance should be much greater than the internal impedance of the source, or vice versa if the signal is carried by current. This results in the minimum power and maximum voltage (or current) out of the source. But what if the goal is to extract maximum energy from the system? To determine the conditions for maximum power out of the system, we consider the Thévenin circuit with its load resistor in Figure 14.38.

To address this question, we assume that R_T is inside the source and cannot be adjusted, that is, it is an internal and inaccessible property of the source. If only R_L can be adjusted, the question becomes: what is the value of R_L that will extract maximum power from the system? The power out of the system is: $P = v_{out}\ i_{out}$ or $P = R_L i_{out}{}^2$. To find the value of the load resistor R_L that will deliver the maximum power to itself, we use the standard calculus trick for maximizing a function: solve for P in terms of R_L, and then take dP/dR_L and set it to zero:

$$P = R_L i^2; \quad i = \frac{V_T}{(R_L + R_T)}; \quad \text{hence,} \quad P = \frac{R_L V_T^2}{(R_L + R_T)^2}$$

$$\text{Solving for } \frac{dP}{dR_L} \text{ by parts:}$$

$$\frac{dP}{dR_L} = \frac{V_T^2 (R_L + R_T)^2 - 2V_T^2 R_L (R_L + R_T)}{(R_L + R_T)^4} = 0 \tag{14.23}$$

$$V_T^2 (R_L + R_T)^2 = 2V_T^2 R_L (R_L + R_T)$$

$$R_L + R_T = 2R_L;$$

$$R_L = R_T$$

So for maximum power out of the system, R_L should equal R_T (or, more generally, $Z_L = Z_T$), a condition known as "impedance matching." Since the power in a resistor is proportional to the resistance ($i^2\ R$) and the two resistors are equal, the power transferred to R_L will be half the total power, and the other half is dissipated by R_T.

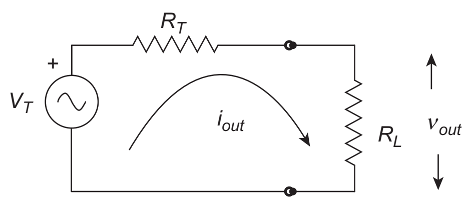


FIGURE 14.38 A Thévenin circuit is shown with a load resistor, R_L. For *minimum* power out of the system, R_L should be much greater than R_T. That way the output current and power approach 0. In this section we seek to determine the value of R_L that will extract maximum power from the Thévenin source.

Equation 14.23 is known as the "Maximum Power Theorem." Using this theorem, it is possible to find the value of load resistance that extracts maximum power from any network. Just convert the network to a Thévenin equivalent and set R_L equal to R_T. Recall that the maximum power theorem applies when R_T is fixed and R_L is varied. If R_T can be adjusted, then just by looking at Figure 14.38 we can see that maximum power will be extracted from the circuit when $R_T = 0$. When sinusoidal or other signals are involved, Equation 14.23 extends directly to impedances.

14.7 MECHANICAL SYSTEMS

All of the concepts described in this chapter are applicable to mechanical systems with only minor modifications. The concepts of equivalent impedances and impedance matching are often used in mechanical systems, particularly in acoustic applications. Of particular value are the concepts of real and ideal sources and real and ideal loads or measurement devices. As mentioned in Chapter 12, an ideal force generator produces a specific force irrespective of the velocity, just as an ideal voltage source produces the required voltage at any current. An ideal force generator will generate the same force at 0.0 velocity, or 10 mph, or 10,000 mph, or even at the speed of light (clearly impossible), if necessary. The force produced by a real force generator will decrease with velocity. This can be expressed in a force−velocity plot (analogous to the $v-i$ plot) as shown in Figure 14.39A.

An ideal velocity (or displacement) generator will produce a specific velocity against any force, be it 1 oz. or 1 ton, but for a real velocity generator, the velocity produced decreases as the force against it increases, Figure 14.39B. Again there is an ambiguity between real force and velocity generators. The device producing the force−velocity curve shown in Figure 14.39C could be interpreted as a nonideal force generator or a nonideal velocity generator: it is not possible to determine its true nature from the force−velocity plot alone.

Ideal measurement devices follow the same guiding principle in mechanical systems: they should extract no energy from the system being measured. For a force-measuring device, a "force transducer," velocity must be zero and position a constant. A constant position

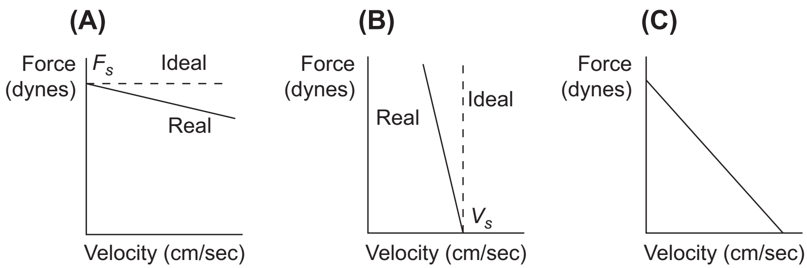


FIGURE 14.39 (A) The force−velocity plot of an ideal (*dashed line*) and a real (*solid line*) force generator. The force produced by a real generator decreases the faster it must move to generate that force. (B) The force−velocity plot of an ideal (*dashed line*) and a real (*solid line*) velocity generator. The velocity produced by a real generator decreases as the opposing force increases. (C) The force−velocity plot of a generator that may be interpreted as either a nonideal force or a nonideal velocity transducer.

condition is also known as an "isometric" condition. So an ideal force transducer requires no movement to make its measurement—it appears as a solid, immobile wall to the system being measured. Since mechanical impedance is defined as F/v in Equation 12.65, if v is zero for all F, then the mechanical impedance is infinite.

For an ideal velocity (or displacement) transducer, the force required to make a measurement would be zero. If F is zero for all v, then the mechanical impedance is zero. The characteristics of ideal mechanical sources and loads are given in Table 14.2 in a fashion analogous to the electrical characteristics in Table 14.1.

The concept of equivalent impedances and sources is useful in determining the alteration produced by a nonideal measurement device or load. The analog of Thévenin and Norton equivalent circuits can also be constructed for mechanical systems. The mechanical analog of a Thévenin circuit is a force generator with a parallel impedance (the configuration is reversed in a mechanical system), Figure 14.40A. Conversely, the mechanical equivalent analog of a Norton circuit is a velocity (or displacement) generator in series with the equivalent impedance, Figure 14.40B. To find the values for either of the two equivalent mechanical systems in Figure 14.40, we use the analog of the v_{oc}–i_{sc} method, but now call it the F_{iso}–$v_{no\ load}$ method: find "isometric" force, the force when velocity is zero (position is constant), and the unloaded velocity, the velocity when no force is applied to the system.

Most practical lumped-parameter mechanical systems are not so complicated and do not usually require reduction to a Thévenin- or Norton-like equivalent circuit. Nonetheless, all of the concepts regarding source and load impedances developed previously apply to mechanical systems. If the output of the mechanical system is a force, the minimum load is one with a large equivalent mechanical impedance, a load that tends to produce a large opposing force

TABLE 14.2 Mechanical Characteristics of Ideal Sources and Loads

	Sources		Measurement Devices (Loads)	
Characteristics	**Force**	**Velocity**	**Force**	**Velocity**
Impedance	0.0 Ω	∞ Ω	∞ Ω	0.0 Ω
Force	F_S	Up to ∞ dyn	$F_{measured}$	0.0 dyn
Velocity	Up to ∞ cm/s	v_S	0.0 cm/s	$v_{measured}$

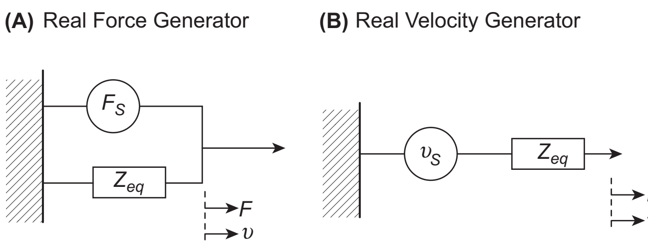


FIGURE 14.40 (A) A mechanical analog of a Thévenin equivalent circuit consisting of an ideal force generator with a parallel mechanical impedance. (B) The mechanical analog of a Norton circuit consisting of an ideal velocity generator with series mechanical impedance. These configurations can be used to determine the effect of loading by a measurement device or by another mechanical system.

and allows very little movement. A solid wall is an example of high mechanical impedance. Conversely, if the output of the mechanical system is a velocity, the minimum load is produced by a system having very little opposition to movement, a small mechanical impedance. Air, or better yet a vacuum, is an example of a low-impedance mechanical system. Finally, if the goal is to transfer maximum power from the source to the load, the mechanical impedances of each should be equal. These principles are explored in the following examples.

EXAMPLE 14.15

The mechanical elements of a real force generator are shown on the left-hand side of Figure 14.41, whereas the mechanical elements of a real force transducer are shown on the right-hand side.

The left-hand side shows a real force generator consisting of an ideal force generator (F_S) in parallel with friction and mass. The right-hand side is a real force transducer consisting of a displacement transducer (marked X) with a parallel spring. This transducer actually measures the displacement of the spring, which is proportional to force ($F = k_e X$).

Find the force that would be measured by an ideal force transducer, that is, the force at the interface that would be produced by the generator if its velocity was zero. The force transducer actually measures displacement of the spring (the usual construction of a force transducer). What is the force that is actually measured by this nonideal transducer? How could this transducer be improved to make a measurement with less error? The system parameters are:

$$k_f = 20 \text{ dyn s/cm}; \text{m} = 5 \text{ g}; k_e = 2400 \text{ dyn/cm}; F_S = 10 \sin(4\text{t}).\text{dyn}$$

Solution: To find the force measured by an ideal force transducer, write the equations for the force generator and set velocity to zero as would be the case for an ideal force transducer. Note that F_S is negative based on its defined direction (the arrow pointing to the left).

$$F_{measured} = -F_S - v\,(j\omega m + kf) \quad \text{if } v = 0$$
$$F_{measured} = -F_S$$

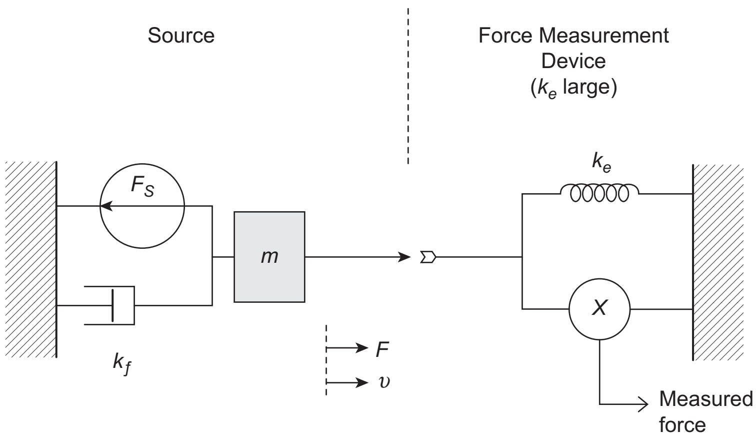


FIGURE 14.41 A realistic force generator on the left is connected to a realistic force transducer on the right. These combined systems are used to examine the effect of loading on a real force generator in Example 14.15.

As expected, the force measured by an ideal force transducer is just the force of the ideal component of the source. If the measurement device or load is ideal, then it does not matter what the internal impedance of the source is, the output of the ideal component is measured at the output. However, when the measurement transducer is attached to a less-than-ideal transducer, the velocity is no longer zero and some of the force is distributed to the internal impedance.

To determine the force out of the force generator under the nonideal load, write the equations for the combined system. Since there is the equivalent of only one node in the combined system, only a single equation will be necessary, but we need to pay attention to the signs.

$$-F_S - v\left(j\omega m + k_f + \frac{k_e}{j\omega}\right) = 0$$

$$-10 - v\left(j4(5) + 20 + \frac{2400}{j4}\right) = -10 - v(j20 + 20 - j600) = 0$$

$$v = \frac{-10}{20 - j580} = \frac{-10}{580.3\angle -88} = -17.2 \times 10^{-3}\angle 88 \ \text{cm/s}$$

$$F_{measured} = vZ_{k_e} = -17.2 \times 10^{-3}\angle 88(-j600) = -10.3\angle -2 \ \text{dyn}$$

Result: Hence the measured force is 3% larger than the actual force. The measured force is larger because of a very small resonance between the mass in the force generator and the elastic element in the transducer. Note that the elastic element is very stiff ($k_e = 2400$ dyn/cm) to make the transducer a good force transducer: the stiffer the elastic element, the closer the velocity will be to zero. To improve the measurement further, this element could be made even stiffer. For example, if the elasticity, k_e, were increased to 9600 dyn/cm, the force measured would be 10.08 $\angle -.5$ dyn, reducing the error to less than 1.0 percent. The determination of improvement created by different transducer loads is presented as a problem at the end of the chapter.

The next example involves the measurement of a velocity generator.

EXAMPLE 14.16

The mechanical elements of a real velocity generator are shown on the left side of Figure 14.42. The left side shows a real velocity generator consisting of an ideal velocity generator (V_S) in series with a friction and an elastic element. The right side is a real velocity transducer consisting of an ideal velocity transducer (marked X, its output is proportional to v_2) with a parallel spring. Assume V_S is a sinusoid with $\omega = 4$ rad/s and $k_f = 20$ dyn s/cm; $k_{e1} = 20$ dyn/cm; $k_{e2} = 2$ dyn/cm.

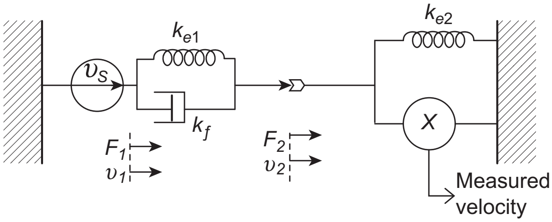


FIGURE 14.42 A realistic velocity generator is connected to a realistic velocity transducer. This configuration is used in Example 14.16 to examine the difference between the measured velocity and that produced by the ideal source.

Find the velocity that is measured by the velocity transducer on the right side of Figure 14.42. Note that this transducer is the same as the force transducer in the last example except the elasticity is very much smaller. Improvement in the accuracy of this transducer is given as one of the problems at the end of the chapter.

Solution: The solution proceeds in exactly the same manner as in the previous problem except that now there are two velocity points. However, since velocity v_1 is equal to V_S, it is not independent and only one equation needs to be solved.

Writing the sum of forces equation about the point indicated by v_2 point:

$$-\left(\frac{k_{e1}}{j\omega}+k_f\right)(v_2-v_1)-\left(\frac{k_{e2}}{j\omega}\right)v_2 = 0 \quad \text{Substituting } v_1 = \mathscr{V}_S \text{ and other variables}$$

$$-\left(\frac{20}{j4}+20\right)(v_2-\mathscr{V}_S)-\left(\frac{2}{j4}\right)v_2 = (j5-20)(v_2-\mathscr{V}_S)+j0.5(v_2) = 0$$

$$(-20+j5.5)v_2-(-20+j5)\mathscr{V}_S = 0 \quad \text{Solving for } v_2$$

$$v_2 = \frac{(-20+j5)\mathscr{V}_S}{-20+j5.5} = \frac{(20.6\angle 165.9)\mathscr{V}_S}{20.7\angle 164.6} = (0.995\angle 1.3)\mathscr{V}_S \text{ cm/s}$$

The measured value of v_2 is very close to that of $\mathscr{V}_S$. That is the value of v_2 that would have been measured by an ideal velocity transducer, one that produced no resistance to movement. The measurement error is small because the impedance of the transducer, although not zero, is still much less than that of the source (i.e., $k_{e2} = .1\ k_{e1}$). The elasticity provides the only resistance to movement in the transducer, so if it is increased the error increases and if it is reduced the error is reduced. The influence of transducer impedance is demonstrated in several of the problems.

In some measurement situations, it is better to match the impedance of the transducer with that of the source. Matching mechanical impedances is particularly important in ultrasound imaging. Ultrasound imaging uses a high-frequency (1 MHz and up) pressure pulse wave that is introduced into the body and reflects off of various internal surfaces. The time of flight for the return signal is used to estimate the depth of a given surface. Using a scanning technique, many individual pulses are directed into the body at different directions and a two-dimensional image is constructed. Because the return signals can be very small, it is important that the maximum energy be sent into the body and maximum energy obtained from the return signals. The following example illustrates the advantage of matching acoustic (i.e., mechanical) impedances in ultrasound imaging.

EXAMPLE 14.17

An ultrasound transducer that uses a barium titinate piezoelectric device is applied to the skin. The transducer is round, with a diameter of 2.5 cm. Use an acoustic impedance of 24.58×10^6 kg-s/m^2 for barium titinate and an acoustic impedance of 1.63×10^6 kg-s/m^2 for the skin. What is the maximum percent power that can be transferred into the skin with and without impedance matching?

Solution: The interface between skin and transducer can be represented as two series mechanical impedances as shown in Figure 14.43. The idea is to transfer maximum power ($F\,v$) to Z_2. We know from the maximum power transfer theory that maximum power will be transferred to Z_2 when

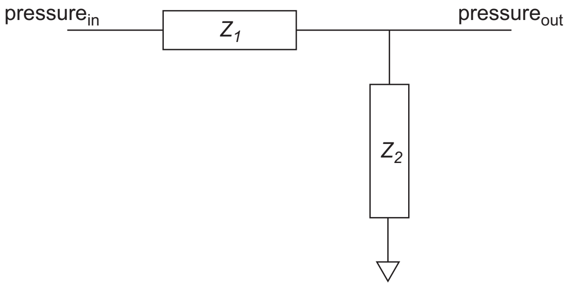


FIGURE 14.43 A model of the piezoelectric–skin interface in terms of acoustic (i.e., mechanical) impedance. Z_1 is the impedance of the transducer and Z_2 the impedance of the skin. In Example 14.16 the power transferred to Z_2 is determined when the two impedances are matched and when they are unmatched.

$Z_1 = Z_2$ (Equation 14.23). The problem also requires us to find how much power is transferred in both the matched and unmatched conditions.

The transducer is applied directly to the skin, so the area of the transducer and the area of the skin are the same and we can substitute pressures for forces since the areas cancel. In an electrical circuit, the two impedances act as a voltage divider, and in a mechanical circuit they act as a force (or pressure) divider:

$$\frac{F_{Z_2}}{F_{in}} = \frac{p_{Z_2}a}{p_{in}a} = \frac{p_{Z_2}}{p_{in}} = \frac{Z_2}{Z_1 + Z_2}$$

where F is forces and p is pressures. The power into the skin is given by $P_{Z2} = p_{Z2}\, v_{Z2}$ where p is the pressure across Z_2 and v is the velocity of Z_2.

$$\frac{P_{Z_2}}{P_{in}} = \frac{p_{Z_2}v}{p_{in}v} = \frac{p_{Z_2}}{p_{in}} = \frac{Z_2}{Z_1 + Z_2} \tag{14.24}$$

Note that the impedances are given in MKS units, not the cgs units used throughout this text. However, since they are used in a ratio, the units cancel so there is no need to convert them to cgs in this problem.

Result: When the impedances are matched, $Z_1 = Z_2$, then half the power is transferred to Z_2 as given by the maximum power transfer theorem, so the percent transferred is 50%. Under the unmatched conditions, the power ratio can be calculated as:

$$\frac{P_{Z_2}}{P_{in}} = \frac{Z_2}{Z_1 + Z_2} = \frac{1.63}{24.58 + 1.63} = 0.062$$

So the power ratio in the unmatched case is 6.2%, as opposed to 50% in the matched case. This approximately eightfold gain in power transferred into the body shows the importance of matching impedances to improve power transfer. Since we have no control over the impedance of the skin, we must adjust the impedance of the ultrasound transducer. In fact, special coatings are applied to the active side of the transducer to match tissue impedance. In addition, a gel having the same impedance is used to improve coupling and ensure impedance matching between the transducer and skin.

14.8 MULTIPLE SOURCES—REVISITED

In Chapter 6 we apply multiple sinusoids at different frequencies to a single input to find the frequency characteristics of a system. Superposition allows us to compute the transfer function for a range of frequencies. It assures us that if these multiple frequencies are applied to the system, the response is the summation of responses to individual frequencies. But what if the sources have both different frequencies and different locations?[4]

Even if the sources have different locations and different frequencies, superposition still can be used to analyze the network. We can solve the problem for each source separately knowing that the total solution will be the algebraic summation of all the partial solutions. We simply turn off all sources but one, solve the problem using standard techniques, and repeat until the problem has been solved for all sources. Then we add all the partial solutions for a final solution.

As stated previously, turning off a source does not mean removing it from the system: it is replaced by its equivalent impedance. Hence, voltage sources are replaced by short circuits and current sources by open circuits (so current sources are actually removed from the circuit). Similarly, force sources become straight connections and velocity sources essentially disappear. The following example uses superposition in conjunction with source equivalent impedance to solve a circuit problem with two sources having different frequencies.

EXAMPLE 14.18

Find the voltage across the 30-Ω resistor in the circuit of Figure 14.44.

Solution: First turn off the right-hand source by replacing it with a short circuit (its internal resistance), solve for the currents through the 30-Ω resistor, and then solve for the voltage across it. Then turn off the left-hand source and repeat the process. Note that the impedances of the inductor and capacitor will be different since the frequency is different. Add the two partial solutions to get the final voltage across the resistor.

Turning off the right-hand source and converting to the phasor domain gives the circuit in Figure 14.45.

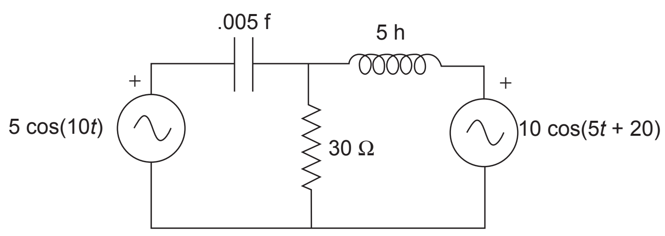


FIGURE 14.44 A two-mesh circuit that has both sources in different locations; each source has a different frequency. This type of problem can be solved using superposition.

[4] If the sources are at the same frequency but different locations, we do not have a problem, as the analysis techniques we have already developed can be used.

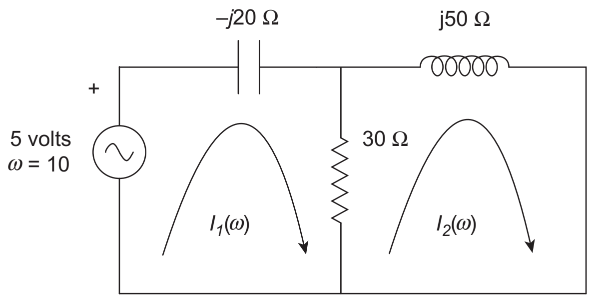


FIGURE 14.45 The circuit of Figure 14.44 with the right-hand source turned off. Since this source is a voltage source, turning it off means replacing it with a short circuit, the equivalent impedance of an ideal voltage source.

Applying KVL leads to a solution for the voltage across the center resistor:

$$\begin{vmatrix} 5 \\ 0 \end{vmatrix} = \begin{vmatrix} 30 - j20 & -30 \\ -30 & 30 + j50 \end{vmatrix} \begin{vmatrix} I_1(\omega) \\ I_2(\omega) \end{vmatrix}$$

Solving (MATLAB was used):

$$I_1(\omega = 10) = .217\angle 17; \quad I_2(\omega = 10) = .112\angle - 42;$$
$$V_R(\omega = 10) = (I_1(\omega) - I_2\,(\omega))R = 5.57\angle 48 \text{ V}$$

Now, turning off the left-hand source and converting to the phasor domain leads to the circuit in Figure 14.46. Note that the impedances are different since the new source has a different frequency. Again apply the standard analysis.

$$\begin{vmatrix} 0 \\ -10\angle 20 \end{vmatrix} = \begin{vmatrix} 30 - j40 & -30 \\ -30 & 30 + j25 \end{vmatrix} \begin{vmatrix} I_1(\omega) \\ I_2(\omega) \end{vmatrix}$$

Solving:

$$I_1(\omega = 5) = 0.274\angle - 156; \quad I_2(\omega = 5) = 0.456\angle 151;$$
$$V_R(\omega = 5) = (I_1(\omega) - I_2(\omega))R = 10.94\angle - 65 \text{ V}$$

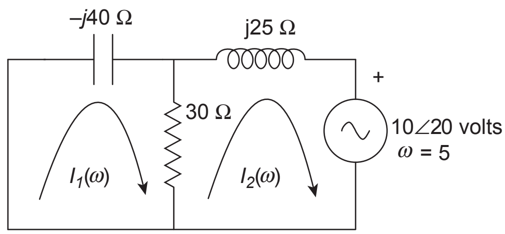


FIGURE 14.46 The circuit of Figure 14.44 with the left-hand source turned off.

The total solution is just the sum of the two partial solutions:

$$v_R(t) = 5.57\cos(10t + 48) + 10.94\cos(5t - 65)\ \text{V}$$

This approach extends directly to any number of sources. It applies equally well to current sources as shown in Problem 20.

14.9 SUMMARY

Even very complicated circuits can be reduced using the rules of network reduction. These rules allow networks containing one or more sources (at the same frequency) and any number of passive elements to be reduced to a single source and a single impedance. This single source–impedance combination could be either a voltage source in series with the impedance, called a Thévenin source, or a current source in parallel with the impedance, a Norton source. Conversion between the two representations is straightforward.

One of the major applications of network reduction is to evaluate the performance of the system when circuits are combined together. The transfer function of each isolated network is determined based on the assumption that the circuit is driven by an ideal source and connected to an ideal load. This can be taken as true if the impedance of the source driving the network is much less than the network's input impedance and the impedance of the load is much greater than the network's output impedance.[5] Network reduction techniques provide a method for determining these input and output impedances.

The ratio of input to output impedance is particularly important when making physiological measurements. Often the goal is to make the input impedance of the measurement device as high as possible to minimally load the process being measured, that is, to draw minimum energy from the process. Sometimes it is desirable to transfer a maximum amount of energy between the process being measured and the measurement system. This is often true if the measurement device must also inject energy into the process to make its measurement. In such situations, an impedance matching strategy is used where the input impedance of the measuring device is adjusted to equal the output impedance of the process being measured.

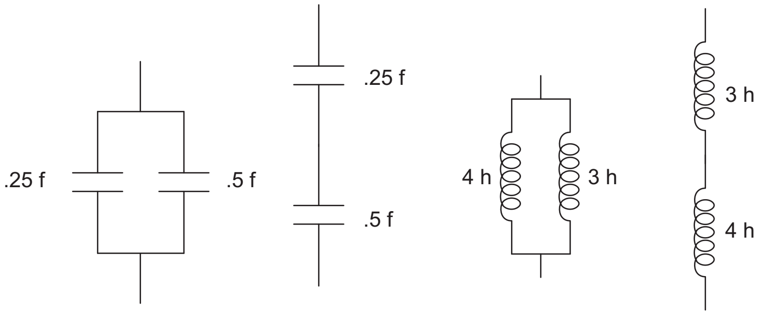
All of the network reduction tools apply equally to mechanical systems. Indeed, one of the major applications of impedance matching in biomedical engineering is in ultrasound imaging, where the ultrasound transducer acoustic impedance must be matched with the acoustic impedance of the tissue.

This chapter concludes with the analysis of networks containing multiple sources at different frequencies. To solve these problems, the effect of each source on the network must be determined separately and the solutions for each source added together. Superposition is an underlying assumption of this summation-of-partial-solutions approach. When solving for the influence of each source on the network, the other sources are turned off by replacing them with their equivalent impedances: voltage sources are replaced by short circuits and current sources are replaced by open circuits.

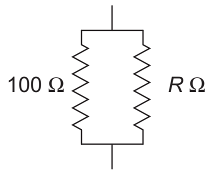
[5]This assumes that the signals are carried as changes in voltage as is usually the case.

PROBLEMS

1. Find the combined values of the following elements.



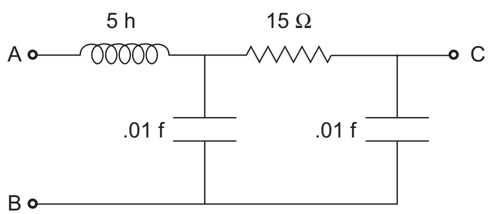
2. Find the value of R so the resistor combination equals 10 Ω. (Hint: use Equation 14.9.)



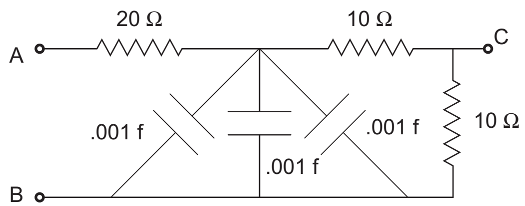
3. A two-terminal element has an impedance of $100\ \angle{-30}\ \Omega$ at $\omega = 2$ rad/s. The element consists of two components in series. What are they: two resistors, two capacitors, a resistor and capacitor, a resistor and inductor, or two inductors? What are their values?

4. An impedance, Z, has a value of $60\angle 25\ \Omega$ at $\omega = 10$ rad/s. What type of element should be added in series to make the combination look like a pure resistance at this frequency? What is the value of the added element at $\omega = 10$ rad/s? What is the value of the combined element at $\omega = 10$ rad/s?

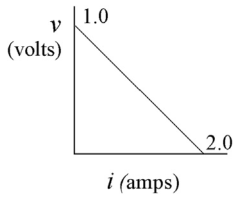
5. Use network reduction to find the equivalent impedance of the following network between terminals A and B.



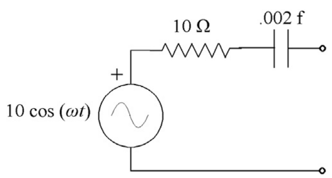
6. Use network reduction to find the equivalent impedance of the network in Problem 5 between the terminals A and C.
7. Find the equivalent impedance of the following network between terminals A and B.



8. Plot the magnitude and phase of the impedance, Z_{eq}, in Example 14.3 over a range of frequencies from $\omega = 0.01$ to 1000 rad/s. Plot both magnitude and phase in log frequency and plot the phase in degrees. Use Bode plot techniques, not MATLAB.
9. The following v–i characteristics were measured on a two-terminal device. Model the device by a Thévenin circuit and a Norton circuit.

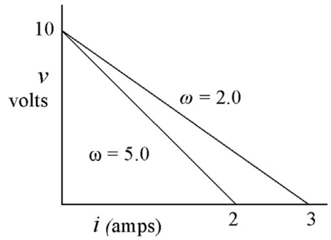


10. Plot the v–i characteristics of this network at two different frequencies $\omega = 2$ and $\omega = 200$ rad/s. Note: the v–i plot is a plot of the voltage magnitude against current magnitude. (Hint: find $|V_T|$ at the two frequencies.)

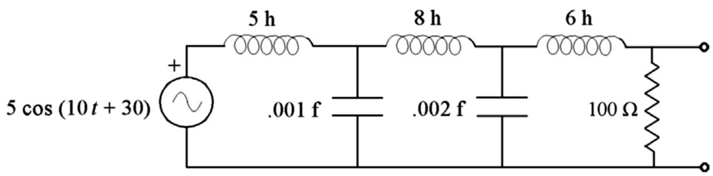


11. Two different resistors were placed across a two-terminal source (a battery) known to contain an ideal voltage source in series with a resistor. Applying two different load resistors to the terminals resulted in two different voltages: when $R = 1000\ \Omega$, $V = 8.5$ V, and when $R = 100\ \Omega$, $V = 8.2$ V. Find the load resistor, R_L, that extracts the maximum power from this source. What is the power dissipated in the load resistor?

12. The following magnitude v–i plot was found for a two-terminal device at the two frequencies shown. Model the device as a Thévenin equivalent. (Hint: to find R_T, use Equation 14.11 generalized for impedances.)

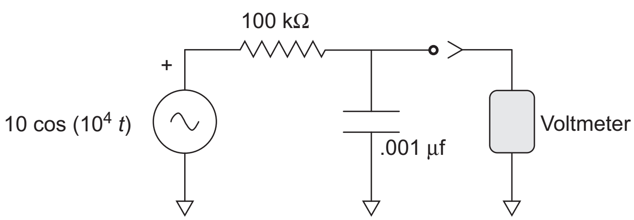


13. Find the Thévenin equivalent circuit of the following network. (Hint: modify the code in Example 14.13.)

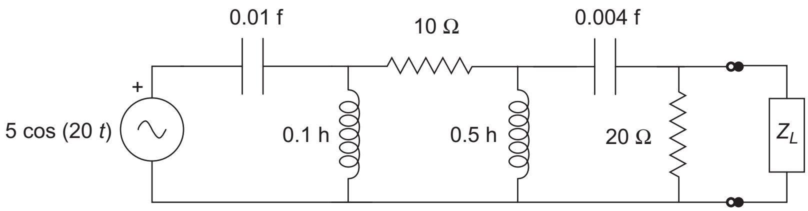


14. For the circuit in Problem 13, further modify the code in Example 14.13 to determine and plot the Norton current source and Norton impedance as a function of frequency. Plot both magnitude and phase of both variables for a range of frequency between 0.1 and 100 Hz.

15. The voltage of the following nonideal voltage source is measured with two voltmeters. One has an internal resistance of 10 MΩ, whereas the other, a cheapie, has an internal resistance of only 100 kΩ. What voltages will be read by the two voltmeters? How do they compare with the true Thévenin voltage? (Assume the voltmeters read peak-to-peak voltage, although voltmeters usually read the root mean square (RMS) voltage.) Note: The components in the Thévenin source have values commonly encountered in real circuits.



16. For the following network, what should the value of Z_L be to extract maximum power from the network? (Hint: use the same approach as in Example 14.13.)
If the voltage source is increased to 15 cos (20t), what should the value of Z_L be to extract maximum power from the network?



17. In the network shown in Problem 16, what is the impedance between nodes A and B assuming Z_L is not attached? Use the approach shown in Example 14.8. Apply a hypothetical 1-V source between these two points and use MATLAB-aided mesh analysis to find the current. Also you need to remove the influence of the 5-V source by turning it off, that is, replacing it by a short circuit. (Note that this can be reduced to a three-mesh problem, but even as a four-mesh problem it requires only a few lines of MATLAB code to solve.)

18. For the mechanical system shown in Example 14.15, find the difference between the measured force, $F_{measured}$, and F_S, the ideal source, if the friction in the source, k_f, is increased from 20 dyn-s/cm to 60 dyn cm/s. Find the difference between $F_{measured}$ and F_S with the increased friction in the source if the elasticity of the transducer is also increased by a factor of 3.

19. For the mechanical system shown in Example 14.16, find the difference between the measured velocity, v_2, and V_S if the elastic coefficient of the transducer, k_{e2}, is doubled, quadrupled, or halved.

20. For the mechanical system shown in Example 14.16, find the difference between the measured velocity, v_2, and V_S if the velocity transducer contains a mass of 4 g.

21. Find the voltage across the 0.01-f capacitor in the following circuit. Note that the two sources are at different frequencies.