CHAPTER

12

Circuit Elements and Circuit Variables

12.1 GOALS OF THIS CHAPTER

In this section of the book we turn our attention to two analog systems: electrical circuits (or networks) and mechanical systems.[1] The overreaching goal of the next four chapters is to find a mathematical representation of signals (or variables) in a circuit or mechanical system and, if appropriate, find the transfer function. There are significant differences between the analog systems described in these chapters and the general systems described in Chapters 6 and 7. The elements appear to be different, but in fact they are described by the same simple calculus equations used for general systems elements. This means we can use the same simplifying approaches developed in Chapters 6 and 7 analyzing these systems in either the frequency domain (i.e., phasor) or Laplace domain. What is different is the way in which the elements interact, so we need to develop a different set of rules to find the transfer function of an analog system. Once the transfer function is determined, the behavior of a mechanical or electrical system can be evaluated by direct application of our time domain, frequency domain, or Laplace domain toolkits (convolution, Bode plots, inverse Fourier transform, and inverse Laplace transform).

This introductory chapter is devoted largely to definitions employed in mechanical and electrical systems: defining the signal variables and the basic elements. The main signal variable in electrical systems is voltage, but another electrical variable, current, can also be important. The analogous mechanical variables are force and velocity. Mechanical and electrical elements are defined using calculus operations and readily slip into Laplace or frequency domain (i.e., phasor) representation.

Choosing between phasor and Laplace analysis for a specific circuit or mechanical system follows the same rules used for the more general systems discussed in Chapters 6 and 7. The Laplace transform is more general as it applies to systems exposed to a wider variety of

[1] The terms "network" and "circuit" are completely interchangeable and I use them more or less randomly. "Analog system" is a general term that includes mechanical and electrical systems as well as most biological systems.

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signals, including transient or step-like signals, or to circuits with nonzero initial conditions. Phasor analysis is easier to use, but applies only to circuits and mechanical systems in steady state with signals that can be decomposed into sinusoids. The basic rules for analyzing a circuit or mechanical system are independent of which technique, Laplace or phasor, is used to handle the resulting differential equations. Of course, these approaches still require that the underlying processes be linear, but electrical elements are surprisingly linear and electrical systems are as close to LTI systems as is found in the real world.

Specific topics of this chapter include:

- Define the signal variables of mechanical and electrical systems.
- Define variables associated with specific elements such as power and energy.
- Describe the general features of mechanical and electrical systems (or models based on mechanical or electrical elements) and show how they differ from systems models described in Chapters 6 and 7.
- Define the basic active and passive elements in electrical systems in the time, frequency (phasor), and Laplace domain.
- Define the basic active and passive elements of mechanical systems in the same three domains.

12.2 SYSTEM VARIABLES: THE SIGNALS OF ELECTRICAL AND MECHANICAL SYSTEMS

Electric and electronic circuits consist of arrangements of basic elements that define relationships between voltages and currents. The mechanical systems covered here are arrangements of mechanical elements that define relationships between force and velocity. In both electric circuits and mechanical systems only two variables, voltage/current and force/velocity, define the behavior of an element.[2] From another perspective, an element, electrical or mechanical, can be viewed as forcing a specific relationship between the two variables. Although voltage and current seem quite different from force and velocity, they have much in common: one variable, voltage or force, is associated with potential energy; the other variable, current or velocity is related to kinetic energy. The potential energy variable may be viewed as the cause of an action, whereas the kinetic energy variable is the effect: voltage across an element causes current to flow and force applied to an element causes it to move. To emphasize the strong relationship between mechanical and electrical systems, both these variables are defined in the next section.

12.2.1 Electrical and Mechanical Variables

For electric circuits, the major variables are voltage and current. Voltage, the potential energy variable, is sometimes called "potential." When voltage is applied to a circuit element

[2]There are also energy considerations—how much energy an element dissipates or supplies—but these variables are associated with a specific element and do not interact with, or influence, other elements in the system.

it causes current to flow through that element. Voltage is the push behind current. It is defined as a potential energy: the energy with respect to charge:

$$v = \frac{dE}{dq}\left(\frac{\mathrm{J}}{\mathrm{C}}\right)\mathrm{V} \tag{12.1}$$

where v is voltage, E is the energy of an electric field, and q is charge. The basic units of V (volts) is J (joules) per C (coulomb). (Slightly different typeface will be used in this text to represent voltage, v, and velocity, υ, to minimize confusion.)

The kinetic energy or flow variable that results from voltage is current. Current is the flow of charge: the differential change in charge with time:

$$i = \frac{dq}{dt}\left(\frac{\mathrm{C}}{\mathrm{s}}\right)\mathrm{A} \tag{12.2}$$

where i is current and q is charge. The basic units of A (amps) is C (coulombs) per s (seconds).

A variable that is uniquely associated with a specific element or an entire system defines the energy used (or supplied) over a given time period. Energy per unit time is termed "power" and for electrical elements is given as:

$$P = \frac{dE}{dt}\left(\frac{\mathrm{J}}{\mathrm{s}}\right)\mathrm{W} \tag{12.3}$$

where W (watts) are defined as J/s (Joules per s), To relate power to the variables v and i, note that from Equation 12.1:

$$dE = vdq \tag{12.4}$$

Substituting this into the definition of energy for electrical elements (Equation 12.3) gives:

$$P = \frac{dE}{dt} = v\left(\frac{dq}{dt}\right) = vi\ \mathrm{W} \tag{12.5}$$

In mechanical systems the two variables are force, which is related to potential energy and velocity, which is related to kinetic energy. The relationship between force and energy can be derived from the basic definition of energy in mechanics:

$$E = \mathrm{Work} = \int F\,dx = Fx\ \mathrm{J} \quad (\text{if } F \text{ constant over } x) \tag{12.6}$$

Solving Equation 12.6 for F by differentiating both sides:

$$F = dE/dx \tag{12.7}$$

where F is force in dynes (dyn) and x is distance in centimeters (cm).

TABLE 12.1 Major Variables in Mechanical and Electrical Systems

Domain	Potential Energy Variable (Units)	Kinetic Energy Variable (Units)
Mechanical	Force, F $F = \frac{dE}{dx}$ (J/cm = dyn)	Velocity, v $v = \frac{dx}{dt}$ (cm/s)
Electrical	Voltage, v $v = \frac{dE}{dq}$ (J/C = V)	Current, i $i = \frac{dq}{dt}$ (C/s = A)

The kinetic energy variable, the variable caused by force, is velocity:

$$v = \frac{dx}{dt} \text{ cm}/s \tag{12.8}$$

Power in mechanical systems can be shown to again be the product of the potential and kinetic energy variables (Equation 12.3). Starting with the basic definition of power (Equation 12.3):

$$P = \frac{dE}{dt}$$

From the definition of force (Equation 12.7):

$$P = \left(\frac{dE}{dx}\right)\left(\frac{dx}{dt}\right) = F\left(\frac{dx}{dt}\right) = Fv \tag{12.9}$$

where P is power, F is force, x is distance, and v is velocity.

Table 12.1 summarizes the variables used to describe the behavior of mechanical and electrical systems. The tools developed in this chapter are first introduced in terms of electrical circuits, and later in this chapter are applied to certain mechanical systems. The mechanical analysis described later in this chapter could be applied to mechanical systems, but for biomedical engineers the most likely application is to analog models of physiological processes that use mechanical elements.

12.2.2 Voltage and Current Definitions

Analyzing a circuit usually means finding the voltages (or currents) in the circuit or finding its transfer function. In electrical systems, the transfer function is almost always a ratio of voltages, specifically, the voltages considered as input and output, $TF = V_{out}/V_{in}$.

Voltage is a relative variable: it is the difference between the voltages at two points. In fact the proper term for voltage is "potential difference" (abbreviated p.d.), but this term is rarely used by engineers. Subscripts are sometimes used to indicate the points from which the potential difference is measured. For example, in Equation 12.10 the notation v_{ba} means "the voltage at point b with respect to point a":

$$v_{ba} = v_b - v_a \tag{12.10}$$

The positive voltage point, point "b" in Equation 12.10, is indicated by a plus sign when drawn on a circuit diagram, as shown in Figure 12.1. Given the position of the plus sign in Figure 12.1, it is common to say that there is a voltage drop from point b to point a (i.e., left to right), or a voltage rise from a to b. By this convention, it is logical that v_{ab} should be the negative of v_{ba}: $v_{ab} = -v_{ba}$. Voltage always has a direction, or "polarity," that is usually indicated by a "+" sign to show the side assumed to have a greater voltage, Figure 12.1.

A source of considerable confusion is that the "+" sign indicates only the point that is assumed to have a more positive value for the purpose of analysis or discussion. Sometimes the plus position is selected arbitrarily. It could be that the voltage polarity is actually the opposite of what was originally assigned. As an example, suppose that b was, in fact, more negative than a, i.e., there is actually a rise in voltage from b to a for the element in Figure 12.1. Even though we have already made the assumption that b was the more positive as indicated by the "+" sign, we do not change our original polarity assignment. We merely state that v_{ba} has a negative value. So a negative voltage does not imply negative potential energy, it is just that the actual polarity is the reverse of that assumed.

By convention, but with some justification as is shown later, the voltage of the earth is assumed to be at 0.0 V, so voltages are often measured with respect to the voltage of the earth, or some common point referred to as "ground." A common ground point is indicated by either of the two symbols shown at the bottom of the simple circuit shown in Figure 12.4. Some conventions use the symbol on the right side to mean a ground that is actually connected to earth, whereas the symbol on the left side indicates a common reference point not necessarily connected to the earth, but still assumed to be 0.0 V. However, this usage is not standardized and the only assumption that can be made with certainty is that both symbols represent a common grounding point that may or may not be connected to earth, but is assumed to be 0.0 V with respect to the rest of the circuit. Hence when a voltage is given with only one subscript, v_a, it is understood that this is voltage at a with respect to ground or a common reference point.

Current is a flow so it must have a direction. This direction is indicated by an arrow as in Figures 12.1 and 12.2, but it is an assumed direction as we often do not yet know the actual direction when we assign the direction. By convention, the direction of the arrow indicates the direction of assumed positive charge flow. In electric or electronic circuits, charge is

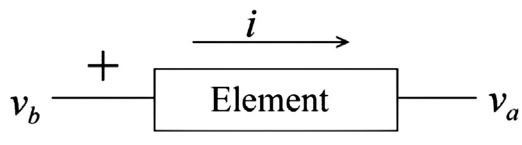


FIGURE 12.1 A generic electric circuit element demonstrating how voltage and current directions are specified. The *straight lines* on either side indicate wires connected to the element.

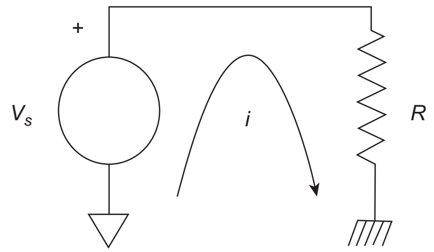


FIGURE 12.2 A simple circuit consisting of a voltage source, V_s, and a resistor R. Two different symbols for grounding points are shown. Sometimes the symbol on the right is taken to mean an actual connection to earth, but this is not standardized.

usually carried by electrons, which have a negative value for charge, so the particles that are actually flowing, the electrons, are flowing in the opposite direction of (assumed) positive charge flow. Nonetheless, the convention of defining positive charge flow was established by Benjamin Franklin before the existence of electrons was known and has not been modified because it really does not matter which direction is taken as positive as long as we are consistent.

As with voltage, it may turn out that positive charge flow is actually in the direction opposite to that indicated by the arrow. Again, we do not change the direction of the arrow, rather we assign the current a negative value. So a negative value of current flow does not mean that some strange positrons or anti-particles are flowing but only that the actual current direction is opposite to our assumed direction.

This approach to voltage polarity and current direction may seem confusing, but it is actually quite liberating. It means that we can make our polarity and direction assignments without worrying about reality, that is, without concern for the voltage polarity or current direction that actually exists in a given circuit. We can make these assignments more or less arbitrarily (there are some rules that must be followed as described later) and, after the circuit is analyzed, allow the positive or negative values to indicate the actual direction or polarity.

12.3 ANALOG SYSTEM VERSUS GENERAL SYSTEMS

Both analog systems and the general systems studied in Chapters 6 and 7 consist of collections of elements. Moreover, we find in the next section that analog elements can be defined by the same basic equations used for general system elements. But the way in which the elements interact is different.

Figure 12.3 shows a generic two-element system and a two-element electric circuit. In the system model, Figure 12.3A, Element 1 acts on Element 2, but Element 2 has no influence on Element 1. If there was such an interaction, it would be explicitly shown as a feedback

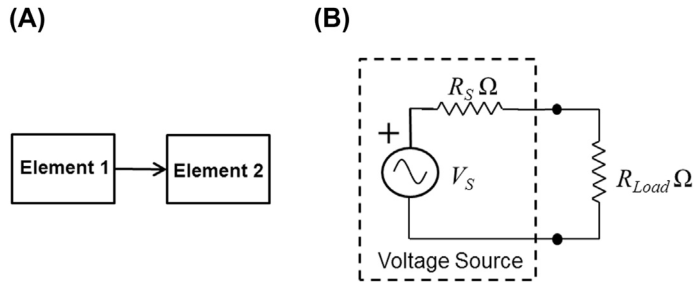


FIGURE 12.3 (A) A generic two-element system. Element 1 influences Element 2, but Element 2 has no influence on Element 1. (B) A typical two-element circuit. The realistic voltage source (*dashed line*) influences the load resistor, R_{Load}, but this resistor also influences the voltage source.

pathway. In the electrical circuit, Figure 12.3B, Element 1 is a realistic battery consisting of an ideal source and a resistor, and the second element, R_{Load}, is a load resistor. When the voltage source is connected to the load resistor it creates a voltage across the resistor, which causes current to flow through this element. But the resistor also has an effect on the voltage source because it draws current, which decreases its output voltage because of its internal resistance, R_S. So there is a hidden feedback pathway between the voltage source and the load resistor.

In systems representations all influences between elements are explicitly shown, but in analog models the interactions between elements are usually implicit. This means that the techniques used to develop equations are different for the two types of systems. In systems models, we simply multiply the individual transfer functions together to find the transfer function for any number of series elements. If feedback is involved, we apply the feedback equation (Equation 6.7) to the combined transfer function of feedforward and feedback pathways. In analog models, the approach to developing and overall transfer function is more complicated, but still consists of a set of rules that, if correctly applied, produce the desired equation. These rules are detailed in the next chapter.

12.4 ELECTRICAL ELEMENTS

The elements as described later are idealizations: true elements only approximate the characteristics described. However, actual electrical elements come quite close to these idealizations, so their shortcomings can usually be ignored, at least with respect to that famous engineering phrase "for all practical purposes."

Electrical elements are defined by the mathematical relationship they impose between the voltage and current. They are divided into two categories based on their energy use: active elements can supply energy to a circuit; passive elements cannot. Active elements do not always supply energy; in some situations they actually absorb energy (imagine charging a battery), but they have the ability to supply energy.

12.4.1 Passive Electrical Elements

Passive elements are divided into two subcategories: those that use up, or dissipate, energy and those that store energy. Only one passive element falls into the first category, the resistor. This element is described first.

12.4.1.1 Energy Users: Resistors

The resistor is the only element in the first group of passive elements: elements that use up energy. Resistors dissipate energy as heat. The defining equation for a resistor, the basic voltage—current relationship, is the classic Ohm's law:

$$v_R = R\, i_R \ \text{V} \tag{12.11}$$

where R is the resistance in volts/amp, better known as ohms (Ω), i is the current in amps (A), and v is the voltage in volts (V). The resistance value of a resistor is a consequence of a property of the material from which it is made, known as resistivity, ρ. The resistance value is determined by this resistivity and the geometry and is given by:

$$R = \rho \frac{l}{A} \ \Omega \tag{12.12}$$

where ρ is the resistivity of the resistor material, l is the length of the material, and A is the area of the material. Table 12.2 shows the resistivity, ρ, of several materials commonly used in electric components.

The power that is dissipated by a resistor can be determined by combining Equation 12.5, the basic power equation for electrical elements, and the resistor-defining equation, Equation 12.11:

$$P = vi = v\left(\frac{v}{R}\right) = \frac{v^2}{R} \ \text{W}$$

or:

$$P = vi = (iR)i = i^2 R \ \text{W} \tag{12.13}$$

TABLE 12.2 Resistivity of Common Conductors and Insulators

Conductors	ρ (Ohm-m)	Insulators	ρ (Ohm-m)
Aluminum	2.74×10^{-8}	Glass	10^{10}–10^{14}
Nickel	7.04×10^{-8}	Lucite	$>10^{13}$
Copper	1.70×10^{-8}	Mica	10^{11}–10^{15}
Silver	1.61×10^{-8}	Quartz	75×10^{16}
Tungsten	5.33×10^{-8}	Teflon	$>10^{13}$

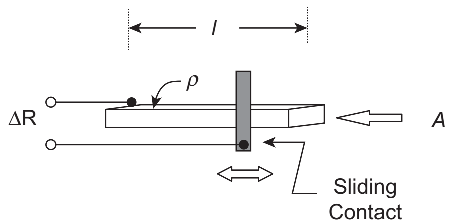


FIGURE 12.4 A variable resistor made by changing the effective length, Δl, of the resistive material.

The voltage—current relationship expressed by Equation 12.11 can also be stated in terms of the current:

$$i = \frac{1}{R}v = Gv \ \text{A} \tag{12.14}$$

The inverse of resistance, R, is termed the "conductance," G, and has the units of mhos (ohms spelled backward, a rare example of engineering humor) and is symbolized by the upside down omega, ℧. Equation 12.12 can be exploited to make a device that varies in resistance, usually by varying the length l, as shown in Figure 12.4. Such a device is termed a "potentiometer" or "pot" for short. The two symbols that are used to indicate a variable resistor are shown in Figure 12.5B.

By convention, power is positive when it is being lost or dissipated. Hence resistors must always have a positive value for power. In fact, one way to define a resistor is to say that it is a device for which $P > 0$ for all t. For P to be positive, the voltage and current must point in the same direction, that is, they must have the same orientation. In other words the current direction must point in the direction of the voltage drop. This polarity restriction is indicated in Figure 12.5A along with the symbol that is used for a resistor in electric circuit diagrams or "schematics." To meet the positive power criterion, the voltage and current polarities must be set so that current flows into the positive side of the resistor as shown in Figure 12.5A. Either the voltage direction (+ side) or the current direction can be chosen arbitrarily, but not both. Once either the voltage polarity or the current direction is selected, the other is fixed by power considerations. The same will be true for other passive elements, but not for source elements. This is because source elements can, and usually do, supply energy, so their associated power usage is usually negative.

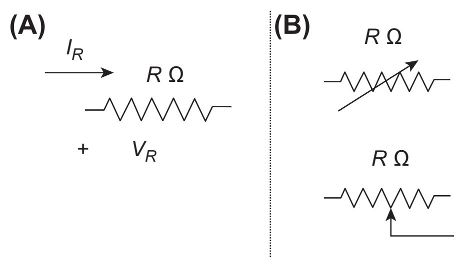


FIGURE 12.5 (A) The symbol for a resistor along with its polarity conventions. For a resistor, as with all passive elements, the current direction must be such that it flows from positive to negative. In other words, current flows into the positive side of the element. (B) Two symbols that denote a variable resistor or potentiometer.

Figure 12.5B shows two symbols used to denote a variable resistor such as shown in Figure 12.4.

EXAMPLE 12.1

Determine the resistance of 100 ft of #14 AWG copper wire.

Solution: A wire of size #14 AWG (American Wire Gauge or B. & S. gauge) has a diameter of 0.064 in. (see Appendix D). The value of ρ for copper is 1.70×10^{-8} Ω-m (Table 12.2). Convert all units to the cgs system and then apply Equation 12.12. To ensure proper unit conversion, we will carry the dimensions into the equation and make sure they cancel to give us the desired dimension. This approach is sometimes known as dimensional analysis and can be very helpful where there are a lot of unit conversions.

$$l = 100\cancel{\text{ft}}\left(\frac{12\cancel{in}}{1\,\text{ft}}\right)\left(\frac{2.54\text{cm}}{\cancel{\text{in}}}\right) = 3048\,\text{cm}$$

$$A = \pi r^2 = \pi\left(\frac{d\ \text{cm}}{2}\right)^2 = \pi\left(\frac{2.54\ \text{cm}}{\cancel{\text{in}}}\frac{0.064\ \cancel{\text{in}}}{2}\right)^2 = 0.0208\,\text{cm}^2$$

$$R = \rho\left(\frac{l}{A}\right) = 1.7x10^{-8}\Omega\cancel{m}\left(\frac{100\cancel{cm}}{1\cancel{m}}\right)\left(\frac{3048\cancel{cm}}{0.0208\cancel{cm}^2}\right) = 0.2491\,\Omega$$

12.4.1.2 *Energy Storage Devises: Inductors and Capacitors*

Energy storage devices can be divided into two classes: "inertial" elements and "capacitive" elements. The corresponding electrical elements are the inductor and capacitor, respectively, and the voltage–current equations for these elements involve differential or integral equations.

12.4.1.2.1 INDUCTOR

Current flowing into an inductor carries energy that is stored in a magnetic field. The voltage across an inductor is the result of a self-induced electromotive force that opposes that voltage and is proportional to the time derivative of the current:

$$v = L\frac{di}{dt} \tag{12.15}$$

where L is the constant of proportionality termed the "inductance" measured in henrys (h). (The henry is actually Weber-turns per amp, or volts per amp/s, and is named for the American physicist Joseph Henry, 1797–1878.) An inductor is simply a coil of wire that utilizes mutual flux coupling (i.e., mutual inductance) between the wire coils. The inductance is related to the magnetic flux, Φ, carried by the inductor and by the geometry of the coil and the number of loops, or "turns," N:

$$L = \frac{N\Phi}{i}\ \text{h} \tag{12.16}$$

This equation is not as useful as the corresponding resistor equation (Equation 12.11) for determining the value of a coil's inductance because the actual flux in the coil depends on the shape of the coil. In practice, coil inductance is determined empirically.

The energy stored can be determined from the equation for power (Equation 12.5) and the voltage—current relationship of the inductor (Equation 12.15):

$$P = vi = Li\left(\frac{di}{dt}\right) = \frac{dE}{dt} \quad \text{solving for } i$$

$$dE = Pdt = Li\left(\frac{di}{dt}\right)dt = Li\ di$$

The total energy stored as current increases from 0 to a value I given by:

$$E = \int dE = L\int_0^I i di = \frac{1}{2}LI^2 \tag{12.17}$$

Later in this chapter we find there is a similarity between the equation for kinetic energy of a mass (Equation 12.55) and the energy in an inductor. Equation 12.17 explains why an inductor is considered an inertial element. It behaves as if the energy is stored as kinetic energy associated with a flow of moving electrons and that is a good way to conceptualize the behavior of an inductor, even though the energy is actually stored in an electromagnetic field. Inductors follow the same polarity rules as resistors. Figure 12.6 shows the symbol for an inductor, a schematic representation of a coil, with the appropriate current—voltage directions.

If the current through an inductor is constant (i.e., "direct current" or "DC"), then there will be no energy stored in the inductor and the voltage across the inductor will be zero, regardless of the amount of steady current flowing through the inductor.[3] The condition when voltage across an element is zero irrespective of the current through the element is known as a "short circuit." Hence an inductor appears as a short circuit to a DC current, a feature that can be used to solve certain electrical circuit problems encountered later.

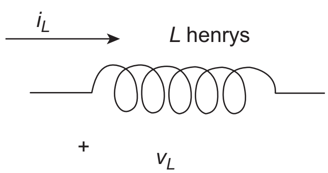


FIGURE 12.6 Symbol for an inductor showing the polarity conventions for this passive element.

[3]The term "DC" originally stood for direct current, but it has been modified to mean "constant value" so it can be applied to either current or voltage: as in "DC current" or "DC voltage."

The v-i relationship of an inductor can also be expressed in terms of current. Solving Equation 12.15 for i:

$$\begin{aligned} v_L &= L\frac{di_L}{dt}; \quad di_L = \frac{1}{L}v_L dt; \quad \int di_L = \int \frac{1}{L} v_L dt \\ i_L &= \frac{1}{L}\int v_L dt \end{aligned} \tag{12.18}$$

The integral of any function will be continuous, even if that function contains a discontinuity as long as that discontinuity is finite. A continuous function is one that does not change instantaneously, i.e., for a continuous function:

$$f(t-) = f(t+) \quad \text{for any } t \tag{12.19}$$

Since the current through an inductor is the integral of the voltage across the inductor (Equation 12.18), the current will be continuous in real situations because any voltage discontinuities that occur will surely be finite. Thus the current through an inductor can change slowly or rapidly (depending on the current), but it can never change in a discontinuous or step-like manner. In mathematical terms, for an inductor:

$$i_L(t-) = i_L(t+) \tag{12.20}$$

Since the current passing through an inductor is always continuous, one of the popular applications of an inductor is to reduce current spikes (i.e., discontinuities). Current that is passed through an inductor will have any spikes "choked off," so an inductor used in the purpose of spike reductions is sometimes called a "choke."

12.4.1.2.2 CAPACITOR

A capacitor also stores energy, in this case in an electromagnetic field created by oppositely charged plates.[4] In the case of a capacitor, the energy stored is proportional to the charge on the capacitor and charge is related to the time integral of current flowing through the capacitor. This gives rise to voltage–current relationships that are the reverse of the relationships for an inductor:

$$v = \frac{1}{C}\int i\,dt \tag{12.21}$$

or solving for i_C:

$$i_C = C\frac{dv}{dt} \tag{12.22}$$

[4]Capacitors are nicknamed "caps" and engineers frequently use that term. Curiously, no such nicknames exist for resistors or inductors, except for the occasional use of "choke" for an inductor as noted earlier.

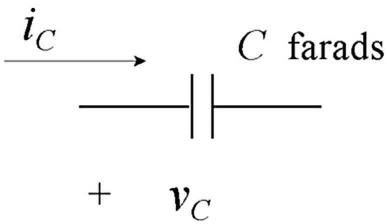


FIGURE 12.7 The symbol for a capacitor showing the polarity conventions.

where C, the "capacitance," is the constant of proportionality and is given in units of "farads," which are coulombs/volt. (The farad is named after Michael Faraday, an English chemist and physicist who, in 1831, discovered electromagnetic induction.) The inverse relationship between the voltage–current equations of inductors and capacitors is an example of "duality," a property that occurs often in electric circuits and in physics in general. The symbol for a capacitor is two short parallel lines reflecting the parallel plates of a typical capacitor, Figure 12.7.

The capacitance is well named because it describes the ability (or capacity) of a capacitor to store (or release) charge without much change in voltage. Stated mathematically, this relationship between charge and voltage is:

$$C = \frac{q}{v}\ \mathrm{F} \tag{12.23}$$

where q is charge in coulombs and v is volts. A large capacitor can take on or release charge, q, with little change in voltage, whereas a small capacitor shows a greater voltage change for a given charge flow. The largest capacitor readily available to us, the earth, is considered to be a near-infinite capacitor: its voltage remains constant (for all practical purposes) no matter how much current flows into or out of it. This is why the earth is a popular ground point or reference voltage; it is always at the same voltage and we all agree that this voltage is 0.0 V.

Most capacitors are constructed from two approximately parallel plates. Sometimes the plates are rolled into a circular tube to reduce volume. The capacitance for such a parallel plate capacitor is given as:

$$C = \frac{q}{v} = \varepsilon\frac{A}{d} \tag{12.24}$$

where A is the area of the plates, d is the distance separating the plates, and ε is a property of the material separating the plates termed the "dielectric constant." Although Equation 12.24 is only an approximation for a real capacitor, it does correctly indicate that capacitance can be increased either by increasing the plate area, A, or by decreasing the plate separation, d. However, if the plates are closely spaced and a high voltage is applied, the material between the plates, the "dielectric material," may break down, allowing current to flow directly between the plates. Sometimes this leads to dramatic failure involving smoke or even a tiny explosion. The device, if it survives, is now a resistor (or short circuit) not a capacitor. So capacitors that must work at high voltages need more material, hence more distance, between their plates. For a given capacitance, increasing the distance, d, means the area, A, must be increased. This leads to a larger physical volume so capacitors that can handle large voltages are physically large, much larger than low-voltage capacitors. Alternatively, there are special dielectrics that can sustain higher voltages with smaller distances between plates. Such capacitors are more expensive, a classic engineering trade-off. Referring again to Equation 12.24, capacitors having larger capacitance values also require more plate area again leading to larger physical volume. For a given dielectric material, the size of a capacitor is related to the product of its capacitance and voltage rating, at least for larger capacitance/voltage sizes.

EXAMPLE 12.2

Calculate the dimensions of a 1-F capacitor. Assume a plate separation of 1.0 mm. with air between the plates.

Solution: Use Equation 12.24 and the dielectric constant for a vacuum. The dielectric constant for a vacuum is $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/N\ m^2}$ and is also used for air.

$$A = \frac{Cd}{\varepsilon_0} = \frac{1\ \mathrm{C}/v(10^{-3}\mathrm{m})}{8.85 \times 10^{-12}\mathrm{C^2/N\ m^2}} = \frac{\dfrac{10^{-3}\ \mathrm{C}}{\mathrm{N\ m/C}}}{8.85 \times 10^{-12}\mathrm{C^2/N\ m^2}} = 1.13 \times 10^{8}\ \mathrm{m^2}$$

This is an area of about 6.5 miles on a side! This large size is related to the units of farads, which are very large for practical capacitors. Typical capacitors are in the microfarads ($1\ \mu\mathrm{F} = 10^{-6}\mathrm{F}$) or picofarads ($1\ \mathrm{pF} = 10^{-12}\ \mathrm{F}$), giving rise to much smaller plate sizes. An example calculating the dimensions of a practical capacitor is given in the problems.

The energy stored in a capacitor can be determined using modified versions of Equations 12.4 and 12.23:

$$v = q/C \text{ and from Equation 12.4:} \quad dE = v dq = \frac{q}{C} dq$$

Hence, for a capacitor holding a specific charge, Q:

$$E = \int dE = \frac{1}{C}\int_0^Q q dq = \frac{1}{2}\frac{Q^2}{C} \quad \text{Substituting } V = \frac{Q}{C}\ \mathrm{J}$$
$$E = \frac{1}{2}CV^2\ \mathrm{J} \tag{12.25}$$

Capacitors in parallel essentially increase the effective size of the capacitor plates, so when two or more capacitors are connected in parallel, their values add. If they are connected in series, their values add as reciprocals. Such series and/or parallel combinations are discussed at length in Chapter 14.

Inductors do not allow an instantaneous change in current; capacitors do not allow an instantaneous change in voltage. Since the voltage across a capacitor is the integral of the

TABLE 12.3 Energy Storage and Response to Discontinuous and Direct Current (DC) Variables in Inductors and Capacitors

Element	Energy Stored	Continuity Property	DC Property
Inductor	$E = \frac{1}{2}LI^2$	Current continuous $i_L(0-) = i_L(0+)$	If i_L = constant (DC current) $v_L = 0$ (short circuit)
Capacitor	$E = \frac{1}{2}CV^2$	Voltage continuous $v_C(0-) = v_C(0+)$	If v_C = constant (DC voltage) $i_C = 0$ (open circuit)

current, capacitor voltage will be continuous based on the same arguments used for inductor current. Thus, for a capacitor:

$$v_c(t-) = v_c(t+) \tag{12.26}$$

It is possible to change the voltage across a capacitor either slowly or rapidly depending on the current, but never instantaneously. For this reason, capacitors are frequently used to reduce voltage spikes just as inductors are sometimes used to reduce current spikes. The fact that the behavior of voltage across a capacitor is similar to the behavior of current through an inductor is another example of duality. Again, this behavior is useful in the solution of certain types of problems encountered later in this text.

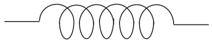
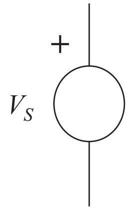
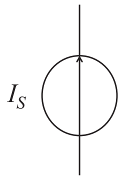
Capacitors and inductors have reciprocal responses to situations where voltages and currents are constant; i.e., DC conditions. Since the current through a capacitor is proportional to the derivative of voltage (Equation 12.22), if the voltage across a capacitor is constant the capacitor current will be zero irrespective of the value of the voltage (the derivative of a constant is zero). An "open circuit" is defined as an element having zero current for any voltage; hence, capacitors appear as open circuits to DC current. For this reason, capacitors are said to "block DC" and are sometimes used for exactly that purpose.

The continuity and DC properties of inductors and capacitors are summarized in Table 12.3. A general summary of passive and active electrical elements is presented later in Table 12.4.

12.4.1.3 Electrical Elements: Reality Check

The equations given earlier for passive electrical elements are idealizations of the actual elements. In fact, real electrical elements do have fairly linear voltage–current characteristics. However, all real electric elements will contain a combination of resistance, inductance, and capacitance irrespective of their primary function. The undesired properties are termed "parasitic" elements. For example, a real resistor will have some inductor- and capacitor-like characteristics, although these will generally be small and can be ignored except at very high frequencies. (Resistors made by winding resistance wire around a core, so-called wire-wound resistors, have a large inductance as might be expected for this coil-like configuration. However, these are rarely used nowadays.) Similarly, real capacitors also closely approximate ideal capacitors except for some parasitic resistance. This parasitic element appears as a large resistance in parallel with the capacitor, Figure 12.8, leading to a small

TABLE 12.4 Electrical Elements—Basic Properties

Element	Units	Equation $v(t) = f\,[i(t)]$	Symbol
Resistor (R)	Ω ~ ohms (volts/amp)	$v(t) = R\ i(t)$	
Inductor (L)	h ~ henry (weber turns/amp)	$v(t) = L\frac{di}{dt}$	
Capacitor (C)	f ~ farad (coulombs/volt)	$v(t) = \frac{1}{C}\int idt$	
Voltage source (V_S)	v ~ volts (joules/coulomb)	$v(t) = V_S(t)$	
Current source (I_S)	a ~ amperes (coulombs/sec)	$i(t) = I_S(t)$	

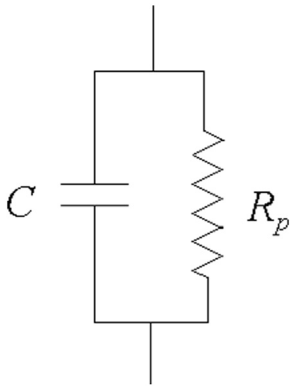


FIGURE 12.8 The schematic of a real capacitor showing the parasitic resistance that would lead to a leakage of current through the capacitor.

"leakage" current through the capacitor. Low-leakage capacitors can be obtained at additional cost with parallel resistances exceeding 12^{12}–12^{14} Ω, resulting in very small leakage currents. Inductors are constructed by winding a wire into coil configuration. Since all wire contains some resistance, and a considerable amount of wire may be used in an inductor, real inductors generally include a fair amount of series resistance. This resistance can be reduced by using wire with a larger diameter (as suggested by Equation 12.11), but at the expense of increased physical size.

In most electrical applications, the errors introduced by real elements can be ignored. It is only under extreme conditions, involving high-frequency signals or the need for very high resistances, that these parasitic contributions need be taken into account. The inductor is the least ideal of the three passive elements; it is also the least used in conventional electronic circuitry, so its shortcomings are not that consequential.

12.4.2 Electrical Elements: Active Elements or Sources

Active elements can supply energy to a system, and in the electrical domain they come in two flavors: voltage sources and current sources. These two devices are somewhat self-explanatory. Voltage sources supply a specific voltage, which may be constant or time varying but is always defined by the element. In the ideal case, the voltage is independent of the current going through the source: a voltage source is concerned only about maintaining its specified voltage; it does not care about what the current is doing.

Voltage polarity is part of the voltage source definition and must be included with the symbol for a voltage source as shown in Figure 12.9. The current through the source can be in either direction (again, the source does not care). If the current is flowing into the positive end of the source, the source is being "charged" and is removing energy from the circuit. If current flows out of the positive side, then the source is supplying energy.

The voltage—current equation for a voltage source is just $v = V_{Source}$. Not only are voltages unconcerned with their currents, they really have no control over those currents, at least in theory. The current through a voltage source depends on the circuit elements that are connected to the source.

The energy supplied or taken up by the source is still given by Equation 12.5: $P = vi$. The voltage source in Figure 12.9 is shown as "grounded," that is, one side is connected to

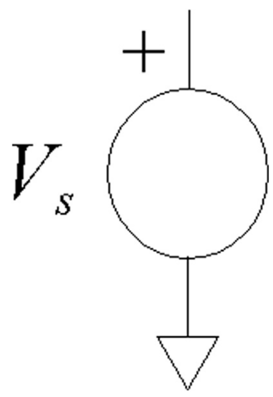


FIGURE 12.9 Schematic representation of a voltage source, *Vs*. This element specifies only the voltage, including the direction or polarity. The current value and direction are unspecified and depend on the rest of the circuit. Voltage sources are often used with one side connected to ground as shown.

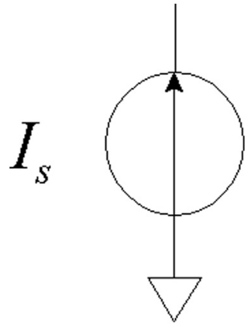


FIGURE 12.10 Schematic representation of a current source, I_s. This element specifies only the current: the voltage value and polarity are unspecified and will be whatever needed to produce the specified current.

ground. Voltage sources are commonly used in this manner and many commercial power supplies have this grounding built in. Voltage sources that are not grounded are sometimes referred to as "floating" sources. A battery could be used as a floating voltage source since neither its positive or negative terminal is necessarily connected to ground.

An ideal current source supplies a specified current, which again can be fixed or time-varying. It cares only about the current running through it. Current sources are less intuitive since current is thought of as an effect (of voltage and circuit elements), not as a cause. One way to think about a current source is that it is really a voltage source whose output voltage is somehow automatically adjusted to produce the desired current. A current source manipulates the cause, voltage, to produce a desired effect, current. The current source does not directly control the voltage across it: it will be whatever it has to be to produce the desired current. Figure 12.10 shows the symbol used to represent an ideal current source.

Current direction is part of the current source specification and is indicated by an arrow, Figure 12.10. Since a current source does care about voltage (except indirectly) it does not specify a voltage polarity. The actual voltage and voltage polarity will be whatever it has to be to produce the desired current.

Again, these are idealizations and real current and voltage sources usually fall short. Real voltage sources do care about the current they have to produce, at least if it gets too large, and their voltages will drop off if the current requirement becomes too high. Similarly, real current sources do care about the voltage across them, and their current output will decrease if the voltage required to produce the desired current gets too large. More realistic representations for voltage and current sources are given in Chapter 14 under the topics of Thévenin and Norton equivalent circuits.

Table 12.4 summarizes the various electrical elements giving the associated units, the defining equation, and the symbol used to represent that element in a circuit diagram.

12.4.3 The Fluid Analogy

One of the reasons analog modeling is popular is that it parallels human intuitive reasoning. To understand a complex notion, we often use metaphors that describe something similar that is easier to comprehend. Some intuitive insight into the characteristics of electrical elements can be made using an analogy based on the flow of a fluid such as water. In this analogy the flow volume of the water would be analogous to the flow of charge in an electric circuit (i.e., current), and the pressure behind that flow would be analogous to voltage. In this analogy, a resistor would be a constriction, or pipe, inserted into the path of water flow. As with a resistor, the flow through this pipe would be linearly dependent on the pressure (voltage) and inversely related to the resistance offered by the constructing pipe. The equivalent of Ohm's law (i.e., $v = i/R$) would be: pressure = flow/resistance. Also as with a resistor, the resistance to flow generated by the pipe would increase linearly with its length and decrease with its cross-sectional area, so the analogy to Equation 12.11 $\left(R = \rho\, l/A\right)$ would be: pipe resistance = constant (length/area).

The fluid analogy to a capacitor would be that of a container with a given cross-sectional area. The pressure at the bottom of the container would be analogous to the voltage across the capacitor, and the water flowing into or out of the container would be analogous to current. As with a capacitor, the pressure at the bottom would be proportional to the height of the water. This pressure or water height would be linearly related to the integral of water flow and inversely related to the area of the container, Figure 12.11.

A container with a large area (i.e., a larger capacity) would be analogous to a large capacitor; it would have the ability to accept larger amounts of water (charge) with little change in bottom pressure (voltage). Conversely, a vessel with a small area would fill quickly so the change in pressure at the bottom would change dramatically with only minor changes in the amount of water in the vessel. Just as in a capacitor, it would be impossible to change the height of the water, and therefore the pressure at the bottom, instantly unless you had an infinite flow of water. With a high flow, you could change the height and related bottom pressure quickly, but not instantaneously.

Water flowing out of the bottom of the vessel would continue to flow until the vessel is empty. This is analogous to fully discharging a capacitor. In fact, even the time course of the outward flow (an ever decreasing exponential) would parallel that of a discharging capacitor. Also, for the pressure at the bottom of the vessel to remain constant, the flow into or out of the vessel would have to be zero, just as the current must be zero for constant capacitor voltage.

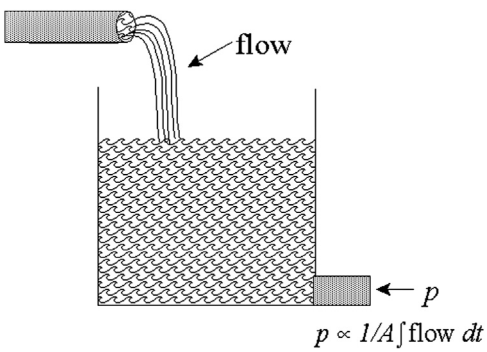


FIGURE 12.11 Water analogy of a capacitor. Water pressure at the bottom is analogous to voltage across a capacitor, and water flow is analogous to the flow of charge, current. The amount of water contained in the vessel is analogous to the charge on a capacitor.

A water container, like a capacitor, stores energy. In a container, the energy is stored as the potential energy of the contained water. Using a dam as an example, the amount of energy stored is proportional to the amount of water contained behind the dam and the pressure squared. A dam, or any other real container, would have a limited height, and if the inflow of water continued for too long it would overflow. This is analogous to exceeding the voltage rating of the capacitor where the inflow of charge would cause the voltage to rise until some type of failure occurred. It is possible to increase the overflow value of a container by increasing its height, but this would lead to an increase in physical size just as in the capacitor. A real container might also leak, in which case water stored in the container would be lost, either rapidly or slowly depending on the size of the leak. This is analogous to the leakage current that exists in all real capacitors. Even if there was no explicit current outflow, eventually all charge on the capacitor will be lost due to leakage and a capacitor's voltage reduced to zero.

In the fluid analogy, the element analogous to an inductor would be a large pipe with negligible resistance to flow, but in which any change in flow would require some pressure just to overcome the inertia of the fluid. This parallel with inertial properties of a fluid demonstrates why an inductor is sometimes referred to as an "inertial element." For water traveling in this large pipe, the change in flow velocity (d (flow)$/dt$) would be proportional to the pressure applied. The proportionality constant would be related to the mass of the water. Hence the relationship between pressure and flow in such an element would be:

$$p = k \text{ flow velocity} = \frac{k\, d(flow)}{dt} \tag{12.27}$$

which is analogous to Equation 12.15, the defining equation for an inductor. Energy would be stored in this pipe as kinetic energy of the moving water.

The greater the applied pressure, the faster the velocity of the water would change, but just as with an inductor, it would not be possible to change the flow of a mass of water instantaneously using finite pressures. Also as with an inductor, it would be difficult to construct a pipe holding a substantial mass of water without some associated resistance to flow analogous to the parasitic resistance found in an inductor.

In the fluid analogy a current source would be an ideal, constant-flow pump. It would generate whatever pressure was required to maintain its specified flow. A voltage source would be similar to a very-large-capacity vessel, such as a dam. It would supply the same pressure stream, no matter how much water was flowing out of it, or even if water was flowing into it, or if there was no flow at all.

12.5 PHASOR ANALYSIS

If a system or element is being driven by sinusoidal signals or signals that can be converted to sinusoids via the Fourier series or Fourier transform, then phasor analysis techniques can be used to analyze the system's response (see Chapter 6, Section 6.3). The defining equations for an inductor (Equation 12.15) and capacitor (Equation 12.21) contain the calculus operations of differentiation and integration, respectively. In Chapter 6, we found that when the system variables were in phasor notation, differentiation was replaced by multiplying by $j\omega$ (Equation 6.15) and integration was replaced by dividing by $j\omega$ (Equation 6.17). If we use phasor equations to define an inductor and capacitor, the associated equations become algebraic like the equation for a resistor (Equation 12.11).

Since phasor analysis will be an important component of circuit analysis, a quick review of Section 6.3 might be worthwhile. In those cases where the variables are not sinusoidal, or cannot be reduced to sinusoids, we again turn to Laplace transforms to convert calculus operations into algebraic operations. The Laplace domain representation of electrical elements is covered in Section 12.6.

12.5.1 Phasor Representation—Electrical Elements

Converting the voltage–current equation for a resistor from time to phasor domain is not difficult, nor is it particularly consequential since the time domain equation (Equation 12.10) is already an algebraic relationship. Accordingly, the conversion is only a matter of restating the voltage and current variables in phasor notation:

$$V(\omega) = RI(\omega) \tag{12.28}$$

Rearranged as a voltage–current ratio:

$$R = \frac{V(\omega)}{I(\omega)} \tag{12.29}$$

Converting the voltage–current equation of an inductor or capacitor to phasor notation is more consequential as the differential or integral relationships then become algebraic. For an inductor the voltage–current equation in the time domain is given in Equation 12.15 and repeated here:

$$v_L = \frac{L^{di(t)}}{dt} \tag{12.30}$$

But in phasor notation, the derivative operation becomes multiplication by $j\omega$: $L\frac{di(t)}{dt} \Leftrightarrow j\omega\, I(\omega)$ so the voltage–current operation for an inductor becomes:

$$V_L(\omega) = Lj\omega\, I(\omega) = j\omega\, LI(\omega) \tag{12.31}$$

Since the calculus operation has been removed it is now possible to rearrange Equation 12.31 to obtain a voltage-to-current ratio similar to that for the resistor (Equation 12.29):

$$\frac{V(\omega)}{I(\omega)} = j\omega L\ \Omega \tag{12.32}$$

The ability to express the voltage–current relationship as a ratio is part of the power of the phasor domain method. Thus the term $j\omega L$ becomes something like the equivalent resistance of an inductor. This resistor-like ratio is termed "impedance," represented by the letter "Z," and has the units of ohms (volts/amp), the same as for a resistor:

$$Z_L(\omega) = \frac{V_L(\omega)}{I_L(\omega)} = j\omega L\ \Omega \tag{12.33}$$

Impedance, the ratio of voltage to current, is not defined for inductors or capacitors in the time domain since the voltage—current relationships for these elements contain integrals or differentials and it is not possible to determine a V/I ratio. Impedance is a function of frequency except for resistors. Often impedance is written simply as Z with the frequency term understood. Since impedance is a generalization of the concept of resistance (it is the V/I ratio for any passive element), the term is often used in discussion of any V/I relationships, even if only resistances are involved. The concept of impedance extends to mechanical and thermal systems as well. For example, in mechanical systems, the impedance of an element would be the ratio of force to velocity defined by the element: $Z(\omega) = \frac{F(\omega)}{v(\omega)}$.

To apply phasor analysis and the concept of impedance to a capacitor, we start with the basic voltage—current equation for a capacitor (Equation 12.21), repeated here:

$$v_C(t) = \frac{1}{C}\int i_c(t)dt \tag{12.34}$$

Noting that integration becomes the operation of dividing by $j\omega$ in the phasor domain:

$$\int i(t)dt \Leftrightarrow \frac{I(\omega)}{j\omega}$$

so the phasor voltage—current equation for a capacitor becomes:

$$V(\omega) = \frac{I(\omega)}{j\omega C} \tag{12.35}$$

The capacitor impedance then becomes:

$$Z_C(\omega) = \frac{V_C(\omega)}{I_C(\omega)} = \frac{1}{j\omega C} = \frac{-j}{\omega C}\ \Omega \tag{12.36}$$

Active elements producing sinusoidal voltages or currents can also be represented in the phasor domain by returning to the original phasor description of sinusoid given as Equation 6.13 in Chapter 6:

$$A\cos(\omega t + \theta) \Leftrightarrow Ae^{j\theta} \tag{12.37}$$

Using this equation, the phasor representation for a voltage source becomes:

$$V_s(t) = V_s\cos(\omega t + \theta) \Leftrightarrow V_s e^{j\theta} \equiv V_s \angle \theta \tag{12.38}$$

and for a current source:

$$I_s(t) = I_s\cos(\omega t + \theta) \Leftrightarrow I_s e^{j\theta} \equiv I_s \angle \theta \tag{12.39}$$

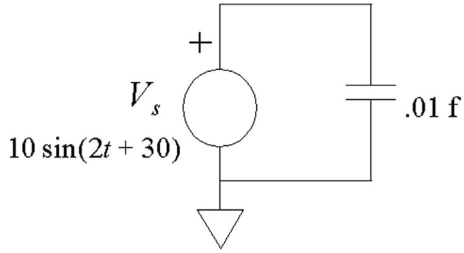


FIGURE 12.12 A simple circuit consisting of a voltage source and a capacitor used in Example 12.3. The two elements are represented by their voltage/current relationships in the phasor domain.

(Recall, capital letters are used for phasor variables.) Usually the frequency, ω, is not explicitly included in the phasor expression of a current or voltage, but it is often shown in its symbolic representation on the circuit diagram as shown in Figure 12.12. Here the frequency is defined in the definition of the voltage source as $\omega = 2$. An elementary use of phasors is demonstrated in the following example.

EXAMPLE 12.3

Find the current through the capacitor in the circuit of Figure 12.12.

Solution: Since the voltage across the capacitor is known (it is the just voltage of the voltage source, V_s), the current through the capacitor can be determined by the phasor extension of Ohm's law: $V(\omega) = I(\omega)\ Z(\omega)$:

Solving Ohm's law for $I_C(\omega)$: $I_C(\omega) = \frac{V_C(\omega)}{Z_C(\omega)}$.

The voltage across the capacitor is the same as the source voltage, $V_C = V_S$:

$$V_c = V_s = 10\sin(2t+30) = 10\sin(2t-60) \Rightarrow V_c(\omega) = V_s(\omega) = 10\angle-60$$

Recall, phasor representation is based on the cosine so the sine is converted to a cosine before converting to phasor representation. Next, we find the phasor notation of the capacitor:

$$Z_c = \frac{1}{j\omega C} = \frac{1}{j2(.01)} = -j50\,\Omega = 50\angle-90\,\Omega$$

Then solving for I_C using Ohm's law:

$$I_C(\omega) = \frac{V_c(\omega)}{Z_c(\omega)} = \frac{10\angle-60}{50\angle-90}$$

Recall, the rule for dividing two complex numbers is to convert them to polar notation (these already are in polar notation) and divide the magnitudes and subtract the denominator angle from the numerator angle (see Appendix E for details.).

$$I_C(\omega) = \frac{10\angle-60}{50\angle-90} = 0.2\angle 30\ \text{A}$$

Converting from the phasor domain to the time domain:

$$i_C(t) = 0.2\cos(2t+30)\ \text{A}$$

The solution to the problem requires only algebra, although it is complex algebra. This is true for all electrical network problems as long as we operate in the phasor (i.e., frequency) domain.

The phasor approach can be used whenever the voltages and currents are sinusoids or can be decomposed to sinusoids. In the latter case, the circuit needs to be solved separately for each sinusoidal frequency applying the law of superposition. Like Fourier series analysis, this can become quite tedious if manual calculations are involved, but such analyses are not that difficult using MATLAB. An example of solving a simple circuit over a large number of frequencies using MATLAB is shown in the following discussion.

EXAMPLE 12.4

The circuit shown in Figure 12.13 was one of the earliest models of the cardiovascular system. This model, the most basic of the "Windkessel" models described in Chapter 1 (Section 1.4.5.1), represents the cardiovascular load on the left heart. The voltage source, $v(t)$, represents the pressure in the aorta and the current, $i(t)$, the flow of blood into the periphery. The resistor represents the resistance to flow of the arterial system and the capacitor represents the compliance of the arteries (i.e., stretchiness of the arteries). Of course the heart output is pulsatile not sinusoidal, so the analysis of this model is better suited to Laplace methods, but a sinusoidal analysis can be used to find the frequency characteristics of the load, that is, the effective impedance (resistance to blood flow), of the vascular system.

Assume that the voltage source produces a sinusoidal series consisting of frequencies between 0.01 and 10 Hz in 0.01-Hz intervals. This will allow us to generate a Bode plot of the impedance characteristics of the peripheral vessels in the Windkessel model. Plot the log of the magnitude impedance against log frequency as used in constructing Bode plots.

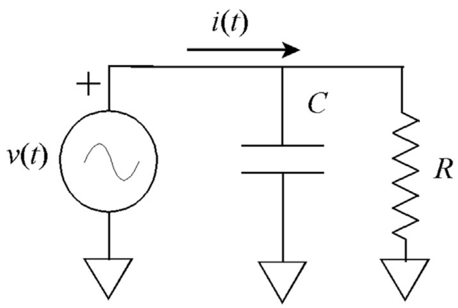


FIGURE 12.13 The least complicated Windkessel representation of the cardiovascular load on the heart. Electrical elements and variables are used to represent analogous cardiovascular elements. The voltage, $v(t)$, represents the pressure generated in the aorta in mmHg; the current, $i(t)$, the blood flowing into the periphery in mL/s; R the resistance in units of mmHg s/mL; and C the effective compliance (stretchiness) of the peripheral vessels in mL/mmHg. In Example 12.4, we determine the frequency characteristics (i.e., Bode plot) for the arteries loading the heart between 0.01 and 10 Hz.

Solution: The peripheral impedance is just $Z(f) = V(f)/I(f)$ where $V(f)$ is the phasor representation of the aortic pressure and $I(f)$ is the phasor notation for the blood flow, $i(t)$. Since the problem gives frequency ranges in hertz instead of radians, we will use f instead of ω. One approach is to apply a $V(f)$ having a known value and solving for the current, $I(f)$. In this case, $V(f)$ will be a sinusoidal series between 0.01 and 10 Hz. The impedance as a function of frequency is the ratio of $V(f)/I(f)$. Since the actual input pressure is arbitrary, we make it a series of cosine waves with amplitudes of 1.0; i.e., $v(t) = \cos(2\pi f t)$ for $f = 0.01-10$ Hz. This leads to a phasor representation of $V(f) = 1.0$.

First convert the elements into phasor notation. The resistor does not change, the voltage source becomes 1.0, and the capacitor would be $1/j\omega C = 1/j2\pi f C$. Note that since the voltage represents pressure it is in mmHg and the current will be in mL/s. Typical values for R and C are: $R = 1.05$ mmHg s/mL (pressure/volume flow) and $C = 1.1$ mL/mmHg (volume/pressure).

In the next chapter, we develop an algorithmic approach to solving any circuit problem, but for now note that the total current flow is just the current through the resistor plus the current through the capacitor. By Ohm's law for the resistor and capacitor (Equation 12.35):

$$V(f) = I_R(f)R; \quad I_R(f) = \frac{V(f)}{R} \quad V(f) = \frac{I_C(f)}{j2\pi f C}; \quad I_C(f) = V(f)(j2\pi f C)$$

So the total current (flow) becomes:

$$I(f) = I_R(f) + I_C(f) = \frac{V(f)}{R} + V(f)(2\pi f C) = V(f)\left(\frac{1}{R} + j2\pi f C\right)$$

Substituting in appropriate values for R and C:

$$I(f) = V(f)\left(\frac{1}{1.05} + j2\pi f 1.1\right) = V(f)(0.95 + j6.9f)$$

Solving for the impedance of the arteries:

$$Z(f) = \frac{V(f)}{I(f)} = \frac{1.0}{0.95 + j6.9f} = \frac{1.05}{1 + j6.55f}$$

Results: From our knowledge of Bode plot primitives (recall Section 6.5.1), this equation has the form of a first-order system with a cutoff frequency of $1/6.55 = 0.153$ Hz.[5] The Bode plot for the magnitude and phase of $Z(f)$ can be plotted using Bode plot primitives (see the generic magnitude plot, Figure 6.13, and phase plot in Figure 6.14). The plots (actually generated using MATLAB) are shown in Figure 12.14. These show that the arterial impedance decreases above 0.152 Hz due to the compliance of the arteries (capacitance in the model). This means that at higher frequencies flow increases for a given pressure, but remember this is a much simplified model of the arterial system.

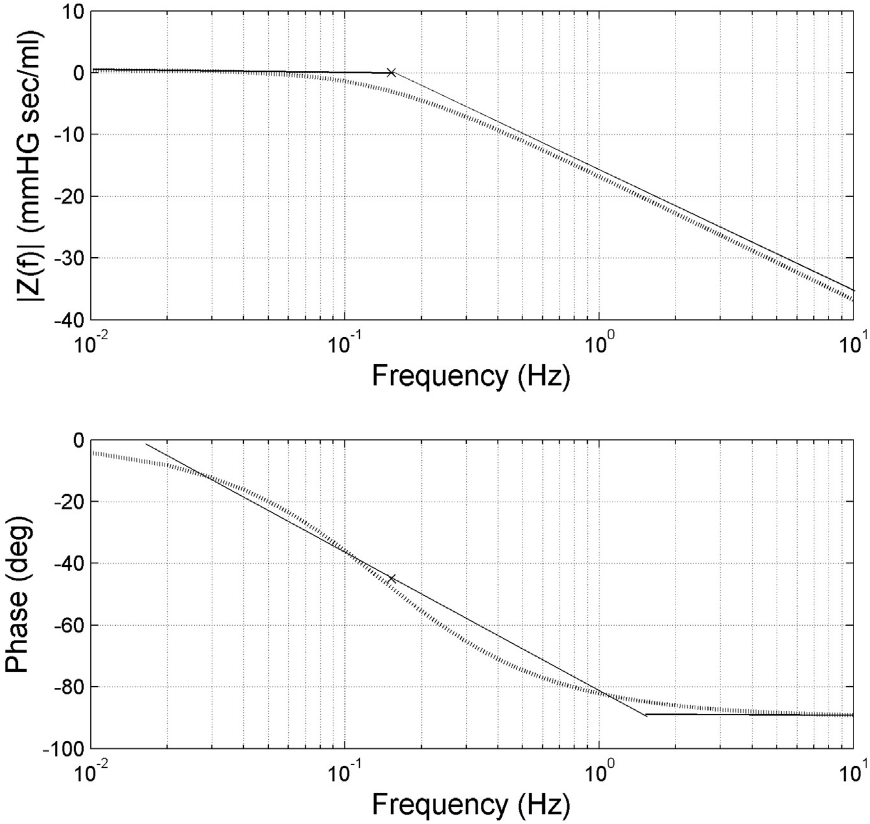


FIGURE 12.14 The impedance of the cardiovascular system as a function of frequency as determined from a simple two-element Windkessel model. The impedance decreases above 0.152 Hz due to the compliance of the arteries (capacitance in the model).

A more complicated version of the Windkessel model with a more realistic pulse-like pressure signal is analyzed in the next chapter.

[5]This is the same as Equation 6.31 except that frequency in now in Hz instead of rad/s. As biomedical engineers, we should be comfortable expressing frequency in either unit.

12.6 LAPLACE DOMAIN—ELECTRICAL ELEMENTS

If the voltages and currents are not sinusoids, periodic, or aperiodic signals, then they must be represented in the Laplace domain. The Laplace transform cannot be used in situations where $t < 0$ since the transform equation (Equation 7.4) diverges for negative time. In the Laplace domain, signals are valid only for $t \geq 0$. With this restriction, analysis in the Laplace domain is a straightforward extension of phasor analysis. Recall that activity before $t = 0$ can be summarized in terms of nonzero initial conditions as shown in Section 12.6.2, but first we define the Laplace domain representation of electrical elements (R, L, and C) assuming zero initial conditions.

12.6.1 Electrical Elements With Zero Initial Conditions

The impedance of a resistor, R, is the same in both the time and Laplace domains since resistors have a constant relationship between voltage and current:

$$V(s) = R\,I(s) \tag{12.40}$$

The impedance of a resistor is $V(\mathrm{s})/I(s)$:

$$Z_R(s) = \frac{V_s}{I_s} = R \tag{12.41}$$

For inductors and capacitors with zero initial conditions the Laplace representation is straightforward: differentiation and integration are represented by the Laplace operators s and $1/s$, respectively. For an inductor, the time derivative in Equation 12.16 is replaced by s and the defining equation becomes:

$$V(s) = sLI(s) \tag{12.42}$$

The Laplace impedance for an inductor is:

$$Z_L(s) = \frac{V(s)}{I(s)} = sL\ \ \Omega \tag{12.43}$$

Impedance in the Laplace domain, like impedance in the phasor domain, is given in ohms. For a capacitor, the time integral is represented by $1/s$ and the defining equation is:

$$V(s) = \frac{1}{Cs} I(s) \tag{12.44}$$

The impedance of a capacitor is:

$$Z_C(S) = \frac{v(S)}{i(S)} = \frac{1}{Cs}\ \ \Omega \tag{12.45}$$

The defining equations and the impedance relationships are summarized for electrical elements in Table 12.5 for the time, phasor, and Laplace domain representations.

12.6.2 Nonzero Initial Conditions

Circuit activity that occurs for $t < 0$ can be included in a Laplace analysis as initial voltages and/or currents. Essentially we are taking all of the past history and boiling it down to a

TABLE 12.5 *V-I* Relationships and Impedances for Electrical Elements

		Phasor Domain		Laplace Domain	
Element	***v/i* Time Domain**	***V-I* Equation**	***Z*(ω)**	***V-I* Equation**	***Z*(*s*)**
Resistor	$v = R\,i$	$V(\omega) = R\,I(\omega)$	$R\ \Omega$	$V(s) = R\,I(s)$	$R\ \Omega$
Inductor	$v = L\frac{di}{dt}$	$V(\omega) = j\omega L\,I(\omega)$	$j\omega L\ \Omega$	$V(s) = sL\,I(s)$	$sL\ \Omega$
Capacitor	$v = \frac{1}{C}\int i dt$	$V(\omega) = \frac{I(\omega)}{j\omega C}$	$\frac{1}{j\omega C}\ \Omega$	$V(s) = \frac{1}{sC}I(s)$	$\frac{1}{sC}\ \Omega$

voltage or current at $t = 0$. Only the passive energy storage elements, L or C, have nonzero initial conditions; specifically, capacitors can have nonzero initial voltages and inductors can have nonzero initial currents.

In an inductor, energy is stored in the form of current flow, so the salient initial condition for an inductor is the current at $t = 0$. This can be obtained based on the past history of the inductor's voltage:

$$I_L(0) = \frac{1}{L}\int_{-\infty}^{0} v_L dt \tag{12.46}$$

An inductor treats the current that flows through it the same way a capacitor treats voltage: the current flowing through an inductor is continuous and cannot change instantaneously. So no matter what change occurs at $t = 0$, the current just before $t = 0$ will be the same as the current immediately after $t = 0$. Stated mathematically: $I_L(0-) = I_L(0+)$.

For an inductor with nonzero initial conditions, consider the equation that defines the derivative operation in the Laplace domain, Equation 7.5, repeated here:

$$\mathcal{L}\frac{dx(t)}{dt} = sX(s) - x(0-) \tag{12.47}$$

To get the equation defining an inductor with an initial current, we start with the basic time domain equation, $v(t) = L\frac{di(t)}{dt}$, and apply Equation 12.47 for the derivative operation. Note that since the inductance L is a constant, its Laplace transform is just L and the Laplace domain equation for an inductor is:

$$\mathcal{L}v(t) = \mathcal{L}\left[L\frac{di(t)}{dt}\right] = L\,\mathcal{L}\left[\frac{di(t)}{dt}\right]$$

Applying Equation 12.47 substituting $i(t)$ for $x(t)$, the Laplace domain equation for the voltage across an inductor becomes:

$$V_L(s) = L(sI(s) - i(0)) = sLI(s) - Li(0) \tag{12.48}$$

So the initial current $i(0)$ adds a second term to the voltage−current relationship of the inductor. Often these values are known, but in some cases they must be calculated from the past history of the system.

Regarding Equation 12.48 it seems impossible to get a simple impedance term, i.e., a single term for the ratio $V(s)/I(s)$. However, there is a clever way to deal with the initial condition term, the $Li(0)$ term, and still retain the concept of impedance. Dissecting the two terms in Equation 12.48, the first term is the impedance, sL, the same as Equation 12.43, and the second term is a constant, $Li(0)$. The second term can be viewed as a constant voltage source with a value of $Li(0)$. It is a strange voltage source, its value being dependent on both the initial current and the inductance, but it does look like a constant voltage source in the Laplace domain.[6] So the symbol for an inductor with a nonzero initial condition in the Laplace domain would actually be a combination of two elements: a Laplace impedance representing the inductor in series with the voltage source representing the initial current condition, Figure 12.15.

For a capacitor, the entire $t < 0$ history can be summarized as a single voltage at $t = 0$ using the basic equation that defines the voltage−current relationship of a capacitor:

$$V_C(t = 0) = \frac{1}{C}\int_{-\infty}^{0} i_C(t)dt \tag{12.48}$$

This equation has the same form as the right-hand term of the Laplace transform of the integration operation, Equation 7.7 repeated here:

$$\mathcal{L}\left[\int_0^T x(t)dt\right] = \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0} x(t)dt \tag{12.49}$$

Substituting in $i(t)$ for $x(t)$ in Equation 12.49 and noting that $1/C$ is a constant, the Laplace equation for the voltage across a capacitor becomes:

$$\mathcal{L}v(t) = \mathcal{L}\left[\frac{1}{C}\int i(t)dt\right] = \frac{1}{sC}I(s) + \frac{1}{sC}\int_{-\infty}^{0} i(t)dt \tag{12.50}$$

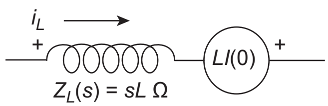


FIGURE 12.15 The Laplace domain representation of an inductor with a nonzero initial current. The inductor becomes two elements: a Laplace domain inductor having an impedance of sL, and a voltage source with a value of $Li(0)$ where $i(0)$ is the initial current. Note the polarity of the voltage source, which is based on the negative sign in Equation 12.48.

[6]Look at it this way: if you set the first term to zero, you are left with $V(s) = Li(0) = V_L$, a constant since both L and $i(0)$ are constants.

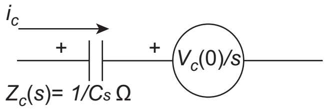


FIGURE 12.16 Laplace domain representation for a capacitor with an initial voltage. The element consists of two components: an impedance element, $1/Cs$, and a voltage source element representing the initial condition $V_C(0)/s$. The polarity of the voltage element is in the same direction as polarity of the impedance element as given by Equation (12.51).

Again the nonzero initial voltage adds a second term to the voltage current relationship. The first term is just the standard impedance of a capacitor, Equation 12.45. The second term is actually a voltage. From the basic time domain definition of a capacitor:

$$\frac{1}{C}\int_{-\infty}^{-0} i(t)dt = V_c(-0)$$

So the second term is just the voltage on a capacitor at $t = 0$ divided by s (i.e., $V_C(0-)/s$). The voltage across a capacitor cannot change instantaneously so $V_C(0-) = V_C(0+) = V_C(0)$ and Equation 12.50 becomes:

$$V(s) = \frac{1}{sC}I(s) + \frac{V_C(0)}{s} \tag{12.51}$$

The Laplace domain representation of a capacitor having an initial voltage, Equation 12.51, can be interpreted as capacitance impedance, $1/sC$, in series with a voltage source. In this case, the voltage source is $V_C(0)/s$. This leads to the combined Laplace elements shown in Figure 12.16. The polarity of the voltage source in this case has the same polarity as the initial voltage on the capacitor.

EXAMPLE 12.5

A 0.1-F capacitor with an initial voltage of 10 V is connected to a 100-Ω resistor at $t = 0$. Find the value of the resistor voltage for $t \geq 0$.

Solution: Use Figure 12.16 to configure a Laplace domain capacitor with nonzero initial conditions. Adding a resistor results in the circuit shown in Figure 12.17.

From Figure 12.17 we see that the voltage source representing the initial condition, $V_C(0)$, equals the voltage across the series resistor and capacitor. By Ohm's law, this voltage must equal the current in the circuit times the combined impedance of resistor and capacitor:

$$V_C = Z_C I(s) + RI(s) = \frac{1}{sC}I(s) + RI(s)$$

$$\frac{10}{s} = \frac{1}{0.1s}I(s) + 100I(S) = I(s)\left(\frac{10}{s} + 100\right)$$

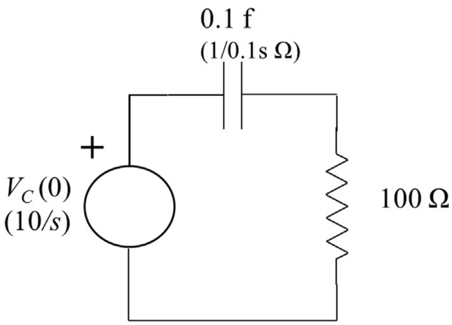


FIGURE 12.17 The network used in Example 12.5 shown in Laplace notation. Note that the 0.1 F capacitor is also shown with its impedance value of $\frac{1}{sC} = \frac{1}{0.1s}$ Ω. We use the ohm symbol to indicate that the capacitance has been converted to impedance.

Solving for current, $I(s)$:

$$I(s) = \frac{10}{s\left(\frac{10}{s} + 100\right)} = \frac{10}{100s + 10} = \frac{0.1}{s + 0.1}$$

To find the voltage across the resistor, again apply Ohm's law:

$$V_R(s) = RI(s) = 100\frac{.1}{s + 0.1} = \frac{10}{s + 0.1}$$

Converting to the time domain using entry #3 of the Laplace transform table in Appendix B gives the required answer:

$$v_R(t) = 10e^{-0.1t}\ \text{V}$$

12.7 SUMMARY: ELECTRICAL ELEMENTS

In the time domain, impedance, the ratio of voltage to current for a component, can be defined only for a resistor (i.e., Ohm's law). In the phasor and Laplace domains, impedance can be defined for inductors and capacitors as well. Using phasors or Laplace notation it is possible to treat these "reactive elements" (inductors and capacitors) as if they were resistors, at least from a mathematical standpoint. This allows us to generalize Ohm's law, $V = I\,Z$, to include inductors and capacitors and to treat them mathematically using only algebra. Applications using this extension of Ohm's law are given in Examples 12.3–12.5.

In the next chapter, rules will be introduced that capitalize on the generalized form of Ohm's law. These rules will lead to a step-by-step process to analyze any network of sources and passive elements no matter how complicated. However, the first step in circuit analysis is always the same as shown in the examples here: convert the electrical elements into their phasor or Laplace representations. This could include the generation of additional elements if the energy storage elements have initial conditions. In the last chapter of this text, the analysis of circuits containing electronic elements will be presented.

12.8 MECHANICAL ELEMENTS

The mechanical properties of many materials often vary across and through the material so that analysis requires an involved mathematical approach known as "continuum mechanics." However, if only the overall behavior of an element or collection of elements is needed, then the properties of each element can be grouped together and a "lumped-parameter" analysis can be performed. An intermediate approach facilitated by high-speed computers is to apply lump-parameter analysis to small segments of the material, and then compute how each of these segments interacts with its neighbors. This approach is often used in biomechanics and is known as "finite element analysis."

Lumped-parameter mechanical analysis is similar to that used for electrical elements and most of the mathematical techniques described earlier and elsewhere in this text can be applied to this type of mechanical analyses. In lumped-parameter mechanical analysis, the major variables are force and velocity. Mechanical elements have well-defined relationships between these variables, a relationship very similar to the voltage−current relationship defined by electrical elements. In mechanical systems, the flow-like variable analogous to current is velocity while the potential energy variable analogous to voltage is force. Mechanical elements can be either active or passive and, as with electrical elements, passive elements can either dissipate or store energy.

12.8.1 Passive Mechanical Elements

Dynamic friction is the only mechanical element that dissipates energy and, as with the resistor, that energy is converted to heat. The force-velocity relationship for a friction element is also similar to a resistor: the force generated by the friction element is proportional to its velocity:

$$F = k_f v \tag{12.52}$$

where k_f is the constant proportionality and is termed "friction," F is force, and v is velocity.

In the "cgs" (centimeters, grams, dynes) metric system used in this text, the unit of force is dynes and the unit of velocity is cm/s, so the units for friction are dyn/cm/s (force in dynes divided by velocity in cm/s) or dyne-s/cm. Another commonly used measurement system is the "mks" (meters, kilograms, seconds) system preferable for systems having larger forces and velocities than those generally found in biological systems. Conversion between the two is straightforward (see Appendix D).

The equation for the power lost as heat in a friction element is analogous to that of a resistor:

$$P = Fv \tag{12.53}$$

The symbol for such a friction element is termed a "dashpot," and is shown in Figure 12.18. Friction is often a parasitic element, but can also arise from a device specifically constructed to

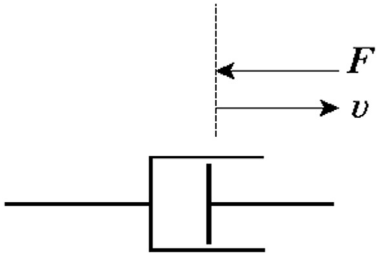


FIGURE 12.18 The schematic representation of a friction element showing the convention for the direction of force and velocity. This element is also referred to as a dashpot.

produce it. Devices designed specifically to produce friction are sometimes made using a piston that moves through a fluid (or air), for example, shock absorbers on a car or some door-closing mechanisms. This construction approach, a moving piston, forms the basis for the schematic representation of the friction element shown in Figure 12.18.

As with passive electrical elements, passive mechanical elements have a specified directional relationship between force and velocity: specifically, the direction of positive force is opposite to that of positive velocity. The direction of one of the variables can be chosen arbitrarily after which the direction of the other variable is determined. These conventions are illustrated in Figure 12.18.

In addition to elements specifically designed to produce friction (such as shock absorbers), friction occurs in association with other elements, just as resistance is unavoidable in other electrical elements (particularly inductors). For example, a mass sliding on a surface would exhibit some friction no matter how smooth the surface. Irrespective of whether friction arises from a dashpot element specifically designed to create friction or is associated with another element, it is usually represented by the dashpot schematic shown in Figure 12.18.

There are two mechanical elements that store energy just as there are two energy-storing electrical elements. The "inertial type" element corresponding to inductance is, not surprisingly, inertia associated with mass. It is termed simply "mass," and is represented by the letter m. The force–velocity relationship associated with mass is a version of Newton's law:

$$F = ma = m\frac{dv}{dt} \tag{12.54}$$

The mass element is schematically represented as a rectangle, again with force and velocity in opposite directions, Figure 12.19.

A mass element stores energy as kinetic energy following the well-known equation for kinetic energy.

$$E = \frac{1}{2}mv^2 \tag{12.55}$$

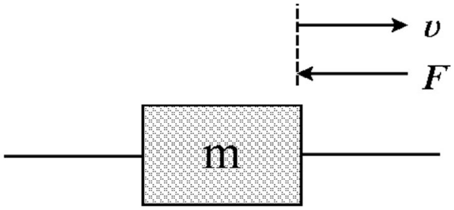


FIGURE 12.19 The schematic representation of a mass element that features the same direction conventions as the friction element.

The parallel between the inertial electrical element and its analogous mechanical element, mass, includes the continuity relationship imposed on the flow variable. Just as current moving through an inductor must be continuous and cannot be changed instantaneously, moving objects tend to continue moving (to paraphrase Newton) so the velocity of a mass cannot be changed instantaneously without applying infinite force. Hence, the velocity of a mass is continuous, so that $v_m(0-) = v_m(0+)$. It is possible to change the force on a mass instantaneously, just as it is possible to change the voltage applied to an inductor instantaneously, but not the velocity.

The mechanical energy storage element analogous to a capacitor is a spring and it has a force−velocity equation that is similar to that of a capacitor:

$$F = k_e x(t) = k_e \int v dt \tag{12.56}$$

where k_e is the "spring constant" in dyn/cm. A related term frequently used is the compliance, C_k, which is just the inverse of the spring constant ($C_k = 1/k_e$), and its use makes the equation of spring and capacitor even more similar to that of a capacitor:

$$F = \frac{1}{C_k} x(t) = \frac{1}{C_k} \int v dt \tag{12.57}$$

Although a spring is analogous to a capacitor, the symbol used for a spring is similar to that used for an inductor as shown in Figure 12.20. Springs look schematically like inductors, but they act like capacitors; no analogy is perfect.

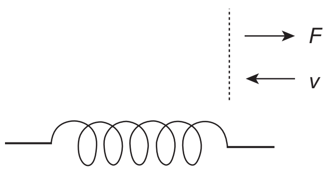


FIGURE 12.20 The symbol for a spring showing the direction conventions for this passive element. Schematically, the spring looks like an inductor: an unfortunate coincidence because it acts like a capacitor.

TABLE 12.6 Energy Storage and Response to Discontinuous and DC Variables in Mass and Elasticity

Element	Energy Stored	Continuity Property	DC Property
Mass	$E = \frac{1}{2}LI^2$	Velocity continuous $v_m(0-) = v_m(0+)$	If v_m = constant (DC velocity) $F = 0$
Elastic element	$E = \frac{1}{2}k_e x^2$	Force continuous $F_e(0-) = F_e(0+)$	If F_e = constant (DC force) $v_m = 0$

As with a capacitor, a spring stores energy as potential energy. A spring that is stretched or compressed generates a force that can do work if allowed to move through a distance. The work or energy stored in a spring is:

$$E = \int F dx = \int k_e x dx = \frac{1}{2} k_e x^2 \tag{12.58}$$

Displacement, x, is analogous to charge, q, in the electrical domain, so the equation for energy stored in a spring is analogous to the equation for energy stored in a capacitor found in the derivation of Equation 12.25 and repeated here:

$$E = \frac{Q^2}{2C} \tag{12.59}$$

As with a capacitor, it is impossible to change the force on a spring instantaneously using finite velocities. This is because force is proportional to length ($F_s = k_e x$) and the length of a spring cannot change instantaneously. Using high velocities, it is possible to change spring force quickly, but not instantaneously; hence a spring force is continuous: $F_S(0-) = F_S\,(0+)$.

Since passive mechanical elements have defining equations similar to those of electrical elements, the same analysis techniques, such as phasor and Laplace analyses, can be applied. Moreover, the rules for analytically describing combinations of elements (i.e., mechanical systems) are similar to those for describing electrical circuits. Table 12.6 is analogous to Table 12.3 and shows the energy, continuity, and DC properties of mass and elasticity.

12.8.2 Elasticity

Elasticity relates to the spring constant of a spring, but because it is such an important component in biomechanics, a few additional definitions related to the spring constant and compliance are presented here. Elasticity is most often distributed through or within a material and is defined by the relationship between "stress" and "strain." Stress is a *normalized force*, one that is normalized by the cross-sectional area:

$$\text{Stress} = \frac{\Delta F}{A} \tag{12.60}$$

Strain is a *normalized stretching* or elongation. Strain is the change in length with respect to the rest length, which is the length the material would assume if no force were applied:

$$\text{Strain} = \frac{\Delta \ell}{\ell} \tag{12.61}$$

The ratio of stress to strain is a normalized measure of the ability of a material to stretch and is given as an elastic coefficient termed Young's modulus:

$$Y_M = \frac{\text{Stress}}{\text{Strain}} = \frac{\frac{\Delta F}{A}}{\frac{\Delta \ell}{\ell}} \tag{12.62}$$

If a material is stretched by a load or weight produced by a mass, m, then the equation for Young's modulus can be written as:

$$Y_M = \frac{\frac{mg}{\pi r^2}}{\frac{\Delta \ell}{\ell}} \tag{12.63}$$

where g is the earth's gravitational constant, 980.665 cm/s^2. Values for Young's modulus for a wide range of materials can be found in traditional references such as the *Handbook of Physics and Chemistry* (CRC Press). Some values for typical materials are shown in Table 12.7. The following examples illustrate applications of Young's modulus and the related equations given earlier.

TABLE 12.7 Young's Modulus of Selected Materials

Material	**Y_M (dyn/cm^2)**
Steel (drawn)	19.22×10^{10}
Copper (wire)	10.12×10^{10}
Aluminum (rolled)	$6.8–7.0 \times 10^{10}$
Nickel	$20.01–21.38 \times 10^{10}$
Constantan	$14.51–14.89 \times 10^{-10}$
Silver (drawn)	7.75×10^{10}
Tungsten (drawn)	35.5×10^{10}

EXAMPLE 12.6

A 10 lb. weight is suspended by a #12 (AWG) wire 12 in. long. How much does the wire stretch?

Solution: To find the new length of the wire use Equation 12.63 and solve for $\Delta\ell$. First, convert all constants to cgs units:

$$m = 10\,\text{lb} = 10\,\cancel{\text{lb}}\left(\frac{1\,\cancel{\text{kg}}}{20.4\,\cancel{\text{lb}}}\right)\left(\frac{1000\,\text{gm}}{1\,\cancel{\text{kg}}}\right) = 490.2\,\text{gm}$$

$$\ell = 10\,\text{in}\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right) = 25.4\,\text{cm}$$

To find the diameter of the 12-gauge (AWG) wire use Table 5 in Appendix D:

$$d = 0.081\,\text{in} \quad \text{From Table 5 in Appendex D}$$

$$r = \frac{d}{2} = \left(\frac{0.081\,\cancel{\text{in}}}{2}\right)\left(\frac{2.54\,\text{cm}}{1\,\cancel{\text{in}}}\right) = 0.103\,\text{cm}$$

Then solve for $\Delta\ell$; use the value of Y_M for copper from Table 12.7:

$$Y_M = \frac{\frac{mg}{A}}{\frac{\Delta\ell}{\ell}} = \frac{\frac{mg}{\pi r^2}}{\frac{\Delta\ell}{\ell}} \qquad \Delta\ell = \frac{mg\ell}{\pi r^2 Y_M} = \frac{(490.2)(980.6)(25.4)}{\pi(0.103)^2(10.12\times10^{10})} = 0.0036\ \text{cm}$$

EXAMPLE 12.7

Find the elastic coefficient of a steel bar with a diameter of 0.5 mm and length of 0.5 m.

Solution: From Equation 12.56: $F = k_e x$, so $k_e = F/x$ where in this case $x = \Delta\ell$. After rearranging Equation 12.62, k_e is found in terms of Young's modulus:

$$k_e = \frac{F}{x} = \frac{F}{\Delta\ell} = \frac{Y_M A}{\ell}$$

Use the dimensions given and the material in Equation 12.62 to find Young's modulus:

$$Y_M = \frac{\frac{\Delta F}{A}}{\frac{\Delta\ell}{\ell}} = \frac{F\ell}{\Delta\ell A}; \quad F = \frac{Y_M A\Delta\ell}{\ell}; \quad k_e = \frac{F}{\Delta\ell} = \frac{Y_M A}{\ell} = \frac{Y_M\pi\left(\frac{d}{2}\right)^2}{\ell}$$

$$k_e = \frac{19.22\times10^{10}\pi\left(\frac{.05}{2}\right)^2}{50} = \frac{3.77\times10^8}{50} = 7.55\times10^6\ \text{dyn/cm}^2$$

12.8.3 Mechanical Sources

Sources supply mechanical energy and can be sources of force, velocity, or displacement. Displacement is another word for "a change in position," and is just the integral of velocity: $x = \int v dt$. As mentioned earlier, displacement is analogous to charge in electrical circuits since $q = \int i dt$, and current is analogous to velocity. Although sources of constant force or constant velocity do occasionally occur in mechanical systems, most sources of mechanical energy are much less ideal than their electrical counterparts. Sometimes a mechanical source can look like either a velocity (or displacement) generator or a force generator depending on the characteristics of the load, that is, the mechanical properties of the elements connected to the source. For example, a muscle contracting under a light, constant load, a so-called isotonic contraction because the force (i.e., "tonus") opposing the contraction is constant (i.e., "iso"), would appear to be a velocity generator, although the velocity would not be constant throughout the contraction. However, if the muscle's end points were not allowed to move, a so-called isometric contraction because the muscle's length (i.e., "metric") is constant (again "iso"), then the muscle would look like a force generator.

In fact, a muscle is neither an ideal force generator nor an ideal velocity generator. An ideal force generator would put out the same force no matter what the conditions; however, the maximum force developed by a muscle depends strongly on its initial length. Figure 12.21 shows the classic "length–tension curve" for skeletal muscle and shows how maximum force

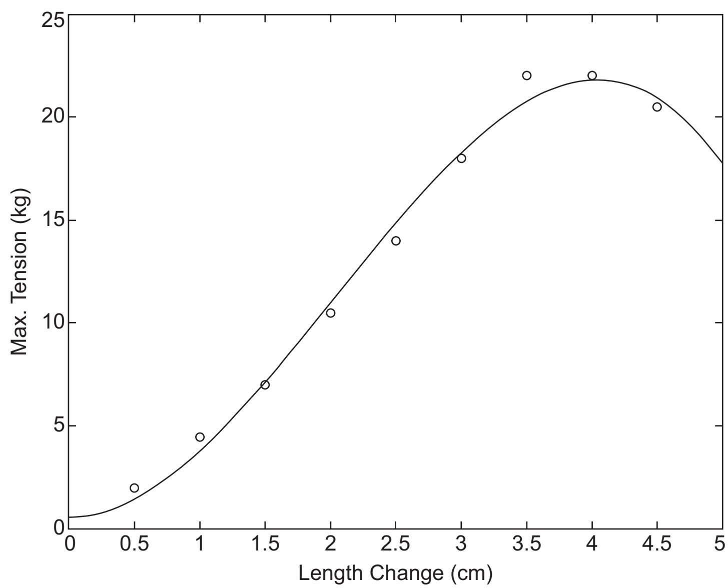


FIGURE 12.21 The length–tension relationship of skeletal muscle: the relationship between the maximum force a muscle can produce depends strongly on its length. An ideal force generator would produce the same force irrespective of its length, or its velocity for that matter. (This curve is based on historical measurements made on the human triceps muscle.)

depends on position with respect to rest length. (Again, the rest length is the position the muscle assumes where there is no force applied to the muscle.) When operating as a velocity generator under constant load, muscle is far from ideal since the velocity generated is highly dependent on the load. As shown by another classic, the "force–velocity curve," as the force resisting the contraction of a muscle is increased its velocity decreases and can even reverse if the opposing force becomes great enough, Figure 12.22. Of course, electrical sources are not ideal either, but they are generally more nearly ideal than mechanical sources. The characteristics of real sources, mechanical and electrical, are explored in Chapter 14.

With these practical considerations in mind, a force generator is usually represented by a circle or simply an F with a directional arrow, Figure 12.23.

A mass placed in a gravitational field looks like a force generator. The force developed by gravitation is in addition to its inertial force described by Equation 12.54. The forces devel-

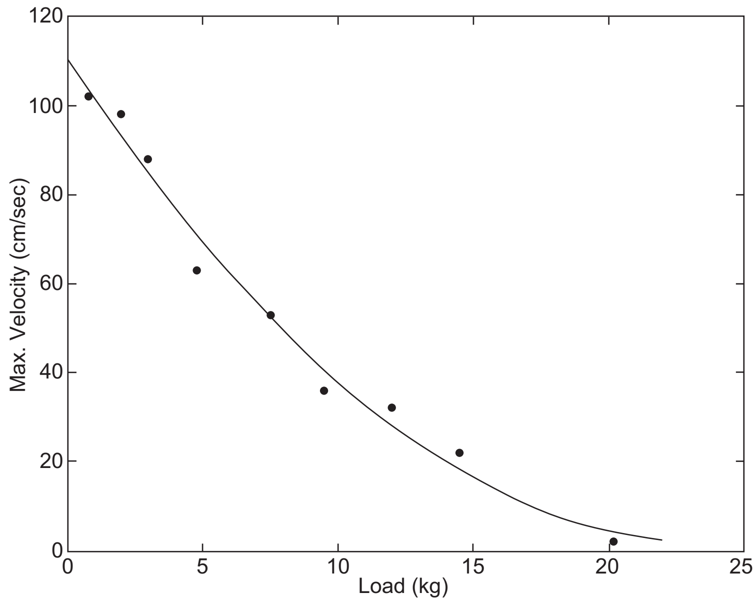


FIGURE 12.22 As a velocity generator, muscle is hardly ideal. As the load increases the maximum velocity does not stay constant as would be expected of an ideal source, but decreases with increasing force and can even reverse direction if the force becomes too high. This is known as the force–velocity characteristics of muscle. (This curve is based on historical measurements made on the human pectoralis major muscle.)

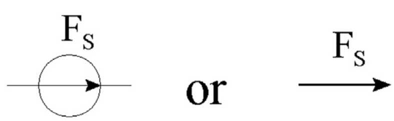


FIGURE 12.23 Two schematic representations of an ideal force generator showing direction of force.

oped by the inertial properties of a mass, or "inertial mass," and its gravitational properties, "gravitational mass," need not necessarily be coupled if they are the result of separate physical mechanisms. However, very careful experiments have shown them to be linked down to very high resolutions, indicating that they are related to the same underlying physics. The force is proportional to the value of the mass and the earth's gravitational constant:

$$F = mg \tag{12.64}$$

where m is the mass in grams and g is the gravitational constant in cm/sec^2 Note that a g-cm/s^2 equals a dyne of force. The average value of g at sea level is 980.665 cm/s^2 Provided the mass does not change significantly in altitude, the force produced by a mass due to gravity is nearly ideal: the force produced is even independent of velocity, although, if it is not moving in a vacuum, a frictional force due to wind resistance would be present.

In some mechanical systems that include mass, the force due to gravity must be considered, whereas in others it is cancelled by some sort of support structure. In Figure 12.24, the system on the left side has a mass supported by a surface (either a frictionless surface or with the friction incorporated in k_f) and only the inertial force defined in Equation 12.54 is considered. In the system on the right-hand side, the mass is under the influence of gravity and produces both an inertial force that is a function of velocity (Equation 12.54) and a gravitational force that is constant and defined by Equation 12.64. This additional force would be represented as a force generator acting in the downward direction with a force of mg.

A velocity or displacement generator would be represented as in Figure 12.23, but the letters used would be either V_S if it were a velocity generator or X_S for a displacement generator. Motors, can be viewed either as sources of rotational velocity or of rotational force (i.e., torque). The schematic representation of a motor is shown in Figure 12.25.

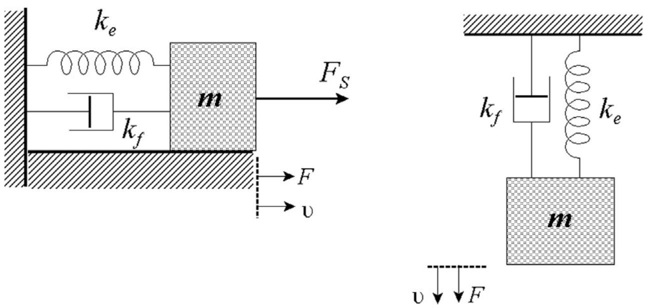


FIGURE 12.24 Two mechanical systems containing mass, m. In the left-hand system, the mass is supported by a surface so the only force involved with this element is the inertial force. In the right-hand system, gravity is acting on the mass so that it produces two forces: a constant force due to gravity (mg) and its inertial force.

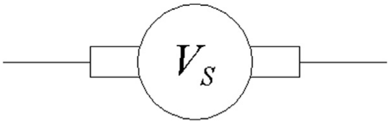


FIGURE 12.25 Symbol used to represent a motor. A motor can be either a force (torque) generator or a velocity (rpm).

12.8.4 Phasor Analysis of Mechanical Systems: Mechanical Impedance

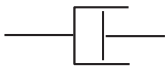
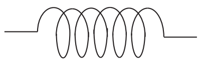
Phasor analysis of mechanical systems is the same as for electrical systems, only the names, and associated letters, change. The mechanical elements, their differential and integral equations, and their phasor representations are summarized in Table 12.8 just as the electrical elements are summarized in Table 12.4. Impedance is defined for mechanical elements as:

$$Z(\omega) = \frac{F(\omega)}{\upsilon(\omega)} \tag{12.65}$$

Mechanical impedance has the units of dyn-cm/s.

The application of phasors to mechanical systems is given in the next example. More complicated systems are presented in the next chapter.

TABLE 12.8 Mechanical Elements

Element (Units)	Equation $F(t) = f\,[\upsilon(t)]$	Phasor Equation Laplace Equation	Impedance $Z(\omega)$ $Z(s)$	Symbol
Friction (k_f) (dyn-s/cm)	$F(t) = k_f \upsilon(t)$	$F(\omega) = k_f\, \upsilon(\omega)$	k_f	
		$F(s) = k_f\, \upsilon(s)$	k_f	
Mass (m) (g)	$F(t) = m\frac{d\upsilon}{dt}$	$F(\omega) = j\omega m\, \upsilon(\omega)$	$j\omega m$	
		$F(s) = sm\, \upsilon(s)$	sm	
Elasticity (k_r) (spring) (dyn/cm)	$F(t) = k_e \int \upsilon\, dt$	$F(\omega) = \frac{k_e}{j\omega}\upsilon(\omega)$	$\frac{k_e}{j\omega}$	
		$F(\omega) = \frac{k_e}{s}\upsilon(\omega)$	$\frac{k_e}{s}$	
Force generator (F_S)	$F(t) = FS(t)$	$F(\omega) = F_S(\omega)$	–	
		$F(s) = F_S(s)$		
Velocity or displacement generator (V_S or X_S)	$\upsilon(t) = VS(t)$	$\upsilon(\omega) = V_S(\omega)$	–	
	$x(t) = XS(t)$	$\upsilon(s) = V_S(s)$		

EXAMPLE 12.8

Find the velocity of the mass in the mechanical system shown in Figure 12.26. The force, F_S, is 5cos(12t) dynes and the mass is 5 g. The mass is supported by a frictionless surface. (Note that force and velocity are defined in the same, arbitrary, direction. Since the force produced by the mass is opposite to the defined velocity direction, it will appear on the opposite side of the equation from F_S.)

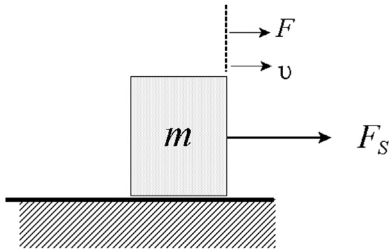


FIGURE 12.26 Mechanical system consisting of a mass with a force applied used in Example 12.8.

Solution: Convert the force to a phasor and apply the appropriate phasor equation from Table 12.8. Solve for $v(\omega)$. Converting the force to phasor notation:

$$5\cos(12t) \quad \Leftrightarrow \quad 5\angle 0 \ \text{dyn}$$

$$F(\omega) = j\omega m v(\omega); \quad v(\omega) = \frac{F(\omega)}{j\omega m}$$

$$v(\omega) = \frac{5\angle 0}{j9(5)} = \frac{5\angle 0}{45\angle 90} = 0.11\angle -90 \ \text{cm/s}$$

Converting back to the time domain (if desired):

$$v(t) = 0.11\cos(12t - 90) = 0.11\sin(12t)\text{cm/s}.$$

12.8.5 Laplace Domain Representations of Mechanical Elements With Nonzero Initial Conditions

The analogy between electrical and mechanical elements holds for initial conditions as well. The two energy storage mechanical elements can have initial conditions that need to be taken into account in the analysis. A mass can have an initial velocity, which will clearly produce a force, and a spring can have a nonzero rest length, which also produces a force.

For a mass, an initial velocity produces a force that is equal to the mass times the initial velocity. Taking the Laplace transform of the equation defining the force–velocity relationship of a mass (Newton's law; Equation 12.54):

$$\mathcal{L}F(t) = \mathcal{L}\left[m\frac{dv(t)}{dt}\right] = m\mathcal{L}\left[\frac{dv(t)}{dt}\right]$$

Applying the Laplace transform equation for the derivative operation, Equation 12.46, the force−velocity relationship of a mass with initial conditions becomes:

$$F(s) = m(sV(s) - v(0)) = smV(s) - mv(0) \tag{12.66}$$

The Laplace representation of a mass with an initial velocity consists of two elements: an impedance term related to the Laplace velocity and a force generator, Figure 12.27A. When solving mechanical systems that contain mass with an initial velocity, the Laplace elements in Figure 12.27A should be used to represent the mass.

The same approach can be used to determine the Laplace representation of a spring with an initial nonzero rest length. Applying the Laplace transform to both sides of the equation defining the force−velocity relationship of a spring (Equation 12.57):

$$\mathcal{L}F(t) = \mathcal{L}\left[k_e \int v dt\right] = k_e \mathcal{L}\left[\int v dt\right]$$

Applying the Laplace transform for an integral operation:

$$F(s) = \frac{k_e}{s}\left(V(s) + \int_{-\infty}^{0} v(t)dt\right) = \frac{k_e}{s}V(s) + \frac{k_e}{s}\int_{-\infty}^{0} v(t)dt$$

The second term is just the initial displacement $\left(x(0) = \int_{-\infty}^{0} v\, dt\right)$ times the spring constant divided by s, so the force−velocity equation becomes:

$$F(s) = \frac{k_e}{s}V(s) + \frac{k_e x(0)}{s}$$

Note that $k_e x(0)$ is also the initial force on the spring. Thus the Laplace representation of a spring having a nonzero rest length again includes two elements: the impedance of the spring in parallel with a force generator having a value of $k_e x(0)/s$, Figure 12.27B.

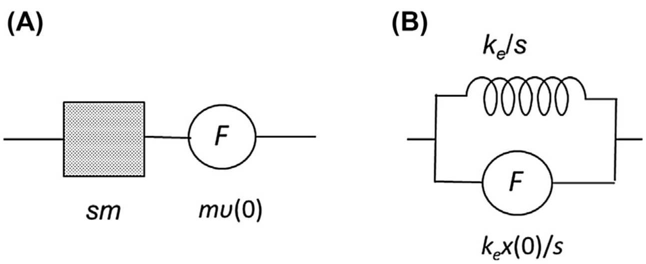


FIGURE 12.27 The Laplace representation of mechanical energy storage elements with nonzero initial conditions. (A) A mass with a nonzero initial velocity is represented in the Laplace domain as a mass impedance plus a series force generator having a value of $mv(0)$. (B) A spring having a nonzero rest length is represented in the Laplace domain as a spring impedance with a parallel force generator of $k_e x(0)/s$, which is also the initial force divided by s.

An example of a mechanical system having energy storage elements with nonzero initial conditions is given next with more examples in the next chapter.

EXAMPLE 12.9

Find the velocity of the mass in Figure 12.28A for $t \geq 0$. Assume the mass is 5 g, the spring constant is 0.4 dyn/cm, and the spring is initially stretched 2 cm beyond its rest length.

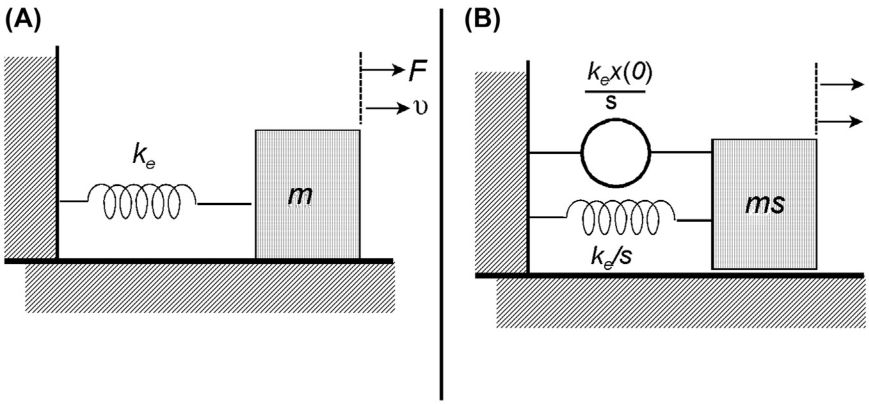


FIGURE 12.28 A mechanical system composed of a mass and a spring used in Example 12.9. The mass is 5 g, the spring constant is 0.4 dyn/cm and the spring is stretched 2 cm beyond its rest length.

Solution: Replace the time variables by their Laplace equivalent and add a force generator to account for the initial condition on the spring, Figure 12.28B. The force generator will have a value of:

$$F_s(0) = \frac{k_e x(0)}{s} = \frac{0.4(2)}{s} = \frac{0.8}{s} \text{ dyn}$$

This initial force is directly applied to both the mass and the spring, so

$$Fs(0) = Fm(s) + Fk(s); \quad \frac{0.8}{s} = msV(s) + \frac{k_e}{s}V(s) = V(s)\left(5s + \frac{0.4}{s}\right)$$

Solving for $V(s)$:

$$V(s) = \frac{\frac{0.8}{s}}{5s + \frac{0.4}{s}} = \frac{0.8}{5s^2 + 0.4} = \frac{0.16}{s^2 + 0.08}$$

This matches entry #6 in the Laplace transform table of Appendix B where $\beta = \sqrt{0.08} = 0.283$.

$$V(s) = 0.565\frac{0.283}{s^2 + 0.283^2} \Leftrightarrow v(t) = 0.565 \sin(0.283t) \text{ dyn/s}$$

The next chapter will explore solutions to more complicated mechanical systems using an algorithmic-like approach that parallels an approach developed for electric circuits.

12.9 SUMMARY

The most complicated electrical and mechanical systems are constructed from a small set of basic elements. These elements fall into two general categories: active elements, which usually supply energy to the system, and passive elements, which either dissipate or store energy. In electrical systems, the passive elements are described and defined by the relationship they enforce between voltage and current. In mechanical systems, the defining relationships are between force and velocity. These relationships are linear and involve only scaling, differentiation, or integration. Active electrical elements supply either a specific voltage and are logically termed voltage sources or a well-defined current and are called current sources. Active mechanical elements are categorized as either sources of force or sources of velocity (or displacement), although mechanical sources are generally far from ideal. All of these elements are defined as idealizations. Many practical elements, particularly electrical elements, approach these idealizations and some of the major deviations have been described.

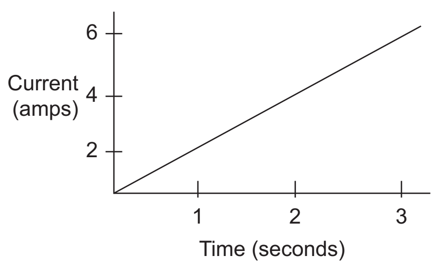
These basic elements are combined to construct electrical and mechanical systems. Since some of the passive elements involve calculus operations, differential equations are required to describe most electrical and mechanical systems. Using phasor or Laplace analysis, it is possible to represent these elements so that only algebra is needed for solution.

PROBLEMS

1. A resistor is constructed of thin copper wire wound into a coil (a "wire-wound" resistor). The wire has a diameter of 1 mm.
   a. How long is the wire required to be to make a resistor of 12 Ω?
   b. If this resistor is connected to a 5-V source, how much power will it dissipate as heat?

2. **a.** A length of size #12 copper wire has a resistance of 0.05 Ω. It is replaced by #16 (AWG) wire. What is the resistance of this new wire?
   b. Assuming both wires carry 2 A of current, what is the power lost in the two wires?

3. The following figure shows the current passing through a 2-h inductor.
   a. What is the voltage drop across the inductor?
   b. What is the energy stored in the inductor after 2 s?



4. The voltage drop across a 10-h inductor is measured as 10 cos (20t) V. What is the current through the inductor?

5. A parallel plate capacitor has a value of 1 μF (10^{-6} F). The separation between the two plates is 0.2 mm. What is the area of the plates?

6. The current waveform shown in Problem 3 passes through a 0.1-F capacitor.

a. What is the equation for the voltage across the capacitor?

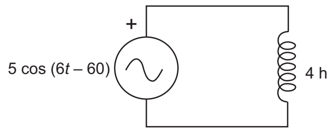
b. What is the charge, q, contained in the capacitor after 2 s (assuming it was unchanged or $t = 0$)?

7. A current of 1 A has been flowing through a 1-F capacitor for 1 s.

a. What is the voltage across the capacitor, and what is the total energy stored in the capacitor?

b. Repeat for a 120-F capacitor.

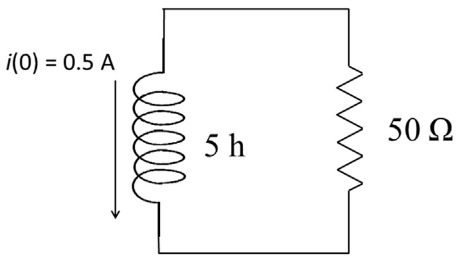
8. In the following circuit, find the value of the current, $i(t)$, through the inductor using the phasor extension of Ohm's law.



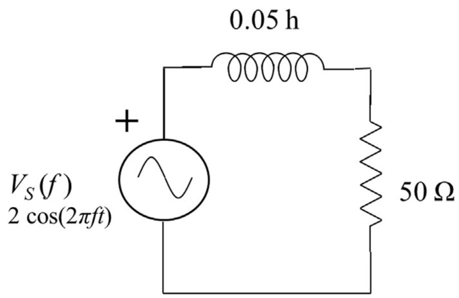
9. The sinusoidal source in Problem 8 is replaced by source that generates a step function at $t = 0$. Find the current, $i(t)$, through the inductor for $t \geq 0$. (Hint: Use the same approach as in Problem 8, but replace the source and inductor by their Laplace representations.)

10. In Problem 8, assume the inductor has an initial current of 2 A going toward ground (i.e., from top to bottom in the schematic). Find the current, $i(t)$, through the inductor for $t \geq 0$. [Hint: The problem is similar except you now have two voltage sources in series. These two sources can be combined into one by algebraic summation (be sure to get the signs right).]

11. The following 5-h inductor has an initial current of 0.5 A in the direction shown. Find the voltage across the 50-Ω resistor. (Hint: This is the same as Example 12.5 except an inductor is used instead of the capacitor. The approach will be the same.)



12. In the following circuit, the voltage source is a sinusoid that varies in frequency between 10 and 10,000 Hz in 10-Hz intervals. Use MATLAB to find the voltage across the resistor for each of the frequencies and plot this voltage as a function of frequency. The plot should be in dB versus log frequency. The resulting curve should look familiar. (Hint: The same current flows thought all three elements. Convert the elements to the phasor representation, write the extended Ohm's law equation, solve for $I(f)$, and solve for $V_R(f)$. In MATLAB, define a frequency vector, `f`, that ranges between 10 and 10,000 in steps of 10 Hz and use `f` to find $V_R(f)$. Since the code that solves for $V_R(f)$ will contain a vector, you need to use the `./`divide operator to perform the division.)



13. In a physiological preparation, the left heart of a frog is replaced by a sinusoidal pump that has a pressure output of $v(t) = \cos(2\pi t)$ mmHg ($f = 1$ Hz). Assuming that the Windkessel model of Figure 12.14 accurately represents the aorta and vasculature system of the frog, what is the resulting blood flow? If the pump frequency is increased to 4 Hz, what is the blood flow?

14. Use MATLAB to find cardiac pressure ($v(t)$ in Figure 12.13) of the Windkessel model used in Example 12.4 to a more realistic waveform of blood flow. In particular, the aortic flow ($i(t)$ in Figure 12.14) should be a periodic function having a period T and defined by:

$$i(t) = \begin{cases} I_o \sin^2\left(\dfrac{\pi t}{T_1}\right) & 0 \le t < \dfrac{T_1}{2} \\ 0 & \dfrac{T_1}{2} \le t < T \end{cases}$$

This function will have the appearance of a sharp half-rectified sine wave. Assume $T_1 = 0.3$ s, the period, T, is 1 s and $I_o = 500$ mL/s. (Hint: Since $i(t)$ is periodic it can be decomposed into a series of sinusoids using the Fourier transform, multiplied by $V(f)/I(f)$, the inverse Fourier transform taken to get the pressure wave $v(t)$. Use a sampling frequency define a 1-s time vector and use it to define $i(t)$. Set values of i(t) above 0.3 s to zero. Because of round off errors, you need to take the real part of $v(t)$ before plotting.)

15. A constant force of 12 dyn is applied to a 5- g mass. The force is initially applied at $t = 0$ when the mass is at rest.

a. At what value of t does the speed of the mass equal 6 dyn/s?

b. What is the energy stored in the mass after 2 s?

16. A force of $12 \cos(6t + 30)$ dyn is applied to a spring having a spring constant of 20 dyn/cm.

a. What is the equation for the velocity of the spring?

b. What is the instantaneous energy stored in the spring at $t = 2.0$ s?

17. A 120-foot length of silver wire having a diameter of 0.02 in. is stretched by 0.5 in. What is the tension (stretching force) on the wire?

18. Use MATLAB to find the velocity of the mass in Example 12.8 for the 5-dyn cosine source where frequency varies from 1 to 40 rad/s in increments of 1 rad/s. Plot the velocity in dB as a function of log frequency in radians. (Hint: The approach is analogous to that of Problem 12. Generate a frequency vector between 1 and 40 using the MATLAB command `w = 1:40`, and then solve for a velocity vector, `vel`, by dividing the source value, 5, by `j*5*w`. Since `w` is a vector you will need to do point-by-point division using the `./` command. Plot the magnitude of the velocity vector.)