Appendix E
Complex Arithmetic

Complex numbers and complex variables consist of two numbers rolled into one. However, they are not completely independent, as most operations affect both numbers in some manner. The two components of a complex number are the real part and the imaginary part, the latter called so because it is multiplied by $\sqrt{-1}$. The imaginary part is denoted by the symbol i in mathematical circles, but the symbol j is used in engineering since i is reserved for current. A typical complex number would be written as: $a + jb$ where a is the real part and jb is the imaginary part.

Complex numbers are visualized as lying on a plane consisting of a real horizontal axis and an imaginary vertical axis, as shown in Figure E.1.

A complex number is one point on the real—imaginary plane and can be represented either in rectangular notation as a real and imaginary coordinate (Figure E.1, dashed lines), or in polar coordinates as a magnitude C and angle θ (Figure E.1, solid line). In this text the polar form is written using a shorthand notation: $C\angle\theta$.

To convert between the two representations, refer to the geometry of Figure E.1. To go from polar to rectangular, apply standard trigonometry:

$$a = C\cos\theta \quad b = C\sin\theta$$

To go in the reverse direction:

$$C^2 = a^2 + b^2$$

$$\tan\theta = \frac{b}{a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

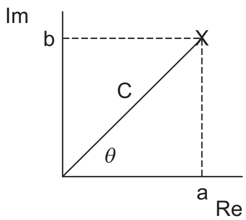

FIGURE E.1 A complex number can be visualized as one point on a plane.

These operations are very useful in complex arithmetic because addition and subtraction are done using the rectangular representation, whereas multiplication and division are easier using the polar form. Care must be taken with regard to signs and the quadrant of the angle, θ.

E.1 ADDITION AND SUBTRACTION

To add two or more complex numbers, add real numbers to real, and imaginary numbers to imaginary:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

Subtraction follows the same strategy:

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

If the numbers are in polar form (or mixed) covert them to rectangular form first.

EXAMPLE E.1

Add the following complex numbers:

$$8\angle -30 + 6\angle 60$$

Solution: Convert both numbers to rectangular form following the above-mentioned rules. Note that the first term is in the fourth quadrant so it will have the general form: $a - jb$.

For the first term:
$a = 8\ \cos(-30)$; $b = 8\ \sin(-30)$;
$a = 6.9$; $b = -4$

For the second term:
$c = 6\ \cos(60) = 3$; $d = 6\ \sin(60) = 5.2$
$\text{Sum} = 6.9 - j4 + 3 + j5.2 = 9.9 + j1.2$

This could then be converted back to polar form if desired.

E.2 MULTIPLICATION AND DIVISION

These arithmetic operations are best done in polar form, although they can be carried out in rectangular notation. For multiplication, multiply magnitudes and add angles:

$$C\angle\theta(D\angle\phi) = CD\angle(\theta + \phi)$$

For division, divide the magnitudes and subtract the angles:

$$\frac{C\angle\theta}{D\angle\phi} = \frac{C}{D}\angle(\theta - \phi)$$

EXAMPLE E.2

Perform the indicated multiplications or divisions:

A. $8\angle -30(6\angle 120)$;
B. $(10 - j6)(1 + j10)$;
C. $\frac{7\angle 305}{6\angle -80}$;
D. $\frac{6\angle 50}{8-j6}$

Solution: For the numbers already in polar coordinates, simply follow the rules given earlier. Otherwise convert to polar form where necessary.

A. $8\angle -30(6\angle 120) = 48\angle 90$
B. $(10 - j6)(5 + j10) = 11.66\angle -31(11.2\angle +63) = 130.5\angle 32$
C. $\frac{7\angle 305}{6\angle -80} = 1.166\angle 385 = 1.166\angle 25$
D. $\frac{6\angle 50}{8-j6} = \frac{6\angle 50}{10\angle -36.9} = 0.6\angle 86.9$

More involved arithmetic operations can call for combinations of these conversions.

EXAMPLE E.3

Add the two complex fractions:

$$\frac{5 + j6}{3 - j7} + \frac{-8 + j6}{-3 - j8}$$

Solution: Convert all the rectangular representations to polar form, carry out the division, then convert back to rectangular form for the addition. When converting from rectangular to polar form, note that each number is in a different quadrant, so care must be taken with the angles.

Evaluate each fraction in turn:

$$\frac{5 + j6}{3 - j7} = \frac{7.8\angle 50}{7.6\angle -67} = 1.03\angle 117 = -0.47 + j0.92$$

$$\frac{-8 + j6}{-3 - j8} = \frac{10\angle 143}{8.5\angle -110} = 1.18\angle 253 = -0.34 - j1.13$$

So the sum becomes:

$$-0.47 + j0.92 - 0.34 - j1.13 = -0.81 - j0.21 = 0.83\angle -165$$

It is a good idea to visualize where each number falls in the real–imaginary plane, or at least what quadrant of the plane, to help keep angles and signs straight.

Multiplication or division by the number j has the effect of rotation of the complex point by $\pm$ 90 degrees. This is apparent if the number j, which is in rectangular form, is converted to polar form: $j = 1\angle 90$. So multiplying or dividing by j adds or subtracts 90 degrees from a number:

$$j(C\angle\theta) = 1\angle 90(C\angle\theta) = C\angle(\theta + 90)$$

$$\frac{C\angle\theta}{j} = \frac{C\angle\theta}{1\angle 90} = C\angle(\theta - 90)$$