CHAPTER

6

Linear Systems in the Frequency Domain: The Transfer Function

6.1 GOALS OF THIS CHAPTER

Just as signals have spectral properties, systems have spectral properties. The difference is that a signal's spectrum describes how energy is distributed over the frequency range of the signal; a system's spectrum (also given as magnitude and phase) shows how a signal's energy distribution is modified as it passes through the system. In other words, the system's spectrum describes the relationship of the output signal's spectrum to the input signal's spectrum. The difference between the output and input signal spectrum is due to the system and is defined by the system's spectrum. Given the input signal's spectrum and the system's spectrum, we can determine the output signal's spectrum. We know there is a direct bilateral relationship between a signal's spectrum and its time-domain representation through the Fourier transform and its inverse, so the output time-domain signal can be constructed by taking the inverse Fourier transform of the output signal's spectrum. So if we know a system's spectrum, we can achieve in the frequency domain the same result that convolution achieves in the time domain: the output of the system to any input.

Why do we need another approach when we already have something that works, and works well? Convolution is a powerful technique for determining the behavior of any system that can be described by an impulse response, but it does not give much insight into the inner workings of these systems. Most systems can be viewed as collections of fundamental elements. In fact, all linear systems can be represented (i.e., modeled) using just a few basic element types. A system's spectrum provides information on the type of elements that make up the system.

We begin this chapter by describing the spectra associated with the different element types. We then show how, when such elements are combined into a system, the system's spectrum can be determined from the spectra of the individual elements. Once we have the system's spectra, we can determine its output to any input. We can also work the other way around: we can estimate the system's spectrum if we have both the input and output

Circuits, Signals, and Systems for Bioengineers
https://doi.org/10.1016/B978-0-12-809395-5.00006-0

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signals spectra. Discovering a system's spectra using the input and output signals is known as "system identification."

Topics covered in this chapter include:

- The basic concept behind systems models and the various types of model elements.
- The response of system elements to sinusoidal stimuli determined using complex sinusoids, an approach called "phasor analysis."
- How and why to construct the system's transfer function (introduced in Section 1.4.5.2).
- The spectral characteristics of basic system elements.
- How to derive the system's frequency response (i.e., spectrum) from the transfer function using an approach termed "Bode plots."
- Linking the transfer function and Fourier transform to find the output to any input signal.
- How to determine the system spectrum given the input and output signals.
- How to estimate the transfer function from the system's spectrum.

6.2 SYSTEMS ANALYSIS MODELS

A systems model is a process-oriented representation that emphasizes the influences, or flow, of information between model elements. A systems model describes how processes interact and what operations these processes perform, but it does not go into details as to how these processes are implemented. The basic element of the system model is a block enclosing a transfer function. Some very general systems-type models are based solely on descriptive (i.e., nonquantitative) elements. The purpose of such models is to show the relationships, or flow of influence, between processes, but they are not amenable to mathematical analysis.

Systems model elements are sometimes referred to as "black boxes" because they are defined only by the transfer function, which is not concerned with, and does not reveal, the inner working of the box. Both the greatest strength and the greatest weakness of system models are found in their capacity to ignore the details of mechanism and emphasize the interactions between the elements.

A mathematical expression defines the input–output relationship of a system element. Since a systems element is just a mathematical relationship, it is represented graphically by a geometrical shape, usually a rectangle, or sometimes a circle when an arithmetic process is involved. Two typical system elements are shown in Figure 6.1. The circle is an arithmetic element that subtracts signal x_2 from signal x_1 as indicated by the plus and minus signs next to the element (If both signs were positive the two signals would be summed.). The rectangular element could represent any general mathematical process. The inputs and outputs of

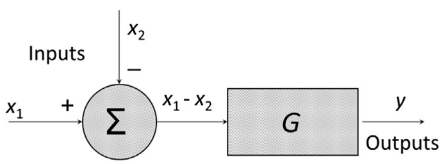


FIGURE 6.1 Typical graphical representation of systems elements. Input and output signals are shown as *arrows* and possible associated variable names.

all elements are signals with a well-defined direction, or flow, or influence. These signals and their direction of influence are shown by lines and arrows connecting the system elements.

The letter G in the right-hand element of Figure 6.1 is a stand-in for the mathematical description of the element: the mathematical operation that converts the input signal into an output signal. Stated mathematically:

$$Output = G(Input) \tag{6.1}$$

Equation 6.1 represents a very general concept: G could be any linear relationship and the terms *Input* and *Output* could be any signal of any complexity. It could represent a single element or an entire system composed of many elements. If it is a black box, how can we know? All we really know is the mathematical relationship. Capital letters are commonly used to represent the mathematical relationship, whereas lower case letters represent the signal.

Rearranging this basic equation, G can be expressed as the ratio of output to input:

$$G = \frac{Output}{Input} \tag{6.2}$$

Equation 6.2 emphasizes that G relates the output of an element to its input; it can be thought of as transferring information from the input to the output giving rise to the term "transfer function." Again, the input–output relationship is the only concern of a system element so the transfer function is a complete description of the element (or an entire system for that matter). Although the transfer function concept is sometimes used very generally, in linear systems analysis it is only an algebraic term (although possibly/probably having complex notation). This implies that linear system elements can be represented as algebraic functions and indeed we show how systems can be represented algebraically even if they contain calculus operators.

Making the transfer function an algebraic function greatly simplifies the analysis of even very complicated systems. For example, if two elements (or systems) are connected together as in Figure 6.2, the transfer function of the paired systems is just the product of the two individual transfer functions:

$$Out_2 = G_2 \times In_2 \text{ but } Out_1 \equiv In_2 = G_1 \times In_1,$$

then combining:

$$Out_2 = G_2 \times (G_1 \times In_1) = G_2 G_1 In_1. \tag{6.3}$$

and the overall transfer function for the two series elements is:

$$\frac{Out_2}{In_1} = G_1 G_2 \tag{6.4}$$

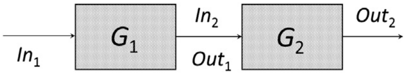


FIGURE 6.2 Two systems or elements connected together with transfer functions G_1 and G_2, respectively. Simple algebra shows that the transfer function of the combined systems is $Out_2/In_1 = G_1G_2$.

Note that the overall transfer function of the combined systems in Figure 6.2 would be the same even if the order of the two systems was reversed. This is a property termed "associativity." We can extend this concept to any number of system elements in series:

$$Output = Input \prod_i G_i \tag{6.5}$$

And the overall transfer function is just the product of the individual series element transfer functions:

$$\frac{Output}{Input} = \prod_i G_i \tag{6.6}$$

The transfer function concept is very powerful and makes determining the input–output characteristics of even the most complex systems easy. In practice, the trick is determining the transfer function of the process of interest. Finding the transfer function of most biological systems is challenging and usually involves extensive empirical observation of the input/output relationship. However, once the relationships for the individual elements have been determined, finding the input/output relationship of the overall system is straightforward.

EXAMPLE 6.1

Find the transfer function for the systems model in Figure 6.3. The mathematical description of each element is either an algebraic term or simply an arithmetic operation. In this system, the element with the Σ identifier does subtraction since one of the input signals has a negative sign. G and H are assumed to be linear algebraic functions so they produce an output that is the product of the function times the input (i.e., Equation 6.2). The system shown is a classic feedback system because the output is coupled back to the input via the lower pathway. In this system, the upper pathway is called the "feedforward pathway" because it moves the signal toward the output. The lower pathway is called the "feedback pathway"; it takes a signal from further along the forward pathway and feeds it back as an input to an element at, or nearer, the system input. Note that the signal in this feedback pathway becomes mixed with the real input signal through the summation operator (actually subtraction, but we still generally call the is summation element). To finish off the terminology lecture, G in the configuration is called the "feedforward gain" and H is called the "feedback gain."

Solution: Generate an algebraic equation based on the configuration of the system and the fact that the output of each process is the input multiplied by the associated gain term (Equation 6.2 $Output = G \times Input$).

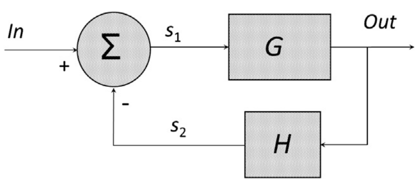


FIGURE 6.3 Systems model used in Example 6.1. This is an example of a classic feedback control system.

For the upper box: $G = \frac{Out}{s_1}$

and for the lower box: $H = \frac{s_2}{Out}$

Rearranging the two equations: $s_1 = \frac{Out}{G}$ and $s_2 = Out\ (H)$

Since: $s_1 = In - s_2$; Substituting: $\frac{Out}{G} = In - Out(H)$

Rearranging: $Out = In(G) - Out(GH)$; $Out(1 + GH) = In(G)$

$$TF = \frac{Out}{In} = \frac{G}{1 + GH} \tag{6.7}$$

Analysis: The solution given in Equation 6.7 is known as the "feedback equation" and will be used in subsequent analyses of more complex systems. In this example, the two elements, G and H, could be anything as long as they can be treated algebraically and we show later how all elements, even calculus operations, can be transformed into algebraic functions.

Figures 6.1 and 6.2 show another important property of systems models. The influence of one process on another is explicitly stated and indicated by the line connecting two processes. This line has a direction usually indicated by an arrow, which implies that the influence or information flow travels only in that direction. If there is also a reverse flow of information, such as in feedback systems, this must be explicitly stated in the form of an additional connecting line showing information flow in the reverse direction.

As described in Chapter 1, analog models represent physiological processes using elements that are, to some degree, analogous to those in the actual processes. Good analog models can represent systems at a lower level, and in greater detail, than system models. Analog models provide better representation of secondary features such as energy use, which is usually similar between analog elements and the components they represent. However, in an analog model, the interaction between components may not be obvious from inspection of the model.

System models emphasize component interaction, particularly information flow. They also provide a clear illustration of a system's overall organization. This can be of great benefit in clarifying the control structure of a complex system. These models explain what an element does with its input or stimulus, but they give no clue as to how it does it. Perhaps the most significant advantage of the systems approach is what it does not represent: it allows the behaviors of biological processors to be quantitatively described without requiring the modeler to know the details of the underlying physiological mechanism. Given our lack of understanding of the details of some biological processes, this can be a considerable blessing.

6.3 THE RESPONSE OF SYSTEM ELEMENTS TO SINUSOIDAL INPUTS: PHASOR ANALYSIS

If the signals or variables in a system are sinusoidal or are converted to sinusoids using the Fourier transform, then a technique known as phasor analysis can be used to convert calculus operations into algebraic operations. As used here, the term "phasor analysis" is considerably more mundane than the name implies: the analysis of phasors such as those used on Star Trek is, unfortunately, beyond the scope of this text. The phasor analysis we will learn about here combines complex representation of sinusoids with the fact that calculus operations (integrations and differentiation) change only the magnitude and phase of a sinusoid. This analysis assumes the signals are in sinusoidal steady state; that is, they always have been and always will be sinusoidal.

There are four basic systems elements: arithmetic operators, scale operation, differentiators, and integrators. Since sinusoidal frequency is not altered by these operations, if the input(s) to a system are sinusoids, then all signals in the system will be sinusoidal at the same frequency (sinusoidal Property 6 in Chapter 3). Then all signals in a linear, time invariant (LTI) system can be described by the same general equation:

$$x(t) = A\cos(\omega t + \theta) = A\cos(2\pi ft + \theta) \tag{6.8}$$

where the values of A and θ can be modified by the system elements, but the value of ω (or f) will be the same throughout the system.

Sinusoids require just three variables for complete description, amplitude, phase, and frequency (sinusoidal Property 2 in Chapter 3). But if the frequency is always the same, then we really only need to keep track of two variables: amplitude[1] and phase. Here is where complex notation sounds like it could be useful since a single complex variable is actually two variables rolled into one (i.e., $a + jb$). A single complex number or variable is all that is needed to describe the amplitude and phase of a sinusoid.

To find how to represent a sinusoid by a single complex variable, we return to the complex representation of sinusoids given by Euler's equation:

$$e^{jx} = \cos x + j\sin x \tag{6.9}$$

or for sinusoidal signals such as Equation 6.8:

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t + \theta) + jA\sin(\omega t + \theta) \tag{6.10}$$

Comparing the basic sinusoid equation, Equation 6.8, with Equation 6.10 indicates that only the real part of e^{jx} is needed to represent a sinusoid:

$$A\cos(\omega t + \theta) = \mathrm{Re}Ae^{j(\omega t+\theta)} = \mathrm{Re}Ae^{j\theta}e^{j\omega t} \tag{6.11}$$

If all variables in an equation contain the real part of the complex sinusoid, the real terms can be dropped. Consider,

If, $\mathrm{Re}Ae^{j\theta_1}e^{j\omega t} = \mathrm{Re}Be^{j\theta_2}e^{j\omega t}$ for all t,

Then, $A = B$ and[2]

$$Ae^{j\theta_1}e^{j\omega t} = Be^{j\theta_2}e^{j\omega t} \tag{6.12}$$

[1]Some engineers use the word "amplitude" when referring to the peak value of the sinusoid and the word "magnitude" when referring to the rms value. I use these words interchangeably.

[2]In general, A does not necessarily equal B when Re A = Re B, but the only way that Re $Ae^{j\omega t}$ can equal the Re $Be^{j\omega t}$ at all values of t is when $A = B$. Appendix E presents a review of complex arithmetic.

Since all variables in a sinusoidally driven LTI system are the same except for amplitude and phase, they will all contain the "Re" operator, and these terms can be removed from the equations as was done in Equation 6.12. They do not actually cancel; they are just unnecessary since the equality stands just as well without them. Similarly, since all variables will be at the same frequency, identical $e^{j\omega t}$ terms will appear in each variable and will cancel once the Re's are dropped. Therefore, the two defining sinusoidal variables in Equation 6.8, A and θ, can be represented by a single complex variable:

$$A\cos(\omega t+\theta)\Leftrightarrow Ae^{j\theta} \tag{6.13}$$

where $Ae^{j\theta}$ is the "phasor" representation of a sinusoid.

Equation 6.13 does not indicate a mathematical equivalence, but a transformation from the standard sinusoidal representation to a complex exponential representation without loss of information. In the phasor representation, the frequency, ω, is not explicitly stated, but is understood to be associated with every variable in the system (sort of a virtual variable). Note that the phasor variable, $Ae^{j\theta}$, is in polar form as opposed to the rectangular form $(a+jb)$ and you may need to convert it to the rectangular representation in some calculations. Since the phasor, $Ae^{j\theta}$, is defined in terms of the cosine (Equation 6.11), sinusoids defined in terms of sine waves must be converted to cosine waves when using this analysis.

If phasors offered only a more succinct representation of a sinusoid, their usefulness would be limited. It is their calculus-friendly behavior that endears them to engineers. To determine the derivative of the phasor representation of a sinusoid, we return to the original complex definition of a sinusoid (i.e., Re $Ae^{j\theta}e^{j\omega t}$):

$$\frac{d(\mathrm{Re}Ae^{j\theta}e^{j\omega t})}{dt}=\left(\mathrm{Re}j\omega Ae^{j\theta}e^{j\omega t}\right) \tag{6.14}$$

The derivative of a sinusoid in complex notation is the original complex variable, but multiplied by $j\omega$. After the Re operators are dropped, the $e^{j\omega t}$'s cancel, and the derivative of a phasor becomes multiplication by $j\omega$. For phasors, the derivative operation reduces to a simple arithmetic operation:

$$\frac{d}{dt}\Leftrightarrow j\omega \tag{6.15}$$

Similarly, integration using the complex definition of a sinusoid:

$$\int \mathrm{Re}Ae^{j\theta}e^{j\omega t}dt=\mathrm{Re}\frac{Ae^{j\theta}e^{j\omega t}}{j\omega} \tag{6.16}$$

Again, integration of a sinusoid in complex notation gives rise to the same complex variable except divided by $j\omega$. Integration applied to a phasor is accomplished by dividing by $j\omega$, again an arithmetic operation:

$$\int dt\Leftrightarrow\frac{1}{j\omega} \tag{6.17}$$

The basic rules of complex arithmetic are covered in Appendix E; however, a few properties of the complex operator j are important enough to be repeated here. Note that $1/j = -j$, since:

$$\frac{1}{j} = \frac{1}{\sqrt{-1}} = \frac{-\sqrt{-1}}{(-\sqrt{-1})(-\sqrt{-1})} = \frac{-\sqrt{-1}}{-(-1)} = -\sqrt{-1} = -j \tag{6.18}$$

So Equation 6.17 could also be written as:

$$\int dt \Leftrightarrow -j/\omega \tag{6.19}$$

Multiplying by j in complex arithmetic is the same as shifting the phase by 90 degrees, which follows directly from Euler's equation:

$$je^{jx} = j(\cos x + j\sin x) = j\cos x - \sin x = -\sin x + j\cos x$$

Substituting in $\cos(x+90)$ for $-\sin x$, and $\sin(x+90)$ for $\cos x$, je^{jx} becomes[3]:

$$je^{jx} = \cos(x+90) + j\sin(x+90)$$

This is the same as e^{jx+90}, which can also be written as:

$$je^{jx} = e^{jx}e^{90} \tag{6.20}$$

Similarly, dividing by j is the equivalent of shifting the phase by -90 degrees:

$$\frac{e^{jx}}{j} = \frac{\cos x + j\sin x}{j} = \frac{\cos x}{j} + \sin x = -j\cos x + \sin x$$

Substituting in $\cos(x-90)$ for $\sin x$, and $\sin(x-90)$ for $-\cos x$:

$$\frac{e^{jx}}{j} = \cos(x-90) + j\sin(x-90) = e^{jx}e^{-90} \tag{6.21}$$

Equations 6.15, 6.17, and 6.19 demonstrate the benefit of representing sinusoids by phasors: the calculus operations of differentiation and integration become the algebraic operations of multiplication and division. Moreover, the transformation used to convert a time-domain sinusoid to a phasor (Equation 6.13), or vice versa, is super easy.

[3]Note that we are using degrees in these equations, although the mathematics would normally be carried out in radians. I feel it is easier to visualize the term "90 degrees" than "$\pi/2$." In any case, the conclusions reached by these equations are the same.

EXAMPLE 6.2

Find the derivative of $x(t) = 10\cos(2t + 20)$ using phasor analysis.

Solution: Convert $x(t)$ to a phasor (represented as $x(j\omega)$). To take the derivative, multiply by $j\omega$, then take convert the resulting phasor back to a sinusoid:

$$10\cos(2t+20) \Leftrightarrow 10e^{j20}$$

$$\frac{dx(j\omega)}{dt} \Leftrightarrow j\omega\left(10e^{j20}\right) = j2\left(10e^{j20}\right) = j20e^{j20}$$

$$j20e^{j20} = 20e^{j20}e^{j90} \Leftrightarrow 20\cos(2t+20+90) = 20\cos(2t+110)$$

Since $\cos(x) = \sin(x + 90) = -\sin(x - 90)$, this can also be written as $-20\sin(2t + 20)$, which is what you would get from standard differentiation. Note that the frequency ($\omega = 2$) is not explicitly stated in the phasor solution (i.e., $20e^{j20}e^{j90}$), but is reinserted when converting back to sinusoidal representation. Again, an explicit representation of frequency is unnecessary since frequency is the same for all elements in the system.

A shorthand notation is common for the phasor description of a sinusoid. Rather than write $Ve^{j\theta}$, we simply write $V\angle\theta$ (stated as "V at an angle of θ"). When a time variable such as $v(t)$ is converted to a phasor variable, it is common to write it as a function of ω using capital letters for the amplitude, i.e., $V(\omega)$. This acknowledges the fact that phasors represent sinusoids at specific frequencies, even though the sinusoidal term, $e^{j\omega t}$, is not explicit in the phasor itself. Also, in phasor analysis, it is common to represent frequency as rad/s (i.e., ω) rather than as Hz, even though Hz is generally used in engineering settings. Putting these conventions together, the time-to-phasor transformation for variable $v(t)$ can be stated as:

$$v(t) \Leftrightarrow V(\omega) = V\angle\theta \tag{6.22}$$

In this notation, the phasor representation of $20\cos(2t + 110)$ would be written as $20\angle110$ rather than $20\ e^{j110}$. Sometimes, the phasor representation of a sinusoid expresses the amplitude of the sinusoid in root mean square (rms) values rather than peak-to-peak values, in which case the phasor representation of $20\cos(2t + 110)$ would be written as $(0.707)\ 20\angle110 = 14.14\angle110$. Here we use peak-to-peak values, but it really does not matter as long as we are consistent.

The phasor approach is an excellent method for simplifying the mathematics of LTI systems. It can be applied to all systems that are driven by sinusoids and, in conjunction with the Fourier transform, systems driven by periodic, or aperiodic, signals.

6.4 THE TRANSFER FUNCTION

The transfer function mathematically transfers the input to an output. The transfer function concept is so compelling that it has been generalized to include many different types of processes or systems with different types of inputs and outputs. In the phasor domain, the transfer function is a minor modification of Equation 6.2:

$$\text{Transfer Function}(\omega) = \frac{Output(\omega)}{Input(\omega)} \tag{6.23}$$

Frequently, both *Input*(ω) and *Output*(ω) are signals measured in volts and represented by symbols such as $v_{in}(\omega)$ and $v_{out}(\omega)$, but for now we continue to use general terms. By strict definition, the transfer function should always be a function of ω, f, or, as shown in the next chapter, the Laplace variable, s, but the idea of expressing the behavior of a process by its transfer function is so powerful that it is sometimes used in a nonmathematical, conceptual sense.

EXAMPLE 6.3

The system shown in Figure 6.8 is a simplified, linearized model of the Guyton–Coleman body fluid balance system presented by Ridout (1991). This model describes how the arterial blood pressure P_A in mmHg responds to a small change in fluid intake, F_{IN}, in mL/min. Find the transfer function for this system, $P_A(\omega)/F_{IN}(\omega)$ using phasor analysis.

Solution: The two elements in the upper path can be combined into a single element using Equation 6.4, which states that the transfer function of the combined element is the product of the two individual transfer functions:

$$G(\omega) = G_1(\omega)G_2(\omega) = \frac{0.06}{j\omega}(16.67) = \frac{1}{j\omega}$$

The resulting system has the same configuration as the feedback system in Figure 6.3, which has already been solved in Example 6.1 (Equation 6.7). We could go through the same algebraic process, but it is easier to use the fact that the system in Figure 6.4 has the same configuration as the feedback system in Figure 6.3, where the transfer function of G is $1/j\omega$ and the feedback gain, H, is 0.05.

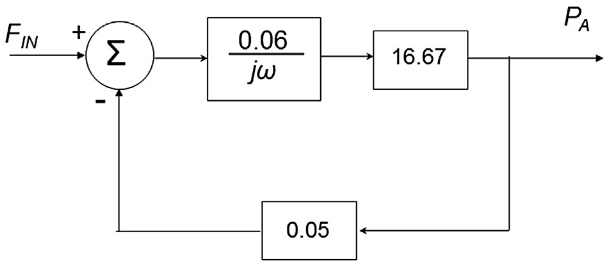


FIGURE 6.4 A simplified linear version of the Guyton–Coleman fluid balance system that relates arterial blood pressure, P_A, to small changes in fluid intake, F_{IN}. The transfer function of this model is developed in Example 6.3.

Substituting the expressions for G and H into the feedback equation, Equation 6.7, gives the transfer function of this system:

$$\frac{Out(\omega)}{In(\omega)} = \frac{G}{1+GH} = \frac{\frac{1}{j\omega}}{1+0.05\left(\frac{1}{j\omega}\right)} = \frac{1}{0.05+j\omega} = \frac{20}{1+j20\omega}\,\frac{\text{mmHg}}{\text{mL/min}}$$

Analysis: This transfer function applies to any input signal at any frequency, ω, as long as it is sinusoidal steady state. Note that the denominator consists of a real and an imaginary part, and that the constant term is normalized to 1.0. This is the common format for transfer function equations in the frequency domain: the lowest power of ω, usually a constant term, is normalized to 1.0.

Systems having transfer functions with a $1 + jk\omega$ term in the denominator, where k is a constant, are called "first-order" systems and are discussed, along with other common systems, later in this chapter. The output of this system to a specific sinusoidal input is determined in the next example.

Many of the transfer functions encountered in this text have the same units for the numerator and denominator terms (e.g., volts/volts), so these transfer functions are dimensionless. However, when this approach is used to represent physiological systems, the numerator and denominator often have different units as is the case here (i.e., the transfer function has the dimensions of mmHg/mm/min).

EXAMPLE 6.4

Find the output of the system in Figure 6.4 if the input signal is $F_{IN}(t) = 0.6\ \sin(0.3t + 20)$ mL/min.

Solution: Since phasors are based on the cosine, we first need to convert $F_{IN}(t)$ to a cosine wave. From Appendix C, Equation 6.4: $\sin(\omega t) = \cos(\omega t - 90)$, so $F_{IN}(t) = 0.6\ \cos(0.3t - 70)$ mL/min.

In phasor notation the input signal is:

$$F_{IN}(t) = 0.6\cos(0.3t - 70) \Leftrightarrow F_{IN}(\omega) = 0.5\angle -70 \ \text{ mL/min}$$

In Example 6.3, we found the transfer function as:

$$\frac{Out(\omega)}{In(\omega)} = \frac{20}{1 + j20\omega}$$

Solving for $Out(\omega)$ and then substituting in $0.5\angle -70$ for $In(\omega)$ and letting $\omega = 0.3$

$$Out(\omega) = \frac{20}{1 + j20\omega} In(\omega) = \frac{20}{1 + j20(0.3)\omega}(0.5\angle -70) = \frac{10\angle -70}{1 + j6\omega}$$

The rest of the problem is just working out the complex arithmetic. To perform division (or multiplication), it is easiest to have a complex number in polar form. The number representing the input, which is also the numerator, is already in polar form ($6\angle -70$), so the denominator needs to be converted to polar form:

$$1 + j6 = \sqrt{1^2 + 6^2}\angle \tan^{-1}\left(\frac{6.1}{1}\right)\left(\frac{360}{2\pi}\right) = 6.08\angle 80$$

Result: Note that $\tan^{-1}(6.1)$ has been converted to degrees by multiplying by $360/2\pi$. Substituting and solving:

$$P_A(\omega) = \frac{10\angle -70}{6.08\angle 80} = 1.64\angle -150 \ \text{ mmHg}$$

Converting to the time domain:

$$P_A(t) = 1.64\cos(0.3t - 150) \ \text{ mmHg}$$

So the arterial pressure response to this input is a sinusoidal variation in blood pressure of 1.64 mmHg. That variation has a phase of −150 degrees, which is shifted 80 degrees from the input phase of −70 degrees (using the cosine form of the input signal). The corresponding time delay between the stimulus and response sinusoid can be found from Equation 2.21:

$$t_d = \frac{\theta}{360f} = \frac{80}{360\left(\frac{0.3}{2\pi}\right)} = 4.65 \text{ min}$$

So the response is delayed 4.65 min from the stimulus because of the long delays in transferring fluid into the blood. As with many physiological systems, generating a sinusoidal stimulus, in this case a sinusoidal variation in fluid intake, is difficult. The next example shows the application of phase analysis to a more complicated system.

EXAMPLE 6.5

Find the transfer function of the system shown in Figure 6.5. Find the time-domain output if the input is 10 cos (20t + 30).

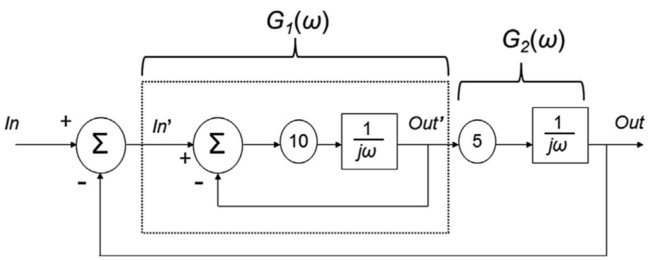


FIGURE 6.5 The system used in Example 6.5.

Solution: Since we do not already have a transfer function for this configuration of elements, we need to find the transfer function from the element equations and then use this transfer function to solve for the output in terms of the input. In all of these problems, the best strategy is to find the equivalent feedforward gain, G, and then apply the feedback equation to find the transfer function. To facilitate this strategy, some internal signals are labeled for reference. We observe that the elements inside the dashed rectangle constitute an internal feedback system with input In' and output Out'. So we can use the basic feedback equation, Equation 6.7, to get the transfer function of this internal subsystem. For this subsystem, the feedforward gain is a combination of the scalar and integrator: $G' = 10\left(\frac{1}{j\omega}\right)$ and $H = 1$. So the transfer function for the subsystem becomes:

$$G_1 = \frac{Out'}{In'} = \frac{\frac{10}{j\omega}}{1 + \frac{10}{j\omega}} = \frac{10}{j\omega + 10} = \frac{1}{1 + j0.1\omega}$$

Following convention, the denominator is normalized so the constant term (or lowest power of ω) is 1. The rest of the feedforward path consists of a scalar and integrator, which can be represented by transfer function G_2.

$$G_2 = 5\left(\frac{1}{j\omega}\right) = \frac{5}{j\omega}$$

The transfer function of the feedforward gain is the product of G_1 and G_2:

$$G = G_1G_2 = \left(\frac{5}{j\omega}\right)\left(\frac{1}{1+j0.1\omega}\right) = \frac{5}{-0.1\omega^2 + j\omega}$$

We apply the feedback equation yet again, this time to find the transfer function of the overall system. Substituting G above into the feedback equation and letting $H = 1$:

$$\frac{Out}{In} = \frac{G}{1+GH} = \frac{\dfrac{5}{-0.1\omega^2 + j\omega}}{1 + \dfrac{5}{-0.1\omega^2 + j\omega}} = \frac{5}{-0.1\omega^2 + j\omega + 5} = \frac{1}{1 - 0.1\omega^2/5 + j\omega/5}$$

Rearranging the denominator to collect the real and imaginary part and normalizing the constant term to 1:

$$\frac{Out}{In} = = \frac{1}{1 - 0.02\omega^2 + j0.2\omega}$$

This type of transfer function is known as a "second-order function," as the polynomial in the denominator includes a second-order frequency term, ω^2.

Finding the output given the input is a matter of working through some complex algebra. In phasor notation the input signal is $10\angle 30$ with $\omega = 20$. The output in phasor notation becomes:

$$Out = TF\ In = \frac{10\angle 30}{1 - 8 + j4} = \frac{10\angle 30}{-7 + j4} = \frac{10\angle 30}{8\angle 150} = 1.24\angle -120$$

Converting to the time domain: $out(t) = 1.24\cos(20t - 120)$.

6.4.1 The Spectrum of a Transfer Function

One of the best ways to examine transfer functions, and the behavior of the systems they represent, follows the approach used for signals: find the spectral characteristics. The transfer function specifies how a system transforms the input signals into output signals as a function of frequency. Specifically, the transfer function shows how the amplitudes and phases of sinusoids at different frequencies are modified as they pass through the system. So plotting the magnitude and phase of the complex transfer function as a function of frequency provides a complete description of how a system modifies the signal passing through it.

The signal spectrum describes a signal's magnitude and phase characteristics as a function of frequency. The system spectrum describes how the system changes signal magnitude and phase as a function of frequency. For example, Figure 6.6 shows the magnitude and phase spectra of some hypothetical system. At the lower frequencies, below around 80 Hz, the magnitude spectrum is 1.0. This means that sinusoids from 0.0 to around 80 Hz pass through the system without a change in amplitude: they emerge from the system at the same amplitude as when they went in. This does not mean that the output will contain energy at those frequencies if there is no energy in the input signal, but if frequencies in that range exist in the input they will be unchanged in the output. At the higher frequencies the system magnitude spectrum diminishes, so any input signal frequencies in that range will be reduced at the output. For example, at around 100 Hz the transfer function has a magnitude value of around

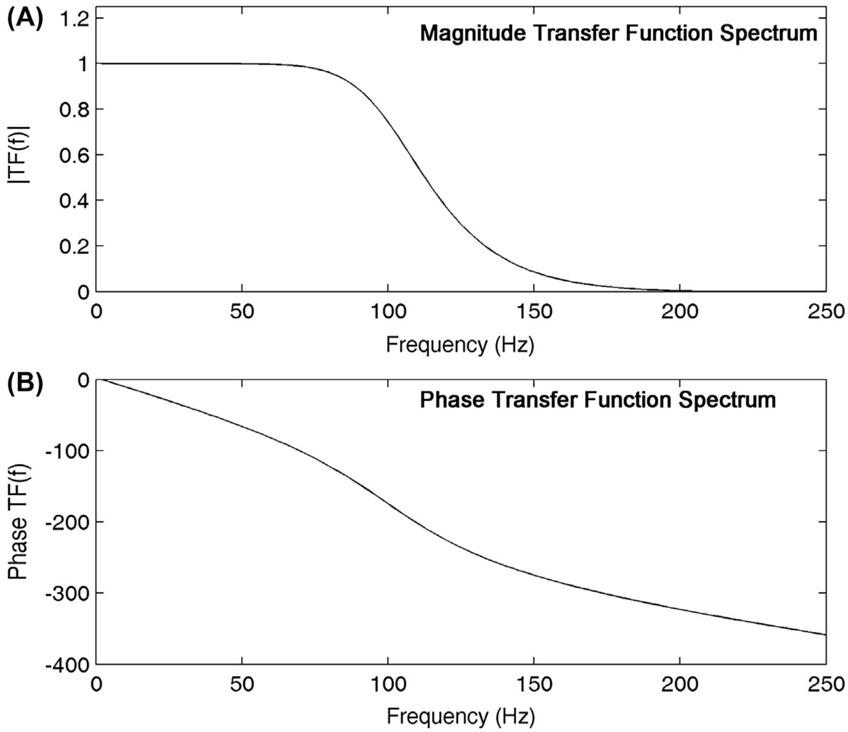


FIGURE 6.6 The magnitude (A) and phase (B) spectra of a hypothetical system. These spectra show the changes over frequency in the magnitude and phase of sinusoids as they pass (are transformed) from input to output.

0.707.[4] That means that the amplitude of a 100 Hz sinusoid will be reduced by a factor of 0.707 as it goes through the system. Moreover, the energy in the output signal will be reduced by half. (Remember energy is proportional to magnitude squared so when the magnitude is reduced by 0.707, the energy is reduced by 0.707^2 or 0.5.)

Following the same logic, unlike the phase spectrum of a signal, the phase plot does not show the phase angle of the system's output; rather it shows the change in the phase angle induced by the system. For this system, sinusoids would emerge from the system with a more negative phase angle, that angle becoming increasingly more negative at higher frequencies.

If the entire spectral range of a system is explored, then the transfer function spectrum provides a complete description of the way in which the system alters all possible input frequencies. Transfer function properties of systems are usually represented as frequency-domain plots because it is easier to conceptualize the operation of a system from spectral plots as in Figure 6.6 than as a complex mathematical function. It is easy to generate the frequency characteristics of any transfer function using MATLAB, as shown in the next two examples.

[4]It may be hard to tell from the figure, but the system really has a cutoff frequency of 100 Hz. The definitions for bandwidth in Section 4.6 apply directly to systems.

EXAMPLE 6.6

Use MATLAB to find and plot the magnitude and phase spectra of the transfer function below (also a second-order transfer function similar to that in Example 6.5):

$$\frac{Out}{In} = = \frac{1}{1 - 0.03\omega^2 + j0.01\omega}$$

Solution: After defining a frequency vector, `w`, the transfer function can be coded directly in MATLAB. The magnitude and phase are determined by applying the `abs` and `angle` routines. We have to guess the frequency range, and increments for `w`. We want a frequency range that will show the most interesting portion of the spectrum. A range of 0.1—1000 rad/s in increments of 0.1 rad/s worked well.

```
% Example 6.6 Use MATLAB to plot a second-order transfer function
%
w = .1:.1:1000;                          % Select frequency range an increment
TF = 1./(1 - .03*w.^2 + .01*j*w);        % Define transfer function
Mag = 20*log10(abs(TF));                 % Magnitude spectrum in dB
Phase = unwrap(angle(TF))*360/(2*pi);  % Phase spectrum in deg
%
semilogx(w,Mag);
..............label and grid, repeat for phase spectrum......
```

Results: Note how easy it is to define the transfer function in MATLAB. It is common to plot the system magnitude spectrum on a log-log scale, that is, in dB (a log scale) versus log frequency (using `semilogx`). The phase spectrum is plotted in degrees versus log frequency. Figure 6.7 shows

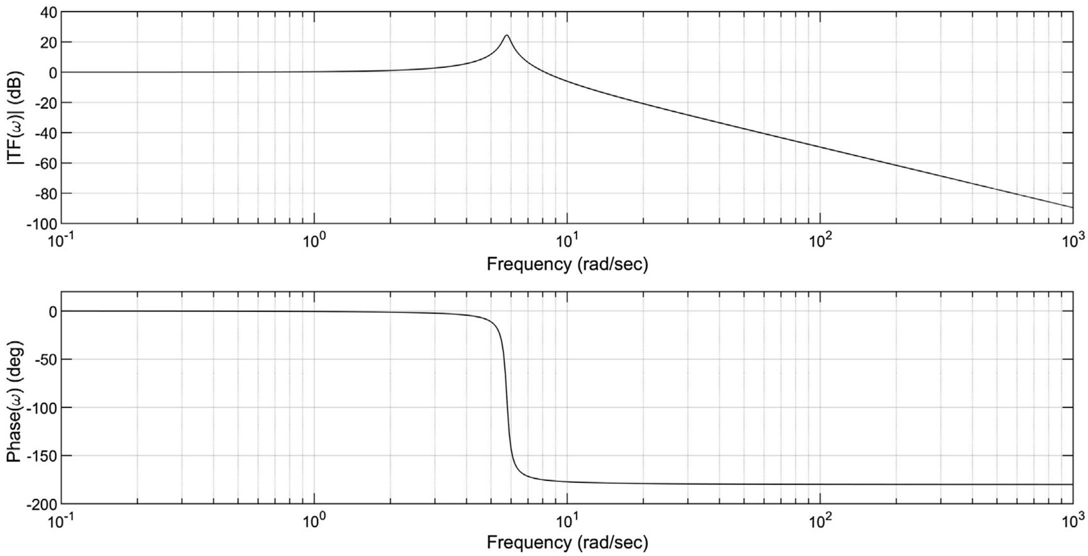


FIGURE 6.7 The magnitude and phase spectra of a second-order system. The magnitude is in dB, the phase in deg, and the frequency in log radians. Note that inputs with frequency components around 6 rad/s are enhanced in the output by 20 dB (i.e., a factor of 10).

the resulting magnitude and phase spectra. Here the plot is in radians, but it would have been easy to convert to frequency in Hz. The magnitude spectrum shows that the amplitude of input sinusoids having frequencies around 6 rad/s will be enhanced by 20 dB (a factor of 10 on a linear scale) in the output. The phase of output sinusoids will be roughly the same as that of input sinusoids up to that frequency, but as input sinusoidal frequencies increase, the output shows a −180-degree phase shift.

EXAMPLE 6.7

The transfer function in this example represents the relationship between the applied external pressure and airway flow for a person on a respirator in an intensive care unit. This transfer function was derived by Koo (2000) for typical lung parameters, but could be modified for specific patients. The original model is in the Laplace domain (described in the next chapter), but has been modified to the frequency domain for this example. We want to find the magnitude and phase spectrum of this airway system. We use MATLAB to plot the magnitude in dB and phase spectrum in deg, both against log frequency, this time in Hz.

$$\frac{Q(\omega)}{P(\omega)} = = \frac{9.52j\omega(1 + j0.0024\omega)}{1 - 0.00025\omega^2 + j0.155\omega}\frac{\text{L/min}}{\text{mmHg}}$$

where $Q(\omega)$ is airway flow in L/min and $P(\omega)$ is the pressure applied by the respirator in mmHg. Traditional engineering systems usually have the same units for the input(s) and output(s), for example, and the electronic system would have the input and the output in volts. These input and output units cancel, so the transfer function is dimensionless. However, transfer functions for biological systems often have inputs and outputs in different units, as is the case here, where the input is a pressure in mmHg and the output is a flow in L/min. In this case the transfer function has a dimension: L/(min mmHg).

Solution: To plot the spectra we follow the same procedure used in the last example. For the frequency vector, we originally tried a range of 0.1 to 1000 rad/s in steps of 0.1 rad, but then extended it to range between 0.001 and 10,000 in steps of 0.001 to better show the lower frequencies. Note that in this model, time is in minutes not seconds, so the plotting frequency needs to be modified accordingly. Since there are 60 s in a minute, the plotting frequency should be divided by 60 (as well as by 2π to convert from rad to Hz).

```
% Example 6.7 Use MATLAB to plot the given transfer function
%
w = .001:0.001:1000;                       % Construct frequency vector
TF = (9.62*j*w.*(1+j*w/420))./(1 - (w.^2)/4000 + .166*j*w);  % Define TF
Mag = 20*log10(abs(TF));                   % Magnitude of TF in dB
Phase = unwrap(angle(TF))*360/(2*pi);      % Phase of TF in deg
semilogx(w/(60*2*pi),Mag);                 % Plot |TF| in log freq in Hz.
..... labels, grid, and repeat for phase spectrum
```

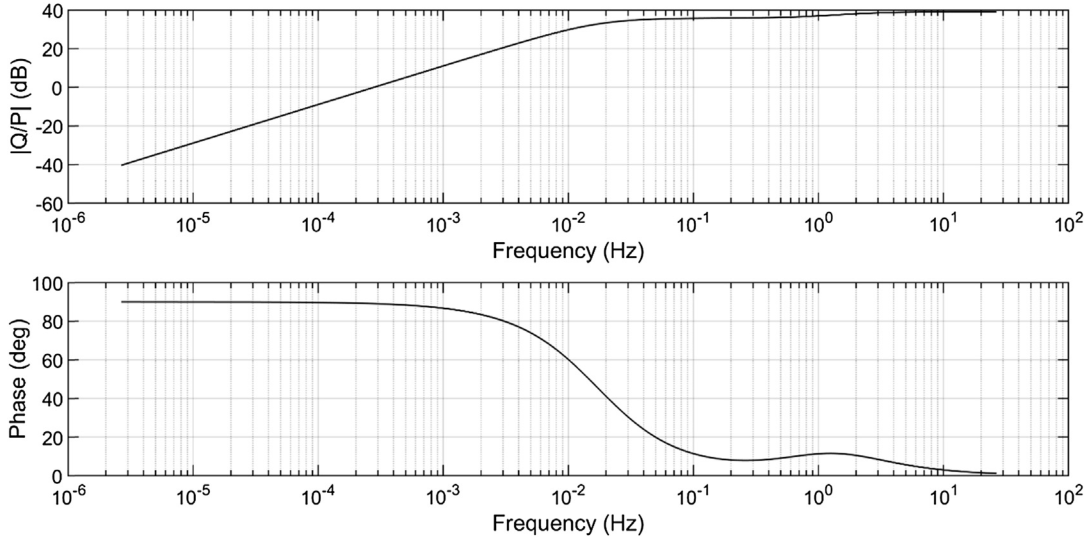


FIGURE 6.8 Magnitude and phase plot of a model that links airway flow to the air pressure in a respirator. The relationship between airway flow and respirator pressure is fairly constant over the range of normal respiratory rates (around 0.18 Hz).

Results: The spectral plots for this respiratory-airway system are shown in Figure 6.8. The flow is constant for a given external pressure for frequencies above 1 Hz (corresponding to 60 breaths/min), but decreases for slower breathing. However, in the normal respiratory range (11 breaths/min = 0.18 Hz), the magnitude spectrum is also constant and nearly the same. This shows that the relationship between airway flow and respiratory flow remains about the same for physiological respiratory rates.

Relegating the problem to the computer is easy but more insight is found by working it out manually. To really understand the frequency characteristics of transfer functions, it is necessary to examine the typical components of a general transfer function. In so doing, we learn how to plot transfer functions without the aid of a computer. More importantly, by examining the component structure of a typical transfer function, we also gain insight into what the transfer function actually represents. This knowledge is often sufficient to allow us to examine a process strictly within the frequency domain and learn enough about the system that we need not bother with time-domain responses.

6.5 THE SPECTRUM OF SYSTEM ELEMENTS: THE BODE PLOT

To dig into the meaning of transfer functions and be able to determine their spectral plots directly (i.e., without using MATLAB), we start by identifying the spectral characteristics of the four basic elements. These spectral characteristics are termed "Bode plot primitives." We then look at a couple of popular configurations of these elements. The arithmetic element (summation and subtraction) does not really have a spectrum, as it performs its operations the same way at all frequencies. Learning the spectra of the other elements is not difficult, but combining the various spectral elements into a spectrum that represents the overall system can be a bit tedious. For engineers, this depth is necessary as it results in increased understanding of systems and their design.

6.5.1 Constant Gain Element

The constant gain element shown in Figure 6.9 has already been introduced in previous examples. Since this element simply scales the input to produce the output, the transfer function for this element is a constant called the "gain." When the gain is greater than 1, the output is larger than the input and vice versa. In electronic systems, a gain element is called an "amplifier."

The transfer function for this element is:

$$TF(\omega) = G \tag{6.24}$$

The output of a gain element is $y(t) = G\,x(t)$ and hence only depends on the instantaneous and current value of t. Since the output does not depend on past values of time, it is called a "memoryless" element.

In the spectrum plot, the transfer function magnitude is usually plotted in dB so the transfer function magnitude equation is:

$$|TF(\omega)|_{dB} = |20 \log TF(\omega)| = 20 \log G \tag{6.25}$$

The magnitude spectrum of this element is a horizontal line at 20 log G. If there are other elements contributing to the spectral plot, it is easiest simply to rescale the vertical axis so that the former zero line equals 20 log G. This rescaling will be shown in the later examples.

The phase of this element is zero since the phase angle of a real constant is zero. The phase spectrum of this element plots as a horizontal line at a level of zero and makes no contribution to the phase plot.

$$\angle TF(\omega) = \angle G = 0 \tag{6.26}$$

6.5.2 Derivative Element

The derivative element, Figure 6.10A, has somewhat more interesting frequency characteristics than the constant gain element. The output of a derivative element depends on both the

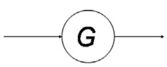


FIGURE 6.9 A constant gain element.

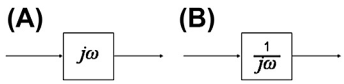


FIGURE 6.10 (A) A derivative element. (B) An integrator element.

current and past input values, so it is an example of an element with memory. This basic element is sometimes referred to as an "isolated zero" for reasons that become apparent later. The transfer function of this element is:

$$TF(\omega) = j\omega \tag{6.27}$$

The magnitude of this transfer function in dB is 20 $\log(|j\omega|) = 20 \log(\omega)$, which is a logarithmic function of ω that plots as a curve on a linear frequency axis, but as a straight line against log frequency. This is another reason to use dB against log frequency in spectral plots. To find the intercept, note that when $\omega = 1$, 20 $\log(\omega = 1)$ equals 0. So the magnitude plot of this transfer function is a straight line that intersects the 0 dB line at $\omega = 1$ rad/s.

To find the slope of this line, note that when $\omega = 10$ the transfer function in dB equals 20 log $(10) = 20$ dB, and when $\omega = 100$ it equals 20 log $(100) = 40$ dB. So for every order of magnitude increase in frequency, there is a 20-dB increase in the value of Equation 6.27. This leads to unusual dimensions for the slope; specifically, 20 dB/decade. These units are due to the logarithmic scaling of the horizontal and vertical axes. The magnitude spectrum plot is shown in Figure 6.11 along with that of the integrator element described next.

The angle of $j\omega$ is +90 degrees irrespective of the value of ω:

$$\angle j\omega = 90 \text{ degrees}. \tag{6.28}$$

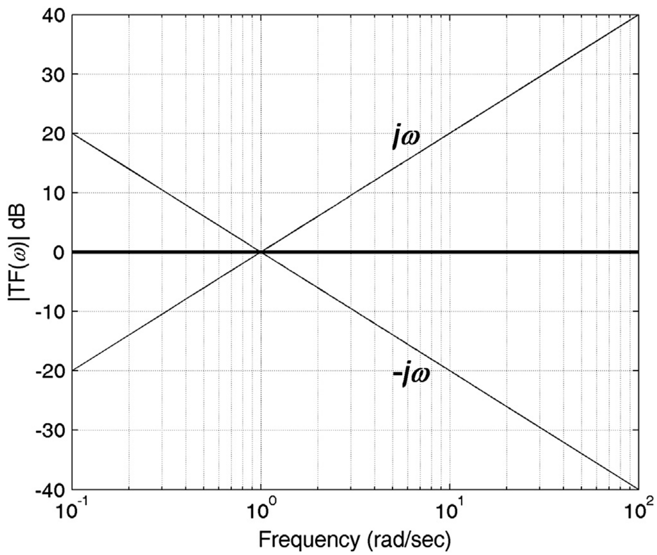


FIGURE 6.11 The upward sloping line is the magnitude spectrum of a derivative element, whereas the downward sloping line is the magnitude spectrum of an integrator element.

So the phase spectrum of this transfer function is just a straight line at +90 degrees. In constructing spectral plots manually, this term is usually not plotted, but is just used to rescale the phase plot after the phase characteristics of the other elements are plotted. (The same approach as used to deal with a gain element in the magnitude plot.)

6.5.3 Integrator Element

The last of the basic elements is the integrator element, Figure 6.10B, with a transfer function that is the inverse of the derivative element:

$$TF(\omega) = \frac{1}{j\omega} \tag{6.29}$$

This element also depends on the current and past values of the input, so is an element with memory. The integrator element is sometimes referred to as an "isolated pole," explained later. The magnitude spectrum in dB is just $20\log\left|\frac{1}{j\omega}\right| = -20\ \log|j\omega| = -20\ \log(\omega)$, which plots as a straight line when the frequency axis is in log ω. The line intercepts 0 dB at $\omega = 1$ since $-20\ \log(1) = 0$ dB. This is similar to the derivative element described earlier, but with the opposite slope: −20 dB/decade. The magnitude plot of this transfer function is shown in Figure 6.11.

The phase spectrum of this transfer function is:

$$\angle\frac{1}{j\omega} = -90\ \text{degrees.} \tag{6.30}$$

Again this is usually not plotted since it is a straight line; rather the phase axis is rescaled after the plot is complete.

6.5.4 First-Order Element

We can now plot both the magnitude and phase spectra of four basic elements, and none of them is particularly exciting. However, if we start putting some of these elements together, we can construct more interesting, and useful, spectra. The "first-order element" can be constructed by placing a negative feedback path around a gain term and an integrator, Figure 6.12A. The gain term, which is just a constant, is given the symbol ω_1 for reasons that will be apparent.

The transfer function for a first-order element can be found easily from the feedback equation, Equation 6.7.

$$TF(\omega) = \frac{G}{1+GH} = \frac{\omega_1\left(\frac{1}{j\omega}\right)}{1+\omega_1\left(\frac{1}{j\omega}\right)} = \frac{\omega_1}{\omega_1 + j\omega} = \frac{1}{1+j\dfrac{\omega}{\omega_1}} \tag{6.31}$$

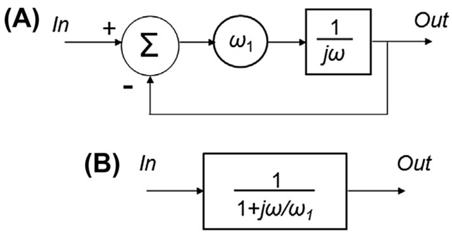


FIGURE 6.12 Constructing a first-order element for a gain term and integrator element in a feedback path. (A) A first-order term as a collection of elements. (B) The transfer function symbol for a first-order element.

This combination is usually presented as a single element as shown in Figure 6.12B. It is called a first-order element because the denominator of its transfer function is a first-order polynomial of the frequency variable, ω. This is also true of the integrator element (Equation 6.32), but it already has a name. Since the first-order element contains an integrator and an integrator element has memory, the first-order element also has memory; its output depends on current and past values of the input.

Finding the magnitude spectrum of the first- and higher-order elements uses graphical techniques based on transfer function asymptotes, an approach originally developed by Hendrik Bode in the 1930s, so the resulting spectral plots are called "Bode plots." First, the high-frequency asymptote and low-frequency asymptotes are plotted. Here high frequency means frequencies where $\omega >> \omega_1$ and low frequency means frequencies where $\omega << \omega_1$. To find the low-frequency asymptote in dB, take the limit as $\omega << \omega_1$ of the absolute value of 20 log of Equation 6.31:

$$|TF(\omega_{low})| = \lim_{\omega \ll \omega_1} 20 \log \left| \frac{1}{1 + j\dfrac{\omega}{\omega_1}} \right| = 20 \log \left(\frac{1}{1 + j0} \right) = 0 \text{ dB} \tag{6.32}$$

The low-frequency asymptote given by Equation 6.32 plots as a horizontal line at 0 dB. The high-frequency asymptote is obtained when $\omega >> \omega_1$:

$$|TF(\omega_{high})| = \lim_{\omega \gg \omega_1} \left[20 \log \left| \frac{1}{1 + j\dfrac{\omega}{\omega_1}} \right| \right] = 20 \log \left| \frac{1}{j\dfrac{\omega}{\omega_1}} \right| = -20 \log \left(\frac{\omega_1}{\omega} \right) \text{ dB} \tag{6.33}$$

The high-frequency asymptote, $-20 \log(\omega_1/\omega)$, is a logarithmic function of ω and plots as a straight line when frequency is plotted against log frequency (again, it is $\log(\omega_1/\omega)$ against $\log(\omega)$). It has the same slope as the integrator element: −20 dB/decade. This line intersects the 0 dB line at $\omega = \omega_1$ since $-20 \log(\omega_1/\omega_1) = -20 \log(1) = 0$ dB.

Often just plotting the asymptotes is enough to give us a general picture of an element's spectrum. Errors between the actual curve and the asymptotes occur when the asymptotic assumptions are no longer true; that is, when the frequency, ω, is neither much greater than, nor much less than, ω_1. The biggest error occurs when ω exactly equals ω_1. At that frequency the magnitude value is:

$$|TF(\omega = \omega_1)| = 20 \log\left|\frac{1}{1+\frac{j\omega_1}{\omega_1}}\right| = 20 \log\left|\frac{1}{1+j}\right| = -20 \log\left(\sqrt{2}\right) = -3 \text{ dB} \tag{6.34}$$

From our definition of bandwidth in Chapter 4, recall that the "cutoff frequency" or "break frequency" occurs when the nominal spectral value is reduced by 3 dB. To approximate the spectrum of the first-order element, we first plot the high- and low-frequency asymptotes as shown in Figure 6.13 (dashed lines). Then we identify the cutoff frequency or −3 dB point, which occurs at $\omega = \omega_1$. Finally, we draw a freehand curve from asymptote to asymptote through the −3 dB, Figure 6.13, solid line. In Figure 6.13, this curve was actually plotted using MATLAB, but you could do almost as well freehand. The deviation of this curve from the two asymptotes is quite small.

The phase plot of the first-order element can be estimated using the same approach: taking the asymptotes and the worst case point, $\omega = \omega_1$. The low-frequency asymptote is:

$$\angle TF(\omega_{low}) = \lim_{\omega \ll \omega_1}\left[\angle\left(\frac{1}{1+j\frac{\omega}{\omega_1}}\right)\right] = \angle -1 = 0 \text{ degrees.} \tag{6.35}$$

This is a straight line at 0 degrees. In this case, the assumption "much much less than" is taken to mean one order of magnitude less, so that the low-frequency asymptote is assumed

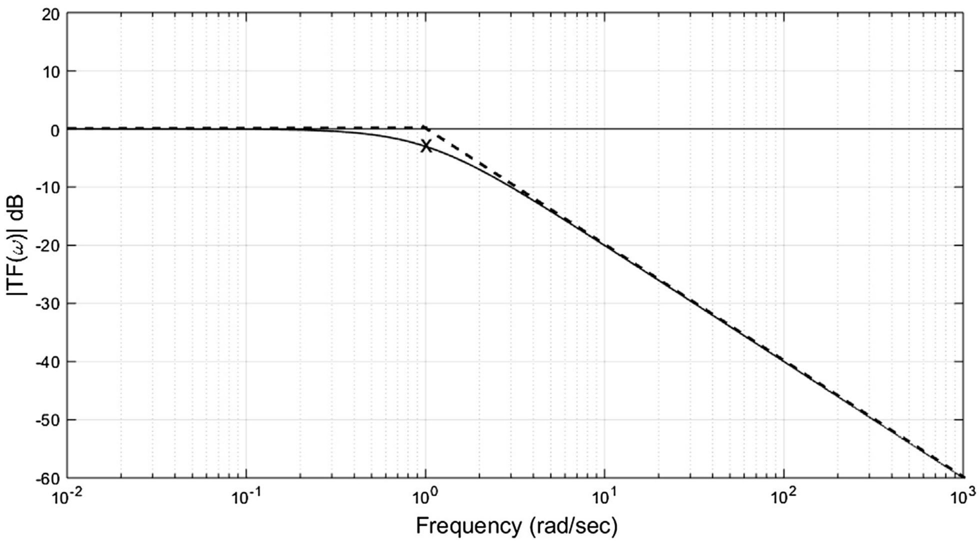


FIGURE 6.13 The magnitude spectrum of a first-order element where $\omega_1 = 1$ rad/s.

to be valid from $0.1\omega_1$ down to zero frequency. The high-frequency asymptote is determined to be -90 degrees in Equation 6.36, and by the same reasoning is assumed to be valid from $10\omega_1$ to all higher frequencies.

$$\angle TF(\omega_{high}) = \lim_{\omega \gg \omega_1}\left[\angle\left(\frac{1}{1+j\dfrac{\omega}{\omega_1}}\right)\right] = \angle\left(\frac{1}{j\dfrac{\omega}{\omega_1}}\right) = -90 \text{ degrees.} \tag{6.36}$$

Again the greatest difference between the asymptotes and the actual curve is when ω equals ω_1:

$$\angle(TF(\omega = \omega_1)) = \angle\left(\frac{1}{1+j\dfrac{\omega_1}{\omega_1}}\right) = \angle\left(\frac{1}{1+j}\right) = -45 \text{ degrees.} \tag{6.37}$$

This value, -45 degrees, is exactly halfway between the high- and low-frequency asymptotes. Usually, a straight line is drawn between the high end of the low-frequency asymptote at $0.1\omega_1$ and the low end of the high-frequency asymptote at $10\omega_1$ passing through 45 degrees. Although the phase curve is nonlinear in this range, the error induced by a straight line approximation is small as shown in Figure 6.14. To help you to plot these spectral curves by hand, the associated files contain semilog.pdf, which can be printed out to produce semilog graph paper.

There is a variation of the transfer function described in Equation 6.31 that, although not as common, is sometimes found as part of the transfer function of other elements. One such variation has the first-order polynomial in the numerator but nothing in the denominator:

$$TF(\omega) = 1 + j\frac{\omega}{\omega_1} \tag{6.38}$$

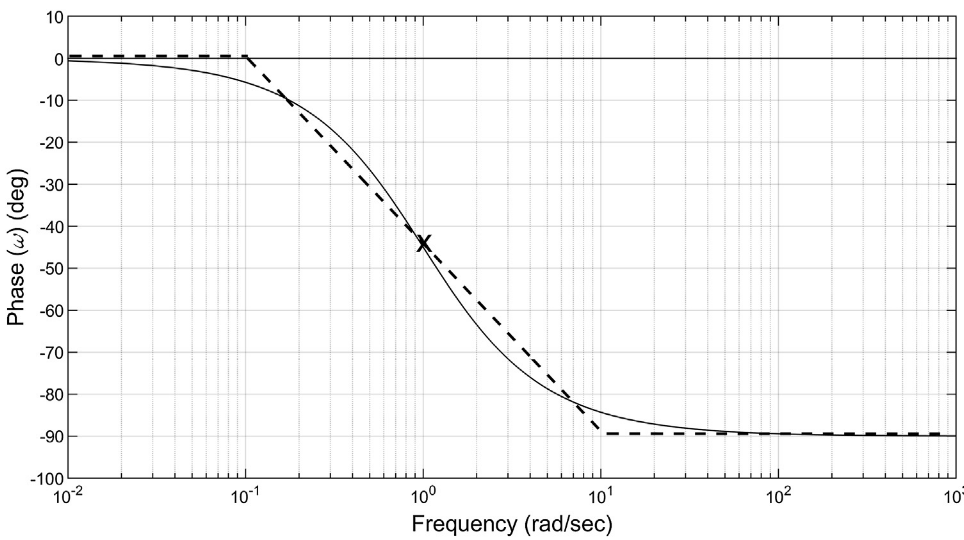


FIGURE 6.14 The phase spectrum of a first-order element where $\omega_1 = 1$ rad/s.

The asymptotes for this transfer function are very similar to that of Equation 6.31; in fact, the low-frequency asymptote is the same.

$$|TF(\omega_{low})| = \lim_{\omega \ll \omega_1} 20 \log\left|1 + j\frac{\omega}{\omega_1}\right| = 20 \log(1) = 0 \text{ dB} \tag{6.39}$$

The high-frequency asymptote has the same intercept, but the slope is positive, not negative:

$$|TF(\omega_{high})| = \lim_{\omega \gg \omega_1} \left[20 \log\left|1 + j\frac{\omega}{\omega_1}\right|\right] = 20 \log\left|j\frac{\omega}{\omega_1}\right| = 20 \log\left(\frac{\omega}{\omega_1}\right) \text{ dB} \tag{6.40}$$

Similarly the value of the transfer function when $\omega = \omega_1$ is +3 dB instead of −3 dB:

$$|TF(\omega = \omega_1)| = 20 \log\left|1 + \frac{j\omega_1}{\omega_1}\right| = 20 \log|1 + j| = 20 \log\left(\sqrt{2}\right) = 3 \text{ dB} \tag{6.41}$$

The phase plots are also the same except that the change in angle with frequency is positive rather than negative. Demonstration of this is left as an exercise in the problems at the end of this chapter. The magnitude and phase spectra of the transfer function given in Equation 6.38 is shown in Figure 6.15.

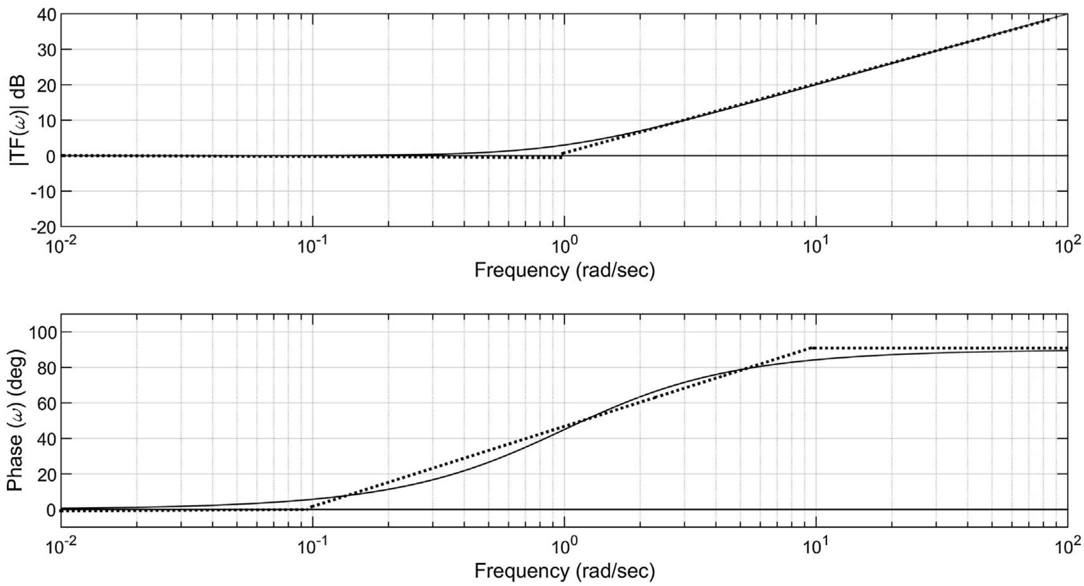


FIGURE 6.15 The magnitude and phase spectra of the transfer function given in Equation 6.38. These spectra are similar to those of the standard first-order process, Figures 6.13 and 6.14, except both curves go up instead of down at the higher frequencies. $\omega_1 = 1$ rad/s.

EXAMPLE 6.8

Plot the magnitude and phase spectrum of the following transfer function:

$$TF(\omega) = \frac{100}{(1 + j0.1\omega)}$$

Solution: This transfer function is actually a combination of two elements: a gain term of 100 and a first-order term where ω_1 is equal to $1/0.1 = 10$. The easiest way to plot the spectrum of this transfer function is to plot the first-order term first, then rescale the vertical axis of the magnitude spectrum so that 0 dB is equal to 20 log(100) = 40 dB. The rescaling accounts for the gain term. The gain term has no influence on the phase plot.

Results: Figure 6.16 shows the resulting magnitude and phase spectral plots. For the magnitude plot the spectrum is drawn freehand through the −3 dB point. For both plots, the asymptotes alone provide a sufficiently accurate picture of the system's spectra.

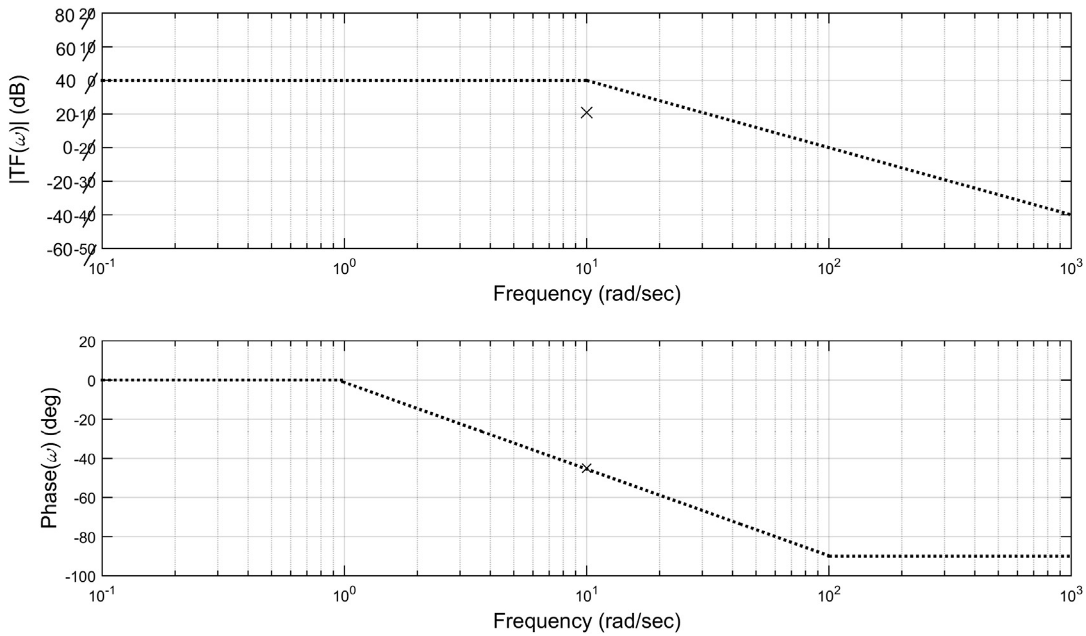


FIGURE 6.16 The magnitude and phase spectra of the system described in Example 6.8. Only the asymptotes are plotted and they provide an adequate picture of the system's spectra.

6.5.5 Second-Order Element

A second-order element has, as you might expect, a transfer function that includes a second-order polynomial of ω. A second-order element contains two integrators so it also has memory. Again that means that its output at any given time depends on both current and past values of the input. One way to construct a second-order system is to put two first-order systems in series, but other configurations also produce a second-order equation as long as they contain two integrators. For example, the system shown in Figure 6.6, and the transfer function plotted in Example 6.6, have a second-order ω terms in the denominator. The equation for a general second-order system has the format:

$$TF(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\frac{2\delta\omega}{\omega_n}} \tag{6.42}$$

where ω_n and δ are constants associated with the second-order polynomial. The format of this second-order equation may seem strange, but the two constants, ω_n and δ, have direct relationships to both the time behavior and the spectrum. The parameter δ is called the "damping factor" and ω_n the "undamped natural frequency." The rational for these strange names comes from the time-domain behavior and is described in Chapter 7. But ω_n and δ also relate directly to a system's spectra; in fact, sometimes they are all we need for an overview of the system spectral characteristics. Determining these constants from a typical second-order equation is the motivation of the next example.

EXAMPLE 6.9

Find the constants ω_n and δ in the following second-order transfer function:

$$TF(\omega) = \frac{1}{1 - .03\omega^2 + j.01\omega} \tag{6.43}$$

Solution/Results: We obtain the values of constants ω_n and δ by equating coefficients between the denominator terms of Equations 6.42 and 6.43:

$$\left(\frac{1}{\omega_n}\right)^2 = 0.03; \quad \omega_n = \frac{1}{\sqrt{0.03}} = 5.77 \text{ rad/s}$$

$$\frac{2\delta}{\omega_n} = 0.1; \quad \delta = \frac{0.01\omega_n}{2} = \frac{(5.77)0.01}{2} = 0.029$$

If the roots of the denominator polynomial are real, it can be factored into two first-order terms, then separated into two first-order transfer functions using partial fraction expansion. This is done in Chapter 7, but for spectral plotting the second-order term will be dealt with as is.

Not surprisingly, the second-order element is more challenging to plot than the previous transfer functions. Basically the same strategy is used as in the first-order function except special care must be taken with the worst case error at frequency $\omega = \omega_n$. We begin by finding the high- and low-frequency asymptotes. These now occur when ω is either much greater or much less than ω_n. When $\omega << \omega_n$, both the ω and the ω^2 terms go to 0 and the denominator goes to 1.0:

$$|TF(\omega_{low})| = \lim_{\omega \ll \omega_n}\left[20\log\left|\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\frac{2\delta\omega}{\omega_n}}\right|\right] = 20\log(1) = 0 \text{ dB} \tag{6.44}$$

The low-frequency asymptote is the same as for the first-order element, namely, a horizontal line at 0 dB.

The high-frequency asymptote occurs when $\omega >> \omega_n$ where the ω^2 term in the denominator dominates:

$$|TF(\omega_{high})| = \lim_{\omega \gg \omega_n} \left[20 \log \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\frac{2\delta\omega}{\omega_n}} \right| \right] = 20 \log \left(\frac{1}{\frac{\omega}{\omega_n}} \right)^2 = -40 \log \left(\frac{\omega_n}{\omega} \right) \text{ dB} \tag{6.45}$$

The high-frequency asymptote is also similar to that of the first-order element, but with double the downward slope: −40 dB/decade instead of −20 dB/decade.

A major difference between the first- and second-order terms occurs when $\omega = \omega_n$:

$$|TF(\omega = \omega_n)| = 20 \log \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\frac{2\delta\omega}{\omega_n}} \right| = 20 \log \left| \frac{1}{j2\delta} \right| = -20 \log(2\delta) \text{ dB} \tag{6.46}$$

The magnitude spectrum at $\omega = \omega_n$ is not a constant, but depends on δ: specifically, −20 log(2δ). In fact, the value of δ can radically alter the shape of the magnitude curve and must be taken into account when plotting. If δ is less than 0.5, then log(2δ) will be negative and the transfer function at ω_n will be positive. Usually the magnitude plot is determined by first plotting the asymptotes, then calculating and plotting the *TF*'s value at $\omega = \omega_n$ using Equation 6.46, and finally drawing a curve freehand through the ω_n point, converging smoothly with the high- and low-frequency asymptotes above and below ω_n. The freehand curve can sometimes take some artistic skill. Nonetheless, the spectrum is wholly determined by the two constants, ω_n and δ. The second-order magnitude plot is shown in Figure 6.17 for several values of δ.

The phase plot of a second-order system is also approached using the asymptote method. For phase angle the high- and low-frequency asymptotes are given as:

$$\angle TF(\omega_{low}) = \lim_{\omega \ll \omega_n} \angle \left[\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\frac{2\delta\omega}{\omega_n}} \right] = \angle 1 = 0 \text{ degrees.} \tag{6.47}$$

$$\angle TF(\omega_{high}) = \lim_{\omega \gg \omega_n} \angle \left[\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\frac{2\delta\omega}{\omega_n}} \right] = \angle \left(\frac{1}{-\omega^2} \right) = -180 \text{ degrees.} \tag{6.48}$$

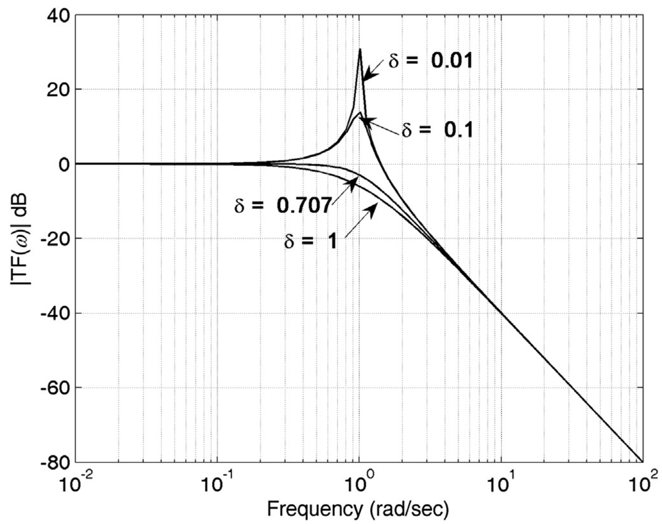


FIGURE 6.17 Magnitude spectra of second-order systems with different values of δ. For all spectra $\omega_n = 1.0$ rad/s.

This is similar to the asymptotes of the first-order process except the high-frequency asymptote is at -180 degrees instead of -90 degrees. The phase angle when $\omega = \omega_n$ can easily be determined:

$$\angle TF(\omega_n) = \angle\left(\frac{1}{1-(\omega_n/\omega_n)^2 + j\dfrac{2\delta\omega_n}{\omega_n}}\right) = \angle\left(\frac{1}{j2\delta}\right) = -90 \text{ degrees.} \tag{6.49}$$

So the phase at $\omega = \omega_n$ is -90 degrees, halfway between the two asymptotes. Unfortunately, the shape of the phase curve between $0.1\omega_n$ and $10\omega_n$ is a function of δ and can no longer be approximated as a straight line except at larger values of δ. Phase curves are shown in Figure 6.18 for the same range of values of δ used in Figure 6.17. The curves for low values of δ have steep transitions between 0 and -180 degrees, whereas the curves for high values of δ have gradual slopes approximating the phase characteristics of a first-order element (except for the 180-degree phase change). Hence if δ is 2.0 or more, a straight line between the low-frequency asymptote at $0.1\omega_n$ and the high-frequency asymptote at $10\omega_n$ works well. If δ is much less than 2.0, the best that can be done using this manual method is to approximate, freehand, the appropriate curve (or approximately appropriate curve) in Figure 6.18.

EXAMPLE 6.10

Plot the magnitude and phase spectra of the second-order transfer function used in Example 6.9 (i.e., Equation 6.43). Confirm the results obtained manually with those obtained from MATLAB.[5]

Solution: The equivalent values of ω_n and δ are found in Example 6.9 to be 5.77 rad/s and 0.029, respectively. The magnitude and phase spectra can be plotted directly using the asymptotes, noting that the magnitude spectrum equals $-20\log(2\delta) = +24.7$ dB when $\omega = \omega_n$.

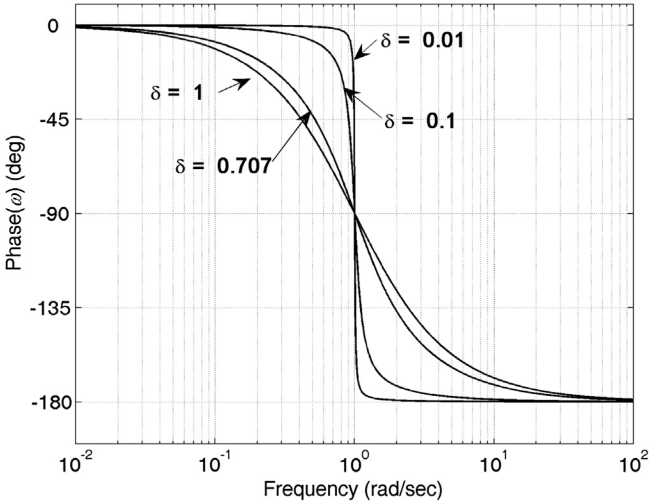


FIGURE 6.18 Phase spectra of second-order systems with different values of δ. The values of δ are the same as used in Figure 6.17 and $\omega_n = 1.0$ rad/s.

To plot the transfer function using MATLAB, first construct the frequency vector. Given that $\omega_n = 6.77$ rad/s, use the frequency vector range between $0.01\omega_n = 0.0677$ and $100\omega_n = 677$ rad/s in increments of, say, 0.1 rad/s. Define the function in MATLAB and for the magnitude spectrum, take the 20 times log magnitude and for the phase spectrum take the angle in degrees. Plot these functions against frequency in radians using a semilog plot.

```
% Example 6.10  Use MATLAB to plot the transfer function given in Equation 6.42
%
w = .067:.1:667;                              % Define frequency vector
wn = 5.77;                                    % Define wn
delta = 0.029;                                % Define delta
TF = 1./(1 - (w/wn).^2 + j*2*delta*w/wn);     % Transfer function
Mag = 20*log10(abs(TF));                      % Magnitude in dB
Phase = angle(TF)*360/(2*pi);                 % Phase in deg
subplot(2,1,1);
  semilogx(w,Mag,'k');                        % Plot as log frequency
  .......labels......
subplot(2,1,2);
  semilogx(w,Phase,'k');
  .......labels.......
```

Results: The asymptotes, the point where $\omega = \omega_n$, and freehand spectra are shown in Figure 6.19, and the results of the MATLAB code are shown in Figure 6.20. The ease of plotting the transfer function using MATLAB is seductive, but the manual method provides more insight into the system. More importantly, it also allows us to go in the reverse direction: from a spectrum to an equivalent transfer function as shown later. This also aids in the design of systems having particular spectral characteristics.

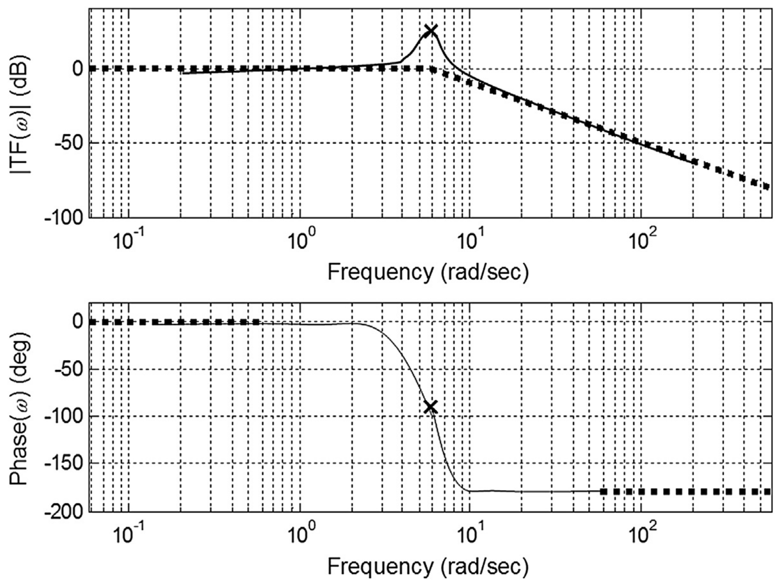


FIGURE 6.19 Magnitude and phase spectra of the system defined by the transfer function equation, Equation 6.42. The *solid line* was drawn freehand using the asymptotes (*dotted lines*) and the points where $\omega = \omega_n$.

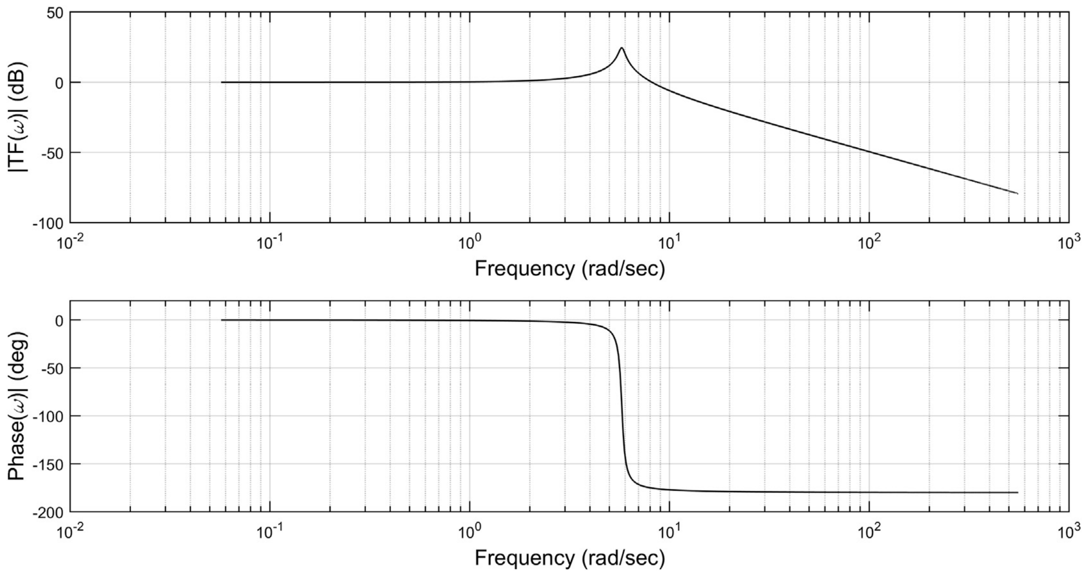


FIGURE 6.20 Magnitude and phase spectra of the system defined by the transfer function equation, Equation 6.43, as determined by MATLAB. These spectra approximately match those constructed manually shown in Figure 6.19.

[5]MATLAB has a function, `bode.m`, that plots the spectral characteristics of any linear system. The system is defined by its transfer using another MATLAB routine, `tf.m`. Both these routines are in the Control System Toolbox. However, as shown here, it is easy to generate Bode plots using standard MATLAB.

Occasionally the second-order term is found in the numerator, usually as part of a more complex transfer function. In this case, both the magnitude and phase spectra are inverted so the phase spectrum goes from 0 to +180 degrees passing through +90 degrees when $\omega = \omega_n$. Modifying the equations to demonstrate this is an exercise in the problems.

Table 6.1 summarizes the various elements, which along with their magnitude and phase characteristics, are known as "Bode plot primitives." This table also lists alternate names for these elements where applicable. Table 6.2 brings together the magnitude and phase characteristics of these Bode plot primitives.

6.6 BODE PLOTS COMBINING MULTIPLE ELEMENTS

We now know how to find and plot the spectra of some basic, and not so basic, systems even without a computer. But what of more complicated systems? We can deal with these using an extension of what we already learned: first use our Bode plot techniques to plot the spectra of individual elements (or systems), then combine those spectra graphically.

As always, we start with the transfer function, then assume it can be factored into combinations describing the elements we have already covered (i.e., the Bode plot primitives summarized in Table 6.2). Under this assumption, the transfer function of any system, no matter how complicated, can be written as:

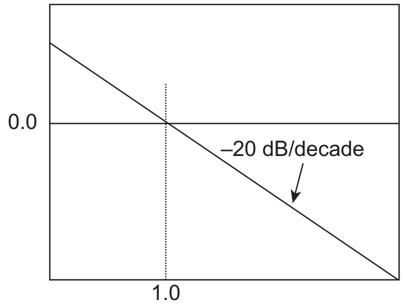
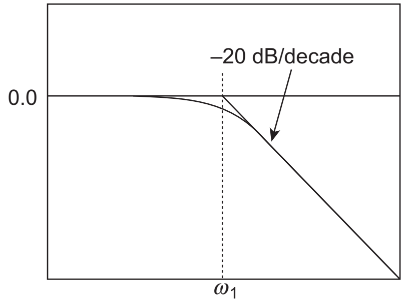
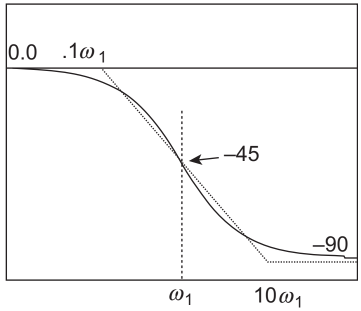
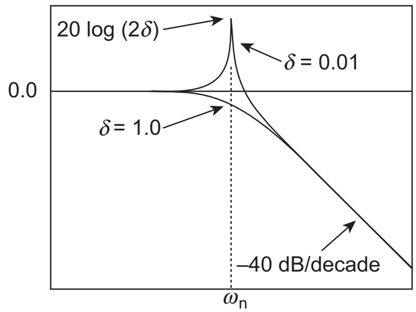
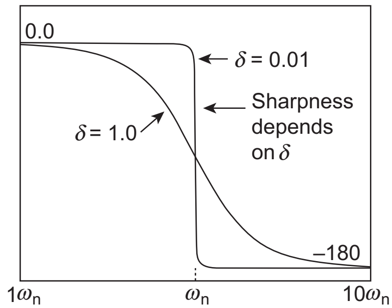
$$TF(\omega) = \frac{Gj\omega\left(1 + j\dfrac{\omega}{\omega_1}\right)\left(1 - \left(\dfrac{\omega}{\omega_{n1}}\right)^2 + j\dfrac{2\delta_1\omega}{\omega_{n1}}\right)\ldots}{j\omega\left(1 + j\dfrac{\omega}{\omega_2}\right)\left(1 - \left(\dfrac{\omega}{\omega_{n2}}\right)^2 + j\dfrac{2\delta_2\omega}{\omega_{n2}}\right)\ldots} \tag{6.50}$$

TABLE 6.1 Transfer Function Elements

Basic Equation	Name(s): Numerator	Name(s): Denominator
G	Constant or gain	–
$j\omega$	Differentiator[a] Isolated zero	Integrator[a] Isolated pole
$1 + j\omega/\omega_1$	Real zero or just zero Lead element	First-order element[a] Real pole or just pole Lag element
$1 - (\omega/\omega_n)^2 + j2\delta\omega/\omega_n$	Complex zeros[b]	Complex poles[b] Second-order element[a]

[a] *Name most commonly used in this text.*
[b] *Depends on the values of δ.*

TABLE 6.2 Bode Plot Primitives

<table>
<tr><th>Denominator Term</th><th>Magnitude Plot</th><th>Phase Plot</th></tr>
<tr><td>Constant Gain</td><td>20 log G</td><td>–</td></tr>
<tr><td>$j\omega$</td><td>
</td><td>−90 degrees</td></tr>
<tr><td>$1 + j\dfrac{\omega}{\omega_1}$</td><td>
</td><td>
</td></tr>
<tr><td>$1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + j\dfrac{2\delta\omega}{\omega_n}$</td><td>
</td><td>
</td></tr>
</table>

Admittedly, polynomial factoring can be tedious, but it is easily accomplished using MATLAB's `roots` routine.[6] An example involving a fourth-order denominator polynomial is found at the end of the next chapter. The constants for first- and second-order elements

[6]Why would we first use MATLAB to factor a higher-order polynomial, then proceed manually with our graphical approach? Why not just plot the system's spectrum straight away in MATLAB? Plotting the transfer function would be much easier, but designing a system to duplicate that transfer function would be much more difficult without knowing the individual elements. You would be unaware of these more basic elements embedded in that higher-order transfer function.

were described earlier. Note that this is a general equation illustrating the various possible element combinations; if the same elements actually appear in both the numerator and denominator (e.g., the $j\omega$ term), of course they would cancel.

To convert Equation 6.50 to magnitude in dB, we take 20 log of the absolute value:

$$TF(\omega) = 20\log\left|\frac{Gj\omega\left(1+j\dfrac{\omega}{\omega_1}\right)\left(1-\left(\dfrac{\omega}{\omega_{n1}}\right)^2+j\dfrac{2\delta_1\omega}{\omega_{n1}}\right)\ldots}{j\omega\left(1+j\dfrac{\omega}{\omega_2}\right)\left(1-\left(\dfrac{\omega}{\omega_{n2}}\right)^2+j\dfrac{2\delta_2\omega}{\omega_{n2}}\right)\ldots}\right| \tag{6.51}$$

Note that multiplication and division in Equation 6.51 become addition and subtraction after taking the log, so in dB Equation 6.51 can be expanded to:

$$\begin{aligned}|TF(\omega)|_{dB} = {} & \overbrace{20\log(G)} + \overbrace{20\log|j\omega|} + \overbrace{20\log|1+j\omega/\omega_1|} + \overbrace{20\log\left|1-(\omega/\omega_{n1})^2+j2\delta_1\omega/\omega_{n1}\right|} + \ldots \\ & \underbrace{-20\log|j\omega|} \underbrace{-20\log|1+j\omega/\omega_2|} \underbrace{-20\log\left|1-(\omega/\omega_{n2})^2+j2\delta_2\omega/\omega_{n2}\right|} + \ldots.\end{aligned} \tag{6.52}$$

where the first line in Equation 6.52 is the expanded version of the numerator and the second line is the expanded version of the denominator. Aside from the constant term, the first and second lines in Equation 6.52 have the same form except for the sign: numerator terms are positive and denominator terms are negative. Each term in the summation is one of the element types described in the last section as emphasized by the horizontal brackets. This shows that the magnitude spectrum of any transfer function can be plotted by plotting each individual element and then adding the spectral curves graphically.

Adding the individual spectra is not as difficult as it first appears. Usually only the asymptotes and a few other important points (such as the value of a second-order term at $\omega = \omega_n$) are plotted, then the overall curve is completed freehand by connecting the asymptotes and critical points. Aiding this procedure is the fact that most real transfer functions do not contain a large number of elements. Although the resulting Bode plot is only approximate and often somewhat crude, it is usually sufficient to represent the general spectral characteristics of the transfer function.

The phase portion of the transfer function can also be dissected into individual components:

$$\angle TF(\omega) = \frac{\angle\left[Gj\omega\left(1+j\dfrac{\omega}{\omega_1}\right)\left(1+\left(\dfrac{\omega}{\omega_{n1}}\right)^2+j\dfrac{2\delta_1\omega}{\omega_{n1}}\right)\right]}{\angle\left[j\omega\left(1+j\dfrac{\omega}{\omega_2}\right)\left(1+\left(\dfrac{\omega}{\omega_{n2}}\right)^2+j\dfrac{2\delta_2\omega}{\omega_{n2}}\right)\right]} \tag{6.53}$$

By the rules of complex arithmetic, the angles of the individual elements simply add if they are in the numerator or subtract if they are in the denominator:

$$\begin{aligned}\angle TF(\omega) = {} & \overbrace{\angle G} + \overbrace{\angle j\omega} + \overbrace{\angle\left(1 + \frac{j\omega}{\omega_1}\right)} + \overbrace{\angle\left(1 + \left(\frac{\omega}{\omega_{n1}}\right)^2 + j\frac{2\delta_1\omega}{\omega_{n1}}\right)} \\ & - \underbrace{\angle j\omega} - \underbrace{\angle\left(1 + j\frac{\omega}{\omega_2}\right)} - \underbrace{\angle\left(1 + \left(\frac{\omega}{\omega_{n2}}\right)^2 + j\frac{2\delta_2\omega}{\omega_{n2}}\right)}\end{aligned} \tag{6.54}$$

So the phase transfer function also consists of individual components (emphasized by the brackets) that correspond to the Bode plot primitives shown in Table 6.2. Again these components add or subtract depending on whether they are in the numerator or denominator of the transfer function. The phase spectrum of any general transfer function can be constructed manually by plotting the spectrum of each component and adding the curves graphically. Again, we usually start with the asymptotes and fill in freehand; like the magnitude spectrum the result is a sometimes crude, but usually adequate, approximation of the phase spectrum. Often only the magnitude plot is of interest and it is not necessary to construct the phase plot.

The next two examples demonstrate the Bode plot approach for transfer functions of increasing complexity.

EXAMPLE 6.11

Plot the magnitude and phase spectra using Bode plot methods for the transfer function:

$$TF(\omega) = \frac{100j\omega}{(1 + j1\omega)(1 + j.1\omega)} \tag{6.55}$$

Solution: The transfer function contains four elements: a constant, an isolated zero (i.e., $j\omega$ in the numerator), and two first-order terms in the denominator. For the magnitude curve, plot the asymptotes for all but the constant term, then add these asymptotes together graphically to get an overall asymptote. Finally, use the constant term to scale the value of the vertical axis. For the phase plot, construct the asymptotes for the two first-order denominator elements, then rescale the axis by $+90$ degrees to account for the $j\omega$ in the numerator. Recall that the constant term does not contribute to the phase plot.

The general form for the two first-order elements is $\frac{1}{1+j\frac{\omega}{\omega_1}}$ where ω_1 is the point where the high- and low-frequency asymptotes intersect. In this transfer function, ω_1 and ω_2 are $1/1 = 1.0$ and $1/0.1 = 10$ rad/s. Figure 6.21 shows the asymptotes obtained for the two first-order primitives plus the $j\omega$ primitive. Note that the asymptotes can be determined directly from the transfer function; it is not necessary to rewrite the equation into the format of Equation 6.54.

Results: Graphically adding the three asymptotes shown in Figure 6.21 gives the curve consisting of three straight lines shown in Figure 6.22. Note that the numerator and denominator asymptotes cancel out at $\omega = 1.0$ rad/s, so the overall asymptote is flat until the additional downward asymptote comes in at $\omega = 10$ rad/s. The actual magnitude transfer function is also shown in Figure 6.23 and closely follows the overall asymptote. A small error (3 dB) is seen at the two breakpoints: $\omega = 1.0$ and $\omega = 10$. A final step in constructing the magnitude curve is to rescale the

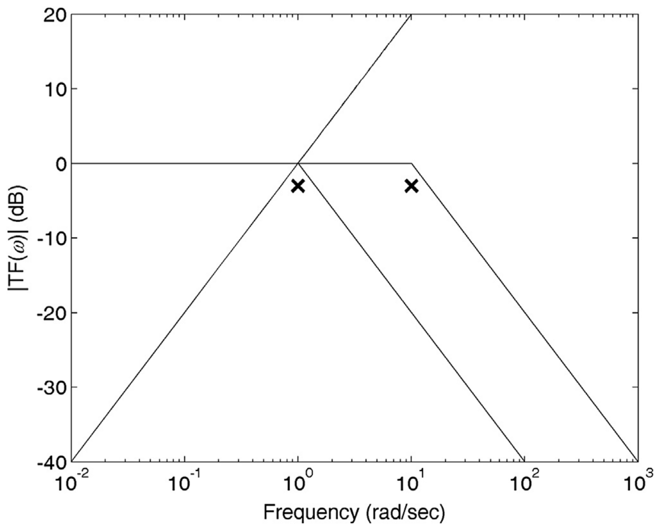


FIGURE 6.21 The magnitude spectrum asymptotes for three of the elements in the transfer function equation, Equation 6.55. Also shown are the two breakpoints where $\omega = \omega_1$ and $\omega = \omega_2$. The next step is to add these three curves graphically resulting in the overall spectrum shown in Figure 6.22.

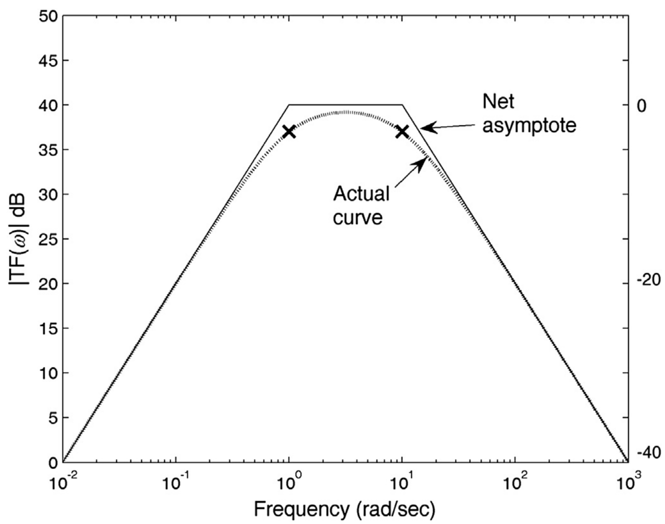


FIGURE 6.22 The *solid line* is the algebraic summation of asymptotes shown in Figure 6.23. The *dotted line* is the actual magnitude spectrum of the transfer function given by Equation 6.5. The right-hand axis shows the original scaling and the left-hand axis shows the rescaling due to the gain term. The actual curve was drawn by MATLAB, but does not differ significantly from the asymptotes. A hand-drawn curve would likely have been just as good.

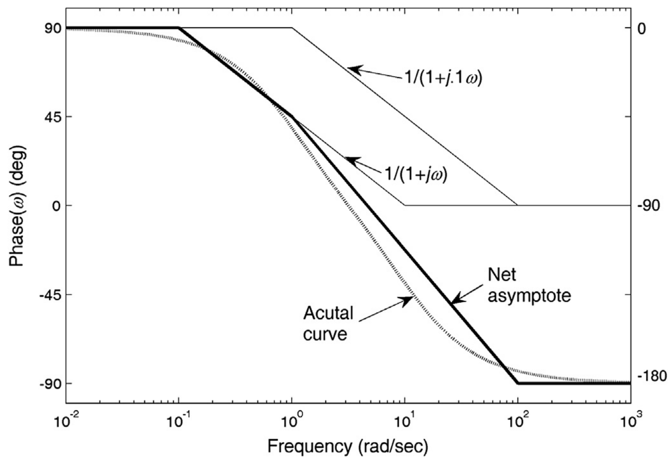


FIGURE 6.23 The *light lines* show the phase spectrum asymptotes for two elements found in the transfer function Equation 6.52. The *heavy line* is the summation of these two asymptotes and the *light curve* is the actual phase spectrum.

vertical axis so that 0 dB corresponds to 20 log (100) = 40 dB. (The original axis values are shown on the right vertical axis, whereas the rescaled values are shown on the left.)

The asymptotes of the phase curve are shown in Figure 6.23 along with the overall asymptote that is obtained by graphical addition. Also shown is the actual phase curve, which, as with the magnitude curve, closely follows the overall asymptote. As a final step the vertical axis of this plot has been rescaled by +90 degrees on the right side to account for the $j\omega$ term in the numerator.

In both the magnitude and phase plots, the actual curves follow the overall asymptote fairly closely for transfer functions that have terms no higher than first order. Tracing freehand through the -3 dB points further improves the match, but often the asymptotes are sufficient. As we see in the next example, this is not true for transfer functions that contain second-order terms, at least when the damping factor, δ, is small.

EXAMPLE 6.12

Find the magnitude and phase spectra using both Bode plot primitives and MATLAB code for the transfer function.

$$TF(\omega) = \frac{10(1 + j2\omega)}{j\omega(1 - 0.04\omega^2 + j0.04\omega)} \tag{6.56}$$

Solution: This transfer function contains four elements: a gain constant, a numerator first-order term, an isolated pole (i.e. $j\omega$) in the denominator, and a second-order term in the denominator. For the magnitude curve, plot the asymptotes for all primitives except the constant, then add these up graphically. Lastly, use the constant term to scale the value of the vertical axis. For the phase curve, plot the asymptotes of the first-order and second-order terms, then rescale the vertical axis by

90 degrees to account for the $j\omega$ term in the denominator. To plot the magnitude asymptotes, it is first necessary to determine ω_1, ω_n, and δ from the associated coefficients:

$$\omega_1 : \frac{1}{\omega_1} = 2; \quad \omega_1 = 0.5 \text{ rad/s}$$

$$\omega_n : \frac{1}{\omega_n^2} = 0.04; \quad \omega_n = \frac{1}{\sqrt{0.04}} = 5 \text{ rad/s}$$

$$\delta : \frac{2\delta}{\omega_n} = 0.04; \quad \delta = \frac{0.04\omega_n}{2} = \frac{0.04(5)}{2} = 0.1$$

$$-20\log(2\delta) = 14 \text{ dB}$$

Results: Note that the second-order term is positive 14 dB when $\omega = \omega_n$. This is because 2δ is less than 1 and so the log is negative and the two negatives make a positive. Using these values and including the asymptotes leads to the magnitude plot shown in Figure 6.24.

The log frequency scale can be a little confusing (particularly without grid lines, which are left off to improve visibility), so the positions of 0.5 and 5.0 rad/s are indicated by arrows in Figure 6.24. The overall asymptote and the actual curve for the magnitude transfer function are shown in Figure 6.25. The vertical axis has been rescaled by 20 dB to account for the constant term in the numerator. Note that the actual magnitude spectrum (light dashed line) goes higher than +14 dB because of the contribution of the first-order numerator term. The net asymptote (heavy dashed line) at $\omega = \omega_n$ is approximately 5 dB, so the actual curve, not including the rescaling, peaks around $14 + 5 = 19$ dB. Adding in the constant 20 dB brings the peak up to 39 dB.

The individual phase asymptotes are shown in Figure 6.26, whereas the overall asymptote and the actual phase curve are shown in Figure 6.27. Note that the actual phase curve is quite different from the overall asymptote, again because the second-order term has a small value for δ. This low

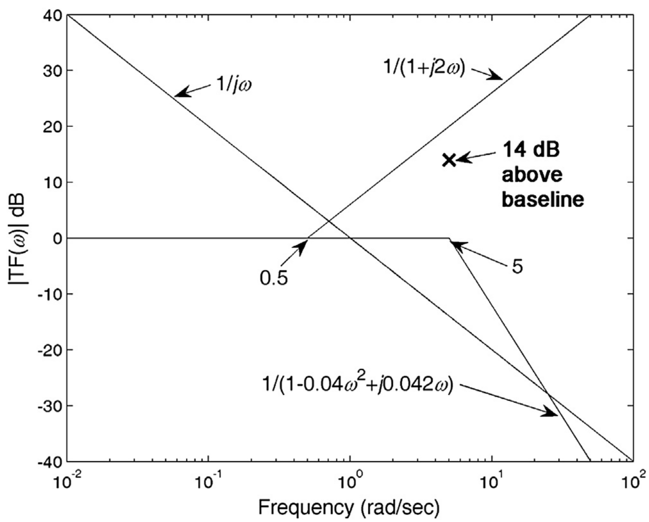


FIGURE 6.24 The magnitude spectrum asymptotes for three of the elements in the transfer function Equation 6.56. Also shown is the point where $\omega = \omega_n$, the undamped natural frequency of the second-order element. The next step is to add these three curves graphically resulting in the overall spectrum shown in Figure 6.25.

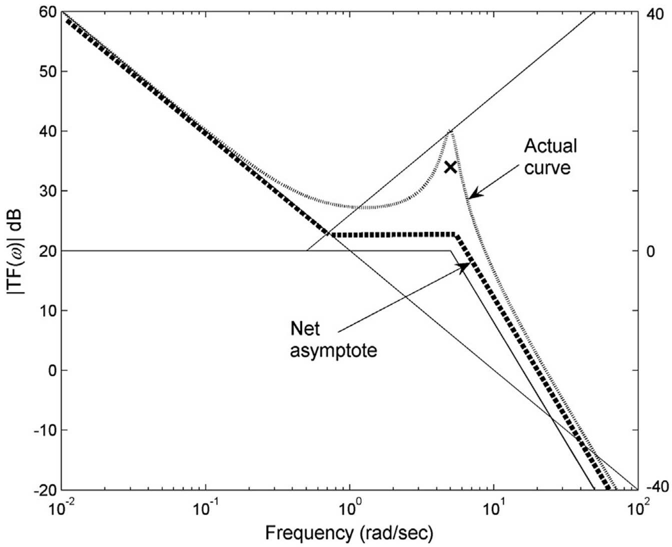


FIGURE 6.25 The *light solid lines* are the asymptotes of the magnitude spectra of the individual elements of Equation 6.55. The *heavy dashed line* is the algebraic sum of the individual asymptotes and the *light dashed line* is the actual magnitude spectrum. Because of the second-order term, the magnitude spectrum deviates significantly from the net asymptote in the region of $\omega = \omega_n$. The left vertical axis has been rescaled by 20 dB to account for the constant gain term.

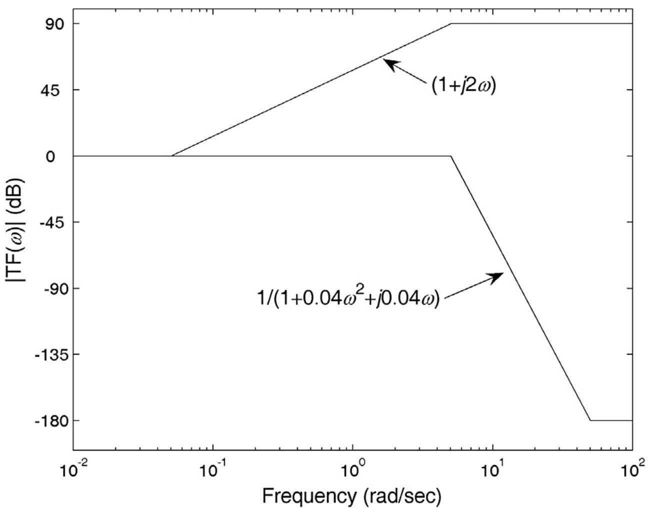


FIGURE 6.26 The asymptotes of the phase spectra of two of the individual elements of Equation 6.56. The algebraic summation of these asymptotes is shown in Figure 6.27.

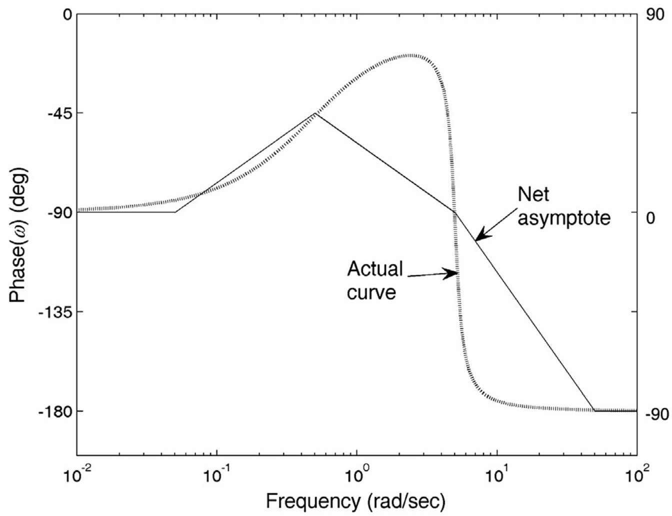


FIGURE 6.27 The *solid line* is the algebraic sum of the individual asymptotes shown in Figure 6.26 and the *light dashed line* is the actual magnitude spectrum. Because of the second-order term, the magnitude spectrum deviates significantly from the net asymptote in the region of $\omega = \omega_n$.

value of δ means the phase curve has a very sharp transition between 0 and -180 degrees and will deviate substantially from the asymptote. The vertical axis is rescaled by -90 degrees to account for the $j\omega$ term in the denominator. The original axis is shown on the right side.

Results, MATLAB: The MATLAB code required for this example is a minor variation of that required in previous examples. Only the range of the frequency vector and the MATLAB statement defining the transfer function have been changed. Note that the point-by-point operators, .* and ./, are required whenever vector terms are multiplied. The plots produced by this code are shown as light dashed lines in Figures 6.26 and 6.27.

```
% Example 6.12  Use MATLAB to plot the transfer function given in this example
%
w = .005:.1:500;
w1 = 0.5;                % First-order cutoff freq (rad)
wn = 5;                  % Second-order undamped resonant freq. (rad)
delta = 0.1;             % Damping factor
TF  =  10*(1+j*w/w1)./(j*w.*(1  -  (w/wn).^2  +  j*2*delta*w/wn));   %  Transfer
function

  ......the rest of the code is the same as Example 6.7.......
```

The code in this example could easily be modified to plot a transfer function of any complexity as some of the problems demonstrate. The Bode plot approach may seem like a lot of effort to achieve a graph that could be done better with a couple of lines of MATLAB code. However, these techniques based on Bode plot primitives provide us with a mental map that links the transfer function to the system's spectrum. These primitives will guide us when we go the other way, from spectrum (perhaps derived experimentally) to transfer function.

6.6.1 Constructing the Transfer Function From the System Spectrum

Bode plot primitives can also be used to derive the transfer function given a desired frequency response as illustrated in the next example.

EXAMPLE 6.13

Find the transfer function of a system that has the magnitude spectrum shown in Figure 6.28.

Solution: We start by identifying some basic features of the spectrum. From our knowledge of Bode plot primitives we note that the high-frequency portion of the curve looks like a second-order element with an undamped natural frequency (i.e., ω_n) of 100 rad/s. The baseline is 20 dB so $G = 10$ and the peak rises 20 dB above the baseline (from 20 to 40 dB), so the damping factor is:

$$-20\log(2\delta) = 20 \text{ dB}; \; -\log(2\delta) = 1.0; \; 1/(2\delta) = 10^1; \; \delta = 1/20 = 0.05$$

So a partial transfer function that accounts for the high-frequency spectrum is:

$$TF(\omega) = \frac{10}{1 - \dfrac{\omega^2}{10^4} + j\dfrac{0.1\omega}{100}}$$

The low-frequency spectrum looks like the lower frequencies of the spectrum in Example 6.11. In that example, the low-frequency curve was constructed by a combination of a $j\omega$ in the numerator and a $(1 + j\omega/\omega_1)$ in the denominator where ω_1 is the cutoff or break frequency. In Figure 6.28, the break frequency appears to be around 0.1 rad/s. This gives the partial transfer function:

$$TF(\omega) = \frac{\dfrac{j\omega}{0.1}}{1 + j\dfrac{\omega}{0.1}}$$

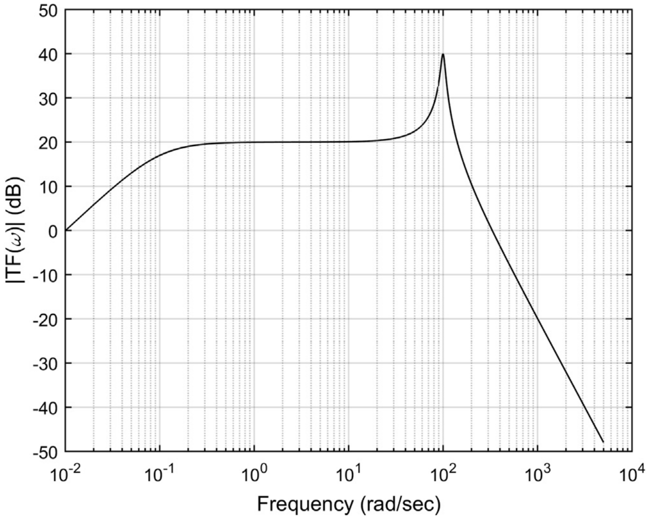


FIGURE 6.28 The magnitude spectrum used in Example 6.13. The goal in this example is to derive a transfer function that will give us this magnitude spectrum.

Result: Combining the two partial transfer functions gives:

$$TF(\omega) = \frac{100j\omega}{(1 + j10\omega)(1 - 10^{-4}\omega^2 + j10^{-3}\omega)} \tag{6.57}$$

Plotting this function will give the spectrum shown in Figure 6.28 as confirmed in one of the problems. Once we derive the transfer function, it is only one more step to design a system that has this transfer function, a step that is touched on later in this book. More relevant to us as bioengineers is that, if we can determine the frequency characteristics of a physiological process experimentally, we can use Bode plot primitives to determine its transfer function. From this transfer function, we can develop a quantitative model for this system. Moreover, we can predict the response of this biosystem to a wide range of input stimuli by combining the transfer function with Fourier decompositions. Examples of this approach are given in the next section.

6.7 THE TRANSFER FUNCTION AND THE FOURIER TRANSFORM

Combining the transfer function with the Fourier transform is a no-brainer: it allows you to predict the output to any signal that can be decomposed into a sinusoidal series. The approach is straightforward: (1) decompose the input signal into a Fourier series using the complex representation; (2) at each frequency, find the output signal by multiplying the input sinusoid by the transfer function; and (3) construct the output waveform from the output sinusoidal series using the inverse Fourier transform. Although this seems like a roundabout approach, it is really easy to do in MATLAB. In Step 1, we keep the complex sinusoidal representation as real and imaginary components rather than converting to magnitude and phase. Then in Step 2, we simply multiply the complex spectrum by the complex transfer function to get a representation of the output sinusoidal series, also complex. For Step 3, we apply the inverse Fourier transform directly to the output of Step 2 to get our output signal. This approach is illustrated in Figure 6.29 and implemented in the next example. More examples are explored in later chapters. Note that this decomposition approach invokes the principle of superposition since it assumes the system responds the same way to each sinusoidal component whether presented in isolation or as a combined signal.

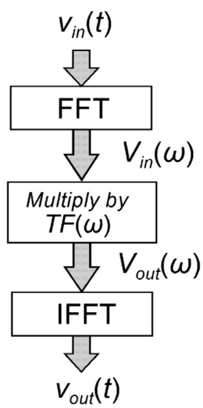


FIGURE 6.29 The three steps involved in predicting the output signal, $v_{out}(t)$, of any system, to any input signal, $v_{in}(t)$, given the transfer function of the system, $TF(\omega)$.

EXAMPLE 6.14

Find the output of the system having the following transfer function when the input is the electroencephalography (EEG) signal shown Figure 1.7 and found as variable `eeg` in file `eeg data.mat`. Plot both input and output signals in both the time and the frequency domains.

$$TF(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{1}{1 - .05\omega^2 + j.1\omega} \tag{6.58}$$

Solution: In MATLAB, Step 1 applies the `fft` command directly to the EEG signal, essentially giving us $V_{in}(\omega)$ as real and imaginary components. Step 2 finds the output from these sinusoids by multiplying their complex representation with the transfer function: $V_{out}(\omega) = TF(\omega)\ V_{in}(\omega)$. In Step 3, this frequency-domain output (still represented as real and imaginary components) is converted back into the time domain using the inverse Fourier transform command, `ifft`. The program plots the original time-domain signal and the calculated system output in both time and frequency domains. It also plots the magnitude spectrum of transfer function, Equation 6.58.

```
% Example 6.14 Apply a specific transfer function to EEG data

%
load eeg_data;                          % Load EEG data (in variable eeg)
N = length(eeg);                        % Get data length
fs = 50;                                % Sample frequency is 60 Hz
t = (1:N)/fs;                           % Construct time vector
f = (0:N-1)*fs/N;                       %  and frequency vector for plotting
Vin = (fft(eeg));                       % Step 1: decompose data
TF = 1./(1 - .002*(2*pi*f).^2 + j*.003*2*pi*f);    % Define TF
Vout = Vin .* TF                        % Step 2, get output
vout = ifft(Vout);                      % Step 3 Convert output to time domain
  .......plot and label system input and output in the time and frequency
domains.......
  .......plot magnitude spectrum of transfer function.........
```

The program loads the data, constructs time and frequency vectors based on the sampling frequency, and plots the time data. Next, the `fft` routine converts the EEG time data to the frequency domain, the transfer function is defined, then multiplied with input signal frequency components. The output of the product of transfer function and the signal (in the frequency domain) is converted back into the time domain using the inverse Fourier transform. Note that each of the steps in Figure 6.29 requires only one line of MATLAB code. The last section plots the magnitude spectra and time data of the input and output signals and the magnitude of the transfer function.

Results: The plots generated by this program are shown in Figures 6.30−6.32. The transfer function indicates that the system is of second order, but the magnitude spectrum, Figure 6.32, shows that it is acting as a narrowband band-pass filter with a peak frequency between 3 and 4 Hz. This operation is evident in the frequency-domain plots of the input and output signals, Figure 6.31, which shows the system reduces both high- and low-frequency components. This emphasizes EEG activity around 3 to 4 Hz and, as shown in the time-domain plots, Figure 6.30, appreciably alters the appearance of the EEG signal.

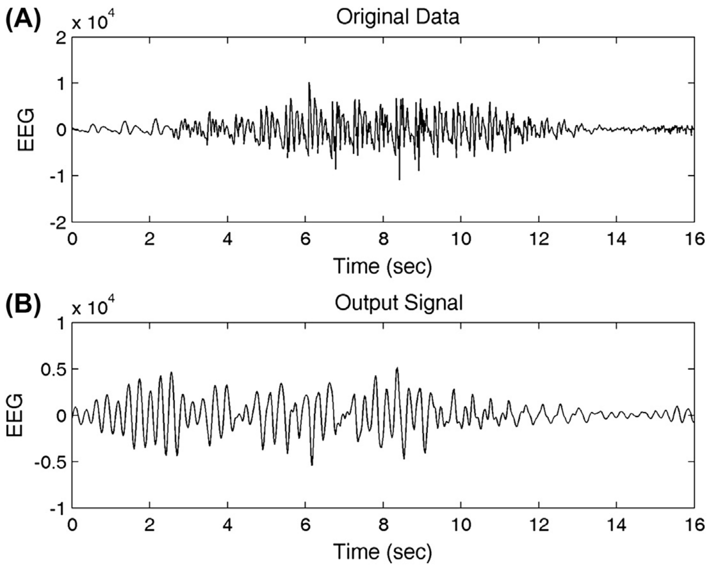


FIGURE 6.30 (A) The electroencephalography signal that is used as input to the system defined by the transfer function in Equation 6.58. (B) The output signal after passing through the system. This signal clearly contains a smaller range of frequencies.

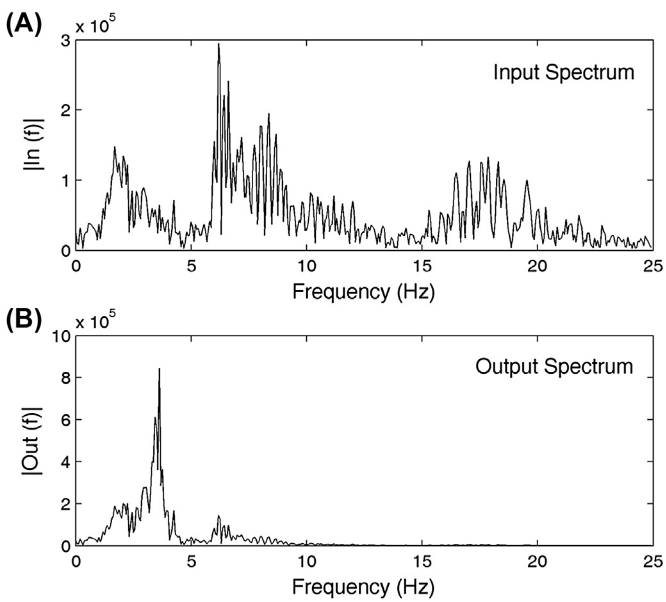


FIGURE 6.31 The magnitude spectra of the input (A) and output signals (B). The action of a filter that reduces the high- and low-frequency components is evident.

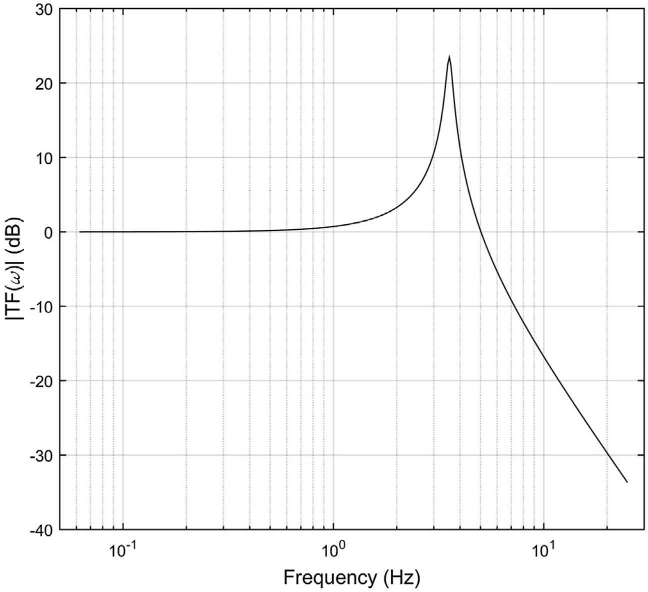


FIGURE 6.32 The magnitude spectrum of the system used in Example 6.14 and defined by the transfer function in Equation 6.58. The system acts as a narrowband filter with a peak frequency between 3 and 4 Hz. The bandwidth of this filter is approximately 0.4 Hz.

6.8 SUMMARY

Linear systems can be represented by differential equations and contain combinations of only four unique elements. If the input signals can be restricted to steady-state sinusoids, then phasor techniques can represent these elements using algebraic equations that are functions only of frequency. Phasor techniques represent steady-state sinusoids as a single complex number. With this representation, the calculus operation of differentiation can be implemented in algebra simply as multiplication by $j\omega$, where $j = \sqrt{-1}$ and ω is frequency in radians. Similarly, integration becomes division by $j\omega$.

Since system elements can be represented by algebraic operations, they can be combined into a single equation termed the transfer function. The transfer function relates the output of a system to the input. As presented here, the transfer function can only deal with signals that are sinusoidal or can be decomposed into sinusoids, but the transfer function concept is extended to a wider class of signals in the next chapter.

The transfer function not only offers a succinct representation of the system, it also gives a direct link to the frequency spectrum of the system. Frequency plots can be constructed directly from the transfer function using MATLAB, or they can be drawn by hand using a graphical technique termed Bode plot methods. With the aid of Bode plot methods, we can also go the other way, from a frequency plot to the corresponding transfer function.

Using the Bode plot approach, we are able to estimate the transfer function of a system given its magnitude spectrum. The transfer function can then be used to predict the system's behavior to other input signals, including signals that might be difficult to generate experimentally. These signals must be suitable for Fourier decomposition, but that applies to all digital signals stored in finite memory. The transfer function concept, coupled with Bode primitives, is a powerful tool for determining the transfer function representation of a wide range of physiological systems and for predicting their behavior to an equally broad range of stimuli.

If a real system can be stimulated and the resultant response measured, the system's spectral response can be used to estimate its transfer function. Alternatively, we can use the system's own natural stimulus as long as we can measure that stimulus and it contains energy over the spectral frequencies of interest. Determining a system's transfer function—based input/output data is an area of ongoing development known as "systems identification," and is briefly covered in the next chapter.

PROBLEMS

1. Assume that the feedback control system presented in Example 6.1 is in a steady-state or static condition. If $G = 100$ and $H = 1$ (i.e., a unity gain feedback control system), find the output if the input equals 1. Find the output if the input is increased to 10. Find the output if the input is 10 and G is increased to 1000. Note how the output is proportional to the input, which accounts for why this system (having the configuration shown) is sometimes termed a "proportional control system."
2. In the system given in Problem 1 with $G = 100$, the input is changed to a signal that smoothly goes from 0.0 to 5.0 in 10 s (i.e., $In(t) = 0.5\ t$ s). What does the output look like? Give the equation for the output. (Note G and H are simple constants so Equation 6.7 still holds.)
3. Convert the following to phasor representation:

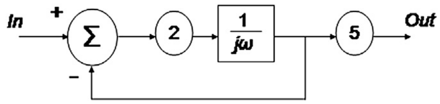
   A. $10\cos(10t)$
   B. $5\sin(5t)$
   C. $6\sin(2t + 60)$
   D. $2\cos(5t) + 4\sin(5t)$
   E. $\int 5\cos(20t)dt$
   F. $\dfrac{d(2\cos(20t + 30))}{dt}$
4. Add the following real, imaginary, and complex numbers to form a single complex number:

$$6, j10,\ 5 + j12,\ 8 + j3,\ 10\angle 0,\ 5\angle -60,\ 1/(j0.1)$$

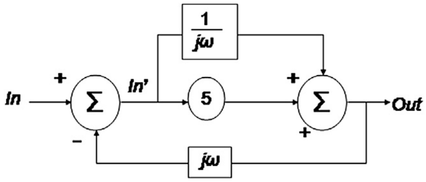
5. Evaluate the following expressions:

   A. $(10+j6) + 10\angle{-30} - 10\angle 30$
   B. $6\angle -120 + \dfrac{5 - j10}{j4}$
   C. $\dfrac{10 + j5}{16 - j6} - \dfrac{8 - j8}{12 + j4}$
   D. $\dfrac{j(6 + j5)(3 - j4)}{(8 + j3)10\angle 260}$

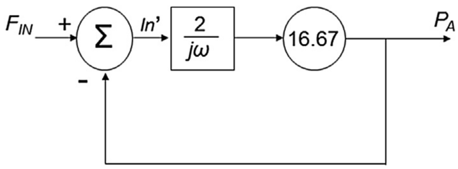
6. Find the transfer function, $Out(\omega)/In(\omega)$, using phasor notation for the following system.



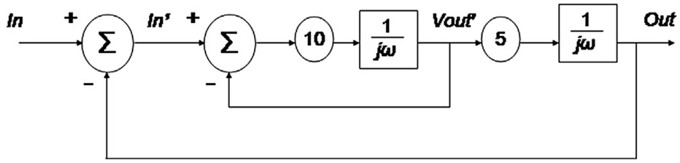
7. Find the transfer function, $Out(\omega)/In(\omega)$, for the following system. Find the time function, $out(t)$, if $in(t) = 5\cos(5t - 30)$. (Hint: You could use the feedback equation to find this transfer function, but the feedforward term contains an arithmetic operator; specifically, a summation operation.)



8. Using the transfer function of the respirator-airway pathway given in Example 6.7, find the airway pressure, $Q(\omega)$, if the respirator produces an external sinusoidal pressure of 2.0 mmHg at 11 breaths per minute. (Hint: Generate the input function $P(\omega)$, then solve using phasor analysis. Recall the transfer function is in minutes, which needs to be taken into account for the calculation of $P(\omega)$.)
9. Find the transfer function of the modified Guyton–Coleman fluid balance system shown below. Find the time function $P_A(t)$ if $F_{IN} = 5\sin(5t-30)$.



10. Find the transfer function of the following circuit. Find the time function, $out(t)$, if $in(t) = 5\sin(5t - 30)$. (Hint: This system contains a feedback system in the feedforward path and the feedback equation must be applied twice.)



11. Plot the magnitude spectra of the transfer functions in Problems 6 and 7. What is the difference between the two?

12. Modify Equations 6.36 and 6.37 to find the equations for the asymptotes and worst case angle for the phase spectrum of the transfer function: $TF(\omega) = 1 + j\frac{\omega}{\omega_1}$.

13. Modify Equations 6.44–6.49 to show that when the second-order term appears in the numerator, the magnitude and phase spectra are inverted. Use MATLAB to plot the magnitude and phase spectra of the second-order transfer function used in Example 6.9 but with the second-order term in the numerator (i.e., $TF(\omega) = 1 - 0.03\omega^2 + j0.1\omega$).

14. Plot the Bode plot (magnitude and phase) for the following transfer function using graphical techniques.

$$TF(\omega) = \frac{100(1 + j.05\omega)}{(1 + j.01\omega)}$$

15. Plot the Bode Plot (magnitude and phase) for the following transfer function using graphical techniques:

$$TF(\omega) = \frac{100(1 + j\omega)}{(1 + j0.005\omega)(1 + j0.0002\omega)}$$

16. Plot the Bode plot (magnitude and phase) of the following transfer function using graphical techniques.

$$TF(\omega) = \frac{10j\omega}{(1 - 0.0001\omega^2 + j0.002\omega)}$$

17. Plot the Bode plot for the transfer function of the respirator-airway system given in Example 6.7 using graphical methods. Compare with the spectrum generated by MATLAB given in Figure 6.8.

18. Use MATLAB to plot the transfer functions (magnitude and phase) given in Problems 14 and 16.

19. Plot the Bode plot (magnitude and phase) of the following transfer function using graphical techniques.

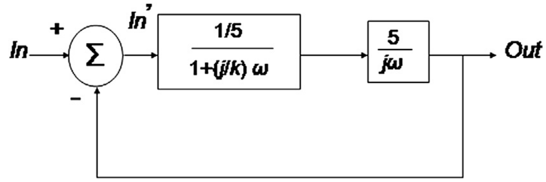
$$TF(\omega) = \frac{100(1 + 0.1j\omega)}{(1 - 0.0004\omega^2 + j0.028\omega)}$$

20. Use MATLAB to plot the transfer functions (magnitude and phase) given in Problems 16 and 19.

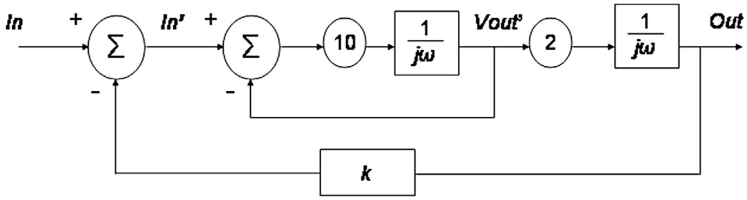
21. In the system used in Problem 10, find the values of the ω_n and δ.

22. For the respirator-airway system of Example 6.7, find the values of ω_n and δ. Remember that time for this model is in minutes (not seconds) so this will have an effect on the value of ω_n. Convert ω_n to Hz and determine the breaths per minute that correspond to that frequency.

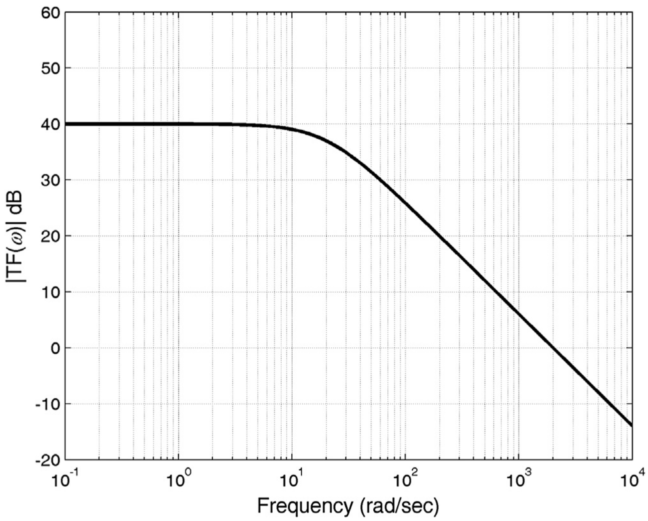
23. In the following system, find the value of k that makes $\omega_n = 10$ rad/s.



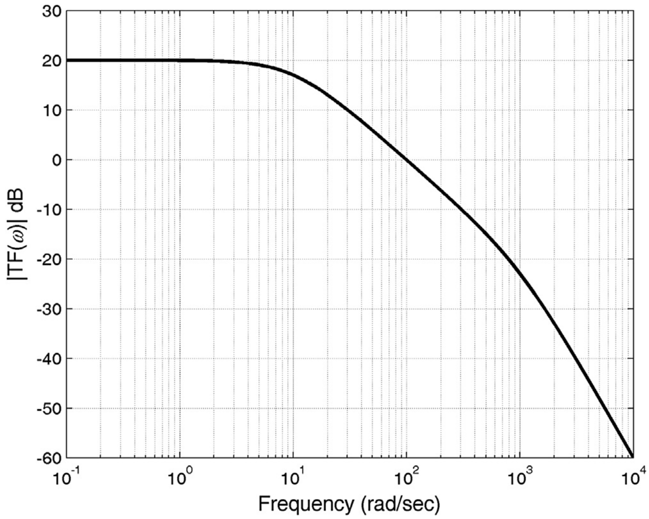
24. In the following system, find the maximum value of k so that the roots of the transfer function are real. As we show in the next chapter, the TF has real roots when the value of k results in $\delta \geq 1$.



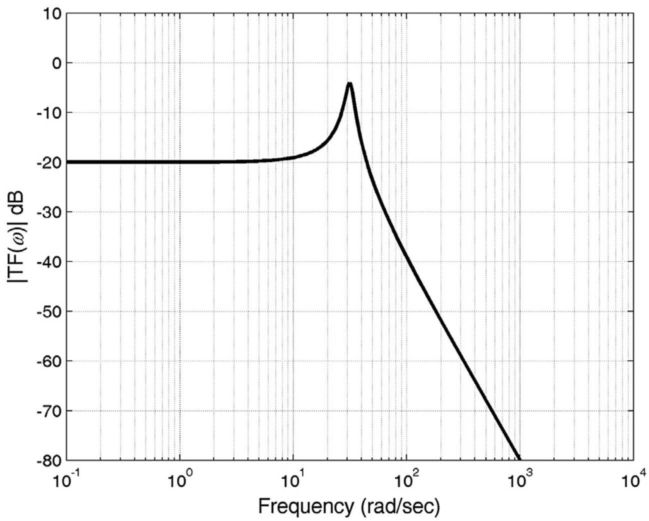
25. Estimate the transfer function that produces the following magnitude spectrum curve. Use MATLAB to plot the phase curve graphically of your estimated transfer function.



26. Find the transfer function that produces the following magnitude frequency curve. Plot the magnitude spectrum of your estimated transfer function using MATLAB and compare. Also plot the phase spectrum using MATLAB.



27. Find the transfer function that produces the following magnitude frequency curve. Plot the magnitude spectrum of your estimated transfer function using MATLAB and compare. Also use MATLAB to plot the phase spectrum.



28. Find the transfer function that produces the following magnitude frequency curve. Plot the magnitude spectrum of your estimated transfer function using MATLAB and compare. Also use MATLAB to plot the phase spectrum.