Appendix A
Derivations

A.1 DERIVATION OF EULER'S FORMULA

Assume a sinusoidal function:

$$x = \cos\theta + j\sin\theta \tag{A.1}$$

(where $j = \sqrt{-1}$ as usual)

Differentiating with respect to θ produces:

$$\frac{dx}{d\theta} = j(\cos\theta + j\sin\theta) = jx \tag{A.2}$$

Separating the variables gives:

$$\frac{dx}{x} = jd\theta \tag{A.3}$$

and integrating both sides gives:

$$\ln x = j\theta + K$$

where K is the constant of integration. To solve for this constant, note that in Equation (A.1): $x = 1$ when $\theta = 0$. Applying this condition to Equation (A.3):

$$\ln 1 = 0 = 0 + K; \quad K = 0;$$

so Equation (A.3) becomes: $\ln x = j\theta$.
or

$$x = e^{j\theta} \tag{A.4}$$

but since x is defined in Equation (A.1) as: $\cos\theta + j\sin\theta$

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{A.5}$$

Alternatively,

$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{A.6}$$

A.2 CONFIRMATION OF THE FOURIER SERIES

Fourier showed that a periodic function of period T can be represented by a series, possibly infinite, of sinusoids, or sine and cosines:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega_o t + b_n \sin n\omega_o t) \tag{A.7}$$

where $\omega_o = 2\pi/T$ and a_n and b_n are the Fourier coefficients.

To derive the Fourier coefficients, let us begin with the a_0 or DC term. Integrating both sides of Equation (A.7) over a full period:

$$\int_0^T x(t)dt = \int_0^T \frac{a_0}{2}dt + \sum_{n=1}^{\infty}\int_0^T (a_n \cos n\omega_o t + b_n \sin n\omega_o t)dt \tag{A.8}$$

For $n = 0$, the second term on the right-hand side is zero since the sum begins at $n = 1$, and the equation becomes

$$\begin{aligned} &\int_0^T x(t)dt = \int_0^T \frac{a_0}{2}dt; \quad \int_0^T x(t)dt = \frac{a_0 T}{2}; \\ &a_0 = \frac{2}{T}\int_0^T x(t)dt \end{aligned} \tag{A.9}$$

To find the other coefficients, multiply both sides of Equation (A.7) by $\cos(m\omega_o t)$, where m is an integer, and again integrate both sides.

$$\begin{aligned} \int_0^T x(t)\cos(m\omega_o t)dt = &\int_0^T \frac{a_0}{2}\cos(m\omega_o t)dt + \sum_{n=1}^{\infty}\int_0^T (a_n \cos(m\omega_o t)\cos n\omega_o t \\ &+ b_n \cos(m\omega_o t)\sin n\omega_o t)dt \end{aligned} \tag{A.10}$$

Rearranging:

$$\begin{aligned} \int_0^T x(t)\cos(m\omega_o t)dt = &\int_0^T \frac{a_0}{2}\cos(m\omega_o t)dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos(m\omega_o t)\cos n\omega_o t\, dt \\ &+ \sum_{n=1}^{\infty} b_n \int_0^T \cos(m\omega_o t)\sin n\omega_o t \end{aligned} \tag{A.11}$$

Since m is an integer, the first and third terms on the right-hand side integrate to zero for all m. The second term also integrates to zero for all m except $m = n$. At $m = n$, the second term becomes:

$$an \int_0^T \cos^2(n\omega_o t)dt = \frac{\pi}{\omega_o}a_n = \frac{T}{2}a_n; \tag{A.12}$$

so that:

$$\begin{aligned}\frac{T}{2}a_n &= \int_0^T x(t)dt \\ a_n &= \int_0^T \cos(n\omega_o t)dt; \quad m = 1, 2, 3, \ldots\end{aligned} \tag{A.13}$$

The bn coefficients are found in a similar fashion except Equation (A.7) is multiplied by $\sin(m\omega_o t)$, then integrated. In this case, all but the third term integrate to zero and the third term is nonzero only for $m = n$.

$$bn \int_0^T \sin^2(n\omega_o t)dt = \frac{\pi}{\omega_o}b_n = \frac{T}{2}b_n;$$

so that:

$$\begin{aligned}\frac{T}{2}b_n &= \int_0^T x(t)dt \\ b_n &= \int_0^T \sin(n\omega_o t)dt; \quad m = 1, 2, 3, \ldots\end{aligned}$$

A.3 DERIVATION OF THE TRANSFER FUNCTION OF A SECOND-ORDER OP AMP FILTER

The op amp circuit for a second-order low-pass filter is shown in Figure A.1.

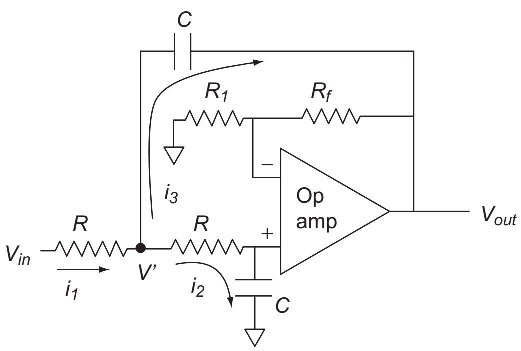

FIGURE A.1 A second-order op amp filter circuit.

This derivation applies not only to the following low-pass version, but also to the high-pass version where the positions of R and C are reversed.

Note that at node V' by KCL: $i_1 - i_2 - i_3 = 0$.

This allows us to write a nodal equation around that node:

$$\frac{V_{in} - V'}{R} - \frac{V' - V_{out}}{\frac{1}{CS}} - i_2 = 0$$

where:

$$i_2 = \frac{V^+}{\frac{1}{Cs}} = V^+ Cs$$

Since the two terminals of an op amp must be at the same voltage, the voltage V^+ must be equal to V^-. Applying the voltage divider equation to the two feedback resistors, V^-, and hence V^+, can be found in terms of V_{out}:

$$V^+ = V^- = V_{out} G$$

where

$$G = \frac{R_1}{R_f + R_1}$$

So i_2 becomes: $V_{out}(Cs)/G$. Substituting i_2 into the nodal equation at V':

$$\frac{V_{in} - V'}{R} - (V_{out} - V')Cs - \frac{V_{out}(Cs)}{G} = 0$$

$$\frac{V_{in}}{R} - \frac{V'}{R} - V'Cs + V_{out}Cs - \frac{V_{out}Cs}{G} = 0$$

$$\frac{V_{in}}{R} - V'\left(\frac{1}{R} - V'Cs\right) + V_{out}Cs\left(1 - \frac{1}{G}\right) = 0$$

Note that V' can also be written in terms of just i_2:

$$V' = i_3\left(R + \frac{1}{CS}\right) = \frac{V_{out}}{G}\left(R + \frac{1}{CS}\right)$$

Substituting this for V' in the nodal equation:

$$\frac{V_{in}}{R} - \frac{V_{out}Cs}{G}\left(\frac{1}{R} + Cs\right)\left(R + \frac{1}{Cs}\right) + V_{out}Cs\left(1 - \frac{1}{G}\right) = 0$$

$$\frac{V_{in}}{R} = \frac{V_{out}Cs}{G}\left(3 + \frac{1}{RCs} + RCs - G\right)$$

Solving for V_{out}/V_{in}:

$$\frac{V_{out}}{V_{in}} = \frac{G}{RCs\left(1 + \dfrac{1}{RC} + RCs - g\right)} = \frac{G}{(RCs)^2 + (3 - G)RCs + 1}$$

$$\frac{V_{out}}{V_{in}} = \frac{\dfrac{G}{(RC)^2}}{S2 + \dfrac{3 - G}{RC}s + \dfrac{1}{(RC)^2}}$$

A.4 DERIVATION OF THE TRANSFER FUNCTION OF AN INSTRUMENTATION AMPLIFIER

The classic circuit for a three-op amp instrumentation amplifier is shown in Figure A.2.

To determine the transfer function, note that the voltage V_{in1} appears on both terminals of op amp 1, whereas V_{in2} appears on both terminals of op amp 2. The voltage out of op amp 1 will be equal to V_{in2} plus the voltage drop across the two resistors, R_2 and R_1:

$$V_{out1} = i_{12}(R_1 + R_2) + V_{in2};$$

but

$$i_{12} = \frac{V_{in1} - V_{in2}}{R_1}$$

$$V_{out1} = \frac{V_{in1} - V_{in2}}{R_1}(R_1 + R_2) + V_{in2}$$

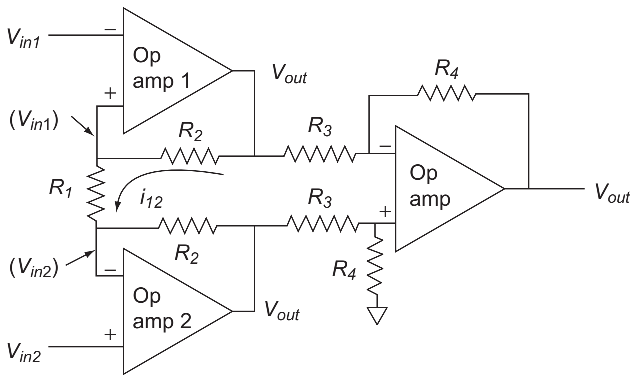

FIGURE A.2 Instrumentation amplifier circuit.

Applying the same logic to op amp 2, its output, V_{out2}, can be written as:

$$
\begin{aligned}
V_{out2} - V_{out1} &= \frac{V_{in1} - V_{in2}}{R_1}(R_1 + R_2)2 - (V_{in1} - V_{in2}) \\
&= \left[\frac{2(R_1 + R_2)}{R_1} - 1\right](V_{in1} - V_{in2}) = \left[\frac{2R_1 + 2R_2 - R_1}{R_1}\right](V_{in1} - V_{in2}) \\
V_{out2} - V_{out1} &= \left(\frac{R_1 + 2R_2}{R_1}\right)(V_{in1} - V_{in2}) \\
V_{out} &= \left(\frac{R_4}{R_3}\right)(V_{out2} - V_{out1}) = \left(\frac{R_4}{R_3}\right)\left(\frac{R_1 + 2R_2}{R_1}\right)(V_{in1} - V_{in2})
\end{aligned}
$$

The overall output, V_{out}, is equal to the difference of $V_{out2} - V_{out1}$ times the gain of the differential amplifier: R_4/R_3.