UNIT-1

CHAPTER-2 (CO1)

1.2.1

Inference Theory of Statement Calculus

The interference theory can be described as the analysis of validity of the formula from the given set of premises.

Structure of an argument

An argument can be defined as a sequence of statements. The argument is a collection of premises and a conclusion. The conclusion is used to indicate the last statement, and premises are used to indicate all the remaining statements. Before the conclusion, the symbol ∴ will be placed. The following syntax is used to show the premises and conclusion:

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Premises: p1, p2, p3, p4, ….., pn
Conclusion: q
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If $(p_1 \land p_2 \land p_3 \land p_4 \land \ldots\ldots \land p_n) \rightarrow q$ indicates a topology, in this case, the argument will be termed as valid otherwise, it will be termed invalid. The following expression is used to show the argument:

1. First premises
2. Second premises
3. Third premises
4. Fourth premises
5. .
6. .
7. Nth premises
8. _____________
9. ∴ Conclusion

Valid Argument:

A valid argument can be described as an argument where if all their premises are true, then their conclusions will also be true.

For example:

1. "If tomorrow is holiday, I will go to mall."
2. "If tomorrow is holiday".
3. ∴ "I will go to mall."

This argument belongs to a form that is described as follows:

1. $P \rightarrow Q$
2. P
3. ____________
4. ∴ Q

The above form is valid. It does not matter what propositions are substituted for the variables. This type of form is known as the valid argument form. From the above definition, if a valid argument form consists

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premises: p₁, p₂, p₃, p₄, ….., pₙ
conclusion: q
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then $(p_1 \wedge p_2 \wedge p_3 \wedge .... \wedge p_k) \rightarrow q$ is a tautology. That means $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.

Rules of Inference

We can construct a more complicated valid argument with the help of using simple arguments, which work as the building blocks. If we are talking about the usage of arguments, then there are some simple arguments that have been established as valid and very important. These types of arguments are known as the Rules of inference. There are various types of Rules of inference, which are described as follows:

1. Modus Ponens

Suppose there are two premises, P and $P \rightarrow Q$. Now, we will derive Q with the help of Modules Ponens like this:

1. $P \rightarrow Q$
2. P
3. ____________
4. ∴ Q

Example:

Suppose $P \rightarrow Q$ = "If we have a bank account, then we can take advantage of this new policy."

P = "We have a bank account."

Therefore, Q = "We can take advantage of this new policy."

2. Modus Tollens

Suppose there are two premises, $P \rightarrow Q$ and $\neg Q$. Now, we will derive $\neg P$ with the help of Modules Tollens like this:

1. $P \rightarrow Q$
2. $\neg Q$
3. ____________
4. $\therefore \neg P$

Example:

Suppose $P \rightarrow Q$ = "If we have a bank account, then we can take advantage of this new policy."

$\neg Q$ = "We cannot take advantage of this new policy."

Therefore, $\neg P$ = "We don't have a bank account."

3. Hypothetical Syllogism

Suppose there are two premises, $P \rightarrow Q$ and $Q \rightarrow R$. Now, we will derive $P \rightarrow R$ with the help of Hypothetical Syllogism like this:

1. $P \rightarrow Q$
2. $Q \rightarrow R$
3. ____________
4. $\therefore P \rightarrow R$

Example:

Suppose $P \rightarrow Q$ = "If my fiancé comes to meet me, I will not go to office."

$Q \rightarrow R$ = "If I will not go to office, I won't require to do office work."

Therefore, $P \rightarrow R$ = "If my fiancé come to meet me, I won't require to do office work."

4. Disjunction Syllogism

Suppose there are two premises $\neg P$ and $P \vee Q$. Now, we will derive Q with the help of Disjunction Syllogism like this:

1. ¬P
2. P ∨ Q
3. ____________
4. ∴ Q

Example:

Suppose ¬P = "Harry birthday cake is not strawberry flavored."

P ∨ Q = "The birthday cake is either red velvet flavored or mixed fruit flavored."

Therefore, Q = "The birthday cake is mixed fruit flavored."

5. Addition

Suppose there is a premise P. Now, we will derive P ∨ Q with the help of Addition like this:

1. P
2. ____________
3. ∴ P ∨ Q

Example:

Suppose P be the proposition, "Harry is a hard working employee" is true

Here Q has the proposition, "Harry is a bad employee".

Therefore, "Either Harry is a hard working employee Or Harry is a bad employee".

6. Simplification:

Suppose there is a premise P ∧ Q. Now, we will derive P with the help of Simplification like this:

1. P∧Q
2. ____________
3. ∴ P

Example:

Suppose P ∧ Q = "Harry is a hard working employee, and he is the best employee in the office".

Therefore, "Harry is a hard working employee".

7. Conjunction

Suppose there are two premises P and Q. Now, we will derive P ∧ Q with the help of conjunction like this:

1. P
2. Q
3. ____________
4. ∴ P ∧ Q

Example:

Suppose P = "Harry is a hard working employee".

Suppose Q = "Harry is the best employee in the office".

Therefore, "Harry is a hard working employee and Harry is the best employee in the office".

8. Resolution

Suppose there are two premises P ∨ Q and ¬P ∨ R. Now, we will derive Q ∨ R with the help of a resolution like this:

1. P ∨ Q
2. ¬ P ∨ R
3. ____________
4. ∴ Q ∨ R

Example:

Suppose P → Q = "If my fiancé comes to meet me, I will not go to office".

¬ P ∨ R = "If my fiancé did not come to met me, I won't require to do office work".

Therefore, Q ∨ R = "Either I will not go to office or I won't require to do office work".

9. Constructive Dilemma

Suppose there are two premises (P → Q) ∧ (R → S) and P ∨ R. Now, we will derive Q ∨ S with the help of a constructive dilemma like this:

1. (P → Q) ∧ (R → S)
2. P ∨ R
3. ____________
4. ∴ Q ∨ S

Example:

Suppose $P \rightarrow Q$ = "If my fiancé will come to meet me, I will not go to office".

$R \rightarrow S$ = "If my relatives will come, I will tell my employees that I will come".

$P \lor R$ = "Either my fiancé will comes to meet me or my relatives will come".

Therefore, $Q \lor S$ = "Either I will not go to office or I will tell my employees that I will come".

10. Destructive Dilemma

Suppose there are two premises $(P \rightarrow Q) \land (R \rightarrow S)$ and $\neg Q \lor \neg S$. Now, we will derive $\neg P \lor \neg R$ with the help of a Destructive dilemma like this:

1. $(P \rightarrow Q) \land (R \rightarrow S)$
2. $\neg Q \lor \neg S$
3. ____________
4. $\therefore \neg P \lor \neg R$

Example:

Suppose $P \rightarrow Q$ = "If my fiancé comes to meet me, I will not go to office".

$R \rightarrow S$ = "If my relatives come, I will tell my employees that I will come".

$\neg Q \lor \neg S$ = "Either I will go to office or I will tell my employees that I will not come".

Therefore, $\neg P \lor \neg R$ = "Either my fiancé will not come to meet me or my relatives will not come".

Rules of Inference with Quantifiers

There are some other rules of inference with quantifier statements, which are described as follows:

1. Universal Instantiation

Suppose there is a premise $\forall x\, P(x)$. Now, we will derive $P(c)$ with the help of Universal Instantiation like this:

1. $\forall x\, P(x)$
2. ____________
3. $\therefore P(c)$, for any c

2. Universal Generalization

Suppose there is a premise P(c) for any arbitrary c. Now, we will derive ∀x P(x) with the help of a Universal generalization like this:

1. P(c) for any arbitrary c
2. ____________
3. ∴ ∀x P(x)

3. Existential Instantiation

Suppose there is a premise ∃x P(x). Now, we will derive P(c) for some element c with the help of Existential Instantiation like this:

1. ∃x P(x)
2. ____________
3. ∴ P(c), for some element c

4. Existential Generalization

Suppose there is a premise P(c) for some element c. Now, we will derive ∃x P(x) with the help of Existential generalization like this:

1. P(c) for some element c
2. ____________
3. ∴ ∃x P(x)

Example of rules of inferences

Example 1:

If my fiancé comes to meet me, then I will be happy.

If my fiancé does not come to meet me, then I will go to office.

If I go to office, then I will complete my work.

Can we conclude, "If I am not happy, then I will complete my work"?

Solution: We will simplify this discussion by identifying propositions and using the variables of propositional to represent them like this:

P := My fiancé come to meet me

Q := I will be happy

R := I will go to office

S := I will complete my work

After putting the above propositional variables, we will get the following premises and conclusion like this:

1. Premises: $P \rightarrow Q$, $\neg P \rightarrow R$, $R \rightarrow S$
2. Conclusion: $\neg Q \rightarrow S$

Now we will use the rules of inference so that we can deduce the conclusion with the help of a given hypothesis like this:

Step	Reason
$P \rightarrow Q$	Premise
$\neg Q \rightarrow \neg P$	Contrapositive of (1)
$\neg P \rightarrow R$	Premise
$\neg Q \rightarrow R$	Hypothetical Syllogism by (2) and (3)
$R \rightarrow S$	Premise
$\neg Q \rightarrow \neg S$	Hypothetical Syllogism by (4) and (5)

Example 2:

An employee in my office has not completed his daily work

Everyone in my office completed his monthly files.

Can we conclude, "Someone who completed his monthly files has not completed his daily work"?

Solution: We will simplify this discussion by identifying propositions and using the variables of propositional to represent them like this:

C(x) := x is a employee in my office

B(x) := x has completed his daily work

P(x) := x completed his monthly files

After putting the above propositional variables, we will get the following premises and conclusion like this:

1. Premises: $\exists x\ (\ C(x) \land \neg B(x)\ )$
2. $\forall x\ (\ C(x) \rightarrow P(x)\ )$
3. Conclusion: $\exists x\ (\ P(x) \land \neg B(x)\ )$

Now we will use the rules of inference for quantifiers so that we can deduce the conclusion with the help of a given hypothesis like this:

No.	Step	Reason
1	$\exists x\ (\ C(x) \land \neg B(x)\ )$	Premise
2	$C(a) \land \neg B(a)$	Existential Instantiation
3	$C(a)$	Simplified by (2)
4	$\forall x\ (\ C(x) \rightarrow P(x)\ )$	Premise
5	$C(a) \rightarrow P(a)$	Universal Instantiation
6	$P(a)$	Modus Ponens by (3) and (5)
7	$\neg B(a)$	Simplified by (2)
8	$P(a) \land \neg B(a)$	Conjunction by (6) and (7)
9	$\exists x\ (\ P(x) \land \neg B(x)\ )$	Existential Generalization

Predicate Calculus

Predicate Logic deals with predicates, which are propositions, consist of variables.

Predicate Logic - Definition

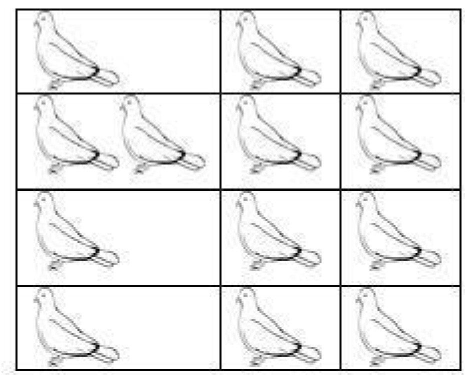
A predicate is an expression of one or more variables determined on some specific domain. A predicate with variables can be made a proposition by either authorizing a value to the variable or by quantifying the variable.

The following are some examples of predicates.
Consider X(a, b, c) denote "a + b + c = 0"

- Consider M(x, y) denote "x is married to y."

The Pigeonhole Principle and Application (CO4)

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.



Theorem –

I) If "A" is the average number of pigeons per hole, where A is not an integer then

- At least one pigeon hole contains **ceil[A]** (smallest integer greater than or equal to A) pigeons
- Remaining pigeon holes contains at most **floor[A]** (largest integer less than or equal to A) pigeons

Or

II) We can say as, if n + 1 objects are put into n boxes, then at least one box contains two or more objects.

The abstract formulation of the principle: Let X and Y be finite sets and let be a function.

- If X has more elements than Y, then f is not one-to-one.
- If X and Y have the same number of elements and f is onto, then f is one-to-one.

- If X and Y have the same number of elements and f is one-to-one, then f is onto.

Pigeonhole principle is one of the simplest but most useful ideas in mathematics. We will see more applications that proof of this theorem.

- **Example – 1:** If (Kn+1) pigeons are kept in n pigeon holes where K is a positive integer, what is the average no. of pigeons per pigeon hole?
  Solution: average number of pigeons per hole = (Kn+1)/n
  = K + 1/n
  Therefore there will be at least one pigeonhole which will contain at least (K+1) pigeons i.e., ceil[K +1/n] and remaining will contain at most K i.e., floor[k+1/n] pigeons.
  i.e., the minimum number of pigeons required to ensure that at least one pigeon hole contains (K+1) pigeons is (Kn+1).
- **Example – 2:** A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?
  Solution: Apply pigeonhole principle.
  No. of colors (pigeonholes) n = 3
  No. of marbles (pigeons) K+1 = 4
  Therefore the minimum no. of marbles required = Kn+1
  By simplifying we get Kn+1 = 10.
  Verification: ceil[Average] is [Kn+1/n] = 4
  [Kn+1/3] = 4
  Kn+1 = 10
  i.e., 3 red + 3 white + 3 blue + 1(red or white or blue) = 10

Pigeonhole principle strong form –
Theorem: Let $q_1, q_2, \ldots, q_n$ be positive integers.
If $q_1 + q_2 + \ldots + q_n – n + 1$ objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, . . ., the nth box contains at least q_n objects.

Application of this theorem is more important, so let us see how we apply this theorem in problem solving.

- **Example – 1:** In a computer science department, a student club can be formed with either 10 members from first year or 8 members from second year or 6 from third year or 4 from final year. What is the minimum no. of students we have to choose randomly from department to ensure that a student club is formed?
  Solution: we can directly apply from the above formula where,
  q_1 =10, q_2 =8, q_3 =6, q_4 =4 and n=4
  Therefore the minimum number of students required to ensure department

club to be formed is
10 + 8 + 6 + 4 – 4 + 1 = 25

- **Example – 2:** A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum no. of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?
  Solution: Here in this we cannot blindly apply pigeon principle. First we will see what happens if we apply above formula directly.
  From the above formula we have get answer 47 because 6 + 8 + 10 + 12 + 15- 5 + 1 = 47
  But it is not correct. In order to get the correct answer we need to include only blue, yellow and white balls because red and green balls are less than 9. But we are picking randomly so we include after we apply pigeon principle.
  i.e., 9 blue + 9 yellow + 9 white – 3 + 1 = 25
  Since we are picking randomly so we can get all the red and green balls before the above 25 balls. Therefore we add 6 red + 8 green + 25 = 39
  We can conclude that in order to pick 9 balls of same color randomly, one has to pick 39 balls from a box.
    - **Example – 1:** If (Kn+1) pigeons are kept in n pigeon holes where K is a positive integer, what is the average no. of pigeons per pigeon hole?

      Solution: average number of pigeons per hole = (Kn+1)/n

      = K + 1/n

      Therefore there will be at least one pigeonhole which will contain at least (K+1) pigeons i.e., ceil[K +1/n] and remaining will contain at most K i.e., floor[k+1/n] pigeons.

      i.e., the minimum number of pigeons required to ensure that at least one pigeon hole contains (K+1) pigeons is (Kn+1).
    - **Example – 2:** A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?

      Solution: Apply pigeonhole principle.

      No. of colors (pigeonholes) n = 3

      No. of marbles (pigeons) K+1 = 4

      Therefore the minimum no. of marbles required = Kn+1

By simplifying we get Kn+1 = 10.

Verification: ceil[Average] is [Kn+1/n] = 4

[Kn+1/3] = 4

Kn+1 = 10

i.e., 3 red + 3 white + 3 blue + 1(red or white or blue) = 10

Pigeonhole principle strong form –

Theorem: Let $q_1, q_2, \ldots, q_n$ be positive integers.

If $q_1 + q_2 + \ldots + q_n - n + 1$ objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, . . ., the nth box contains at least q_n objects.

Application of this theorem is more important, so let us see how we apply this theorem in problem solving.

- **Example – 1:** In a computer science department, a student club can be formed with either 10 members from first year or 8 members from second year or 6 from third year or 4 from final year. What is the minimum no. of students we have to choose randomly from department to ensure that a student club is formed?

  Solution: we can directly apply from the above formula where,

  $q_1 = 10$, $q_2 = 8$, $q_3 = 6$, $q_4 = 4$ and n=4

  Therefore the minimum number of students required to ensure department club to be formed is

  10 + 8 + 6 + 4 – 4 + 1 = 25
- **Example – 2:** A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum no. of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

  Solution: Here in this we cannot blindly apply pigeon principle. First we will see what happens if we apply above formula directly.

From the above formula we have get answer 47 because 6 + 8 + 10 + 12 + 15- 5 + 1 = 47

But it is not correct. In order to get the correct answer we need to include only blue, yellow and white balls because red and green balls are less than 9. But we are picking randomly so we include after we apply pigeon principle.

i.e., 9 blue + 9 yellow + 9 white – 3 + 1 = 25

Since we are picking randomly so we can get all the red and green balls before the above 25 balls. Therefore we add 6 red + 8 green + 25 = 39

We can conclude that in order to pick 9 balls of same color randomly, one has to pick 39 balls from a box.

MCQ

1. Which rule of inference is used in each of these arguments, “If it is Wednesday, then the Smartmart will be crowded. It is Wednesday. Thus, the Smartmart is crowded.”
a) Modus tollens
b) Modus ponens
c) Disjunctive syllogism
d) Simplification
View Answer
Answer: b
Explanation: $(M \wedge (M \rightarrow N)) \rightarrow N$ is Modus ponens.

2. Which rule of inference is used in each of these arguments, “If it hailstoday, the local office will be closed. The local office is not closed today. Thus, it did not hailed today.”
a) Modus tollens
b) Conjunction
c) Hypothetical syllogism
d) Simplification
View Answer
Answer: a
Explanation: $(\neg N \wedge (M \rightarrow N)) \rightarrow \neg M$ is Modus tollens.

3. Which rule of inference is used, ”Bhavika will work in an enterprise this summer. Therefore, this summer Bhavika will work in an enterprise or he will go to beach.”
a) Simplification
b) Conjunction
c) Addition
d) Disjunctive syllogism
View Answer
Answer: c
Explanation: $p \to (p \lor q)$ argument is ‘Addition’.

4. What rule of inference is used here?
“It is cloudy and drizzling now. Therefore, it is cloudy now.”
a) Addition
b) Simplification
c) Resolution
d) Conjunction
View Answer
Answer: b
Explanation: $(p \land q) \to p$ argument is Simplification.

5. What rule of inference is used in this argument?
“If I go for a balanced diet, then I will be fit. If I will be fit, then I will remain healthy. Therefore, if I go for a balanced diet, then I will remain healthy.”
a) Modus tollens
b) Modus ponens
c) Disjunctive syllogism
d) Hypothetical syllogism
View Answer
Answer: d
Explanation: $((p \to q) \land (q \to r)) \to (p \to r)$ argument is ‘Hypothetical syllogism’.

6. What rules of inference are used in this argument?
“All students in this science class has taken a course in physics” and “Marry is a student in this class” imply the conclusion “Marry has taken a course in physics.”
a) Universal instantiation
b) Universal generalization
c) Existential instantiation
d) Existential generalization
View Answer
Answer: a
Explanation: $\forall x P(x)$, $\therefore P(c)$ Universal instantiation.

7. What rules of inference are used in this argument?
“It is either colder than Himalaya today or the pollution is harmful. It is hotter than Himalaya today. Therefore, the pollution is harmful.”
a) Conjunction

b) Modus ponens
c) Disjunctive syllogism
d) Hypothetical syllogism
View Answer
Answer: c
Explanation: $((p \lor q) \land \neg p) \to q$ argument is Disjunctive syllogism.

8. The premises $(p \land q) \lor r$ and $r \to s$ imply which of the conclusion?
a) $p \lor r$
b) $p \lor s$
c) $p \lor q$
d) $q \lor r$
View Answer
Answer: b
Explanation: The premises $(p \land q) \lor r$ has two clauses: $p \lor r$, and $q \lor r$. We can also replace $r \to s$ with the equivalent clause $r \lor s$. Using the two clauses $p \lor r$ and $r \lor s$, we can conclude $p \lor s$.

9. What rules of inference are used in this argument?
"Jay is an awesome student. Jay is also a good dancer. Therefore, Jay is an awesome student and a good dancer."
a) Conjunction
b) Modus ponens
c) Disjunctive syllogism
d) Simplification
View Answer
Answer: a
Explanation: $((p) \land (q)) \to (p \land q)$ argument is conjunction.

10. "Parul is out for a trip or it is not snowing" and "It is snowing or Raju is playing chess" imply that ___________
a) Parul is out for trip
b) Raju is playing chess
c) Parul is out for a trip and Raju is playing chess
d) Parul is out for a trip or Raju is playing chess
View Answer
Answer: d
Explanation: Let p be "It is snowing," q be "Parul is out for a trip," and r the proposition "Raju is playing chess." The hypotheses as $\neg p \lor q$ and $p \lor r$, respectively. Using resolution, the proposition $q \lor r$ is, "Parul is out for a trip or Raju is playing chess."

Bottom of Form

Relevant Books

Textbooks

- C.L. Liu “Elements of Discrete Mathematics". McGraw Hill, 3rd Edition.
- Santha,"Discrete Mathematics with Graph Theory, Cengage Learning, 1st Edition.

Reference Books

- B. Kolaman, and R.C. Busby, “Discrete Mathematical Structures”, PHI, 1st Edition.
- Gersting, L. Judith “Mathematical Structures for computer Science”, Computer Science Press.

Links for e-book:

http://discrete.openmathbooks.org/pdfs/dmoi-tablet.pdf

References:

http://www2.fiit.stuba.sk/~kvasnicka/Mathematics%20for%20Informatics/Rosen_Discrete_Mathematics_and_Its_Applications_7th_Edition.pdf

https://www.javatpoint.com/inference-theory-in-discrete-mathematics

vedio links

https://www.youtube.com/watch?v=E7IKE7V5A90&list=RDQM38u8C5RzuhQ&index=2

https://www.youtube.com/watch?v=p2b2Vb-cYCs&list=PLBlnK6fEyqRhqJPDXcvYlLfXPh37L89g3