UNIT-1

Chapter 1(CO1)

Statements and notations

A proposition or statement is a declarative sentence that is either true or false (but not both). For instance, the following are propositions: “Paris is in France” (true), “London is in Denmark” (false), “2 < 4” (true), “4 = 7 (false)”. However the following are not propositions: “what is your name?” (this is a question), “do your homework” (this is a command), “this sentence is false” (neither true nor false), “x is an even number” (it depends on what x represents),

“Socrates” (it is not even a sentence). The truth or falsehood of a proposition is called its truth value.

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Predicate

A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

The following are some examples of predicates −

- Let E(x, y) denote "x = y"
- Let X(a, b, c) denote "a + b + c = 0"
- Let M(x, y) denote "x is married to y"

Well Formed Formula

Well Formed Formula (wff) is a predicate holding any of the following −

- All propositional constants and propositional variables are wffs
- If x is a variable and Y is a wff, $\forall$ x Y and $\exists$ x Y are also wff
- Truth value and false values are wffs
- Each atomic formula is a wff
- All connectives connecting wffs are wffs

Quantifiers

The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic − Universal Quantifier and Existential Quantifier.

Universal Quantifier

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol $\forall$.

∀ x P(x) is read as for every value of x, P(x) is true.

Example − "Man is mortal" can be transformed into the propositional form ∀ x P(x) where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol ∃.

∃ x P(x) is read as for some values of x, P(x) is true.

Example − "Some people are dishonest" can be transformed into the propositional form ∃ x P(x) where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Connectives:

A Logical Connective is a symbol which is used to connect two or more propositional or predicate logics in such a manner that resultant logic depends only on the input logics and the meaning of the connective used.

Generally there are five connectives which are −

- OR (∨)
- AND (∧)
- Negation/ NOT (¬)
- Implication / if-then (→)
- If and only if (⇔)

Logical Operation and Truth Table

OR (∨) − The OR operation of two propositions A and B (written as A ∨ B) is true if at least any of the propositional variable A or B is true.

The truth table is as follows −

A	B	A ∨ B
True	True	True
True	False	True
False	True	True
False	False	False

AND (∧) − The AND operation of two propositions A and B (written as A ∧BA∧B) is true if both the propositional variable A and B is true.

The truth table is as follows −

A	B	A ∧ B
True	True	True
True	False	False
False	True	False
False	False	False

Negation (¬) − The negation of a proposition A (written as ¬ A) is false when A is true and is true when A is false.

The truth table is as follows −

A	¬ A
True	False
False	True

Implication / if-then (→) − An implication A → B is the proposition "if A, then B". It is false if A is true and B is false. The rest cases are true.

The truth table is as follows −

A	B	A → B
True	True	True
True	False	False
False	True	True
False	False	True

If and only if (⇔) − A ⇔ B is bi-conditional logical connective which is true when p and q are same, i.e. both are false or both are true.

The truth table is as follows −

A	B	A ⇔ B
True	True	True
True	False	False
False	True	False
False	False	True

XOR Operation

This operation states, the input values should be exactly True or exactly False. The symbol for XOR is (⊻)

A	B	A ⊻B
True	True	False
True	False	True
False	True	True
False	False	False

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N AND (↑) – NAND negation of AND.↑

The truth table is as follows −

A	B	A ↑ B
True	True	False
True	False	True
False	True	True
False	False	True

NOR (↓) It is negation of logical OR connector

A	B	A ↓B
True	True	False
True	False	False
False	True	False
False	False	True

Tautologies

A Tautology is a formula which is always true for every value of its propositional variables.

Example − Prove [(A→B)∧A]→B is a tautology

The truth table is as follows −

A	B	A → B	(A → B) ∧ A	[(A → B) ∧ A] → B
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

As we can see every value of [(A→B)∧A]→B is "True", it is a tautology.

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example − Prove (A∨B)∧[(¬A)∧(¬B)] is a contradiction

The truth table is as follows −

A	B	A ∨ B	¬ A	¬ B	(¬ A) ∧ (¬ B)	(A ∨ B) ∧ [(¬ A) ∧ (¬ B)]
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of (A∨B)∧[(¬A)∧(¬B)] is “False”, it is a contradiction.

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example − Prove (A∨B)∧(¬A) a contingency

The truth table is as follows −

A	B	A ∨ B	¬ A	(A ∨ B) ∧ (¬ A)
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of (A∨B)∧(¬A) has both “True” and “False”, it is a contingency.

Algebra of Proposition

Logical implication-

What is logical implication?

Logical implication is a type of relationship between two statements or sentences. The relation translates verbally into "logically implies" or the logical connective "if/then" and is symbolized by a double-lined arrow pointing toward the right (=>).

In logic, implication is relationship between different propositions where the second proposition is a logical consequence of the first. For instance, if A and B represent semantic statements, then AB means "A implies B" or "If A, then B." The word "implies" is used in the strongest possible sense.

How does logical implication work?

As an example of logical implication, suppose the sentences A and B are assigned as follows:

A = The sky is overcast.

B = The sun is not visible.

In the above example, AB is a true statement, assuming we are at the surface of the earth, below the cloud layer. However, the statement BA is not necessarily true because it might be a clear night. In this case, logical implication does not work both ways. However, the sense of logical implication is reversed if the negation of both statements is used. That is, (AB) = (-B-A)

Using the above sentences as examples, we can say that if the sun is visible, then the sky is not overcast. This conditional statement is always true. In fact, there is logical equivalence between the two statements AB and -B-A.

Propositional Equivalences

Two statements X and Y are logically equivalent if any of the following two conditions hold −

- The truth tables of each statement have the same truth values.

The bi-conditional statement X⇔Y is a tautology.

Laws of Logical Equivalence

Idempotent Law:

In the idempotent law, we only use a single statement. According to this law, if we combine two same statements with the symbol ∧(and) and ∨(or), then the resultant statement will be the statement itself. Suppose there is a compound statement P. The following notation is used to indicate the idempotent law:

P ∨ P ? P

P ∧ P ? P

Commutative Laws:

The two statements are used to show the commutative law. According to this law, if we combine two statements with the symbol ∧(and) or ∨(or), then the resultant statement will be the same even if we change the position of the statements. Suppose there are two statements, P and Q. The proposition of these statements will be false when both statements P and Q are false. In all the other cases, it will be true. The following notation is used to indicate the commutative law:

- P ∨ Q ? Q ∨ P

- $P \land Q ? Q \land P$

Associative Law:

- The three statements are used to show the associative law. According to this law, if we combine three statements with the help of brackets by the symbol $\land$(and) or $\lor$(or), then the resultant statement will be the same even if we change the order of brackets. That means this law is independent of grouping or association. Suppose there are three statements P, Q and R. The proposition of these statements will be false when P, Q and R are false. In all the other cases, it will be true. The following notation is used to indicate the associative law:
- $P \lor (Q \lor R) ? (P \lor Q) \lor R$
- $P \land (Q \land R) ? (P \land Q) \land R$

Same as Distributive Law

$P \lor (Q \land R) ? (P \lor Q) \land (P \lor R)$
$P \land (Q \lor R) ? (P \land Q) \lor (P \land R)$

Complement Law

- $P \lor \neg P ? T$ and $P \land \neg P ? F$
- $\neg T ? F$ and $\neg F ? T$

De Morgan's Law:

- The two statements are used to show De Morgan's law. According to this law, if we combine two statements with the symbol $\land$(AND) and then do the negation of these combined statements, then the resultant statement will be the same even if we combine the negation of both statements separately with the symbol $\lor$(OR). Suppose there are two compound statements, P and Q. The following notation is used to indicate De Morgan's Law:

- $\neg(P \land Q) ? \neg P \lor \neg Q$
- $\neg(P \lor Q) ? \neg P \land \neg Q$
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Normal Forms

The problem of finding whether a given statement is tautology or contradiction or satisfiable in a finite number of steps is called the Decision Problem. For Decision Problem, construction of truth table may not be practical always. We consider an alternate procedure known as the reduction to normal forms.

There are two such forms:

1. Disjunctive Normal Form (DNF)
2. Conjunctive Normal Form

Disjunctive Normal Form (DNF)

If p, q are two statements, then "p or q" is a compound statement, denoted by p ∨ q and referred as the disjunction of p and q. The disjunction of p and q is true whenever at least one of the two statements is true, and it is false only when both p and q are false

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Example: - if p is "4 is a positive integer" and q is "√5 is a rational number", then p ∨ q is true as statement p is true, although statement q is false.

Conjunctive Normal Form:

If p, q are two statements, then "p and q" is a compound statement, denoted by p ∧ q and referred as the conjunction of p and q. The conjunction of p and q is true only when both p and q are true, otherwise, it is false

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Functionally complete set of connectives

• A set of logical connectives is called functionally complete if every boolean expression is equivalent to one involving only these connectives.

• The set { ¬, ∨, ∧ } is functionally complete. – Every boolean expression can be turned into a CNF, which involves only ¬, ∨, and ∧.

• The sets { ¬, ∨ } and { ¬, ∧ } are functionally complete. – By the above result and de Morgan's laws.

• { nand } and { nor } are functionally complete.

Example: if statement p is "6<7" and statement q is "-3>-4" then the conjunction of p and q is true as both p and q are true statements.

Testing by 2nd method (Bi-conditionality)

A	B	¬ (A ∨ B)	[(¬ A) ∧ (¬ B)]	[¬ (A ∨ B)] ⇔ [(¬ A) ∧ (¬ B)]
True	True	False	False	True
True	False	False	False	True
False	True	False	False	True
False	False	True	True	True

As [¬(A∨B)]⇔[(¬A)∧(¬B)] is a tautology, the statements are equivalent.

Inverse, Converse, and Contra-positive

Implication / if-then (→) is also called a conditional statement. It has two parts −

- Hypothesis, p
- Conclusion, q

As mentioned earlier, it is denoted as p→q.

Example of Conditional Statement − "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis, p, and "you will not be punished" is the conclusion, q.

Inverse − An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If not p, then not q". Thus, the inverse of p→q is ¬p→¬q.

Example − The inverse of "If you do your homework, you will not be punished" is "If you do not do your homework, you will be punished."

Converse − The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If p, then q", the converse will be "If q, then p". The converse of p→q is q→p.

Example − The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do your homework".

Contra-positive − The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is "If p, then q", the contra-positive will be "If not q, then not p". The contra-positive of p→q is ¬q→¬p.

Example − The Contra-positive of " If you do your homework, you will not be punished" is "If you are punished, you did not do your homework".

Duality Principle

Duality principle states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true. If dual of any statement is the statement itself, it is said self-dual statement.

Example − The dual of $(A \cap B) \cup C$ is $(A \cup B) \cap C$.

Contradictions 1.3.5

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example − Prove $(A \lor B) \land [(\neg A) \land (\neg B)]$ is a contradiction

The truth table is as follows −

A	**B**	**A ∨ B**	**¬ A**	**¬ B**	**(¬ A) ∧ (¬ B)**	**(A ∨ B) ∧ [(¬ A) ∧ (¬ B)]**
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of $(A \lor B) \land [(\neg A) \land (\neg B)]$ is "False", it is a contradiction.

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example − Prove $(A \lor B) \land (\neg A)$ a contingency

The truth table is as follows −

A	B	$A \vee B$	$\neg A$	$(A \vee B) \wedge (\neg A)$
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of $(A \vee B) \wedge (\neg A)$ has both "True" and "False", it is a contingency.

Inverse, Converse, and Contra-positive

Implication / if-then ($\rightarrow$) is also called a conditional statement. It has two parts −

- Hypothesis, p
- Conclusion, q

As mentioned earlier, it is denoted as $p \rightarrow q$.

Example of Conditional Statement − "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis, p, and "you will not be punished" is the conclusion, q.

Inverse − An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If not p, then not q". Thus the inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

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Converse − The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If p, then q", the converse will be "If q, then p". The converse of $p \rightarrow q$ is $q \rightarrow p$.

Example − The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do your homework".

Contra-positive − The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is "If p, then q", the contra-positive will be "If not q, then not p". The contra-positive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

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Duality principle states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true. If dual of any statement is the statement itself, it is said self-dual statement.

Example − The dual of $(A \cap B) \cup C$ is $(A \cup B) \cap C$.

Normal Forms

We can convert any proposition in two normal forms −

- Conjunctive normal form
- Disjunctive normal form

Examples

- $(A \vee B) \wedge (A \vee C) \wedge (B \vee C \vee D)$
- $(P \cup Q) \cap (Q \cup R)$

Disjunctive Normal Form

A compound statement is in disjunctive normal form if it is obtained by operating OR among variables (negation of variables included) connected with ANDs. In terms of set operations, it is a compound statement obtained by Union among variables connected with Intersections.

Relevant Books

Textbooks

- C.L. Liu “Elements of Discrete Mathematics". McGraw Hill, 3rd Edition.
- Santha,"Discrete Mathematics with Graph Theory, Cengage Learning, 1st Edition.

Reference Books

- B. Kolaman, and R.C. Busby, “Discrete Mathematical Structures”, PHI, 1st Edition.
- Gersting, L. Judith “Mathematical Structures for computer Science”, Computer Science Press.

Links for e-book:

- http://discrete.openmathbooks.org/pdfs/dmoi-tablet.pdf

web link:

1. https://www.isical.ac.in/~sush/Discrete-maths-2014/Principle%20of%20inclusion%20and%20exclusion.pdf
2. https://www.slideserve.com/kayla/inclusion-exclusion-principle
3. https://brilliant.org/wiki/principle-of-inclusion-and-exclusion-pie/
4. https://www.brainkart.com/article/Exercise-12-2--Mathematical-Logic_41293/
5.

Video link:
https://www.youtube.com/watch?v=p2b2Vb-cYCs&list=PLBlnK6fEyqRhqJPDXcvYlLfXPh37L89g3