ARDEN'S THEOREM ($FA \rightarrow RE$) ①

If P and Q are two Regular Expressions over Σ, and if P does not contain ε, then the following equation in R given by $R = Q + RP$ has a unique solution $R = QP^*$.   104.

Proof:

$$R = Q + RP \quad\text{——— (i)}$$

Replace R with QP^*

$$R = Q + QP^* P \quad\text{——— (ii)}$$

$$R = Q(\varepsilon + P^* P) \quad\text{——— } (\because \varepsilon + R^* R = R^*)$$

$$\Rightarrow Q P^*$$

Proof:

this is the unique solution.

$$R = Q + RP \quad\text{——— (i)}$$

Replace R with $Q + RP$

$$R = Q + (Q + RP) P$$

$$\Rightarrow Q + QP + RP^2 \Rightarrow Q + QP + Q + [Q + RP] P^2$$

$$\Rightarrow \quad Q + QP + QP^2 + RP^3$$

$$\dots\dots\dots$$

$$= Q + QP + QP^2 + \dots QP^n + RP^{m+1} \quad \xrightarrow{\text{Replace with}} \quad \left[R = QP^* \right]$$

$$= Q + QP + QP^2 + \dots QP^n + QP^* P^{m+1}$$

$$= Q \left[\varepsilon + P + P^2 + \dots P^n + P^* P^{n+1} \right]$$

$$\underbrace{\qquad\qquad}_{P^*}$$

$$= Q P^*$$

Example Proof using Identities

Ques: Prove that $(1+00^*1) + (1+00^*1)(0+10^*1)(0+10^*1)^*(0+10^*1)$
is equal to $0^*1(0+10^*1)^*$.

Ans: $LHS = (1+00^*1) + (1+00^*1)(0+10^*1)(0+10^*1)^*(0+10^*1)$

$\Rightarrow (1+00^*1)\left[\varepsilon + (0+10^*1)(0+10^*1)^*(0+10^*1)\right]$ $\boxed{\therefore\ \varepsilon + R^*R = R^*}$

$\Rightarrow (1+00^*1)(0+10^*1)^*$

$\Rightarrow (\varepsilon.1+00^*1)(0+10^*1)^*$ $\boxed{\therefore\ \varepsilon.R = R}$

$\Leftarrow (\varepsilon+00^*)1(0+10^*1)^*$ $\boxed{\varepsilon + RR^* = R^*}$

$\Leftarrow 0^*1(0+10^*1)^*$