Mini-Project #3: Physical Balance Models

Math 4334: Mathematical Modeling
Dr. Scott Norris

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Terminal Velocity. The file "raindrop-velocity-small.xlsx" contains careful measurements on the terminal velocity of falling droplets with diameters between 0mm and 3mm performed in 1949 by Gunn and Kinzer[1]. We will attempt to determine a physical model for this data using the methods from this unit. Note that the size and speed of the raindrops varies considerably, so that we might expect a wide range of Reynolds numbers.

(a) Look up the density and viscosity of air, and determine the Reynolds number of both the smallest and largest raindrops. Use the *diameter* of the drop as the characteristic length. [Note that the data are given in non-standard units. You will need to convert to standard units to determine the Reynolds numbers.]

(b) Obtain a model for terminal velocity in the regime of *small* Reynolds numbers ($\mathrm{Re} \ll 1$), assuming spherical droplets, and assuming that the drag force satisfies $F_D = k_1 \mu v l$ as predicted in class. Determine the coefficient k_1 of this model by fitting it to the first 3 data points. Then look up the drag coefficient of a sphere in the viscous drag case (search for "Stokes drag on a sphere"). Compare your fitted coefficient to the stated value, and your fitted model to the data for very small droplets. Discuss.

(c) Obtain a model for terminal velocity in the regime of *large* Reynolds numbers ($\mathrm{Re} \gg 1$), again assuming spherical droplets, and now assuming that the drag force satisfies $F_D = k_2 \rho v^2 l^2$ as predicted in class. Determine the coefficient k_2 of this model by fitting it to the last three data points. Then look up the drag coefficient of a sphere in the inertial drag case (search for "drag on a sphere"). Compare your fitted coefficient to the stated value, and your fitted model to the the data for very large droplets. Discuss.

To obtain a single model valid in both regimes, we will need an expression for the drag coefficient that spans a wide range of Reynolds number. You might hope that the result is simply a linear superposition of the two results from above: i.e.

$$F_D = k_1 \mu v l + k_2 \rho v^2 l^2$$

but your hope would be disappointed – *behavior for intermediate parameter values is rarely so simple.*

[1] Ross Gunn and Gilbert D. Kinzer. *The Terminal Velocity of Fall for Water Droplets in Stagnant Air.* Journal of Meteorology, vol. 6 (1949), pp. 243-248.

Instead, expressions that interpolate between the viscous and inertial regimes must be obtained either empirically, or through careful computational modeling. The results are often summarized with empirical formulas that look completely random, but do a decent job of connecting the two regimes. For instance, for spheres, at Reynolds numbers between about 0.01 and 10,000, a fairly accurate empirical approximation for $C_d\,(\mathrm{Re})$ is[2]

$$C_d\,(\mathrm{Re}) \approx \frac{24}{\mathrm{Re}} + \frac{3e^{-.002\,\mathrm{Re}}}{(\mathrm{Re})^{0.4}} + 0.47 \tag{1}$$

(d) Combine the expression (1) with a force balance expression to yield a *nonlinear equation* for the terminal velocity. Note that Re contains factors of both d and v, and so you won't be able to solve explicitly for the velocity v. Instead, use the numerical root-finder scipy.optimize.fsolve() to predict the terminal velocity associated with each drop size. Compare the predictions obtained using this empirical/numerical approach with those you found in limiting cases above.

(e) Create a single log/log plot containing:

- **(i)** the terminal velocity data itself
- **(ii)** the small-Re predictions you obtained in part (b), over diameters for which $\mathrm{Re} < 0.2$
- **(iii)** the large-Re predictions you obtained in part (c), over diameters for which $\mathrm{Re} > 5$.
- **(iv)** empirical/numerical predictions that you obtained in part (d), for all diameters

[2] http://www.thermopedia.com/content/707/