# MATH 4334: Mathematical Modeling (HW #3a)

Reece Iriye

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## Exercise 1

### (a)

We start with defining a dimensionless product and using the quantities that can be found in a spring-mass system.

$$\Pi = T^{\alpha} m^{\beta} l^{\gamma} k^{\delta} (\Delta x)^{\varepsilon}$$
$$[\Pi] = [s]^{\alpha} [kg]^{\beta} [m]^{\gamma} [\frac{kg \cdot m}{s^2}]^{\delta} [m]^{\varepsilon}$$
$$[\Pi] = [s]^{\alpha - 2\delta} [kg]^{\beta + \delta} [m]^{\gamma + \delta + \varepsilon}$$

$\delta$ and $\varepsilon$ are free variables. So, $\alpha$, $\beta$, and $\gamma$ can be written in the following way:

$$2 \cdot \delta = \alpha$$
$$-\delta = \beta$$
$$-\delta - \varepsilon = \gamma$$

For $\Pi_1$, WLOG set $\varepsilon = -\frac{1}{2}$, $\delta = \frac{1}{2}$. Using the equations above, $\alpha = 1$, $\beta = -\frac{1}{2}$, and $\gamma = 0$.

Thus,

$$\begin{aligned} \Pi_1 &= T \cdot m^{-1/2} \cdot l^0 \cdot k^{1/2} \cdot (\Delta x)^{-1/2} \\ &= T \sqrt{\frac{k}{m \cdot \Delta x}} \end{aligned}$$

For $\Pi_2$, WLOG set $\delta = 0$, $\varepsilon = 1$. We then continue on to find that $\alpha = 0$, $\beta = 0$, and $\gamma = -1$.

Thus,

$$\Pi_2 = \frac{\Delta x}{l}$$

### (b)

$$\begin{aligned} & f(\Pi_2) = \Pi_1 \\ \implies & f(\frac{\Delta x}{l}) = T \sqrt{\frac{k}{m \cdot \Delta x}} \\ \implies & f(\frac{\Delta x}{l}) \cdot \sqrt{\frac{m \cdot \Delta x}{k}} = T \end{aligned}$$

As $f(\Pi_2) \rightarrow q_1$, then $T \rightarrow q_1 \cdot \sqrt{\frac{m \cdot \Delta x}{k}}$.

**(c)**

The period of a pendulum, $T_{pendulum} = 2\pi\sqrt{\frac{L}{g}}$, and the period of a spring-mass $T_{spring} = q_1\sqrt{\frac{m \cdot \Delta x}{k}}$ share several characteristics. The parts of each term not in the square root $2\pi$ and $q_1$ serve a similar purpose in scaling the period. The fractions inside the square root both exist but contain different terms in each equation. The period of the spring-mass system contains the product of mass and initial displacement as the numerator and the spring constant as the denominator. On the other hand, the square root in the period of the pendulum equation contains of the length of the pendulum string in the numerator and the acceleration of gravity coefficient in the denominator.

**(d)**

Minimizing $\Pi_3$ should be easier to accomplish with the period of a spring, because the spring constant can potentially be cancelled out due to the $\sqrt{\frac{m \cdot \Delta x}{k}}$ term. "Old" clocks are related to this because the rotation of a clock is periodic in nature and controlled by a spring-mass system designed for clock ticking to occur with a period of 1 second.

# Exercise 2

### (a)

We start with defining a dimensionless product and using the quantities that can be found in understanding ocean waves.

$$
\begin{aligned}
\Pi &= v^{\alpha} \cdot \lambda^{\beta} \cdot h^{\gamma} \cdot \rho^{\delta} \cdot g^{\varepsilon} \\
[\Pi] &= m^{\alpha} \cdot s^{-\alpha} \cdot m^{\beta} \cdot m^{\gamma} \cdot kg^{\delta} \cdot m^{-3\delta} \cdot m^{\varepsilon} \cdot s^{-2\varepsilon} \\
[\Pi] &= m^{\alpha+\beta+\gamma-3\delta+\varepsilon} \cdot s^{-\alpha-2\varepsilon} \cdot kg^{\delta}
\end{aligned}
$$

We assert that $\alpha$ and $\beta$ are free variables. The following statements can be inferred from the unit set-up of $[\Pi]$ above.

$$
\begin{aligned}
-\frac{1}{2}\alpha - \beta &= \gamma \\
\delta &= 0 \\
-\frac{1}{2}\alpha &= \varepsilon
\end{aligned}
$$

For $\Pi_1$, WLOG set $\alpha = 1, \beta = 0$. Using our definitions and the equations above, we conclude that $\gamma = -\frac{1}{2}$, $\delta = 0$, and $\varepsilon = -\frac{1}{2}$.

Thus,

$$
\begin{aligned}
\Pi_1 &= v \cdot h^{-1/2} \cdot g^{-1/2} \\
&= v\sqrt{\frac{1}{g \cdot h}}.
\end{aligned}
$$

For $\Pi_2$, WLOG set $\alpha = 0, \beta = 1$. Using our definitions and what we already know, we conclude that $\gamma = 1$, $\delta = 0$, and $\varepsilon = 0$.

Thus,

$$
\begin{aligned}
\Pi_2 &= v^0 \cdot \lambda^1 \cdot h^{-1} \cdot \rho^0 \cdot g^0 \\
&= \frac{\lambda}{h}.
\end{aligned}
$$

### (b)

$$
\begin{aligned}
& f(\Pi_2) = \Pi_1 \\
\implies & f\left(\frac{\lambda}{h}\right) = v\sqrt{\frac{1}{g \cdot h}} \\
\implies & f\left(\frac{\lambda}{h}\right)\sqrt{g \cdot h} = v.
\end{aligned}
$$

The role of water density remains unclear, and it seems like it is not incorporated into the equation in the theoretical scenario. The role of the fraction of wave length and ocean depth remain unclear, because it is incorporated into the function but we cannot really how that term plays a role. We just know that it plays a role together as a fraction in determining wave speed.

**(c)**

By omitting depth, we achieve the following.

$$\Pi = v^\alpha \cdot \lambda^\beta \cdot \rho^\delta \cdot g^\varepsilon$$
$$[\Pi] = m^{\alpha+\beta-3\delta+\varepsilon} \cdot s^{-\alpha-2\varepsilon} \cdot kg^\delta$$

I assert that $\alpha$ is a free variable, and all other greek letters can be written here in terms of $\alpha$.

$$\beta = -\frac{\alpha}{2}$$
$$\delta = 0$$
$$\varepsilon = -\frac{\alpha}{2}$$

WLOG set $\alpha = 1$. We can then infer that $\beta = -\frac{1}{2}$, $\delta = 0$, and $\varepsilon = -\frac{1}{2}$.

Thus,

$$\begin{aligned}\Pi &= v \cdot \lambda^{-1/2} \cdot \rho^0 \cdot g^{-1/2} \\ &= v\sqrt{\frac{1}{g\lambda}}.\end{aligned}$$

By Buckingham's Theorem, $v \cdot \sqrt{\frac{1}{g\lambda}} = k_1$. So, $v = k_1\sqrt{g\lambda}$.

**(d)**

Smaller ripples with smaller wavelengths in the ocean have smaller velocities in comparison to large waves. This is because when omitting depth as we did in this case, the velocity of a wave increases by a factor of a constant $c$ when the wavelength is increased by a factor of $c^2$.

**(e)**

By omitting wavelength, we achieve the following representation of $\Pi$.

$$\begin{aligned}\Pi &= v^\alpha \cdot h^\gamma \rho^\delta g^\varepsilon \\ &= [\Pi] = m^{\alpha+\gamma-3\delta+\varepsilon} \cdot s^{-\alpha-2\varepsilon} kg^\delta.\end{aligned}$$

I assert that $\alpha$ is a free variable, and all of the other Greek letters in the exponents can be represented in terms of $\alpha$.

$$\gamma = -\frac{\alpha}{2}$$
$$\delta = 0$$
$$\varepsilon = -\frac{\alpha}{2}$$

WLOG set $\alpha = 1$. Based on this assertion, it can be inferred that $\gamma = -\frac{1}{2}$, $\delta = 0$, and $\varepsilon = -\frac{1}{2}$. Thus, $\Pi = v\sqrt{\frac{1}{gh}}$.

By Buckingham's Theorem and the Implicit Function Theorem, $k_2 = v\sqrt{\frac{1}{gh}}$, which means that $v = k_2\sqrt{gh}$.

**(f)**

The speed of waves as they approach the shoreline approaches 0 as the depth gradually approaches 0.

**(bonus)**

In the case of a tsunami, as seen in the relationship described in part (f), the front of the crest, being in shallower water as it approaches the shore, would move slower than the back of the crest. This difference in speed can cause the height of the crest to increase dramatically because they begin to overlap onto each other and the amplitude of the wave increases, leading to the high waves associated with tsunamis.