MATH 4334: Mathematical Modeling (HW #4b)

Reece Iriye

November 15, 2023

Exercise 1

Exercise 1 omitted.

Exercise 2

(a) $\frac{dx}{dt} = 1 + Ax + x^2$

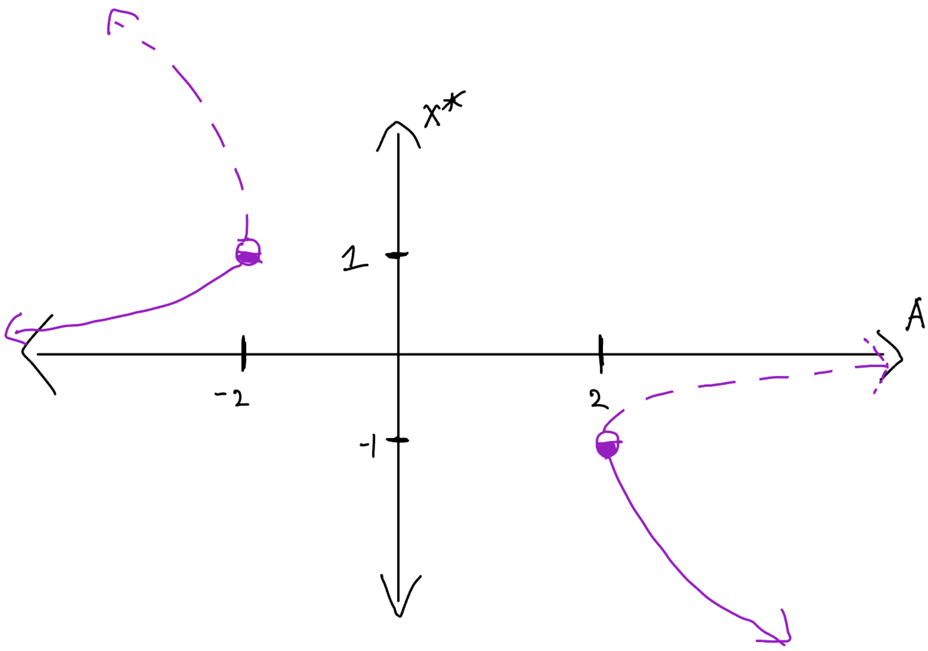
In finding where $\frac{dx}{dt} = 0$, we use the quadratic formula to arrive at $x^* = \frac{-A \pm \sqrt{A^2-4}}{2}$.

When $A \in (-2, 2)$, there does not exist an equilibrium point.

When $A \in \{-2, 2\}$, there exists only 1 equilibrium point. That equilibrium point is $x^* = 1$ when $A = -2$ and $x^* = -1$ when $A = 2$.

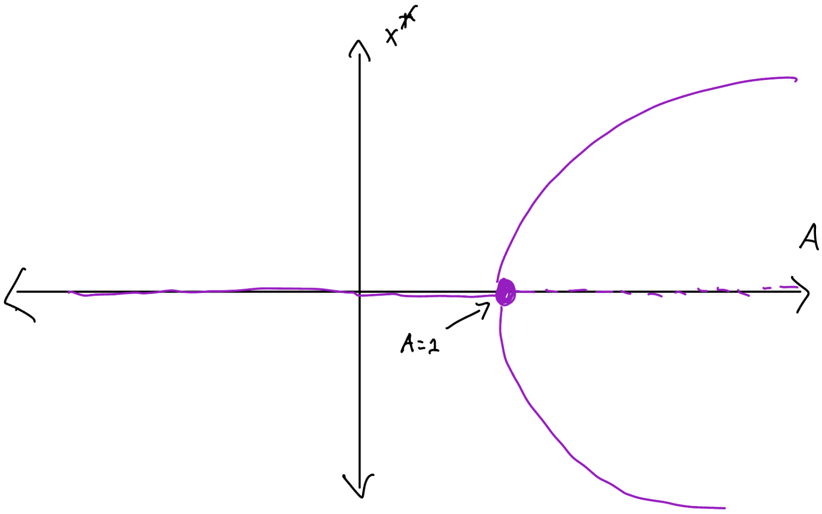
When $A \in (-\infty, -2) \cup (2, \infty)$, there exists 2 equilibrium points at $x^* = \frac{-A \pm \sqrt{A^2-4}}{2}$.

The bifurcation diagram is below and depicts two saddle node bifurcations at $A = -2$ and $A = 2$. They are saddle node bifurcations because 2 equillibria crash in and make one and then disappear.



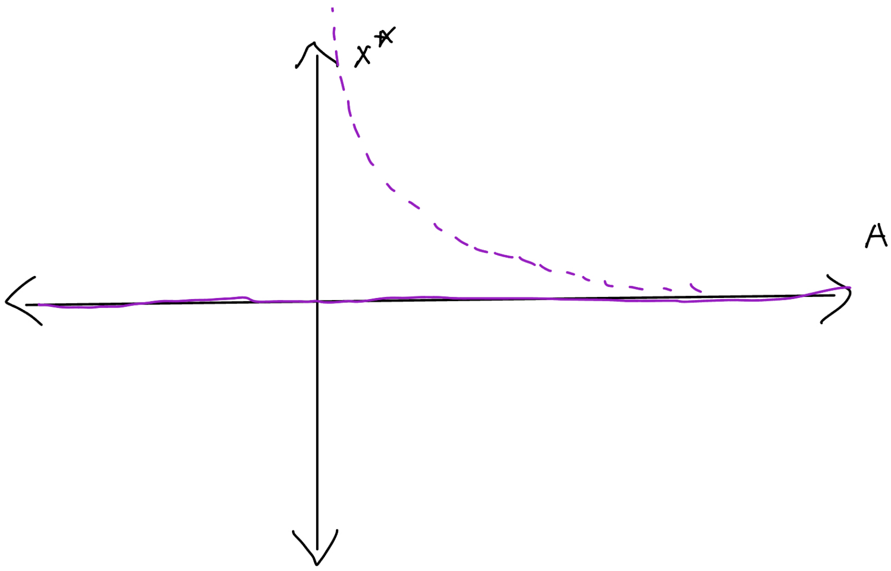
(b) $\frac{dx}{dt} = Ax - \sinh(x)$

When $A \leq 1$, there exists only a single equilibrium point at $x^* = 0$, and the equilibrium point is stable for all these values of A. When $A > 1$, however, two extra equilibrium points emerge to the left and right of it. The leftmost equilibrium point is stable, the center equilibrium point at x^* is unstable, and the rightmost equilibrium point is stable. All these points exist where $\sinh(x^*) = Ax^*$. The bifurcation diagram depicts a pitchfork bifurcation at $A = 1$, because the single equilibrium point is maintained, but two extra equilibria equidistant from the center equilibrium also emerge and branch out as A increases.



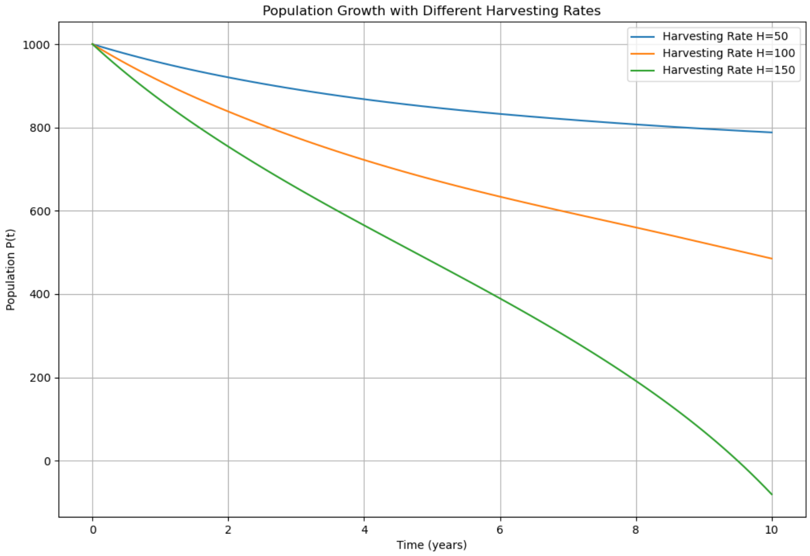
(c) $\frac{dx}{dt} = Ax - \ln(1 + x)$

When $A \leq 0$, there exists only a single equilibrium point at $x^* = 0$, and the equilibrium point is stable for all these values of A. However, once $A > 0$, an extra equilibrium point emerges far greater than 0 then approaches 0 as A increases. The bifurcation diagram depicts a transcritical bifurcation at $A = 0$, as the behavior does not change whatsoever of the equilibrium point at $x = 0$, but another point appears.



Exercise 3

(a)



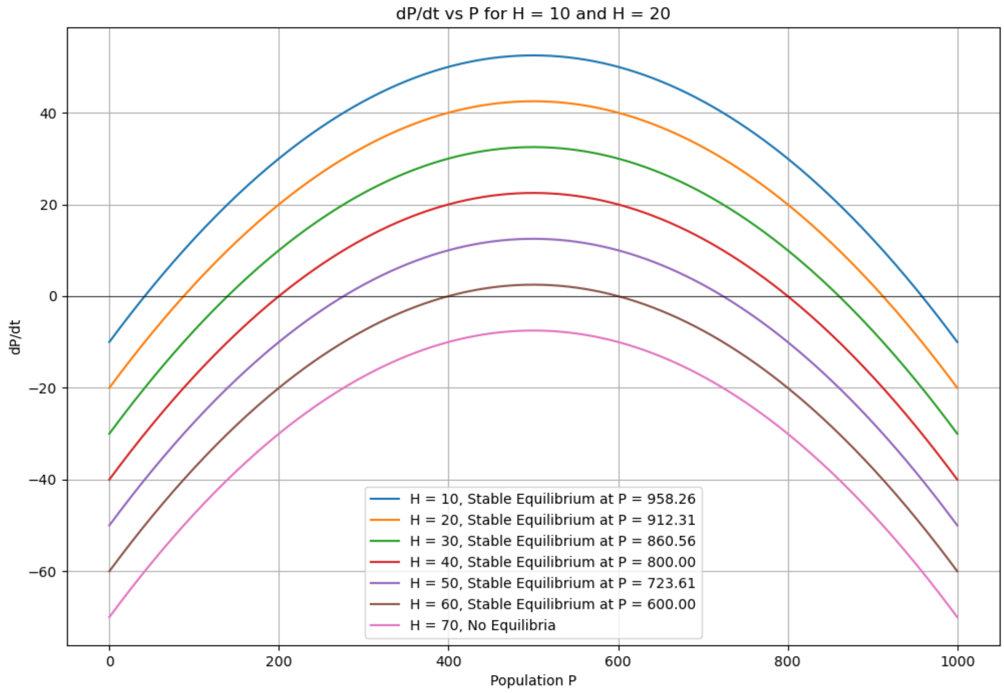
The plot illustrates the population dynamics of a fish species over a 10-year period with varying harvesting rates, given that the initial population is set at the carrying capacity (K = 1000).

At $H = 50$, the population decreases initially but then stabilizes at a level below the carrying capacity. This indicates that the ecosystem can sustain a moderate harvesting rate without collapsing, as the population tends to a new equilibrium where the growth rate compensates for both the natural deaths and the harvesting rate.

At $H = 100$, the population declines at a faster rate than with $H = 50$ and reaches a lower equilibrium. This suggests that increasing the harvesting rate puts more pressure on the population, leading to a lower sustainable population size.

At $H = 150$, the population is in sharp decline, suggesting that the harvesting rate is unsustainable. The population decreases continuously over the 10-year period, indicating that the harvesting rate exceeds the natural replenishment rate of the population. Without intervention, the fish population could be driven to very low numbers or potentially to extinction.

(b)

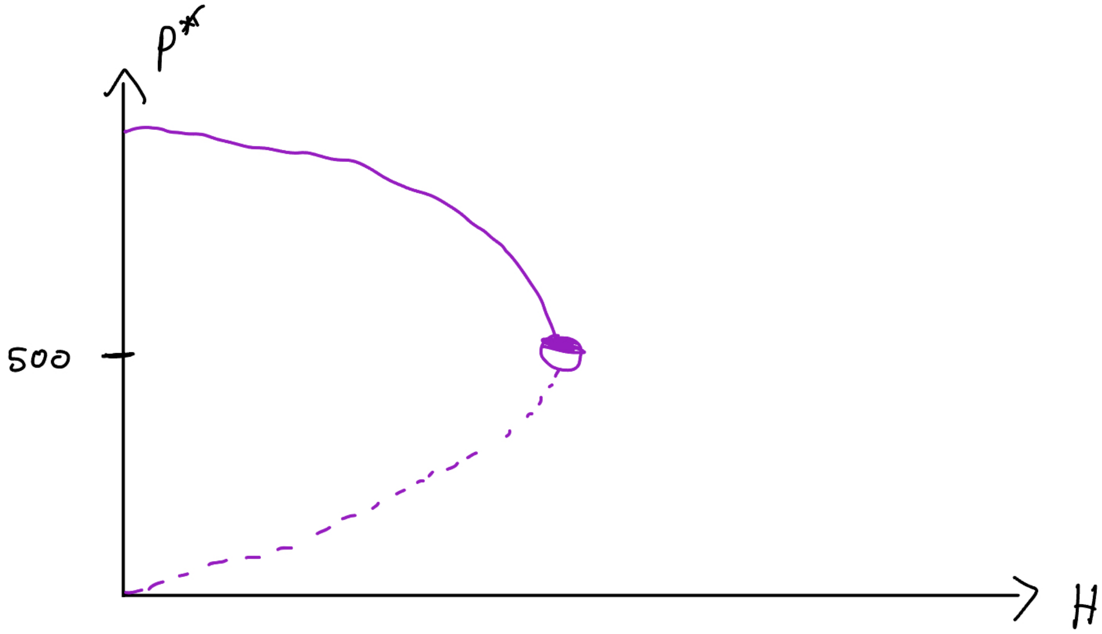


Each curve corresponds to a different harvesting rate, with the rate of population change peaking at a certain population size before declining as the population approaches the carrying capacity ($K = 1000$). The carrying capacity is the maximum population size that the environment can sustain indefinitely.

For lower harvesting rates ($H = 10, 20, 30$), there are stable equilibrium points where the population growth rate, $\frac{dP}{dt}$ crosses the horizontal axis from positive to negative. These points on the right of the curve peaks where $\frac{dP}{dt} = 0$ represent population sizes at which the population is stable – neither growing nor declining. The population will naturally move towards these stable points over time. At H = 10 and H = 20, the stable equilibrium population sizes are high, indicating that the environment can sustain a substantial population even with ongoing harvesting. If the population, however, starts lower than the left equilibrium point on each curve, extinction will occur.

As the harvesting rate increases ($H = 40, 50, 60$), the stable equilibrium points shift to lower population sizes, indicating that higher harvesting pressure reduces the sustainable population size. When the harvesting rate reaches a certain threshold ($H = 70$ in this case), there are no longer any equilibrium points. This indicates a critical situation where the harvesting rate exceeds the population's natural growth rate, and thus the population is destined to decline towards extinction. At this stage, the population cannot sustain itself, as the death rate due to harvesting surpasses the natural birth rate.

(c)



The bifurcation diagram provided shows how the stable P population level changes as a function of the harvesting rate H. Initially, for lower values of H, the population level is stable and high, close to the carrying capacity. As H increases, the stable population level decreases gradually, reflecting the impact of increased harvesting on the population. The diagram features a critical point where the stable and unstable branches meet, which is indicative of a saddle-node bifurcation. Beyond this critical harvesting rate, the stable population branch disappears, suggesting that no stable population level is possible the species would be driven to extinction due to over-harvesting. This point marks a threshold beyond which the system cannot recover, emphasizing the balance between sustainable harvesting and population viability. The diagram underlines the importance of setting a harvesting rate below this critical threshold to ensure the long-term sustainability of the fish population.

(d)

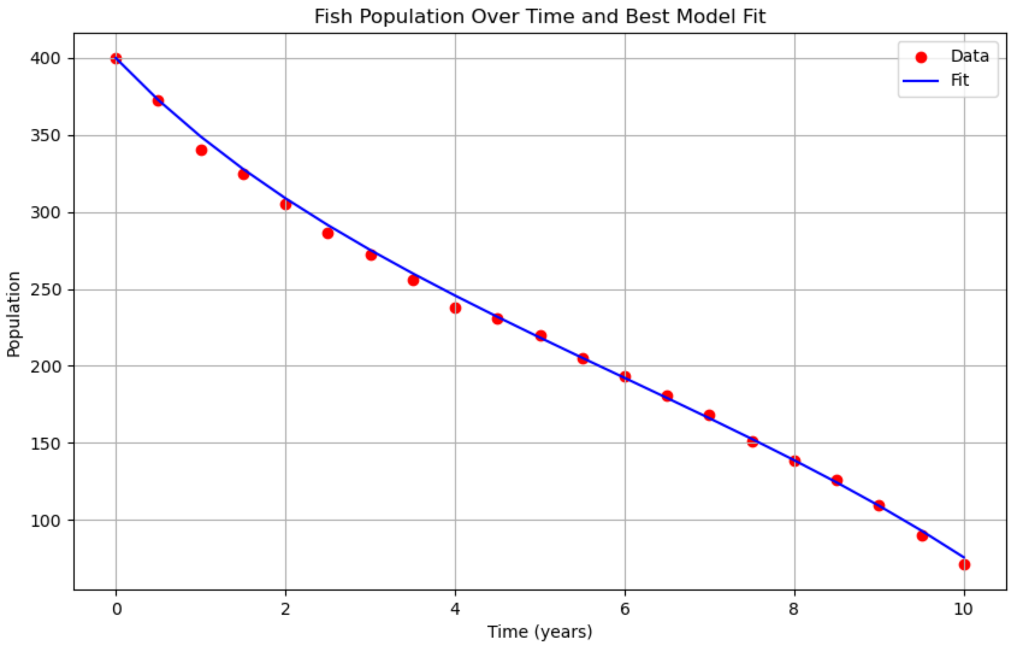
$$
\begin{aligned}
0 &= rP(1-\frac{P}{k}) - H \\
&\implies rP - \frac{r}{k}P^2 - H \\
&\implies -\frac{r}{k}P^2 + rP - H \\
&\implies P = \frac{r \pm \sqrt{r^2 - \frac{4Hr}{k}}}{2r}
\end{aligned}
$$

Thus, the maximum stable harvesting rate H can be found when the discriminant equals 0.

Thus,

$$
\begin{aligned}
r^2 - \frac{4Hr}{k} &= 0 \\
\implies r^2 &= \frac{4Hr}{k} \\
\implies H &= \frac{kr}{4}
\end{aligned}
$$

(e)



Best parameters:

r: 0.28947368421052631

K: 384.2105263157895

H: 53.84615384615384

The provided best-fit parameters from the model fitting suggest that the current harvesting practices are unsustainable. The estimated value of the harvesting rate H at approximately 53.85 is nearly twice as high as the maximum sustainable rate calculated by $\frac{kr}{4} = 27.8$. This means that the population is being depleted faster than it can replenish, and if these harvesting rates continue, the fishery is indeed on a trajectory towards collapse.

The model's projection emphasizes the urgency of revising current fishing policies. Reducing the harvesting rate to at or below the maximum sustainable rate is not merely a recommendation but a necessity to prevent the extinction of the fish population. Sustainable practices ensure that the population remains above a certain threshold as it stays at an equilibrium point. The fate of the population being lethal under current harvesting rates is a wake-up call for immediate action.

(f)

The decline and hopeful recovery of the Atlantic cod fisheries as detailed in the Wikipedia excerpt echo the dynamics captured by the population model with harvesting H. The historical overfishing of cod, exacerbated by advances in fishing technology, parallels the scenario where H is too high, leading to a catastrophic drop in the fish population. This real-world example underscores the model's premise that overharvesting beyond a certain threshold is unsustainable and can lead to population collapse.

The introduction of equipment that expanded fishing capabilities in the 1950s and 1960s led to a situation where the harvesting rate H far exceeded the sustainable limit calculated by the model. The consequent depletion of not just the commercial fish but also the "bycatch" species critical to the ecosystem's health illustrates the destructive potential of ignoring ecological equilibrium as predicted by the model. The dramatic decline of the cod stocks to 1% of historical levels by 1992 is a stark reminder of what occurs when the harvesting rate H in the model surpasses the maximum sustainable rate, here quantified as $\frac{kr}{4}$.

The slow recovery of cod populations signal a reduction in H to levels that may allow for a return to sustainable equilibrium, aligning with the model's indication that significant reductions in harvesting rates are required to prevent extinction. It also showcases how the population did not dip down to the point of no return but was instead put back on track by how the harvesting rate was changed. The model not only offers a theoretical framework for understanding such ecological crises but also provides a quantifiable benchmark for corrective measures. As with the Atlantic cod, the model emphasizes the critical need for regulatory actions to align harvesting practices with ecological realities to ensure the long-term viability of vital fisheries and their surrounding ecosystems.