Homework #4b: Nonlinear ODEs

Math 4334: Mathematical Modeling
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November 7, 2023

Problem #1. Model construction and fitting. The file 'nonlinear-ode-data-1.xlsx' contains data for the evolution of the variable $x(t)$ for four different initial conditions $x(0) = \{x_{01}, x_{02}, x_{03}, x_{04}\}$. Assuming that the same nonlinear ODE governs the evolution of all four data sets:

(a) perform an empirical phase line study using all four data sets

(b) construct a model based on the results of the study in (a)

(c) fit your model parameters to all four data sets, separately

(d) based on the four fits you obtained, estimate the governing ODE.

(e) How many fixed points does the resulting system have? Where? What kinds?

Problem #2. Bifurcations. For each of the nonlinear ODEs listed, analyze the phase line for different values of A, and construct a bifurcation diagram. In each case, identify the value of A at which a bifurcation occurs, and classify the bifurcation.

(a) $\frac{dx}{dt} = 1 + Ax + x^2$

(b) $\frac{dx}{dt} = Ax - \sinh(x)$

(c) $\frac{dx}{dt} = Ax - \ln(1 + x)$

Problem #3. Population Growth Revisited. Suppose the population of a fish species in its natural habitat satisfies

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - H \qquad P(0) = K \tag{1}$$

where r is the annual growth rate, K is the carrying capacity, and H is the harvesting rate.

(a) Let $r = 0.25$ and $K = 1000$. Create a single plot containing $P(t), \quad t \in [0, 10]$ for each of the following: $H = \{50, 100, 150\}$. Discuss.

(b) Sketch the graph of $\frac{dP}{dt}$ vs P for several nonnegative values of H, including zero. For some H slightly larger than zero, sketch solutions $x(t)$ based on your phase line.

(c) What happens to the phase line as H continues increasing? Construct a bifurcation diagram illustrating the location and type of fixed points vs. H. Discuss the difference in system behavior for H below and above this value.

(d) Find the maximum sustainable harvesting rate H, in terms of r and K (hint: this occurs at the bifurcation point).

(e) The file hw-06-fishing.csv contains a hypothetical population of fish in a fishery (in thousands) versus time (in years). Fit the model in Equation (1) to this data set using the methods discussed in class. What fit parameters do you find? According to the model you have created, and the findings above, what is the future fate of the fishery? What action(s) do you recommend?

(f) Read the Wikipedia article on the Cod population of the northwest Atlantic ocean (https://en.wikipedia.org/wiki/Collaps
Discuss this history in the context of what you learned from your model.