Homework #4a: Linear ODEs

Math 4334: Mathematical Modeling
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Problem #1. Personal Finance. Recall that the model for the growth of an investment is

$$\frac{dM}{dt} = rM + d, \qquad\qquad M(0) = M_0.$$

where r is the interest rate, and D is the net deposit rate, and M_0 is the initial balance. Also recall that the solution to this equation was

$$M(t) = M_0 e^{rt} + \tfrac{d}{r}\left(e^{rt} - 1\right)$$

Let's apply this solution to a sequence of life decisions associated with a hypothetical student entering college.

(a) Suppose that the student pursues undergraduate and graduate degrees over 7 years, and over this time has to borrow \$25k per year at an interest rate of 6%. What is the student's total debt at the end of these 7 years? (assume $M_0 = 0$, $d = -\$25\text{k}$).

(b) Suppose that the student now enters the workforce (at age 25), and begins paying off their debt. If the worker wishes to pay off their debt within 15 years (i.e. by age 40), what must be their annual payment rate d?

(c) Once the debt is paid off, our worker continues depositing the amount d into a retirement savings account, where it earns an average of 3% per year. How long until the account value reaches \$500k?

(d) Suppose that, upon reaching this balance, the worker begins thinking about retirement. Her goal is to keep saving \$d per year until she retires, after which she will begin withdrawing \$50k per year. If the student plans to live to age 85, how many years longer must she continue working? (we again assume an interest rate of 3%)

Problem 2: Virus Concentrations in a Room. In this problem we will construct a simple model for the concentration $C(t)$ of aerosolized virus particles in a room full of people talking (say, a classroom, or a bar). This problem starts out as a classic mixing problem – we consider a room with volume V, and assume that a ventilation system brings in fresh air with a (hopefully zero) virus concentration C_{in} at a rate F_{in}, and removes stale air with a (possibly nonzero) virus concentration C_{out} at a rate F_{out}. However, if any of the people in the room are infected and contagious, we must add an extra effect to our model – a *source* of additional particles, produced at a constant (i.e. absolute) rate R as the infected person coughs, talks, or simply breathes. This leads to an ODE of the form

$$\frac{dC}{dt} = \frac{1}{V}\left[F_{\text{in}} C_{\text{in}} - F_{\text{out}} C_{\text{out}} + R\right]$$

(a) Assume that

- $F_{\text{in}} = F_{\text{out}} = F$ (i.e. the air enters and exits the room at the same rate F),
- $C(0) = C_{\text{in}} = 0$ (i.e. initial and incoming air have no virus particles)
- $C_{\text{out}} = C(t)$ (i.e. the air in the room is perfectly well-mixed)

and then obtain a solution to this ODE. Then identify the *steady state* concentration that emerges in the limit $t \to \infty$.

(b) By critically examining the solution of your ODE and its steady-state limit, identify ways that the steady virus concentration can be reduced.answer the following questions by referring to appropriate parts of the solution structure:

- What parameter might be affected by wearing masks? How does this affect the viral exposure?
- What parameter might be affected by social distancing? How does this affect the viral exposure?
- What parameter might be affected by opening windows? How does this affect the viral exposure?

Problem 3: Water Treatment Plants. In this problem we will construct a simple model for a sewage treatment plant. This problem *also* starts out as a classic mixing problem, in that sewage enters the plant at a rate F_{in} with initial contaminant concentration C_{in}, spends time in a well-mixed aeration pond of volume V, and then leaves the pond at a rate F_{out}, with reduced contaminant concentration C_{out}. However, the pond is filled with bacteria that convert hazardous chemicals such as ammonia into safer chemicals such as nitrates. They do this through a metabolic reaction that removes contaminant mass at a rate proportional to the existing mass, $M(t)$, with (relative) rate coefficient r. This leads to an ODE of the form

$$\frac{dC}{dt} = \frac{1}{V}\left[F_{\text{in}}C_{\text{in}} - F_{\text{out}}C_{\text{out}}\right] - rC$$

(a) Assume that

- $F_{\text{in}} = F_{\text{out}} = F$ (i.e. water enters and exits the pond at the same rate F),
- $C(0) = C_{\text{in}} > 0$ (i.e. initial and incoming water contains hazardous chemicals)
- $C_{\text{out}} = C(t)$ (i.e. the water in the pond is perfectly well-mixed)

and then obtain a solution to this ODE. Then identify the *steady state* concentration that emerges in the limit $t \to \infty$.

(b) By critically examining the solution of your ODE and its steady-state limit, answer the following questions by referring to appropriate parts of the solution structure:

- What is the likely effect of increasing the pond volume V?
- What is the likely effect of increasing the flow rate F?
- What is the likely effect of increasing the reaction rate r?

(c) If you have ever seen a sewage treatment plant from an airplane (or searched Google Images for "sewage treatment plant"), you may have noticed that instead of one large pond, it consists of many small ponds. These ponds are "daisy chained" together, with the output from the first pond serving as the input to the second, and so on. Assuming that each pond is at steady state, find an expression for the outgoing contaminant concentration from a chain of N ponds of volume V_0, and compare this to the outgoing contaminant concentration from a single pond of volume $N \times V_0$. Why are multiple small ponds better for reaching very low outgoing contaminant concentrations?