MATH 4334: Mathematical Modeling (HW #4a)

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November 6, 2023

Exercise 1

(a) The following quantities are set.

$$\begin{aligned} M_0 &= 0 \\ d &= -25000 \\ r &= 0.06 \\ t &= 7 \end{aligned}$$

So, $M(7) = \frac{-25000}{0.06}(e^{0.06(7)}-1) = -217483.9815$, which means that she will be in debt \$217,483.98.

(b)

The following quantities are set.

$$\begin{aligned} M_0 = M(7) &= -217483.9815 \\ d &=? \\ t &= 15 \\ r &= 0.06 \\ M(15) &= 0 \end{aligned}$$

So, this means that the equation $0 = M(7)e^{.06\cdot 15} + \frac{d}{.06}(e^{.06\cdot 15} - 1)$. So,

$$\begin{aligned} d &= \frac{217483.9815 \cdot e^{.06\cdot 15}(.06)}{e^{.06\cdot 15} - 1} \\ &= 21989.16706 \end{aligned}$$

(c)

We start now with the $d = 21989.16706$ yearly down-payment quantity.

$$\begin{aligned} & 500000 = -217483.9815 \cdot e^{.03\cdot t} + \frac{21989.16706}{0.03}(e^{.03\cdot t} - 1) \\ \implies & \frac{500000 + 732972.2352}{515488.2538} = e^{0.03\cdot t} \\ \implies & \ln 2.39185 = 0.03t \\ \implies & t = 29.0689 \end{aligned}$$

This value, however incorporates the 15 year gap between 25 and 40, so subtracting 15 from t, we find that around 14 years after turning 40, the account balance becomes \$500,000.

(d)

We begin setting up this equation using the year quantity from **(c)** but also with acknowledging the difference in years from where she reaches \$500,000 to when she plans to live to at age 85. The time difference there is $85 - 54.0689 = 30.981$. So, she should plan to hit zero at the point where $0 = \frac{-50000}{0.03} e^{0.03 \cdot 30.981} - 217483.9815 e^{0.03t} e^{0.03 \cdot 15} + 732972.2353 e^{0.03t} - 732972.2353$.

Simplifying,

$$\begin{aligned}
\frac{\frac{50000}{0.03} e^{0.03 \cdot 30.981} + 732972.2353}{217483.9815 e^{0.05 \cdot 15} - 732972.2353} &= e^{0.03 \cdot t} \\
\implies t &= \frac{\ln\left(\frac{\left(\frac{50000}{0.03} e^{0.03 \cdot 30.981}\right) + (732972.2353)}{217483.9815 e^{0.03 \cdot 15} - 732972.2353}\right)}{0.03} \\
\implies t &= 72.82.
\end{aligned}$$

She must continue to work until she is 72 years old in order for the money to even out.

Exercise 2

$$
\begin{aligned}
\frac{dC}{dt} &= \frac{1}{V}(-F \cdot C(t) + R) \\
\implies \frac{dC}{dt} &= -\frac{F}{V}C(t) + \frac{R}{V} \\
\implies \frac{dC}{dt} &+ \frac{F}{V}C(t) = \frac{R}{V}
\end{aligned}
$$

Let $p(t) = \frac{F}{V}$, $q(t) = \frac{R}{V}$. Then, $I(t) = e^{\int F/vdt} = e^{\frac{F}{V}t}$. Multiplying the integrating factor on both sides of the original equation:

$$
\begin{aligned}
& e^{\frac{F}{V}t} \cdot \frac{dC}{dt} + \frac{F}{V}C(t) \cdot e^{\frac{F}{V}t} = \frac{R}{V} \cdot e^{\frac{F}{V}t} \\
\implies & \frac{d}{dt}(C(t) \cdot e^{\frac{F}{V}t}) = \frac{R}{V}e^{\frac{F}{V}t} && \text{by the product rule} \\
\implies & \int \frac{d}{dt}(C(t) \cdot e^{\frac{F}{V}t})dt = \int \frac{R}{V}e^{\frac{F}{V}t}dt \\
\implies & C(t) \cdot e^{\frac{F}{V}t} = \frac{R}{F}e^{\frac{F}{V}t} + A \\
\implies & C(t) = \frac{R}{F} + Ae^{-\frac{F}{V}t} \\
\implies & C(t) = \frac{R}{F} - \frac{R}{F}e^{-\frac{F}{V}t} && \text{as the constant } A = -\frac{R}{F}
\end{aligned}
$$

As $t \to \infty$, $C(t) \to \frac{R}{F}$. This is the steady state concentration that emerges.

(b)

Wearing masks affects the parameter R, the rate of virus particles being added to the room. Masks can significantly reduce the emission of aerosolized particles from infected individuals, thereby reducing the value of R. This directly reduces the steady-state viral concentration, as R appears in the numerator of the expression.

Social distancing may indirectly affect the parameter R as well. By increasing the physical distance between individuals, the effective rate at which virus particles reach the well-mixed air of the room can be reduced, as there is a lower probability of particles from an infected individual spreading to others. While not directly in the model, social distancing can be thought of as reducing the effective R by likely decreasing the particles dispersing throughout the room.

Opening windows would increase the parameter F, the rate at which fresh air enters the room and stale air is removed. Increasing F would increase the denominator of the expression, which leads to a lower steady-state concentration of virus particles. This is probably the most direct and effective way to reduce the steady-state concentration according to the model, as it increases the rate of removal of virus particles from the room.

Exercise 3

Substituting the given assumptions, we get:

$$\begin{aligned}\frac{dC}{dt} &= \frac{F}{V}C_{in} - \frac{F}{V}C(t) - rC(t) \\ \implies \frac{dC}{dt} + (\frac{F}{V} + r)C(t) &= \frac{F}{V}C_{in}\end{aligned}$$

Let $p(t) = \frac{F}{V} + r$. Then, $I(t) = e^{\int (\frac{F}{V}+r)dt} = e^{(\frac{F}{V}+r)t}$. Multiplying the integration factor on both sides of the equation above, we get the following:

$$\begin{aligned}\frac{dC}{dt} \cdot e^{(\frac{F}{V}+r)t} + (\frac{F}{V} + r)C(t) \cdot e^{(\frac{F}{V}+r)t} &= \frac{F}{V}C_{in} \cdot e^{(\frac{F}{V}+r)t} \\ \implies \frac{d}{dt}(C(t) \cdot e^{(\frac{F}{V}+r)t}) &= \frac{F}{V}C_{in} \cdot e^{(\frac{F}{V}+r)t} \\ \implies \int \frac{d}{dt}(C(t) \cdot e^{(\frac{F}{V}+r)t})dt &= \int \frac{F}{V}C_{in} \cdot e^{(\frac{F}{V}+r)t}dt \\ \implies C(t) \cdot e^{(\frac{F}{V}+r)t} &= \frac{F}{V}C_{in}\frac{e^{(\frac{F}{V}+r)\cdot t}}{\frac{F}{V}+r} + A \\ \implies C(t) &= \frac{F}{V}C_{in}\frac{1}{\frac{F}{V}+r} + Ae^{-(\frac{F}{V}+r)\cdot t}.\end{aligned}$$

Finding A,

$$\begin{aligned}C_{in} &= \frac{F}{V}C_{in}\frac{1}{\frac{F}{V}+r} + A \\ \implies A &= C_{in}(\frac{rV}{F+rV})\end{aligned}$$

So,

$$C(t) = \frac{F}{V}C_{in}\frac{1}{\frac{F}{V}+r} + C_{in}(\frac{rV}{F+rV})e^{-(\frac{F}{V}+r)\cdot t}$$

As $t \to \infty$, $C(t) \to \frac{F}{V}C_{in}\frac{1}{\frac{F}{V}+r} = C_{in}(\frac{F}{F+rV})$, which represents the steady state concentration.

(b)

As V increases, the denominator of the steady-state concentration term $F + rV$ increases, which causes the overall steady-state concentration to decrease. A larger pond volume allows for more time for the bacteria to break down contaminants, leading to cleaner water being outputted from the treatment plant.

Increasing F increases both the numerator and the denominator of the fraction that represents the steady-state concentration. Thus, assuming that the $rV \geq 0$, since it doesn't make intuitive sense otherwise, as $F \to \infty$, the steady-state concentration approaches 1, and the effect of the rV term is diluted. Increasing the flow rate allows for an increase in the overall contaminant concentration.

An increase in r means that contaminants are being broken down more quickly by the bacteria. This will increase the denominator $F + rV$ more significantly than the numerator, resulting in a lower steady-state concentration, indicating a cleaner output from the treatment plant due to the faster reaction rates causing faster clean-up.

(c)

Define C_{ss} as the steady-state concentration $C_{in}(\frac{F}{F+rV_0})$.

For a chain of N ponds, each with volume V_0, the concentration at the output of the first pond becomes the input concentration for the second pond, and so on. Thus, for the i^{th} pond in the series, the input concentration $C_{in}^{(i)}$ is the output concentration of the $(i-1)^{th}$ pond $C_{ss}^{(i-1)}$. We then achieve the relationship $C_{ss}^{(i)} = C_{ss}^{(i-1)}(\frac{F}{F+rV_0})$. If we repeat this for N daisy-chained ponds, we get the following formula:

$$C_{ss}^{(N)} = C_{in}(\frac{F}{F + rV_0})^N$$

In comparison, the steady-state concentration of a large pond would be $C_{ss\_large} = C_{in}(\frac{F}{F+r(V_0 \times N)})$.

It is clear that the exponential formula for the steady-state concentration would be much larger than the equation with the large pond, because it's the product of repeated multiplications of a fraction less than one. This is a result of the compound effect of sequential processing, where each pond has the opportunity to reduce the concentration further. This compounding effect does not occur in a single large pond, where the process happens only once. Therefore, the sequential ponds can drive the concentration down more effectively than a single pond of equivalent total volume.