Projection and least squares

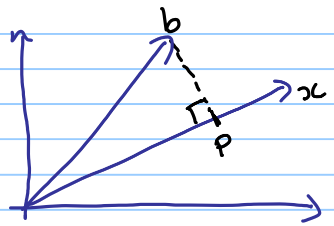
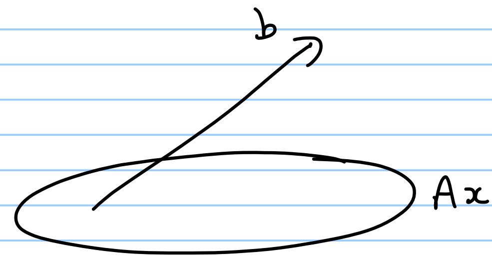


Fig (1)



Want to project b onto the line through x, or, more generally onto the column space of a matrix A.

① Why project?

② Given a basis for a subspace S (e.g., spanned by cols of A), is there an easy to calculate the projection p of b onto S?

On ①: Suppose we are given (x_1, b_1) - - - - (x_n, b_n)

e.g. $$\begin{aligned} 2x &= b_1 \\ 3x &= b_2 \\ 4x &= b_3 \end{aligned}$$ or $$\begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \\ 2x + 4y &= 6 \end{aligned}$$

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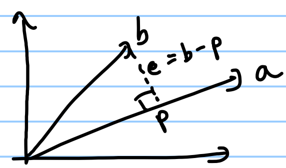
"These are inconsistent" $\Rightarrow$ No solution that satisfies this system of equations

Matrix view: $Ax = b$

Inconsistent if $b \notin C(A)$

In such situations, it makes sense to project b onto $C(A)$.

Projection onto a line:



$$p = \hat{x} a$$

$$e = b - p = b - \hat{x} a$$

$$e \perp a$$

$$(b - \hat{x} a) \perp a$$

$a^T (b - \hat{x} a) = 0$ leading to $\hat{x} = \dfrac{a^T b}{a^T a} \Rightarrow p = \hat{x} a = \left(\dfrac{a^T b}{a^T a}\right) a$

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Cauchy - Schwarz inequality:

$$\|e\|^2 = \|b - p\|^2 \geq 0$$

$$\left\| b - \frac{a^T b}{a^T a} a \right\|^2 = b^T b - 2 \frac{(a^T b)^2}{a^T a} + \left(\frac{a^T b}{a^T a} \right)^2 a^T a$$

$$= \frac{(b^T b)(a^T a) - (a^T b)^2}{(a^T a)} \geq 0$$

$$\Rightarrow (b^T b)(a^T a) \geq (a^T b)^2$$

(or) $\boxed{|a^T b| \leq \|a\| \, \|b\|}$ → Cauchy - Schwarz inequality

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Projection matrix: Recall $p = \left(\frac{a^T b}{a^T a} \right) a = \left(\frac{a a^T}{a^T a} \right) b$

Let $\mathbb{P} = \frac{a a^T}{a^T a}$. Then, projection of b onto a is $\mathbb{P} b$

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To project any vector b, just left multiply by the projection matrix 'P'.

Example:

$$a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Projection matrix is $P = \dfrac{a\,a^T}{a^T a} = \dfrac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

Observe that (i) P is symmetric

(ii) $P^2 = P$ i.e., $P^2 b = Pb$ (idea: Pb is already on the line through a. So, another round of projection won't change it)

(iii) Column space $C(P)$ = line through $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Null space $N(P)$ = plane orthogonal to $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Rank $r(P) = 1$

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(iv) $a = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ $\mathbb{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

check this