# Minimum Cost Spanning Trees

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python
Week 5

## Examples

#### Roads

- District hit by cyclone, roads are damaged
- Government sets to work to restore roads
- Priority is to ensure that all parts of the district can be reached
- What set of roads should be restored first?

## Examples

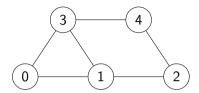
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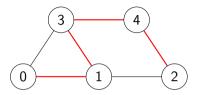
### Fibre optic cables

- Internet service provider has a network of fibre optic cables
- Wants to ensure redundancy against cable faults
- Lay secondary cables in parallel to first
- What is the minimum number of cables to be doubled up so that entire network is connected via redundant links?

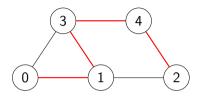
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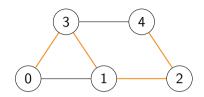
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- Recall that a minimally connected graph is a tree
  - Adding an edge to a tree creates a loop
  - Removing an edge disconnects the graph



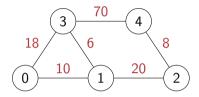
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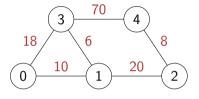
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- More than one spanning tree, in general



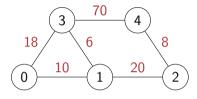
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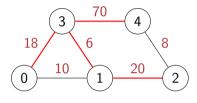
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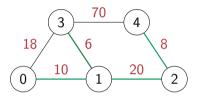
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Definition A tree is a connected acylic graph.

### Fact 1

A tree on n vertices has exactly n-1 edges

- Initially, one single component
- Deleting edge (i, j) must split component
  - Otherwise, there is still a path from i to j, combine with (i, j) to form cycle
- Each edge deletion creates one more component
- Deleting n-1 edges creates n components, each an isolated vertex

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- Deleting *n* − 1 edges creates *n* components, each an isolated vertex

#### Fact 2

Adding an edge to a tree must create a cycle.

- Suppose we add an edge (i, j)
- Tree is connected, so there is already a path from i to j
- The new edge (i, j) combined with this path from i to j forms a cycle

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### Fact 3

In a tree, every pair of vertices is connected by a unique path.

If there are two paths from i to j, there must be a cycle



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### Observation

Any two of the following facts about a graph G implies the third

- *G* is connected
- *G* is acyclic
- G has n-1 edges

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- Two natural strategies
- Start with the smallest edge and "grow" a tree
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- Scan the edges in ascending order of weight to connect components without forming cycles
  - Kruskal's algorithm