Sum of Independent Random Variables

$$X, Y$$

$$Z = X + Y$$

$$f_Z(z) = \underbrace{\int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx}_{\text{Convolution Operation}}$$

Examples

$$f_Z(z) = ?$$

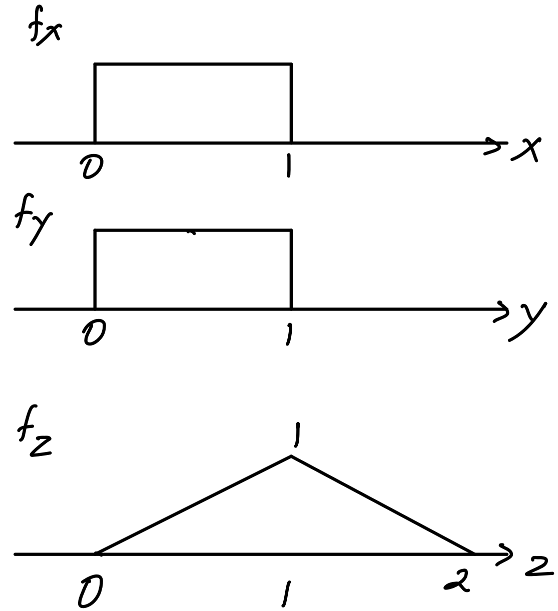
Case 1: $0 \le z \le 1$

$$f_Z(z) = \int_0^1 f_Y(z-x)\,dx$$

$$= \int_0^z 1\,dx = z$$

Case 2: $1 \le z \le 2$

$$f_Z(z) = \int_0^1 f_Y(z-x)\,dx \qquad \int_{z-1}^1 dx = 2 - z$$



Max of Independent Random Variables

$$Z = \max(X, Y)$$

$$F_Z(z) = P(Z \leq z)$$

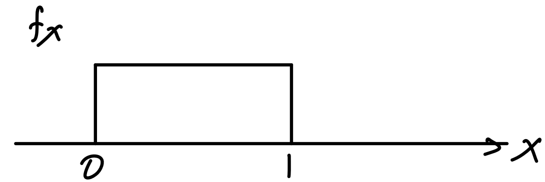
$$= P(X \leq z, Y \leq z)$$

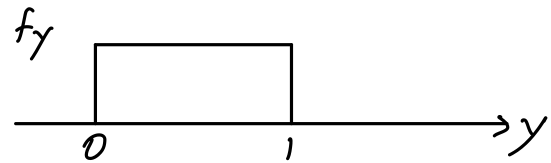
$$= P(X \leq z) \cdot P(Y \leq z)$$

$$= F_X(z) \cdot F_Y(z)$$

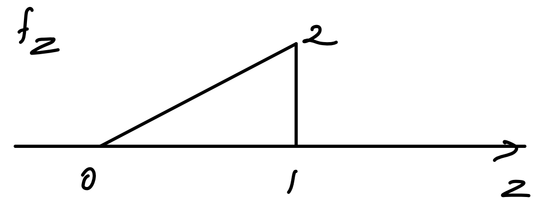
Examples

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \in [0,1] \\ 1 & \text{if } x \geqslant 1 \end{cases}$$





$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ z^2 & \text{if } z \in [0,1] \\ 1 & \text{if } z > 1 \end{cases}$$



$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$f_Z(z) = \begin{cases} 0 & \text{if } z \notin [0,1] \\ 2z & \text{o.w} \end{cases}$$

Min of Independent Random Variables

$$Z = \min(X, Y)$$

$$F_Z(z) = P(Z \le z)$$

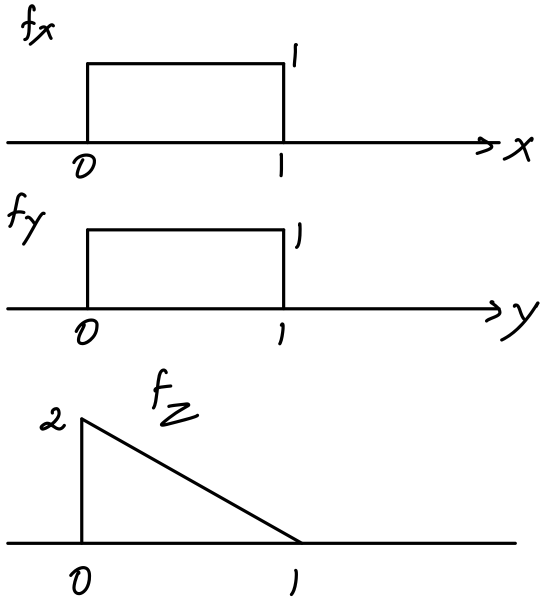
$$= P(X \le z \cup Y \le z)$$

$$= 1 - P(X \geq z, Y \geq z)$$

$$= 1 - P(X \geq z)\,P(Y \geq z)$$

$$= 1 - (1 - F_X(z))(1 - F_Y(z))$$

Examples



Covariance and Correlation

$$X, Y$$

$$\operatorname{cov}[X, Y] = E\left[(X - EX)(Y - EY)\right]$$

$$= E[XY] - (EX)(EY)$$

$$\rho[X, Y] = \frac{\operatorname{cov}[X, Y]}{\sqrt{\operatorname{var}[X] \cdot \operatorname{var}[Y]}}$$

Independent $\Rightarrow$ Uncorrelated

Covariance and Correlation

$$X \sim \text{Unif}([-1, 1])$$

$$Y = X^2$$

$$EXY = EX^3$$

$$= \int_{-\infty}^{\infty} x^3 f_X(x)\, dx$$

$$= \int_{-1}^{1} x^3\, dx = 0$$

$$EXY = EX \cdot EY = 0$$

Covariance and Correlation

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

$$Cov[X] = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots & Cov[X_1, X_n] \\ \vdots & & & \\ Cov[X_n, X_1] & \cdots & & Var[X_n] \end{bmatrix}$$