Tutorial
on
Orthogonality, projection and least squares method

Course: MACHINE LEARNING FOUNDATIONS

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Example 1

Let $S = \{(1, 2, 4, 0)^T, (-2, 3, -1, 0)^T, (0, 2, 6, -1)^T\}$. Which pair(s) of vectors in this given set are orthogonal?

Solution

$u.v = -2 + 6 - 4$
$= 0$ ✓

$v.w = 6 - 6$
$= 0$ ✓

$\therefore$ (u, v) and (v, w) are orthogonal pairs.

$$S = \{\underset{u}{(1, 2, 4, 0)^T}, \underset{v}{(-2, 3, -1, 0)^T}, \underset{w}{(0, 2, 6, -1)^T}\}$$

$u.w = 4 + 24$
$= 28$ ✗

Example 2

i. Find the projection matrix for $a = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 3 \end{bmatrix}$.

ii. Obtain the projection of $b = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 5 \end{bmatrix}$ onto a and compute the error.

Solution

i) Projection matrix of $a = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 3 \end{pmatrix}$

$$P = \frac{aa^T}{a^Ta}$$

$$aa^T = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 4 & 6 \\ -2 & 1 & -2 & -3 \\ 4 & -2 & 4 & 6 \\ 6 & -3 & 6 & 9 \end{pmatrix}$$

$$a^Ta = \begin{pmatrix} 2 & -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \\ 3 \end{pmatrix} = 4 + 1 + 4 + 9 = 18$$

$$P = \begin{pmatrix} 2/9 & -1/9 & 2/9 & 1/3 \\ -1/9 & 1/18 & -1/9 & -1/6 \\ 2/9 & -1/9 & 2/9 & 1/3 \\ 1/3 & -1/6 & 1/3 & 1/2 \end{pmatrix}$$

ii) Projection of $b = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}$ onto a

$$p = P \cdot b = \begin{pmatrix} 10/9 \\ -5/9 \\ 10/9 \\ 5/3 \end{pmatrix}$$

Compute error, $e = b - p$

$$e = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 10/9 \\ -5/9 \\ 10/9 \\ 5/3 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -1 \\ 32 \\ 28 \\ 30 \end{pmatrix}$$

Least squares method

- To determine the ‘line of best fit’ for a set of data.
- How?
- By minimizing the sum of the offsets or residuals of points from the plotted line.
- Represents general trend of the data.
- Used for regression analysis.

Example 3

Build a model that studies the relationship between x and y given in the Table below using least squares method.

x	1	2	3	4	5
y	2.6	3.4	7.1	10.2	13.5

Solution

- Least Squares method: $\boldsymbol{A^T A \hat{x}} = \boldsymbol{A}^T\mathrm{b}$ ✓

$$\hat{x} = \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix} \begin{matrix} \rightarrow \text{slope} \\ \rightarrow \text{intercept} \end{matrix}$$

$$A = \begin{array}{c} \begin{array}{cc} x & \end{array} \\ \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \end{array} \quad b = \begin{array}{c} y \\ \begin{bmatrix} 2.6 \\ 3.4 \\ 7.1 \\ 10.2 \\ 13.5 \end{bmatrix} \end{array}$$

Find A^TA and A^Tb

$$A^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2.6 \\ 3.4 \\ 7.1 \\ 10.2 \\ 13.5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 139 \\ 36.8 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix} = \begin{bmatrix} 139 \\ 36.8 \end{bmatrix}$$

$$55\hat{\theta}' + 15\hat{\theta}'' = 139$$

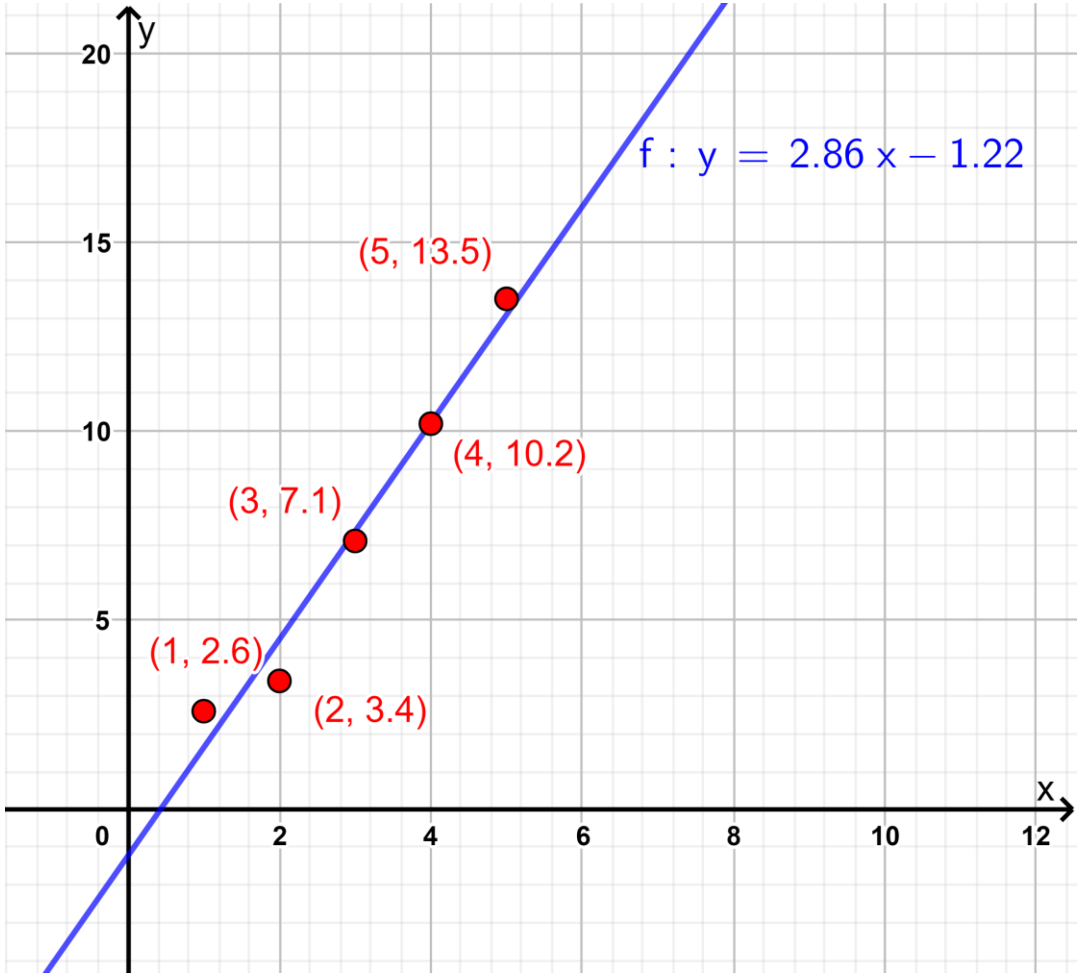
$$15\hat{\theta}' + 5\hat{\theta}'' = 36.8$$

Solving these equations;

$$\hat{\theta}' = 2.86$$

$$\hat{\theta}'' = -1.22$$

Best line through the given data is : $y = 2.86\,x\ - 1.22$



Thank You