

# Outline

- Sets and Functions
  - Notations
  - Logic
  - Graphs and visualisations.
- **Univariate Calculus**
  - **Continuity and differentiability**
  - Derivatives and Linear approximations
  - Applications/Advanced rules
- Multivariate Calculus
  - Lines and planes in high dimensional space.
  - Partial derivatives
  - Gradients
  - Linear approximations and Alternate gradient interpretations
  - Applications/Advanced rules

# Continuity of Functions

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

is continuous at  $x^* \in \mathbb{R}$  if for all sequences  $x_1, x_2, \dots$  converging to  $x^*$  we have that  $f(x_i)$  converges to  $f(x^*)$

$$\lim_{i \rightarrow \infty} x_i = x^* \Rightarrow \lim_{i \rightarrow \infty} f(x_i) = f(x^*)$$

$$\lim_{x \rightarrow x^*} f(x) = f(x^*)$$

E.g 1 :  $f(x) = x^2$ ,  $x^* = 2$

$$x_i : \quad 3, 2.5, 2.25, \dots \quad \rightarrow 2$$

$$f(x_i) : \quad 9, 6.25, 5.0625, \dots \quad \rightarrow 4$$

# Continuity of Functions

e.g 2 :  $f(x) = \text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$

$$x^* = 0$$

$$x_i : 1, \frac{1}{2}, \frac{1}{4}, \dots \rightarrow 0$$

$$f(x_i) : 1, 1, 1, \dots \rightarrow 1$$

$$x_i : -1, -\frac{1}{2}, -\frac{1}{4}, \dots \rightarrow 0$$

$$f(x_i) : -1, -1, -1, \dots \rightarrow -1$$

# Continuity of Functions

$$\text{E.g. 3 : } f(x) = \begin{cases} 2x+1 & \text{if } x > 1 \\ 3 & \text{if } x \leq 1 \end{cases}$$

$$\text{E.g. 3 } f(x) = \frac{1}{x},$$

$$x_i : 1, \frac{1}{2}, \frac{1}{4}, \dots$$

$$f(x_i) : 1, 2, 4, 8$$

$$\text{E.g. 3 } f(x) = \cos\left(\frac{1}{x}\right)$$

$$x_i : 1, \frac{1}{2}, \frac{1}{4}, \dots$$

$$f(x_i) : \cos(1), \cos(2), \cos(4),$$

# Differentiability of Functions

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $x^* \in \mathbb{R}$  if  $\lim_{x \rightarrow x^*} \frac{f(x) - f(x^*)}{x - x^*}$  exists.

$\swarrow$

$f'(x^*)$

$f$  is NOT  
continuous  
at  $x^*$

$\Rightarrow$   $f$  is NOT  
differentiable  
at  $x^*$ .

E.g. 1  $f(x) = |x|$

$x_i : 1, \frac{1}{2}, \frac{1}{4}, \dots$

$\rightarrow 0$

$\frac{f(x_i) - f(0)}{x_i} : 1, 1, 1$

$\rightarrow 1$

$x_i : -1, -\frac{1}{2}, -\frac{1}{4}, \dots$   
 $: -1, -1, -1$

# Differentiability of Functions

Ex. 7 : 2

$$f(x) = \begin{cases} 4x+2 & \text{if } x \geq 2 \\ 2x+8 & \text{if } x < 2 \end{cases}$$

Ex. 8 : 3

$$f(x) = \begin{cases} 4x+2 & \text{if } x \geq 2 \\ 2x+6 & \text{if } x < 2 \end{cases}$$

$$\lim_{x \rightarrow 2+} \frac{f(x) - f(2)}{x - 2} = 4$$

$$\lim_{x \rightarrow 2-} \frac{f(x) - f(2)}{x - 2} = 2$$

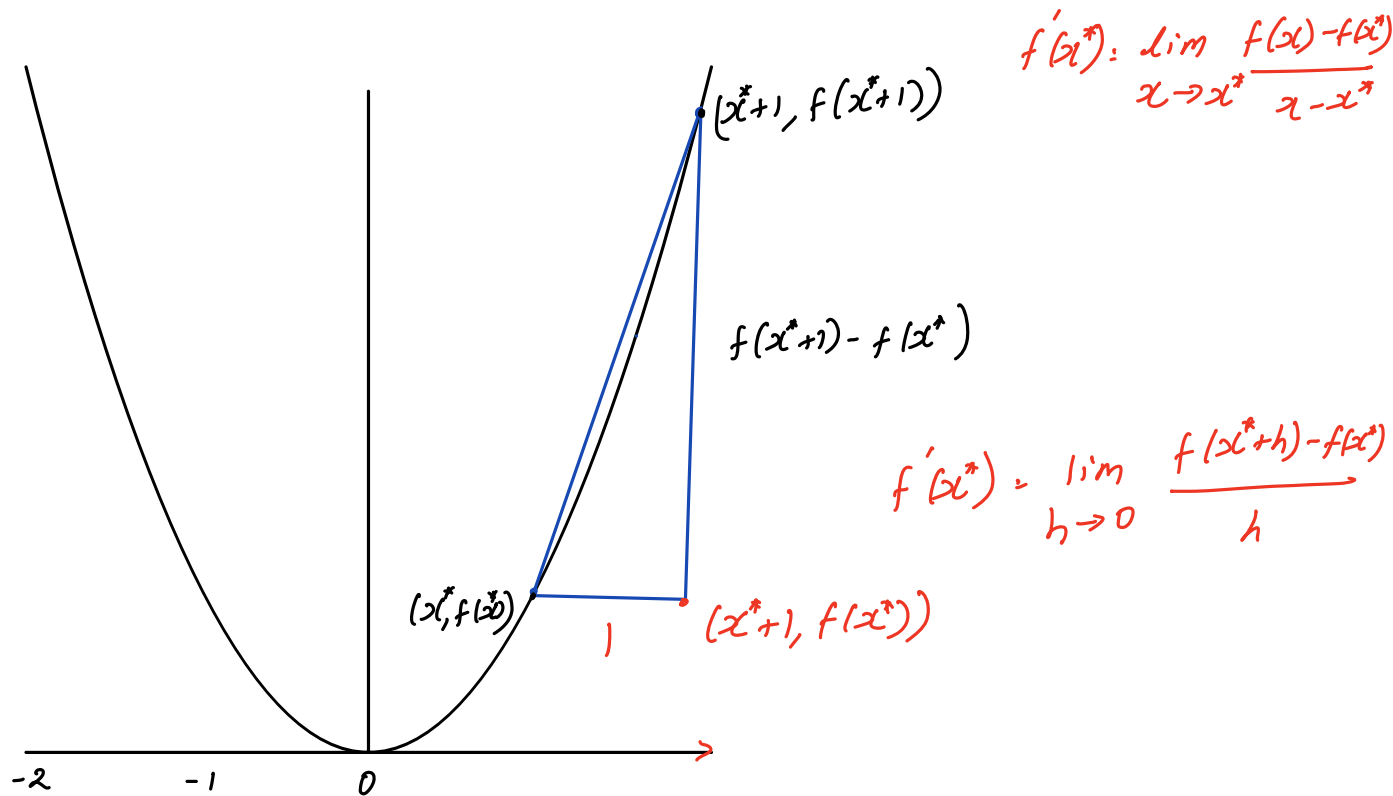
# Differentiability of Functions

E.g. 4:

$$f(x) = \begin{cases} 4x+2 & \text{if } x \geq 2 \\ x^2+6 & \text{if } x < 2 \end{cases}$$

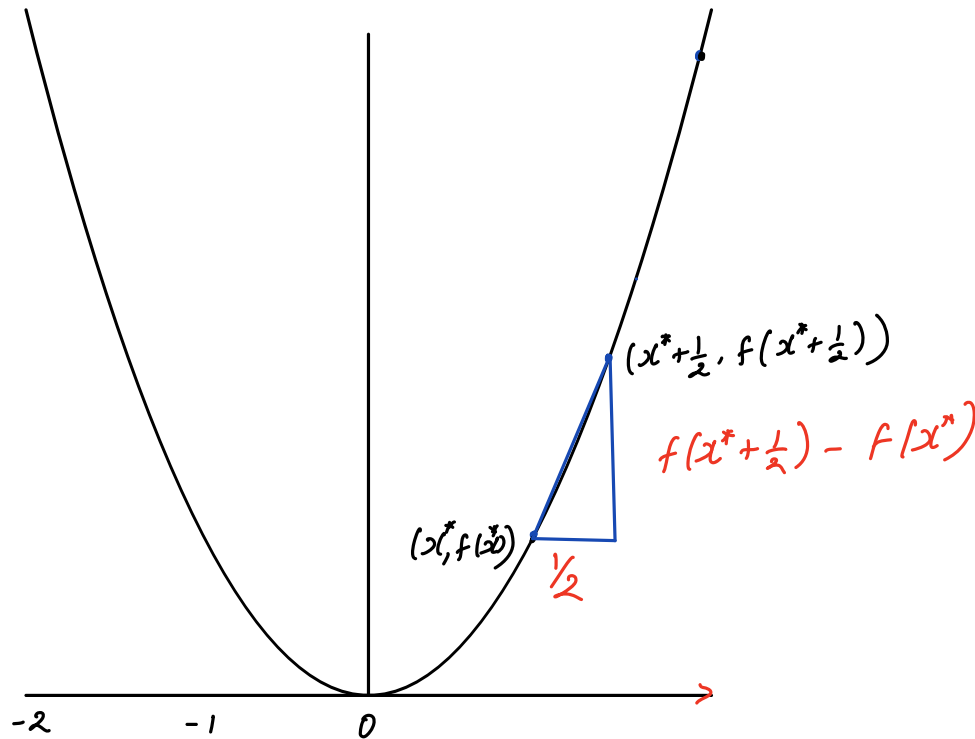
$$\lim_{x \rightarrow 2^+} = \lim_{x \rightarrow 2^-} = \frac{f(x) - f(2)}{x - 2} = 4$$

# Differentiability of Functions

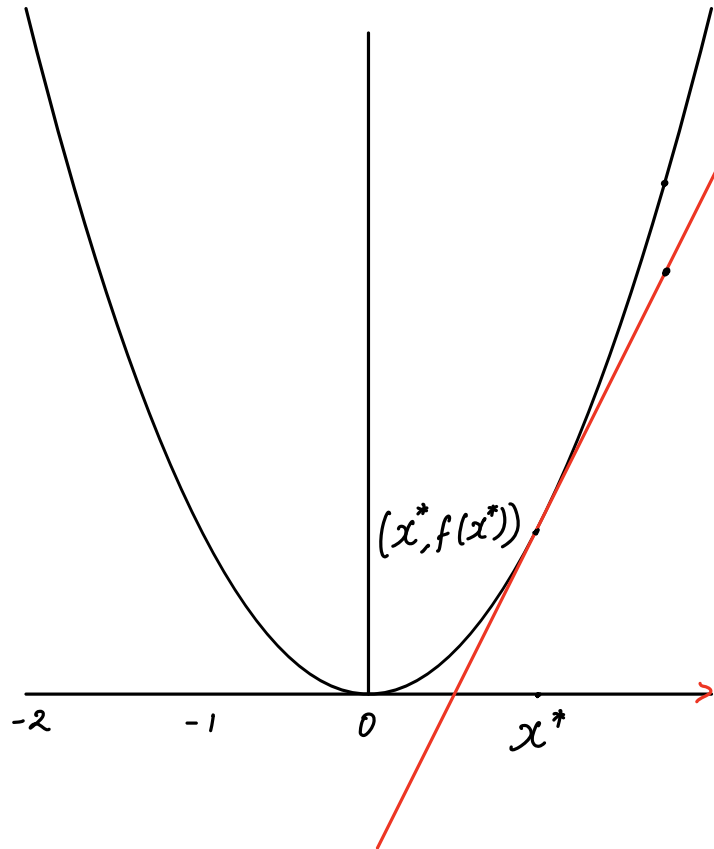




# Differentiability of Functions



# Differentiability of Functions



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