Note Title 5/4/2021

Example: Fibonacci sequence

$0, 1, 1, 2, 3, 5, \ldots \qquad \{F_k\}$

$$F_{k+2} = F_{k+1} + F_k$$

Q: What is F_{100}? A linear algebra solution to this question uses diagonalization.

System of equations: $F_{k+2} = F_{k+1} + F_k$

$F_{k+1} = F_{k+1}$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}, \qquad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k \iff \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad u_0 \xrightarrow{A} u_1 \xrightarrow{A} u_2 \xrightarrow{A} u_3 \longrightarrow \ldots$

$u_k = A^k u_0$ is a solution to $u_{k+1} = A u_k$

In a general scenario, where A is $n \times n$, u_k is $n \times 1$

If A is diagonalizable, i.e., A has "n" independent eigenvectors, then

$u_0 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ since $\{x_1 \dots x_n\}$ is a basis

$Au_0 = c_1 A x_1 + \dots + c_n A x_n$
$= c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n$

$A^2 u_0 = A(c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n)$
$= c_1 \lambda_1^2 x_1 + \dots + c_n \lambda_n^2 x_n$

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n$

Back to Fibonacci: to find u_k, we need to check if $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable & find its eigenvectors, if yes.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ characteristic equation: $\lambda^2 - \lambda - 1 = 0$

$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$

$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 x_1 + c_2 x_2$

Lets find the eigenvectors of A:

$$(A-\lambda I)x = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} x = 0$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$(A-\lambda_1 I)\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} = \begin{bmatrix} \lambda_1^2 - \lambda_1 - 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ since λ_1 is a root of $\lambda^2 - \lambda - 1 = 0$

Similarly, $(A-\lambda_2 I)\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} = 0$

So, x_1 and x_2 are the eigenvectors corresponding to λ_1 and λ_2

Writing u_0 as a linear combination of x_1, x_2:

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \Rightarrow c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = -\frac{1}{\sqrt{5}}$$

$$u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$$

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{<1}^k$$

as k increases $\left(\frac{1-\sqrt{5}}{2}\right)^k$ becomes negligible

A good enough approximation $F_k \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k$

(or) $F_{100} \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{100} = \frac{1}{\sqrt{5}} (1.618)^{100}$

Bottom line: Diagonalizability can be used to understand linear recurrence relations.