Course: Machine Learning - Foundations
Week 5: Test questions

1. (1 point) Consider two non-zero vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$. Suppose the inner product between x and y obeys commutative property (i.e., $x \cdot y = y \cdot x$), it implies that

   A. y must be a conjugate transpose of x

   B. y is equal to x

   C. y must be orthogonal to x

   D. y must be a scalar (possibly complex) multiple of x

   Answer: C

   For orthogonal vectors dot product is zero.

2. (1 point) The inner product of two distinct vectors x and y that are drawn randomly from $\mathbb{C}^{100}$ is $0.8 - 0.37i$.The vector x is scaled by a scalar $1 - 2i$ to obtain a new vector z, then the inner product between z and y is

   A. $0.06 - 1.97i$

   B. $1.54 - 1.23i$

   C. $1.54 + 1.23i$

   D. $0.8 - 0.37i$

   E. Not possible to calculate

   Answer: C

   $x.y = \bar{x}^T y$
   $z = cx$
   $z.y = \bar{c}\bar{x}^T y$
   $z.y = \bar{c}x.y = 1.54 + 1.23i$

3. (1 point) Select the correct statement(s).The Eigen-value decomposition for the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

   A. doesn't exist over $\mathbb{R}$ but exists over $\mathbb{C}$

   B. doesn't exist over $\mathbb{C}$ but exists over $\mathbb{R}$

   C. neither exists over $\mathbb{R}$ nor exists over $\mathbb{C}$

   D. exists over both $\mathbb{C}$ and $\mathbb{R}$

   Answer: A

   Eigenvalues of A are complex.

4. (1 point) Consider the complex matrix $S = \begin{bmatrix} 1 & 1+i & -2-2i \\ 1-i & 1 & -i \\ -2+2i & i & 1 \end{bmatrix}$. The matrix is

    A. Hermitian and Symmetric

    B. Symmetric but not Hermitian

    C. Neithet Hermitian nor Symmetric

    D. Hermitian but not Symmetric

Answer: D
$S^T \neq S$
$S^* = S$

5. (1 point) Suppose that an unitary matrix U is multiplied by a diagonal matrix D with $d_{ii} \in \mathbb{R}$, then the resultant matrix will always be unitary. The statement is

    A. True

    B. False

Answer: A
Let $A = UD$
Where, U=unitary matrix
D=Diagonal matrix
$A^* = (UD)^*$
$AA^* = (UD)(UD)^*$
$AA^* = (UDD^*U^*)$

6. (3 points) The eigenvectors of matrix $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ are

    A. $\begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$

    B. $\begin{bmatrix} 1 \\ 1-2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$

    C. $\begin{bmatrix} -1 \\ 1-2i \\ -1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$

    D. $\begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1-3i \\ 2-i \\ -5 \end{bmatrix}$

Answer: A

$|A - \lambda I| = 0$

$\lambda^3 - 3\lambda^2 - 16\lambda - 12 = 0$

$\lambda = -1, -2, 6$

For $\lambda = -1$

Eigenvector, $v_1 = \begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix}$

For $\lambda = -2$

Eigenvector, $v_2 = \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}$

For $\lambda = 6$

Eigenvector, $v_3 = \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$

7. (1 point) A matrix $A = \frac{1}{2}\begin{bmatrix} k+i & \sqrt{2} \\ k-i & \sqrt{2}i \end{bmatrix}$ is unitary if k is

   A. $\frac{1}{2}$

   B. 1

   C. -$\frac{1}{2}$

   D. -1

   E. ± 1

   F. $\pm\frac{1}{2}$

Answer: B

$$\frac{1}{2}\begin{bmatrix} k+i & \sqrt{2} \\ k-i & \sqrt{2}i \end{bmatrix} \frac{1}{2}\begin{bmatrix} k-i & k+i \\ \sqrt{2} & -\sqrt{2}i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} k^2+3 & k^2-1+2i(k-1) \\ k^2-1-2i(k-1) & k^2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$k = 1$

8. (3 points) A matrix $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$ can be written as $A = UDU^*$, where U is a unitary matrix and D is a diagonal matrix. Then, U and D, respectively, are

   A. $U = \begin{bmatrix} 1+i & \sqrt{2} \\ \sqrt{2} & 1-i \end{bmatrix}$, $D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

   B. $U = \begin{bmatrix} -1+i & \sqrt{2} \\ \sqrt{2} & -1-i \end{bmatrix}$, $D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

C. $U = \begin{bmatrix} 1+i & \sqrt{2} \\ \sqrt{2} & 1-i \end{bmatrix}$, $D = \begin{bmatrix} -1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

D. $U = \begin{bmatrix} 1-i & \sqrt{2} \\ \sqrt{-2} & 1-i \end{bmatrix}$, $D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

Answer: A

To find eigenvalues, $|A - \lambda I| = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda = 1 + \sqrt{2}, 1 - \sqrt{2}$

Find eigenvectors

For $\lambda = 1 + \sqrt{2}$,

$v_1 = \begin{bmatrix} 1+i \\ \sqrt{2} \end{bmatrix}$

For $\lambda = 1 - \sqrt{2}$,

$v_2 = \begin{bmatrix} \sqrt{2} \\ 1-i \end{bmatrix}$

$U = [v_1 \quad v_2]$

$D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

9. (2 points) The matrix $Z = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ has

A. only real eigenvalues.

B. two real and two complex eigenvalue.

C. three real and one complex eigenvalues.

D. all complex eigenvalues

Answer: B

$\lambda = 1, -1, i, -i$

10. (1 point) (Multiple select) Which of the following matrices is/are unitary?

A. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

B. $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

C. $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

D. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

E. $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Answer: C, D
Check $UU^* = I$

11. (2 points) Let U and V be two unitary matrices. Then

    1. UV is unitary.
    2. $U + V$ is unitary.

    A. Both statements are true.
    B. Both statements are false.
    C. 1. is false.
    D. 2. is false.

Answer: D
Addition of two unitary matrices may not be unitary.

12. (2 points) (Multiple select) Which of the following is/are eigenvectors of the Hermitian matrix $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

    A. $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$
    B. $\begin{bmatrix} -2-2i \\ 2 \end{bmatrix}$
    C. $\begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$
    D. $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$
    E. All of these.

Answer: E
To find eigenvalues, $|A - \lambda I| = 0$
$\lambda^2 - 3\lambda = 0$
$\lambda = 0, 3$

For $\lambda = 0$
Eigenvector, $v_1 = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$
For $\lambda = 3$
Eigenvector, $v_2 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$