Divide and Conquer: Closest Pair of Points

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Programming, Data Structures and Algorithms using Python

Week 8

Recall: Video Game

- Several objects on screen
- Basic step: find closest pair of objects

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- Use divide and conquer

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  - Assume no two points have same x or y coordinate
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  - ...or modify the algorithm slightly!
- Brute force
  - Compute $d(p_i, p_j)$ for every pair of points
  - $O(n^2)$

Finding the closest pair of points

In 1 dimension

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- Given n 1D points $x_1, x_2, \ldots, x_n$, find the closest pair
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- How to do this efficiently?

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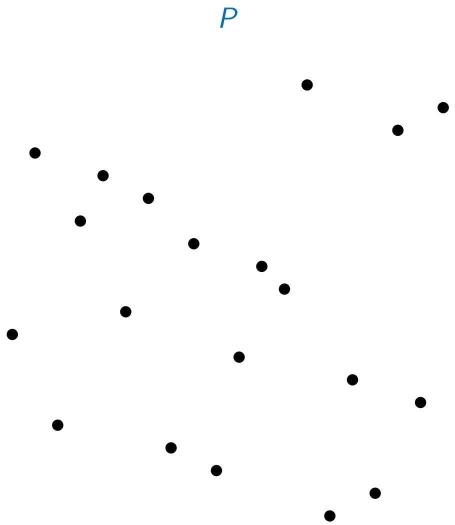
- Given n points $P = \{p_1, p_2, \ldots, p_n\}$, compute
  - P_x, P sorted by x-coordinate
  - P_y, P sorted by y-coordinate



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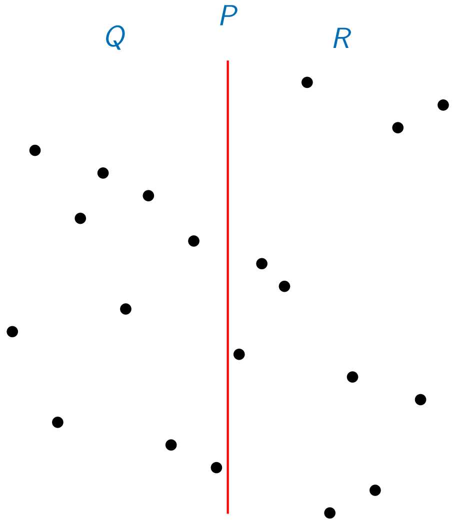

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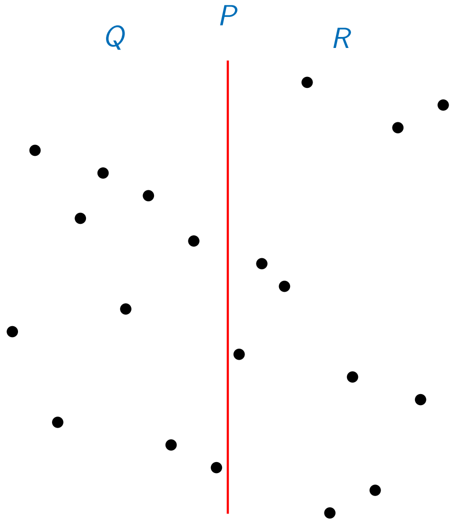

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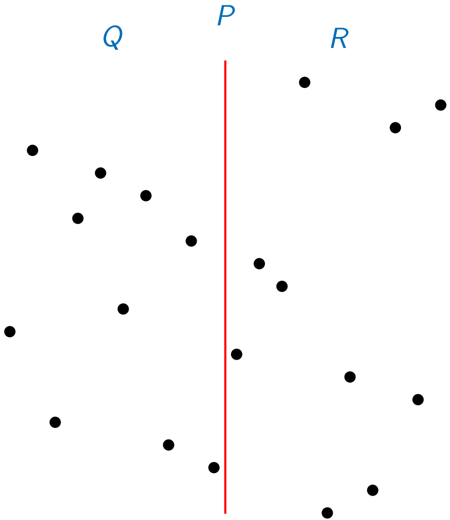

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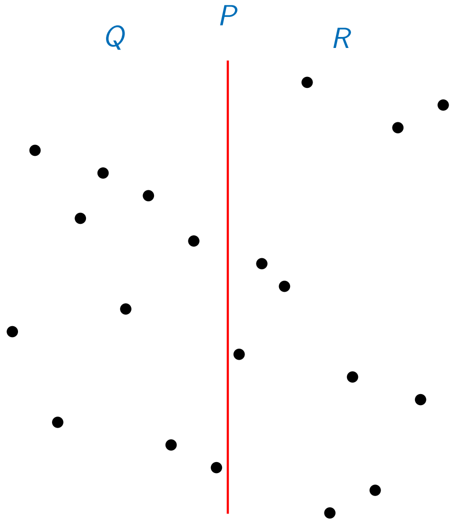
- Let x_R be smallest x coordinate in R



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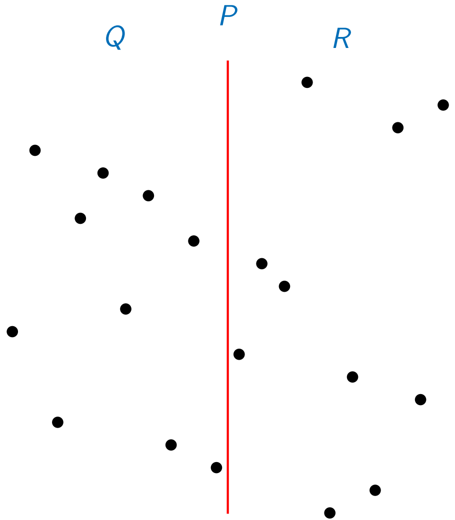

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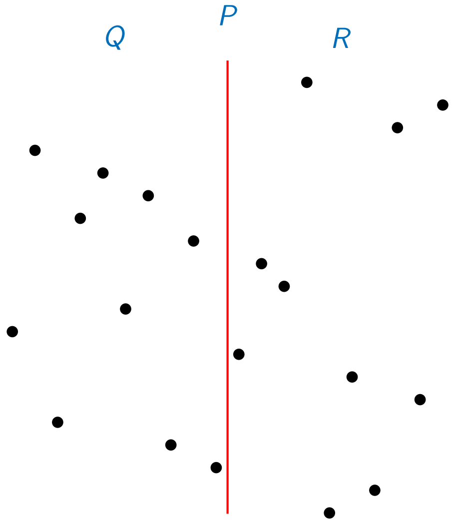
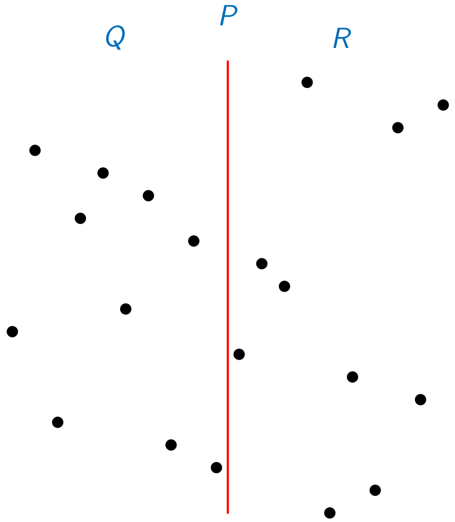
- All of this can be done in $O(n)$



Divide and conquer

- Want to compute `ClosestPair`(P_x, P_y)



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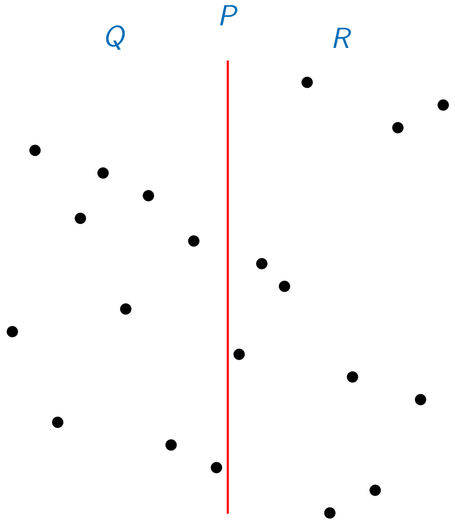

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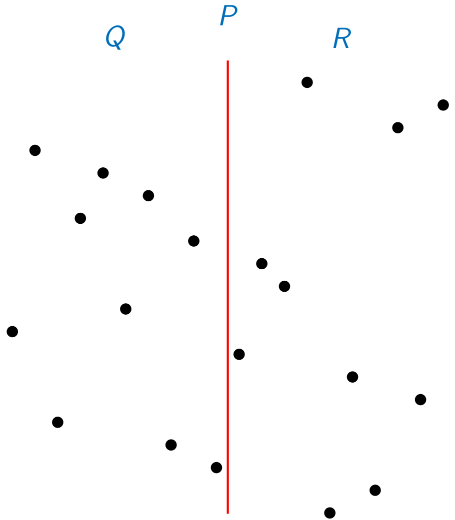

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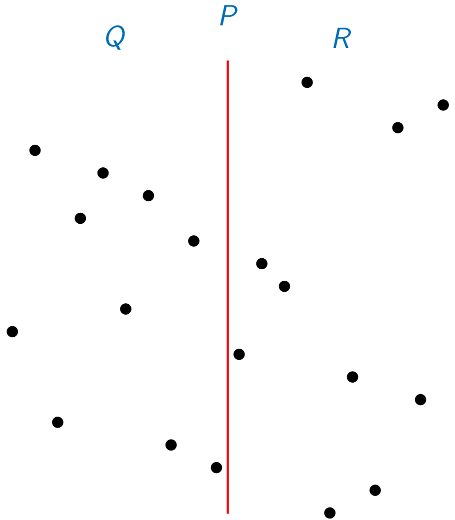
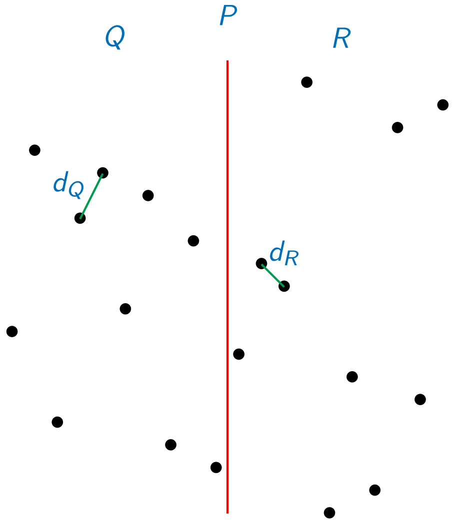
- How to combine these recursive solutions?



Combining solutions

- Let d_Q, d_R be closest distances in Q, R, respectively



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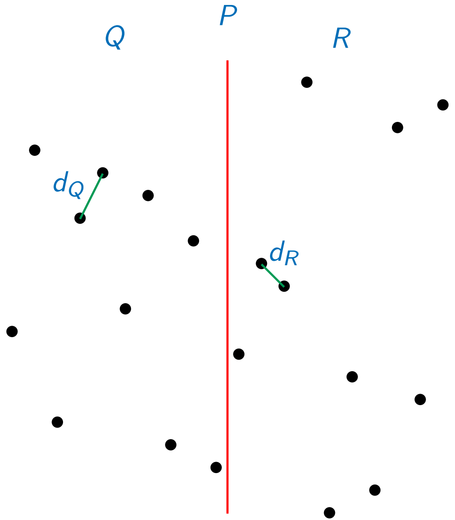
- Set $\delta = \min(d_Q, d_R)$



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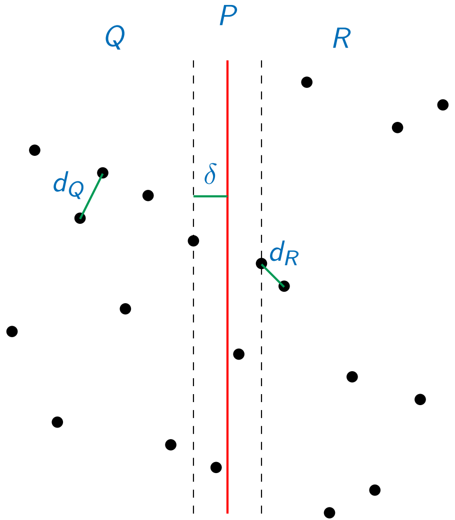

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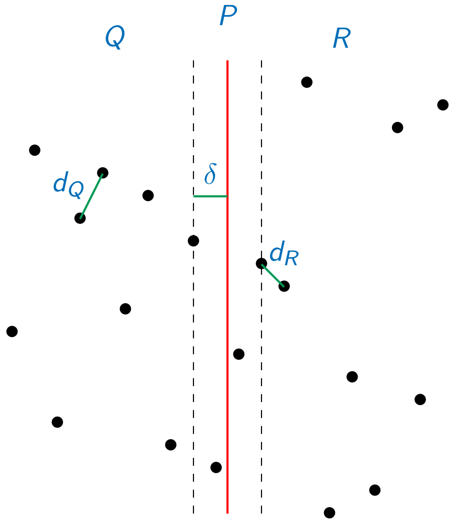
- Only need to consider points within distance δ on either side of the separator
- No pair outside this band can be closer than δ



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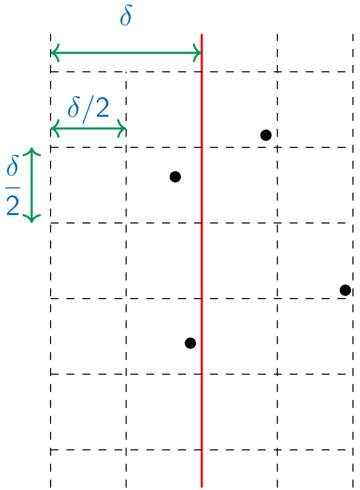

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- Divide the distance δ band into boxes of side $\delta/2$
- Cannot have two points inside the same box
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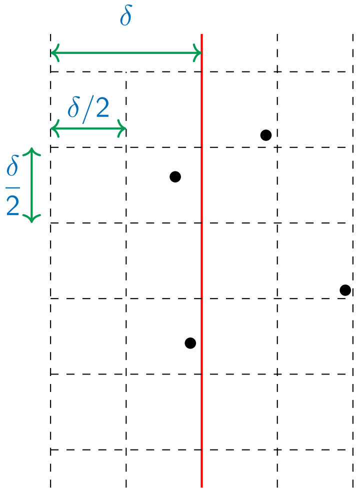

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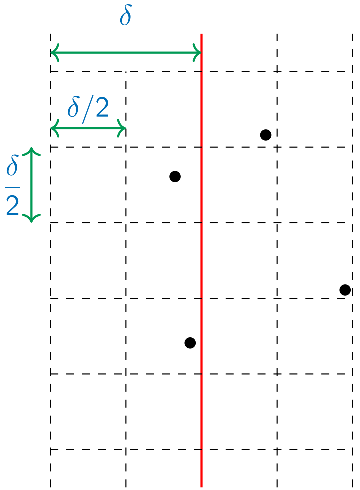
  - Check each point against 15 others



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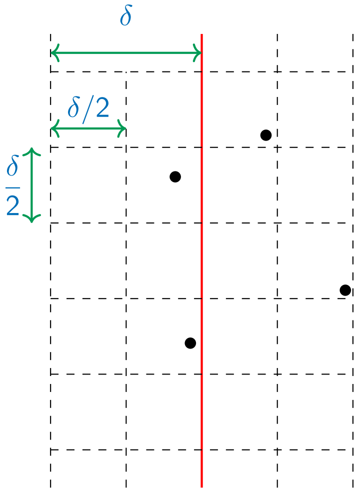

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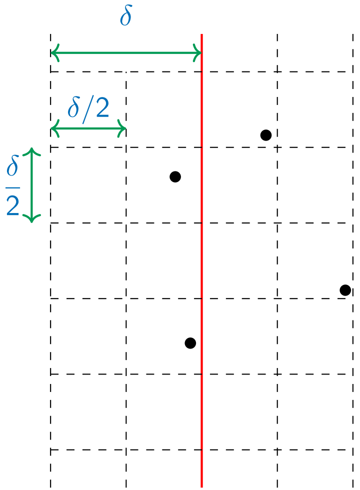

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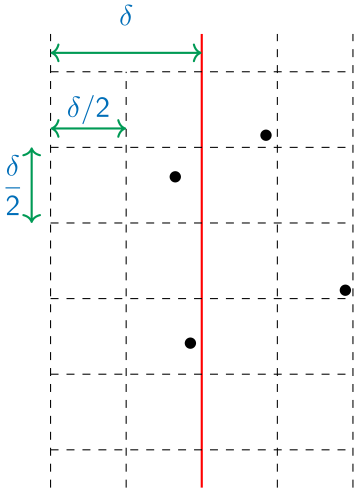
- Linear scan



Algorithm and analysis

Pseudocode

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def ClosestPair(Px,Py):

  if len(Px) <= 3:
    compute pairwise distances
    return closest pair and distance

  Construct (Qx,Qy), (Rx,Ry)

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  (r1,r2,dR) = ClosestPair(Rx,Ry)

  Construct Sy from Qy,Ry
  Scan Sy, find (s1,s2,dS)

  return (q1,q2,dQ), (r1,r2,QR), (s1,s2,dS)
  depending on which of dQ, dR, dS is minimum
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- Overall, $O(n \log n)$