Main result:

A is a real symmetric $n \times n$ matrix. Then,

1. Eigenvalues of A are real
2. Eigenvectors corresponding to different eigenvalues are linearly independent
3. A is orthogonally diagonalizable

this means

$\exists\, Q$ satisfying $Q^T Q = I$ such that $\Rightarrow Q^{-1} = Q^T$

$$A = Q \Lambda Q^T$$

$$= \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - & x_1^T & - \\ - & x_2^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix}$$

Remark: ① A is diagonalizable + $\underbrace{\text{orthogonal matrix for diagonalization}}_{Q^T Q = I}$ $\Rightarrow$ orthogonally diagonalizable

$A = Q \Lambda Q^{-1}$ $\qquad$ $A = Q \Lambda Q^T$

② Any real matrix is not necessarily diagonalizable, but a real symmetric is diagonalizable.

e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ not diagonalizable

Example: $A = \begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix}$

Eigenvalues: $\lambda_1 = -3, \quad \lambda_2 = 2$

Eigenvalues: $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$q_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad q_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \qquad \text{Check: } Q^T Q = I$$

Then, $$Q \Lambda Q^T = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix} = A$$

Check this.