Note Title 6/18/2021

Positive definiteness

Consider the function $\boxed{f(x,y) = 2x^2 + 4xy + y^2}$

At a stationary point, the first derivatives vanish

$$\frac{\partial f}{\partial x} = 4x + 4y = 0 \quad , \quad \frac{\partial f}{\partial y} = 4x + 2y = 0$$

So, $(x,y) = (0,0)$ both $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are zero.

Hence, $(0,0)$ is a stationary point of f.

Question: Whether $(0,0)$ is a minima/maxima/saddle point?

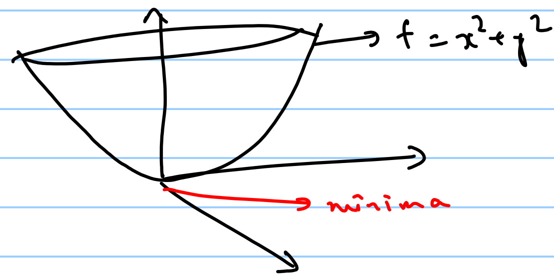
Answer: Check the second derivatives at $(0,0)$.

$$\frac{\partial^2 f}{\partial x^2} = 4 \ , \quad \frac{\partial^2 f}{\partial x \partial y} = 4 \ , \quad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \Rightarrow \quad f \text{ has a minima at } (0,0)$$

Remark: Every quadratic function of the form $ax^2+2bxy+cy^2$ has a stationary point at $(0,0)$

"Positive definite":

A function f that vanishes at $(0,0)$ and is strictly positive at other points is called "positive definite", and is denoted by $\boxed{\text{"}f>0\text{"}}$



Question! What conditions on a,b,c ensure $f(x,y)=ax^2+2bxy+cy^2$ is positive definite?

Necessary conditions:

(I) If $f>0$ (f is positive definite), then $a>0$ (why? Look at the value of f at $(1,0)$)

(II) If $f > 0$, then $c > 0$ (why? Look at the value of f at $(0,1)$)

Are (I) and (II) enough to ensure $f > 0$? NO

Example: $f(x,y) = x^2 - 10xy + y^2$ (Look at the value of f at $(1,1)$)

What extra condition would allow us to infer $f > 0$?

$$f(x,y) = ax^2 + 2bxy + cy^2 = a\underbrace{\left(x + \frac{b}{a}y\right)^2}_{\geq 0} + \left(c - \frac{b^2}{a}\right)\underbrace{y^2}_{\geq 0}$$

These two factors decide if $f > 0$ or not

(III) If $f > 0$, then $ac > b^2$

Combining all three conditions, we have

$f(x,y) = ax^2 + 2bxy + cy^2$ is positive definite "if and only if" $a > 0$ and $ac > b^2$

Remarks: ① If $ac = b^2$, then $f(x,y) = ax^2 + 2bxy + by^2$ is $\begin{cases} \text{positive semi-definite if } a > 0 \\ \text{negative semi-definite if } a < 0 \end{cases}$

② We have a saddle point at $(0,0)$ if $ac < b^2$.

Connection to linear algebra:

$$ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Then $ax^2 + 2bxy + cy^2 = v^T A v$

In general, we look at $v^T A v$ in $\mathbb{R}^n$

$$[x_1 \; \cdots \; x_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Let $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad f(v) = a_{11} x_1^2 + 2a_{12} x_1 x_2 + \cdots \cdots + a_{nn} x_n^2$

At $v = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, $f(v) = 0 \Rightarrow \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is a stationary point of f.

Examples to check if f has a minima/maxima/saddle at origin

① $f(x,y) = 2x^2 + 4xy + y^2$ $\leftarrow$ saddle point at origin since $ac = 2 < b^2 = 4$, $A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$

② $f(x,y) = 2xy$ $\leftarrow$ saddle at origin. $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

③ $f(x_1, x_2, x_3) = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

At the origin, f has a minimum $\leftarrow$ N.W.