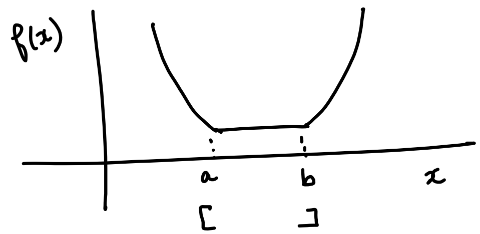
- If $f$ is convex, then all local minima are also global minima.

- 


  $[a,b]$

  • Set of all global minima of a convex function is a convex set.

- Necessary and sufficient conditions for optimality of convex functions.

  $$\boxed{\min_{x} f(x)}$$

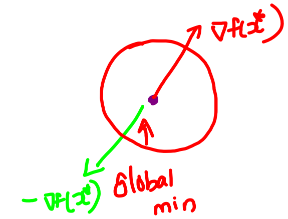
  convex, differentiable.

**Thm.** Let $f$ be a differentiable, convex function from $\mathbb{R}^d \to \mathbb{R}$.
$x^* \in \mathbb{R}^d$ is a global minimum of $f$ if and only if $\boxed{\nabla f(x^*) = 0}$

## Proof sketch

**Necessary part** $\Rightarrow$

$\boxed{\text{If } x^* \text{ is global minimum} \Rightarrow \nabla f(x^*) = 0.}$ ?




$x^*$ is a global minimum $\Rightarrow$ $x^*$ is a local minimum

Say $\nabla f(x^*) \neq 0.$ $\Rightarrow$ There exists a direction $(-\nabla f(x^*))$ and a small step size s.t

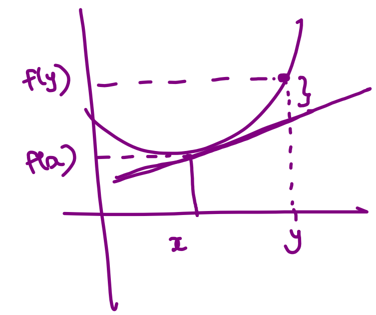
$$f(x^* - \eta \nabla f(x^*)) < f(x^*).$$

$\Rightarrow$ Contradicts the assumption that $x^*$ is the global minima.

$\Leftarrow$ If $\exists\, x^*$ s.t $\nabla f(x^*) = 0 \Rightarrow x^*$ is a global minima.

By definition of convexity

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \forall\, x, y.$$




$$\Rightarrow \quad f(y) \geq f(x^*) + \underbrace{\nabla f(x^*)^T (y - x)}_{0} \quad \forall y$$

$(\because \nabla f(x^*) = 0$ by assumption$)$

$$\Rightarrow \quad f(y) \geq f(x^*) \quad \forall y \quad \Rightarrow x^* \text{ is a global minima!}$$

$\square$

## Some Additional properties of convex functions

**Property 1:** If $f:\mathbb{R}^d \to \mathbb{R}$ and $g:\mathbb{R}^d \to \mathbb{R}$ are both convex functions.

then $h(x) := f(x) + g(x)$ $\forall x$ is also a convex function.

**Proof:** Fix $\lambda \in [0,1]$

$$h(\lambda x + (1-\lambda)y) = f(\lambda x + (1-\lambda)y) + g(\lambda x + (1-\lambda)y) \quad [\text{by defn of } h]$$

$$\leq \lambda f(x) + (1-\lambda) f(y) + \lambda g(x) + (1-\lambda) g(y) \quad [\text{by convexity of } f \text{ and } g]$$

$$= \lambda (f(x) + g(x)) + (1-\lambda)(f(y) + g(y))$$

$$= \lambda h(x) + (1-\lambda) h(y) \qquad \square$$

$$\boxed{\text{Sums of convex functions is convex}}$$

## Property 2 : Compositions

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex and non-decreasing function.

$f(x) \geq f(y)$
$\forall\ x \geq y$

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a convex function.

Define $h := f \circ g$ . $\qquad h(x) = f(\boxed{g(x)})$

$h : \mathbb{R}^d \to \mathbb{R}$

Claim: $h$ is convex!

Fix $\lambda \in [0,1]$ ; $\boxed{h(\lambda x + (1-\lambda) y)} = f(\underbrace{\boxed{g(\lambda x + (1-\lambda) y)}}_{z}) \quad z \leq \boxed{\lambda g(x) + (1-\lambda) g(y)} \quad z'$

$\leq f(\lambda \overset{x'}{g(x)} + (1-\lambda) \overset{y'}{g(y)})$ [by convexity of $g$ and by non-decreasing property of $f$]

Composition of convex with convex + non-decreasing is convex.

$$\leq \lambda f(g(x)) + (1-\lambda) f(g(y))$$

$$= \lambda h(x) + (1-\lambda) h(y).$$

$\square$.

**Property 3:** Let $f: \mathbb{R} \to \mathbb{R}$ be convex and $g: \mathbb{R}^d \to \mathbb{R}$ be linear $h = f \circ g$. Then $h$ is convex

Proof: Fix $\lambda \in [0,1]$. $h(\lambda x + (1-\lambda) y) = f(\underbrace{g(\lambda x + (1-\lambda) y)})$

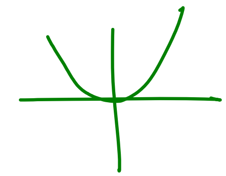
$$= f(\lambda g(x) + (1-\lambda) g(y)) \quad [\text{by linearity of } g]$$

$$\boxed{\leq} \lambda f(g(x)) + (1-\lambda) f(g(y)) = \lambda h(x) + (1-\lambda) h(y)$$

[Convexity of $f$]

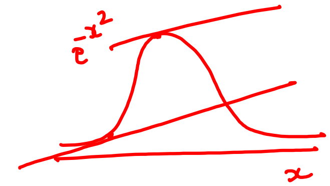
Note: In general if $f$ and $g$ are convex, then $h = f \circ g$ may not be convex!

$g(x) = x^2 \rightarrow$ Convex

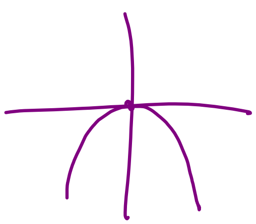


$f(x) = e^{\boxed{-x}} \rightarrow$ Convex.

Is $f \circ g$ Convex? $\quad f \circ g(x) = e^{-x^2}$ . is not Convex.




Exercise: Find similar properties with $g \rightarrow$ concave. and $f \rightarrow$ non-increasing Convex.



$g(x) = -x^2$ is Concave.

$g$ is Concave iff $f = -g$ is convex.