Course: Machine Learning - Foundations
Week 5: Practice questions

1. (1 point) The complex conjugate of matrix $A = \begin{bmatrix} 1-i & 1-3i \\ 6+4i & 35-2i \end{bmatrix}$ is

   A. $\begin{bmatrix} 1-i & 1-3i \\ 6+4i & 35-2i \end{bmatrix}$

   B. $\begin{bmatrix} 1+i & 1+3i \\ 6-4i & 35+2i \end{bmatrix}$

   C. $\begin{bmatrix} -1+i & -1-3i \\ -6+4i & -35-2i \end{bmatrix}$

   D. $\begin{bmatrix} 1-i & 1-3i \\ 6-4i & 35-2i \end{bmatrix}$

   Answer: B
   Take complex conjugate of each term in the matrix.

2. (1 point) The complex conjugate transpose of matrix $A = \begin{bmatrix} 3-2i & 5+i \\ 1+4i & 7-2i \end{bmatrix}$ is

   A. $\begin{bmatrix} 7+i & 5+41 \\ 3-i & 3-2i \end{bmatrix}$

   B. $\begin{bmatrix} 5-i & 3-4i \\ 1+i & 7+2i \end{bmatrix}$

   C. $\begin{bmatrix} 3+i & 5-i \\ 1+4i & 7-2i \end{bmatrix}$

   D. $\begin{bmatrix} 3+2i & 1-4i \\ 5-i & 7+2i \end{bmatrix}$

   Answer: D
   First Take complex conjugate of each term in the matrix and then take transpose the matrix.
   Or
   First take transpose of the matrix and then take complex conjugate of each term.

3. (1 point) The inner product of $x = \begin{bmatrix} 1-i \\ 2i \end{bmatrix}$ and $y = \begin{bmatrix} -1-i \\ i \end{bmatrix}$ is

   A. $7-6i$

   B. $4-4i$

   C. $2-2i$

   D. $3+4i$

Answer: C

$x.y = \bar{x}^T y$

$\bar{x}^T y = \begin{bmatrix} 1+i & -2i \end{bmatrix} \begin{bmatrix} -1-i \\ i \end{bmatrix} = 2-2i$

4. (1 point) The square of length of the vector $x = \begin{bmatrix} 2-i \\ 4-i \end{bmatrix}$ is

    A. 16

    B. 17

    C. 31

    D. 22

Answer: D

$L^2 = \bar{x}^T x$

$\bar{x}^T x = \begin{bmatrix} 2+i & 4+i \end{bmatrix} \begin{bmatrix} 2-i \\ 4-i \end{bmatrix} = 22$

5. (1 point) The matrix $A = \begin{bmatrix} \frac{(1+i)}{\sqrt{3}} & \frac{(1+i)}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{2i}{\sqrt{6}} \end{bmatrix}$ is unitary.

    A. True

    B. False

Answer: B

$$AA^* = \begin{bmatrix} \frac{(1+i)}{\sqrt{3}} & \frac{(1+i)}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{2i}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{(1-i)}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \\ \frac{(1-i)}{\sqrt{6}} & \frac{-2i}{\sqrt{6}} \end{bmatrix} \neq I$$

6. (1 point) The matrix $Z = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is Hermitian.

    A. True

    B. False

Answer: True

$Z = Z^*$

7. (1 point) (Multiple select) Which of the following matrices are Hermitian?

A. $\begin{bmatrix} 1 & 3-i \\ 3+i & i \end{bmatrix}$

B. $\begin{bmatrix} 0 & 3-2i \\ 3-2i & 4 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$

D. $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$

Answer: C, D
For a Hermitian matrix A, $A = A^*$

8. (1 point) The eigenvalues of matrix $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ are

A. -1,-6 and 2

B. 1, -6 and -2

C. 1, 6 and 2

D. -1, 6 and -2

Answer: D
$|A - \lambda I| = 0$
$\lambda^3 - 3\lambda^2 - 16\lambda - 12 = 0$
$\lambda = -1, -2, 6$

9. (2 points) The matrix $A = k\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ is unitary if k is

A. $\frac{1}{2}$

B. 1

C. $\frac{1}{4}$

D. $\frac{1}{8}$

Answer: A
$k\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} k\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$k^2\begin{bmatrix} 2(1-i^2) & 2(1+i^2) \\ 2(1+i^2) & 2(1-i^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$k^2 = \dfrac{1}{4}$
$k = \dfrac{1}{2}, -\dfrac{1}{2}$

10. (2 points) The matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & \sqrt{k} \\ 1-i & \sqrt{k}i \end{bmatrix}$ is unitary if k is

   A. $\frac{1}{2}$

   B. 1

   C. 2

   D. $\frac{1}{4}$

Answer: C

$\dfrac{1}{2}\begin{bmatrix} 1+i & \sqrt{k} \\ 1-i & \sqrt{k}i \end{bmatrix} \dfrac{1}{2}\begin{bmatrix} 1-i & 1+i \\ \sqrt{k} & -\sqrt{k}i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\dfrac{1}{4}\begin{bmatrix} k+2 & (2-k)i \\ (k-2)i & 2(k+2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$k - 2 = 0$

$k = 2$

11. (3 points) A matrix $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$ can be written as $A = UDU^*$, where U is a unitary matrix and D is a diagonal matrix. Then, U and D respectively are

   A. $U = \begin{bmatrix} \dfrac{-1+i}{\sqrt{3}} & \dfrac{1-i}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{6}} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

   B. $U = \begin{bmatrix} \dfrac{-1+i}{\sqrt{3}} & \dfrac{-1+i}{6} \\ \dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{6}} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

   C. $U = \begin{bmatrix} \dfrac{-1+i}{\sqrt{3}} & \dfrac{-1+i}{6} \\ \dfrac{-1}{\sqrt{3}} & \dfrac{-2}{\sqrt{6}} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

   D. $U = \begin{bmatrix} \dfrac{-1+i}{\sqrt{3}} & \dfrac{-1+i}{6} \\ \dfrac{-1}{\sqrt{3}} & \dfrac{-2}{\sqrt{6}} \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$

Answer: A

To find eigenvalues, $|A - \lambda I| = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda = 1, 4$

Find eigenvectors

For $\lambda = 1$,

$v_1 = \begin{bmatrix} -1+i \\ -1 \end{bmatrix}$

For $\lambda = 4$,
$v_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$
$V_1 = \frac{1}{\sqrt{3}} v_1$
$V_2 = \frac{1}{\sqrt{6}} v_2$
$U = [v_1 \quad v_2]$
$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

12. (1 point) (Multiple select) Which of the following matrices is/are unitary?

A. $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

B. $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

C. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

D. $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

E. $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Answer: E
Check $UU^* = I$

13. (1 point) Let U and V be two symmetric matrices. Consider the following statements:

1. UV is symmetric.
2. $U + V$ is symmetric.

Then,

A. both statements are true.

B. both statements are false.

C. 1. is false.

D. 2. is false.

Answer: C
Product of two symmetric matrices may not be symmetric.

14. (1 point) The singular values of a matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ are

    A. 1, 5

    B. 3, 4

    C. 2, 5

    D. 1, 3

Answer: D
$A^T A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$
$|A^T A - \lambda I| = 0$
$(9 - \lambda)(1 - \lambda) = 0$
$\lambda = 1, 9$
$\sigma = \sqrt{\lambda}$
$\sigma = 1, 3$

15. (1 point) The correct SVD of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is

    A. $A = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

    B. $A = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{6}} & \frac{1}{24} \end{bmatrix}$

    C. $A = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

D. $A = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Answer: D

16. (1 point) Find the singular values for matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

A. 1.618, -0.618

B. 1.618, 0.618

C. 2.618, 0.382

D. 2.618, -0.382

Answer: B

$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$|A^T A - \lambda I| = 0$

$\lambda^2 - 3\lambda + 1 = 0$

$\lambda = 0.382, 2.618$

$\sigma = \sqrt{\lambda}$

$\sigma = 0.618, 1.618$

17. (1 point) The singular value decomposition of matrix $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ is

A. $\begin{bmatrix} 0.645 & -0.53 \\ 0.826 & 0.414 \end{bmatrix} \begin{bmatrix} 2.56 & 0 \\ 0 & 1.56 \end{bmatrix} \begin{bmatrix} 0.826 & -0.644 \\ 0.644 & 0.826 \end{bmatrix}^T$

B. $\begin{bmatrix} 0.645 & -0.53 \\ -0.826 & 0.414 \end{bmatrix} \begin{bmatrix} 2.56 & 0 \\ 0 & 1.56 \end{bmatrix} \begin{bmatrix} 0.826 & -0.644 \\ 0.644 & 0.826 \end{bmatrix}^T$

C. $\begin{bmatrix} 0.645 & -0.53 \\ 0.826 & 0.414 \end{bmatrix} \begin{bmatrix} -2.56 & 0 \\ 0 & 1.56 \end{bmatrix} \begin{bmatrix} 0.826 & -0.644 \\ 0.644 & 0.826 \end{bmatrix}^T$

D. $\begin{bmatrix} -0.645 & -0.53 \\ 0.826 & 0.414 \end{bmatrix} \begin{bmatrix} 2.56 & 0 \\ 0 & 1.56 \end{bmatrix} \begin{bmatrix} 0.826 & -0.644 \\ 0.644 & 0.826 \end{bmatrix}^T$

Answer: A