Linear Programming

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Programming, Data Structures and Algorithms using Python

Week 11

Optimization problems

- Many computational tasks involve optimization
  - Shortest path
  - Minimum cost spanning tree
  - Longest common subsequence

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Linear programming

- Constraints and objective to be optimized are linear functions
  - Constraints: $a_1x_1 + a_2x_2 + \cdots + a_mx_m \leq K$, $b_1x_1 + b_2x_2 + \cdots + b_mx_m \geq L$, ...
  - Objective: $c_1x_1 + c_2x_2 + \cdots + c_mx_m$

Example: Maximize profits

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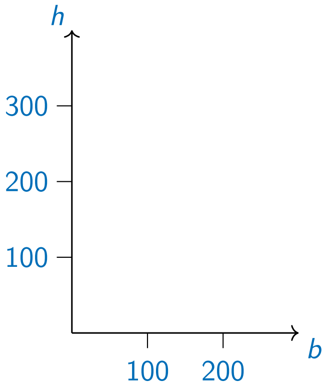

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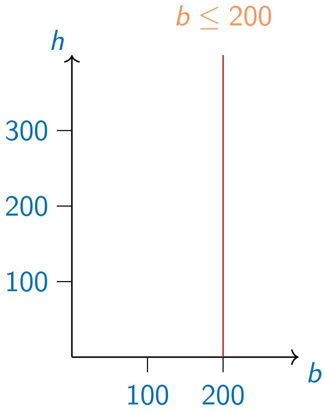

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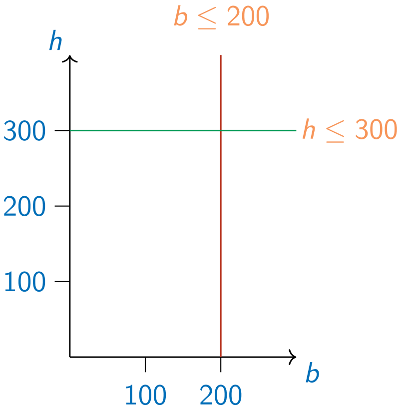

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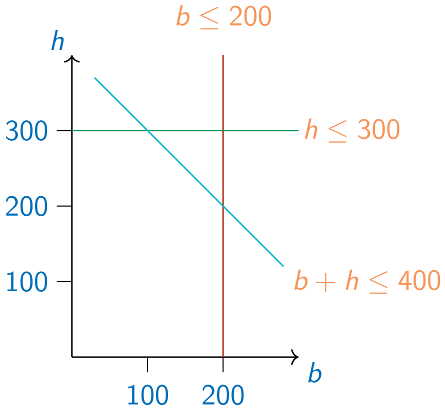

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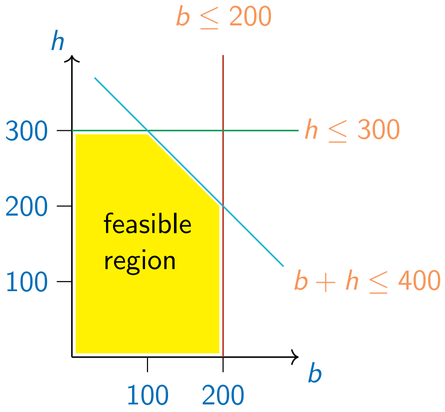

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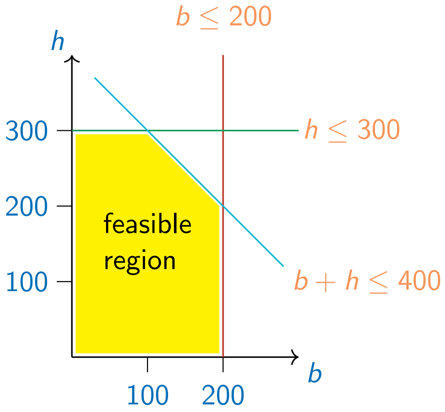

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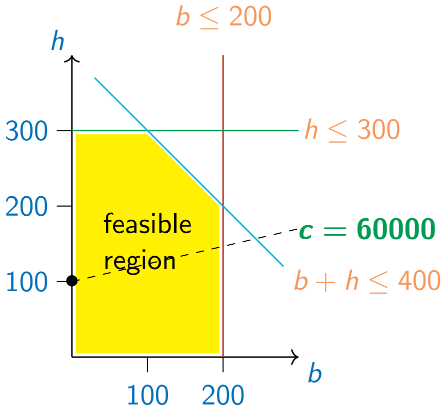

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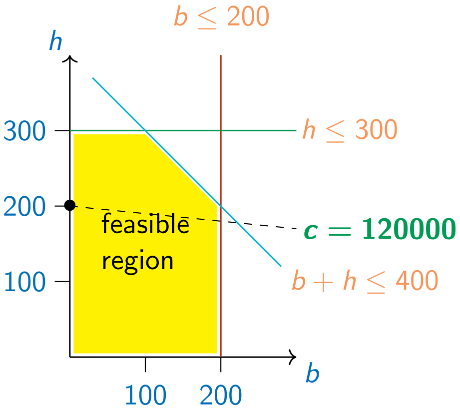

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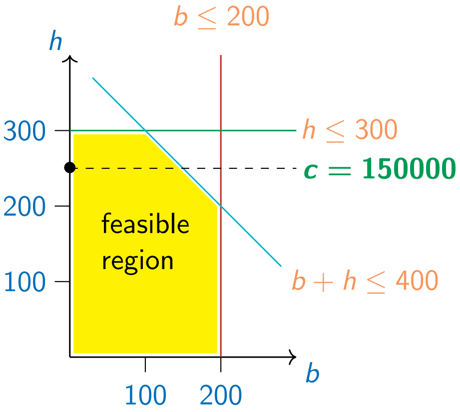

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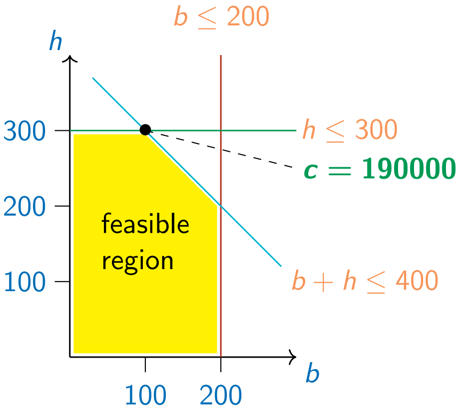

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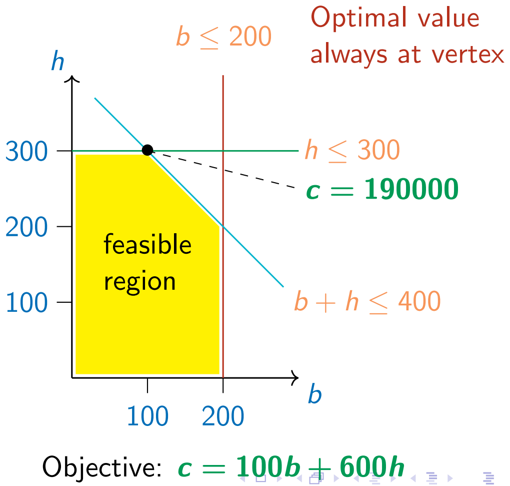

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Simplex algorithm

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Existence of solutions

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- May be empty — constraints are unsatisfiable, no solutions
- May be unbounded — no upper/lower limit on objective

Example, extended

Grandiose Sweets adds almond rasmalai

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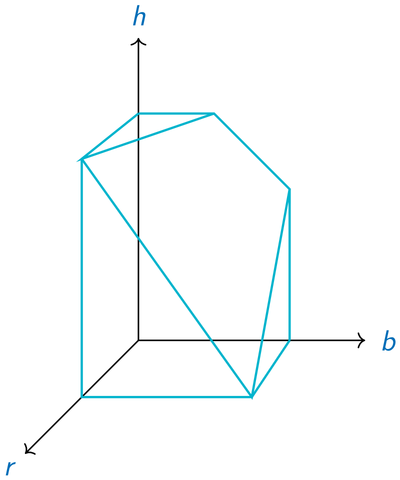

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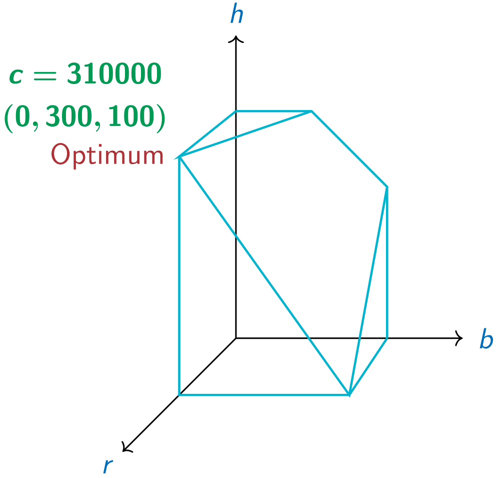
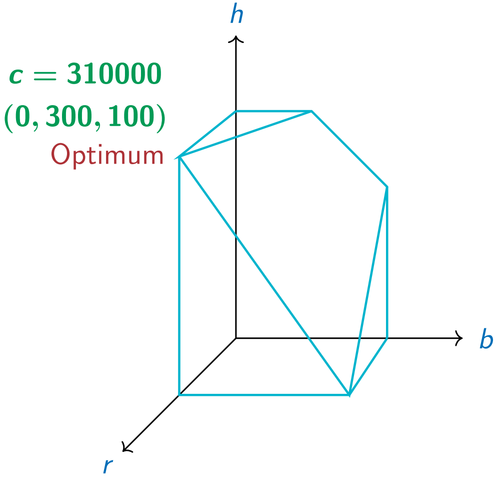


Example, extended

- Why is $(0, 300, 100)$ optimal?



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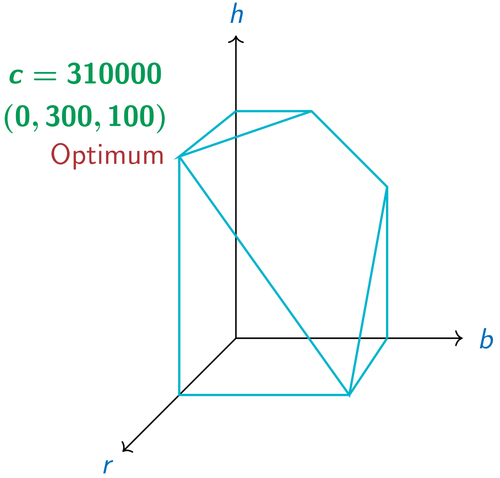
- Profit is $100b + 600h + 1300r$



Example, extended

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

  (A) $h \leq 300$

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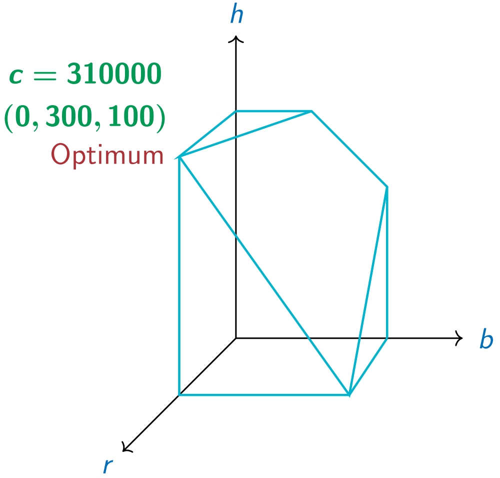

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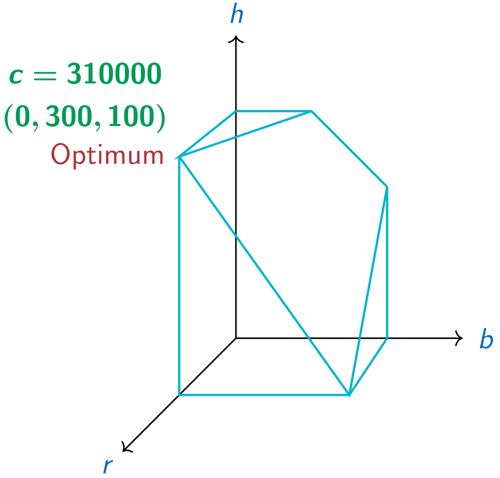
- Combine as
  $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$



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- Combine as
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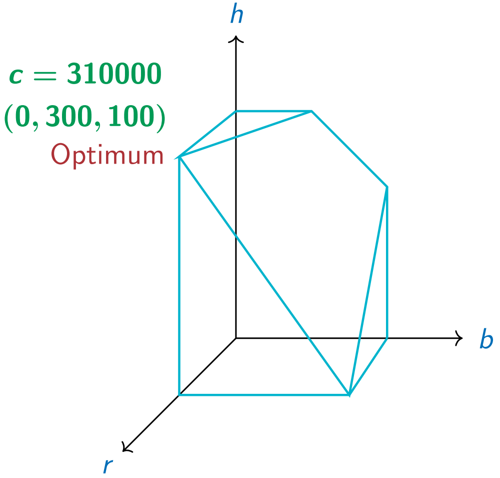
- Result is
  $100b + 600h + 1300r \leq 310000$



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- Consider the following constraints
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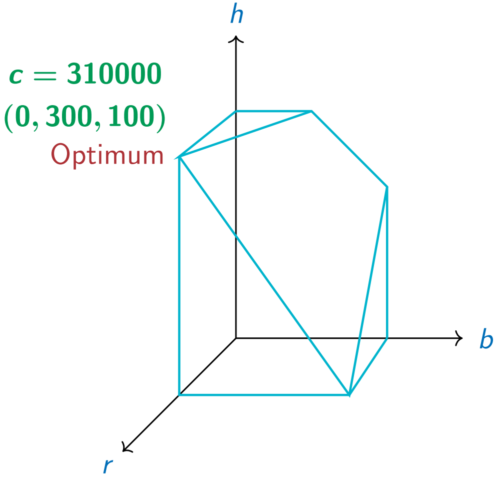
- Combine as
  $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is
  $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit



LP Duality

- We derived an upper bound on the objective through a linear combination of constraints

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

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- Combine as $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit

LP Duality

- We derived an upper bound on the objective through a linear combination of constraints
- This is always possible!

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

  (A) $h \leq 300$

  (B) $b + h + r \leq 400$

  (C) $h + 3r \leq 600$
- Combine as $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit

LP Duality

- We derived an upper bound on the objective through a linear combination of constraints
- This is always possible!
- Dual LP problem
  - Minimize linear combination of constraints
  - Variables are multipliers for the linear combination
  - Implicit constraint: multipliers are non-negative
  - Optimum solution solves both the original (primal) and the dual LP

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

  (A) $h \leq 300$

  (B) $b + h + r \leq 400$

  (C) $h + 3r \leq 600$
- Combine as $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit