Q1) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

Characteristic polynomial :

$$\lambda^2 - 10\lambda + 9 = 0$$

Upon solving : $\lambda = 9, 1$

$$\therefore \sigma_1 = \sqrt{9} = 3$$

$$\sigma_2 = \sqrt{1} = 1$$

Option (D) = 1, 3

Q2) $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3\times3}$ $\qquad A = \underset{3\times3}{Q_1} \underset{3\times3}{\Sigma} \underset{3\times3}{Q_2^T}$

$$A^TA = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

Characteristic polynomial for A^TA :

$$\lambda^3 - [\text{trace}(A^TA)]\lambda^2 + \left(\begin{array}{c}\Sigma \text{ Minors of} \\ \text{diagonal elements} \\ \text{of } A^TA\end{array}\right)\lambda - |A^TA| = 0$$

$$\lambda^3 - 10\lambda^2 + \{8+4+4\}\lambda - \{2(8) - 2\sqrt{2}(4\sqrt{2})\} = 0$$

$$= \lambda^3 - 10\lambda^2 + 16\lambda = 0$$

Upon solving : $\boxed{\lambda = 0, 8, 2}$

for $\lambda = 8$

$[A^TA - 8I]X = 0$

$$\left\{\begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & -2 & 2 \\ 0 & 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-6x + 2\sqrt{2}y = 0$

$2\sqrt{2}x - 2y + 2z = 0$

$2y - 6z = 0$

let $z = t$

$2y = 6z$

$\boxed{y = 3t}$

$-6x + 2\sqrt{2}(3t) = 0$

$-\cancel{6}x = -\cancel{6}\sqrt{2}t$

$\boxed{x = \sqrt{2}t}$

$$\therefore V_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} \sqrt{2} \\ 3 \\ 1 \end{bmatrix} \xRightarrow[\text{Normalizing}]{\text{Upon}} t\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix} = 1\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2\sqrt{3}} \end{bmatrix}$$

(for $t = 1$)

for $\lambda = 2$, $[A^TA - 2I]X = 0$

$$\begin{bmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2\sqrt{2}y = 0$

$\boxed{y = 0}$

$2\sqrt{2}x + 4y + 2z = 0$

let $z = k \Rightarrow x = \frac{-k}{\sqrt{2}}$

$$V_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1 \end{bmatrix} \xrightarrow[\text{Normalizing}]{\text{Upon}} k \begin{bmatrix} -1/\sqrt{3} \\ 0 \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

for $k = -1$

$$\boxed{V_2 = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -2/\sqrt{6} \end{bmatrix}}$$

for $\lambda = 0 \Rightarrow \boxed{A^TA} X = 0$

$$\begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + 2\sqrt{2}y = 0$

$2\sqrt{2}x + 6y + 2z = 0$

$2y + 2z = 0$

let $z = l$

$2y = -2l \Rightarrow y = -l$

$2x + 2\sqrt{2}(-l) = 0$

$\Rightarrow x = \sqrt{2}l$

$$\therefore V_3 = l \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix} \xrightarrow[\text{Normalizing}]{\text{Upon}} l \begin{bmatrix} 1/\sqrt{2} \\ -1/2 \\ 1/2 \end{bmatrix}$$

for $l = 1$

$$V_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

$$\left\{ \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{3}{\sqrt{12}} \right\}$$

$$\left\{ \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{4\times 3}} = \frac{1}{\sqrt{12}} \right\}$$

$$\phi_2^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{+1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$y_1 = \frac{A x_1}{\sigma_1}$$

$$= \frac{1}{2\sqrt{2}} \left\{ \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{3}/2 \\ 1/\sqrt{12} \end{bmatrix} \right\} = \frac{1}{\sqrt{2}} \begin{bmatrix} 4/\sqrt{2} \\ \frac{4\sqrt{3}}{3} \\ \frac{4}{\sqrt{12}} \end{bmatrix}$$

$$\boxed{y_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}}$$

$$y_2 = \frac{A x_2}{\sigma_2}$$

$$= \frac{1}{\sqrt{2}} \left\{ \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} \\ 0 \\ 2/\sqrt{6} \end{bmatrix} \right\} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 2/\sqrt{6} \end{bmatrix}$$

$$\boxed{y_2 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}}$$

Since $\sigma_3 = 0$, $\frac{1}{\sigma_3}$ will be indeterminate

let $u_3 = [a \quad b \quad c]$ such that $\langle u_1 \cdot u_3 \rangle = \langle u_2 \cdot u_3 \rangle = 0$

$$[a \quad b \quad c] \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = 0$$

$$\frac{a}{\sqrt{6}} + \frac{2b}{\sqrt{6}} + \frac{c}{\sqrt{6}} = 0$$

$$\boxed{a + 2b + c = 0} \quad — 1$$

$$[a \quad b \quad c] \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = 0$$

$$\Rightarrow \boxed{a - b + c = 0} \quad — 2$$

let $b = k_1$, $c = k_2$

In 1 $\quad a + 2k_1 + k_2 = 0$

$$a = -2k_1 - k_2 \quad — 3$$

In 2 $\quad a = b - c = k_1 - k_2 \quad — 4$

Equating 3 & 4

$$-2k_1 - \cancel{k_2} = k_1 - \cancel{k_2}$$

$$3k_1 = 0$$

$$\therefore \boxed{k_1 = 0}$$

$$\boxed{a = -k_2}$$

$$\boxed{b = 0}$$

$$\boxed{c = k_2}$$

$$y_3 = k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[\text{upon Normalizing}]{} k_2 \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

for $k_2 = -1$: $y_3 = \begin{bmatrix} +1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$$\therefore Q_1 = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

Q3.) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Characteristic polynomial of A^TA:

$$\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3+\sqrt{5}}{2} = 2.618 \; ; \; \frac{3-\sqrt{5}}{2} = 0.382$$

{ Upon solving using quadratic equation
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ }

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{2.618} = \underline{\underline{1.618}}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{0.382} = \underline{\underline{0.618}}$$

Option (B)

Q4.) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$; $A^T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\lambda^2 - 9\lambda + 16 = 0$$

$$\lambda = \frac{9 \pm \sqrt{17}}{2}$$

$\sigma_1 = 2.56$, $\sigma_2 = 1.56$.

$$\Sigma = \begin{bmatrix} 2.56 & 0 \\ 0 & 1.56 \end{bmatrix}$$

for $\lambda = \frac{9+\sqrt{17}}{2} = 6.56155$:

$$\left[A^TA - 6.56155\,I\right]X = 0$$

$$\begin{bmatrix} -1.56 & 2 \\ 2 & -2.56 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-1.56x + 2y = 0 \quad (\times -2.56)$

$2x - 2.56y = 0 \quad (\times 2)$

$\Downarrow$

$4x - 2(2.56)y = 0$

$4x = 2(2.56)y$

let $y = k$

$\underline{x = 1.28k}$

$\therefore v_1 = k\begin{bmatrix} 1.28 \\ 1 \end{bmatrix} \xrightarrow[\text{Normalizing}]{\text{Upon}} \begin{bmatrix} 0.788 \\ 0.615 \end{bmatrix}$

for $\lambda = \frac{9-\sqrt{17}}{2}$, $\left[A^TA - \left(\frac{9-\sqrt{17}}{2}\right)I\right]X = 0$

$$\begin{bmatrix} 2.56 & 2 \\ 2 & 1.56 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2.56x + 2y = 0$

$2x + 1.56y = 0$.

for $y = k$

$$2.56x = -2k$$

$$\underline{\underline{x = -0.78k}}$$

$$V_2 = k\begin{bmatrix} -0.78 \\ 1 \end{bmatrix} \xrightarrow[\text{Normalizing}]{\text{Upon}} \begin{bmatrix} -0.615 \\ 0.788 \end{bmatrix}$$

$$Q_2 = \begin{array}{cc} \begin{bmatrix} 0.788 & -0.615 \\ 0.615 & 0.788 \end{bmatrix} \\ \quad x_1 \qquad\qquad x_2 \end{array}$$

$$y_1 = \frac{Ax_1}{\sigma_1} = \frac{1}{2.56}\left\{\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0.788 \\ 0.615 \end{bmatrix}\right\} = \frac{1}{2.56}\begin{bmatrix} 1.576 \\ 2.018 \end{bmatrix}$$

$$\boxed{y_1 = \begin{bmatrix} 0.615 \\ 0.788 \end{bmatrix}}$$

$$y_2 = \frac{Ax_2}{\sigma_2} = \frac{1}{1.56}\left\{\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -0.615 \\ 0.788 \end{bmatrix}\right\} = \frac{1}{1.56}\begin{bmatrix} -1.23 \\ 0.961 \end{bmatrix}$$

$$\boxed{y_2 = \begin{bmatrix} -0.788 \\ 0.615 \end{bmatrix}}$$

$$\boxed{A = \begin{bmatrix} 0.615 & -0.788 \\ 0.788 & 0.615 \end{bmatrix}\begin{bmatrix} 2.56 & 0 \\ 0 & 1.56 \end{bmatrix}\begin{bmatrix} 0.788 & -0.615 \\ 0.615 & 0.788 \end{bmatrix}^T}$$

Q4) $f(x,y) = 2xy + y^2$

$f_x = 2y = 0$

$\boxed{y=0}$ $\longrightarrow$ Substituting $y=0$.

$f_y = 2x + \cancel{2y}^{0} = 0$

$2x = 0$

$\boxed{x=0}$.

$\therefore$ Stationary point $= (0,0)$

Option (B)

Q5) $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

Eigenvalue test :

$\lambda_1, \lambda_2, \lambda_3 > 0$ = Postive definite

$\lambda_1, \lambda_2, \lambda_3 \geq 0$ & atleast one $\lambda = 0$ = Positive Semidefinite

$\lambda_1, \lambda_2, \lambda_3 < 0$ = Negative definite

$\lambda_1, \lambda_2, \lambda_3 \leq 0$ & atleast one $\lambda = 0$ = Negative Semidefinite

Some λ +ve, other λ values -ve = Indefinite

Characteristic polynomial for $A_{3\times3}$:

$\lambda^3 - 8\lambda^2 + \{3+7+7\}\lambda - 6 = 0$

$\lambda^3 - 8\lambda^2 + 17\lambda - 6 = 0$

$\left.\begin{matrix} \lambda_1 = 0.438, \\ \lambda_2 = 4.56, \\ \lambda_3 = 3 \end{matrix}\right\} > 0$ $\therefore$ A is positive definite

Option (A)

Q6) $A = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a = 6 > 0$

$ac - b^2 = 6(1) - 2(2) = 6 - 4 = 2 > 0$

Since $a > 0$ & $ac - b^2 > 0$

$\therefore$ A is positive definite Option (A)

Q7.) $f(x,y) = 4 + x^3 + y^3 - 3xy$

$$f_x = 3x^2 - 3y = 0$$
$$f_y = 3y^2 - 3x = 0$$
$$x^2 = y \;\&\; y^2 = x$$
$$\therefore x = y = 1.$$

Hence ; function $f(x,y)$ has stationary point at $(1,1)$

<u>Option(A)</u>

Q8.) Coefficients of:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x^2 & \frac{xy}{2} & \frac{xz}{2} \\ \frac{xy}{2} & y^2 & \frac{yz}{2} \\ \frac{zx}{2} & \frac{zy}{2} & z^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Sym. Sym. Sym.

Compare with : $x^2 - z^2 + 2yz + 2zx \; (+ 0y^2 + 0xy)$

Initially :

Coefficient of $x^2 = 1$

Coefficient of $y^2 = 0$

Coefficient of $z^2 = -1$

$$\begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix}$$

Only present in <u>Option (D)</u>.

Q9.) $f(x,y) = -3x^2 - 6xy - 6y^2$

to check at point $(0,0)$

$f_x = -6x - 6y = -6(0) - 6(0) = 0$

$f_y = -6x - 12y = -6(0) - 12(0) = 0$

Second-derivative test :

$$\left.\begin{aligned} f_{xx} &= -6 \\ f_{yy} &= -12 \end{aligned}\right\} < 0$$

$\therefore$ Point $(0,0)$ is MAXIMA

Option (A)

Q10.) $A = \begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a = 6 > 0$

$ac - b^2 = (6)(4) - (5)(5) = 24 - 25 = -1 < 0$

Since $ac - b^2 < 0$, Hence it is INDEFINITE

(i.e., Neither positive definite ~~none~~ positive semi-definite)

Q11.) $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -7 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$(-6-\lambda)(-5-\lambda)(-7-\lambda) = 0$

Eigenvalues are : $-6, -5, -7 < 0$

Since $\lambda_1, \lambda_2', \lambda_2$ all are < 0

$\therefore$ A is Negative definite.

Q12.) We know that:

Trace = Sum of eigenvalues $(\lambda_1+\lambda_2)$

Determinant = Product of Eigenvalues $(\lambda_1 * \lambda_2)$

Given: $\lambda_1+\lambda_2 = 6$

$\lambda_1 * \lambda_2 = 8$

$$(\lambda_1-\lambda_2)^2 = (\lambda_1+\lambda_2)^2 - 4(\lambda_1 * \lambda_2)$$

$$(\lambda_1-\lambda_2)^2 = (6)^2 - 4(8)$$

$$= 36 - 32$$

$$(\lambda_1-\lambda_2)^2 = 4$$

$$\therefore \lambda_1-\lambda_2 = 2, -2$$

$$\begin{array}{r} \lambda_1 + \cancel{\lambda_2} = 6 \\ (+)\ \lambda_1 - \cancel{\lambda_2} = 2 \\ \hline 2\lambda_1 = 8 \\ \boxed{\lambda_1 = 4} \end{array} \quad \text{(OR)} \quad \begin{array}{r} \lambda_1 + \cancel{\lambda_2} = 6 \\ (+)\ \lambda_1 - \cancel{\lambda_2} = -2 \\ \hline 2\lambda_1 = 4 \\ \boxed{\lambda_1 = 2} \end{array}$$

$$\begin{array}{l} \lambda_1 + \lambda_2 = 6 \\ 4 + \lambda_2 = 6 \\ \therefore \boxed{\lambda_2 = 2} \end{array} \qquad \begin{array}{l} \lambda_1 + \lambda_2 = 6 \\ +2 + \lambda_2 = 6 \\ \boxed{\lambda_2 = 4} \end{array}$$

$\therefore$ The Eigenvalues are $4, 2 > 0$.

Hence, A is Positive definite.