Answer: A

Suppose, the eigen values of the matrix are λ_1, λ_2.

Trace $= \lambda_1 + \lambda_2 = 6$, Determinant $= \lambda_1 * \lambda_2 = 8$ This indicates both λ_1, λ_2 are positive values. Therefore, the matrix is a positive definite matrix.

Questions 10-15 are based on common data

Consider the data points x_1, x_2, x_3 to answer the following questions.

$x_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

10. (1 point) The mean vector of the data points x_1, x_2, x_3 is

    A. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

    B. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

    C. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

    D. $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Answer: B

Mean vector $= \bar{X} = \Sigma_{i=1}^{n} \frac{1}{n} x_i = \frac{1}{3} \begin{bmatrix} (0+1+2) \\ (2+1+0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

11. (2 points) The covariance matrix $C = \frac{1}{n} \Sigma_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T$ for the data points x_1, x_2, x_3 is

    A. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

    B. $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

    C. $\begin{bmatrix} 0.67 & -0.67 \\ -0.67 & 0.67 \end{bmatrix}$

    D. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answer: C

$C = \frac{1}{3}(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}) = \frac{1}{3} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

12. (2 points) The eigenvalues of the covariance matrix $C = \frac{1}{n} \Sigma_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T$ are

A. 0.5, 0.5
B. 1, 1
C. 4, 0
D. 0, 0

Answer: C

Characteristics equation:

$\begin{bmatrix} \frac{2}{3}-\lambda & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda \end{bmatrix} I = 0$

The determinant of the obtained matrix is $\lambda(\lambda-4)=0$

Eigenvalues:

The roots are $\lambda_1 = \frac{4}{3}, \lambda_2 = 0$

Eigenvectors:

$\lambda_1 = 4, \begin{bmatrix} \frac{2}{3}-\lambda & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} \end{bmatrix}$

The null space of this matrix is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, Corresponding eigenvector is, $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0, \begin{bmatrix} \frac{2}{3}-\lambda & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$

The null space of this matrix is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, Corresponding eigenvector is, $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

13. (2 points) The eigenvectors of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are (Note: The eigenvectors should be arranged in the descending order of eigenvalues from left to right in the matrix.)

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

D. $\begin{bmatrix} -0.7 & 0.7 \\ 0.7 & 0.7 \end{bmatrix}$

Answer: D

Refer the solution of the previous question.

14. (2 points) The data points x_1, x_2, x_3 are projected onto the one dimensional space using PCA as points z_1, z_2, z_3 respectively.

A. $z_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $z_2 = \begin{bmatrix}1\\1\end{bmatrix}$, $z_3 = \begin{bmatrix}1\\1\end{bmatrix}$

B. $z_1 = \begin{bmatrix}0.5\\0.5\end{bmatrix}$, $z_2 = \begin{bmatrix}0\\0\end{bmatrix}$, $z_3 = \begin{bmatrix}-0.5\\-0.5\end{bmatrix}$

C. $z_1 = \begin{bmatrix}0\\2\end{bmatrix}$, $z_2 = \begin{bmatrix}1\\1\end{bmatrix}$, $z_3 = \begin{bmatrix}2\\0\end{bmatrix}$

D. $z_1 = \begin{bmatrix}-1\\1\end{bmatrix}$, $z_2 = \begin{bmatrix}0\\0\end{bmatrix}$, $z_3 = \begin{bmatrix}1\\-1\end{bmatrix}$

Answer: D

$\lambda_1 = 4$, $u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$

$z_1 = \frac{1}{\sqrt{2}}(\begin{bmatrix}0 & 2\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix})\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$z_2 = \frac{1}{\sqrt{2}}(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix})\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$z_3 = \frac{1}{\sqrt{2}}(\begin{bmatrix}2 & 0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix})\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

15. (1 point) The approximation error J is given by $\Sigma_{i=1}^{n}||x_i - z_i||^2$. What could be the possible value of the reconstruction error?

A. 1

B. 2

C. 10

D. 20

Answer: B

Reconstruction error, $J = \frac{1}{n}\Sigma_{i=1}^{n}||x_i - z_i||^2 = \frac{1}{3}[(1^2+1^2)+(1^2+1^2)+(1^2+1^2)] = 2$