Course: Machine Learning - Foundations
Practice Questions - Solution
Lecture Details: Week [redacted]

1. (1 point) Given if $x_1, x_2, \frac{x_1+x_2}{2} \in S$ then $\frac{3}{4}x_1 + \frac{1}{4}x_2 \in S$. Is this a true statement?
   - A. Yes
   - B. No

Answer: A

$x_1, x_2, \frac{x_1+x_2}{2} \in S$ then $\frac{(x_1+\frac{x_1+x_2}{2})}{2} \in S \implies \frac{3}{4}x_1 + \frac{1}{4}x_2 \in S$

It is clear that if the set is midpoint convex, then the set is a convex set.

By definition, For a convex set S, if $x_1, x_2 \in S \implies \lambda x_1 + (1-\lambda)x_2 \in S, \lambda \in [0,1]$

2. (1 point) Which of the following is a convex function?
   - A. $f(x) = ax + b$ over $\mathbb{R}$ where $a, b \in \mathbb{R}$
   - B. $f(x) = e^{ax}$ over $\mathbb{R}$ where $a \in \mathbb{R}$
   - C. $f(x) = x^2$ over $\mathbb{R}$
   - D. $f(x) = x^3$ over $\mathbb{R}$

Answer: A, B, C

1. $f(x) = ax + b$ over $\mathbb{R}$ where $a, b \in \mathbb{R}$

$\frac{\partial f(x)}{\partial x} = a$, $\frac{\partial^2 f(x)}{\partial x^2} = 0$,

The second order partial derivative is non-negative. Hence, the function is a convex function.

2. $f(x) = e^{ax}$ over $\mathbb{R}$ where $a \in \mathbb{R}$

$\frac{\partial f(x)}{\partial x} = ae^{ax}$, $\frac{\partial^2 f(x)}{\partial x^2} = a^2e^{ax} \geq 0 \ \forall \ x \in \mathbb{R}$,

The second order partial derivative is non-negative (positive curvature). Hence, the function is a convex function.

3. $f(x) = x^2$ over $\mathbb{R}$

$\frac{\partial f(x)}{\partial x} = 2x$, $\frac{\partial^2 f(x)}{\partial x^2} = 2 \geq 0$,

The second order partial derivative is non-negative (positive curvature). Hence, the function is a convex function.

4. $f(x) = x^3$ over $\mathbb{R}$

$\frac{\partial f(x)}{\partial x} = 3x^2$, $\frac{\partial^2 f(x)}{\partial x^2} = 6x$,

The second order partial derivative depends on the x and can be negatie or positive. Hence, the function is a non-convex function in nature.

3. (1 point) What is the value of a, the function $f : \mathbb{R} \to \mathbb{R}, f(x, y) = ax^4 + 8y$ is a convex function

   A. $a > 0$

   B. $a < 1$

   C. $a \geq 1$

   D. None of these

Answer: A

$f : \mathbb{R} \to \mathbb{R}, f(x, y) = ax^4 + 8y$

$f_x = \frac{\partial f(x,y)}{\partial x} = 4ax^3$, $f_{xx} = \frac{\partial^2 f(x,y)}{\partial x^2} = 12ax^2$,

$f_y = \frac{\partial f(x,y)}{\partial y} = 8$, $f_{yy} = \frac{\partial^2 f(x,y)}{\partial y^2} = 0$,

$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = 0$,

The hessian matrix, $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12ax^2 & 0 \\ 0 & 0 \end{bmatrix}$

The determinant of the hessian matrix, $D = f_{xx}f_{yy} - {f_{xy}}^2 = 0$

For the function to be a convex function, the second order partial derivative with respect to x should be positive, in other words $f_{xx} > 0$

For this to be true, $a > 0$

4. (1 point) Which of the following hessian matrix corresponds to the convex function?

   A. $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

   B. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

   C. $\begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}$

   D. $\begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix}$

Answer: B

The hessian matrix is denoted as, $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$

A function $f(x, y)$ is convex when $f_{xx} > 0$ and $D = f_{xx}f_{yy} - {f_{xy}}^2 \geq 0$

5. (1 point) Function $f : \mathbb{R}^d \to \mathbb{R}, f(x) = x^T Ax$ is a convex function if

   A. A is positive definite matrix

   B. A is positive semi-definite matrix

C. A is negative definite matrix

D. A is negative semi-definite matrix

Answer: A, B

The hessian matrix is denoted as, $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$

A function $f(x, y)$ is positive semi-definite when $f_{xx} > 0$ and $D = f_{xx}f_{yy} - {f_{xy}}^2 \geq 0$

A function $f(x, y)$ is positive definite when $f_{xx} > 0$ and $D = f_{xx}f_{yy} - {f_{xy}}^2 > 0$

In both cases, the function fulfills the criteria of convexity.

6. (1 point) A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}^n$

A. Hessian matrix is positive definite

B. Hessian matrix is positive semi-definite

C. Hessian matrix is negative definite

D. Hessian matrix is negative semi-definite

Answer: A, B

Please refer to the previous solution.

7. (1 point) Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the linear approximation of a function f at the point $(x + \epsilon d)$ is:

A. $f(x) + \epsilon\ d^T \nabla f(x)$

B. $f(x) + \epsilon\ \nabla f(x)$

C. $f(x) + d^T \nabla f(x)$

D. None of these

Answer: A

Please refer to the lecture videos.

8. (1 point) (multiple select) A function in one variable is said to be convex function if it has:

A. Positive curvature

B. Negative curvature

C. Non-positive curvature

D. Non-negative curvature

Answer: A, D

Since the function is convex, its second order partial derivative will be non-negative. Hence, both (A) and (D) are true answer.

9. (1 point) What is the relationship between eigenvalues of the hessian matrix of twice differentiable convex function?
   - A. All eigenvalues are non-negative
   - B. Eigenvalues are both positive and negative
   - C. All eigenvalues are non-positive
   - D. There is no relationship

Answer: A

For a positive definite function, the eigen values are always positive. For a positive semi-definite function, the eigen values are always non-negative.

For a convex function, the determinant of the hessian matrix is non-negative and positive semi-definite (or definite).

Therefore, in case of convex function option (A) is the true answer.

10. (1 point) A batch of cookies requires 4 cups of flour, and a cake requires 7 cups of flour. What would be the constraint limiting the amount of cookies(a) and cakes(b) that can be made with 50 cups of flour.
   - A. $4a + 7b \leq 50$
   - B. $7a + 4b \leq 50$
   - C. $11(a + b) \leq 50$
   - D. $4a.7b \leq 50$

Answer: A

Since we need min 4 cup for cookies $(a) \Rightarrow 4a$.

We need min 7 cup for cake $(b) \Rightarrow 7b$

and Max amt of flour available is$\Rightarrow$ 50 cup. Hence equation becomes $4a + 7b \leq 50$

11. (1 point) If objective function which is to be minimised is $f(x, y, z) = x + z$ and the constrained equation is $g(x, y, z) = x^2 + y^2 + z^2 = 1$. The point where minimum value occurs will be
   - A. $(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$
   - B. $(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$
   - C. $(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$

D. $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

Answer: C

Given

$f(x, y, z) = x + z$

$g(x, y, z) = x^2 + y^2 + z^2 = 1$

To get critical point we need to solve $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$:

using above we get following equation:

(i) diff w.r.t $x \Rightarrow 1 = 2\lambda x$

(ii) diff w.r.t $y \Rightarrow 0 = 2\lambda y$

(iii) diff w.r.t $z \Rightarrow 1 = 2\lambda z$

using above we get: $x = z = \frac{1}{2\lambda}$ and $y = 0$

Substituting above in $x^2 + y^2 + z^2 = 1$ we get critical point as

$(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

Here $f(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}) \leq f(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

and since constrained equation shows a sphere,so:

$f(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$ is constrained minimum point.

and $f(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ is constrained maximum point.

12. (1 point) If objective function which is to be maximized is $f(x, y, z) = x + z$ and the constrained equation is $g(x, y, z) = x^2 + y^2 + z^2 = 1$. The point where maximum value occurs will be

    A. $(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$

    B. $(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

    C. $(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$

    D. $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

Answer: D

Refer Previous solution

13. (1 point) Find the points on the surface $y^2 = 1 + xz$ that are closest to the origin.

    A. $(0, -1, 0)$

    B. $(1, 1, 1)$

    C. $(0, 0, 0)$

    D. $(0, 2, 0)$

E. $(1, 2, 0)$

Answer: A

To get the closest point on the surface from a point we can create distance function and try to minimise them.

Since, coordinate of origin = $(0, 0, 0)$

Our objective function will be distance between them, so:

$d = \sqrt{(x-0)^2 + (y^2 - 0) + (z-0)^2}$

Hence,

objective function= $f(x, y, z) = x^2 + y^2 + z^2$

constrained equation $g(x, y, z) = y^2 - 1 - xz = 0$

To get critical point we need to solve $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$:

using above we get following equation:

(i) diff w.r.t $x \Rightarrow 2x = -\lambda z$

(ii) diff w.r.t $y \Rightarrow 2y = 2\lambda y \rightarrow \lambda = 1$

(iii) diff w.r.t $z \Rightarrow 2z = -\lambda x$

using above we get: $x = z = 0$

and By putting $x = z = 0$ in $y^2 - 1 - xz = 0$ we get $y = 1, -1$

so Point closet to origin are $(0, 1, 0)$ and $(0, -1, 0)$

14. (1 point) The minimum value of the function $f(x, y) = xy^2$ on the circle $x^2 + y^2 = 1$ is (correct upto two decimal places) ______.

Answer: 0.39, Range 0.00 to 0.50

Given $f(x, y) = xy^2$, $g(x, y) = x^2 + y^2 = 1$

$\nabla f(x, y) = \begin{bmatrix} y^2 \\ 2xy \end{bmatrix}$, $\nabla g(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

We find the values of $x, y\lambda$ that simultaneously satisfy the equations to get the extreme points

$\nabla f(x, y) = \lambda \nabla g(x, y)$ and, $g(x, y) = x^2 + y^2 = 1$

Solving, $\nabla f(x, y) = \lambda \nabla g(x, y)$

$$\implies \begin{bmatrix} y^2 \\ 2xy \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \implies x = \lambda, 0, \; y = \sqrt{2}\lambda, 0$$

The point (0,0) does not lie on the circle, $g(x, y) = x^2 + y^2 = 1$.

Solving, $g(x, y) = x^2 + y^2 = 1 \implies \lambda^2 + 2\lambda^2 = 1 \implies \lambda = \pm\sqrt{3}/3$

Therefore, the extreme point coordinates will be,

$(x_1, y_1) = (\lambda, \sqrt{2}\lambda) = (\sqrt{3}/3, \sqrt{6}/3),\ f(\sqrt{3}/3, \sqrt{6}/3) = 0.39,$
$(x_2, y_2) = (-\lambda, \sqrt{2}\lambda) = (-\sqrt{3}/3, \sqrt{6}/3),\ f(\sqrt{3}/3, \sqrt{6}/3) = -0.39,$
$(x_3, y_3) = (\lambda, -\sqrt{2}\lambda) = (\sqrt{3}/3, -\sqrt{6}/3),\ f(-\sqrt{3}/3, -\sqrt{6}/3) = 0.39$
$(x_2, y_2) = (\lambda, \sqrt{2}\lambda) = (-\sqrt{3}/3, -\sqrt{6}/3),\ f(-\sqrt{3}/3, -\sqrt{6}/3) = -0.39$
We can see the function f(x,y) has a minimum at the points $(-\sqrt{3}/3, \sqrt{6}/3), (-\sqrt{3}/3, -\sqrt{6}/3)$.

15. (1 point) (multiple select) The minimum value of the function $f(x, y) = xy^2$ on the circle $x^2 + y^2 = 1$ occurs at the below points:

    A. $(\sqrt{3}/3, \sqrt{6}/3)$
    B. $(-\sqrt{3}/3, \sqrt{6}/3)$
    C. $(\sqrt{3}/3, -\sqrt{6}/3)$
    D. $(-\sqrt{3}/3, -\sqrt{6}/3)$

Answer: C, D

Refer to the solution of the previous question