Shortest Paths in Weighted Graphs

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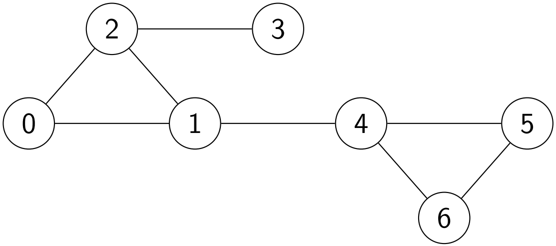
Programming, Data Structures and Algorithms using Python

Week 5

Weighted graphs

- Recall that BFS explores a graph level by level
- BFS computes shortest path, in terms of number of edges, to every reachable vertex



Weighted graphs

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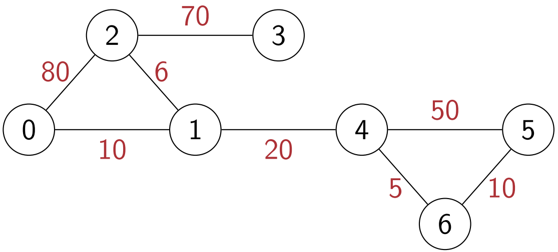
- BFS computes shortest path, in terms of number of edges, to every reachable vertex
- May assign values to edges
  - Cost, time, distance, ...
  - Weighted graph
- $G = (V, E)$, $W : E \to \mathbb{R}$



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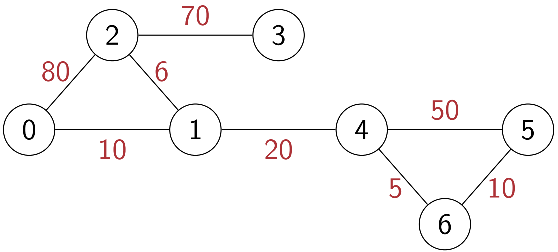
- $G = (V, E)$, $W : E \to \mathbb{R}$
- Adjacency matrix

  Record weights along with edge information — weight is always 0 if no edge



	0	1	2	3	4	5	6
0	(0,0)	(1,10)	(1,80)	(0,0)	(0,0)	(0,0)	(0,0)
1	(1,10)	(0,0)	(1,6)	(0,0)	(1,20)	(0,0)	(0,0)
2	(1,80)	(1,6)	(0,0)	(1,70)	(0,0)	(0,0)	(0,0)
3	(0,0)	(0,0)	(1,70)	(0,0)	(0,0)	(0,0)	(0,0)
4	(0,0)	(1,20)	(0,0)	(0,0)	(0,0)	(1,50)	(1,5)
5	(0,0)	(0,0)	(0,0)	(0,0)	(1,50)	(0,0)	(1,10)
6	(0,0)	(0,0)	(0,0)	(0,0)	(1,5)	(1,10)	(0,0)

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- Adjacency list

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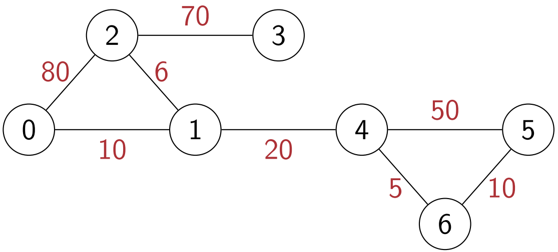

0	[(1,10),(2,80)]
1	[(0,10),(2,6),(4,20)]
2	[(0,80),(1,6),(3,70)]
3	[(2,70)]
4	[(1,20),(5,50),(6,5)]
5	[(4,50),(6,10)]
6	[(4,5),(5,10)]

Shortest paths in weighted graphs

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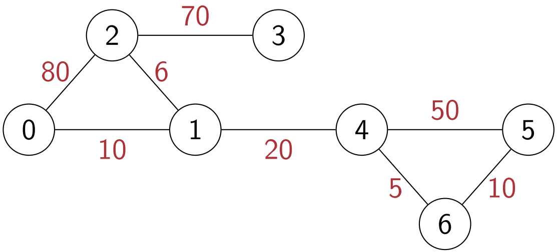
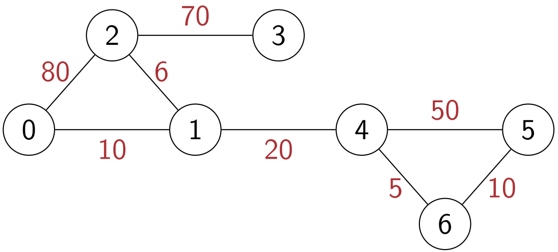


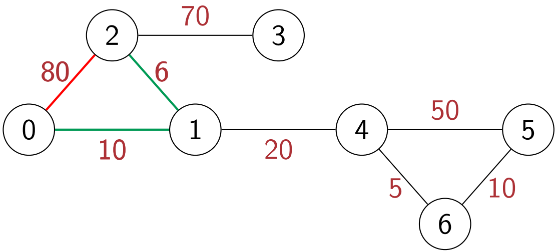
Shortest paths in weighted graphs

- BFS computes shortest path, in terms of number of edges, to every reachable vertex
- In a weighted graph, add up the weights along a path



Shortest paths in weighted graphs

- BFS computes shortest path, in terms of number of edges, to every reachable vertex
- In a weighted graph, add up the weights along a path
- Weighted shortest path need not have minimum number of edges
  - Shortest path from 0 to 2 is via 1



Shortest path problems

Single source shortest paths

- Find shortest paths from a fixed vertex to every other vertex
- Transport finished product from factory (single source) to all retail outlets
- Courier company delivers items from distribution centre (single source) to addressees

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All pairs shortest paths

- Find shortest paths between every pair of vertices i and j
- Optimal airline, railway, road routes between cities

Negative edge weights

Negative edge weights

- Can negative edge weights be meaningful?
- Taxi driver trying to head home at the end of the day
  - Roads with few customers, drive empty (positive weight)
  - Roads with many customers, make profit (negative weight)
  - Find a route toward home that minimizes the cost

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Negative cycles

- A negative cycle is one whose weight is negative
  - Sum of the weights of edges that make up the cycle
- By repeatedly traversing a negative cycle, total cost keeps decreasing
- If a graph has a negative cycle, shortest paths are not defined
- Without negative cycles, we can compute shortest paths even if some weights are negative

Summary

- In a weighted graph, each edge has a cost
  - Entries in adjacency matrix capture edge weights
- Length of a path is the sum of the weights
  - Shortest path in a weighted graph need not be minimum in terms of number of edges
- Different shortest path problems
  - Single source — from one designated vertex to all others
  - All-pairs — between every pair of vertices
- Negative edge weights
  - Should not have negative cycles
  - Without negative cycles, shortest paths still well defined