Divide and Conquer: Quick Select

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Programming, Data Structures and Algorithms using Python

Week 8

Selection

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- Median — $k = n/2$
  - If we can find median in $O(n)$, quicksort becomes $O(n \log n)$

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  - Case 2: `return(pivot)`
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def MoM(L): # Median of medians

  if len(L) <= 5:
    L.sort()
    return(L[len(L)//2])

  # Construct list of block medians
  M = []

  for i in range(0,len(L),5):
    X = L[i:i+5]
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  return(MoM(M))
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- Divide L into blocks of 5
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- Recursively apply the process to M
- What can we guarantee about `MoM(L)`?

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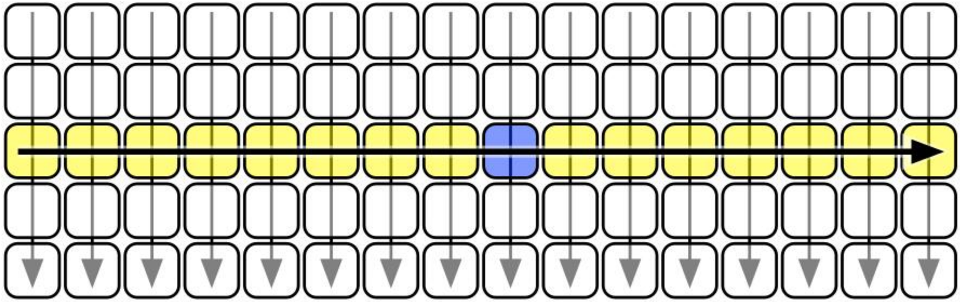

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- We can visualize the blocks as follows
- Each block of 5 is arranged in ascending order, top to bottom
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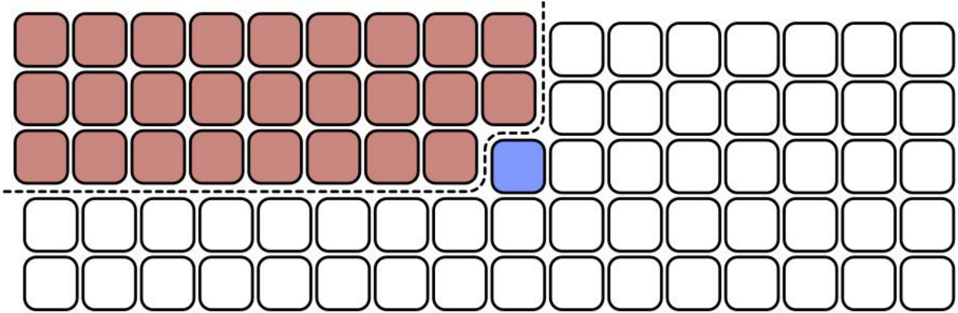
- The median of block medians lies between $3len(L)/10$ and $7len(L)/10$



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- Can also use `MoM` to make quicksort $O(n \log n)$

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- The median of medians algorithm is due to Manuel Blum, Robert Floyd, Vaughn Pratt, Ron Rivest and Robert Tarjan, 1973

Acknowledgment
Illustrations from *Algorithms* by Jeff Erickson, `https://jeffe.cs.illinois.edu/teaching/algorithms/`