Outline for Chapter 6 : Probability

6.1 : Discrete Random Variables

6.2 : Continuous Random Variables

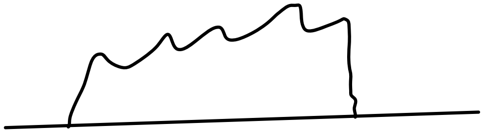
1. Random variables
2. **Expectation, Variance**
3. Multiple Random Variables
4. Uniform, Exponential, Normal
5. Convergence in probability. Laws of large numbers: Markov, Chebyshev, Hoeffding, Central limit.

6.3 : Maximum Likelihood and other advanced topics

Expectation

$$X : \Omega \rightarrow \mathbb{R}$$

$$E[X] : \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$



Expectation

Properties:

i) $E[X+Y] = EX + EY$

ii) $Y = g(X)$

$$E[Y] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx$$

Examples

Ex.

$X \sim \text{Unif}([a,b])$

$$f_X(x) = \begin{cases} 1/(b-a) & x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_a^b \frac{1}{b-a} \cdot x \cdot dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \frac{1}{2} \cdot \frac{1}{b-a} \cdot (b^2 - a^2)$$

$$= \frac{b+a}{2}$$

Examples

Ex. 2

$$f_X(x) = \begin{cases} x/2 & \text{if } x \in [0,2] \\ 0 & \text{o.w} \end{cases}$$

$$E[X] = \int_0^2 x \cdot \frac{x}{2} \cdot dx$$

$$= \frac{1}{2} \cdot \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{4}{3}$$

Variance

$$\operatorname{Var}[X] = E\left[(X - EX)^2\right]$$

$$= EX^2 - (EX)^2$$

$$\sqrt{\operatorname{Var}[X]} = SD[X]$$

<u>Properties</u>

i) $\operatorname{Var}[X+Y] \neq \operatorname{Var}[X] + \operatorname{Var}[Y]$

ii) $\operatorname{Var}[aX] = a^2 \operatorname{Var}[X]$

(iii) $\operatorname{Var}[X] \geq 0$

Examples

Ex. 1

$$f_X(x) = \begin{cases} 1/(b-a) & x \in [a,b] \\ 0 & \text{o.w.} \end{cases}$$

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \cdot \left[\frac{x^3}{3} \right]_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} = \frac{1}{3}\left(b^2 + a^2 + ab\right)$$

$$(EX)^2 = \frac{(b+a)^2}{4} \; ; \; Var[X] = \frac{(b-a)^2}{12}$$

Examples

Conditional and Total Expectation

$X \, , \quad A$

$$E[X|A] = \int_{-\infty}^{\infty} x \cdot f_{X|A}(x) \, dx$$

$$E[X] = E[X|A] \cdot P(A) + E[X|A^c] \, P(A^c)$$

Examples

Waiting for bus.

X = Waiting time

A = Arrival at Bus stop is earlier than 7:15.

$$f_{X/A}(x) = \begin{cases} 1/5 & \text{if } x \in [0,5] \\ 0 & \text{o.w} \end{cases}$$

$E[X/A] = 2.5$ min

$$f_{X|A^c}(x) = \begin{cases} 1/15 & \text{if } x \in [0, 15] \\ 0 & \text{o.w} \end{cases}$$

$$E[X|A^c] = 7.5 \text{ min}$$

$$P(A) = \frac{5}{20} = \frac{1}{4}$$

$$\boxed{E[X]} = \frac{1}{4} \cdot (2.5) + \frac{3}{4}(7.5) = \frac{5 + 45}{8} = \frac{50}{8}$$

$$= \frac{25}{4} \text{ min}$$