# Greedy Algorithms: Huffman Coding

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 7

# Efficient communication

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Morse Code

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  - $aa$

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  - Use binary strings of length 5
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Morse Code

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- Decode 0101
  - *aa* , *etet*, *aet*, *eta*?
- Use pauses between letters
  - Like adding a third symbol for encoding

# Prefix codes

- Encoding of $x$, $E(x)$, is not a prefix of $E(y)$ for any $x$, $y$
  - In Morse Code, $E(e) = 0$ is a prefix of $E(a) = 01$

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$0 \quad 0 \quad 1 \mid 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1$

$c$

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$$c \qquad e \qquad c$$

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$c$ $e$ $c$ $a$

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  - Fraction of occurrences of $x$ over large text corpus
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- Average number of bits per letter in encoding

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- Suppose we have the following frequencies for our earlier example

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- Average number of bits per letter is 2.23
  - $(0.32 \cdot 2) + (0.25 \cdot 2) + (0.20 \cdot 2)$ $+(0.18 \cdot 3) + (0.05 \cdot 3)$
- Given a set of letters $A$ and frequences $f(x)$ for each $x$, produce the most efficient prefix code possible
  - Minimize $ABL(A)$ — Average Bits per Letter

# Codes as trees

- Encoding can be viewed as a binary tree

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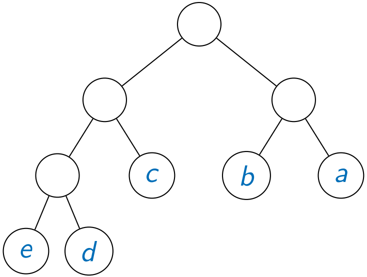

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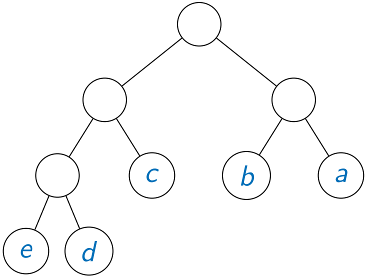

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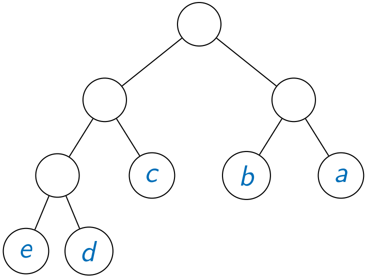

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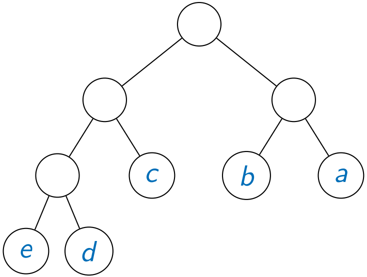
- Letters are at the leaves
- Path to leaf describes encoding — 0 is left, 1 is right
- Prefix code — no internal nodes encode letters

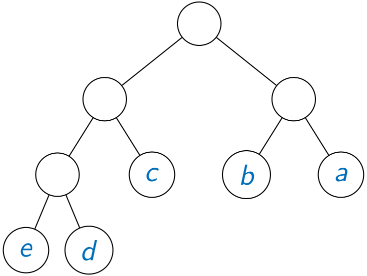


# Codes as trees

**Claim 1**

Any optimal prefix code produces a full tree

- Full tree Each node has 0 or 2 children



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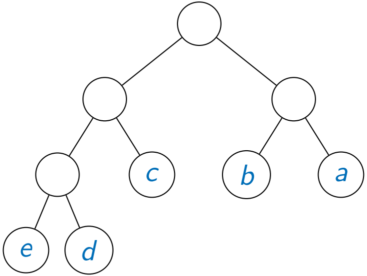
- Full tree: Each node has 0 or 2 children
- For a node with only one child, “promote” child to get a shorter tree



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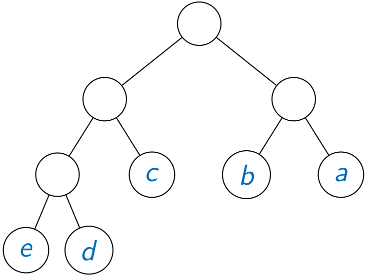
**Claim 2**

In an optimal tree, if leaf labelled $x$ is at smaller depth (higher) than leaf labelled $y$, $f(x) \geq f(y)$



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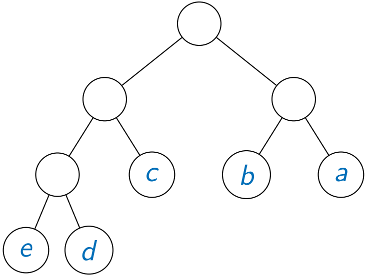
- If $f(y) > f(x)$, exchange labels, improve tree



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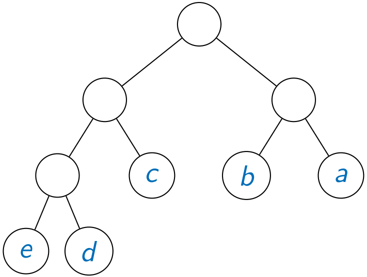
**Claim 3**

In an optimal tree, for any leaf at maximum depth, its sibling is also a leaf $x$



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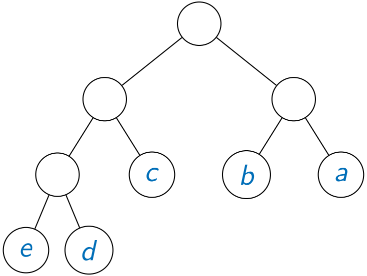

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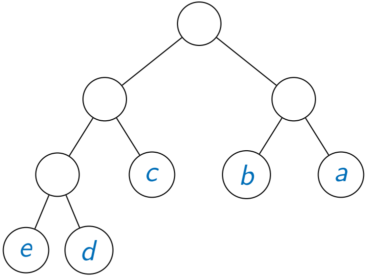
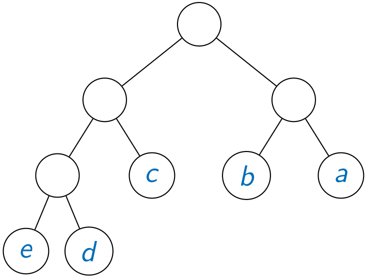


# Codes as trees

**Claim 3**

In an optimal tree, for any leaf at maximum depth, its sibling is also a leaf $x$

- If not, the sibling of this leaf has children
- There is a leaf at lower depth
- The leaf we started with is not a maximum depth!



# A recursive algorithm

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- From Claim 3, leaves at maximum depth occur in pairs

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- From Claim 2, these must have lowest frequencies

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# A recursive algorithm

- From Claim 3, leaves at maximum depth occur in pairs
- From Claim 2, these must have lowest frequencies
- Pick $x$, $y$ with smallest $f(x)$, $f(y)$
- Assign $x$, $y$ to lowest pair of leaves (left/right does not matter)

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- Pick $x$, $y$ with smallest $f(x)$, $f(y)$
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Any optimal prefix code produces a full tree

**Claim 2**

In an optimal tree, if leaf labelled $x$ is at smaller depth (higher) than leaf labelled $y$, $f(x) \geq f(y)$

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In an optimal tree, for any leaf at maximum depth, its sibling is also a leaf $x$

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- Huffman coding — David E Huffman

# Huffman's algorithm

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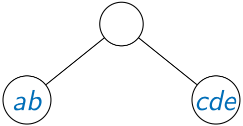

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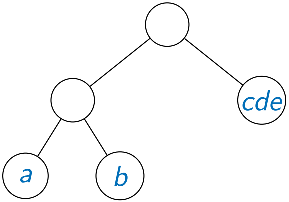

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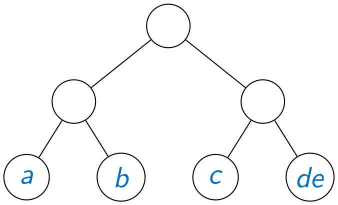

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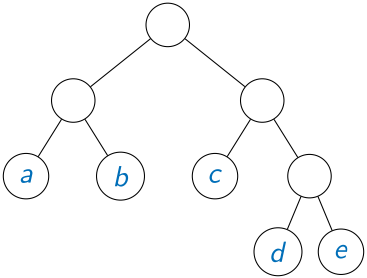

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- Net increase is $f(x) + f(y)$, which is $f(xy)$

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- Hence $ABL(T) \leq ABL(S)$, a contradiction

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  - Complexity is $O(k^2)$
- Instead, maintain frequencies in an heap
  - Extracting two minimum frequency letters and adding back compound letter are both $O(\log k)$
  - Complexity drops to $O(k \log k)$

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- Huffman was a graduate student in Fano's class and discovered his algorithm during the course