Positive Definite Functions and Matrices

Tutorial

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Quadratic function

- Function: $f(x,y) = ax^2 + 2bxy + cy^2$
- In quadratic form:

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Another notation:

$$f(x,y) = v^T A v, \text{where } A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, v = \begin{bmatrix} x \\ y \end{bmatrix}$$

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Partial derivatives of a function

- First order partial derivatives at the point (p, q):

$$f_x = \frac{\partial f}{\partial x}(p, q)$$
$$f_y = \frac{\partial f}{\partial y}(p, q)$$

- Second order partial derivatives at the point (p, q):

$$\boxed{f_{xx} = \frac{\partial f^2}{\partial x^2}(p, q)}$$

$$\boxed{f_{yy} = \frac{\partial f^2}{\partial y^2}(p, q)}$$

$$\boxed{f_{xy} = \frac{\partial f^2}{\partial xy}(p, q)}$$

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Stationary point of a function

- At a stationary point (p, q): First order derivative vanishes

$$f_x = 0, \ f_y = 0$$

- Determinant:

$$D(p,q) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- Second partial derivative test:

Stationary Points	Condition
Minima	$f_{xx} > 0, D(p,q) > 0$
Maxima	$f_{xx} < 0, D(p,q) > 0$
Saddle	$D(p,q) < 0$
Inconclusive	$D(p,q) = 0$

Table: Decision table

Definiteness of a nxn real symmetric matrix

- Trace and Determinant of a nxn matrix A:
  Trace(A) = sum of all eigen values of A
  det(A) = product of all eigen values of A
- Definiteness of the matrix A for all $x \neq 0$ in $\mathbb{R}^n$:

Definiteness	Function form	Eigen Values
Positive definite	$f = x^T A x > 0$	All positive
Positive semi-definite	$f = x^T A x \geq 0$	Non-negative
Negative definite	$f = x^T A x < 0$	All negative
Negative semi-definite	$f = x^T A x \leq 0$	Non-positive
Indefinite	Both $f > 0, f < 0$	Both +ve and -ve

Table: Decision table

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Definiteness of a 2x2 real symmetric matrix

- Quadratic form of a 2x2 real symmetric matrix A:

$$f(x,y) = v^T Av, \text{where } A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \; v = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Check for definiteness:

Definiteness	Function	Condition
Positive definite	$f(x,y) > 0$	$a > 0, ac - b^2 > 0$
Positive semi-definite	$f(x,y) \geq 0$	$a > 0, ac - b^2 = 0$
Negative Definite	$f(x,y) < 0$	$a < 0, ac - b^2 > 0$
Negative semi-definite	$f(x,y) \leq 0$	$a < 0, ac - b^2 = 0$
Indefinite	$f(x,y) > 0, f(x,y) < 0$	$ac - b^2 < 0$

Table: Decision table

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Example 1

Consider the function, $f(x, y) = 4x^2 + 4xy + 2y^2$
In quadratic form:

$$f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Example 1

Check for stationary point:

- First order derivatives: $f_x = 8x + 4y, f_y = 4x + 4y$
- First order derivatives at point (0,0): $f_x = 0, f_y = 0$
- This means the point (0, 0) is an stationary point for $f(x, y)$

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Example 1

Check for type of a stationary point:

- Second order partial derivatives, $f_{xx} = 8, f_{xy} = 0, f_{yy} = 4$
- The determinant, $D = f_{xx}f_{yy} - {f_{xy}}^2 = 8 * 4 - 0^2 = 4$
- Since, $f_{xx} > 0$ and $D > 0,$
- The function has a minima at the point $(0, 0)$

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Example 1

Checking definiteness:

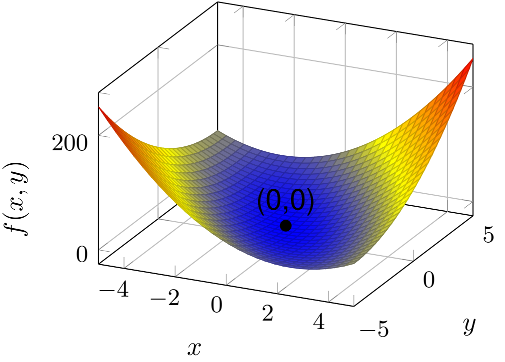
Graphical test

$f(x,y) = v^T A v > 0$ for all

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, $f(x,y)$ is a positive definite function.

3D plot: $f(x,y) = 4x^2 + 4xy + 2y^2$



Example 1

Checking definiteness:

Determinant test

For the function, $a = 4, b = 2, c = 2$
Here $a > 0, ac - b^2 > 0,$
Therefore, $f(x, y)$ is a positive definite function.

Eigenvalue test

For the matrix, $\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$
The eigenvalues $3 - \sqrt{5}, 3 - \sqrt{5}$ are positive.
Therefore, $f(x, y)$ is a positive definite function.

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Example 2

Consider the function, $f(x, y) = x^2 - y^2$
In quadratic form:

$$f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Example 2

Check for stationary point:

- First order derivatives: $f_x = 2x, f_y = -2y$
- First order derivatives at point (0,0): $f_x = 0, f_y = 0$
- This means the point (0, 0) is an stationary point for $f(x, y)$

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Example 2

Check for type of a stationary point:

- Second order partial derivatives, $f_{xx} = 2, f_{xy} = 0, f_{yy} = -2$
- The determinant, $D = f_{xx} f_{yy} - {f_{xy}}^2 = 2 * -2 - 0^2 = -4$
- Since, $f_{xx} > 0$ and $D < 0$,
- The function has a saddle point at $(0, 0)$

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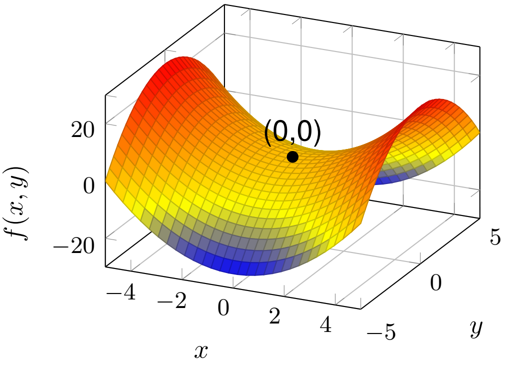
Example 2

3D plot: $f(x, y) = x^2 - y^2$



Checking definiteness

Graphical test

$f(x, y) > 0$ at the point $(1, 0)$ and $f(x, y) < 0$ at the point $(0, 1)$

Therefore, $f(x, y)$ is an indefinite function.

Example 2

Determinant test

For the function, $a = 1, b = 0, c = -1$
Here, $ac - b^2 < 0$, Therefore $f(x, y)$ is an indefinite function.

Eigenvalue test

For the matrix, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
The eigenvalues $-1, 1$ are both positive and negative.
Therefore, $f(x, y)$ is an indefinite function.

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Thank you

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