Spectral theorem   A Hermitian matrix "A" is unitarily diagonalizable, i.e.,

$\exists$ unitary U s.t. $U^* A U = D$ → diagonal matrix with real numbers.

Proof: From Schur's theorem,

$$U^* A U = T$$

$$T^* = U^* A^* U = U^* A U = T$$

$A = A^*$

$T \leftarrow$ upper triangular matrix, $T^* \leftarrow$ lower triangular matrix

$$T = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & 0 & * & \\ & & & \ddots * \end{bmatrix} \qquad T^* = \begin{bmatrix} * & & & 0 \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{bmatrix}$$

$T = T^* \Rightarrow T$ is diagonal

So, $U^* A U = T$, where T is a diagonal matrix

$$T = T^* \Rightarrow T_{ii} = T^*_{ii} = \overline{T_{ii}}$$

$\Rightarrow T_{ii}$ is a real number.

So, $\quad U^* A U = T$, with T denoting a diagonal matrix with real entries $\blacksquare$

Example:

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} \qquad A^* = A$$

$$p(\lambda) = (\lambda - 1)(\lambda - 4)$$

Eigenvalues $\lambda_1 = 1, \qquad \lambda_2 = 4$

Eigenvectors $\quad z_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}, \qquad z_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

$$z_1 \cdot z_2 = \bar{z}_1^T z_2 = 0$$

Normalize z_1, z_2 to obtain

$$u_1 = \begin{bmatrix} \frac{-1+i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \qquad u_2 = \begin{bmatrix} \frac{1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Let $U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$

Then, $$U^* A U = D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Important corollary to the diagonalization result is the following:

Spectral Theorem

A real symmetric matrix A is "orthogonally diagonalizable", i.e., there exists a matrix Q s.t.

$$\underbrace{Q^T A Q = D}_{}, \quad Q^T Q = I$$

D → diagonal matrix with real entries

Proof: From the claim for Hermitian matrices, we have

$$U^* A U = U^{-1} A U = D$$

Columns of U are the eigenvectors of A

A is real symmetric $\Rightarrow$ eigenvalues of A are real

$$(A - \lambda I)x = 0$$

$$\Downarrow$$

Solving leads to a real vector x

$$\Downarrow$$

U is a real matrix

or $U^* = \bar{U}^T = U^T = U^{-1}$

Hence, $U^T A U = D$ with $U^T U = D$

Remark: Hermitian $\Rightarrow$ unitarily diagonalizable

$\not\Leftarrow$

Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad A^* \neq A$

Eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$ $\Rightarrow$ A is diagonalizable

Eigenvectors $z_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\quad z_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Normalizing $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$

$U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$ $\quad U^* A U = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$