Divide and Conquer: Recursion Trees

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Programming, Data Structures and Algorithms using Python

Week 8

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- Is there a uniform way to compute the asymptotic expression for $T(n)$?

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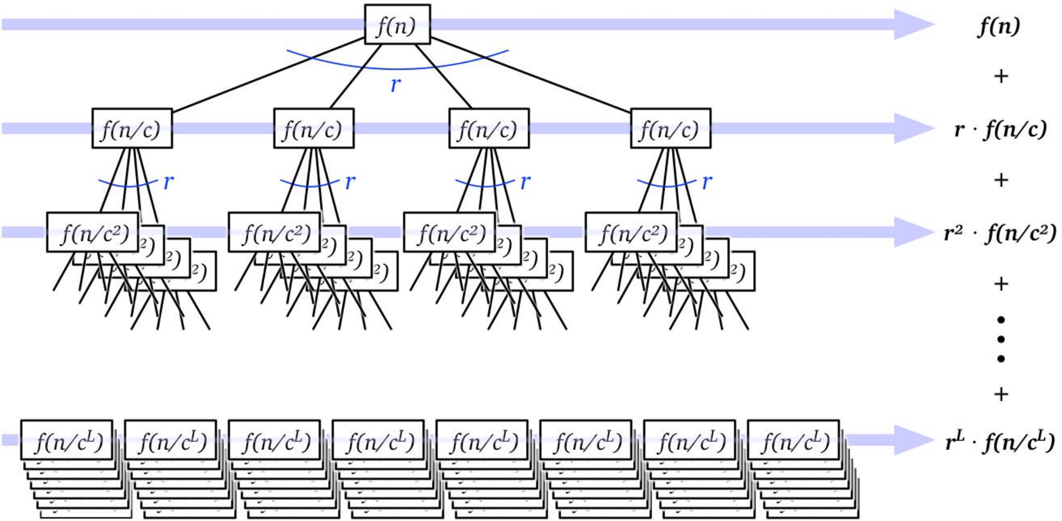
- Root has r children, each (recursively) the root of a tree for $T(n/c)$
- Each node at level d has value $f(n/c^d)$
  - Assume, for simplicity, that n was a power of c

Recursion tree for $T(n) = rT(n/c) + f(n)$



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- Number of leaves is r^L
  - Last term in the level by level sum is $r^L \cdot f(1) = r^{\log_c n} \cdot 1 = n^{\log_c r}$
  - Recall that $a^{\log_b c} = c^{\log_b a}$

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- Increasing Series grows exponentially, each term a constant factor larger than previous term
  - Leaves dominate the sum, $T(n) = O(n^{\log_c r})$

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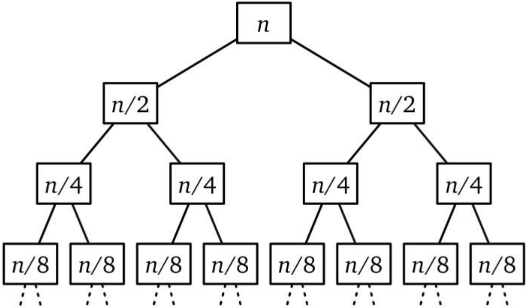

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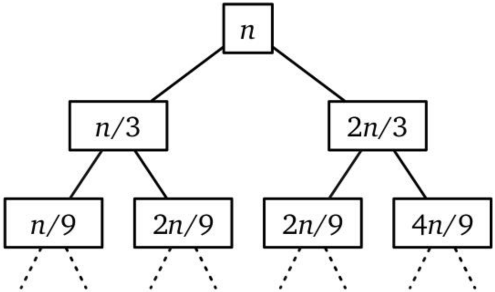

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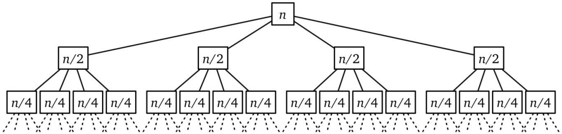

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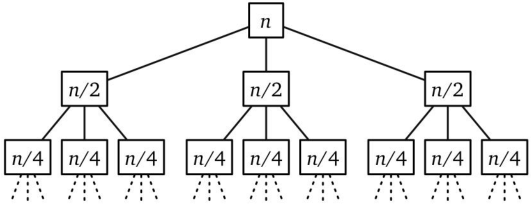

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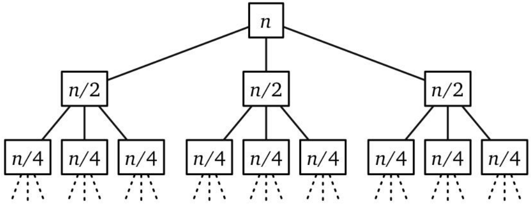

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