Example: $A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}$ Find SVD of A.

Is matrix A diagonalizable? No.

Eigenvalue of A: $\sqrt{2}$ (repeated)

$$A - \sqrt{2} I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A & there aren't any more linearly independent eigenvectors of A.

Finding the SVD of $\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}$

Procedure: (i) Find eigenvalues & eigenvectors of $A^T A$
Using eigenvectors, form an orthonormal basis of $\mathbb{R}^2$

(ii) Use $\sigma_1 y_1 = A x_1$, $\sigma_2 y_2 = A x_2$, where $\{x_1, x_2\}$ is the basis from step (i)

to find y_1, y_2

(iii) $Q_1 = \begin{bmatrix} | & | \\ y_1 & y_2 \\ | & | \end{bmatrix}$, $Q_2 = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$

where λ_1, λ_2 are eigenvalues of $A^T A$

Step (i) $A^T A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 1$

Singular values: $\sigma_1 = \sqrt{\lambda_1} = 2$, $\sigma_2 = \sqrt{\lambda_2} = 1$

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvectors of A^TA:

$$A^TA - 4I = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \quad \text{Eigenvector is } \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \text{ for } \lambda_1 = 4$$

$$A^TA - I = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \quad \text{Eigenvector is } \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} \text{ for } \lambda_2 = 1$$

Normalizing eigenvectors, we obtain $x_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$, $x_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$

$$Q_2 = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$$

Step (ii) Use $\sigma_1 y_1 = Ax_1$ & $\sigma_2 y_2 = Ax_2$

to obtain $y_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$, $y_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$

$$Q_1 = \begin{bmatrix} | & | \\ y_1 & y_2 \\ | & | \end{bmatrix}$$

Step (iii) $Q_2 = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}$ $Q_1 = \begin{bmatrix} | & | \\ y_1 & y_2 \\ | & | \end{bmatrix}$ $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Check: $Q_1 \Sigma Q_2^T = A$