# Heaps

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 6

# Priority queue

- Need to maintain a collection of items with priorities to optimise the following operations
- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
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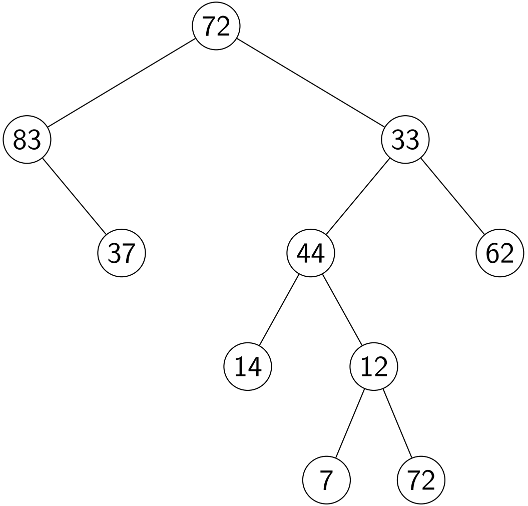
- Maintaining as a list incurs cost $O(N^2)$ across $N$ inserts and deletions
- Using a $\sqrt{N} \times \sqrt{N}$ array reduces the cost to $O(\sqrt{N})$ per operations
  - $O(N\sqrt{N})$ across $N$ inserts and deletions

# Binary trees

- Values are stored as nodes in a rooted tree
- Each node has up to two children
  - Left child and right child
  - Order is important
- Other than the root, each node has a unique parent
- Leaf node — no children
- Size — number of nodes
- Height — number of levels



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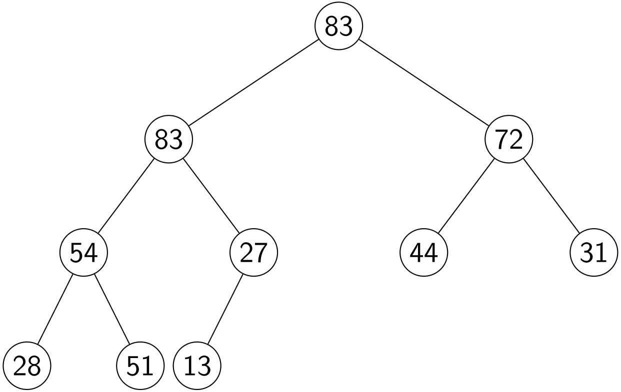
- The value at each node is at least as big the values of its children
    - max-heap



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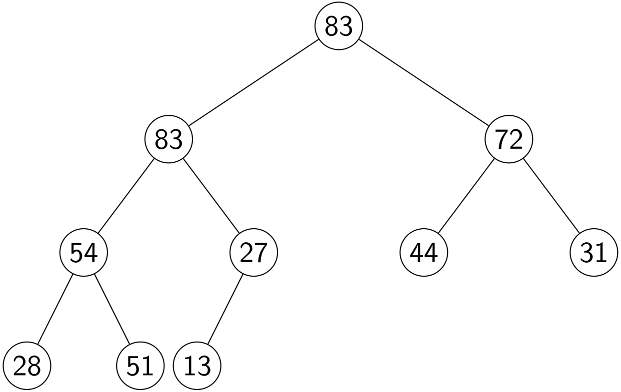

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- The value at each node is at least as big the values of its children
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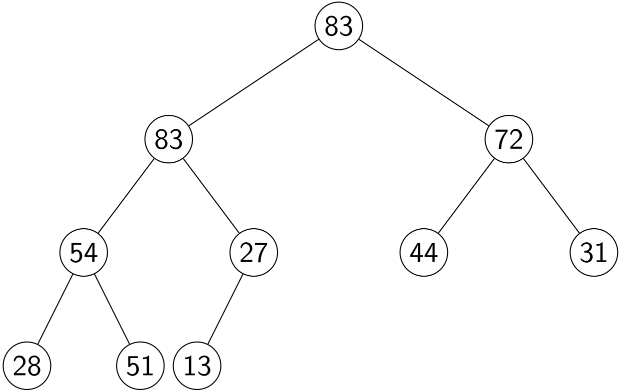
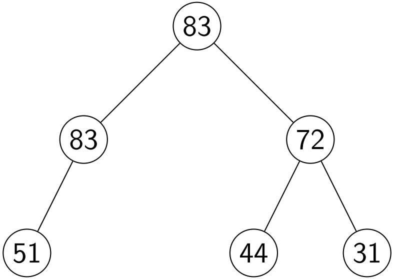
- Binary tree on the right is an example of a heap
- Root always has the largest value
  - By induction, because of the max-heap property



# Non-examples

No "holes" allowed



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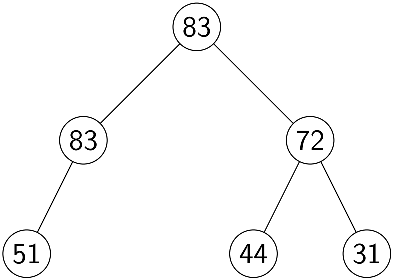
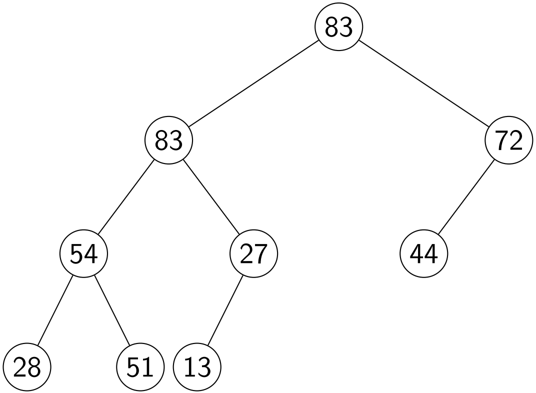
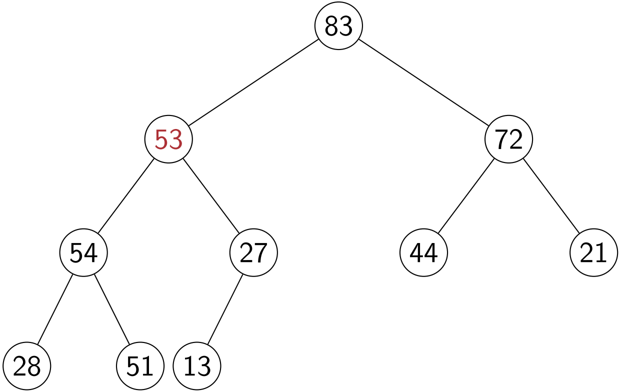



Cannot leave a level incomplete



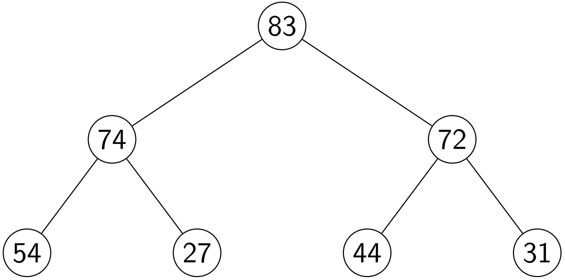
# Non-examples

Heap property is violated



# insert()

- `insert(77)`



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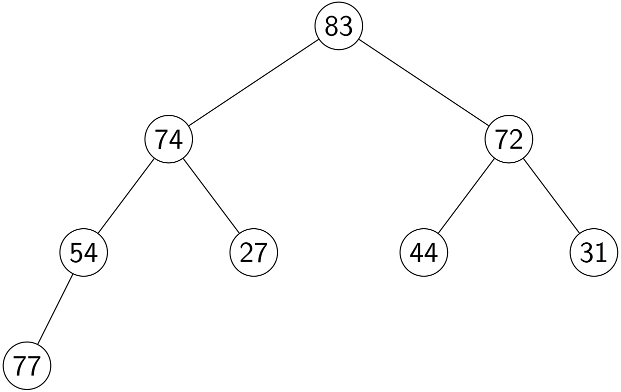

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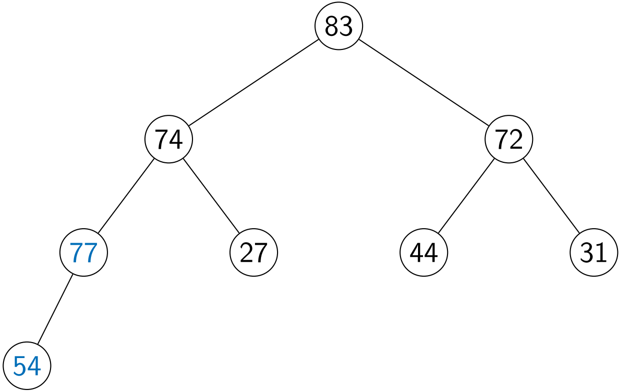

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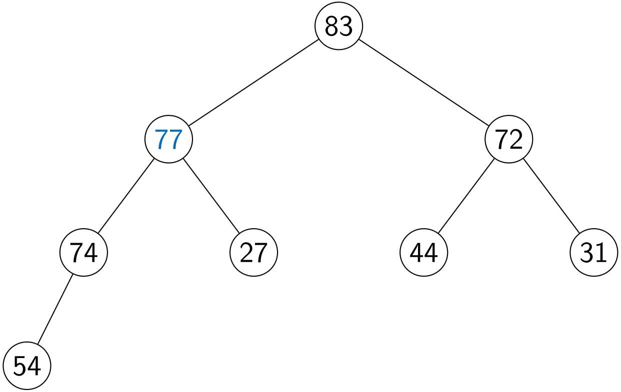

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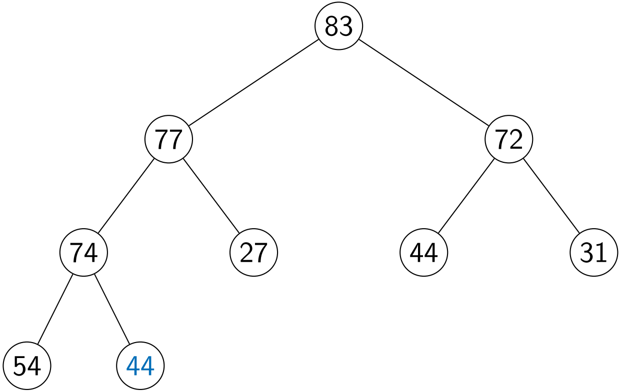

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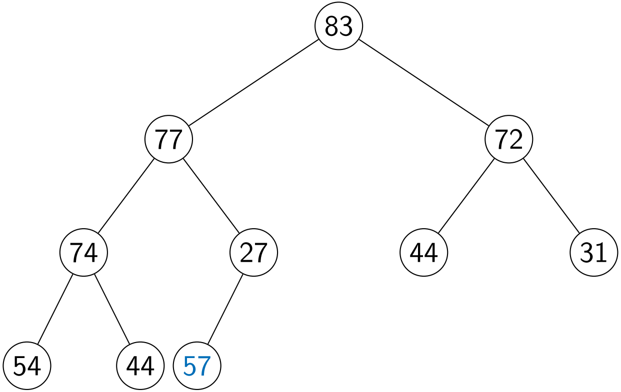

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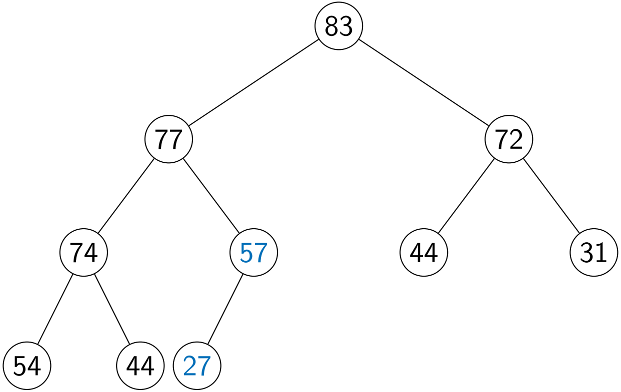

# Complexity of `insert()`

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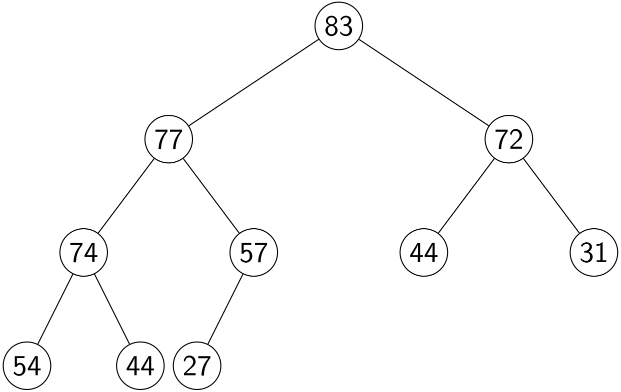

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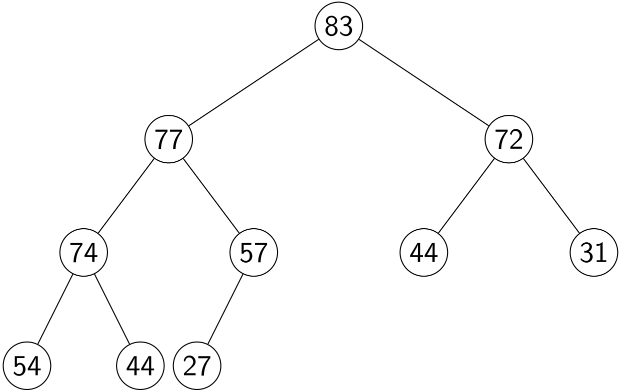

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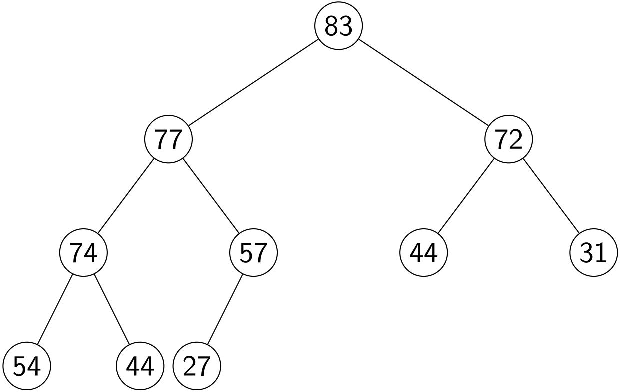

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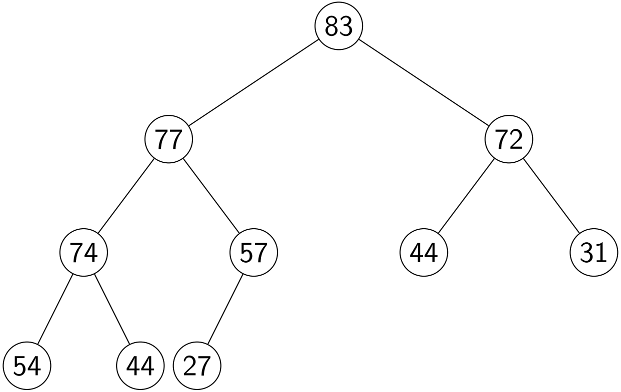

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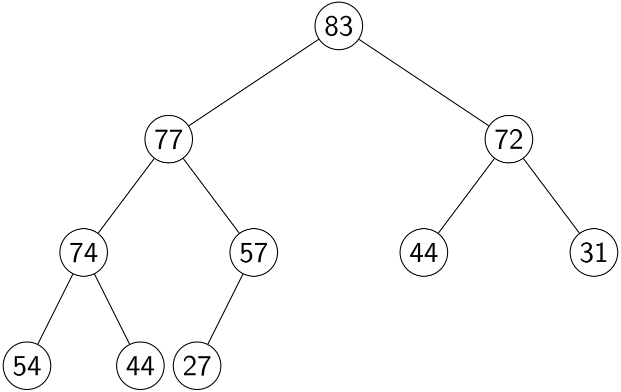

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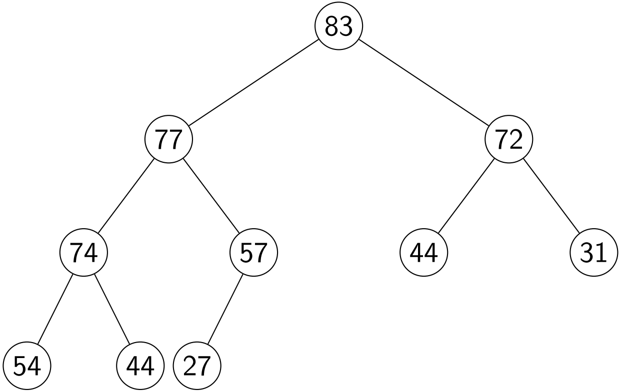

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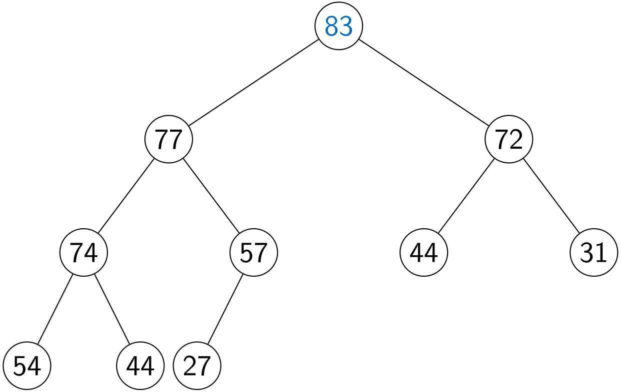

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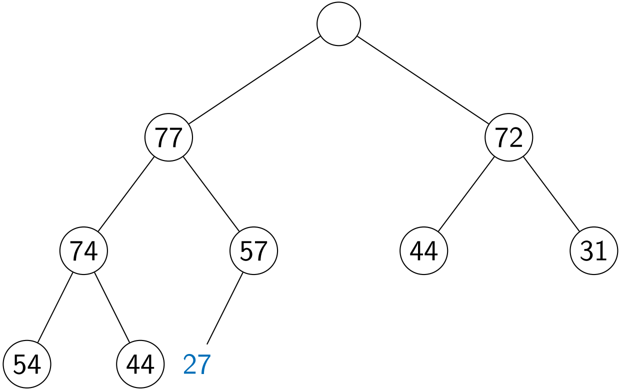

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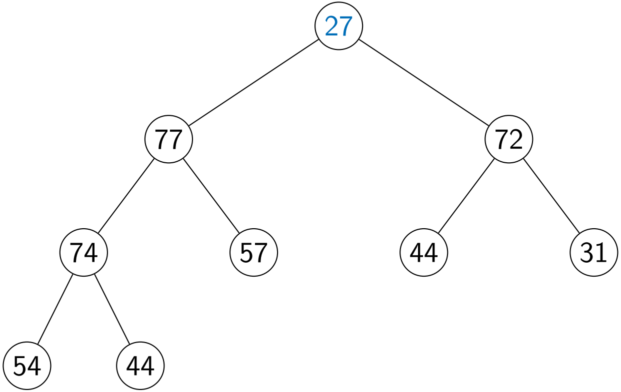

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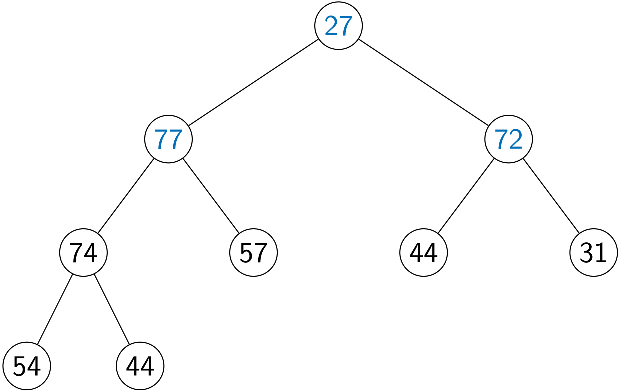

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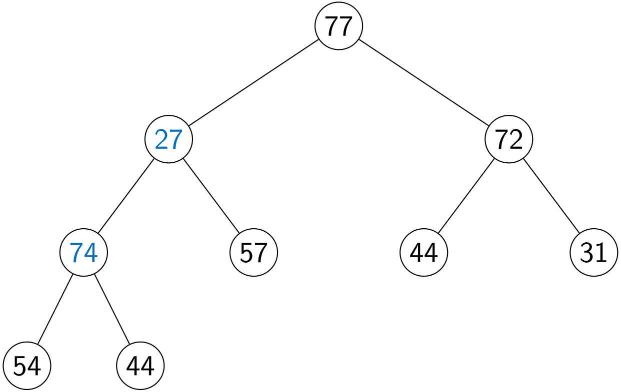

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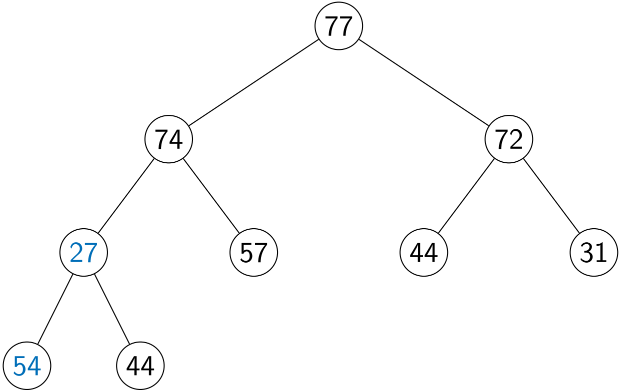

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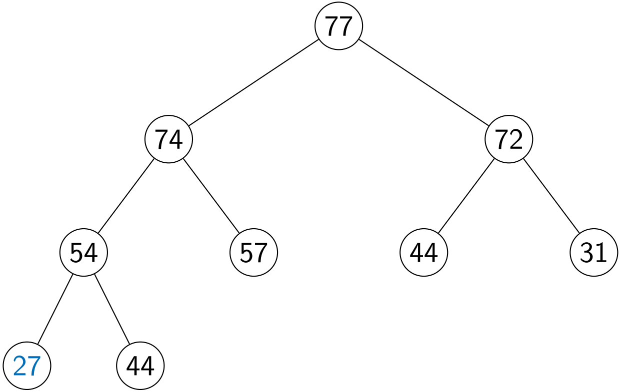
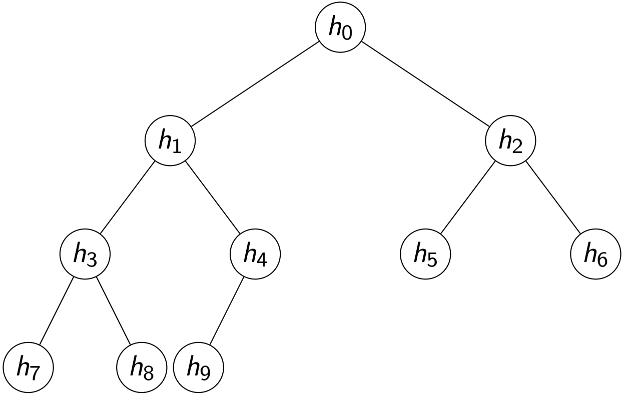


# Implementation

- Number the nodes top to bottom left right
- Store as a list
  `H = [h0,h1,h2,...,h9]`
- Children of `H[i]` are at `H[2*i+1]`, `H[2*i+2]`
- Parent of `H[i]` is at `H[(i-1)//2]`, for `i > 0`



# Building a heap — `heapify()`

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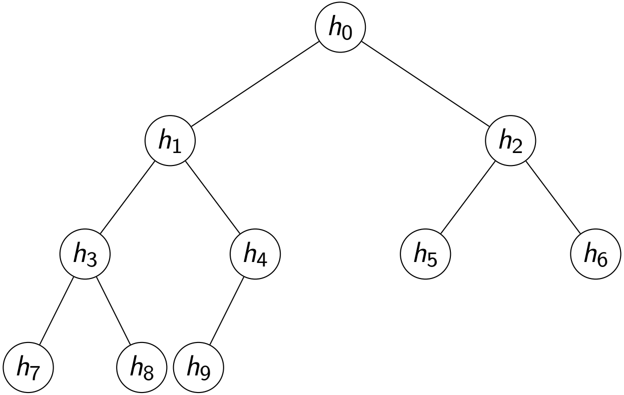
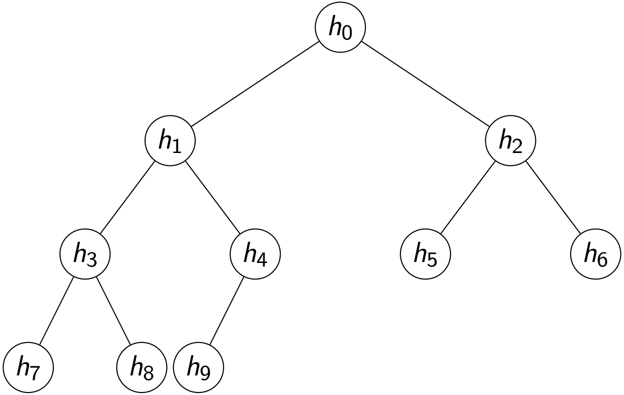


# Building a heap — `heapify()`

- Convert a list `[v0,v1,...,vN]` into a heap
- Simple strategy
  - Start with an empty heap
  - Repeatedly apply `insert(vj)`
  - Total time is $O(N \log N)$



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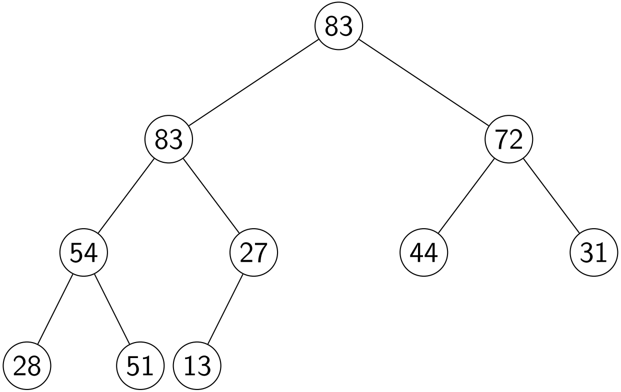
- Cost turns out to be $O(n)$

# Summary

- Heaps are a tree implementation of priority queues
  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - `heapify()` builds a heap in $O(N)$



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  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - `heapify()` builds a heap in $O(N)$
- Can invert the heap condition
  - Each node is smaller than its children
  - min-heap
  - `delete_min()` rather than `delete_max()`