Analysis of Merge Sort

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Programming, Data Structures and Algorithms using Python

Week 2

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done
- Otherwise
  - Sort `A[:n//2]` into `L`
  - Sort `A[n//2:]` into `R`
  - Merge `L` and `R` into `B`

Merging two sorted lists `A` and `B` into `C`

- If `A` is empty, copy `B` into `C`
- If `B` is empty, copy `A` into `C`
- Otherwise, compare first elements of `A` and `B`
  - Move the smaller of the two to `C`
- Repeat till all elements of `A` and `B` have been moved

Analysing merge

- Merge `A` of length m, `B` of length n

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def merge(A,B):
  (m,n) = (len(A),len(B))
  (C,i,j,k) = ([],0,0,0)
  while k < m+n:
    if i == m:
      C.extend(B[j:])
      k = k + (n-j)
    elif j == n:
      C.extend(A[i:])
      k = k + (m-i)
    elif A[i] < B[j]:
      C.append(A[i])
      (i,k) = (i+1,k+1)
    else:
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- Recall that $m + n \leq 2(\max(m, n))$
- If $m \approx n$, `merge` take time $O(n)$

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Analysing mergesort

- Let $T(n)$ be the time taken for input of size n
  - For simplicity, assume $n = 2^k$ for some k

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def mergesort(A):
  n = len(A)

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  L = mergesort(A[:n//2])
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- Recurrence
  - $T(0) = T(1) = 1$
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    - Solve two subproblems of size $n/2$
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- Unwind the recurrence to solve

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- Hence $T(n)$ is $O(n \log n)$

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- Inherently recursive
  - Recursive calls and returns are expensive