Positive definite matrices

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if $a > 0$, $ac - b^2 > 0$

corresponding function is $f(v) = v^T A v$

$v = \begin{bmatrix} x \\ y \end{bmatrix}$

Note: If $a > 0$ and $ac - b^2 > 0$, then both eigenvalues (say λ_1, λ_2) of A are positive.

To see this: $ac - b^2 = \det(A) = \lambda_1 \lambda_2 > 0$, $\text{Trace}(A) = \lambda_1 + \lambda_2 = a + c > 0$

$\lambda_1 \lambda_2 > 0$ and $\lambda_1 + \lambda_2 > 0$

$\Rightarrow \lambda_1 > 0, \ \lambda_2 > 0$

Def: A real-symmetric $n \times n$ matrix A is positive definite if

$v^T A v > 0 \quad \forall\, v \in \mathbb{R}^n, v \neq 0$ $\rightarrow$ Condition (i)

Condition (i) is equivalent to (ii) All eigenvalues of A are > 0.

Proof of equivalence:

(i) $\Rightarrow$ (ii): Suppose (i) holds

$$A x = \lambda x$$
$$x^T A x = x^T \lambda x = \lambda \|x\|^2$$

$x \neq 0$, $x^T A x > 0$ (from (i)) $\Rightarrow \lambda > 0$ since $x^T A x = \lambda \|x\|^2 > 0$ and $x \neq 0$.

(ii) $\Rightarrow$ (i) : A is real symmetric $\overset{\text{Spectral Theorem}}{\Rightarrow}$ there exists an orthonormal basis of eigenvectors, say $\{x_1, \ldots, x_n\}$

Any $x \in \mathbb{R}^n$ can be written as
$$x = c_1 x_1 + \cdots + c_n x_n$$

$$Ax = c_1 A x_1 + \cdots + c_n A x_n$$
$$= c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n$$
(Since $x_1, \ldots, x_n$ are eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$)

$$x^T A x = (c_1 x_1 + \cdots + c_n x_n)^T (c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n)$$
$$= c_1^2 \lambda_1 + \cdots + c_n^2 \lambda_n$$
(since $\|x_i\|^2 = 1$ and $x_i^T x_j = 0 \;\forall i \neq j$)

> 0 since $\lambda_i > 0 \;\forall i$ by condition (ii)

$\Rightarrow$ $x^T A x > 0$ and condition (i) holds.