Diagonalization of Hermitian matrices

Note Title 10-06-2021

Recap: $A^* = \overline{A}^T$

Matrix A is Hermitian if $A^* = A$

Matrix U is Unitary if $U^* U = I$ & U is square.

Def: A matrix "A" is unitarily diagonalizable if there exists a unitary matrix U s.t.

$A = U \Lambda U^*$, where Λ is a diagonal matrix.

To show: A Hermitian matrix is unitarily diagonalizable.

Approach: ① Show that any $n \times n$ matrix is similar to an upper triangular matrix, i.e.,

$A = U T U^*$

unitary

upper-triangular matrix

② Using ①, we show that a Hermitian matrix is unitarily diagonalizable

Schur's theorem:

Any $n \times n$ matrix A is similar to an upper-triangular matrix T, i.e., there exists a unitary matrix U s.t.

$$A = U T U^*$$

Proof for $n=3$:

Let $p(\lambda)$ be the characteristic polynomial of A.

Let λ_1 be a root of $p(\lambda)$. Let z_1 be the corresponding eigenvector.

Extend $\{z_1\}$ to a basis, and make it orthonormal.

Let $\{z_1, u, v\}$ be the orthonormal basis. We have $\|z_1\| = \|u\| = \|v\| = 0$

$z_1^* u = z_1^* v = u^* v = 0$

Let $U_1 = \begin{bmatrix} | & | & | \\ z_1 & u & v \\ | & | & | \end{bmatrix}$ $\quad AU_1 = A \begin{bmatrix} | & | & | \\ z_1 & u & v \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1 z_1 & Au & Av \\ | & | & | \end{bmatrix}$

since $Az_1 = \lambda_1 z_1$

$$U_1^* A U_1 = \begin{bmatrix} - \bar{z}_1^T - \\ - \bar{u}^T - \\ - \bar{v}^T - \end{bmatrix} \begin{bmatrix} | & | & | \\ \lambda_1 z_1 & Au & Av \\ | & | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & * & * \\ 0 & & \\ 0 & & B \end{bmatrix}$$

Some entries that are not necessarily zero

Why?

(2×2) matrix

(i) $\bar{z}_1^T \lambda_1 z_1 = \lambda_1 \bar{z}_1^T z_1 = \lambda_1$; (ii) $\bar{u}^T z_1 = 0$ and $\bar{v}^T z_1 = 0$

So, $U_1^* A U_1 = \begin{bmatrix} \lambda_1 & * & * \\ 0 & & \\ 0 & & B \end{bmatrix}$

Repeat the procedure with B to get an eigenvalue λ_2 of B & a unitary matrix P s.t.

$$P^* B P = \begin{bmatrix} \lambda_2 & * \\ 0 & \lambda_3 \end{bmatrix}$$

Let $U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{P}} \\ 0 & & \end{bmatrix}$. Then, $U_2^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{P^*} \\ 0 & & \end{bmatrix}$

$$U_2^* U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{since} \quad P^* P = I.$$

Consider $U_2^* (U_1^* \; A \; U_1) U_2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{P^*} \\ 0 & & \end{bmatrix} \begin{bmatrix} \lambda_1 & * & * \\ 0 & \multicolumn{2}{c}{B} \\ 0 & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{P} \\ 0 & & \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & * & * \\ 0 & \multicolumn{2}{c}{P^* B P} \\ 0 & & \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{\text{upper-triangular}} = T$$

So, $\quad U_2^* U_1^* A U_1 U_2 = T$

Set $\quad U = U_1 U_2$

$$U^* = (U_1 U_2)^* = U_2^* U_1^* = U_2^{-1} U_1^{-1} = (U_1 U_2)^{-1} = U^{-1}$$

So, U is unitary and

$$U^* A U = U_2^* U_1^* A U_1 U_2 = T$$

or $\quad A = U T U^*$. $\quad$ Hence proved ■

Remark! The proof can be easily extended to a general $n > 3$.

Example: $\quad A = \begin{bmatrix} 5 & 8 & 16 \\ 5 & 0 & 9 \\ -3 & -5 & -10 \end{bmatrix}$ $\quad$ "Upper triangliaze A"

Characteristic polynomial: $p(\lambda) = -(\lambda-1)(\lambda+3)^2$

$$\lambda_1 = 1, \quad \lambda_2 = -3$$

Eigenvector $\quad z_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

$$\{z_1, e_1, e_2\} = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \leftarrow \text{basis for } \mathbb{R}^3$$

$\downarrow$ Gram Schmidt procedure to obtain an orthonormal basis

$$U_1 = \begin{bmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U_1^* A U_1 = U_1^T A U_1 = \begin{bmatrix} 1 & -8\sqrt{2} & -12\sqrt{3} \\ 0 & -3 & 0 \\ 0 & \sqrt{6} & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 0 \\ \sqrt{6} & -3 \end{bmatrix}$$

Find an eigenvalue of B.

$\lambda_2 = -3$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ eigenvector corresponding to λ_2

$$\{e_2\} \xrightarrow[\text{basis of } \mathbb{R}^2]{\text{extend to}} \{e_2, e_1\} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Unitary matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$P^* B P = P^T B P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ \sqrt{6} & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & \sqrt{6} \\ 0 & -3 \end{bmatrix}$$

Let $U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & P & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$U_2^* U_1^* A U_1 U_2 = U_2^T (U_1^T A U_1) U_2$$

$$= \begin{bmatrix} 1 & -8\sqrt{2} & -12\sqrt{3} \\ 0 & -3 & \sqrt{6} \\ 0 & 0 & -3 \end{bmatrix} \longrightarrow \text{upper triangular matrix}$$

$U = U_1 U_2$, we have $U^* A U = T$