Machine Learning Foundations

Week-5 Revision

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Complex vectors

$x \in \mathbb{C}^n$

$$x = \begin{bmatrix} 3-2i \\ -2+i \\ -4-3i \end{bmatrix} \in \mathbb{C}^3$$

$y \in \mathbb{C}^n$

$$y = \begin{bmatrix} -2+4i \\ 5-i \\ -2i \end{bmatrix} \in \mathbb{C}^3$$

Operations:

Addition

$$z = x + y \in \mathbb{C}^n$$

$$z = \begin{bmatrix} 1+2i \\ 3 \\ -4-5i \end{bmatrix} \in \mathbb{C}^3$$

Conjugate

$$\overline{z} = \begin{bmatrix} 1-2i \\ 3 \\ -4+5i \end{bmatrix}$$

Inner Product $\boxed{x \cdot y = \overline{x}^T y \in \mathbb{C}}$

$$x = \begin{bmatrix} 3-2i \\ -2+i \\ -4-3i \end{bmatrix} \quad y = \begin{bmatrix} -2+4i \\ 5-i \\ -2i \end{bmatrix} \quad \overline{x}^T = [3+2i, -2-i, -4+3i] \begin{bmatrix} -2+4i \\ 5-i \\ -2i \end{bmatrix}$$

$$= -19+13i$$

Properties

1. $x \cdot y = \overline{y \cdot x}$
2. $(x+y) \cdot z = x \cdot z + y \cdot z$
3. $x \cdot cy = c(x \cdot y)$
4. $cx \cdot y = \overline{c}(x \cdot y)$
5. $x \cdot x = ||x||^2$

$cx \cdot cy = |c|(x \cdot y)$, true or false ?

Complex Matrices

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2i & 3+3i \\ 3+3i & 5i \end{bmatrix}$$

Hermitian if: $\boxed{A^* = \overline{A}^T = \overline{A^T}}$ $\qquad$ $\boxed{x \cdot y = x^* y \in \mathbb{C}}$

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$$ is Hermitian

$$B = \begin{bmatrix} 2i & 3+3i \\ 3+3i & 5i \end{bmatrix}$$ is not Hermitian

$$C = \begin{bmatrix} 2i & 3+3i \\ 3-3i & 5i \end{bmatrix}$$ is not Hermitian

Properties of Hermitian Matrices

1. All Eigenvalues λ_i are real.

2. Eigenvectors are orthogonal if $\lambda_i \neq \lambda_j$ for $i \neq j$

Finding complex eigenvectors:

Consider the matrix $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$. Find the complex eigenvector for the eigenvalue $\lambda = 8$

$$N[A - \lambda I] = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix}$$

$$R_2 = R_2 + \tfrac{1}{2}(1+i)R_1$$

$$= \begin{bmatrix} -6 & 3-3i \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 3-3i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-6x_1 + (3-3i)x_2 = 0$$

$$-2x_1 + (1-1i)x_2 = 0$$

$$2x_1 = (1-1i)x_2$$

$$\therefore x = \begin{bmatrix} 1 \\ 1+1i \end{bmatrix} = c \begin{bmatrix} 1 \\ 1+1i \end{bmatrix}$$

$$\boxed{x = 1 - i \begin{bmatrix} 1 \\ 1+1i \end{bmatrix} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}}$$

Unitary Matrices

Real Case:

$$Q^T Q = I$$

Complex Case:

$$U^* U = I$$

$$U = \begin{bmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{bmatrix}$$

$$U^T = \begin{bmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{bmatrix}$$

$$U * U^T = \begin{bmatrix} cos^2(t) + sin^2(t) & cos(t)sin(t) - sin(t)cos(t) \\ sin(t)cos(t) - cos(t)sin(t) & sin^2(t) + cos^2(t) \end{bmatrix}$$

$$U * U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Properties:

It preserves the length and angle of vectors!

Therefore, eigenvalues are $|\lambda_i| = 1$

Eigenvectors are orthogonal if $\lambda_i \neq \lambda_j$ for $i \neq j$

Unitary matrices are need not be necessarily Hermitian.

There exists unitary matrix that diagonalizes a Hermitian matrix.

Diagonalization of Hermitian Matrices

Schur's Theorem

Any $n \times n$ matrix is **similar** to upper triangular matrix T, that is $A = UTU^*$

Example:

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \qquad \lambda_1 = -2, \lambda_2 = 3 \qquad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 7 \\ -1 & -2 \end{bmatrix}$$

If we do U^*AU, will it be triangular or diagonal matrix?

No, why?

It is not an orthonormal matrix!

So how to find orthogonal matrix?

Gram-Schmidt process:

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \qquad \lambda_1 = -2, \lambda_2 = 3 \qquad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

Find a vector orthogonal to $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (you could have picked x_2 as well)

$$q_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$q_2 = x_2 - (x_2 \cdot q_1)\frac{q_1}{q_1 \cdot q_1}$$

$$q_2 = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

If we do U^*AU, will it be triangular or diagonal matrix?

$$U^*AU = \begin{bmatrix} -2 & 9 \\ 0 & 3 \end{bmatrix}$$

What if we have considered different vector instead of x_2 during orthogonalization?

Gram-Schmidt process:

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \qquad \lambda_1 = -2, \lambda_2 = 3 \qquad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

https://www.geogebra.org/material/iframe/id/vsb7b4pw/width/700/height/500/border/888888/sfsb/true/smb/false/stb/false/stbh/false/ai/false/asb/false/sri/false/rc/false/ld/false/sdz/false/ctl/false

For a 2 x 2 case, the orthogonal vector to a given vector x_1 is unique!

So, it doesn't matter which vector you use as x_2

This is my claim!

Gram-Schmidt process:

For a matrix of size 3×3, if we have only one eigenvector x_1, then there are many possible orthogonal vectors based on the direction of x_2

Could you reason, why? (I hope, no need for geogebra :-))

Therefore, schur decomposition is not unique!

Spectral Theorem

Any Hermitian matrix is **similar** to diagonal matrix D, that is $A = UDU^*$

Singular Value Decomposition (SVD)

Any matrix A can be diagonalized as $A = Q_1 \Sigma Q_2^T$, where $Q_1 = eig(AA^T)$ and $Q_2 = eig(A^T A)$

No problem in computation steps as long as none of the **singular values** are zero.

If any of the singular value is zero, we need to bring GS process to create unitary matrices.

Add-on

If SVD is used for PCA, then Singular values represent the variance of the data. Higher the singular value, higher the variance!. (Watch once again the Image compression tutorial in week-5, keeping this in mind)