Applications of BFS and DFS

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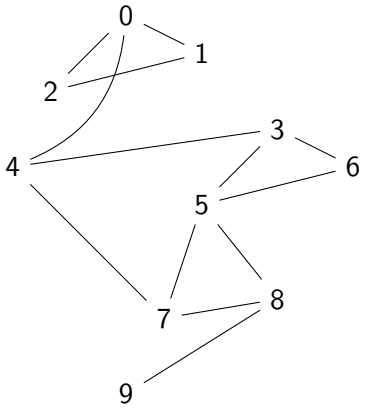
Programming, Data Structures and Algorithms using Python

Week 4

BFS and DFS

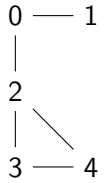
- BFS and DFS systematically compute reachability in graphs
- BFS works level by level
  - Discovers shortest paths in terms of number of edges
- DFS explores a vertex as soon as it is visited neighbours
  - Suspend a vertex while exploring its neighbours
  - DFS numbering describes the order in which vertices are explored
- Beyond reachability, what can we find out about a graph using BFS/DFS?



Connectivity

- An undirected graph is connected if every vertex is reachable from every other vertex

Connected Graph



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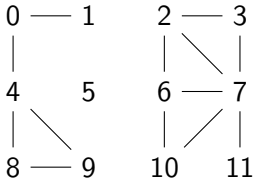

Connectivity

- An undirected graph is connected if every vertex is reachable from every other vertex
- In a disconnected graph, we can identify the connected components

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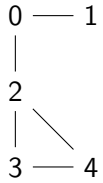

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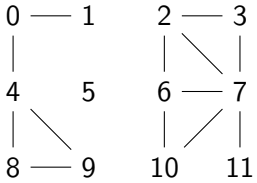

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  - Maximal subsets of vertices that are connected

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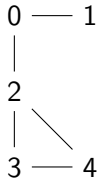

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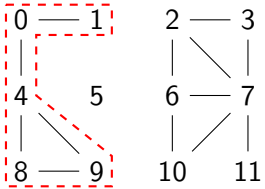

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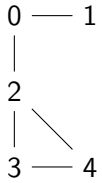

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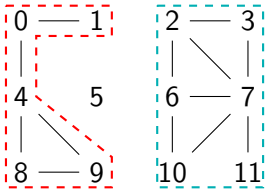
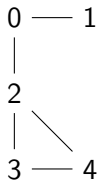


Connectivity

- An undirected graph is connected if every vertex is reachable from every other vertex
- In a disconnected graph, we can identify the connected components
  - Maximal subsets of vertices that are connected
  - Isolated vertices are trivial components

Connected Graph



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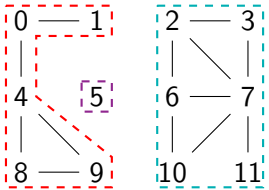
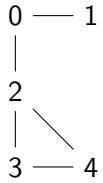


Identifying connected components

- Assign each vertex a component number

Connected Graph



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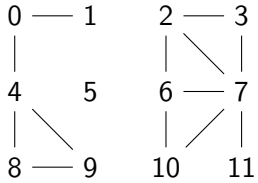
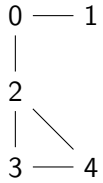


Identifying connected components

- Assign each vertex a component number
- Start BFS/DFS from vertex 0

Connected Graph



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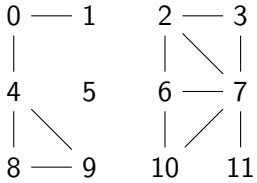

Identifying connected components

- Assign each vertex a component number
- Start BFS/DFS from vertex 0
  - Initialize component number to 0

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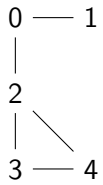

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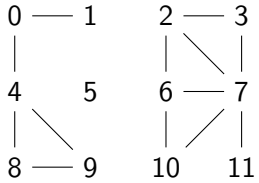
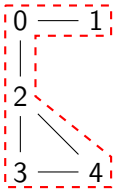


Identifying connected components

- Assign each vertex a component number
- Start BFS/DFS from vertex 0
  - Initialize component number to 0
  - All visited nodes form a connected component

Connected Graph



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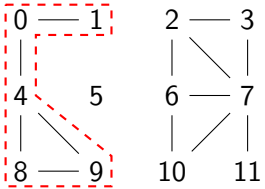

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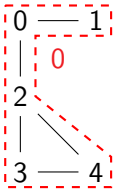

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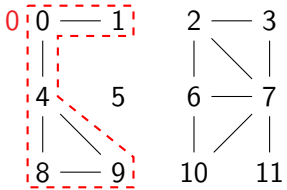

Identifying connected components

- Assign each vertex a component number
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- Pick smallest unvisited node j

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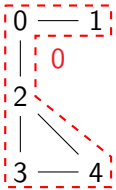

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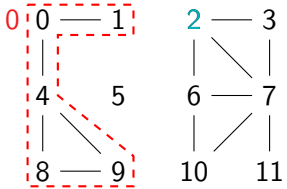

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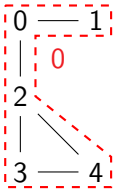

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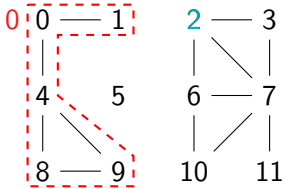

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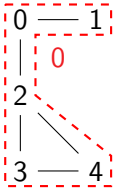

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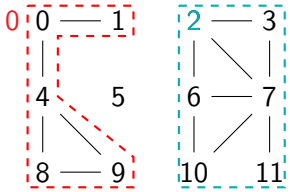

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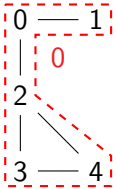

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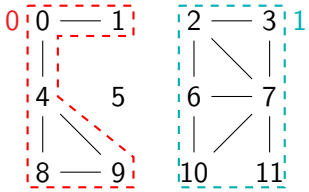

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- Repeat until all nodes are visited

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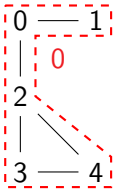

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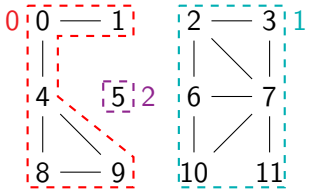

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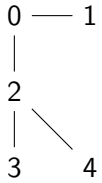
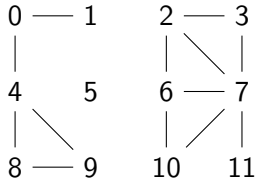
```
def Components(AList):
  component = {}
  for i in AList.keys():
    component[i] = -1

  (compid,seen) = (0,0)

  while seen <= max(AList.keys()):
    startv = min([i for i in AList.keys()
                  if component[i] == -1])
    visited = BFSList(AList,startv)
    for i in visited.keys():
      if visited[i]:
        seen = seen + 1
        component[i] = compid
    compid = compid + 1

  return(component)
```

Detecting cycles

- A cycle is a path (technically, a walk) that starts and ends at the same vertex





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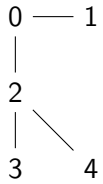
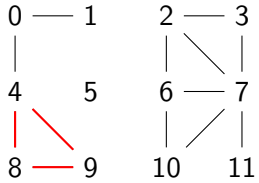
  - $4 - 8 - 9 - 4$ is a cycle





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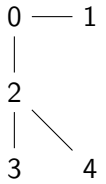

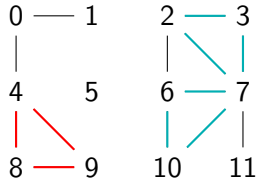

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- A cycle is a path (technically, a walk) that starts and ends at the same vertex
  - $4 - 8 - 9 - 4$ is a cycle
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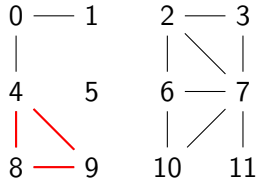
  - Cycle should not repeat edges: $i - j - i$ is not a cycle, e.g., $2 - 4 - 2$



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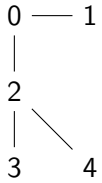

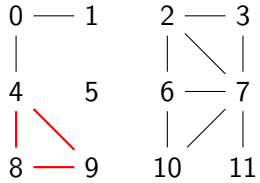

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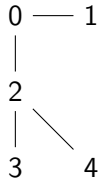
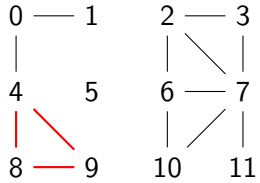
  - Cycle should not repeat edges: $i - j - i$ is not a cycle, e.g., $2 - 4 - 2$
  - Simple cycle — only repeated vertices are start and end
- A graph is acyclic if it has no cycles

Acyclic Graph



Graph with cycles



BFS tree

- Edges explored by BFS form a tree
  - Technically, one tree per component
  - Collection of trees is a forest

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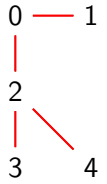
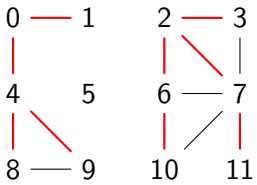


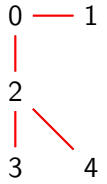
Graph with cycles



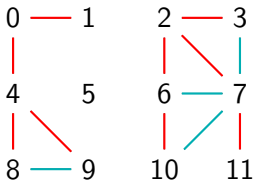
BFS tree

- Edges explored by BFS form a tree
  - Technically, one tree per component
  - Collection of trees is a forest
- Any non-tree edge creates a cycle
  - Detect cycles by searching for non-tree edges

Acyclic Graph

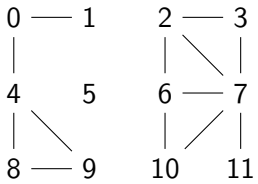


Graph with cycles



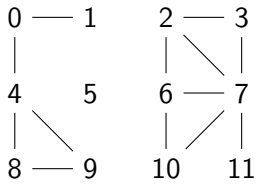
DFS tree

- Maintain a DFS counter, initially 0



DFS tree

- Maintain a DFS counter, initially 0
- Increment counter each time we start and finish exploring a node



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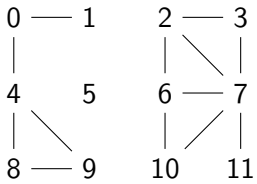

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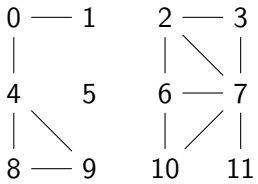

pre

0 0

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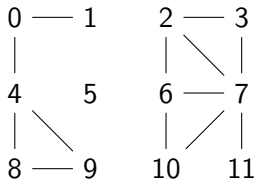

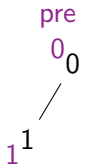

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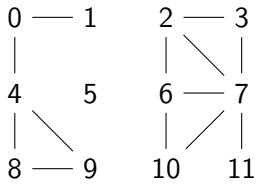

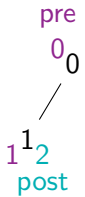

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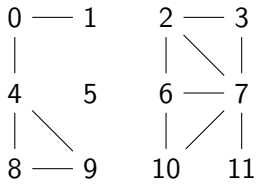

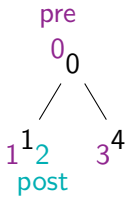

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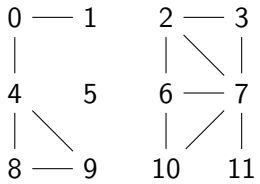

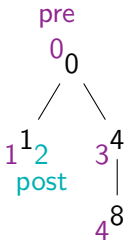

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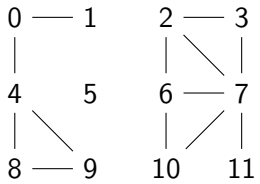

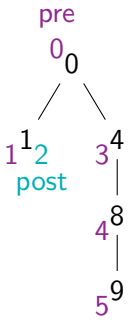

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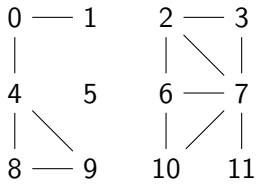

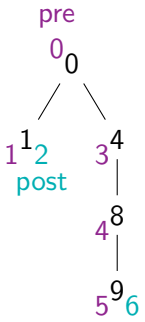

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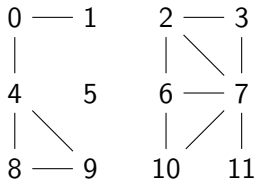

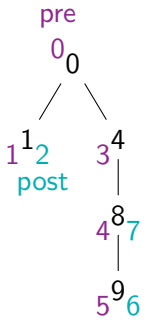

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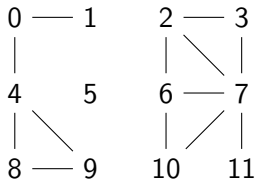

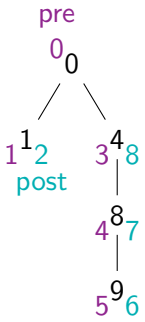

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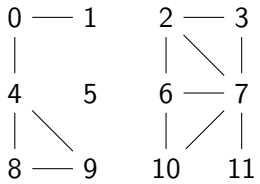

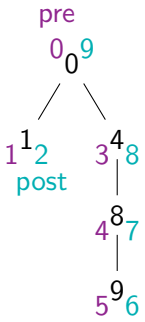

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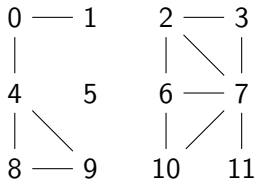

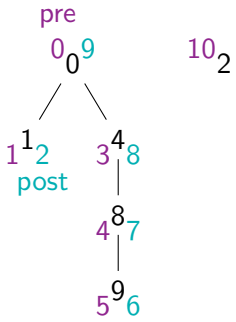

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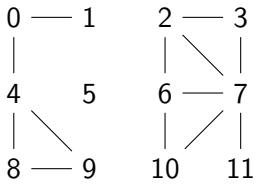

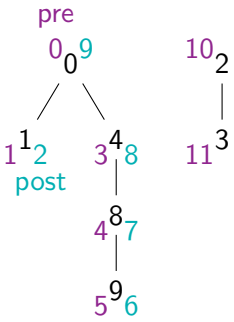

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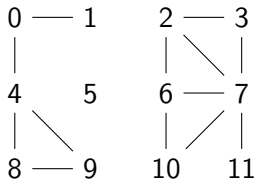

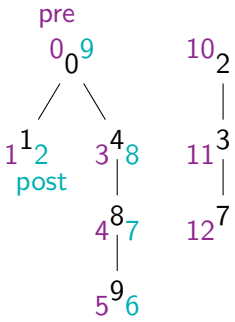

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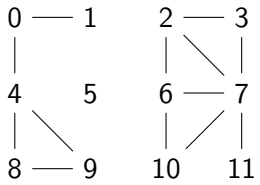

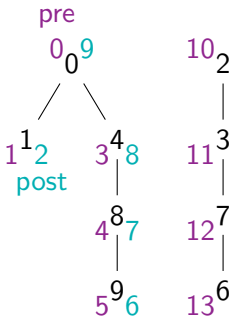

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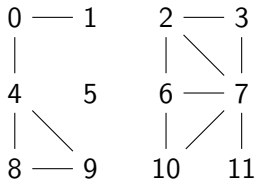

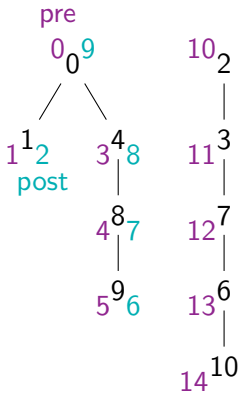

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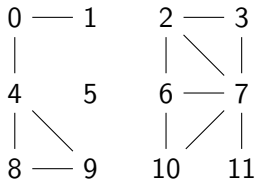

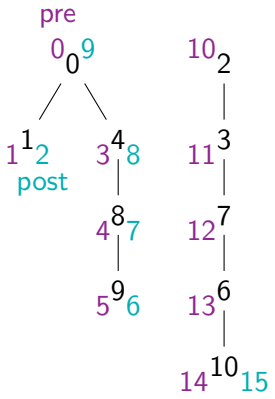

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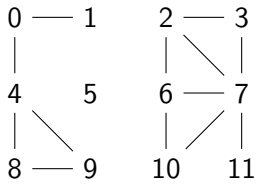

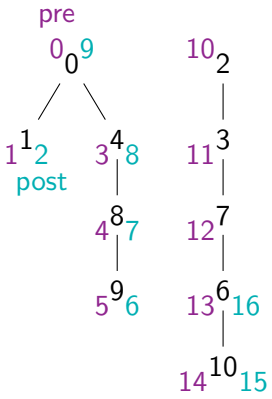

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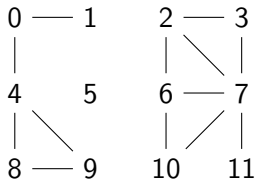

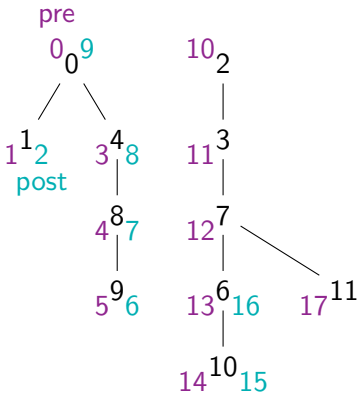

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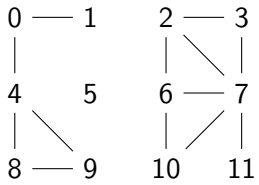

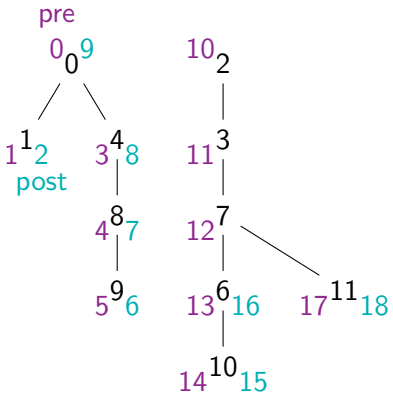

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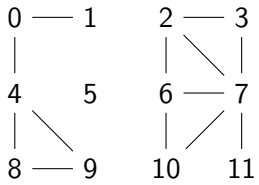

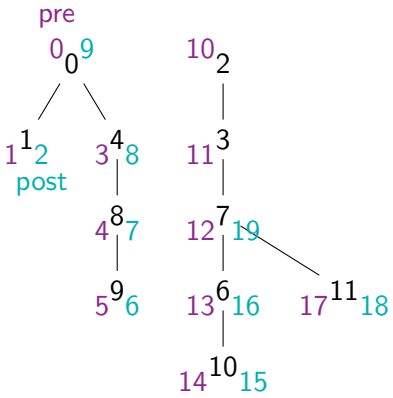

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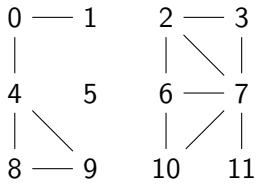

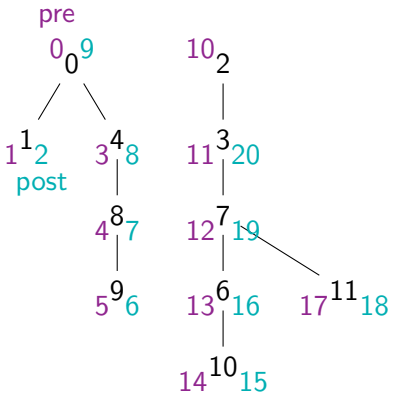

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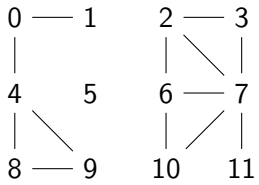

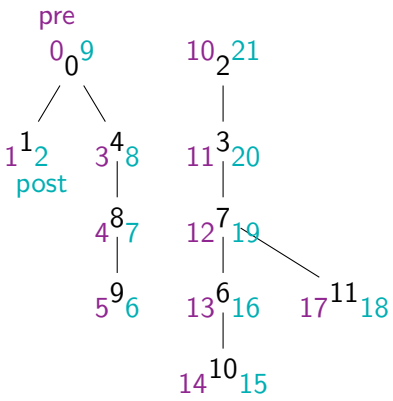

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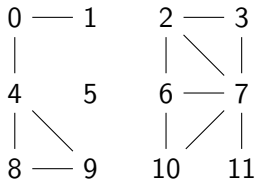
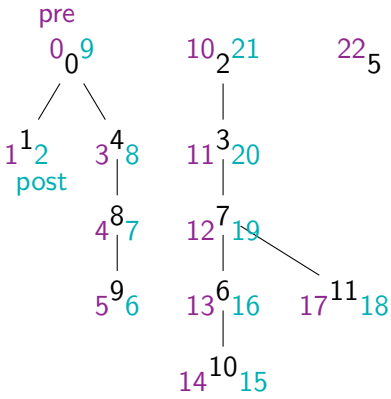
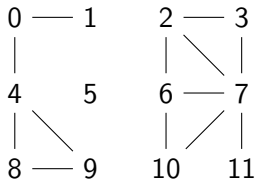
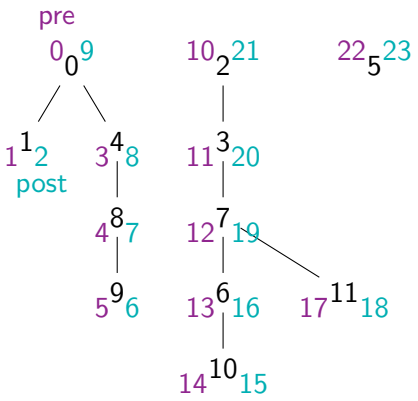




DFS tree

- Maintain a DFS counter, initially 0
- Increment counter each time we start and finish exploring a node
- Each vertex is assigned an entry number (pre) and exit number (post)





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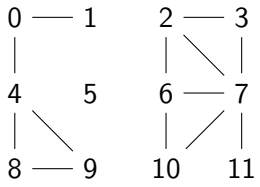
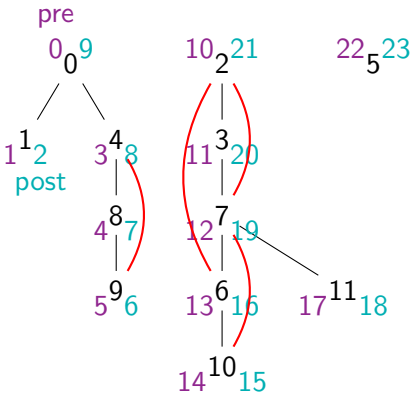


- As before, non-tree edges generate cycles



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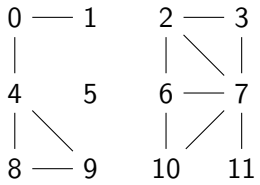


- As before, non-tree edges generate cycles
- To compute `pre` and `post` pass counter via recursive DFS calls

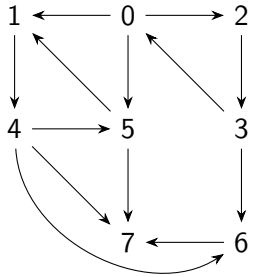
```
(visited,pre,post) = ({},{},{})

def DFSInitPrePost(AList):
  # Initialization
  for i in AList.keys():
    visited[i] = False
    pre[i],post[i]) = (-1,-1)
  return

def DFSPrePost(AList,v,count):
  visited[v] = True
  pre[v] = count
  count = count+1
  for k in AList[v]:
    if (not visited[k]):
      count = DFSPrePost(AList,k,count)
  post[v] = count
  count = count+1
  return(count)
```

Directed cycles

- In a directed graph, a cycle must follow same direction
  - $0 \rightarrow 2 \rightarrow 3 \rightarrow 0$ is a cycle
  - $0 \rightarrow 5 \rightarrow 1 \leftarrow 0$ is not



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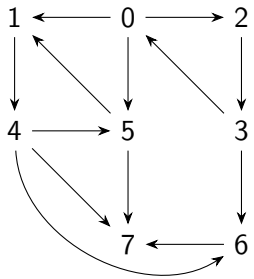

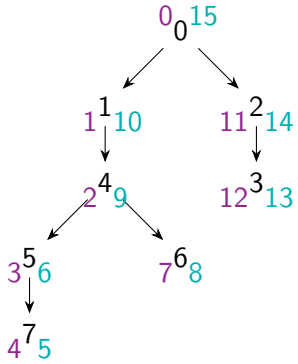

Directed cycles

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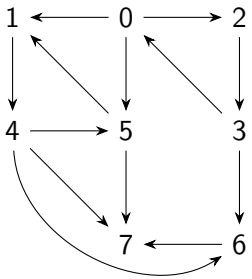
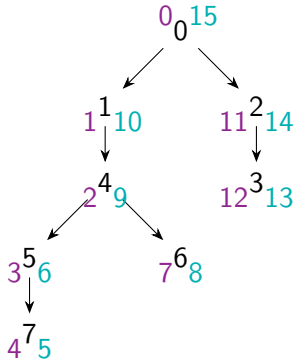
- Tree edges





Directed cycles

- In a directed graph, a cycle must follow same direction
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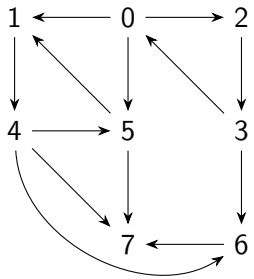
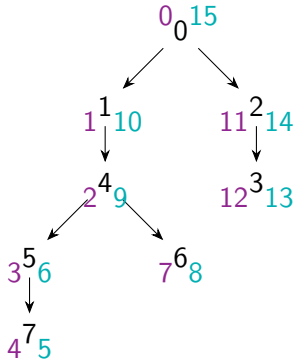
- Tree edges
- Different types of non-tree edges





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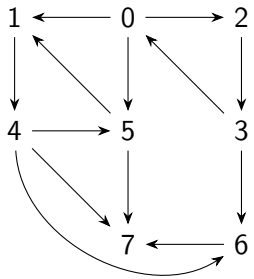
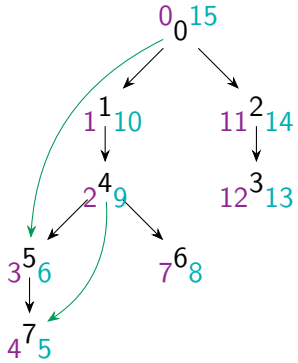
- Tree edges
- Different types of non-tree edges
  - Forward edges





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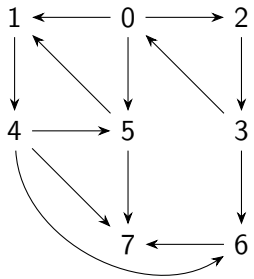

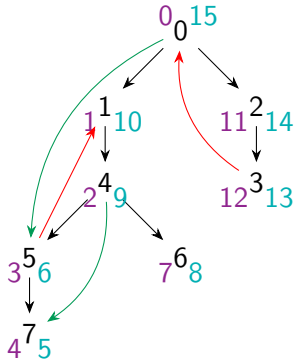

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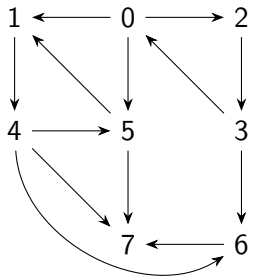

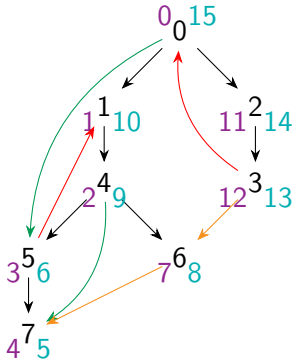

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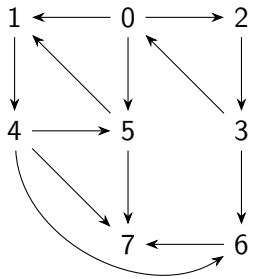
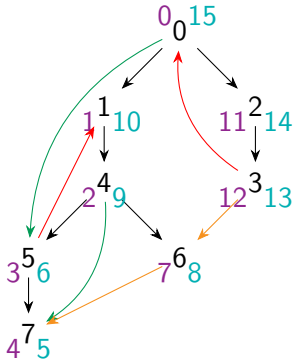
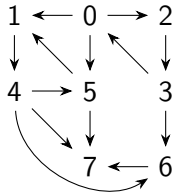


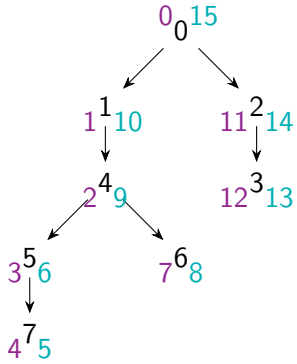
- Only back edges correspond to cycles



Classifying non-tree edges in directed graphs

- Use pre/post numbers





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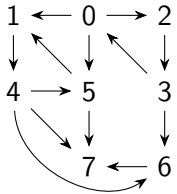
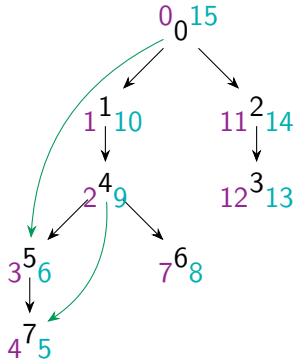
- Use pre/post numbers
- Tree edge/forward edge (u, v)

  Interval $[\text{pre}(u), \text{post}(u)]$ contains $[\text{pre}(v), \text{post}(v)]$





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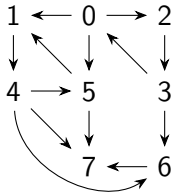

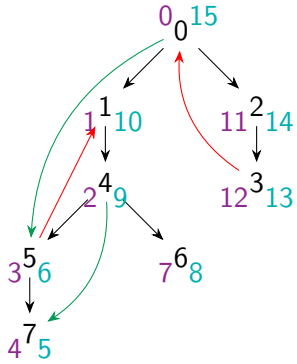

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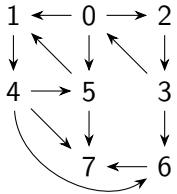
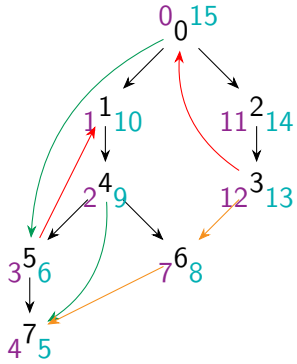
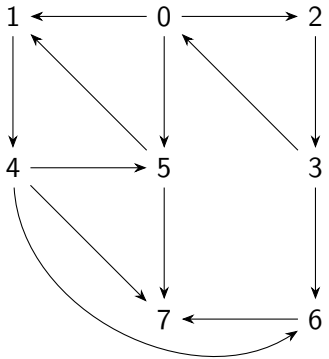
  Intervals $[\text{pre}(u), \text{post}(u)]$ and $[\text{pre}(v), \text{post}(v)]$ are disjoint





Connectivity in directed graphs

- Take directions into account



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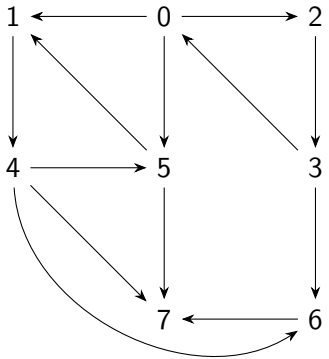

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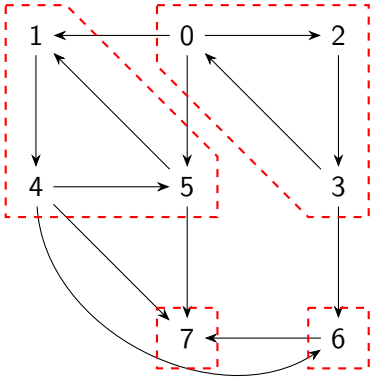

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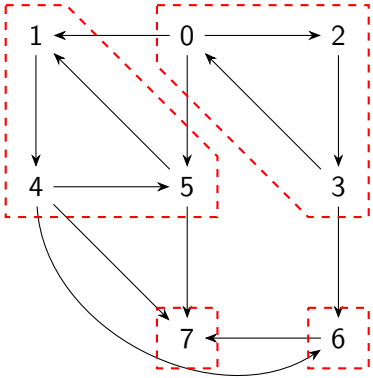

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- DFS numbering can also be used to identify other features such as articulation points (cut vertices) and bridges (cut edges)
- Directed acyclic graphs are useful for representing dependencies
  - Given course prerequisites, find a valid sequence to complete a programme