Unitary matrices:

Def: A matrix is unitary if it is square, and has orthonormal columns

Real case: $Q^T Q = I \iff Q$ is orthogonal and $Q^{-1} = Q^T$

$\iff \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$ then $q_i^T q_j = 0 \quad \forall i \neq j$ & $\|q_i\| = 1 \quad \forall i = 1 \ldots n$

Complex case: $U^* U = I \iff U$ is unitary, and $U^{-1} = U^*$

Example! ① $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ Check that U is unitary.

② $U = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ ———————— " ————————

Properties of unitary matrices: Let U be a unitary matrix, i.e., $U^* U = I$

(I) "Length unchanged" $\|Ux\| = \|x\|$

Pf: $Ux \cdot Uy = (Ux)^* Uy = x^* U^* U y = x^* y = x \cdot y$

$\Rightarrow \|Ux\|^2 = Ux \cdot Ux = x \cdot x = \|x\|^2$ $\blacksquare$

(II) Eigenvalues of a unitary matrix U have absolute value 1, i.e.,
if λ is an eigenvalue of U, then $|\lambda| = 1$.

Pf: $Ux = \lambda x$, $x \neq 0$. Also, $\|Ux\| = \|x\| \Rightarrow \|\lambda x\| = \|x\| \Rightarrow |\lambda| \, \|x\| = \|x\| \Rightarrow |\lambda| = 1$ since $\|x\| \neq 0$.

Another Pf: $Ux \cdot Ux = \lambda x \cdot \lambda x = \lambda\bar{\lambda}\,(x\cdot x) \underset{\uparrow}{=} (x\cdot x) \implies \lambda\bar{\lambda}=1 \implies |\lambda|=1.$

Since $\|Ux\|=\|x\|$

$\blacksquare$

III) Eigenvectors corresponding to different eigenvalues of a unitary matrix U are orthogonal.

Pf: $Ux = \lambda_1 x$, $\quad Uy = \lambda_2 y$, $\quad \lambda_1 \neq \lambda_2$

$$x\cdot y = Ux \cdot Uy = (\lambda_1 x)\cdot(\lambda_2 y) = \bar{\lambda}_1\lambda_2\,(x\cdot y)$$

$$\implies (\bar{\lambda}_1\lambda_2 - 1)(x\cdot y) = 0 \quad \text{—— } (*)$$

This would imply $x\cdot y = 0$ if we show $\bar{\lambda}_1\lambda_2 \neq 1$.

Suppose $\bar{\lambda}_1\lambda_2 = 1$.

Then, $\lambda_1\bar{\lambda}_1\lambda_2 = \lambda_1$

Using $|\lambda_1|=1$ or $\lambda_1\bar{\lambda}_1 = 1$, we get $\lambda_2 = \lambda_1$, a contradiction

So, $\bar{\lambda}_1\lambda_2 \neq 1$ implying $x\cdot y = 0$ from $(*)$

$\blacksquare$

Coming next: For a Hermitian matrix A, we can find a unitary matrix U s.t.

$A = U \Lambda U^*$, Λ is a diagonal matrix with eigenvalues of A.

Real Case: For a real symmetric matrix A, we can find an orthogonal matrix Q ($Q^TQ = I$) s.t.

Spectral Theorem

$$A = Q \Lambda Q^T$$