Similarity and diagonalization

Note Title 5/4/2021

Def: A matrix A is diagonalizable if $\exists$ an invertible matrix S such that

$$S^{-1} A S = \Lambda$$

diagonal matrix

"Matrix being diagonalizable is tied to having enough independent eigenvectors"

Suppose A is a $n \times n$ matrix with n linearly inpendent eigenvectors, say $\{x_1, \ldots x_n\}$ with corresponding eigenvalues $\{\lambda_1 \ldots \lambda_n\}$.
Then, A is diagonalizable. Why?

Let $S = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$

Is S invertible? Yes, because $\text{rank}(S) = n$ because $\{x_1, \ldots x_n\}$ is a linearly independent set

$$AS = A\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Ax_1 & \cdots & Ax_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \\ & & \end{bmatrix} = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= S\Lambda, \text{ where } \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

So, $\quad AS = S\Lambda \iff S^{-1}AS = \Lambda$

Claim: If λ_1, λ_2 are the eigenvalues with corresponding eigenvectors x_1, x_2, and $\lambda_1 \neq \lambda_2$.

Then, $\{x_1, x_2\}$ is a linearly independent set.

Pf:

Suppose $c_1 x_1 + c_2 x_2 = 0$ —— ①

$A(c_1 x_1 + c_2 x_2) = 0$

$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = 0$ —— ②

$c_1(\lambda_1 - \lambda_2) x_1 = 0$ $\leftarrow$ ② $- \lambda_2$ ①

$\lambda_1 \neq \lambda_2$, $x_1 \neq 0 \Rightarrow c_1 = 0$

Similarly, $c_2 = 0$

$\Rightarrow \{x_1, x_2\}$ is a linearly independent set

Extension of this claim: If $\lambda_1, \lambda_2, \ldots \lambda_n$ are distinct, then their eigenvectors $\{x_1 \ldots x_n\}$ are linearly independent

Pf: H.W.

Example: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$

Characteristic equation: $\lambda^2 - 4\lambda - 5 = 0$

Roots: $\lambda_1 = 5$, $\lambda_2 = -1$

Question: Is A diagonalizable? Yes, because it has 2 distinct eigenvalues

Finding eigenvectors: ① $\lambda_1 = 5$

$$A - \lambda_1 I = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A - \lambda_1 I)$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

② $\lambda_2 = -1$

$$A - \lambda_2 I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \in N(A - \lambda_2 I)$$

$$x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

Check: $S^{-1} A S = \Lambda$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

Remarks:

① $S^{-1}AS = \Lambda$ or $A = S\Lambda S^{-1}$

Is S unique? NO why? If x_1 is an eigenvector, then so is cx_1

So, the columns of S can be scaled to get S' such that $S'^{-1}AS' = \Lambda$

② $A = S\Lambda S^{-1}$

Suppose col. 1 of S is y

Col. 1 of $S\Lambda = \lambda_1 y$, Col. 1 of $AS = Ay$

We know $AS = S\Lambda$

$Ay = \lambda_1 y \Rightarrow \lambda_1$ is an eigenvalue & y is an eigenvector of A

③ **Powers of A:** Suppose λ is an eigenvalue and x is an eigenvector of A

$A^2 x = A(Ax) = \lambda A x = \lambda^2 x$

Suppose $S^{-1}AS = \Lambda$ Question: Is $S^{-1}A^2S = \Lambda^2$? Yes.

$$(S^{-1}AS)(S^{-1}AS) = (\Lambda)(\Lambda)$$
$$S^{-1}A^2S = \Lambda^2$$

The argument above works for a general $k \geq 1$, i.e.,

$$S^{-1}A^kS = \Lambda^k$$

(4) "Not all matrices are diagonalizable"

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Check: A does not have 2 linearly independent eigenvectors.