Machine Learning Foundations

Chapter 2: Calculus

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Outline

- **Sets and Functions**
  - **Notations**
  - **Logic**
  - **Graphs and visualisations.**
- Univariate Calculus
  - Continuity and differentiability
  - Derivatives and Linear approximations
  - Applications/Advanced rules
- Multivariate Calculus
  - Lines and planes in high dimensional space.
  - Partial derivatives
  - Gradients
  - Linear approximations and Alternate gradient interpretations
  - Applications/Advanced rules

Sets

$\mathbb{R}$ - set of real numbers

$\mathbb{R}_+$ - set of positive reals including 0

$\mathbb{Z}$ - set of Integers

$\mathbb{Z}_+$ - " " +ve Inte

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

$(a,b) : \{x \in \mathbb{R} : a < x < b\}$

$\mathbb{R}^d$: set of d-dimensional vectors $= \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$

$\begin{pmatrix} 1 \\ 2 \\ 3.3 \end{pmatrix} \in \mathbb{R}^3$

$[a,b]^d : \{x \in \mathbb{R}^d : x_i \in [a,b] \quad i \in \{1,2,\ldots d\}\}$

Metric Spaces

$\mathbb{R}^d$: $D(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2}$

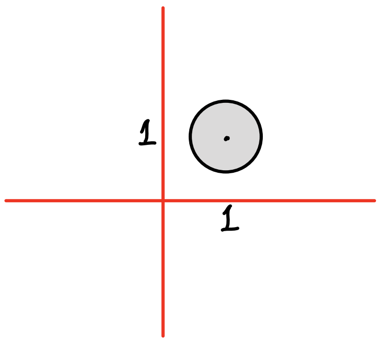
$x \in \mathbb{R}^d$. $B(x, \epsilon) = \{ y \in \mathbb{R}^d : D(x, y) < \epsilon \}$

$\bar{B}(x, \epsilon) = \{ y \in \mathbb{R}^d : D(x, y) \leq \epsilon \}$

$d = 2$

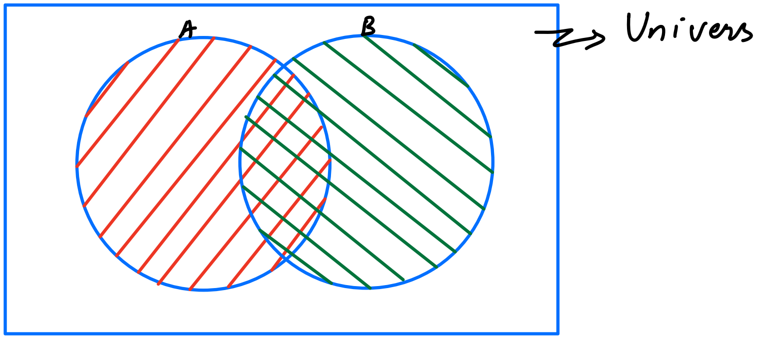
$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 0.5 \right)$



Sets and Logic

U = Universe



$A \cup B$, $A \cap B$, A^c, B^c

$A^c = U \setminus A$

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

$U = [0, 10]$

$A = [2, 5]$, $B = [4, 7]$; $A \cup B = [2, 7]$, $A \cap B = [4, 5]$

$(A \cup B)^c = [0, 2) \cup (7, 10] = A^c \cap B^c$

as $A^c = [0, 2) \cup (5, 10]$, $B^c = [0, 4) \cup (7, 10]$

$\forall$ For all $\qquad \Rightarrow$ Implies $\qquad A \Rightarrow B$

$\exists$ There exists $\qquad \Leftrightarrow$ Equivalent $\qquad A \Leftrightarrow B$

Sequences

$x_1, x_2, \ldots$

where $x_i \in \mathbb{R}^d$

$\lim_{i \to \infty} x_i = x^*$ $\qquad$ $x_i \to x^*$

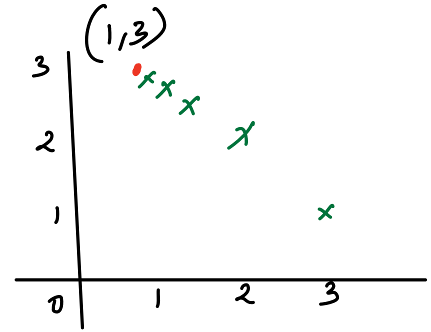
$\Updownarrow$

$\forall\ \epsilon > 0,\ \exists\ N$ s.t

$x_n \in B(x^*, \epsilon)\ \forall\ n \geq N$

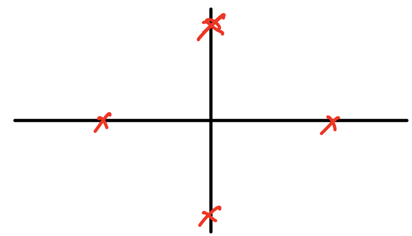
Example sequence 1

$$x_n = \left(1 + \frac{4}{2^n},\ 3 - \frac{4}{2^n}\right)$$



Example sequence 2

$$x_n = \left(\cos \frac{\pi}{2} n,\ \sin \frac{\pi}{2} n\right)$$



Sequences

Example:

(i) $x_i \in \mathbb{R}$; $x_n = 1+n$

(ii) $x_i \in \mathbb{R}^2$; $x_n = \left(\frac{1}{2^n} \cos\left(\frac{\pi}{2} n\right), \frac{1}{2^n} \sin\left(\frac{\pi}{2} n\right) \right)$

(iii) $x_i \in \mathbb{R}^2$; $x_n = \left(\frac{1}{2^n} \cos\left(\frac{\pi}{2} n\right), \sin\left(\frac{\pi}{2} n\right) \right)$

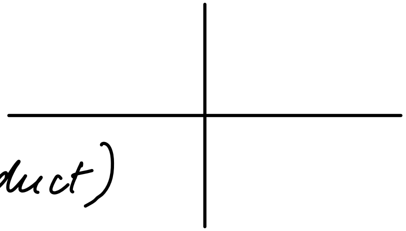
Vector Spaces

If V is a vector space

$u \in V, \quad v \in V \qquad \alpha, \beta \in \mathbb{R}$

$\alpha u + \beta v \in V$

- $\mathbb{R}^d$ is a vector space.



- $x \cdot y = x^T y = \sum_{i=1}^{d} x_i y_i$ (dot product)
- $\|x\|^2 = x \cdot x = x^T x = \sum_{i=1}^{d} x_i^2$
- x & y are perpendicular/orthogonal

$x \cdot y = x^T y = \sum x_i y_i = 0$

Functions and Graphs

$$f : \underbrace{A}_{\text{Domain}} \to \underbrace{B}_{\text{co-domain}}$$

1- dimensional function

$$f : \mathbb{R} \to \mathbb{R}$$

d - dimensional functions

$$f : \mathbb{R}^d \to \mathbb{R}$$

$$G_f \subseteq \mathbb{R}^{d+1}$$

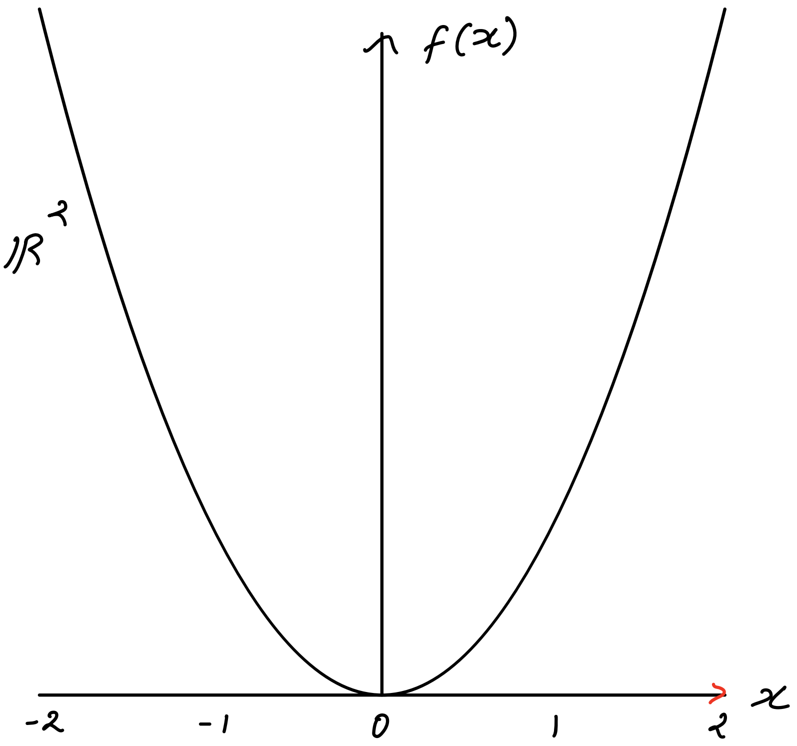
$$G_f = \{ (x, f(x)) : x \in \mathbb{R}^d \}$$

Plots of 1-dimensional Functions

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$G_f = \{ (x, x^2) : x \in \mathbb{R} \} \subseteq \mathbb{R}^2$



Contour Plots of 2-dimensional Functions

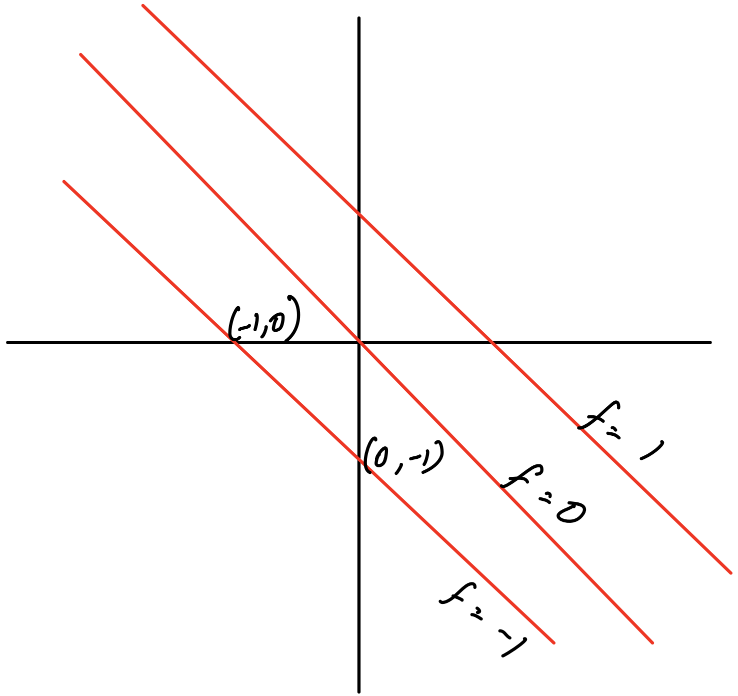
$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(x) = x_1 + x_2$$

$$\text{Values} = \{-1, 0, 1\}$$

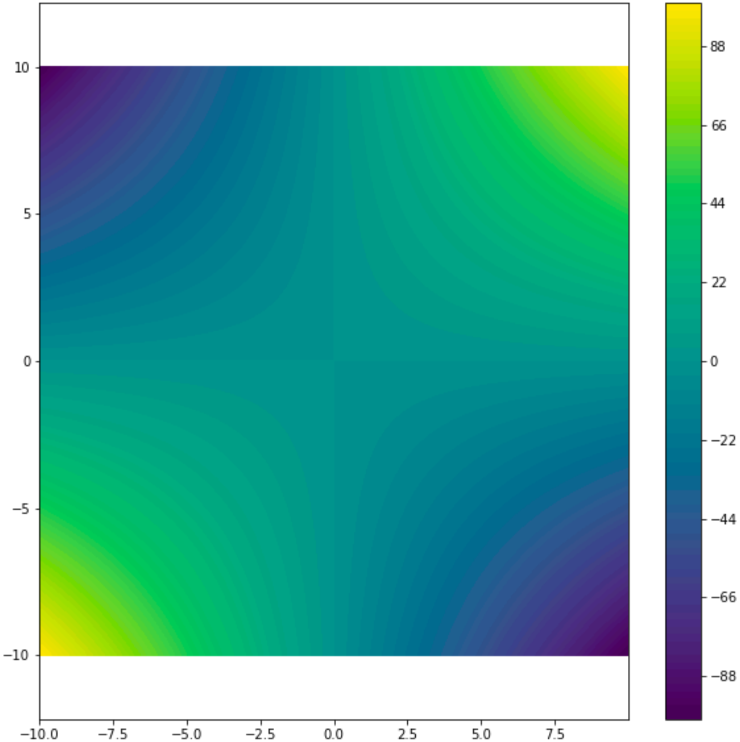
$$f(x) = -1 \Rightarrow x_1 = -x_2 - 1$$

$$f(x) = 0 \Rightarrow x_1 = -x_2$$

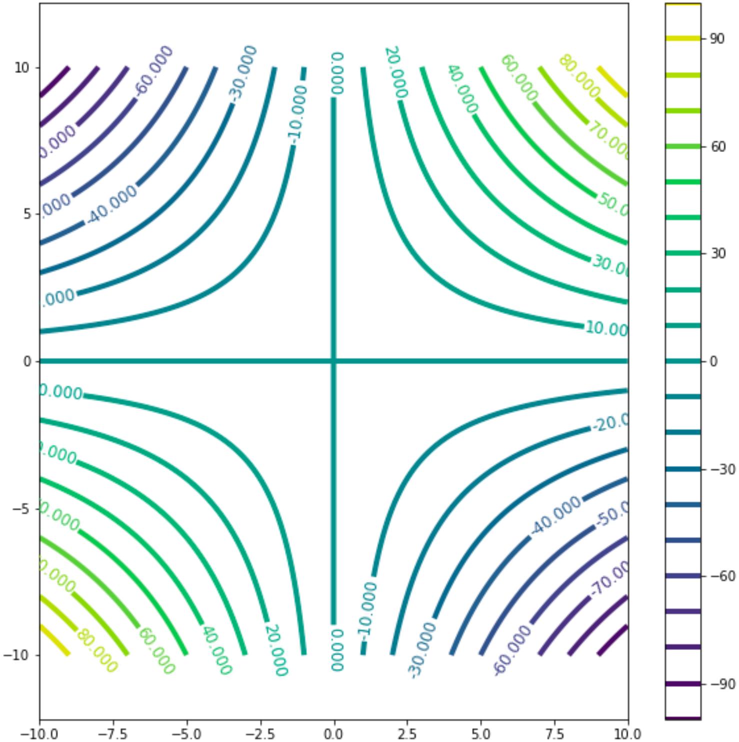


Contour Plots of 2-dimensional Functions

$f(x) = x_1 x_2$



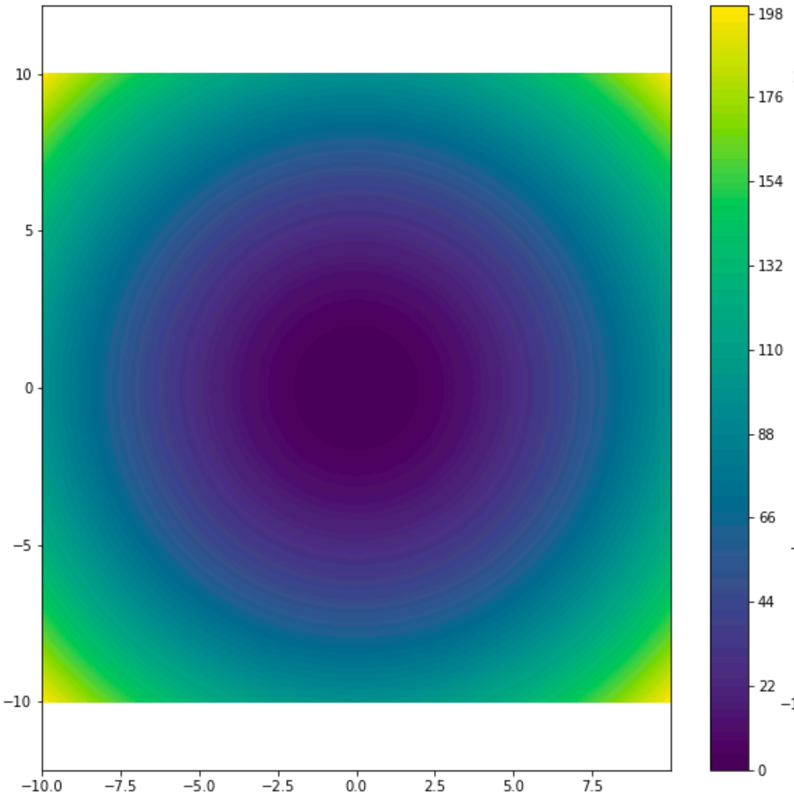
Heat map



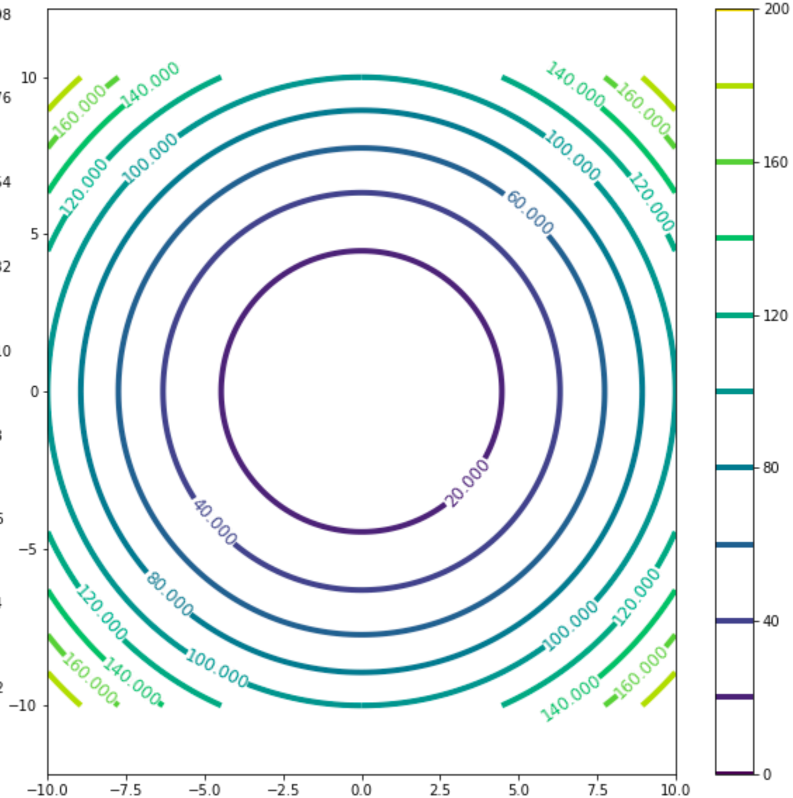
Contour map

Contour Plots of 2-dimensional Functions

$f(x_1, x_2) = x_1^2 + x_2^2$



Heat map



contour map.

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