Minimum Cost Spanning Trees: Kruskal's Algorithm

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Programming, Data Structures and Algorithms using Python

Week 5

Minimum cost spanning tree (MCST)

- Weighted undirected graph,
  $G = (V, E), W : E \to \mathbb{R}$
  - G assumed to be connected
- Find a minimum cost spanning tree
  - Tree connecting all vertices in V

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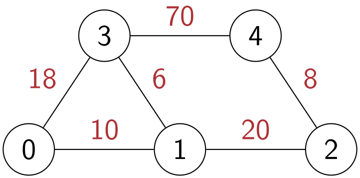

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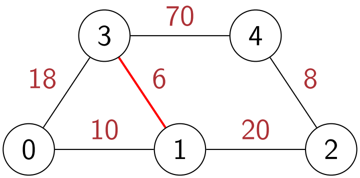

- Start with smallest edge, $(1, 3)$

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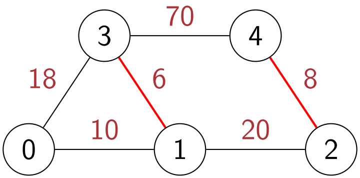

- Start with smallest edge, $(1, 3)$
- Add next smallest edge, $(2, 4)$

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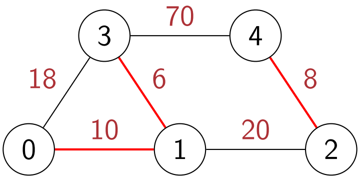

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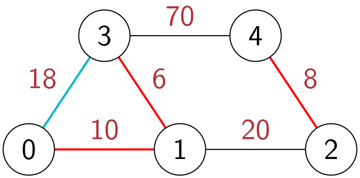

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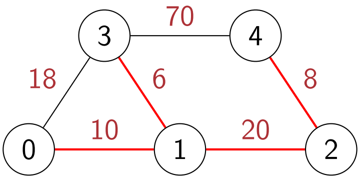

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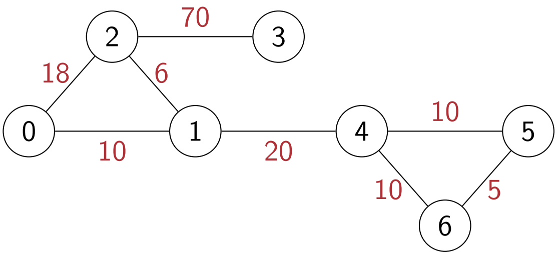

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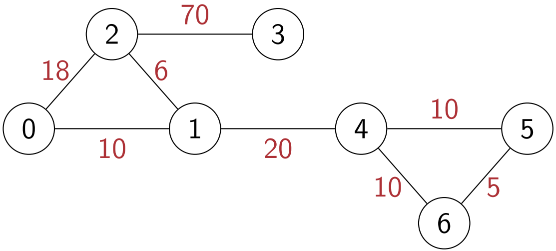

Sort E as

$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Set $TE = \emptyset$

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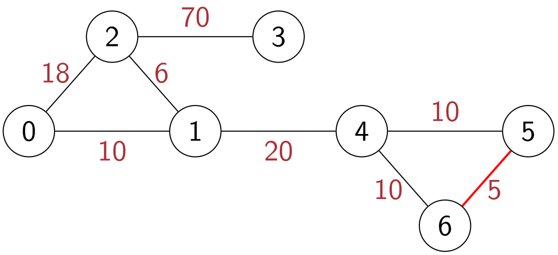

Sort E as
$\{(5,6), (1,2), (0,1), (4,5), (4,6), (0,2), (1,4), (2,3)\}$

Add $(5,6)$

Set $TE = \{(5,6)\}$

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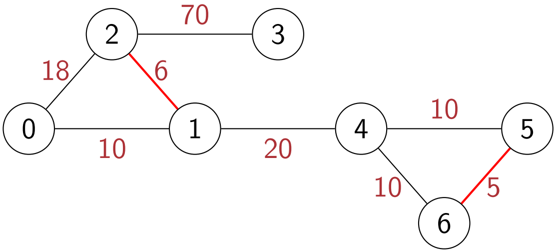

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Add $(1,2)$

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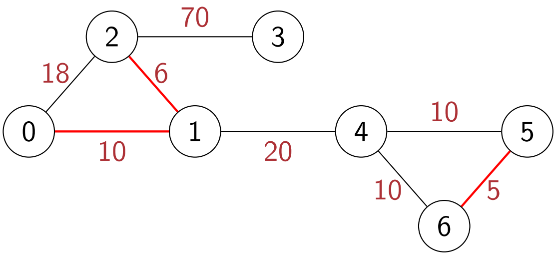

Sort E as
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Add $(0,1)$

Set $TE = \{(5,6),(1,2),(0,1)\}$

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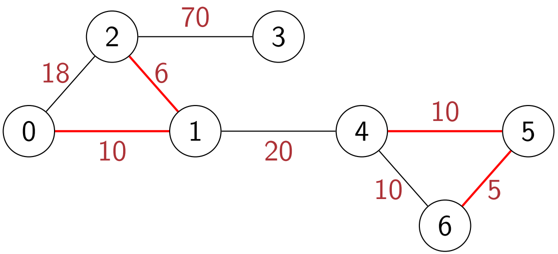

Sort E as
$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(4,5)$

Set $TE = \{(5,6),(1,2),(0,1),(4,5)\}$

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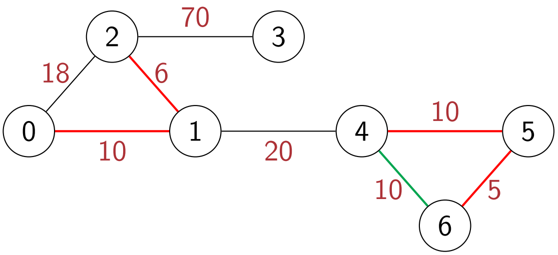

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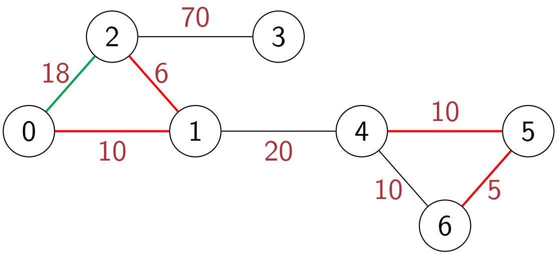

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Skip $(0,2)$

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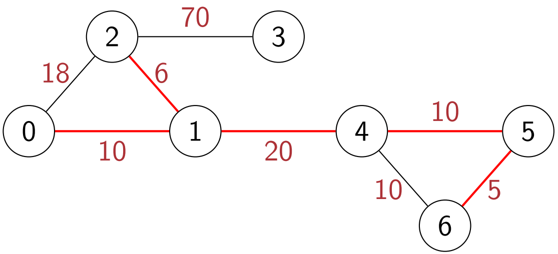

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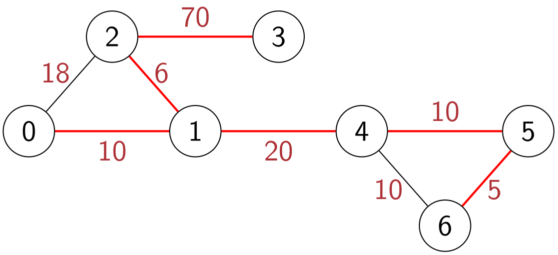

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Add $(2,3)$

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Correctness of Krusksal's algorithm

Minimum Separator Lemma

- Let V be partitioned into two non-empty sets U and $W = V \setminus U$
- Let $e = (u, w)$ be the minimum cost edge with $u \in U$, $w \in W$
- Every MCST must include e

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- Adding $e = (u, w)$ merges components of u and w
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- Let U be component of u, W be $V \setminus U$
  - U, W form a partition of V with $u \in U$ and $w \in W$
  - Since we are scanning edges in ascending order of cost, e is minimum cost edge connecting U and W, so it must be part of any MCST

Implementing Kruskal's algorithm

- Collect edges in a list as `(d,u,v)`
  - Weight as first component for easy sorting

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def kruskal(WList):
    (edges,component,TE) = ([],{},[])
    for u in WList.keys():
        # Weight as first component to sort easily
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    for (d,u,v) in edges:
        if component[u] != component[v]:
            TE.append((u,v))
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Implementing Kruskal's algorithm

- Collect edges in a list as `(d,u,v)`
  - Weight as first component for easy sorting
- Main challenge is to keep track of connected components
  - Dictionary to record component of each vertex
  - Initially each vertex is an isolated component
  - When we add an edge `(u,v)`, merge the components of `u` and `v`

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- Sorting the edges is $O(m \log m)$
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  - Each time we add a tree edge, we have to merge components — $O(n)$ scan
  - $n - 1$ tree edges, so this is done $O(n)$ times

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- Overall, $O(n^2)$

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- Complexity is $O(n^2)$
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- Data structure to maintain collection of disjoint sets
  - `find(v)` — return set containing `v`
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- Efficient union-find brings complexity down to $O(m \log n)$

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Summary

- Kruskal's algorithm builds an MCST bottom up
  - Start with n components, each an isolated vertex
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- In general, there may be a very large number of minimum cost spanning trees