Greedy Algorithms: Minimizing Lateness

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Programming, Data Structures and Algorithms using Python

Week 7

Greedy Algorithms

- Make a sequence of local choices to achieve a global optimum
- Never go back and revise an earlier decision
- How to prove that local choices achieve global optimum?

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- Greedy solution and optimal may not be identical — interval scheduling
- Incrementally show that the greedy solution is at least as good as an optimal one
- The greedy algorithm "stays ahead" of the optimal

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- Incrementally show that the greedy solution is at least as good as an optimal one
- The greedy algorithm "stays ahead" of the optimal

Strategy 2

- Greedy solution and optimal have a common structure
- Transform the optimal solution to match the greedy one, preserving optimality

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- Goal Minimize the maximum lateness

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  - $T(1) = 1$, $D(1) = 100$
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Strategy 3 Schedule requests in increasing order of deadlines — $D(i)$

- This works, but how do you prove it is correct?

Correctness

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  - $D(1) \leq D(2) \leq \cdots \leq D(n)$

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  - Job n starts at $S(n) = F(n-1)$, ends at $F(n) = S(n) + T(n)$

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  - 3D printer is continuously in use from $S(1) = 0$ to $F(n)$
  - No idle time

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- Eliminate idle time by shifting jobs earlier
- Can only reduce lateness

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Exchange argument

- Let O be some other optimal schedule
- Transform O to be identical to greedy schedule $1, 2, \ldots, n$

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- No inversions, idle time — only difference can be the order of requests with same deadline
- Reordering jobs with same deadline produces same lateness

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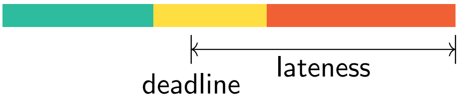

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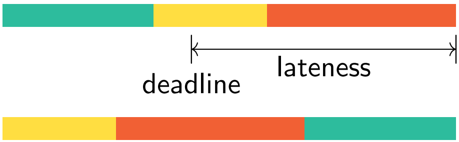

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A. If O has an inversion, there are jobs i, j such that j is scheduled immediately after i and $D(j) < D(i)$
   - Find the first point where deadline decreases

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Let O be an optimal schedule with no idle time

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B. Swap i and j to get one less inversion
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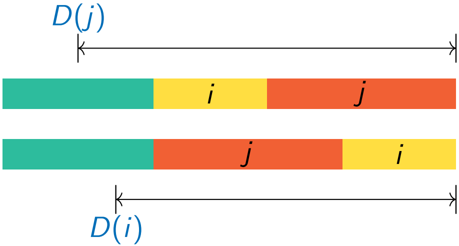
   - Recall that $D(j) < D(i)$



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- From $\boxed{C}$ we can remove each adjacent inversion without increasing lateness

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- From $\boxed{C}$ we can remove each adjacent inversion without increasing lateness
- At most $n(n-1)/2$ inversions in O to start with

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Let O be an optimal schedule with no idle time

- A. If O has an inversion, there are jobs i, j such that j is scheduled immediately after i and $D(j) < D(i)$
- B. Swap i and j to get one less inversion
- C. Swapping i and j does not increase lateness of O

- From $\boxed{C}$ we can remove each adjacent inversion without increasing lateness
- At most $n(n-1)/2$ inversions in O to start with
- Repeatedly remove adjacent inversions to get an optimal schedule with no inversions, no idle time

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  - Sort the requests by $D(i)$ — $O(n \log n)$
  - Read off schedule in sorted order — $O(n)$

Summary

- Schedule requests with start times $T(i)$, deadlines $D(i)$, to minimize maximum lateness
- Simple greedy algorithm with complexity $O(n \log n)$
  - Sort the requests by $D(i)$ — $O(n \log n)$
  - Read off schedule in sorted order — $O(n)$
- Correctness follows from an "exchange argument"
  - Consider any optimal solution O
  - Transform it, step by step, to be equal to the greedy solution