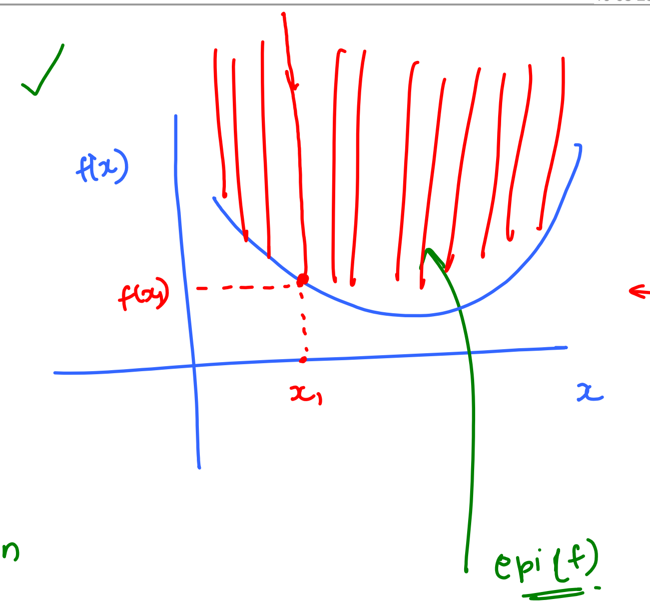
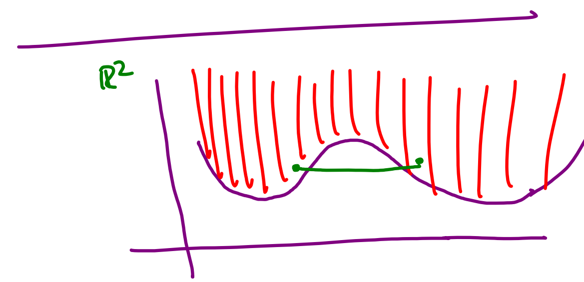
Convex functions

$f : \mathbb{R}^d \to \mathbb{R}.$ any convex set.



$$\text{epi}(f) = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+1} \; : \; z \geq f(x) \right\}$$



Definition :- A function $f : \mathbb{R}^d \to \mathbb{R}$ is a convex function if $\text{epi}(f) \subseteq \mathbb{R}^{d+1}$ is a convex set.

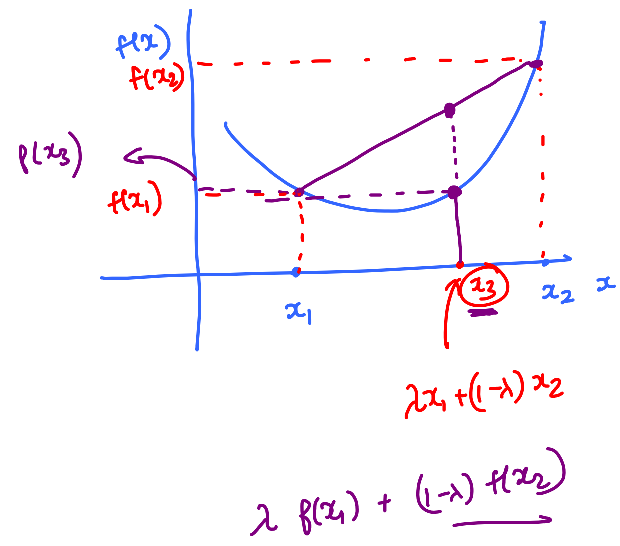
Note Title 16-08-2021

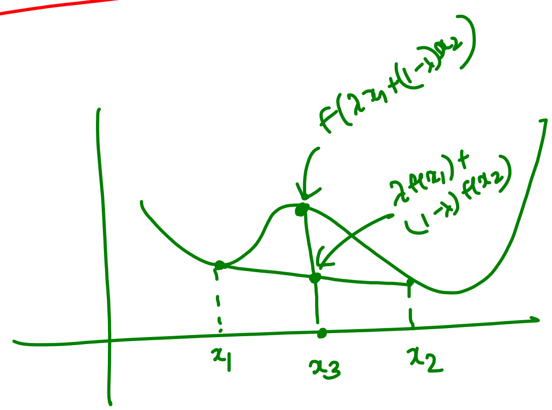
Alternate definition (2)

A function $f: \mathbb{R}^d \to \mathbb{R}$ is convex iff

$\forall\ x_1, x_2 \in \mathbb{R}^d$ and all $\lambda \in [0,1]$

$$\boxed{f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)}$$





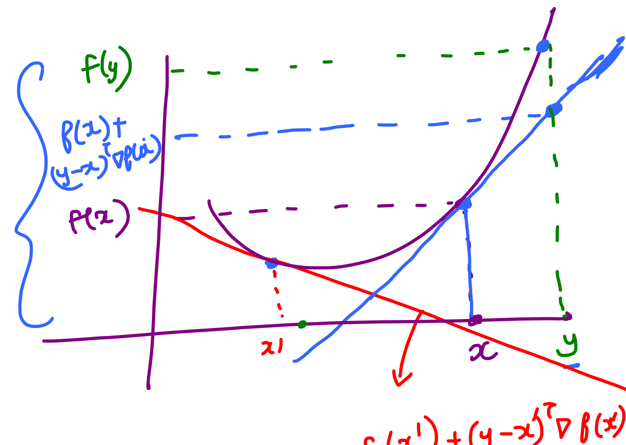
$\Rightarrow$ Not convex.

Definition ③

→ Assume $f: \mathbb{R}^d \to \mathbb{R}$ is differentiable.

f is convex if and only if

$$\boxed{f(y) \geq f(x) + (y-x)^T \nabla f(x)}$$



$$f(x+\epsilon d) = f(x) + \epsilon d^T \nabla f(x) + \underbrace{\cdots}_{\text{higher order terms.}}$$

$$f(y) \geq \underbrace{f(x) + (y-x)^T \nabla f(x)}_{} + \boxed{\cdots}$$

$$f(x) + \epsilon d^T \nabla f(x)$$

$y = x + \epsilon d.$

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Definition 4

f is twice differentiable.

$f: \mathbb{R}^d \to \mathbb{R}.$

$f(x_1, \dots, x_d) \to \mathbb{R}.$

$H \in \mathbb{R}^{d \times d}$

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

f is convex if and only if

H is a p.s.d matrix

$\hookrightarrow$ positive-semi-definite.

eigenvalues $(H) \geq 0.$

Example:

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$f''(x) = 2 > 0 \Rightarrow \text{f is convex!}$$