Dynamic Programming

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Programming, Data Structures and Algorithms using Python

Week 9

Inductive definitions

- Factorial
  - $fact(0) = 1$
  - $fact(n) = n \times fact(n-1)$

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Inductive definitions . . . recursive programs

- Factorial
  - $fact(0) = 1$
  - $fact(n) = n \times fact(n-1)$

```
def fact(n):
  if n <= 0:
    return(1)
  else:
    return(n * fact(n-1))
```

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```
def isort(l):
  if l == []:
    return(l)
  else:
    return(insert(isort(l[:-1]),l[-1]))
```

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- $isort([x_0, x_1, \ldots, x_{n-1}])$ is a subproblem of $isort([x_0, x_1, \ldots, x_n])$
  - So is $isort([x_i, \ldots, x_j])$ for any $0 \leq i < j \leq n$
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Interval scheduling

- IIT Madras has a special video classroom for delivering online lectures
- Different teachers want to book the classroom
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- Slots may overlap, so not all bookings can be honoured
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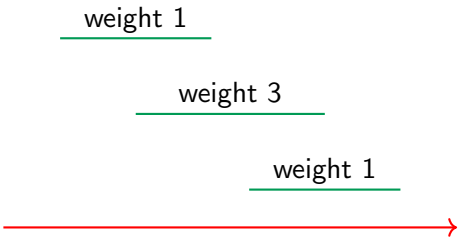

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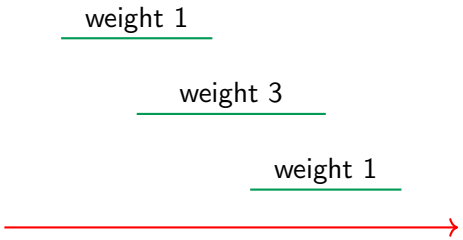

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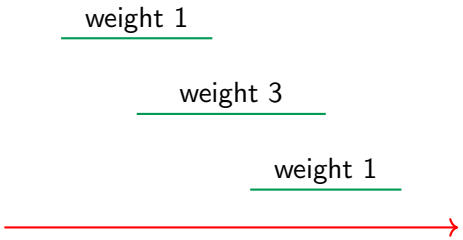

- Search for another greedy strategy that works ...
- ... or look for an inductive solution that is "obviously" correct

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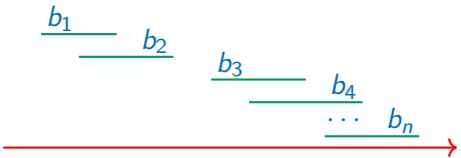
- b_1 is in conflict with b_2, but both are compatible with $b_3, b_4, \ldots, b_n$



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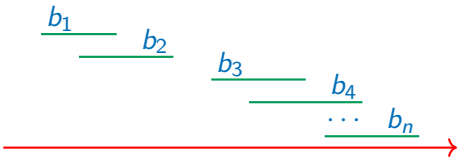

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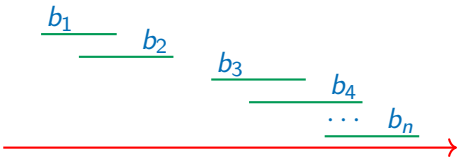

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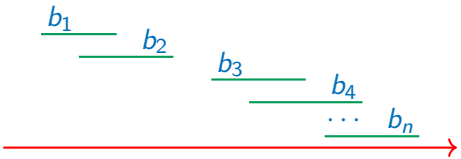

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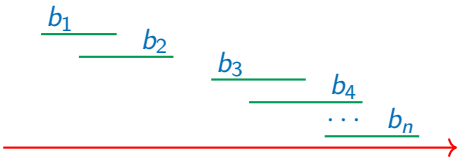

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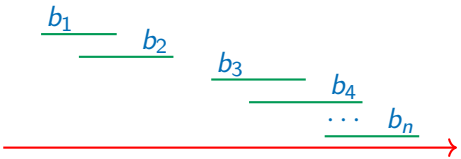

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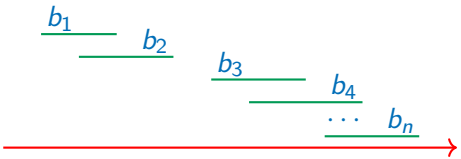

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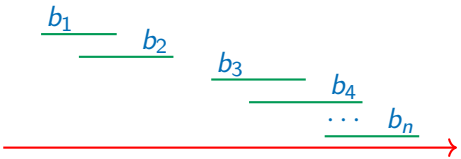

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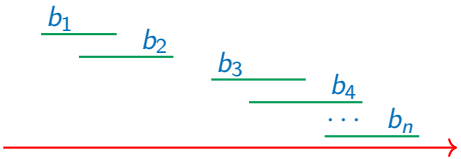

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- Memoization and dynamic programming