Relation between Primal and the Dual problem.

$x^* \rightarrow$ $\boxed{\min_{x} \left[\max_{\lambda \geq 0} L(x, \lambda) \right]}$

Primal

$\boxed{\max_{\lambda \geq 0} \min_{x} L(x, \lambda)} \leftarrow \lambda^*$

DUAL

$$J(x) = \begin{cases} \underline{f(x)} & \text{if } \underline{h(x) \leq 0} \\ \infty & \text{otherwise} \end{cases}$$

For any $\underline{\lambda \geq 0}$

$$\underline{L(x, \lambda)} = f(x) + \underline{\lambda}\, \underline{h(x)} \qquad \leq \underline{f(x)} \quad \text{if } h(x) \leq 0$$

$$\leq \infty$$

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Fix $\lambda \geq 0$ $\quad L(x,\lambda) \leq J(x) \quad \forall x$

$$\min_x L(x,\lambda) \leq \min_x J(x) = f(x^*)$$

solution of the primal problem

$$\boxed{\max_{\lambda \geq 0} \min_x L(x,\lambda) \leq f(x^*)}$$

Primal objective.

$$\max_{\lambda \geq 0} [g(\lambda)] = \boxed{g(\lambda^*) \leq f(x^*)}$$

Value at dual optimum $\leq$ value at primal optimum.

WEAK - DUALITY

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If f, h are convex, then STRONG-DUALITY holds!

$\uparrow$ Objective $\uparrow$ Constraint

[upto some regularity]

$$\min_{x} \left[\max_{\lambda \geq 0} L(x, \lambda) \right] \quad (\text{or}) \quad \underbrace{\max_{\lambda \geq 0} \left[\min_{x} L(x, \lambda) \right]}$$

Assume f, h are convex $\Rightarrow$ Strong duality holds.

Let x^*, λ^* are the primal and dual optimal solutions

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$$x^* = \underset{x}{\text{argmin}} \left[\underbrace{\max_{\lambda \geq 0} f(x) + \lambda h(x)} \right] \quad ; \quad \lambda^* = \underset{\lambda \geq 0}{\text{argmax}} \left[\underbrace{\min_{x} f(x) + \underline{\lambda} h(x)}_{g(\lambda)} \right]$$

By strong duality.

$$\underline{f(x^*)} = g(\lambda^*)$$

$$= \underline{\min_{x} \left[\underbrace{f(x) + \lambda^* h(x)} \right]}$$

$$\Rightarrow \boxed{\nabla f(x^*) + \lambda^* \nabla h(x^*) = 0} \quad — \quad c1$$

$$f(x^*) = g(\lambda^*) \quad [\text{by strong duality}]$$

$$= \underbrace{\min_{x} f(x) + \lambda^* h(x)}$$

$$f(x^*) \leq \underbrace{f(x^*) + \lambda^* h(x^*)} \leq f(x^*)$$

$$\Rightarrow \boxed{\lambda^* h(x^*) = 0} \longrightarrow C2$$

Putting it all together

f, h are convex $\Rightarrow$ Strong duality

$\Downarrow$

x^*, λ^* must satisfy. [modulo Regularity conditions]

(a) $\nabla f(x^*) + \lambda^* \nabla h(x^*) = 0$ [Stationarity condition]. ✓

(b) $\lambda^* h(x^*) = 0$ [Complementary slackness condition] ✓

(c) $h(x^*) \leq 0$ [Primal feasibility]. ✓

(d) $\lambda^* \geq 0$ [Dual feasibility] ✓

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[Sufficiency part] In general if (x^*, λ^*) satisfies the above conditions. $\Rightarrow$ Local optima.

[and some regularity conditions]

KKT conditions [Karush-Kuhn-Tucker]

$$\min f(x)$$
$$h_i(x) \le 0 \quad \forall i = 1, \dots m$$
$$\ell_j(x) = 0 \quad \forall j = 1, \dots, n$$

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{n} v_j \ell_j(x)$$

x: vector, $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

(a) $\nabla f(x^*) + \sum_{i=1}^{m} u_i^* \nabla h_i(x^*) + \sum_{j=1}^{n} v_j^* \nabla \ell_j(x^*) = 0 \in$ vector. $\hookrightarrow$ [Stationarity]

(b) $u_i^* h_i(x^*) = 0 \quad \forall i$ [C.S condition]

(c) $h_i(x^*) \le 0 \quad \forall i$; $\ell_j(x^*) = 0 \quad \forall j$ [Primal Feasibility]

(d) $u_i^* \ge 0 \quad \forall i$ [Dual feasibility]