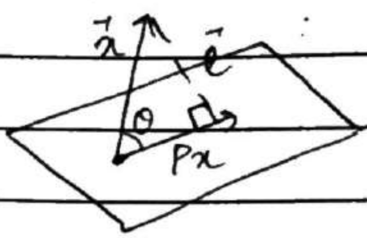
Q1.) Projection matrix P : projection onto a plane



i) $Px = x$ for any x that lies on the plane

if taking $\cos(\theta) = \frac{Px}{x} = 1$ → here x must be NON-ZERO

Since $Px = x$.

then $\boxed{\theta = 0^\circ}$

hence $\vec{x}$ lies in column space of P

$\Rightarrow \boxed{\lambda = 1}$ and any vector x in plane is eigenvector.

ii) $Px = 0$ if $x \perp^r$ to plane.

if taking $\cos(\theta) = \frac{Px}{x} = \frac{0}{x} = 0$

given $x \neq 0$.

then $\cos(\theta) = \frac{0}{x} = 0$

$\boxed{\theta = \cos^{-1}(0) = 90^\circ}$

hence $\vec{x}$ orthogonal to column space of P.

So $\boxed{\lambda = 0}$ and any $x \perp^r$ to plane is eigenvector.

Summary: $Px = \lambda x$ (given $\vec{x} \neq 0$)

$Px = \boxed{1}x$ if x lies in column space of P

$Px = \boxed{0}x$ if $x \perp^r$ to column space of P

Q2.) Properties of eigenvalues when $Ax = \lambda x$

Trace of matrix A : sum of eigenvalues

Determinant of matrix A : product of eigenvalues.

Since $\lambda_1 = 0$, $\lambda_2 = 3$

$$\text{Determinant}(A_{2\times2}) = \lambda_1 \cdot \lambda_2 = 0 \times 3 = \boxed{0}$$

$\therefore$ Ans: Option B (0)

Q3.) As above mentioned:

$$\text{Trace}(A_{2\times2}) = \lambda_1 + \lambda_2 = 4 \quad \text{— (1)}$$

$$\text{Det}(A_{2\times2}) = \lambda_1 * \lambda_2 = 3 \quad \text{— (2)}$$

From (1), $\lambda_2 = 4 - \lambda_1$.

Substituting in (2), we get

$$\lambda_1 * (4 - \lambda_1) = 3$$

$$4\lambda_1 - (\lambda_1)^2 = 3$$

Rearranging the terms:

$$\lambda_1^2 - 4\lambda_1 + 3 = 0$$

$$\lambda_1^2 - \lambda_1 - 3\lambda_1 + 3 = 0$$

$$\lambda_1(\lambda_1 - 1) - 3(\lambda_1 - 1) = 0$$

$$(\lambda_1 - 1)(\lambda_1 - 3) = 0$$

Case (1) if $\underline{\lambda_1 = 1}$;

$$\lambda_2 = 4 - \lambda_1 = 4 - 1 = \boxed{3}$$

Case (2) if $\underline{\lambda_1 = 3}$

$$\lambda_2 = 4 - \lambda_1 = 4 - 3 = \boxed{1}$$

$\therefore$ Eigenvalues of A are $\boxed{1, 3}$

Ans = Option B $(1, 3)$

Q4.) Given : $\lambda_1 = -1$; $\lambda_2 = 3$; $\lambda_3 = 4$

Trace of matrix = Sum of Eigenvalues

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3$

Trace = $-1 + 3 + 4$

$\boxed{\text{Trace} = 6}$

Q5.) Given: $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 4$

Determinant of matrix = Product of Eigenvalues

$\Rightarrow \lambda_1 * \lambda_2 * \lambda_3$

$\underline{\text{Det}} = (-1) * (3) * (4)$

$\boxed{\text{Det} = -12}$

Q6.) General way to find Characteristic polynomial :

determinant $(A - \lambda I) = 0$ (OR)

$|A - \lambda I| = 0$

For a 2x2 matrix $\Rightarrow$

$$\lambda^2 - [\text{trace}(A)]\lambda + \det(A) = 0$$

For a 3x3 matrix $\Rightarrow$

$$\lambda^3 - [\text{trace}(A)]\lambda^2 + \begin{bmatrix}\text{Minors of} \\ \text{diagonal of A}\end{bmatrix}\lambda - \det(A) = 0$$

trace - sum of diagonal elements

For $A_{2\times 2} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Characteristic polynomial :

$$\lambda^2 - [\text{trace}(A)]\lambda + \det(A) = 0$$

$$\lambda^2 - (3+1)\lambda + (3*1 - 2*2) = 0$$

$$\boxed{\lambda^2 - 4\lambda - 1 = 0}$$

Which corresponds to Option (A).

Q7) Eigenvalues of $A_{2\times2} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

is found by solving its characteristic polynomial as derived in (Q6).

$$\lambda^2 - 4\lambda - 1 = 0 \Rightarrow a\lambda^2 + b\lambda + c = 0$$

Using Quadratic formula: $\left\{ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right\}$

$$\lambda = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$

$$\lambda = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm 2\sqrt{5}}{2}$$

$$\lambda = 2 \pm \sqrt{5}$$

$\therefore$ Eigenvalues of A are $2+\sqrt{5}$; $2-\sqrt{5}$ corresponding to option (A)

Q8) We know ~~that~~

$$A^k x = \lambda^k x$$

Proof: $Ax = \lambda x$

$$A \cdot Ax = A \cdot \lambda x$$

$$A^2 x = \lambda \cdot Ax$$

$$A^2 x = \lambda \cdot (\lambda x)$$

$$\Rightarrow A^2 x = \lambda^2 x$$

generalizing this :

$$A^k x = \lambda^k x$$

If $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$

1) $Ax_1 = \lambda_1 x_1$

$A^2 x_1 = \lambda_1^2 x_1 = (1)^2 x_1 = \boxed{1} x_1$

2) $Ax_2 = \lambda_2 x_2$

$A^2 x_2 = (-2)^2 x_2 = \boxed{4} x_2$

3) $Ax_3 = \lambda_3 x_3$

$A^2 x_3 = \lambda_3^2 x_3 = (3)^2 x_3 = \boxed{9} x_3$

$\therefore$ Eigenvalues of A^2 are 1, 4 and 9 corresponding to option (B).

Q9) We know that:

$$F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \underbrace{\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k}_{\text{negligible for large } k}$$

In this question, since $k = 90$ (large) so we ignore the second term.

$$\therefore \boxed{F_{90} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{90}}$$ which corresponds to option (A).

Q10) If x is an eigenvector of matrix A, then

$$Ax = \lambda x$$

where λ is the eigenvalue that can be used as a scaling factor, such that

i) A can stretch x
ii) A can shrink x
iii) A CANNOT change direction of x since λ by which x is scaled is a scalar quantity and hence cannot change direction of x.

$\therefore$ A CANNOT change basis of x.

So correct options correspond to options 1, 2 and 4.

Q11) Properties of Symmetric matrices:

1) They have real eigenvalues.

2) They have orthogonal eigenvectors

If λ_1, λ_2 are eigenvalues of A with corresponding eigenvectors x_1, x_2 and $\lambda_1 \neq \lambda_2$ (i.e, eigenvalues of A are distinct), then $\{x_1, x_2\}$ is a linearly independent set.

Proof: Suppose $c_1 x_1 + c_2 x_2 = 0$

$$A(c_1 x_1 + c_2 x_2) = 0$$

$$c_1(Ax_1) + c_2(Ax_2) = 0$$

$$c_1(\lambda_1 x_1) + c_2(\lambda_2 x_2) = 0$$

$$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = 0$$

$$(-) \quad c_1 x_1 + c_2 x_2 = 0 \quad * (\lambda_2)$$

gives

$$c_1 \lambda_1 x_1 + \cancel{c_2 \lambda_2 x_2} = 0$$

$$(-) \quad c_1 \lambda_2 x_1 + \cancel{c_2 \lambda_2 x_2} = 0$$

$$c_1 \lambda_1 x_1 - c_1 \lambda_2 x_1 = 0$$

$$c_1(\lambda_1 - \lambda_2)x_1 = 0$$

Since $\lambda_1 \neq \lambda_2$ and $x_1 \neq 0$.

$\therefore c_1 = 0$

Similarly, $c_2 = 0$

$\therefore \{x_1, x_2\}$ is a linearly independent set.

Generalizing this claim: If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are all distinct, then eigenvectors $\{x_1, x_2, x_3, \ldots, x_n\}$

corresponding to those eigenvalues are also forming a linearly independent set of vectors

. If $S^{-1}AS = \Lambda$ then A is diagonizable.
(Given an invertible matrix S exists).

<u>Proof</u>: Since S is an invertible matrix, it is a square matrix with all vectors (columns) linearly independent off each other

{ else if linearly dependent columns in S, then $\det(S)$ or $|S| = 0$ that makes it non-invertible as $S^{-1} = \frac{1}{|S|} \operatorname{adj}(S)$ ~~...~~ $|S| \to 0$ not defined }

$\therefore$ S is a full rank matrix [$\operatorname{rank}(S) = n$]

$$AS = A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ax_1 & Ax_2 & \cdots & Ax_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \text{zeros} \\ & \ddots & \\ \text{zeros} & & \lambda_n \end{bmatrix}$$

$= S\Lambda$

diagonal matrix where diagonals have eigenvalues and all other values are zero.

$AS = S\Lambda$

$\rightarrow \boxed{S^{-1}AS = \Lambda}$ $\therefore$ <u>A is diagonizable</u>

Since A is a real symmetric matrix, its eigenvectors are orthogonal hence A is orthogonally diagonizable.

Summary :

If A is a real symmetric matrix :

1) Eigenvalues are real
2) Eigenvectors corresponding to distinct eigenvalues are linearly independent
3) A is orthogonally diagonizable

$$\boxed{\exists\, Q \text{ satisfying } Q^T Q = I \text{ such } \cancel{\text{that}}\ A = Q\Lambda Q^T}$$

$\therefore$ All the above options are TRUE.

Q12.) Characteristic polynomial for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is :

$$\lambda^2 - (\text{trace}(A))\lambda + |A| = 0$$

$$\lambda^2 - (3+1)\lambda + (1*3 - 4*2) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda^2 + \lambda - 5\lambda - 5 = 0$$

$$\lambda(\lambda+1) - 5(\lambda+1) = 0$$

$$(\lambda+1)(\lambda-5) = 0$$

$$\therefore \boxed{\lambda = -1, 5}$$

Q13.) Characteristic polynomial for $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ are :

$$\lambda^2 - (2+1)\lambda + (2*1 - 2*3) = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\lambda^2 + \lambda - 4\lambda - 4 = 0$$

$$\lambda(\lambda+1) - 4(\lambda+1) = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$$\therefore \boxed{\lambda = -1, 4}$$

1) for $\lambda = -1$

$$(A - \lambda I)x = 0$$

$$\left\{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 - \frac{3}{2}R_1$$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$

let $\boxed{x_2 = k}$

$$2x_1 + 2k = 0$$

$$2x_1 = -2k$$

$$\boxed{x_1 = -k}$$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \times (-1)$$

$$x = k_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

2) for $\lambda = 4$

$$(A - \lambda I)\vec{x} = 0$$

$$\left\{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 = R_2 + R_1 \quad \longrightarrow \quad \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 = 0$$

let $\boxed{x_2 = k}$; $-3x_1 + 2k = 0$

$$-3x_1 = -2k$$

$$\therefore \boxed{x_1 = \frac{2}{3}k}$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \quad (\times 3)$$

$$= k_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore \boxed{v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

Hence, Eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Q14) <u>Permutation Matrix :</u>

<u>Matrices obtained by "permutation" or shuffling of rows of Identity matrix</u> They have a single '1' in any row or column.

<u>Ex</u> : 2x2 Permutation matrix .

<u>With I</u> $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$

$\lambda^2 - (1+1)\lambda + (1*1 - 0*0) = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda^2 - \lambda - \lambda + 1 = 0$

$\lambda(\lambda - 1) - 1(\lambda - 1) = 0$

$(\lambda - 1)^2 = 0$

$\therefore \boxed{\lambda = 1}$

<u>$\lambda_1 = \lambda_2 = 1$</u>

<u>With</u> $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow$

$\lambda^2 - (0+0)\lambda + (0*0 - 1*1) = 0$

$\lambda^2 - 1 = 0$

$\lambda^2 = 1$

$\boxed{\lambda = \pm 1}$

<u>$\lambda_1 = 1 , \lambda_2 = -1$</u>

<u>Since eigenvalue λ taking value $\boxed{1}$ is common in both cases of permutation matrix</u>;

"λ will always take 1 as one of its eigenvalues in any permutation matrix."

$\therefore$ A permutation matrix 'A' will always have 1 as its eigenvalue, others may or may not be -1.

The statement ALWAYS TRUE is that 1 is an eigenvalue of A.

Q15)

	x	y	
(x_1)	0	0	(y_1)
(x_2)	1.5	1.5	(y_2)
(x_3)	4	1	(y_3)

For 2^{nd} degree polynomial : $y = \theta_0 + \theta_1 x + \theta_2 x^2$

$$y_1 = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2$$
$$y_2 = \theta_0 + \theta_1 x_2 + \theta_2 x_2^2$$
$$y_3 = \theta_0 + \theta_1 x_3 + \theta_2 x_3^2$$

$$y = \phi(x) \cdot \theta^T \quad \text{or} \quad \theta^T = [\phi(x)]^{-1} y.$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} x_1^0 & x_1^1 & x_1^2 \\ x_2^0 & x_2^1 & x_2^2 \\ x_3^0 & x_3 & x_3^2 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.5 & 2.25 \\ 1 & 4 & 16 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1.5 \\ 1 \end{bmatrix}$$

$\Downarrow$

First find Inverse

let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.5 & 2.25 \\ 1 & 4 & 16 \end{bmatrix}$; $A^{-1} = \frac{1}{|A|} \text{adjoint}(A)$.

$$\det(A) = |A| = 1\left[(1.5)(16) - (4)(2.25)\right] - 0 + 0$$

$$|A| = \boxed{15}$$

Cofactors for adjoint(A):

$C_{11} = +\begin{vmatrix} 1.5 & 2.25 \\ 4 & 16 \end{vmatrix} = 15$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 2.25 \\ 1 & 16 \end{vmatrix} = -13.75$

$C_{13} = +\begin{vmatrix} 1 & 1.5 \\ 1 & 4 \end{vmatrix} = 2.5$

$C_{21} = (-1)^3\begin{vmatrix} 0 & 0 \\ 4 & 16 \end{vmatrix} = 0$

$C_{22} = +\begin{vmatrix} 1 & 0 \\ 1 & 16 \end{vmatrix} = 16$

$C_{23} = (-1)^5\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = -4$

$C_{31} = +\begin{vmatrix} 0 & 0 \\ 1.5 & 2.25 \end{vmatrix} = 0$

$C_{32} = (-1)^5\begin{vmatrix} 1 & 0 \\ 1 & 2.25 \end{vmatrix} = -2.25$

$C_{33} = +\begin{vmatrix} 1 & 0 \\ 1 & 1.5 \end{vmatrix} = +1.5$

$$C = \begin{bmatrix} 15 & -13.75 & 2.5 \\ 0 & 16 & -4 \\ 0 & -2.25 & 1.5 \end{bmatrix}$$

STEPS FOR COFACTOR

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Minors:

M_{11} = leave row 1 & column 1 & find determinant

(i.e., for M_{ij} = ignore row i & column j)

$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$M_{11} = a_{22} \cdot a_{33} - a_{23} \cdot a_{32}$

$C_{11} = (-1)^{1+1}(a_{22} \cdot a_{33} - a_{23} \cdot a_{32})$

$\left\{ \text{i.e., } C_{ij} = (-1)^{i+j}(M_{ij}) \right\}$

Cofactor Matrix $= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

Adjoint Matrix $= C^T$

$= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

$$\text{adj}(A) = C^T = \begin{bmatrix} 15 & 0 & 0 \\ -13.75 & 16 & -2.25 \\ 2.5 & -4 & 1.5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{15}\begin{bmatrix} 15 & 0 & 0 \\ -13.75 & 16 & -2.25 \\ 2.5 & -4 & 1.5 \end{bmatrix}$$

Coming back ...

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = (A^{-1})_{3\times3}\begin{bmatrix} 0 \\ 1.5 \\ 1 \end{bmatrix}_{3\times1}$$

$$= \frac{1}{15}\begin{bmatrix} 15 & 0 & 0 \\ -13.75 & 16 & -2.25 \\ 2.5 & -4 & 1.5 \end{bmatrix}_{3\times3}\begin{bmatrix} 0 \\ 1.5 \\ 1 \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 0 \\ 21.75 \\ -4.5 \end{bmatrix}_{3\times1}$$

$\therefore$ Equating both sides we get :

$$\theta_0 = \frac{0}{15} = \boxed{0}$$

$$\theta_1 = \frac{21.75}{15} = \boxed{1.45}$$

Final answer:

$$\boxed{y = -0.3x^2 + 1.45x}$$

$$\theta_2 = \frac{-4.5}{15} = \boxed{-0.3}$$

$$\therefore y = \theta_0 + \theta_1 x + \theta_2 x^2 = \boxed{0 + 1.45x - 0.3x^2}$$