Machine Learning Foundations

Chapter 6: Probability

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Outline for Chapter 6 : Probability

6.1 : Discrete Random Variables

6.2 : Continuous Random Variables

6.3 : Maximum Likelihood and other advanced topics

Outline for Chapter 6 : Probability

6.1 : Discrete Random Variables

1. **Probability space**
2. Conditioning
3. Random variables
4. Expectation and Variance
5. Multiple Random Variables
6. Bernoulli, Binomial, Poisson and Geometric RVs

6.2 : Continuous Random Variables

6.3 : Maximum Likelihood and other advanced topics

Chapter 6.1.1 : Probability Space

Sample space

Events as sets

- Axioms of sigma-algebra
- Single coin toss example, Single die throw example, Double coin toss example, Double dice throw, Pick card example,

Probability measure

- Axioms of probability
- Examples of valid and invalid probability measures
- Basic probability calculus(complements, disjoint unions, in-ex-clusion)
- Counting : 2 die throw P(sum=5), prob of getting a,a,a from banana. 3!3!/6!

Sample Space

Experiment

Sample space Ω = set of elementary outcomes

Coin toss expt. $\Omega = \{H, T\}$

Events as sets

- Events are subsets of the sample space.
- collection of all events of interest as $\mathcal{F}$

$\mathcal{F}$ = Power set of Ω = $\{0,1\}^{\Omega}$

CTE :

$\mathcal{F} = \{ \emptyset, \{H\}, \{T\}, \{H,T\} \}$

Examples

(ii) DTE : $\Omega = \{1,2,3,4,5,6\}$

$\mathcal{F} = \{0,1\}^{\Omega}$ $\quad |\mathcal{F}| = 2^6$

Sigma-Algebra

(iii) DCTE $\quad \Omega = \{HH, HT, TH, TT\}$

$\mathcal{F} = \{0,1\}^{\Omega}$ $\quad |\mathcal{F}| = 2^4$

(iii) DDTE $\quad \Omega = \{1,2,3,4,5,6\}^2$

$\mathcal{F} = \{0,1\}^{\Omega}$, $|\mathcal{F}| = 2^{36}$

iv) CPE $\quad \Omega = \{AS, AC, AD, AH, \cdots 2S, \cdots 2H\}$

$\mathcal{F} = \{0,1\}^{\Omega}$ $\quad |\mathcal{F}| = 2^{52}$

Examples

Probability Measure

$(\Omega, \mathcal{F}, P)$

$P : \mathcal{F} \to [0,1]$

Axioms of Probability:

(i) $P(\Omega) = 1$

(ii) $P(A) \geq 0$

(iii) $A_1, A_2 \cdots A_n$ disjoint sets $A_i \subseteq \Omega$

$$P(A_1 \cup A_2 \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Examples of Probability Measure

: $\Omega = \{H, T\}$ $\quad F = \{0,1\}^{\Omega}$

$P(\emptyset) = 0$

$P(\{H\}) = P(\{T\}) = 0.5$

$P(\{H, T\}) = 1$

$\{H, T\} = \{H\} \cup \{T\}$

$P(\{H, T\}) = P(\{H\}) + P(\{T\})$

Examples of Probability Measure

DCTE : $\Omega = \{HH, HT, TH, TT\}$

$$\mathcal{F} = \{0,1\}^{\Omega}$$

$$P(A) = \frac{|A|}{4}$$

$$P(\{HH, HT, TT\}) = \frac{3}{4}$$

DDTE : $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}.$

$$\mathcal{F} = \{0,1\}^{\Omega}$$

$$P(A) = \frac{|A|}{36}$$

Examples of Probability Measure

DCTE : $\Omega = \{HH, HT, TH, TT\}$

$\mathcal{F} = \{0,1\}^{\Omega}$

$P(A) = 1(HH \in A)$

$P(\emptyset) = 0$

$P(\Omega) = 1$

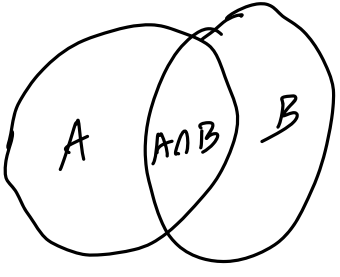
$P(\{HH\}) = 1$; $P(\{HT\}) = 0$,

$P(\{TH\}) = 0$; $P(\{TT\}) = 0$

Basic Properties

i) $P(A^C) = 1 - P(A)$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A \cup B = A \cup (B \setminus A)$

$P(A \cup B) = P(A) + P(B \setminus A)$

$B \setminus A = B \setminus (A \cap B)$

$P(B \setminus A \cap B) = P(B) - P(A \cap B)$

Inclusion Exclusion

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$P(A_1 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{\substack{i,j \\ i<j}} P(A_i \cap A_j) + \sum_{\substack{i,j,k \\ i<j<k}} P(A_i \cap A_j \cap A_k)$$

$$(-1)^{n} \cdot P(A_1 \cap A_2 \cdots \cap A_n)$$

Counting

$$P(A) = \frac{|A|}{|S|}$$

DDTE

$$P(A) = \frac{|A|}{36}$$

$$P(\text{sum of the dice} = 6) = \frac{\#\text{ of outcomes where sum} = 6}{36}$$

$$= \frac{5}{36}$$

Counting

B, A, N, A, N, A

Prob (3 picked pieces are A, A, A) =

$|\Omega| = 6!$

# of Outcomes : $3!\,3!$

$$P(\text{Event}) = \frac{3!\,3!}{6!}$$

$(\Omega, \mathcal{F}, P)$

$P : \mathcal{F} \to [0,1]$