Divide and Conquer: Counting Inversions

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Programming, Data Structures and Algorithms using Python

Week 8

Divide and Conquer

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- Combine these subproblem solutions efficiently

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  - Rearrange into lower and upper partitions, sort each partition separately
  - Place pivot between sorted lower and upper partitions

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- Other examples?

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- You and your friend rank 5 movies, {A, B, C, D, E}
  - Your ranking: D, B, C, A, E
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- For each pair of movies, compare preferences
  - You rank B above C, so does your friend
  - You rank D above B, your friend ranks B above D

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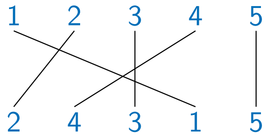

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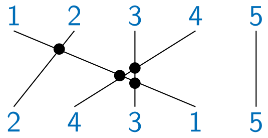

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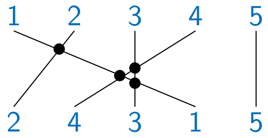

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- Brute force — check every (i, j), $O(n^2)$

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- Adapt merge sort
- Recursively sort and count inversions in L and R
- Count inversions while merging — merge and count

Divide and conquer

Merge and count

- Merge $L = [i_1, i_2, \ldots, i_{n/2}]$ and $R = [i_{n/2+1}, i_{n/2+2}, \ldots, i_n]$, sorted

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  - If we add i_m from R to the output, i_m is smaller than elements currently in L
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```
def mergeAndCount(A,B):
  (m,n) = (len(A),len(B))
  (C,i,j,k,count) = ([],0,0,0,0)
  while k < m+n:
    if i == m:
      C.append(B[j])
      (j,k) = (j+1,k+1)
    elif j == n:
      C.append(A[i])
      (i,k) = (i+1,k+1)
    elif A[i] < B[j]:
      C.append(A[i])
      (i,k) = (i+1,k+1)
    else:
      C.append(B[j])
      (j,k,count) = (j+1,k+1,count+(m-i))
  return(C,count)
```

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  - i_m is hence inverted with respect to elements currently in L
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- `sortAndCount` is merge sort with `mergeAndCount`

```
def sortAndCount(A):
  n = len(A)

  if n <= 1:
     return(A,0)

  (L,countL) = sortAndCount(A[:n//2])
  (R,countR) = sortAndCount(A[n//2:])

  (B,countB) = mergeAndCount(L,R)

  return(B,countL+countR+countB)
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  - Number ranges from 0 to $n(n-1)/2$
- We are counting them efficiently without enumerating each one