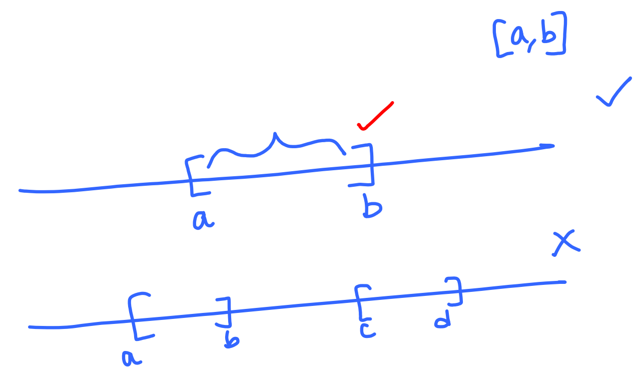
Introduction to Convexity

Convex set

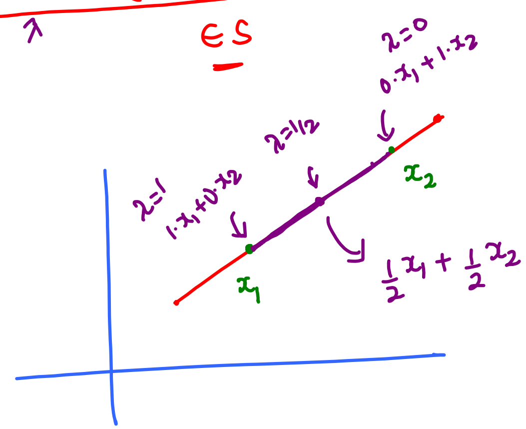
A set $S \subseteq \mathbb{R}^d$ is a convex set

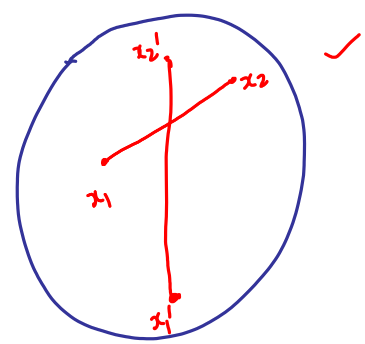
if $\forall x_1, x_2 \in S$, then

$$\rightarrow \boxed{\lambda x_1 + (1-\lambda) x_2} \in S \quad \forall \lambda \in [0,1]$$

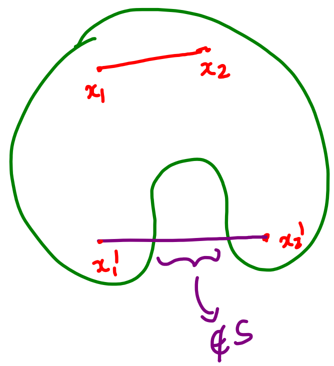


Examples.





Example:



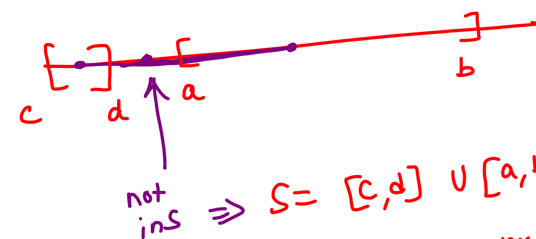
$S \subseteq \mathbb{R}^2$

$\Rightarrow$ Not convex.

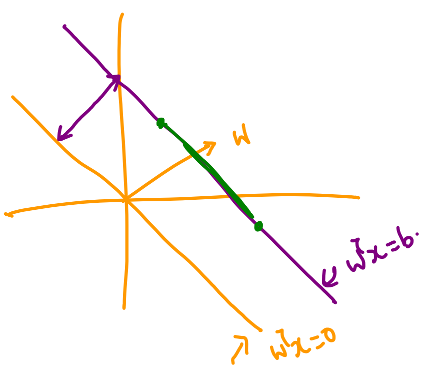
$S \subseteq \mathbb{R}$

$a \le b$

$[a,b]$



$\Rightarrow S = [c,d] \cup [a,b]$ is Not convex.



Hyper plane

$S \subseteq \mathbb{R}^d$

$$\{ x : w^T x = b \}$$

Claim: hyperplanes are convex sets.

Proof:

$x_1 \in S$, $x_2 \in S$

$\Downarrow$ $\Downarrow$

$w^T x_1 = b$ $w^T x_2 = b$

$$w^T(\lambda x_1 + (1-\lambda)x_2)$$
$$= \lambda w^T x_1 + (1-\lambda) w^T x_2$$
$$= \lambda b + (1-\lambda) b$$
$$= b \Rightarrow \in S$$

Note Title 16-08-2021

Exercise: Prove Half-spaces are convex.

$$S = \{ x \in \mathbb{R}^d : w^T x \leq b \}$$
$\subseteq \mathbb{R}^d$

Property of convex set:

Intersection of convex sets is convex.

Let $S_1, S_2 \subseteq \mathbb{R}^d$ be convex sets. Let

$$S_{12} = S_1 \cap S_2 = \{ x : x \in S_1, x \in S_2 \}$$

$x_1, x_2 \in S_{12}$ $\quad$ $\lambda x_1 + (1-\lambda) x_2 \in S_{12}$?

$\in S_1$, $\in S_2$ } convexity of S_1 & S_2 $\Rightarrow \in S_1 \cap S_2$

$\square$