Method of Lagrange multiplier

$$\boxed{\begin{aligned} &\min_{x} \; f(x) \\ &g(x) = 0 \end{aligned}}$$

1. $g(x^*) = 0$
2. $\boxed{\nabla f(x^*) = -\lambda \nabla g(x^*)}$ for some λ. [no sign constraint].

Example:

$$\left.\begin{aligned} f(x_1, x_2) &= x_1^2 + 2x_2 + 4x_2^2 \\ g(x_1, x_2) &= x_1^2 + x_2^2 - 1 \end{aligned}\right\}$$

$$\nabla f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ 2 + 8x_2 \end{bmatrix} \; ; \quad \nabla g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2x_1 \\ 2+8x_2 \end{bmatrix}}_{\nabla f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)} = -\lambda \underbrace{\begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}}_{\nabla g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)} \Rightarrow \begin{aligned} 2x_1 &= -\lambda\, 2x_1 \quad — (1) \\ 2+8x_2 &= -\lambda(2x_2) \quad — (2). \end{aligned}$$

$$(1) \Rightarrow \quad 2x_1 + \lambda 2x_1 = 0 \quad \Rightarrow \quad 2x_1(1+\lambda) = 0$$

$$\Downarrow$$

either $x_1 = 0$ (or) $\lambda = -1$

Case 1: $\boxed{\lambda = -1}$

$$2+8x_2 = -(-1)\, 2x_2 \quad \Rightarrow \quad 2+8x_2 = 2x_2 \quad \Rightarrow \quad -6x_2 = 2$$

$$\Rightarrow \quad \boxed{x_2 = -1/3}$$

$$x_1^2 + x_2^2 = 1 \quad \Rightarrow \quad x_1^2 = 1 - 1/9 \quad = \quad x_1^2 = 8/9 \quad \Rightarrow \quad x_1 = \left\{ +\frac{\sqrt{8}}{3}, -\frac{\sqrt{8}}{3} \right\}$$

$x_2 = -1/3$; $x_1 = \frac{\sqrt{8}}{3}$ or $-\frac{\sqrt{8}}{3}$ $\Rightarrow$ $\underbrace{\left\{ \begin{bmatrix} \sqrt{8}/3 \\ -1/3 \end{bmatrix}, \begin{bmatrix} -\sqrt{8}/3 \\ -1/3 \end{bmatrix} \right\}}$ $\rightarrow$ minimizers.

maximizer

Case 2: $x_1 = 0$

$x_1^2 + x_2^2 = 1 \Rightarrow x_2^2 = 1 \Rightarrow x_2 = 1$ or -1 $\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

To find $\boxed{\begin{array}{l} \min_x \; f(x) \\ g(x) = 0 \end{array}}$, substitute each potential solution into f

$f(x) = x_1^2 + 2x_2 + 4x_2^2$

$f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 6$ ✓

$f\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = 2$ ✓

$f\left(\begin{bmatrix} \sqrt{8}/3 \\ -1/3 \end{bmatrix}\right) = \frac{8}{9} - \frac{2}{3} + \frac{4}{9}$

$= \frac{4}{3} - \frac{2}{3} = 2/3$ ✓

$f\left(\begin{bmatrix} -\sqrt{8}/3 \\ -1/3 \end{bmatrix}\right) = 2/3$ ✓

→ In general is it possible to solve the system of equations that satisfy the Lagrange equation? May not be true!

PROJECTED GRADIENT DESCENT

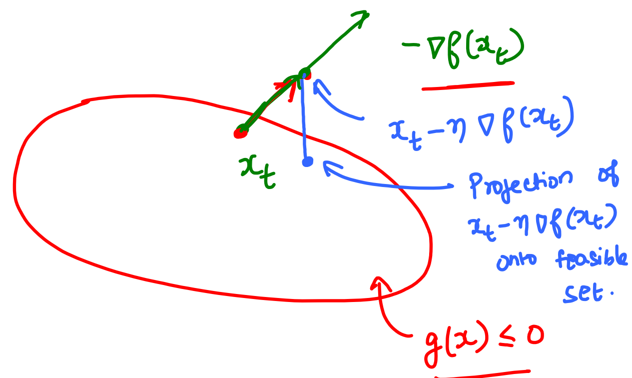
$x_0 \leftarrow$ initialization

for $t = 1, \ldots, T$

$$x_{t+1} = \underbrace{\Pi\Big(\underbrace{x_t - \eta \nabla f(x_t)}_{\text{gradient step}}\Big)}_{\text{Projection step.}}$$

end.

CONVEX Constraint Sets



Projection operator.

$$\Pi(\underline{x}) = \min_{y \in S} \|x - y\|_2^2$$

$S = \{y : g(y) \leq 0\}$