Using Heaps in Algorithms

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Programming, Data Structures and Algorithms using Python

Week 6

Priority queues and heaps

- Priority queues support the following operations
  - `insert()`
  - `delete_max()` or `delete_min()`
- Heaps are a tree based implementation of priority queues
  - `insert()`, `delete_max()` / `delete_min()` are both $O(\log n)$
  - `heapify()` builds a heap from a list/array in time $O(n)$
- Heap can be represented as a list/array
  - Simple index arithmetic to find parent and children of a node
- What more do we need to use a heap in an algorithm?

Dijkstra's algorithm

- Maintain two dictionaries with vertices as keys
  - `visited`, initially `False` for all `v`
  - `distance`, initially `infinity` for all `v`
- Set `distance[s]` to `0`
- Repeat, until all reachable vertices are visited
  - Find unvisited vertex `nextv` with minimum distance
  - Set `visited[nextv]` to `True`
  - Recompute `distance[v]` for every neighbour `v` of `nextv`

```
def dijkstra(WMat,s):
  (rows,cols,x) = WMat.shape
  infinity = np.max(WMat)*rows+1
  (visited,distance) = ({},{})
  for v in range(rows):
    (visited[v],distance[v]) = (False,infinity)
  distance[s] = 0
  for u in range(rows):
    nextd = min([distance[v] for v in range(rows)
                 if not visited[v]])
    nextvlist = [v for v in range(rows)
                 if (not visited[v]) and
                    distance[v] == nextd]
    if nextvlist == []:
      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for v in range(cols):
      if WMat[nextv,v,0] == 1 and (not visited[v]):
        distance[v] = min(distance[v],distance[nextv]
                                      +WMat[nextv,v,1])
  return(distance)
```

Dijkstra's algorithm

Bottleneck

- Find unvisited vertex j with minimum distance
  - Naive implementation requires an $O(n)$ scan

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- Find unvisited vertex j with minimum distance
  - Naive implementation requires an $O(n)$ scan
- Maintain unvisited vertices as a min-heap
  - `delete_min()` in $O(\log n)$ time
- But, also need to update distances of neighbours
  - Unvisited neighbours' distances are inside the min-heap
  - Updating a value is not a basic heap operation

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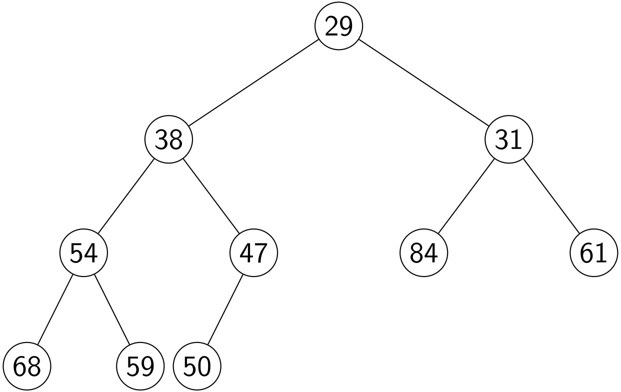
Updating values in a min-heap

- Change 54 to 35



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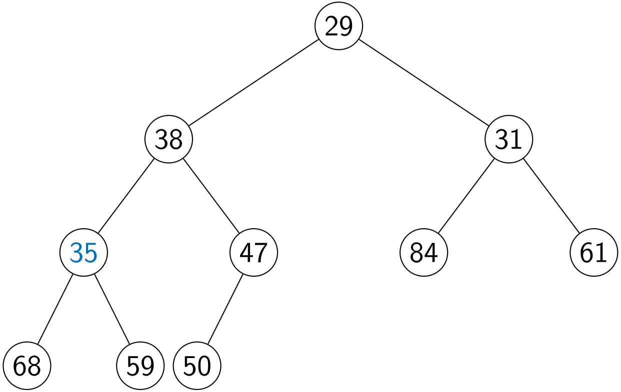
- Change 54 to 35
  - Reducing a value can create a violation with parent



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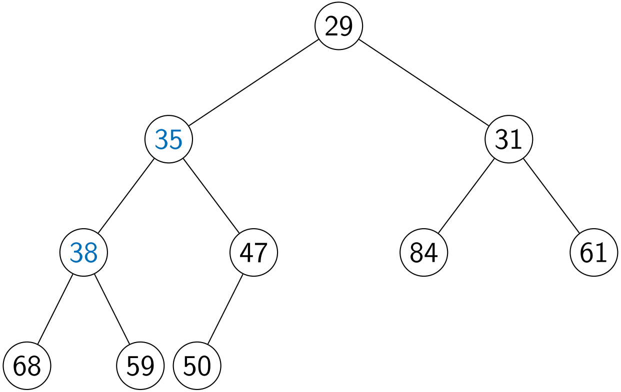
  - Swap upwards to restore heap, similar to `insert()`



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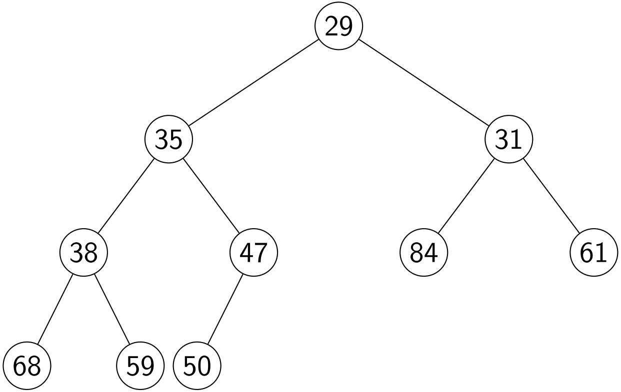

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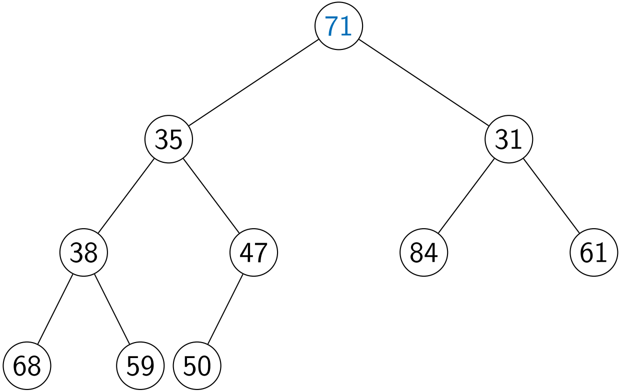

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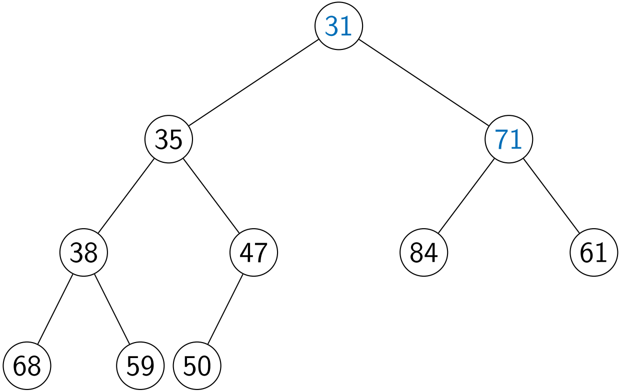

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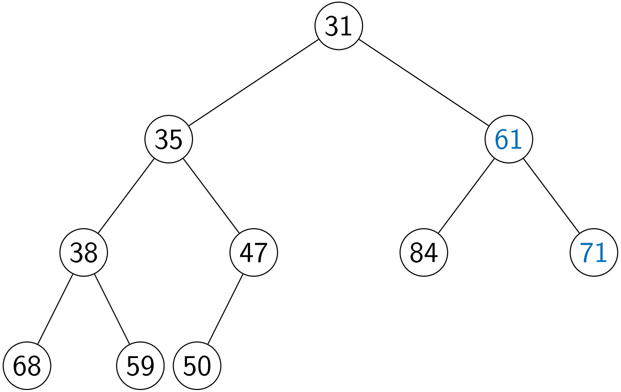

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- Both updates are $O(\log n)$
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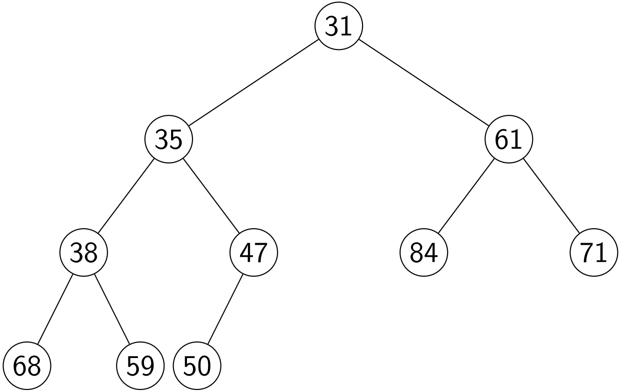

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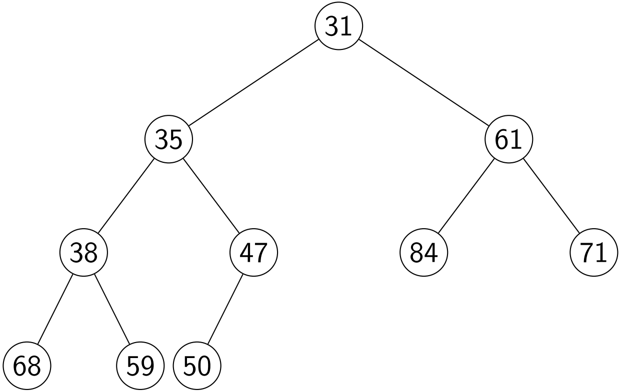
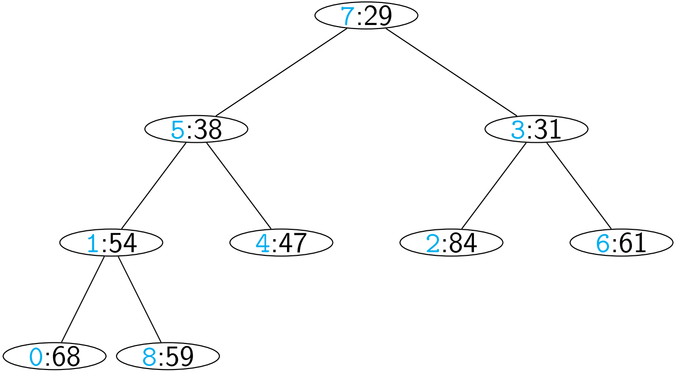
  - Are we done?
- Locate the node to update?



Updating values in a min-heap

- Maintain two additional dictionaries
  - Vertices are `{0,1,...,n-1}`
  - Heap positions are $\{0, 1, \ldots, n-1\}$
  - `VtoH` maps vertices to heap positions
  - `HtoV` maps heap positions to vertices



VtoH

0	1	2	3	4	5	6	7	8
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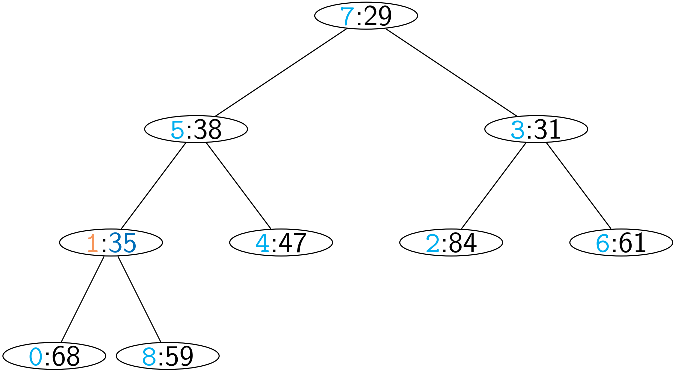

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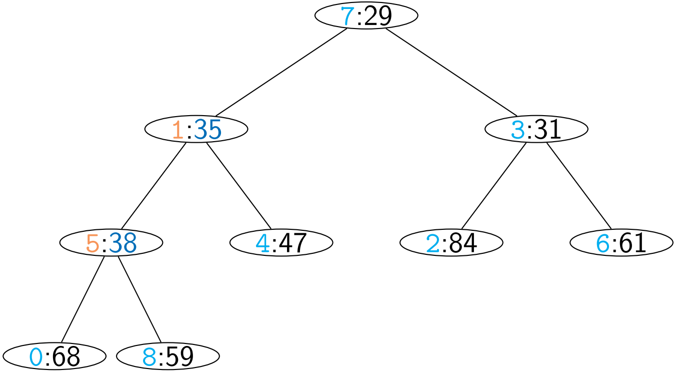
- Update node 1 to 35
- Update `VtoH` and `HtoV` each time we swap values in the heap



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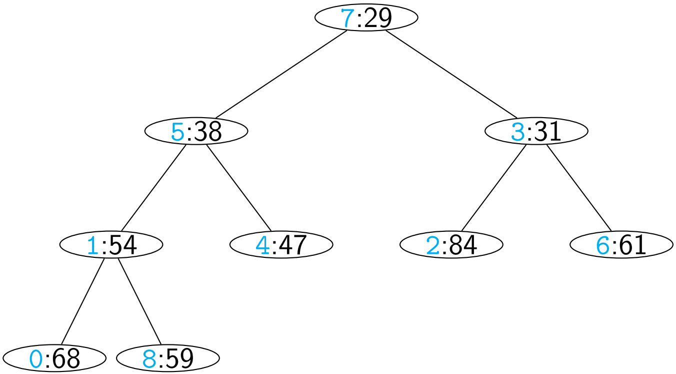
  - Identifying next vertex to visit is $O(\log n)$
  - Updating distance takes $O(\log n)$ per neighbour
  - Adjacency list — proportionally to degree



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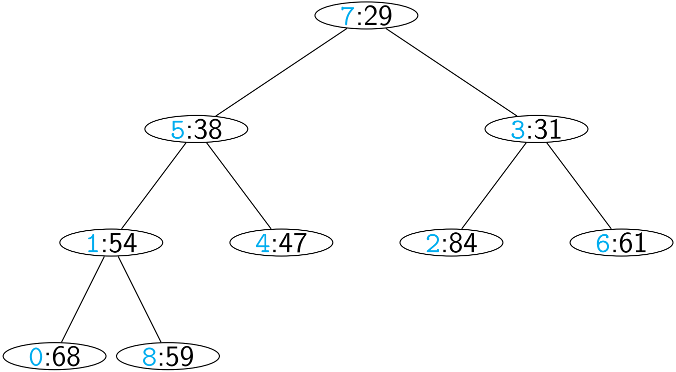
- Cumulatively
  - $O(n \log n)$ to identify vertices to visit across n iterations
  - $O(m \log n)$ distance updates overall



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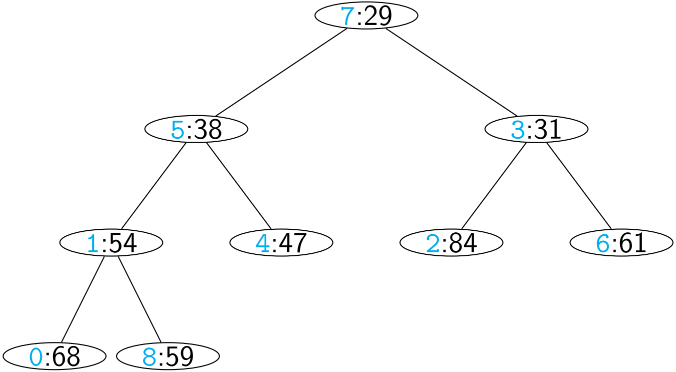
- Overall $O((m+n)\log n)$



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- In place $O(n \log n)$ sort

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- Need to maintain additional pointers to map values to heap positions and vice versa
- With this extended notion of heap, Dijkstra's algorithm complexity improves from $O(n^2)$ to $O((m+n)\log n)$
- In a similar way, improve Prim's algorithm to $O((m+n)\log n)$
- Heaps can also be used to sort a list in place in $O(n \log n)$