String Matching: Knuth-Morris-Pratt algorithm

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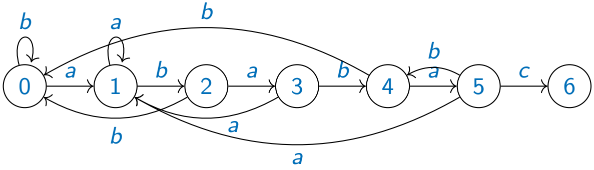
Programming, Data Structures and Algorithms using Python

Week 10

Pattern matching using automata

- Finite state automaton for pattern `p`
  - States $\{0, 1, \dots, m\}$
  - State i denotes match upto `p[:i]`
  - Transition $i \xrightarrow{a} j$ descibes how to update the match on reading `a`
- Start scanning text in initial state 0
- In state i, read `t[j]`, take the transition labelled `t[j]`
- If we reach the final state m, we have found a full match for `p`
- Single scan of `t` suffices



Processing *abababac*

$$0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{a} 5 \xrightarrow{b} 4 \xrightarrow{a} 5 \xrightarrow{c} 6$$

Knuth-Morris-Pratt algorithm

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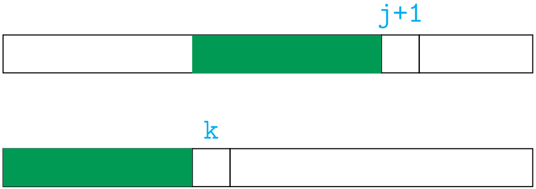

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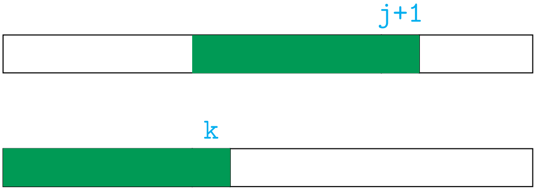

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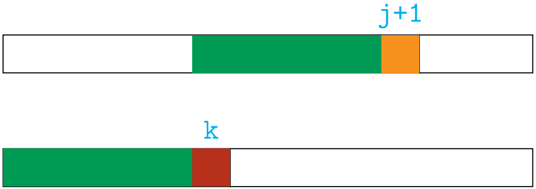

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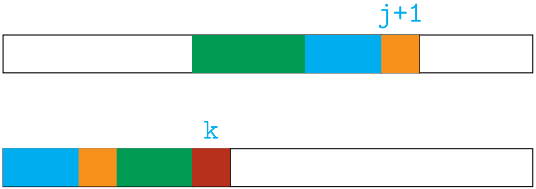

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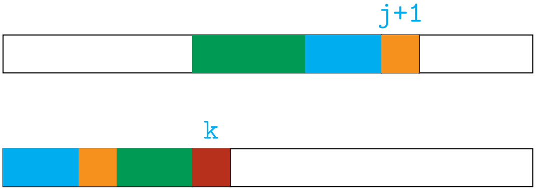
- Usually refer to `match` as failure function `fail`
  - Where to fall back if match fails



Computing the fail function

- Initialize fail[j] = 0 for all j

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- If we don't find a nontrivial prefix to extend, retain fail[j] = 0, move to next position

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- But we also have iterations where `k` decreases and `j` is unchanged
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- Overall `k` is incremented at most $m-1$ times
- Hence overall complexity is $O(m)$

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Implementing string search using fail function

- Scan t from beginning

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def find_kmp(t, p):
  n,m = len(t),len(p)
  if m == 0:
    return 0 # pattern is empty
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  j = 0 # index into text
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  while j < n:
    if t[j] == p[k]: # matched p[0:k+1]
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- This finds first match, modify to find all matches

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Summary

- The Knuth, Morris, Pratt algorithm efficiently computes the automaton describing prefix matches in the pattern `p`
- Complexity of preprocessing the `fail` function is $O(m)$
- After preprocessing, can check matches in the text `t` in $O(n)$
- Overall, KMP algorithm works in time $O(m + n)$
- However, the Boyer-Moore algorithm can be faster in practice, skipping many positions