Course: Machine Learning - Foundations
Graded Questions - Solution
Lecture Details: Week 8

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1. (1 point) Points $(0,0),(5,0),(5,5),(5,0)$ forms a convex hull. Which of the following points are the part of this convex hull?

   A. (1,1)

   B. (1,-1)

   C. (-1,1)

   D. (-1,-1)

   Answer: A

   Let $(x_1, y_1) = (0,0), (x_2, y_2) = (5,0), (x_3, y_3) = (5,5), (x_4, y_4) = (5,0)$ Any point on the convex hull of these points will be the the set S such that

   $\{S = (x, y) \mid x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4, y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0,1]\}$

   The convex hull of these 4 points forms a square and any point inside or on the square will be the part of the convex hull.

   The point (1,1) lies inside this square, while (1,-1), (-1,1) and (-1,=1) lies outside this square.

2. (1 point) Given S is a convex set and the points $x_1, x_2, x_3, x_4 \in S$. Which of the following points must be the part of convex set S:

   A. $0.1x_1 + 0.2x_2 + 0.3x_3 + 0.4x_4$

   B. $-0.1x_1 + -0.2x_2 + 0.6x_3 + 0.7x_4$

   C. $0.1x_1 + 0.1^2x_2 + 0.1^3x_3 + 0.1^3x_4$

   D. $0.25x_1 + 0.25x_2 + 0.25x_3 + 0.25x_4$

   Answer: A, D

   The convex combination of the points will always be the part of the convex set and is known as convex hull.

   For the convex combination, the coefficients should be non-negative and should sum to 1. Therefore , options (A) and (D) are correct.

3. (1 point) Which of the following is a convex function in $\mathbb{R}^2$?

   A. $f(x) = x^2 + y^2$

   B. $f(x) = -x^2 - y^2$

   C. $f(x) = x^2 - y^2$

D. None of these

Answer: A

$f(x) = x^2 + y^2 = v^T A v = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

where $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v = \begin{bmatrix} x \\ y \end{bmatrix}$

Here, $a > 0, ac - b^2 = 1.1 - 0^2 = 1 > 0$, This shows the matrix A is a positive definite matrix. Therefore, the function is a convex function.

4. (1 point) What is the boundary value of x so that the function $(x-3)^3 + (y+1)^2$ to remain convex?

   A. $x \geq 1$

   B. $x \geq 2$

   C. $x \geq 3$

   D. None of these

Answer: C

$f(x, y) = (x-3)^3 + (y+1)^2$

First order partial derivatives, $f_x = 3(x-3)^2, f_y = 2(y+1)$

Second order partial derivatives, $f_{xx} = 6(x-3), f_{xy} = 0, f_{yy} = 2$

The Second order partial derivatives with respect to x, $f_{xx} = 6(x-3)$ changes sign at the point where $x = 3$.

When $x \geq 3, f_{xx} \geq 0$ otherwise $f_{xx} < 0$

The hessian matrix will be,

$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6(x-3) & 0 \\ 0 & 2 \end{bmatrix} = 12(x-3)$

D will be non negative when $x \geq 3$ and the function remains convex.

5. (1 point) Select the most appropriate linear approximation of $f(x, y) = x^2 + y^2$ at a small step 0.02 in the direction (1,2) from the point (3,2)

   A. 11.02

   B. 10.28

   C. 9.98

   D. None of these

Answer: A

$\epsilon = 0.02, d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \nabla f(x,y)_{(3,2)} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}_{(3,2)} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

The linear approximation of a function f at the point $(x + \epsilon d)$ is

$f(x,y) + \epsilon\ d^T \nabla f(x,y) = (3^2 + 1^2) + 0.02 * \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 10 + 0.02 * 14 = 10.28$

6. (1 point) The minimum value of $f(x,y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$ is ______.

Answer: 27.00, Range 26.50 to 27.50

Given $f(x,y) = x^2 + 4y^2 - 2x + 8y,\ g(x,y) = x + 2y = 7$

$\nabla f(x,y) = \begin{bmatrix} 2x - 2 \\ 8y + 8 \end{bmatrix}, \nabla g(x,y) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

We find the values of x, y, λ that simultaneously satisfy the equations to get the extreme points $\nabla f(x,y) = \lambda \nabla g(x,y)$ and, $g(x,y) = x + 2y = 7$

Solving, $\nabla f(x,y) = \lambda \nabla g(x,y)$

$\implies \begin{bmatrix} 2x - 2 \\ 8y + 8 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies x = (\lambda + 2)/2, y = (\lambda - 4)/4$

Solving, $g(x,y) = x + 2y = 7 \implies (\lambda + 2)/2 + 2 * (\lambda - 4)/4 = 7 \implies 2\lambda - 2 = 14 \implies \lambda = 8$

Therefore, the extreme point coordinates will be, $(x_1, y_1) = ((\lambda+2)/2, (\lambda-4)/4) = (5, 1)$

$f(5,1) = 5^2 + 4 * 1^2 - 2 * 5 + 8 * 1 = 25 + 4 - 10 + 8 = 27$

Taking 2 neighbouring point on the line $g(x,y) = x + 2y = 7$,

$f(3,2) = 3^2 + 4 * 2^2 - 2 * 3 + 8 * 2 = 9 + 16 - 6 + 16 = 35$

$f(1,3) = 1^2 + 4 * 3^2 - 2 * 1 + 8 * 3 = 2 + 36 - 2 + 24 = 60$

We can see the function f(x,y) has a minimum the point (5,1).

7. (1 point) The the minimum value of $f(x,y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$ occurs at the below point:

   A. (5,5)
   B. (-5,5)
   C. (1,5)
   D. (5,1)

Answer: D

Refer to the solution of the previous question

8. (1 point) The distance of the plane $x + y - 2z = 6$ from the origin is ____.

   Answer: 2.45, Range 2.30 to 2.60

   Given Squared Distance, $d^2 = f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x + y - 2z = 6$

   $\nabla f(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$, $\nabla g(x, y, z) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

   We find the values of x, y, z, λ that simultaneously satisfy the equations to get the extreme points $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and, $g(x, y, z) = x + y - 2z = 6$

   Solving, $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

   $$\implies \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \implies x = \lambda/2, y = \lambda/2, z = -\lambda$$

   Solving, $g(x, y, z) = x + y - 2z = 6 \implies \lambda/2 + \lambda/2 + 2\lambda = 6 \implies \lambda = 2$

   Therefore, the extreme point coordinates will be, $(x_1, y_1, z_1) = (\lambda/2, \lambda/2, -\lambda) = (1, 1, -2)$

   $f(1, 1, -2) = 1^2 + 1^2 + (-2)^2 = 1 + 1 + 4 = 6$,

   Taking 2 neighbouring point on the line $g(x, y, z) = x + y - 2z = 6$,

   $f(0, 0, -3) = 0^2 + 0^2 + (-3)^2 = 0 + 0 + 9 = 9$

   $f(2, 2, -1) = 2^2 + 2^2 + (-1)^2 = 4 + 4 + 1 = 9$

   We can see the function f(x,y,z) has a minimum at the point (1, 1, -2). The minimum distance of the plane $x + y - 2z = 6$ from the origin is, $d = \sqrt{6} = 2.45$

9. (1 point) The point on the plane $x + y - 2z = 6$ that is closest to the origin is

   A. $(0, 0, 0)$
   B. $(1, 1, 1)$
   C. $(-1, 1, 2)$
   D. $(1, 2, 0)$
   E. $(1, 1, -2)$

   Answer: E

   Refer to the solution of the previous question

10. (1 point) A box (cuboid shaped) is to made out of the cardboard with the total area of 24 cm^2. The maximum volume occupied by the box will be ____.

    Answer: 8, Range 7.50 to 8.50

    Given Volume, $V = f(x, y, z) = xyz$,

    Surface area, $g(x, y, z) = 2xy + 2yz + 2zx = 24 \implies g(x, y, z) = xy + yz + zx = 12$

$$\nabla f(x,y,z) = \begin{bmatrix} yz \\ zx \\ xy \end{bmatrix}, \nabla g(x,y,z) = \begin{bmatrix} y+z \\ z+x \\ x+y \end{bmatrix}$$

We find the values of x, y, z, λ that simultaneously satisfy the equations to get the extreme points $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and, $g(x,y,z) = xy + yz + zx = 12$

Solving, $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$$\implies \begin{bmatrix} yz \\ zx \\ xy \end{bmatrix} = \lambda \begin{bmatrix} y+z \\ z+x \\ x+y \end{bmatrix} \implies xy + yz = yz + xz = xz + yz = xyz/\lambda \implies x = y = z = 0, 2\lambda$$

Solving, $g(x,y,z) = xy + yz + zx = 12 \implies x = y = z = 2\lambda = \pm 2$

Since, x, y, z are length. This can not be a negative quantity. Therefore, the extreme point coordinates will be,

$(x_1, y_1, z_1) = (2\lambda, 2\lambda, 2\lambda) = (2, 2, 2)$, Volume, $V = f(2,2,2) = 2 * 2 * 2 = 8$

$(x_2, y_2, z_2) = (0, 0, 0)$, Volume, $V = f(0,0,0) = 0 * 0 * 0 = 0$

The box has maximum volume when it is a cube with edge $2cm$, and the maximum volume is $8cm^3$

11. (1 point) You are planning to setup a manufacturing business where the revenue (r) is a function of labour units (l), material units (m) and fixed cost (c), $r = l.m^2 + 2c$. You have an annual budget (b) of 1004 million rupees, $b = 2l + 16m + c$ to run the business. What would be maximum revenue that can be generated from the business in million rupees under the optimum combination of labour units (l), material units (m) and fixed cost(c).

    A. 1944

    B. 2036

    C. 2072

    D. 2080

Answer: A

Given

Revenue, $r = f(l,m,c) = l.m^2 + 2c$, Budget constraint, $g(l,m,c) = 2l + 16m + c = 1004$

$$\nabla f(x,y,z) = \begin{bmatrix} m^2 \\ 2lm \\ 2 \end{bmatrix}, \nabla g(x,y,z) = \begin{bmatrix} 2 \\ 16 \\ 1 \end{bmatrix}$$

We find the values of x, y, z, λ that simultaneously satisfy the equations to get the extreme points $\nabla f(l,m,c) = \lambda \nabla g(x,y,z)$ and, $g(l,m,c) = 2l + 16m + c = 1004$

Solving, $\nabla f(l,m,c) = \lambda \nabla g(x,y,z)$

$$\implies \begin{bmatrix} m^2 \\ 2lm \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 16 \\ 1 \end{bmatrix} \implies m = \pm 2, l = \pm 8, \lambda = 2$$

Solving, $g(l, m, c) = 2l + 16m + c = 1004 \implies c = 1004 - 2l - 16m$

Therefore, the extreme point coordinates will be,

$(l_1, m_1, c_1) = (8, 2, 956)$, Revenue, $r = f(8, 2, 956) = 8 * (2)^2 + 2 * 956 = 1944$

$(l_2, m_2, c_2) = (-8, -2, 1052)$, Revenue, $r = f(-8, -2, 1052) = -8 * (-2)^2 + 2 * 1052 = 2072$

Since, the configurations can not be negative. So, $l \geq 0, m \geq 0, c \geq 0$

We can see the maximum revenue is achieved under the configuration $l = 8, m = 2, c = 956$. The maximum revenue is 1944.

12. (1 point) The distance of the point on the sphere $x^2 + y^2 + z^2 = 3$ closest to the point (2,2,2) is ______.

Answer: 1.73, Range 1.50 to 2.00

Given Squared Distance, $d^2 = f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-2)^2$, $g(x, y, z) = x^2 + y^2 + z^2 = 3$

$$\nabla f(x, y, z) = \begin{bmatrix} 2(x-2) \\ 2(y-2) \\ 2(z-2) \end{bmatrix}, \nabla g(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

We find the values of x, y, z, λ that simultaneously satisfy the equations to get the extreme points $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and, $g(x, y, z) = x^2 + y^2 + z^2 = 3$

Solving, $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

$$\implies \begin{bmatrix} 2(x-2) \\ 2(y-2) \\ 2(z-2) \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \implies x = y = z = 2/(1-\lambda)$$

Solving, $g(x, y, z) = x^2 + y^2 + z^2 = 3 \implies 12/(1-\lambda)^2 = 3 \implies \lambda = 3, -1$

Therefore, the extreme point coordinates will be,

$(x_1, y_1, z_1) = (1, 1, 1)$, $d = \sqrt{(1-2)^2 + (1-2)^2 + (1-2)^2} = \sqrt{3} = 1.73$

$(x_2, y_2, z_2) = (-1, -1, -1)$, $d = \sqrt{((-1-2)^2 + (-1-2)^2 + (-1-2)^2} = 3\sqrt{3} = 5.20$

Taking 2 neighbouring point on the line $g(x, y, z) = x + y - 2z = 6$,

$(x_3, y_3, z_3) = (1, 1, -1)$, $d = \sqrt{(1-2)^2 + (1-2)^2 + (-1-2)^2} = \sqrt{11} = 3.32$

$(x_3, y_3, z_3) = (1, -1, -1)$, $d = \sqrt{(1-2)^2 + (-1-2)^2 + (-1-2)^2} = \sqrt{3} = 4.39$

We can see the point on the sphere closest to the point (2,2,2) is (1, 1, 1) and farthest from the point (2,2,2) is (-1, -1, -1)

13. (1 point) The distance of the point on the sphere $x^2 + y^2 + z^2 = 3$ farthest from the point (2,2,2) is ______.

Answer: 5.20, Range 5.00 to 5.50

Refer to the solution of the previous question