Linear Algebra

Four fundamental subspaces of a given matrix A

Column Space $C(A)$:

$$A = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$$

$C(A) = \text{span}(u_1, \ldots, u_n) = \{$ linear combinations of the vectors $u_1, \ldots, u_n \}$

Solving $Ax=b$:

For what b does $Ax=b$ have a solution?
$b \in C(A)$

Let's look at an example

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$Ax = b$ is solvable for $b \in C(A)$.

Does every vector in $\mathbb{R}^4$ belong to $C(A)$?

Some vectors in $\mathbb{R}^4$ are not in $C(A)$, because $Ax=b$ is "4 equations in 3 unknowns"

For the example above, col 3 = col 1 + col 2

So, $C(A) = \text{span}(\text{col}1, \text{col}2)$

$\Rightarrow$ $C(A)$ is a two-dimensional subspace of $\mathbb{R}^4$

Null space $N(A)$

$$N(A) = \{ x \mid Ax = 0 \}$$

Why is $N(A)$ a subspace?

$x_1, x_2 \in N(A)$

Then $Ax_1 = 0$, $Ax_2 = 0$

$\Rightarrow A(x_1 + x_2) = 0$

$\Rightarrow x_1 + x_2 \in N(A)$

$x \in N(A)$, α is a scalar

$A(\alpha x) = \alpha Ax$

$= 0$

So, $\alpha x \in N(A)$

Hence, $N(A)$ is a subspace.

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Find an x s.t. $Ax = 0$

$\Leftrightarrow$ a linear combination of the columns of A should result in the zero vector.

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in N(A)$$

So, $N(A)$ is a line in $\mathbb{R}^3$

Remark 1: If A is invertible, then $N(A)$ has "zero" only,

and $C(A)$ is the whole space.

In this case, $Ax = b$ has a unique solution $x = A^{-1}b$

Else, $N(A)$ has $x_n \neq 0$ and $Ax = b$ solutions are of the form $x = x_p + x_n$

$Ax_p = b$, $Ax_n = 0$

Remark 2: Can we Gaussian elimination to find the Null space of a matrix A
i.e., solve $Ax=0$.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{call this } U}$$

pivots

$$Ux=0$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

col 1, col 3 → pivot cols, col 2, col 4 → free variables

Set $x_2 = 1$, $x_4 = 0 \Rightarrow$ $2x_3 + 4x_4 = 0 \Rightarrow x_3 = 0$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \Rightarrow x_1 = -2$

$$u = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$$

Set $x_2 = 0$, $x_4 = 1 \Rightarrow$ $2x_3 + 4x_4 = 0 \Rightarrow x_3 = -2$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$\Rightarrow x_1 - 4 + 2 = 0 \Rightarrow x_1 = 2$

$$v = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

$N(A) = \text{span}(u, v)$

Rank = number of pivot columns. In the example above, rank(A) = 2

Nullity = number of free variables. In the example above, nullity(A) = 2

Rank = dim(C(A)) , Nullity = dim(N(A)).

If A has n columns, then $\boxed{\text{rank} + \text{nullity} = n}$ ⟺ rank(A) = r , nullity(A) = n − r

Row space R(A): column space of A^T ⟺ span of rows of A

$R(A) = C(A^T)$

Important fact: col rank = dim(C(A)) , row-rank = dim(R(A))

col rank = row rank

So far, we looked at $C(A)$, $N(A)$, $R(A)$
$(= C(A^T))$

Left Null space $N(A^T) = \{ y \mid A^T y = 0 \} = \{ y \mid y^T A = 0 \}$

For a $m \times n$ matrix A,

$$\underbrace{\begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}}_{\text{a linear combination of rows leading to zero vector}}$$

Remark: ① A is a $m \times n$ matrix

$$\dim(C(A)) + \dim(N(A)) = \text{number of columns of } A = n$$

$$\text{"}r\text{"} + \text{"}n-r\text{"} = n$$

② $$\dim C(A^T) + \dim(N(A^T)) = \text{number of rows} = m$$

$\dim(C(A^T)) = r \Rightarrow$ $$r + \dim(N(A^T)) = m \Rightarrow \dim(N(A^T)) = m - r$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

want $y^T A = 0$

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \in N(A^T)$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad m = n = 2$$

$C(A)$ = line through $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \text{rank}(A) = r = 1$

$N(A)$ = line through $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow \text{nullity}(A) = n - r = 1$

$R(A) = C(A^T) =$ line through $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\dim(R(A)) = r = 1$

Left Null space $N(A^T) =$ line through $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $\dim(N(A^T)) = 1$

Home work: Work out the four fundamental spaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$ for

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$