Linear Programming: Production Planning

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Programming, Data Structures and Algorithms using Python

Week 11

Linear programming

- Constraints and objective to be optimized are linear functions
  - Constraints: $a_1x_1 + a_2x_2 + \cdots + a_mx_m \leq K$, $b_1x_1 + b_2x_2 + \cdots + b_mx_m \geq L$, ...
  - Objective: $c_1x_1 + c_2x_2 + \cdots + c_mx_m$
- Defines a convex feasible region

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Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop

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Simplex algorithm

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- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

LP duality

- Can always construct a linear combination of constraints that tightly captures upper bound on objective function
- Dual LP problem
  - Minimize linear combination of constraints
  - Variables are multipliers for the linear combination
  - Implicit constraint: multipliers are non-negative
  - Optimum solution solves both the original (primal) and the dual LP

Production planning

Handwoven carpets

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  - Each produces 20 carpets a month
  - Salary Rs $20,000$
  - Labour cost: Rs 1000 per carpet

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  - $d_1, d_2, \ldots, d_{12}$, demand from January to December

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Coping with varying demand

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Coping with varying demand

- Overtime
  - Pay 80% extra
  - Overtime limit is 30% per worker

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- Overtime
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  - Overtime limit is 30% per worker
- Hiring and firing
  - Hiring costs Rs 3200 per worker
  - Firing costs Rs 4000 per worker

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Coping with varying demand

- Overtime
  - Pay 80% extra
  - Overtime limit is 30% per worker
- Hiring and firing
  - Hiring costs Rs 3200 per worker
  - Firing costs Rs 4000 per worker
- Make surplus and store
  - Costs Rs 80 per carpet

Formulate a linear program

- 30 employees, each 20 carpets a month, salary Rs $20,000$, Rs 1000 per carpet
- Monthly demand $d_1, d_2, \ldots, d_{12}$
- Overtime: pay 80% extra, overtime limit is 30% per worker
- Hiring cost Rs 3200, firing cost Rs 4000
- Surplus storage cost: Rs 80 per carpet

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  - $s_0 = 0$

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  - $s_0 = 0$
- 72 variables, plus w_0, s_0

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Constraints

- All variables are nonnegative
  - $w_i, x_i, o_i, h_i, f_i, s_i \geq 0$

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Constraints

- All variables are nonnegative
  - $w_i, x_i, o_i, h_i, f_i, s_i \geq 0$
- Carpets made = regular + overtime
  - $x_i = 20w_i + o_i$

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  - $x_i = 20w_i + o_i$
- Number of workers match hiring/firing
  - $w_i = w_{i-1} + h_i - f_i$

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- Number of workers match hiring/firing
  - $w_i = w_{i-1} + h_i - f_i$
- Number of stored carpets connected to earlier stock, production, demand
  - $s_i = s_{i-1} + x_i - d_i$
- Overtime production at most 6 carpets per worker (30% of regular production)
  - $o_i \leq 6w_i$

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Constraints

- $w_0 = 30$, $s_0 = 0$

For each $i \in \{1, 2, \ldots, 12\}$

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Objective

- Minimize the cost

$$20000(w_1 + w_2 + \cdots + w_{12}) +$$
$$3200(h_1 + h_2 + \cdots + h_{12}) +$$
$$4000(f_1 + f_2 + \cdots + f_{12}) +$$
$$80(s_1 + s_2 + \cdots + s_{12}) +$$
$$1800(o_1 + o_2 + \cdots + o_{12})$$

Solving the linear program

- Run Simplex and find a solution

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- Round off to 10 or 11 and recompute cost

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- If values are “large”, rounding does not affect quality of solution much

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- Values are "small", need more care when rounding
- Insisting on integer solutions makes the problem computationally intractable

  Integer Linear Programming