Course: Machine Learning - Foundations
Practice Questions - Solution
Lecture Details: Week 6

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1. (1 point) The function $f(x, y) = 2xy + y^2$

   A. has no stationary point

   B. has a stationary point at (0, 0)

   C. has a stationary point at (1, 1)

   D. has a stationary point at (-1, -1)

   Answer: B

   $f(x, y) = 2xy + y^2$,

   $f_x = \frac{\partial f}{\partial x} = 2y, f_y = \frac{\partial f}{\partial y} = 2x + 2y$

   Since, f_x, f_y are 0 at (0, 0). The origin is an stationary point for the function.

2. (1 point) The matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is

   A. positive definite

   B. positive semi-definite

   C. negative definite

   D. negative semi-definite

   Answer: A

   The eigenvalues are 3, $\frac{5+\sqrt{17}}{2}, \frac{5-\sqrt{17}}{2}$. Since, all eigenvalues are positive for the matrix, it is positive definite.

3. (1 point) The matrix $A = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}$ is positive definite.

   A. True

   B. False

   Answer: A

   $f(x, y) = v^T Av = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6x^2 + y^2$ where $v = \begin{bmatrix} x \\ y \end{bmatrix}$

   The function is always positive except where $v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   This indicates the matrix is positive definite.

4. (1 point) The function $f(x, y) = 4 + x^3 + y^3 - 3xy$ has a stationary point at
    A. (1, 1)
    B. (1, 2)
    C. (-1, 2)
    D. (2, -1)

Answer: A

$f(x, y) = 4 + x^3 + y^3 - 3xy$,
$f_x = \frac{\partial f}{\partial x} = 3x^2 - 3y, f_y = \frac{\partial f}{\partial y} = 3y^2 - 3x$
Since, f_x, f_y are 0 at (1, 1). The point is an stationary point for the function.

5. (1 point) The correct representation of $x^2 - z^2 + 2yz + 2xz$ in the matrix form is
    A. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
    B. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
    C. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
    D. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Answer: D

Perform matrix multiplication of the two adjacent matrix at a time and check which option is giving function described in the question.

6. (2 points) Given a function $f(x, y) = -3x^2 - 6xy - 6y^2$, the point $(0, 0)$ is a _________.
    A. maxima.
    B. minima.
    C. saddle point.
    D. none of these

Answer: A

$f(x, y) = -3x^2 - 6xy - 6y^2$,
$f_x = \frac{\partial f}{\partial x} = -6x - 6y, f_y = \frac{\partial f}{\partial y} = -6x - 12y$

Since, f_x, f_y are 0 at (0, 0). The origin is an stationary point for the function.
$f_{xx} = \frac{\partial f^2}{\partial x^2} = -6, f_{xy} = \frac{\partial f^2}{\partial x \partial y} = -6, f_{yy} = \frac{\partial f}{\partial y^2} = -12$
Since $f_{xx} < 0, D = f_{xx}f_{yy} - f_{xy}^2 = -6 * -12 - (-6)^2 = 36 > 0$, the point (0, 0) is a maxima.

7. (1 point) The matrix $\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$ is

   A. positive definite.

   B. positive semi-definite.

   C. neither positive definite nor positive semi-definite.

   D. can not be determined

Answer: C

a = 6, b = 5, c = 4, Since $a > 0, ac - b^2 = 6 * 4 - 5^2 = -1 < 0$, the matrix is neither positive definite nor positive semi-definite.

8. (1 point) (Multiple select) Select all the statements that are true about the matrix

$$A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

   A. A is positive definite.

   B. A is positive semi-definite.

   C. A is neither positive definite nor positive semi-definite.

   D. can not be determined

Answer: C

This is a diagonal matrix. For a diagonal matrix, the eigenvalues are elements on its principal diagonal. Since all the eigenvalues (diagonal elements) are negative, the matrix is a negative definite matrix.

9. (1 point) A matrix 2x2 A has determinant 8 and trace 6. Which of the following are true about the matrix?

   A. A is positive definite.

   B. A is positive semi-definite.

   C. A is neither positive definite nor positive semi-definite.

   D. Can not be determined

Answer: A

Suppose, the eigen values of the matrix are λ_1, λ_2.

Trace = $\lambda_1 + \lambda_2 = 6$, Determinant = $\lambda_1 * \lambda_2 = 8$ This indicates both λ_1, λ_2 are positive values. Therefore, the matrix is a positive definite matrix.

Questions 10-15 are based on common data

Consider the data points x_1, x_2, x_3 to answer the following questions.

$x_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

10. (1 point) The mean vector of the data points x_1, x_2, x_3 is

    A. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

    B. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

    C. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

    D. $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Answer: B

Mean vector = $\bar{X} = \Sigma_{i=1}^{n} \frac{1}{n} x_i = \frac{1}{3} \begin{bmatrix} (0+1+2) \\ (2+1+0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

11. (2 points) The covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ for the data points x_1, x_2, x_3 is

    A. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

    B. $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

    C. $\begin{bmatrix} 0.67 & -0.67 \\ -0.67 & 0.67 \end{bmatrix}$

    D. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answer: C

$C = \frac{1}{3}(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}) = \frac{1}{3} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

12. (2 points) The eigenvalues of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are

A. 0.5, 0.5

B. 1, 1

C. 4, 0

D. 0, 0

Answer: C

Characteristics equation:

$\begin{bmatrix} 2-\lambda & -2 \\ -2 & 2-\lambda \end{bmatrix} I = 0$

The determinant of the obtained matrix is $\lambda(\lambda - 4) = 0$

Eigenvalues:

The roots are $\lambda_1 = 4, \lambda_2 = 0$

Eigenvectors:

$\lambda_1 = 4, \begin{bmatrix} 2-\lambda & -2 \\ -2 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$

The null space of this matrix is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, Corresponding eigenvector is, $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0, \begin{bmatrix} 2-\lambda & -2 \\ -2 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

The null space of this matrix is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, Corresponding eigenvector is, $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

13. (2 points) The eigenvectors of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are (Note: The eigenvectors should be arranged in the descending order of eigenvalues from left to right in the matrix.)

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

D. $\begin{bmatrix} -0.7 & 0.7 \\ 0.7 & 0.7 \end{bmatrix}$

Answer: D

Refer the solution of the previous question.

14. (2 points) The data points x_1, x_2, x_3 are projected onto the one dimensional space using PCA as points z_1, z_2, z_3 respectively.

A. $z_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $z_2 = \begin{bmatrix}1\\1\end{bmatrix}$, $z_3 = \begin{bmatrix}1\\1\end{bmatrix}$

B. $z_1 = \begin{bmatrix}0.5\\0.5\end{bmatrix}$, $z_2 = \begin{bmatrix}0\\0\end{bmatrix}$, $z_3 = \begin{bmatrix}-0.5\\-0.5\end{bmatrix}$

C. $z_1 = \begin{bmatrix}0\\2\end{bmatrix}$, $z_2 = \begin{bmatrix}1\\1\end{bmatrix}$, $z_3 = \begin{bmatrix}2\\0\end{bmatrix}$

D. $z_1 = \begin{bmatrix}-1\\1\end{bmatrix}$, $z_2 = \begin{bmatrix}0\\0\end{bmatrix}$, $z_3 = \begin{bmatrix}1\\-1\end{bmatrix}$

Answer: D

$\lambda_1 = 4$, $u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$

$z_1 = \frac{1}{\sqrt{2}}(\begin{bmatrix}0 & 2\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix})\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$z_2 = \frac{1}{\sqrt{2}}(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix})\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$z_3 = \frac{1}{\sqrt{2}}(\begin{bmatrix}2 & 0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix})\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

15. (1 point) The approximation error J is given by $\Sigma_{i=1}^{n}||x_i - z_i||^2$. What could be the possible value of the reconstruction error?

A. 1

B. 2

C. 10

D. 20

Answer: B

Reconstruction error, $J = \frac{1}{n}\Sigma_{i=1}^{n}||x_i - z_i||^2 = \frac{1}{3}[(1^2+1^2)+(1^2+1^2)+(1^2+1^2)] = 2$