Machine Learning Foundations

Chapter 6: Probability

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Outline for Chapter 6 : Probability

6.1 : Discrete Random Variables

6.2 : Continuous Random Variables

6.3 : Maximum Likelihood and other advanced topics

Outline for Chapter 6 : Probability

6.1 : Discrete Random Variables

1. Probability space
2. **Conditioning**
3. Random variables
4. Expectation and Variance
5. Multiple Random Variables
6. Bernoulli, Binomial, Poisson and Geometric RVs

6.2 : Continuous Random Variables

6.3 : Maximum Likelihood and other advanced topics

Chapter 6.1.2 : Conditioning

Conditional probability

e.g.: Two die throw experiment. P(sum =7l first=5),

Total probability

e.g. Two urn experiment with white and black balls (9W1B, 5W5B)

Independence

e.g. Two die throw experiment Sum=7, First=5, second=3

Bayes Rule

e.g. Two urn exp P(Urn1 is chosenlBall is white)

e.g P(COVID l PCR=True)

Conditional Probability

$A, B \subseteq \Omega$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability Examples

DDTE:

$$\Omega = \{1,2,3,4,5,6\}^2$$

$$\mathcal{F} = \{0,1\}^{\Omega}$$

$$P(A) = \frac{|A|}{36}$$

$$P(\text{Sum} = 7 \mid \text{First} = 5) = \frac{P(\text{Sum} = 7 \ \& \ \text{First} = 5)}{P(\text{First} = 5)}$$

$$= \frac{1/36}{6/36} = \frac{1}{6}$$

Conditional Probability Examples

$$P(\text{First} = 5 \mid \text{sum} = 8) = \frac{P(\text{First} = 5 \ \& \ \text{sum} = 8)}{P(\text{sum} = 8)}$$

$$= \frac{1/36}{5/36} = \frac{1}{5}$$

Total Probability Law

$B_1, B_2, \ldots B_n$ are mutually exclusive & exhaustive

$\forall i, j$, $i \neq j$: $B_i \cap B_j = \emptyset$ $\Rightarrow$ ME

$B_1 \cup B_2 \cdots \cup B_n = \Omega$ $\Rightarrow$ Exhaustive.

$A \subseteq \Omega$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$$

$$= P(A \mid B_1) P(B_1) + \cdots + P(A \mid B_n) P(B_n)$$

Total Probability Law Example

2UE : 2 Buckets 10 balls each

1^{st} bucket : 9W , 1B

2^{nd} bucket : 5W, 5B

P(White Ball is picked)

A : White ball is picked

B_1 : Urn 1 is chosen

B_2 : Urn 2 is chosen

$$P(A) = P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2)$$

$$= \frac{9}{10}\cdot\frac{1}{2} + \frac{5}{10}\cdot\frac{1}{2}$$

Independence

$A, B \subseteq \Omega$

A, B are independent events $\iff P(A \cap B) = P(A) \cdot P(B)$

Independence Examples

DDTE : $P(A) = \frac{|A|}{36}$

A = Sum is 7

B = First is 5

C = Second is 3

D = Sum = 4

$P(B \cap D) = 0$

$P(B) = \frac{1}{6}$

$P(D) = \frac{3}{36}$

Bayes Rule

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B \mid A)\, P(A) + P(B \mid A^c)\, P(A^c)}$$

Bayes Rule Examples

2UE

$P(\text{Urn 1 was chosen} \mid \text{Ball is white})$

$A \to$ Ball is white

$B_1 \to$ Urn 1 $\qquad B_2 \to$ Urn 2

$P(B_1 \mid A) = ?$

$P(A \mid B_1) = 9/10$

$P(B_1) = 1/2$

$P(A \mid B_1^c) = 5/10$

$$P(B_1 \mid A) = \frac{9/10 \cdot 1/2}{9/10 \cdot 1/2 + 5/10 \cdot 1/2}$$

$$= \frac{9}{14}$$

Bayes Rule Examples

C is the event that the chosen person has COVID

T is the event the chosen person's test is positive.

$$P(C \mid T) = ?$$

Test is such that

$$P(T \mid C) = 0.9 \qquad P(C) = 1/100$$

$$P(T^c \mid C^c) = 0.9$$

$$P(C|T) = ?$$

Test is such that

$$P(T|C) = 0.9 \qquad P(C) = 1/100$$

$$P(T^c|C^c) = 0.9$$

$$P(C|T) = \frac{P(T|C)\,P(C)}{P(T|C)\,P(C) + P(T|C^c)\,P(C^c)}$$

$$= \frac{(0.9)(1/100)}{0.9(1/100) + 0.1\,(99/100)} = \frac{0.9}{0.9 + 9.9} =$$