Course: Machine Learning - Foundations
Practice Questions
Lecture Details: Week 9

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1. (1 point) Let $f(x) = -2x^2 + 5$. At $x = -3$, is $f(x)$ increasing or decreasing?

   A. increasing

   B. decreasing

   Answer: A

   If the first derivative of $f(x)$ at any point x is positive or negative, then we call $f(x)$ as increasing or decreasing respectively.
   $f'(x) = -4x$, at $x = -3$ we get $f'(-3) = 12$. Hence increasing.

2. (1 point) For a function $f(x) = -x + 2x^2$, the global minimum occurs at

   A. $x = -0.25$

   B. $x = -0.5$

   C. $x = 0.5$

   D. $x = 0.25$

   Answer: D

   Global minimum occurs at a point x when $\nabla(f(x)) = 0$.
   $-1 + 4x = 0$
   $x = 0.25$

3. (1 point) (Multiple select) Consider two convex functions $f(x) = x^2$ and $g(x) = e^{3x^2}$. Choose the correct convex function(s) that is a resultant of combination of $f(x)$ and $g(x)$.

   A. $h(x) = x^2 + e^{3x^2}$

   B. $h(x) = x^2 e^{-3x^2}$

   C. $h(x) = x^2 e^{3x^2}$

   D. $h(x) = x^2 - e^{3x^2}$

   Answer: A,C

   $h(x) = f(x) + g(x) = x^2 + e^{3x^2}$ is a convex function $h(x) = f(x) \times g(x) = x^2 e^{3x^2}$ is a convex function

4. (1 point) Consider two functions $g(x) = 2x - 3$ and $f(x) = x - 10ln(5x)$. Select the true statement.

   A. $h = fog$ is a convex function.

   B. $h = fog$ is a concave function.

Answer: A

Since $f(x)$ is a convex function and g(x) is a linear function, $h = fog$ is also a convex function.

(Common data for Q5-Q7) Given below is a set of data points and their labels.

X	y
[1,0]	1.5
[2,1]	2.9
[3,2]	3.4
[4,2]	3.8
[5,3]	5.3

To perform linear regression on this data set, the sum of squares error with respect to w is to be minimized.

5. (1 point) Which of the following is the optimal w^* computed using the analytical method?

   A. $\begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$

   B. $\begin{bmatrix} 1.255 \\ -0.317 \end{bmatrix}$

   C. $\begin{bmatrix} 1.512 \\ 0.004 \end{bmatrix}$

   D. $\begin{bmatrix} -1.5 \\ 0 \end{bmatrix}$

Answer: B

optimal $w^* = (X^T X)^{-1}(X^T y)$

$$X \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 55 & 31 \\ 31 & 18 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 59.2 \\ 33.2 \end{bmatrix}$$

$$w^* = \begin{bmatrix} 1.255 \\ -0.317 \end{bmatrix}$$

6. (1 point) Let w^1 be initialized to $\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$. Gradient descent optimization is used to find the value of optimal w^*. For the first iteration $t = 1$, which of the following is the gradient computed with respect to w^1?

A. $\begin{bmatrix} 50.6 \\ 28.3 \end{bmatrix}$

B. $\begin{bmatrix} 8.6 \\ 4.9 \end{bmatrix}$

C. $\begin{bmatrix} -50.6 \\ -28.3 \end{bmatrix}$

D. $\begin{bmatrix} -8.6 \\ -4.9 \end{bmatrix}$

Answer: C

Gradient $\nabla f(w) = (X^T X)w - X^T y$

$\nabla f(w) = \begin{bmatrix} -50.6 \\ -28.3 \end{bmatrix}$

7. (1 point) Using the gradient descent update equation with a learning rate $\eta_t = 0.1$, compute the value of w at $t = 2$.

A. $\begin{bmatrix} 5.16 \\ -2.93 \end{bmatrix}$

B. $\begin{bmatrix} 5.16 \\ 2.93 \end{bmatrix}$

C. $\begin{bmatrix} 5.5 \\ 3.5 \end{bmatrix}$

D. $\begin{bmatrix} 5.5 \\ -3.5 \end{bmatrix}$

Answer: B

Given $w^1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$

update equation: $w^2 = w^1 - \eta_t \nabla f(w^1)$

$w^2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} - 0.1 \begin{bmatrix} -50.6 \\ -28.3 \end{bmatrix}$

$w^2 = \begin{bmatrix} 5.16 \\ 2.93 \end{bmatrix}$

(Common data for Q8-Q10) A rectangle has a perimeter of 20 m. Using the Lagrange multiplier method, find the height and width of the rectangle which results in maximum area.

8. (1 point) What is the optimal height?

Answer: 5

range: 4.5, 5,5

9. (1 point) What is the optimal width?

   Answer: 5

   range: 4.5, 5,5

10. (1 point) Enter the value of Lagrange multiplier.

    Answer: 2.5

    The optimization problem can be stated as follows:

    $$\max f(x, y) = xy$$

    subject to

    $$2x + 2y = 20$$

    Solve the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$
    $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$
    $y = 2\lambda$ and $x = 2\lambda$
    Therefore, $\frac{y}{2} = \lambda = \frac{x}{2}$ will lead to $x = y$.
    Substituting $x = y$ in $2x + 2y = 20$ we get $y = 5$. Therefore $x = 5$ and $\lambda = 2.5$.

11. (1 point) (Multiple select) Which of the following statements about primal and dual problems is (are) true?.

    A. Dual of dual is primal.

    B. If either the primal or dual problem has an infeasible solution, then the value of the objective function of the other is unbounded.

    C. If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.

    D. If one of the variables in the primal has unrestricted sign, the corresponding constraint in the dual is satisfied with equality.

    Answer: A,B,C,D

    (Common data for Q12, Q13) Rahul is a consumer who wants to maximize his utility subject to some constraints. He consumes two goods x and y and the utility function is the product of x and y. His budget is Rs.1000. The per unit price of goods x and y are Rs.15 and Rs.20 respectively.

12. (1 point) Choose the correct optimization problem.

    A. maximize $x + y$ subject to $(15x)(20y) = 1000$

    B. maximize xy subject to $15x + 20y = 1000$

    C. minimize xy subject to $15x + 20y = 1000$

    D. minimize $x + y$ subject to $(15x)(20y) = 1000$

    Answer: B

    Utility function to maximixe is xy
    constraint is the budget: $15x + 20y = 1000$

13. (1 point) Choose the equivalent Lagrange function for the problem.

A. $L(x, y, z) = x + y - \lambda(15x + 20y - 1000)$

B. $L(x, y, z) = x + y + \lambda(15x + 20y + 1000)$

C. $L(x, y, z) = xy + \lambda(15x + 20y - 1000)$

D. $L(x, y, z) = xy - \lambda(15x + 20y + 1000)$

Answer: C

Lagrange function $= f(x) + \lambda h(x) = xy + \lambda(15x + 20y - 1000)$

14. (2 points) Minimize the function $f = x_1^2 + 60x_1 + x_2^2$ subject to the constraints $g_1 = x_1 - 80 \geq 0$ and $g_2 = x_1 + x_2 - 120 \geq 0$ using KKT conditions. Which of the following is the optimal solution set?

A. $[x_1^*, x_2^*] = [80, 40]$

B. $[x_1^*, x_2^*] = [-80, -40]$

C. $[x_1^*, x_2^*] = [45, 75]$

D. $[x_1^*, x_2^*] = [-45, -75]$

Answer: A

The KKT conditions for the problem with multiple variables are given as follows:
(i) Stationary condition

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{2} u_j \frac{\partial g_j}{\partial x_i} = 0, i = 1, 2$$

Therefore, we get

$$2x_1 + 60 + u_1 + u_2 = 0 \tag{1}$$

$$2x_2 + u_2 = 0 \tag{2}$$

(ii) Complementary slackness condition

$$u_i g_i = 0, i = 1, 2$$

Therefore, we get

$$u_1(x_1 - 80) = 0 \tag{3}$$

$$u_2(x_1 + x_2 - 120) = 0 \tag{4}$$

(iii) Primal feasibility condition

$$g_i \geq 0$$

Therefore, we get

$$x_1 - 80 \geq 0 \tag{5}$$

$$x_1 + x_2 - 120 \geq 0 \tag{6}$$

(iv) Dual feasibility condition

$$u_i \geq 0$$

Therefore, we get

$$x_1 - 80 \geq 0 \tag{7}$$

$$x_1 + x_2 - 120 \geq 0 \tag{8}$$

From (3), either $u_1 = 0$ or $x_1 = 80$
Case (i): Let $u_1 = 0$
Substitute in (1);

$$2x_1 + 60 + u_2 = 0$$

$$x_1 = -\frac{u_2}{2} - 30 \tag{9}$$

Substitute in (2);

$$2x_2 + u_2 = 0$$

$$x_2 = -\frac{u_2}{2} \tag{10}$$

Substitute in (9) and (10) in (4);

$$u_2(-\frac{u_2}{2} - 30 - \frac{u_2}{2} - 120) = 0$$

$$u_2(-u_2 - 150) = 0$$

From this, $u_2 = 0$ or $u_2 = -150$
Using $u_2 = 0$ in (9) and (1) we get, $x_1 = -30$ and $x_2 = 0$ respectively. This violates (5) and (6).
Using $u_2 = -150$ in (9) and (10) we get, $x_1 = 45$ and $x_2 = 75$, respectively. This violates (5).
Case (ii): Let $x_1 = 80$, substitute in (1)

$$2(80) + 60 + u_1 + u_2 = 0$$

$$u_2 = -u_1 - 220 \tag{11}$$

Substitute (11) in (2):

$$u_1 = 2x_2 - 220 \tag{12}$$

Using (12) in (11) we get $u_2 = -2x_2$. Using $x_1 = 80$ and $u_2 = -2x_2$ in (4) we get

$$-2x_2(80 + x_2 - 120) = 0$$

$$x_2(x_2 - 40) = 0 \tag{13}$$

From this, either $x_2 = 0$ or $x_2 = 40$.
Case (ii-a): Let $x_2 = 0$, then from (12) we get $u_1 = -220$.
Substituting $x_1 = 80$, $x_2 = 0$ and $u_1 = -220$ in (6) will violate the condition.
Case (ii-b): Let $x_2 = 40$
Substituting $x_1 = 80$, $x_2 = 40$ in (5) and (6), the conditions are satisfied.
Substituting $x_1 = 80$, $x_2 = 40$ in (11) and (12), we get $u_1 = -140$ and $u_2 = -80$. These values satisfy conditions (7) and (8).
Thus the optimal solution is $[x_1^*, x_2^*] = [80, 40]$ because it satisfies the KKT conditions.