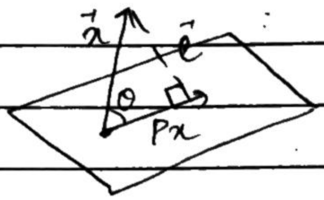
Q1.) Projection matrix P: projection onto a plane



i) $Px = x$ for any x that lies on the plane

if taking $\cos(\theta) = \frac{Px}{x} = 1$ → here x must be NON-ZERO

Since $Px = x$.

then $\boxed{\theta = 0°}$

hence $\vec{x}$ lies in column space of P.

$\Rightarrow \boxed{\lambda = 1}$ and any vector x in plane is eigenvector.

ii) $Px = 0$ if x $\perp^r$ to plane.

if taking $\cos(\theta) = \frac{Px}{x} = \frac{0}{x} = 0$

given $x \neq 0$.

then $\cos(\theta) = \frac{0}{x} = 0$.

$\boxed{\theta = \cos^{-1}(0) = 90°}$

hence $\vec{x}$ orthogonal to column space of P.

So $\boxed{\lambda = 0}$ and any x $\perp^r$ to plane is eigenvector.

Summary: $Px = \lambda x$ (given $\vec{x} \neq 0$)

$Px = \boxed{1}x$ if x lies in column space of P

$Px = \boxed{0}x$ if x $\perp^r$ to column space of P

Q2.) If 'A' is a Real Symmetric Matrix :

1) Eigenvalues of 'A' are always Real

2) Eigenvectors corresponding to distinct eigenvalues form a linearly independent set.

3) A is orthogonally diagonizable

{A is diagonizable if $\exists$ S such that $S^{-1}AS = \lambda$ where S is an invertible matrix}

{A is orthogonally diagonizable if $\exists Q$ such that $Q^TQ = I$ or $Q^{-1} = Q^T$ so that $A = Q\lambda Q^T$}

Hence, statements 1, 3 and 5 are true based on above properties of Real Symmetric Matrices.

Q3.) Suppose $c_1x_1 + c_2x_2 = 0 \quad (\times A)$

$$A(c_1x_1 + c_2x_2) = 0$$

$$c_1(Ax_1) + c_2(Ax_2) = 0$$

$$c_1(\lambda_1 x_1) + c_2(\lambda_2 x_2) = 0$$ {since $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$}

$$c_1\lambda_1 x_1 + c_2\lambda_2 x_2 = 0$$

$$(-)\ c_1x_1 + c_2x_2 = 0 \quad (\times \lambda_2)$$

gives

$$c_1\lambda_1 x_1 + \cancel{c_2\lambda_2 x_2} = 0$$

$$(-)\ c_1\lambda_2 x_1 + \cancel{c_2\lambda_2 x_2} = 0$$

$$c_1x_1(\lambda_1 - \lambda_2) = 0 \quad \text{—— (1)}$$

Here, assumption is λ_1 & λ_2 are eigenvalues of A with corresponding eigenvectors x_1 & x_2 and $\lambda_1 \neq \lambda_2$. (i.e., distinct eigenvalues)

$\therefore$ If $\lambda_1 \neq \lambda_2$ & $x_1 \neq 0 \Rightarrow \boxed{c_1 = 0}$ {in eq. (1)}

Similarly $\boxed{c_2 = 0}$

$\therefore \{x_1, x_2\}$ from a linearly independent set

Hence, eigenvectors of corresponding distinct eigenvalues of a matrix are linearly independent [Option (A)].

Q4:) By properties of eigenvalues:

Det(A) or $|A|$ = Product of eigenvalues

$|A| = \lambda_1 * \lambda_2 * \lambda_3$ [for a 3x3 matrix A]

Given: $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$

$|A| = (1) * (-2) * (3)$

$\boxed{|A| = -6}$

Ans = Option (D)

Q5:) By properties of eigenvalues:

Det(A) or $|A|$ = Product of eigenvalues

Trace (A) = Sum of eigenvalues

Given: det (A) = -6 ; trace (A) = -1

for a 2x2 matrix A

To find: λ_1 & λ_2

Solution: $-6 = \lambda_1 * \lambda_2$ — (1)

$-1 = \lambda_1 + \lambda_2$ — (2)

from (2), $\boxed{\lambda_2 = -1 - \lambda_1}$ — (3)

Substituting (3) in (1), we get:

$\lambda_1 * (-1 - \lambda_1) = -6$

$-\lambda_1 - \lambda_1^2 = -6$

$\lambda_1^2 + \lambda_1 - 6 = 0$

$\lambda_1^2 + 3\lambda_1 - 2\lambda_1 - 6 = 0$

$\lambda_1(\lambda_1 + 3) - 2(\lambda_1 + 3) = 0$

$(\lambda_1 + 3)(\lambda_1 - 2) = 0$

Gives $\boxed{\lambda_1 = -3}$ or $\boxed{\lambda_1 = 2}$

for $\lambda_1 = -3$:

$\lambda_2 = -1 - \lambda_1$

$= -1 - (-3)$

$= -1 + 3$

$\boxed{\lambda_2 = 2}$

for $\lambda_1 = 2$:

$\lambda_2 = -1 - \lambda_1$

$= -1 - (2)$

$\boxed{\lambda_2 = -3}$.

$\therefore$ Eigenvalues of A are $\boxed{2, -3}$.

Option (C)

Q6.) By properties of Eigenvalues :

trace (matrix) = Sum of eigenvalues.

Given : $\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 4$

trace $= +\lambda_1 + \lambda_2 + \lambda_3$

$= -1 + 0 + 4$

$\boxed{\text{trace} = 3}$

Q7.) By properties of Eigenvalues :

det(matrix) = Product of eigenvalues

Given : $\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 4$

det (matrix) $= \lambda_1 * \lambda_2 * \lambda_3$

$= (-1) * (0) * (4)$

$\boxed{\det = 0}$

Q8.) Characteristic polynomial for a 2x2 matrix :

$\lambda^2 - \boxed{\text{trace (matrix)}}\,\lambda + \det(\text{matrix}) = 0$

General way :

$\det(A - \lambda I) = 0$.

for $A_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$:

$\rightarrow$ trace = sum of diagonal elements

$\lambda^2 - (1+3)\lambda + (1*3 - 1*1) = 0$

$\boxed{\lambda^2 - 4\lambda + 2 = 0}$

Option (E).

Q9.) Characteristic polynomial of $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

$$\lambda^2 - (1+3)\lambda + (1*3 - 2*1) = 0$$

$$\lambda^2 - 4\lambda + 1 = 0$$

$$\Updownarrow$$

$$a\lambda^2 - b\lambda + c = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(Quadratic formula)}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$\boxed{\lambda = 2 \pm \sqrt{3}}$$

$\therefore \boxed{\lambda_1 = 2 + \sqrt{3}, \ \lambda_2 = 2 - \sqrt{3}}$ are the

Eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

Q10.) We know ~~that~~ $S^{-1} A^k S = \Lambda^k$

or $\boxed{A^k x = \lambda^k x}$

1) $A x_1 = \lambda_1 x_1$

$A x_1 = (0) x_1$

$\Rightarrow A^3 x_1 = \lambda_1^3 x_1 = (0)^3 x_1 = \boxed{0} x_1$

2) $A x_2 = \lambda_2 x_2$

$\Rightarrow A^3 x_2 = \lambda_2^3 x_2 = (-1)^3 x_2 = \boxed{-1} x_2$

3) $A x_3 = \lambda_3 x_3$

$\Rightarrow A^3 x_3 = \lambda_3^3 x_3 = (5)^3 x_3 = \boxed{125} x_3$.

$\therefore$ Eigenvalues of A^3 are $\boxed{0, -1, 125}$

Option (B)

Q11.) $F_k \quad \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$.

For $k : 110$, being a very large value, the second term in F_k becomes negligible and is ignored.

$$\therefore \boxed{F_{110} : \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{110}} \Rightarrow \text{option (A)}$$

Q12.) a) If A : $m \times n$ matrix then

$\text{rank}(A) + \text{nullity}(A) = n$ [Rank-nullity theorem]

Nullity is dimension of kernel where all vectors v are of form $Av = 0 = 0v$.

$\therefore \text{Rank}(A) = n - \left(\text{dimension of eigenspace corresponding to } 0\right)$

If A has 'r' non-zero eigenvalues: $\text{rank}(A) \leq r$.

It can be lesser than 'r' in case some of the non-zero eigenvalues are identical, generating eigenvectors that can be expressed as linear combination of the others.

Eigenvectors corresponding to distinct eigenvalues are linearly-independent and since rank of a matrix is max number of linearly independent vectors, computation of rank only considers linearly independent eigenvectors generated by distinct eigenvalues.

Hence, option (A) is <u>TRUE</u>

b) If one of the eigenvalues of A are zero,

$|A|$: product of eigenvalues

and multiplying any number of eigenvalues of A with zero leads to answer being 0.

$\therefore |A| = 0$.

Hence, option (B) is FALSE.

c) $Ax = \lambda x$ [multiply both sides by scalar 'k']

$\Downarrow$

$k \cdot Ax = k \cdot \lambda x$

$\boxed{A(kx) = \lambda(kx)}$

If x is an eigenvector of A, then so are all the multiples of x.

Hence, option (C) is TRUE.

d) If 0 is an eigenvalue of A,

as stated in b) $|A| = 0$.

Since $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$; if $|A| = 0$, A^{-1} is

undefined and hence, A cannot be inverted

Hence, option (D) is TRUE.

False Statements include option (B) $\Rightarrow$ Correct ans : (B).

Q13) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Characteristic polynomial : $\lambda^2 - (1+4)\lambda + (1*4 - 2*2) = 0$

$\therefore \lambda^2 - 5\lambda = 0$

$\lambda(\lambda - 5) = 0$

giving $\boxed{\lambda = 0, 5}$

Options (A, D).

Q14) for $\lambda = 0$,

$$[A - \lambda I] X = 0$$

$$\left\{ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0$$

let $\boxed{x_2 = k}$,

$$\boxed{x_1 = -2k}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \end{bmatrix} \longrightarrow \text{option (B)}$$

for $\lambda = 5$,

$$[A - \lambda I] X = 0 \Rightarrow \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow R_2 \to R_2 + \tfrac{1}{2}(R_1)$$

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 2x_2 = 0$$

let $\boxed{x_2 = k}$,

$$-4x_1 = -2k$$

$$\boxed{x_1 = \tfrac{1}{2}k}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} (\times 2)$$

$$x = k_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \text{option (A)}$$

Eigenvectors are $\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$

Q15.) $P^{-1}AP = \lambda = \begin{bmatrix}-1 & 0 & 3\\ 0 & 3 & 8\\ 0 & 0 & 4\end{bmatrix}$

Diagonal values of $P^{-1}AP$ correspond to λ values of A.

$\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = 4$

We know ~~that~~ $A^k x = \lambda^k x$

Eigenvalues of A^2 are:

1) $Ax_1 = \lambda_1 x_1$ where $\lambda_1 = -1$

$A^2 x_1 = \lambda_1^2 x_1 = (-1)^2 x_1 = \boxed{1}\, x_1$

2) $Ax_2 = \lambda_2 x_2$ where $\lambda_2 = 3$

$A^2 x_2 = \lambda_2^2 x_2 = (3)^2 x_2 = \boxed{9}\, x_2$

3) $Ax_3 = \lambda_3 x_3$ where $\lambda_3 = 4$

$A^2 x_3 = \lambda_3^2 x_3 = (4)^2 x_3 = \boxed{16}\, x_3$.

$\boxed{1, 9, 16}$ $\Rightarrow$ Option A, C, D.

Q16.) Best second degree polynomial.

x	y
0	0
1.3	1.5
4	1.2

$$y = \theta_0 + \theta_1 x + \theta_2 x^2$$

$$y_1 = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2 \qquad (y_1 = 0, x_1 = 0)$$
$$y_2 = \theta_0 + \theta_1 x_2 + \theta_2 x_2^2 . \qquad (y_2 = 1.5, x_2 = 1.3)$$
$$y_3 = \theta_0 + \theta_1 x_3 + \theta_2 x_3^2 \qquad (y_3 = 1.2, x_3 = 4).$$

$$y = \phi(x) \cdot \theta^T$$
$$\text{or } \theta^T = (\phi(x))^{-1} y$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.3 & 1.69 \\ 1 & 4 & 16 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1.5 \\ 1.2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A).$$
↳ Transpose of Cofactor matrix.

$$\det(A) = 1((1.3)(16) - 4(1.69)) - 0 + 0$$
$$= 14.04$$

$$C_{11} = \begin{vmatrix} 1.3 & 1.69 \\ 4 & 16 \end{vmatrix} = 14.04 \ ; \quad C_{12} = \begin{vmatrix} 1 & 1.69 \\ 1 & 16 \end{vmatrix} = -14.31$$

$$C_{13} = \begin{vmatrix} 1 & 1.3 \\ 1 & 4 \end{vmatrix} = 2.7 \ ; \quad C_{21} = \begin{vmatrix} 0 & 0 \\ 4 & 16 \end{vmatrix} = 0 \ ; \quad C_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 16 \end{vmatrix} = 16$$

$$C_{23} = \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = -4 \ ; \quad C_{31} = \begin{vmatrix} 0 & 0 \\ 1.3 & 1.69 \end{vmatrix} = 0 \ ; \quad C_{32} = \begin{vmatrix} 1 & 0 \\ 1 & 1.69 \end{vmatrix} = -1.69$$

$$C_{33} = \begin{vmatrix} 1 & 0 \\ 1 & 1.3 \end{vmatrix} = 1.3$$

$$\operatorname{adj}(A) = \begin{bmatrix} 14.04 & 0 & 0 \\ -14.201 & 16 & -1.69 \\ 2.7 & -4 & 1.3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -53/52 & 400/351 & \frac{-13}{108} \\ \frac{5}{26} & -\frac{100}{351} & \frac{5}{54} \end{bmatrix}$$

$$A^{-1}y = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-53}{52} & \frac{400}{351} & \frac{-13}{108} \\ \frac{5}{26} & \frac{-100}{351} & \frac{5}{54} \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \\ 1.2 \end{bmatrix}$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1831}{1170} \\ -\frac{37}{117} \end{bmatrix}$$

$\theta_0 = 0$; $\theta_1 = 1.564957$; $\theta_2 = -0.31624$

$$\therefore \boxed{y = 0 + 1.564957x - 0.31624x^2}$$