Tutorial
on
Four fundamental vector subspaces

Course: MACHINE LEARNING FOUNDATIONS

Ms. E. Amrutha

(Tutorial Instructor)

IIT Madras
ONLINE DEGREE

Four fundamental subspaces

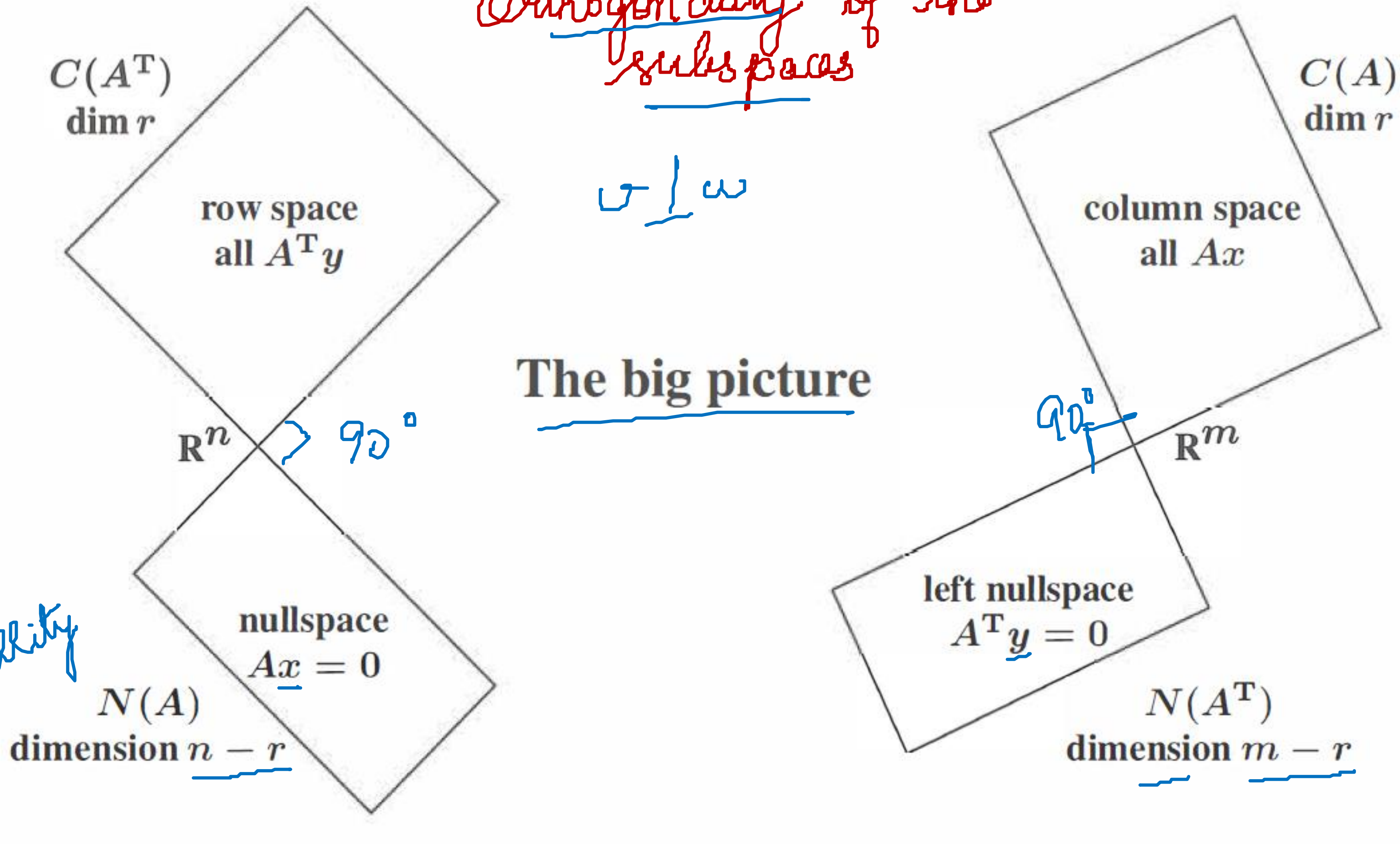
Suppose **A** is a $m \times n$ matrix.

$$A = \begin{array}{cc} & n \\ & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \Bigg\} m$$

1. The column space is C(**A**), a subspace of $\mathbb{R}^m$.
2. The row space is C($\mathbf{A}^T$), a subspace of $\mathbb{R}^n$.
3. The nullspace is N(**A**), a subspace of $\mathbb{R}^n$. $\boxed{Ax = 0}$
4. The left nullspace is N($\mathbf{A}^T$), a subspace of $\mathbb{R}^m$. $\boxed{A^T y = 0}$

Orthogonality of the subspaces

$v \perp w$



Dimension ?

Rank ?

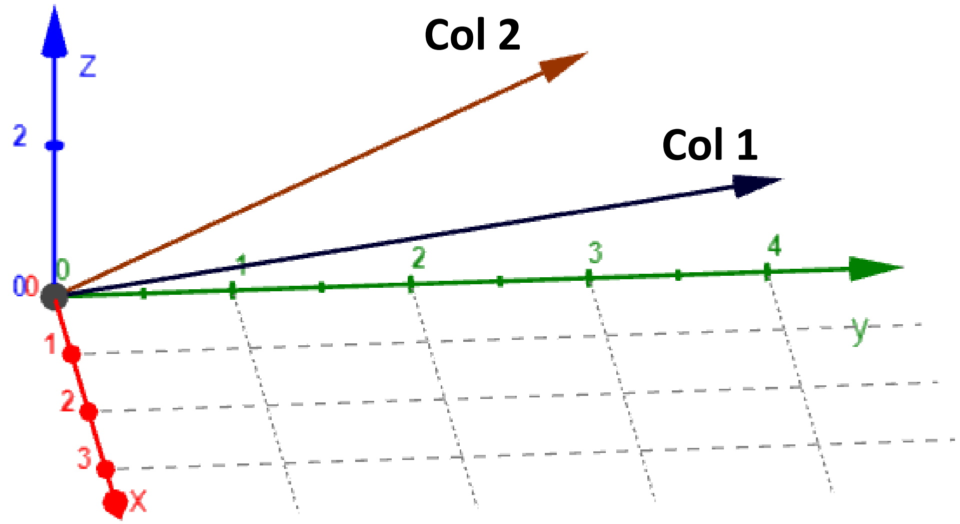
Rank nullity

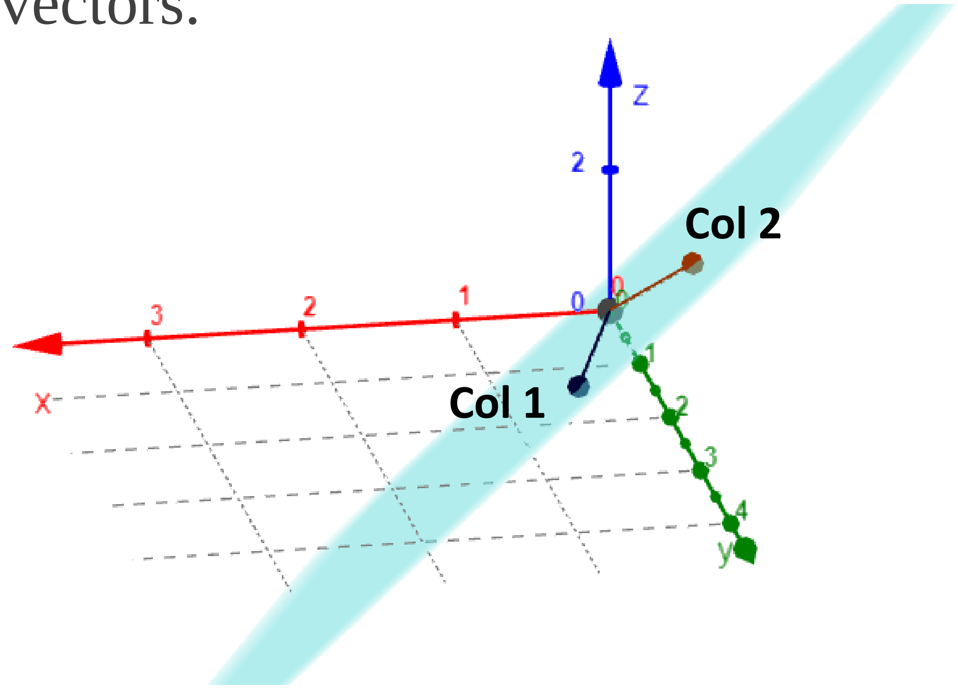
$\boxed{\text{rank} + \text{nullity} = n}$

(Image Source: Book - 'Introduction to Linear Algebra' by Gilbert Strang – Fifth Edition)

Column space visualization

- Consider a matrix: $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$
- C(**A**): Linear combinations of the column vectors.





https://www.geogebra.org/m/xk4qpm7c

Example 1

Obtain the four fundamental spaces of **A** and find its rank and nullity.

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 & 10 \\ 2 & 0 & 2 & 4 & -2 \\ 2 & 2 & 4 & 0 & 2 \\ 4 & 4 & 8 & 12 & 8 \end{bmatrix}$$

Column space

pivot

$$\begin{bmatrix} 2 & 4 & 6 & 8 & 10 \\ 2 & 0 & 2 & 4 & -2 \\ 2 & 2 & 4 & 0 & 2 \\ 4 & 4 & 8 & 12 & 8 \end{bmatrix}$$

Gaussian elimination

Row echelon form

$R_2 - R_1$

$R_3 - R_1$

$R_4 - 2R_1$

$$\begin{pmatrix} 2 & 4 & 6 & 8 & 10 \\ 0 & -4 & -4 & -4 & -12 \\ 0 & -2 & -2 & -8 & -8 \\ 0 & -4 & -4 & -4 & -12 \end{pmatrix}$$

$R_3 - \frac{1}{2}R_2$

$R_4 - R_2$

column space

$$C(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 0 \\ 12 \end{bmatrix} \right\}$$

Pivots

$$\begin{pmatrix} \boxed{2} & 4 & 6 & 8 & 10 \\ 0 & \boxed{-4} & -4 & -4 & -12 \\ 0 & 0 & 0 & \boxed{-6} & -2 \\ 0 & 0 & 0 & 6 & 0 \end{pmatrix}$$

Row echelon form

$\boxed{\text{rank} = 3} \Rightarrow \dim C(A)$

Null space

$$N(x) = \{x \mid Ax = 0\} \to Ux = 0$$

$$\begin{bmatrix} 2 & 4 & 6 & 8 & 10 \\ 0 & -4 & -4 & -4 & -12 \\ 0 & 0 & 0 & -6 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the homogenous system of linear equations are:

$2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 = 0$ — (1)

$-4x_2 - 4x_3 - 4x_4 - 12x_5 = 0$ — (2)

$-6x_4 - 2x_5 = 0$ — (3)

Set $x_3 = 1$; $x_5 = 0$

$x_4 = 0$

Sub x_3, x_4 & x_5 in (2)

$-4x_2 - 4 = 0$

$x_2 = -1$

Sub x_2, x_3, x_4, x_5 in (1)

$2x_1 + 4(-1) + 6 = 0$

$x_1 = -1$

$$u = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

set $x_3 = 0$; $x_5 = 1$

$$v = \begin{bmatrix} 5/3 \\ -8/3 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

$N(A) = \text{span}\{u, v\}$

$\dim N(A) = 2 \Rightarrow$ nullity

Row space $\rightarrow C(A^T)$

pivot

$$A^T = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 4 & 0 & 2 & 4 \\ 6 & 2 & 4 & 8 \\ 8 & 4 & 0 & 12 \\ 10 & -2 & 2 & 8 \end{bmatrix}$$

$R_2 - 2R_1$
$R_3 - 3R_1$
$R_4 - 4R_1$
$R_5 - 5R_1$

$$\begin{pmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & -4 & -2 & -4 \\ 0 & -4 & -8 & -4 \\ 0 & -12 & -8 & -12 \end{pmatrix} \xrightarrow[\substack{R_4 - R_2 \\ R_5 - 3R_2}]{R_3 - R_2} \begin{pmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_5} \begin{pmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_4 - 3R_3$

pivots

$$\begin{pmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(A)$

$\downarrow$

$$\text{span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$\dim R(A) = 3$

$\downarrow$

r

Left null space

$N(A^T) = \{ y \mid A^T y = 0 \}$

$$\begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The system of linear equations are :

$2y_1 + 2y_2 + 2y_3 + 4y_4 = 0$ —— (1)

$-4y_2 - 2y_3 - 4y_4 = 0$ —— (2)

$-2y_3 = 0$

$y_3 = 0$

Set $y_4 = 1$

$-4y_2 - 4 = 0$

$y_2 = -1$

$2y_1 + 2(-1) + 4(1) = 0$

$y_1 = -1$

$$y = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Left nullspace of A

$\downarrow$

$\text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$