Eigenvalues and eigenvectors

Note Title 4/26/2021

Motivation: Ordinary differential equations

$(*) \rightarrow \dfrac{dv}{dt} = 4v - 5w, \quad v = 8 \text{ at } t=0, \quad \dfrac{dw}{dt} = 2v - 3w, \quad w = 5 \text{ at } t=0$

$$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \qquad u(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$(*)$ can be re-written as

$$\frac{du}{dt} = Au, \text{ where } A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

and $u = u(0)$ at $t=0$.

Simple case of a single equation: $\dfrac{du}{dt} = au$ —— $(**)$

Solution: $u(t) = e^{at} u(0)$ $\leftarrow$ easy to see that this is a solution of $(**)$

$$(**)\begin{cases} \text{is unstable if } a>0 & (e^{at}\to\infty \text{ as } t\to\infty) \\ \text{is neutrally stable if } a=0 \\ \text{is stable if } a<0 \end{cases}$$

Case of a 2d-parameter:

Want solutions of the following form for the system (*):

$$v(t)= e^{\lambda t} y \quad , \quad w(t) = e^{\lambda t} z$$

(or) $\qquad u(t) = e^{\lambda t} x$, where $x = \begin{bmatrix} y \\ z \end{bmatrix}$

If we substitute this in $\frac{du}{dt} = Au$, we obtain

$$\lambda e^{\lambda t} y = 4 e^{\lambda t} y - 5 e^{\lambda t} z$$
$$\lambda e^{\lambda t} z = 2 e^{\lambda t} y - 3 e^{\lambda t} z$$

Take out $e^{\lambda t}$, we get

$$\boxed{\begin{aligned} 4y - 5z &= \lambda y \\ 2y - 3z &= \lambda z \end{aligned}} \iff \boxed{Ax = \lambda x} \longrightarrow \text{eigenvalue equation}$$

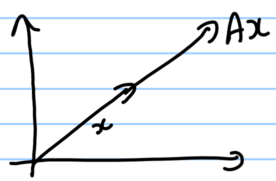
eigenvalue

eigenvector (non-trivial) $x \neq 0$

Message: Can solve $\frac{du}{dt} = Au$ using solutions of the form $\boxed{u(t) = e^{\lambda t}x}$ if $\boxed{Ax = \lambda x}$ can be solved.

Eigenvalues & eigenvectors:

For a matrix A, λ is a eigenvalue & $x \neq 0$ is a eigenvector if $Ax = \lambda x$.



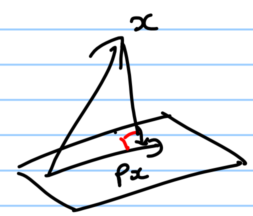
If x is an eigenvector, then A either stretches it or shrinks it, but A does not change the direction of x.

Case $\lambda = 0$: $Ax = 0 \implies x \in N(A)$

Examples:

1. Projection matrix P

e.g. project onto a plane



(i) $Px = x$ for any x in the plane.

$\Rightarrow$ $\lambda = 1$ is an eigenvalue & any x in the plane is an eigenvector

(ii) Consider an x that is perpendicular to the plane

$$Px = 0$$

So, $\lambda = 0$ is another eigenvalue & any $x \perp$ plane is an eigenvector.

2. Permutation matrix $\rightarrow$ take the identity matrix & shuffle its rows.

e.g. $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Bx = x$ for $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Bx = -x$ for $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Finding the eigenvalues:

$$Ax = \lambda x \iff (A - \lambda I)x = 0 \iff (A - \lambda I) \text{ is singular}$$

$$\iff \boxed{\det(A - \lambda I) = 0}$$

Characteristic polynomial of a matrix A.

If A is $n \times n$, then the characteristic polynomial is of degree n

$$(a_{11} - \lambda) \cdots\cdots (a_{nn} - \lambda) = 0$$

"n" roots of characteristic polynomial = eigenvalues

For eigenvectors: Given λ, want $(A - \lambda I)x = 0$

i.e., $x \in \underbrace{N(A - \lambda I)}$

find this to obtain an eigenvector corresponding to eigenvalue λ

(* can use Gaussian elimination)

Example: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda_1 + \lambda_2 = 6, \quad \lambda_1 \lambda_2 = 8$ (Check: $\lambda_1 + \lambda_2 = \text{trace}(A) =$ Sum of diagonal entries.

$\lambda_1 \lambda_2 = \det(A) =$ product of eigenvalues)

Eigenvalues: $\lambda_1 = 4, \quad \lambda_2 = 2$

$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad (A - 4I)x = 0$ for $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(A - 2I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad (A - 2I)x = 0$ for $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The eigenvectors are linearly independent

Variation! Suppose $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad B + 3I = A$

$$Ax = (B+3I)x = \lambda x + 3x = (\lambda+3)x$$

H.W. Eigenvectors of B = eigenvectors of A.

Remarks:

1. Suppose $Ax = \lambda_1 x$ and $Bx = \lambda_2 x$

Then, Can we claim $(A+B)x \overset{?}{=} (\lambda_1+\lambda_2)x$

NO: Since the x's need not be the same for A & B

ie, $Ax_1 = \lambda_1 x_1$ & $Bx_2 = \lambda_2 x_2$ with x_1 not necessarily equal to x_2.

2. Symmetric matrix has "real" eigenvalues.

A matrix whose eigenvalues are not real: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(A-\lambda I) = \lambda^2+1$$

$$\lambda_1 = i, \quad \lambda_2 = -i$$

3 A 2x2 matrix where we do not get "two independent" eigenvectors:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 3$$

$$(A-\lambda I) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ is a eigenvector &

there is no other eigenvector that is linearly independent of x.