Note Title 10-06-2021

Hermitian and Unitary matrices

Hermitian matrix:

Matrix A is <u>Hermitian</u> if $A^* = A$

Example: $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ $\quad A^* = \bar{A}^T = \begin{bmatrix} 2 & 3+3i \\ 3-3i & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} = A$

Note: Diagonal entries of a Hermitian matrix are real.

Properties of Hermitian matrices:

(I) If A is Hermitian, then all eigenvalues of A are real.

Consider the example matrix A given above. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3-3i|^2$

$= \lambda^2 - 7\lambda - 8 = (\lambda-8)(\lambda+1)$

So, the eigenvalues of A are $\lambda_1 = 8$ and $\lambda_2 = -1$

Proof of (I): Suppose $Ax = \lambda x$, $x \neq 0$, $\lambda \neq 0$.

$$(Ax)^* = (\lambda x)^* = \underset{\text{conjugate of } \lambda}{\bar{\lambda}}\, x^*$$

$$\Leftrightarrow \quad x^* A^* = \bar{\lambda} x^*$$

$$\Leftrightarrow \quad x^* A^* x = \bar{\lambda} x^* x$$

use $A^* = A$ $\quad \Leftrightarrow \quad x^* A x = \bar{\lambda} x^* x$

use $Ax = \lambda x$ $\quad \Leftrightarrow \quad x^* \lambda x = \bar{\lambda} x^* x$

$$\Leftrightarrow \quad \lambda x^* x = \bar{\lambda} x^* x$$

$\Leftrightarrow \quad \lambda = \bar{\lambda}$ since $x^* x \neq 0$ (since x is an eigenvector (or) $x \neq 0$)

So, λ is real.

(II) If A is Hermitian, then eigenvectors corresponding to different eigenvalues are orthogonal, i.e.,

If $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$, $\lambda_1 \neq \lambda_2$, then $x \cdot y = \bar{x}^T y = 0$.

Example: $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ with eigenvalues $8, -1$

Let's find the eigenvectors of A.

$$(A-8I)x = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{leading to} \quad x = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$(A+I)y = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{leading to} \quad y = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$$

$$x \cdot y = \bar{x}^T y = \begin{bmatrix} 1 & 1-i \end{bmatrix}\begin{bmatrix} 1-i \\ -1 \end{bmatrix} = 0$$

So, eigenvectors corresponding to 8 and −1 eigenvalues are orthogonal

Proof of (II): We have $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$, $\lambda_1 \neq \lambda_2$

To show: $x \cdot y = 0$

$$x \cdot Ay = x \cdot \lambda_2 y = \lambda_2 (x \cdot y) \quad \text{— (1)}$$

$$x \cdot Ay = x^* A y \underset{A=A^*}{=} x^* A^* y = (Ax)^* y = (\lambda_1 x)^* y = \bar{\lambda}_1 (x \cdot y) \underset{\bar{\lambda}_1 = \lambda \text{ from part (I)}}{=} \lambda_1 (x \cdot y) \quad \text{— (2)}$$

$x \cdot Ay = \lambda_2 (x \cdot y) = \lambda_1 (x \cdot y)$

Since $\lambda_1 \neq \lambda_2$, we have $x \cdot y = x^* y = 0$ $\blacksquare$

Remark:

① The equivalent of Hermitian matrices in the "real" case is "real symmetric matrices"

All "real symmetric" matrices are Hermitian

② If no eigenvalue is repeated ($\Leftrightarrow$ we have "n" distinct eigenvalues for a $n \times n$ matrix A), then A is diagonalizable. Why? Use property (II) to obtain "n" linearly independent eigenvectors, and use these eigenvectors to diagonalize the given matrix.

H.W. If x and y are orthogonal, show that $\{x, y\}$ is a linearly independent set.

In the example $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$, we had two distinct eigenvalues $\{8, -1\}$ with eigenvectors $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}, \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$

So, A is diagonalizable. In particular,

$$A = \underbrace{\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}}_{\substack{\text{a matrix with} \\ \text{eigenvectors as} \\ \text{columns}}} \underbrace{\begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{\text{a diagonal} \\ \text{matrix with} \\ \text{eigenvalues}}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}^{-1}$$