Outline for Chapter 6 : Probability

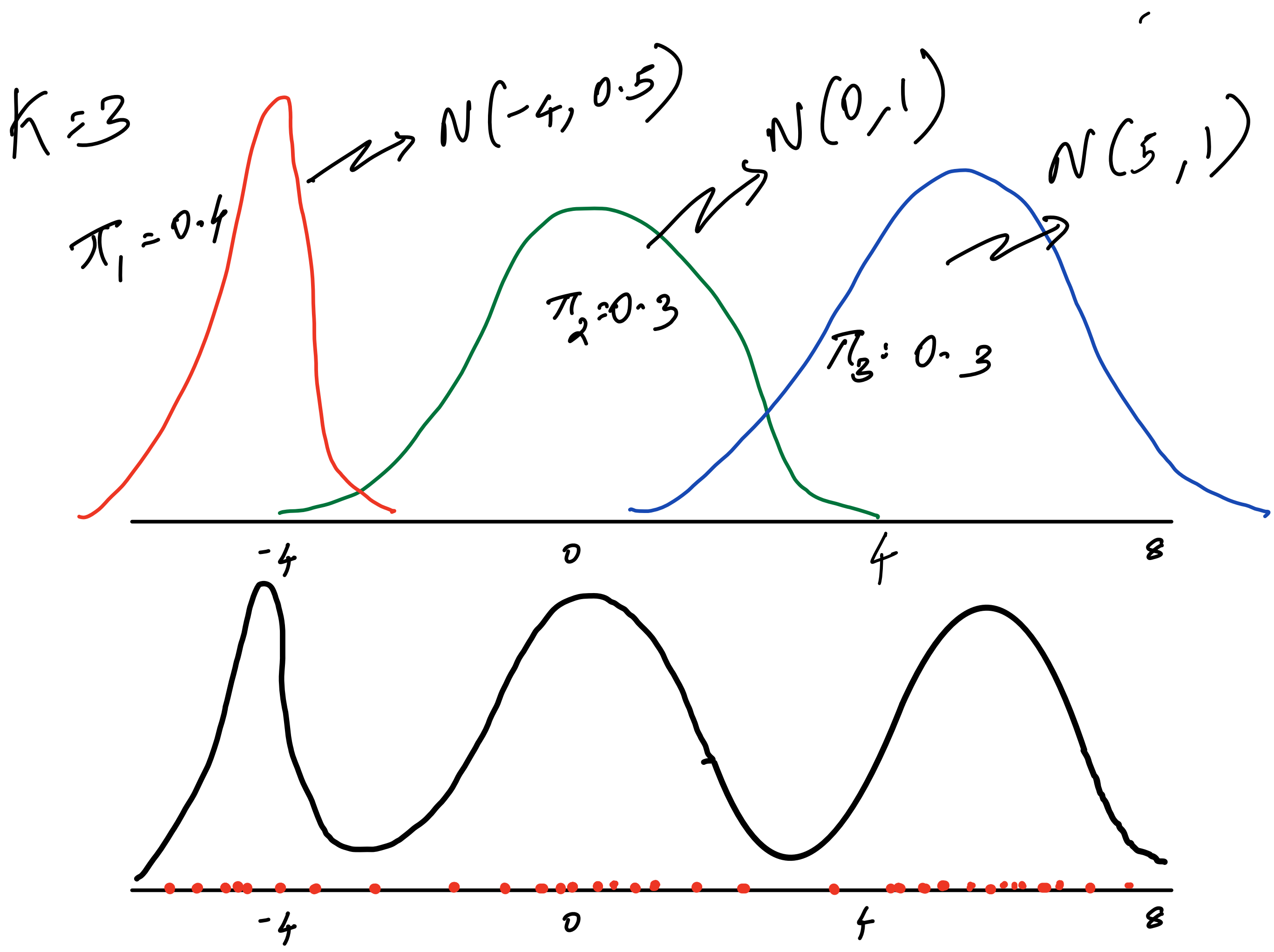
6.1 : Discrete Random Variables

6.2 : Continuous Random Variables

6.3 : Advanced topics

1. Bivariate and Multivariate normal
2. Estimation of parameters using ML
3. **Gaussian Mixture Models and Expectation Maximisation**
4. Law of large numbers

The Gaussian Mixture Model

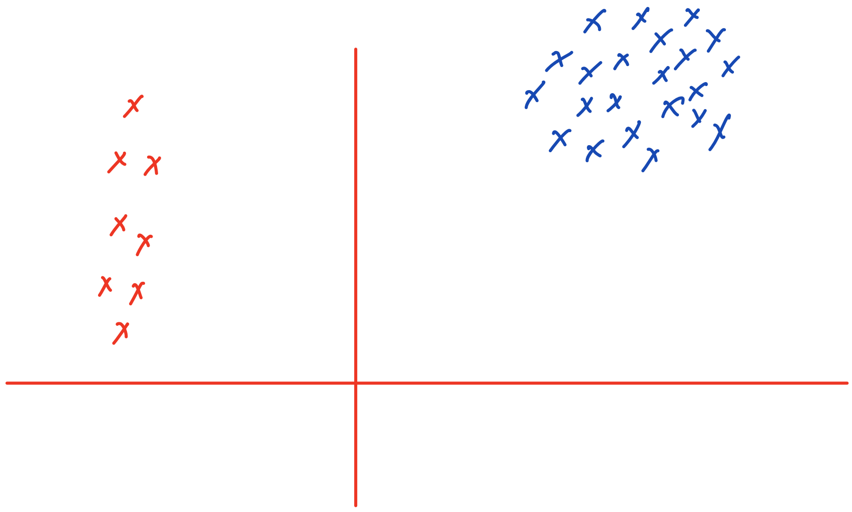


The Gaussian Mixture Model

$$f_X(x) = \sum_{k=1}^{K} \pi_k \, N(x \mid \mu_k, \Sigma_k)$$

$$= \sum_{k=1}^{K} \pi_k \frac{1}{(2\pi)^{d/2} \sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu_k)^T \Sigma_k^{-1} (x-\mu_k)\right)$$

The Gaussian Mixture Model



The Gaussian Mixture Model

$$P(Z=k) = \pi_k$$

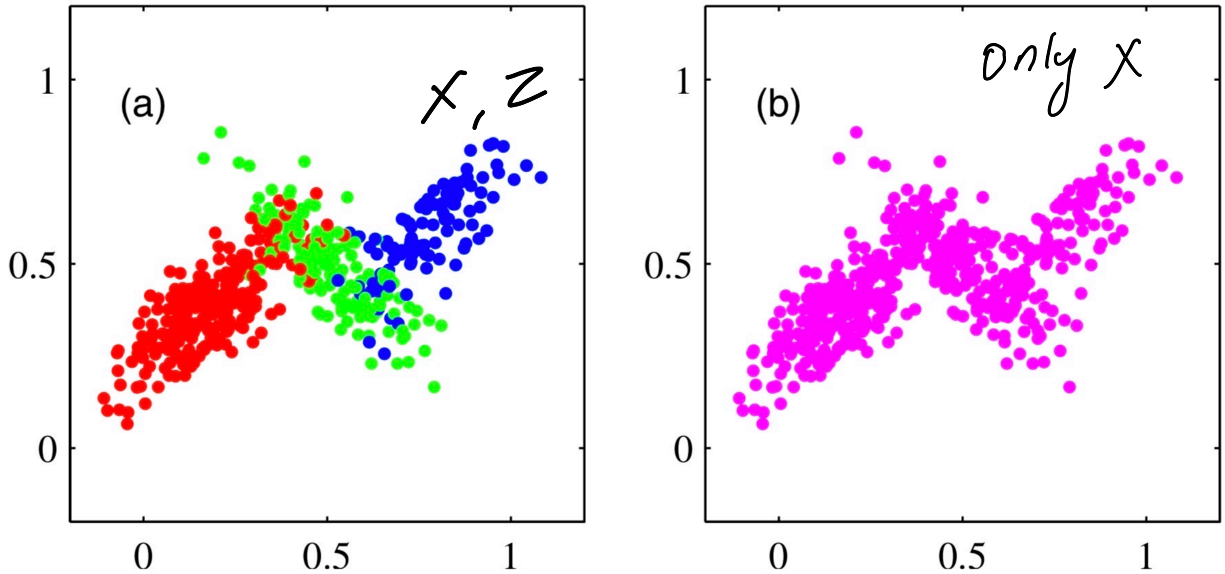
$$X \mid Z=k \sim N(\mu_k, \Sigma_k)$$

$$P(X) = \sum_Z P(X, Z)$$

$$= \sum_{k=1}^{K} P(X \mid Z=k) \cdot P(Z=k)$$

$$= \sum_{k=1} \pi_k \, N(x \mid \mu_k, \Sigma_k)$$

The Gaussian Mixture Model



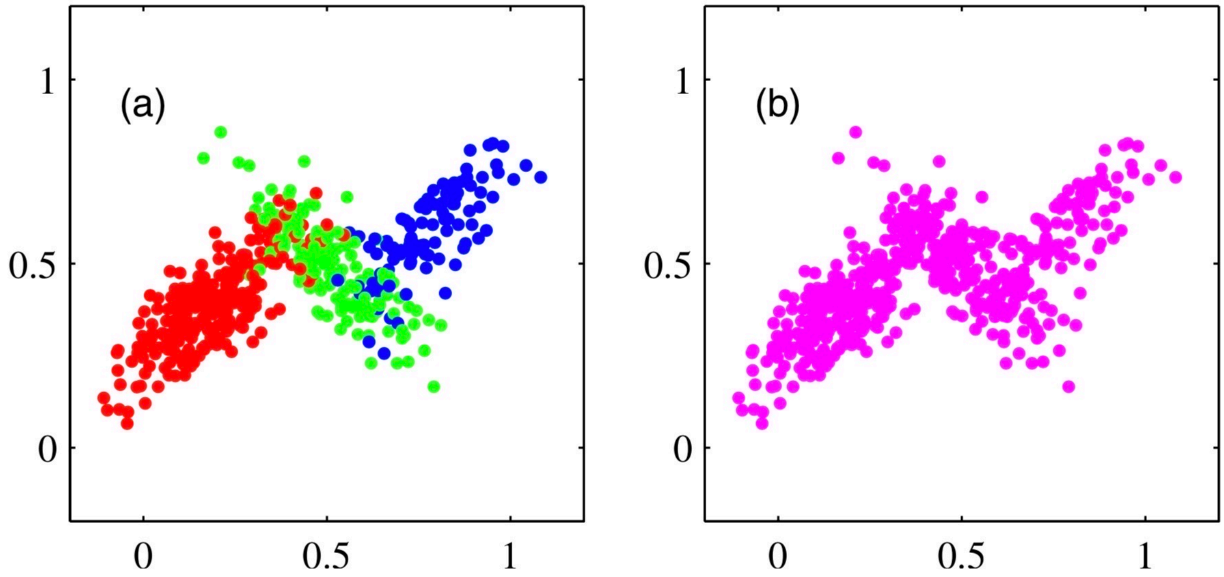
Maximum Likelihood Estimation

Data: $\{x_1, \ldots, x_N\}$

$$P(\text{Data} \mid \pi_1 \ldots \pi_K, \mu_1 \ldots \mu_K, \Sigma_1 \ldots \Sigma_K)$$

$$= \prod_{n=1}^{N} \left(\sum_{k=1}^{K} \pi_k \mathcal{N}(x_i \mid \mu_k, \Sigma_k) \right)$$

Chicken and Egg Problem



Cluster Responsibilities

E step. Evaluate the responsibilities using the current parameter

$$P(Z=k \mid X=x_n) = \gamma(z_{nk}) = \frac{\pi_k \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{j=1}^{K} \pi_j \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)}.$$

$$= \frac{P(X=x_n \mid Z=k)\, P(Z=k)}{\sum_j P(X=x_n \mid Z=j)\, P(Z=j)}$$

Parameter Updates

M step. Re-estimate the parameters using the current responsibiliti

$$
\begin{aligned}
\boldsymbol{\mu}_k^{\text{new}} &= \frac{1}{N_k}\sum_{n=1}^{N}\gamma(z_{nk})\mathbf{x}_n \\
\boldsymbol{\Sigma}_k^{\text{new}} &= \frac{1}{N_k}\sum_{n=1}^{N}\gamma(z_{nk})\left(\mathbf{x}_n - \boldsymbol{\mu}_k^{\text{new}}\right)\left(\mathbf{x}_n - \boldsymbol{\mu}_k^{\text{new}}\right)^{\mathrm{T}} \\
\pi_k^{\text{new}} &= \frac{N_k}{N}
\end{aligned}
$$

where

$$N_k = \sum_{n=1}^{N} \gamma(z_{nk})$$

Illustration