

## Unitary matrices:

Def: A matrix is unitary if it is square, and has orthonormal columns

Real case:  $Q^T Q = I \Leftrightarrow Q$  is orthogonal and  $Q^{-1} = Q^T$

$$\Leftrightarrow \underbrace{\begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}}_{\text{columns}} \text{ then } v_i^T v_j = 0 \ \forall i \neq j \ \& \ \|v_i\| = 1 \ \forall i = 1 \dots n$$

Complex case:  $U^* U = I \Leftrightarrow U$  is unitary, and  $U^{-1} = U^*$

Example! ①  $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Check that  $U$  is unitary.

②  $U = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

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Properties of unitary matrices: Let  $U$  be a unitary matrix, i.e.,  $U^* U = I$

(I) "Length unchanged"  $\|Ux\| = \|x\|$

Pf:  $Ux \cdot Uy = (Ux)^* Uy = x^* U^* Uy = x^* y = x \cdot y$

$\Rightarrow \|Ux\|^2 = Ux \cdot Ux = x \cdot x = \|x\|^2$

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(II) Eigenvalues of a unitary matrix  $U$  have absolute value 1, i.e.,  
If  $\lambda$  is an eigenvalue of  $U$ , then  $|\lambda| = 1$ .

Pf:  $Ux = \lambda x, x \neq 0$ . Also,  $\|Ux\| = \|x\| \Rightarrow \|\lambda x\| = \|x\| \Rightarrow |\lambda| \|x\| = \|x\| \Rightarrow |\lambda| = 1$  since  $\|x\| \neq 0$ .

Another pf:  $Ux \cdot Ux = \lambda x \cdot \lambda x = \lambda \bar{\lambda} (x \cdot x) \stackrel{\uparrow}{=} (x \cdot x) \Rightarrow \lambda \bar{\lambda} = 1 \Rightarrow |\lambda| = 1.$   
 Since  $\|Ux\| = \|x\|$

III) Eigenvectors corresponding to different eigenvalues of a unitary matrix  $U$  are orthogonal.

pf:  $Ux = \lambda_1 x, \quad Uy = \lambda_2 y, \quad \lambda_1 \neq \lambda_2$   
 $x \cdot y = Ux \cdot Uy = (\lambda_1 x) \cdot (\lambda_2 y) = \bar{\lambda}_1 \lambda_2 (x \cdot y)$   
 $\Rightarrow (\bar{\lambda}_1 \lambda_2 - 1)(x \cdot y) = 0 \quad (*)$

This would imply  $x \cdot y = 0$  if we show  $\bar{\lambda}_1 \lambda_2 \neq 1.$

Suppose  $\bar{\lambda}_1 \lambda_2 = 1.$

Then,  $\lambda_1 \bar{\lambda}_1 \lambda_2 = \lambda_1$

Using  $|\lambda_1| = 1$  or  $\lambda_1 \bar{\lambda}_1 = 1$ , we get  $\lambda_2 = \lambda_1$ , a contradiction

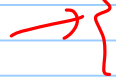
So,  $\bar{\lambda}_1 \lambda_2 \neq 1$  implying  $x \cdot y = 0$  from  $(*)$

Coming next: For a Hermitian matrix  $A$ , we can find a unitary matrix  $U$  s.t.

$A = U \Lambda U^*$ ,  $\Lambda$  is a diagonal matrix with eigenvalues of  $A$ .

Real case! For a real symmetric matrix  $A$ , we can find an orthogonal matrix  $Q$  ( $Q^T Q = I$ ) s.t.

Spectral  
Theorem



$$A = Q \Lambda Q^T$$