Tutorial on Linear Programming, Karush-Kuhn-Tucker Conditions, Relationship between primal and dual problem

Course: MACHINE LEARNING FOUNDATIONS

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Linear Programming (LP)

- Linear programming is a subclass of convex optimization problem.
- Both the constraints and the objective function are linear functions.
- It is about solving systems of linear inequalities.

Example

- Consider the following linear program.

$$\text{minimize} \quad \underbrace{3x_1 + x_2}_{f(x)} \qquad \text{subject to} \begin{cases} x_1 - x_2 + 4 \leq 0 & g_1(x) \\ -3x_1 + 2x_2 + 10 \leq 0 & g_2(x) \\ x_1, x_2 \geq 0 \end{cases}$$

Karush-Kuhn-Tucker Conditions

- Stationarity $\boxed{\nabla f(x) + \sum_{i=1}^{n} u_i \nabla g(x) + \sum_{j=1}^{m} v_j \nabla h(x) = 0}$
- Complementary slackness $\boxed{u_i g_i = 0} \ \forall i$
- Primal feasibility $\boxed{g_i(\mathrm{x}) \leq 0} \ \forall i$
- Dual feasibility $\boxed{u_i \geq 0} \ \forall i$

$$\textit{minimize} \quad 3x_1 + x_2$$

$$\textit{subject to} \begin{cases} x_1 - x_2 + 4 \le 0 \\ -3x_1 + 2x_2 + 10 \le 0 \\ x_1, x_2 \ge 0 \end{cases}$$

Stationarity conditions

$$3 + u_1 - 3u_2 = 0 \quad \text{— } 1$$

$$1 - u_1 + 2u_2 = 0 \quad \text{— } 2$$

Complementary slackness conditions

$$u_1(x_1 - x_2 + 4) = 0 \quad \text{— } 3$$

$$u_2(-3x_1 + 2x_2 + 10) = 0 \quad \text{— } 4$$

Primal feasibility conditions

$$x_1 - x_2 + 4 \le 0$$

$$-3x_1 + 2x_2 + 10 \le 0$$

Dual feasibility conditions

$$u_1, u_2 \ge 0$$

From 1 $\quad u_1 = 3u_2 - 3$

Substituting in 2

$$1 - 3u_2 + 3 + 2u_2 = 0$$

$\boxed{u_2 = 4}$ Substituting in 1 $\boxed{u_1 = 9}$

From 3

$$x_1 - x_2 + 4 = 0$$

$$x_1 = x_2 - 4 \quad \text{— } 5$$

Substituting in 4

$$u_2(-3(x_2 - 4) + 2x_2 + 10) = 0$$

$$u_2(-x_2 + 22) = 0 \qquad \boxed{x_2 = 22} \quad \text{— } 6$$

Substituting 6 in 5

$$x_1 = 22 - 4$$

$$\boxed{x_1 = 18}$$

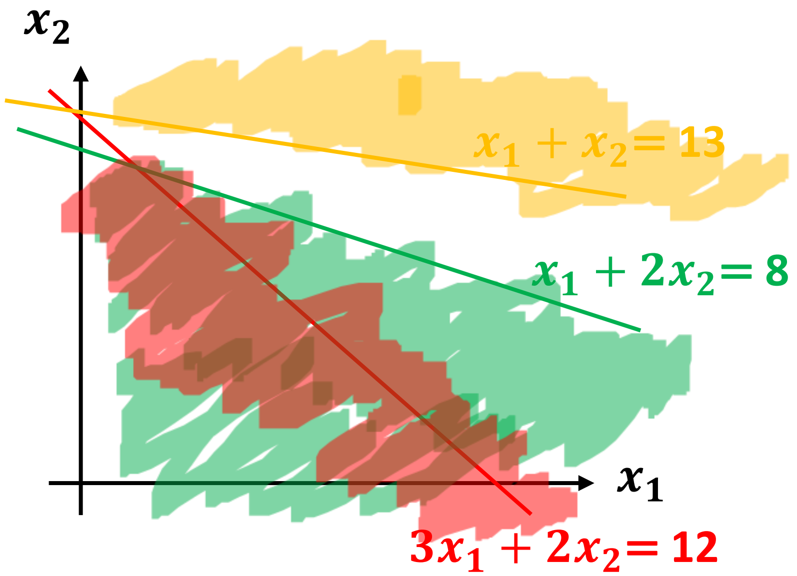
Minimum value of objective function is

$$\boxed{f(x_1, x_2) = 3 \times 18 + 22 = 76}$$

Types of solution possibilities for LP

$$\min \quad x_1 + x_2$$

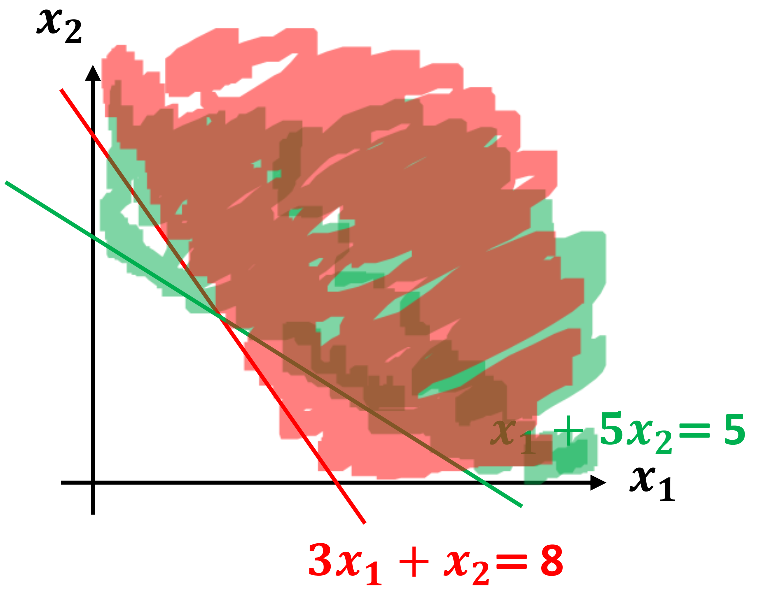
$$s.t \begin{cases} x_1 + 2x_2 \leq 8 \\ 3x_1 + 2x_2 \leq 12 \\ x_1 + 3x_2 \geq 13 \end{cases}$$



‘Infeasible’

$$\max \quad 3x_1 + 4x_2$$

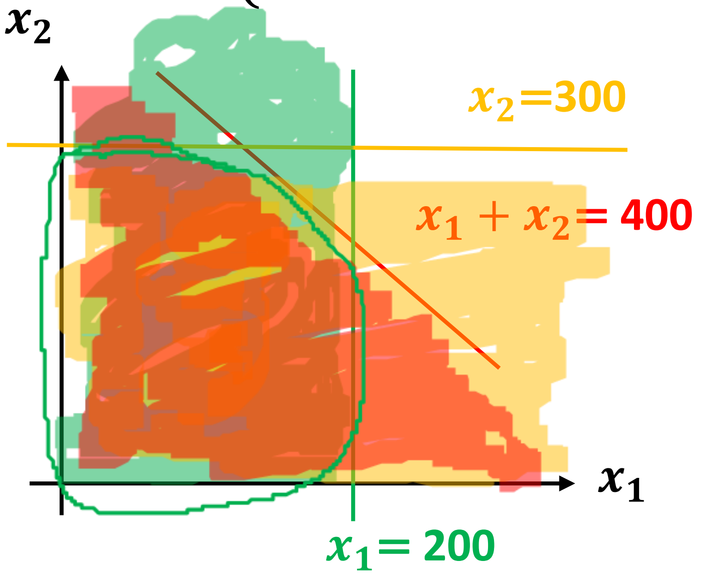
$$s.t \begin{cases} x_1 + x_2 \geq 5 \\ 3x_1 + x_2 \geq 8 \\ x_1, x_2 \geq 0 \end{cases}$$



‘Unbounded’

$$\max \quad x_1 + 6x_2$$

$$s.t \begin{cases} x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{cases}$$



‘Feasible’

Duality

- In linear programming, duality implies that each linear programming problem can be analyzed in two different ways but would have equivalent solutions.
- For any linear program (LP), there is a closely related LP called the dual.
- Duality relates to the inversion of a maximization problem into a minimization problem, or vice-versa, through a change of variables based on Lagrange Multipliers and / or Karush-Kuhn Tucker (KKT) multipliers.

Interpretation of the dual: 'Diet problem'

A student wants to purchase a snack from a bakery to meet certain dietary requirements by choosing the best combination of brownies and cheesecake. The student is following some new diet trend which requires her to eat at least 6 oz of chocolate, 8 oz of cream cheese, and 10 oz of sugar. The cost of 1 piece of brownie and one piece of cake is 50 cts and 80 cts respectively. Her goal is to satisfy these requirements at **minimal cost**.

Ingredients needed			
	3 oz	2 oz	2 oz
	0 oz	4 oz	5 oz
Requirements	**6 oz**	**10 oz**	**8 oz**

‘Primal problem’

$$\min \quad 50x_1 + 80x_2$$

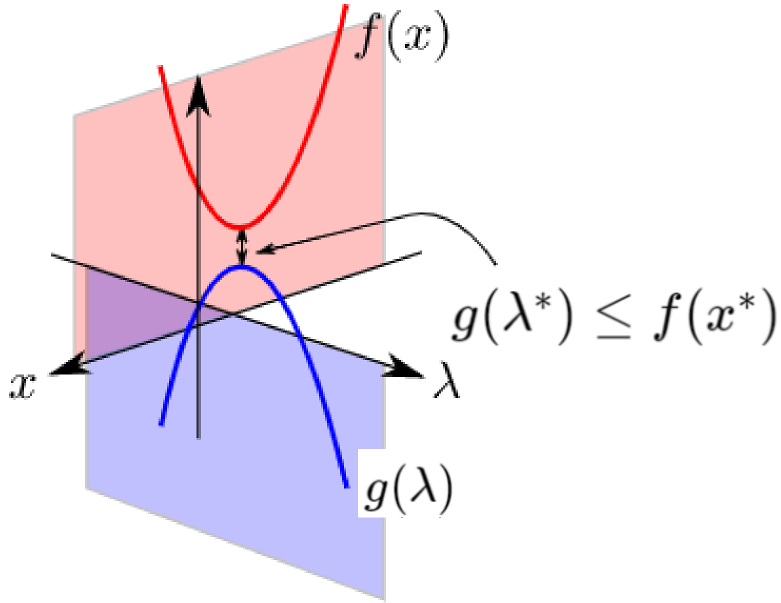
$$s.t \begin{cases} 3x_1 + 0x_2 \geq 6 & \leftarrow y_1 \\ 2x_1 + 4x_2 \geq 10 & \leftarrow y_2 \\ 2x_1 + 5x_2 \geq 8 & \leftarrow y_3 \\ x_1, x_2 \geq 0 \end{cases}$$

‘Dual problem’

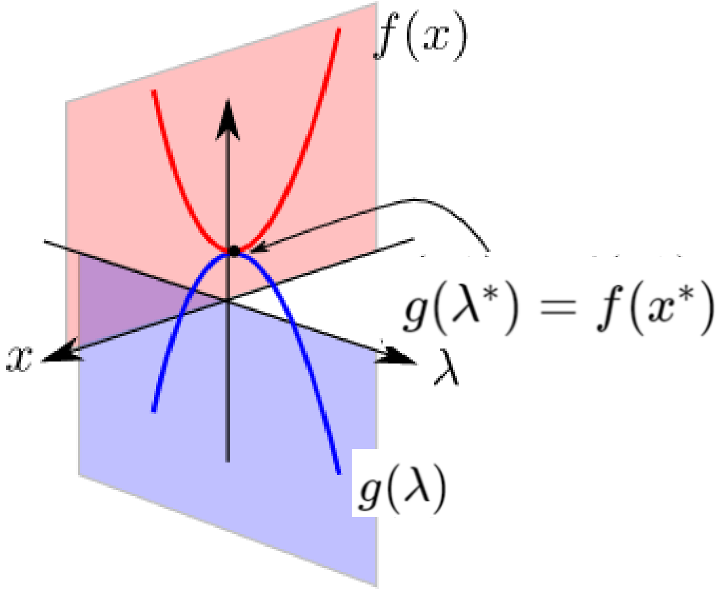
$$\max \quad \underline{6y_1 + 10y_2 + 8y_3}$$

$$s.t \begin{cases} 3y_1 + 2y_2 + 2y_3 \leq 50 \\ 0y_1 + 4y_2 + 5y_3 \leq 80 \\ y_1, y_2, y_3 \geq 0 \end{cases}$$

Weak and strong duality



weak duality



strong duality

Primal/dual solution possibilities

		Primal		
		Finite optimal	**Unbounded**	**Feasible**
Dual	**Finite optimal**	Possible	Impossible	Impossible
	Unbounded	Impossible	Impossible	Possible
	Feasible	Impossible	Possible	Possible

Thank you...!