Q1.) a) if $y = x^*$

$x \cdot y = x^*(x^*) = (x^*)^2$

$y \cdot x = (x^*)^* x = x \cdot x = x^2$

Since $x^2 \neq (x^*)^2$ $(x \neq x^*)$

$\therefore x \cdot y \neq y \cdot x$ (×)

b) if $y = x$

LHS: $x \cdot y = x^* y = x^* x$

RHS: $y \cdot x = y^* x = x^* x$

LHS = RHS

$\therefore x \cdot y = y \cdot x$ (✓)

c) if y is orthogonal to x

LHS: $x \cdot y = 0$

RHS: $y \cdot x = 0$

Since if y is orthogonal to $x \Rightarrow y \cdot x = 0$

then x is also orthogonal to $y \Rightarrow x \cdot y = 0$

$\therefore$ LHS = RHS

Hence $x \cdot y = y \cdot x$ (✓)

iv) if y = scalar (complex) multiple of x

Let $y = kx$ (scalar & complex multiple of x)

$x \cdot y = x \cdot (kx) = k(x \cdot x)$

$y \cdot x = (kx) \cdot x = \bar{k}(x \cdot x)$

Since k is complex,

k is not necessarily equal to $\bar{k}$.

Hence $x \cdot y \neq y \cdot x$

Option B, C

Q2) $x \cdot y = 0.8 - 0.37i$

c = scaling factor of $x = 1-2i$

$z = cx$

$$z \cdot y = (cx) \cdot y = \bar{c}(x \cdot y) = (\overline{1-2i})(0.8 - 0.37i)$$
$$= (1+2i)(0.8 - 0.37i)$$
$$= 0.8 - 0.37i + 1.6i + 0.74$$
$$= \boxed{1.54 + 1.23i} \Rightarrow \text{Option (C)}.$$

Q3) Eigenvalue decomposition is the process of decomposing a matrix into its eigenvalues and eigenvectors

Characteristic polynomial of A:

$$\lambda^2 - 0\lambda + (0 - (1)(-1)) = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda^2 = -1 \qquad \Rightarrow \text{Option (A)}$$
$$\boxed{\lambda = i}. \quad \text{as } i = \sqrt{-1}$$
$$\lambda_1 = \lambda_2 = i$$

Since the eigenvalues are complex (equal to i)

Hence, Eigenvalue decomposition for matrix A doesn't exist over $\mathbb{R}$ (Real numbers) but exists over $\mathbb{C}$ (complex numbers).

Q4) Symmetric Matrix : $S^T = S$

Hermitian Matrix : $S^* = S$.

$$S^* = \begin{bmatrix} 1 & 1+i & -2-2i \\ 1-i & 1 & -i \\ -2+2i & i & 1 \end{bmatrix} = S$$

$\therefore$ S is Hermitian.

$$S^T = \begin{bmatrix} 1 & 1-i & -2+2i \\ 1+i & 1 & i \\ -2-2i & -i & 1 \end{bmatrix} \neq S$$

$\therefore$ S is NOT SYMMETRIC.

Option (D) is correct.

Q5) U is Unitary $\Rightarrow$ $U^*U = UU^* = I$.

To check if : $D \cdot U$ is Unitary

Since D is a diagonal matrix with $d_{ii} \in \mathbb{R}$ (real entries)

hence $D = \overline{D}$

$\boxed{\text{Check if } (DU)(DU^*) = I}$

$\Rightarrow (DU)(U^*D^*)$

$\Rightarrow D \cdot (UU^*) \cdot D^*$

$\Rightarrow DID^*$

$\Rightarrow D \cdot D^* = D \cdot \overline{D}^T = D \cdot D^T$

If $D \cdot D^T = I$ then

$(D \cdot U)(DU^*) = I$

which implies $D \cdot U$ is Unitary.

Answer : True if $DD^T = I$.

Q6.) $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$

Characteristic polynomial of 3x3 Matrix :

$$\lambda^3 - [\text{trace}(A)]\lambda^2 + \left(\underset{\text{elements of } A}{\Sigma \text{Minors of diagonal}}\right)\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 + \left\{ \begin{vmatrix} 0 & 1-i \\ 1+i & 0 \end{vmatrix} + \begin{vmatrix} 3 & -3i \\ 3i & 0 \end{vmatrix} + \begin{vmatrix} 3 & 2-i \\ 2+i & 0 \end{vmatrix} \right\}\lambda$$

$$- \left\{ 3\left[-(1-i)(1+i)\right] - (2-i)\left[-3i(1-i)\right] - 3i(2+i)(1+i) \right\}$$

$$\Rightarrow \lambda^3 - 3\lambda^2 + (-2-9-5)\lambda - \left[3(-2) - (2-i)[-3i-3] - 3i(2+i)(1+i)\right]$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 16\lambda - 12 = 0$$

Upon solving above equation, we get

$$\boxed{\lambda = 6, -1, -2}$$

for $\lambda = 6$, $[A - 6I]X = 0$

$$\begin{bmatrix} -3 & 2-i & -3i \\ 2+i & -6 & 1-i \\ 3i & 1+i & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x + (2-i)y - (3i)z = 0 \quad \times (i)$$

$$(2+i)x - 6y + (1-i)z = 0$$

$$(3i)x + (1+i)y - 6z = 0$$

$$\cancel{-3ix} + (1+2i)y + 3z = 0$$

$$(+)\ \cancel{+3ix} + (1+i)y - 6z = 0$$

$$(2+3i)y - 3z = 0$$

$$\boxed{y = \frac{3z}{(2+3i)} \times \frac{(2-3i)}{(2-3i)} = \frac{(6-9i)z}{13}}$$

$$(2+i)x - 6y + (1-i)z = 0 \quad \times (1+i)$$
$$3i\,x + (1+i)y - 6z = 0 \quad \times 6$$

$$\Downarrow$$

$$(1+3i)x - \cancel{6(1+i)y} + 2z = 0$$
$$(+)\ 18i\,x + \cancel{6(1+i)y} - 36z = 0$$

$$(1+21i)x - 34z = 0$$

$$\therefore x = \frac{34z}{1+21i} \times \frac{(1-21i)}{(1-21i)} = \left\{\frac{(34-714i)z}{442}\right\} \div 34$$

$$\boxed{x = \frac{(1-21i)z}{13}}$$

$\therefore$ if $z = k$,

$$y = \left(\frac{6}{13} - \frac{9i}{13}\right)k \; ; \quad x = \left(\frac{1}{13} - \frac{21i}{13}\right)k.$$

$$\therefore V_1 = k\begin{bmatrix} \frac{1}{13} - \frac{21i}{13} \\ \frac{6}{13} - \frac{9i}{13} \\ 1 \end{bmatrix} \times 13$$

$$\boxed{V_1 = k_1 \begin{bmatrix} 1-2i \\ 6-9i \\ 13 \end{bmatrix}}$$

for $\lambda = -1$, $[A-(-I)]X = 0$

$$\begin{bmatrix} 4 & 2-i & -3i \\ 2+i & 1 & 1-i \\ 3i & 1+i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x + (2-i)y - 3iz = 0 \quad -(1)$$
$$(2+i)x + y + (1-i)z = 0 \quad -(2)$$
$$(3i)x + (1+i)y + z = 0. \quad -(3)$$

Taking eq. (1) & (2) :

Multiply $(2+i)$ to eq. (1) & (-4) to eq. (2)

$$\cancel{4(2+i)x} + 5y + (3-6i)z = 0$$
$$(+) \quad \cancel{-4(2+i)x} - 4y + (-4+4i)z = 0$$
$$y + (-1-2i)z = 0$$
$$\boxed{y = (1+2i)z}.$$

Taking eq. (1) & (3) :

Multiply $(1+i)$ to eq. (1) & $(-2+i)$ to eq (3)

$$4(1+i)x + \cancel{(2-i)(1+i)y} + (3-3i)z = 0$$
$$(+) \quad (-3-6i)x - \cancel{(2-i)(1+i)y} + (-2+i)z = 0$$
$$(1-2i)x + (1-2i)z = 0$$
$$\boxed{x = -z}$$

let $z = k$,

then $x = -k$; $y = (1+2i)k$

$$\boxed{v_2 = k \begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix}}$$

for $\lambda = -2$

$$[A+2I]X = 0$$

$$\begin{bmatrix} 5 & 2-i & -3i \\ 2+i & 2 & 1-i \\ 3i & 1+i & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x + (2-i)y - (3i)z = 0 \quad \times(-2)$$
$$(2+i)x + 2y + (1-i)z = 0 \quad \times(2-i)$$
$$\Downarrow$$
$$-10x - \cancel{2(2-i)y} + (6i)z = 0$$
$$(+) \quad 5x + \cancel{2(2-i)y} + (1-3i)z = 0$$

$$-5x + (1+3i)z = 0$$

$$\boxed{x = \left(\frac{1+3i}{5}\right)z.}$$

$$(2+i)x + 2y + (1-i)z = 0 \quad \times(-3i)$$
$$(3i)x + (1+i)y + 2z = 0 \quad \times(2+i)$$
$$\Downarrow$$
$$\cancel{-3i(2+i)x} - (6i)y + (-3-3i)z = 0$$
$$(+) \quad \cancel{3i(2+i)x} + (1+3i)y + (4+2i)z = 0$$

$$(1-3i)y + (1-i)z = 0$$

$$\boxed{y = \left(\frac{-2-i}{5}\right)z}$$

let $z = k$, then

$$x = \left(\frac{1}{5} + \frac{3i}{5}\right)k \; ; \quad y = \left(-\frac{2}{5} - \frac{1i}{5}\right)k$$

$$\boxed{V_3 = k\begin{bmatrix} \frac{1}{5} + \frac{3i}{5} \\ -\frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix} \times 5 \Rightarrow k_1 \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}}$$

Corresponding to Option (A).

Q7) $A = \frac{1}{2}\begin{bmatrix} k+i & \sqrt{2} \\ k-i & \sqrt{2}i \end{bmatrix}$; $A^* = \frac{1}{2}\begin{bmatrix} k-i & k+i \\ \sqrt{2} & -\sqrt{2}i \end{bmatrix}$

A matrix is Unitary if $A^*A = I$.

$$A^*A = \frac{1}{2}\begin{bmatrix} k-i & k+i \\ \sqrt{2} & -\sqrt{2}i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} k+i & \sqrt{2} \\ k-i & \sqrt{2}i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2k^2+2 & \sqrt{2}(k-i+ik-1) \\ \sqrt{2}(k+i-ik-1) & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{k^2+1}{2} & k+1+i(k-1) \\ k+1-i(k+1) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating both sides:

$$\frac{k^2+1}{2} = 1$$

$$k^2+1 = 2$$

$$k^2 = 1$$

$$\therefore \boxed{k = \sqrt{1} = 1} \Rightarrow \text{Option}(B)$$

Q8. $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$

Characteristic polynomial:

$$\lambda^2 - 2\lambda + [1-(1+i)(1-i)] = 0$$

$$\lambda^2 - 2\lambda - 1 = 0$$

Upon solving, $\boxed{\lambda_1 = 1+\sqrt{2},\ \lambda_2 = 1-\sqrt{2}}$

$$D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$$

for $x = 1+\sqrt{2}$

$$\left[A - (1+\sqrt{2})I\right]X = 0$$

$$\begin{bmatrix} -\sqrt{2} & 1+i \\ 1-i & -\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

① $\left.\begin{array}{l} -\sqrt{2}x + (1+i)y = 0 \\ (1-i)x - \sqrt{2}y = 0 \end{array}\right\}$ Both are same

let $y = k$.

Using ①:

$$\sqrt{2}x = (1+i)k$$

$$x = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)k$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\ 1 \end{bmatrix}$$

Upon Normalizing

$$k\begin{bmatrix} \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

taking $\frac{1}{2}$ common:

$$\boxed{X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i \\ \sqrt{2} \end{bmatrix}}$$

for $X = 1-\sqrt{2}$

$$\left[A - (1-\sqrt{2})I\right]X = 0$$

$$\begin{bmatrix} \sqrt{2} & 1+i \\ 1-i & \sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

② $\left.\begin{array}{l} \sqrt{2}x + (1+i)y = 0 \\ (1-i)x + \sqrt{2}y = 0 \end{array}\right\}$ Both are same

Let $y = k$

Using ②:

$$\sqrt{2}x = -(1+i)k$$

$$x = \left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)k$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} \frac{-1-i}{\sqrt{2}} \\ 1 \end{bmatrix}$$

Upon Normalizing

$$k\begin{bmatrix} \frac{-1-i}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

taking $\frac{1}{2}$ common:

$$\boxed{X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1-i \\ \sqrt{2} \end{bmatrix}}$$

$$\therefore \boxed{U = \frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ \sqrt{2} & \sqrt{2} \end{bmatrix}}$$

Q9-) $\det(A-\lambda I)=0$

$$\begin{vmatrix} -\lambda & 0 & 0 & -1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix}$$

$$\Big\downarrow C_1 \to C_1 - \lambda C_4$$

$$\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 1+\lambda^2 & 0 & 0 & -\lambda \end{vmatrix} \xrightarrow[\text{along } R_1]{\text{Expanding}} \begin{vmatrix} 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ \lambda^2+1 & 0 & 0 \end{vmatrix}$$

$$= -(-\lambda)\left[-(-\lambda)(\lambda^2+1)\right] + 1(-\lambda^2-1) = 0$$

$$= \lambda(\lambda(\lambda^2+1)) - \lambda^2 - 1 = 0$$

$$= \lambda^2(\lambda^2+1) - \lambda^2 - 1 = 0$$

$$= \lambda^4 + \cancel{\lambda^2} - \cancel{\lambda^2} - 1 = 0$$

$$= \lambda^4 - 1 = 0$$

$$(\lambda^2-1)(\lambda^2+1) = 0$$

$$(\lambda+1)(\lambda-1)(\lambda^2+1) = 0$$

Eigenvalues are : $\lambda = -1, +1, \pm i$

✓

Two real, Two complex eigenvalues

Option (B)

Q10.) $U^*U = I$ if a matrix U is Unitary.

a) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & -2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 1 \end{bmatrix} \neq I$

b) $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ \sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \sin^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 1 \end{bmatrix} \neq I$

c) $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cancel{\cos\theta\sin\theta} + \cancel{\sin\theta\cos\theta} \\ -\cancel{\cos\theta\sin\theta} + \cancel{\cos\theta\sin\theta} & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ (✓)

d) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cancel{\cos\theta\sin\theta} - \cancel{\sin\theta\cos\theta} \\ \cancel{\sin\theta\cos\theta} - \cancel{\cos\theta\sin\theta} & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ (✓)

e) $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1 \end{bmatrix} \neq I$

$\therefore$ Correct options are Option $\underline{C, D}$.

Q11.) a) U & V are Unitary matrices.

We know that: $U^*U = I$

$V^*V = I$

$$(UV)^*(UV) = (V^*U^*)(UV) = V^*(U^*U)V$$
$$= V^*(I)V$$
$$= V^*V$$
$$= I$$

$\therefore$ UV will always be Unitary.

b) Proof by contradiction:

Let $U = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$; $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$U^*U = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$V^*V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Z = U + V = \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$$

$$Z^*Z = \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq I$$

$\therefore$ U+V is NOT ALWAYS Unitary.

Option D (2 is false)

Q12.) $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

Characteristic polynomial:

$$\lambda^2 - 3\lambda + \left[2 - \cancel{(1+i)(1-i)}\right] = 0$$

$$\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda - 3) = 0$$

$$\boxed{\lambda = 3, 0}$$

for $\lambda = 3$: $[A - 3I]X = 0$

$$\begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} ① \quad -2x + (1+i)y &= 0 \\ (1-i)x - y &= 0 \end{aligned}\right\} \text{Both are same}$$

let $y = k$

From ① : $2x = (1+i)k$

$$\therefore x = \left(\frac{1}{2} + \frac{i}{2}\right)k.$$

$$\boxed{V_1 = k \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1+i \\ 2 \end{bmatrix}}$$

for $\lambda = 0$: $[A]X = 0$

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} ② \quad x + (1+i)y &= 0 \\ (1-i)x + 2y &= 0 \end{aligned}\right\} \text{Both are same.}$$

let $y = k$

From ② : $x = (-1-i)k$.

$$\boxed{V_2 = k \begin{bmatrix} -1-i \\ 1 \end{bmatrix}}$$

If $k = 2$

$$\boxed{V_2 = 2 \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -2-2i \\ 2 \end{bmatrix}}$$

Hence, all options are correct.