Note Title 16-08-2021

Exercise: Prove Half-spaces are convex.

$$S = \{ x \in \mathbb{R}^d : w^T x \leq b \}$$
$\subseteq \mathbb{R}^d$

Property of convex set:

Intersection of convex sets is convex.

Let $S_1, S_2 \subseteq \mathbb{R}^d$ be convex sets. Let

$$S_{12} = S_1 \cap S_2 = \{ x : x \in S_1, x \in S_2 \}$$

$x_1, x_2 \in S_{12}$

$$\underbrace{\lambda x_1 + (1-\lambda) x_2}_{} \in S_{12} \ ?$$

$\left.\begin{array}{l} \in S_1 \\ \in S_2 \end{array}\right\}$ convexity of S_1 & S_2 $\Rightarrow \in S_1 \cap S_2$

$\blacksquare$

Note Title 16-08-2021

Example: $S = \{ x \in \mathbb{R}^d : Ax = b \}$ $A \in \mathbb{R}^{m \times d}$ $b \in \mathbb{R}^m$ is convex ?

$$\underbrace{\begin{bmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — a_m — \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

$$\{ x : \underbrace{a_1^T x = b_1}_{\text{Convex}}, \underbrace{a_2^T x = b_2}_{\text{Convex}}, \dots, \underbrace{a_m^T x = b_m}_{\text{Convex}} \}$$

$\Rightarrow$ S is convex [Intersection of hyperplanes which we know are convex sets].

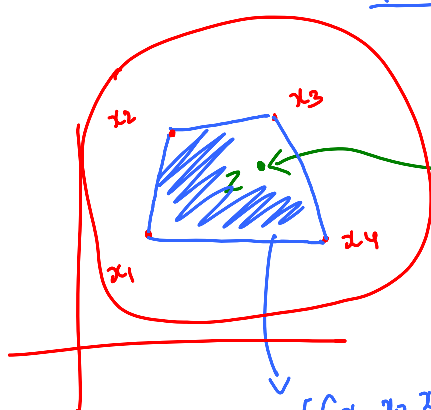
Convex combinations

Let $\boxed{S = \{x_1, \dots, x_n\}} \subseteq \mathbb{R}^d$. Then we will say $z \in \mathbb{R}^d$ to be a $\underbrace{\text{convex combination}}$ of points in S if $\exists\, \lambda_1, \dots, \lambda_n$ s.t $\boxed{\lambda_i \geq 0}$, $\boxed{\sum_{i=1}^{n} \lambda_i = 1}$

s.t $\boxed{z = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n.}$

Exercise:

Prove that $CH(\{x_1, \dots, x_n\})$ a convex set.



$CH(\{x_1, x_2, x_3, x_4\})$

$$z = \underset{0.8}{0.1}\, x_1 + \underset{0.2}{0.3}\, x_2 + \underset{0}{0.5}\, x_3 + \underset{0}{0.1}\, x_4$$

$$\underbrace{\text{Convex-Hull}(S)}_{\{x_1, \dots, x_n\}} = \left\{ z : z = \sum_{i=1}^{n} \lambda_i x_i \text{ for some } \lambda_1, \dots, \lambda_n \geq 0,\ \sum \lambda_i = 1 \right\}.$$

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Alternate definition of convex hulls :

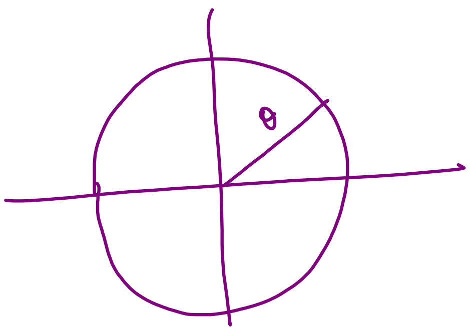
Convex hull $\{x_1, \ldots, x_n\}$ as the intersection of all convex sets that contain $\{x_1, \ldots, x_n\}$.

Exercise: Show the two definitions of convex hull are equivalent.

Euclidean Balls in $\mathbb{R}^d$

$$B = \left\{ x : \|x\|_2 \le \theta \right\}$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{d} x_i^2}$$



Exercise: Show B is convex.