WEEK 3: REVISION

Contents

1. Four Fundamental Subspaces

Suppose **A** is a $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

1. The column space is C(**A**), a subspace of $\mathbb{R}^m$.
2. The row space is C($\mathbf{A}^T$), a subspace of $\mathbb{R}^n$.
3. The nullspace is N(**A**), a subspace of $\mathbb{R}^n$. $Ax = 0$
4. The left nullspace is N($\mathbf{A}^T$), a subspace of $\mathbb{R}^m$. $A^T y = 0$

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} \boxed{1} & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 8 & -4 \\ 2 & 4 & -2 \end{bmatrix} \begin{matrix} R_2 - 4R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot

Column space is C(**A**) span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ $\begin{pmatrix} \boxed{1} & 4 & 2 \\ 0 & \boxed{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row space is C(A^T) span $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$

$$x_1 + 4x_2 + 2x_3 = 0$$

$x_2 = 1, x_3 = 0$ $\quad x_1 + 4 = 0 \quad x_1 = -4$

$x_2 = 0, x_3 = 1$ $\quad x_1 + 2 = 0 \quad x_1 = -2$

Nullspace is N(**A**) span $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Left nullspace is N(A^T) span $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$y_1 + 2y_2 - y_3 = 0$$

$y_2 = 1, y_3 = 0$ $\quad y_1 + 2 = 0 \quad y_1 = -2$

$y_2 = 0, y_3 = 1$ $\quad y_1 - 1 = 0 \quad y_1 = 1$

Solution to Ax = b

- Find the condition on (b_1, b_2, b_3) for $\underline{Ax = b}$ to be solvable, if

$\in C(A)$

$[A\ b]$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$b_3 + b_2 + b_1 = 0$

$\boxed{b_3 = -(b_2 + b_1)}$

$$\left[\begin{array}{ccc} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{array}$$

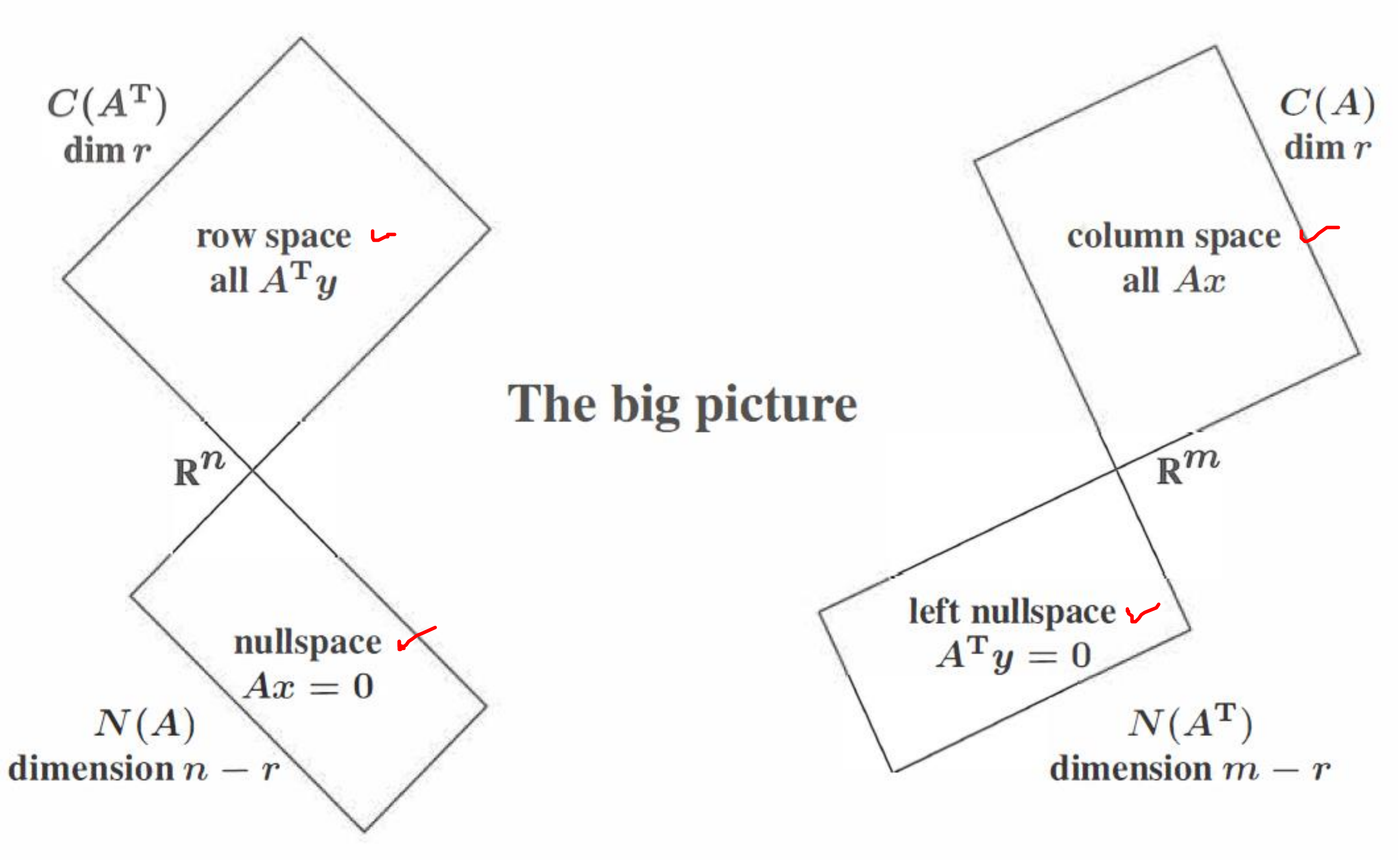
$$\left[\begin{array}{ccc} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{array}\right] R_3 + R_2 \rightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & \underbrace{b_3 + b_2 + b_1}_{=0} \end{array}\right] \Rightarrow 0$$

$b \in C(A)$

"All zero rows" $\rightarrow b \in C(A)$

2. Orthogonal Vectors and Subspaces

$x \cdot y = 0$

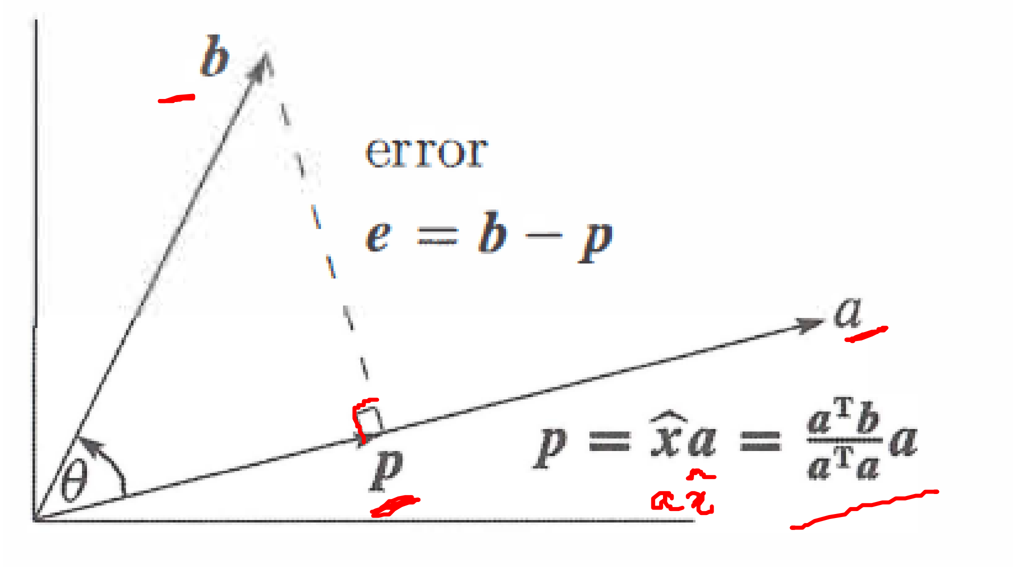


rank(A) + nullity(A) = n

$\dim(C(A^T)) + \dim(N(A^T)) = m$

3. Projections

- The projection p of b onto a line:



Projection matrix of vector a, $\mathbb{P} = \frac{aa^T}{a^Ta}$

$$\boxed{\begin{array}{l} P^2 = P \\ P \text{ is symmetric} \end{array}}$$

- Projection matrix of $a = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ $\quad \mathbb{P} = \frac{aa^T}{a^T a}$

$$aa^T = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 & -1 \\ -3 & 9 & -6 & 3 \\ 2 & -6 & 4 & -2 \\ -1 & 3 & -2 & 1 \end{bmatrix} \quad 4 \times 4$$

$$a^T a = \begin{bmatrix} -1 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix} = 15$$

$$\mathbb{P} = \frac{1}{15} \begin{bmatrix} 1 & -3 & 2 & -1 \\ -3 & 9 & -6 & 3 \\ 2 & -6 & 4 & -2 \\ -1 & 3 & -2 & 1 \end{bmatrix}$$

Projection of $b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ onto a:

$$p = \mathbb{P} * b$$

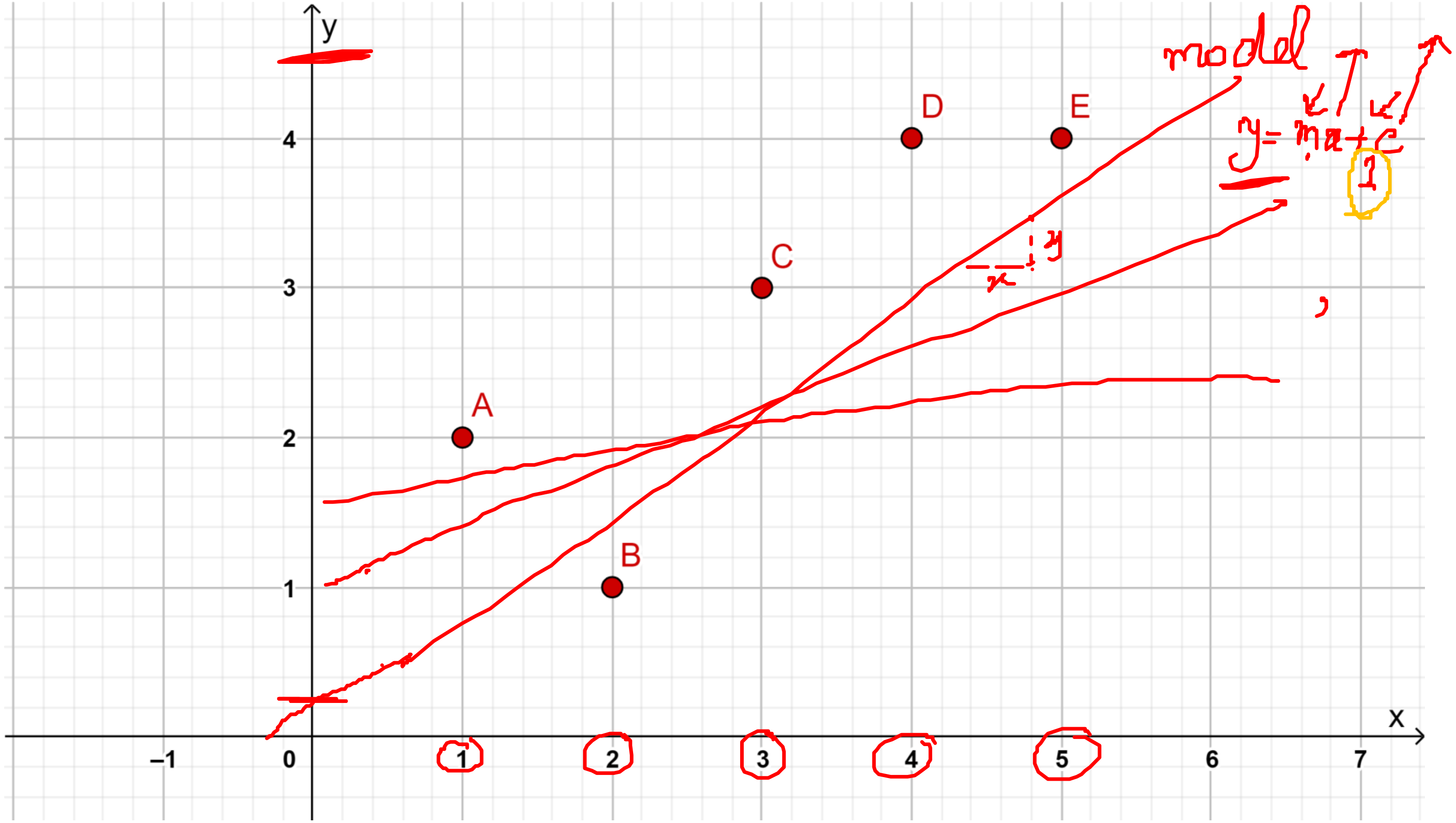
$$p = \frac{1}{15} \begin{bmatrix} 1 & -3 & 2 & -1 \\ -3 & 9 & -6 & 3 \\ 2 & -6 & 4 & -2 \\ -1 & 3 & -2 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 2 \\ -6 \\ 4 \\ -2 \end{bmatrix}$$

Least Squares approximations

- It often happens that $Ax = \boldsymbol{b}$ has no solution.
- The usual reason is: *too many equations.*
- The matrix A has more rows than columns.
- There are more equations than unknowns (m is greater than n).
- Then columns span a small part of m-dimensional space.
- We cannot always get the error $e = b - Ax$ down to zero. When e is zero, x is an exact solution to $Ax=b$.
- When the length of e is as small as possible, $\hat{x}$ is a least squares solution.

- Least Squares method: Solving $\boxed{\boldsymbol{A}^T\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^T\mathrm{b}}$ we get $\hat{x}$: $\begin{bmatrix} \text{slope} \\ \text{intercept} \end{bmatrix}$

Least Squares approximations



x	y
1	2
2	1
3	3
4	4
5	4

$$(A^T A)\hat{x} = (A^T b)$$

$\downarrow$

coefficients

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

$$p = \frac{a^T b}{a^T a}$$

$$A^T A\hat{x} = A^T\mathbf{b}$$

$y = mx + \overset{1}{c}$

coefficient matrix

$$A = \begin{array}{cc} x & c \\ \left[\begin{matrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{matrix}\right] \end{array} \quad A^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{array}{c} y \\ \left[\begin{matrix} 2 \\ 1 \\ 3 \\ 4 \\ 4 \end{matrix}\right] \end{array} \qquad \boxed{\hat{x} = \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix}}$$

$y = \overset{1}{c} + mx$

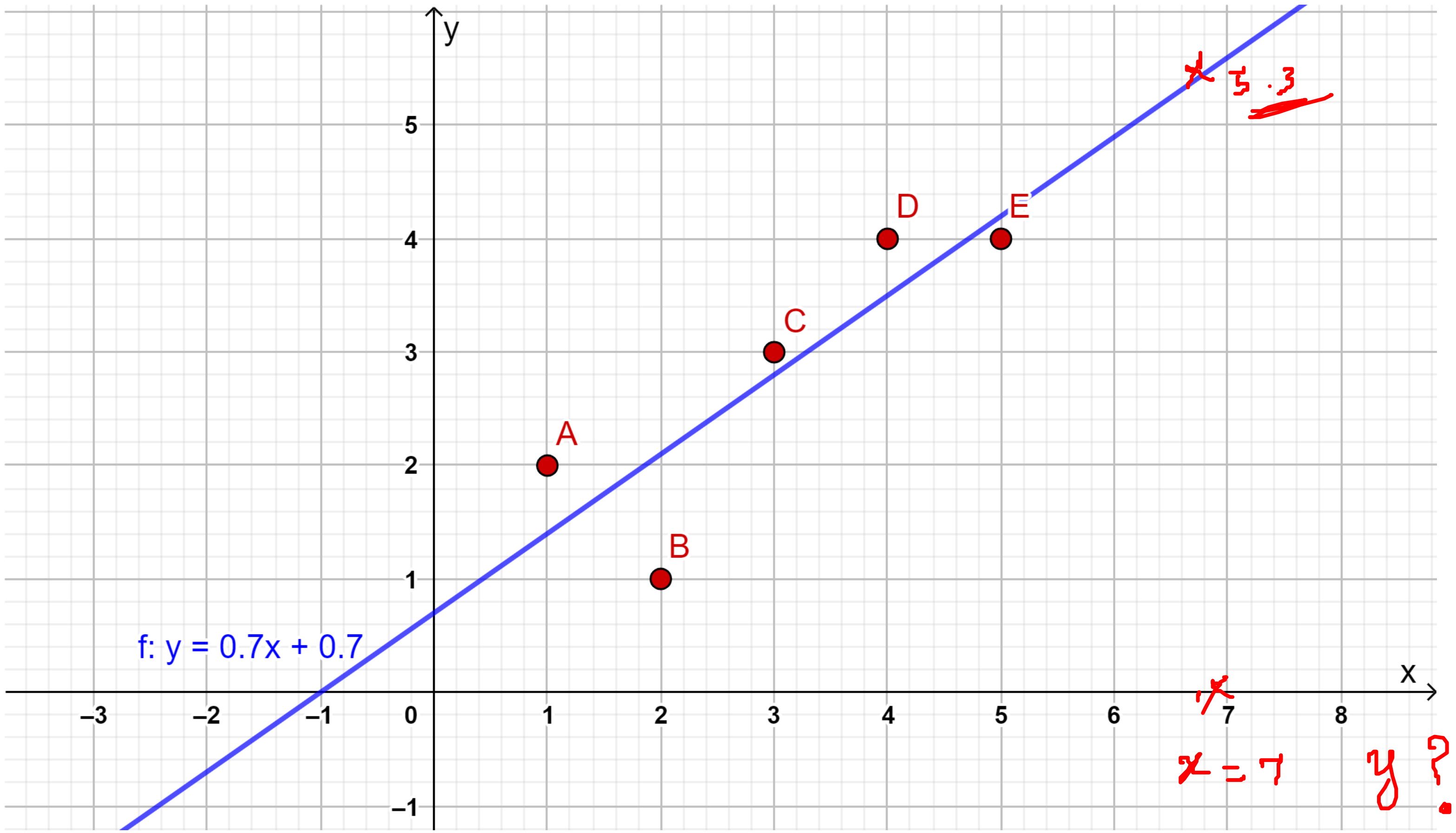
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix} = \begin{bmatrix} 49 \\ 14 \end{bmatrix}$$

Solving this we get, $\hat{x} = \begin{array}{cl} \left[\begin{matrix} 0.7 \\ 0.7 \end{matrix}\right] & \begin{matrix} \to \text{slope} \\ \to \text{intercept} \end{matrix} \end{array}$

Best fit line: $\boldsymbol{y = 0.7x + 0.7}$

Best fit line: $y = 0.7x + 0.7$



THANK YOU