Intractability: Checking Algorithms

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Programming, Data Structures and Algorithms using Python

Week 11

Efficient algorithms

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- Unfortunately not
- For a large class of "natural" problems, no shortcut is known to exist

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- For factorization
  - I is N
  - S is $\{p, q\}$
  - C involves verifying that $pq = N$

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- Assign suitable values {True,False} to $x_1, x_2, x_3, \ldots$ so that the formula evaluates to True

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$$

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- Now there is no satisfying assignment

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- Each clause forces a unique valuation
- Try each clause in sequence

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- Now, given a solution S, we can check it
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- Find optimum K — test different values using binary search

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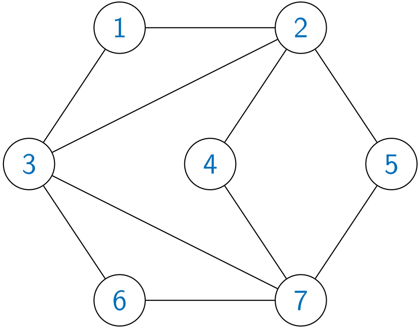

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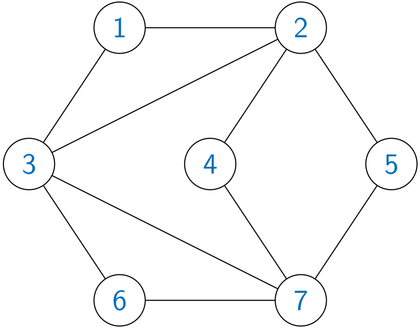

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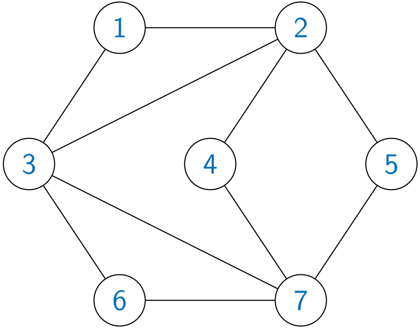

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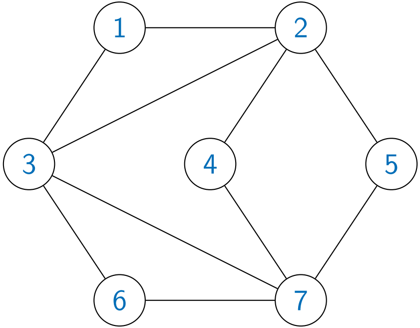

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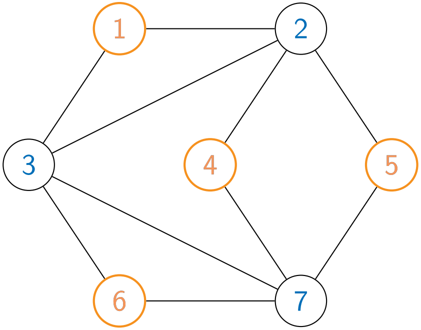

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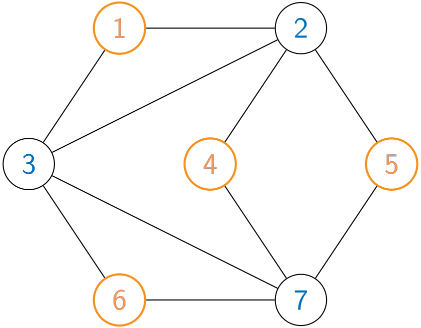
- Checking version: Is there an independent set of size K?



Vertex cover

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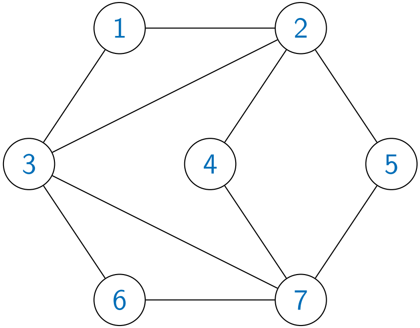

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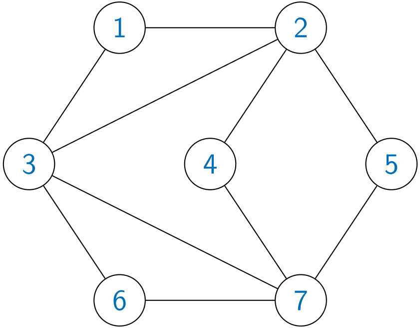

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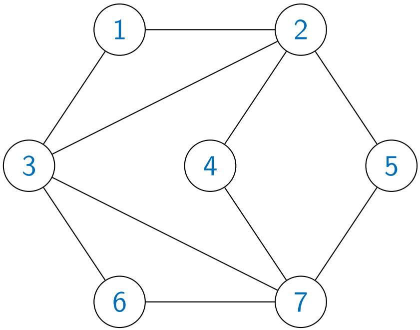

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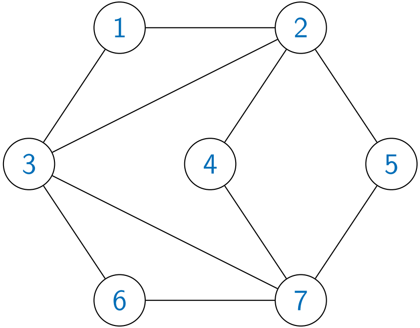

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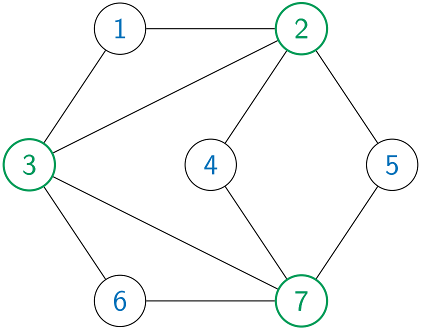

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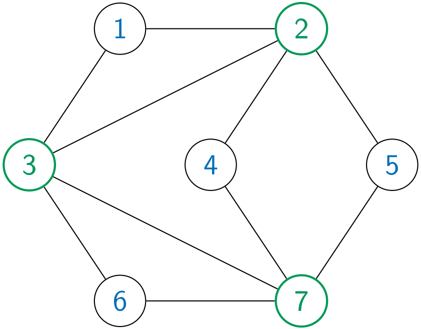

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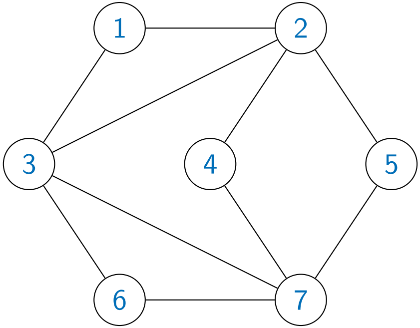

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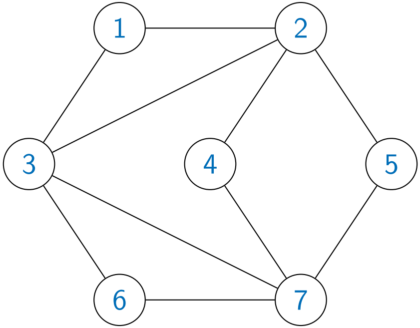

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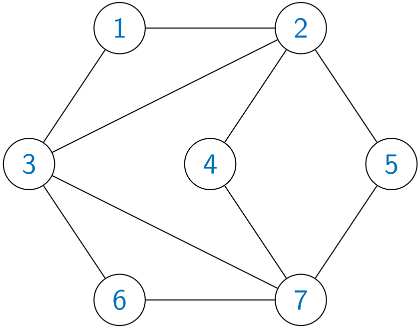
($\Leftarrow$) For any edge (u, v), at least one endpoint is in $V \setminus U$, so there are no edges (u, v) within U



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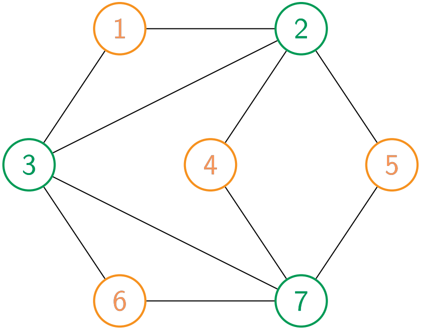
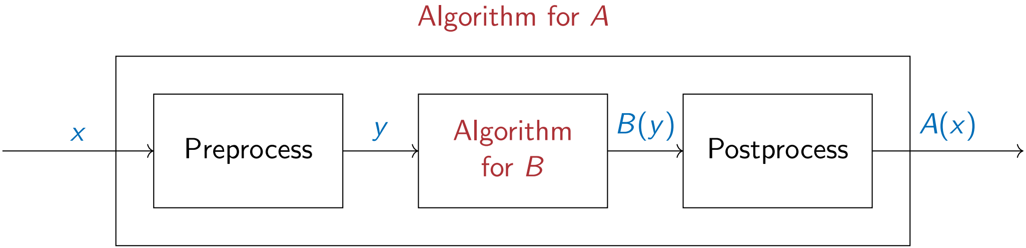
($\Rightarrow$) Every edge (u, v) has at most one end point in U, so at least one end point in $V \setminus U$

($\Leftarrow$) For any edge (u, v), at least one endpoint is in $V \setminus U$, so there are no edges (u, v) within U



Reductions

- Independent set and vertex cover reduce to each other



Reductions

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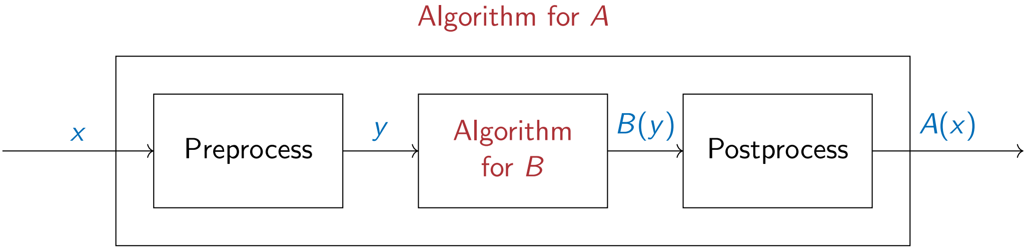
- Recall: if A reduces to B and A is intractable, so is B



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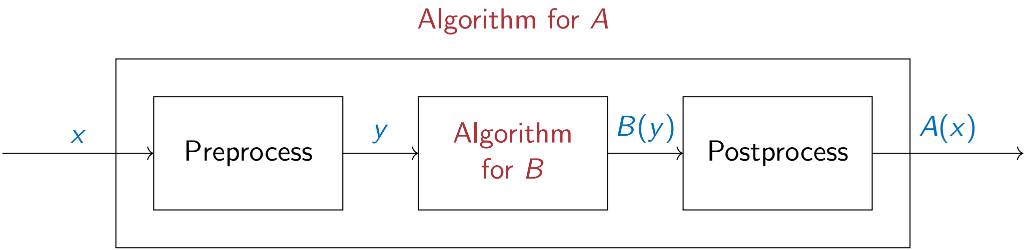

Reductions

- Independent set and vertex cover reduce to each other
- Recall: if A reduces to B and A is intractable, so is B
- Many pairs of checkable problems are inter-reducible
- All "equally" hard