Single Source Shortest Paths

Madhavan Mukund

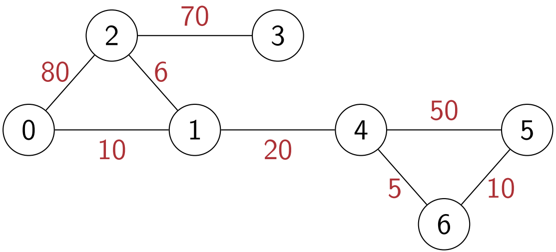
https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 5

Single source shortest paths

- Weighted graph:
  - $G = (V, E)$
  - $W : E \to \mathbb{R}$
- Single source shortest paths
  - Find shortest paths from a fixed vertex to every other vertex



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  - $G = (V, E)$
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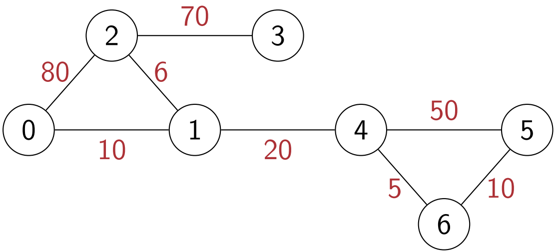
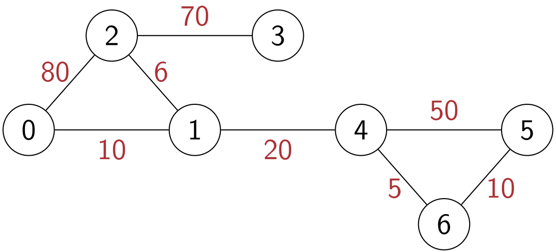
  - Find shortest paths from a fixed vertex to every other vertex
- Assume, for now, that edge weights are all non-negative



Single source shortest paths

- Compute shortest paths from 0 to all other vertices



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- Compute shortest paths from 0 to all other vertices
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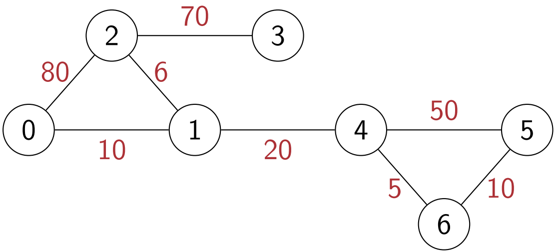

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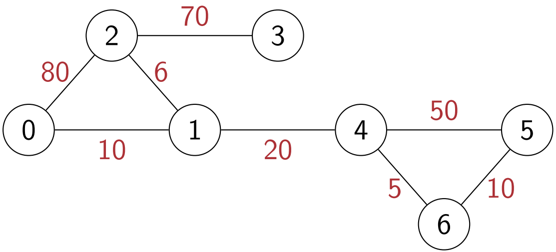

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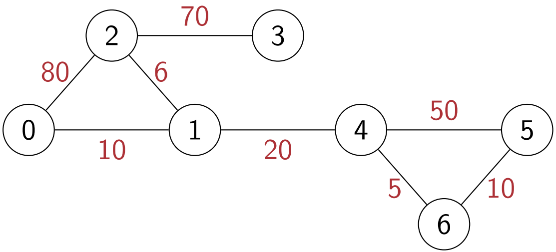

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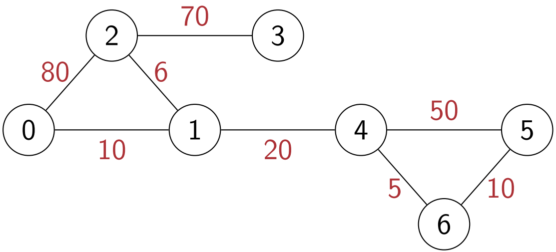

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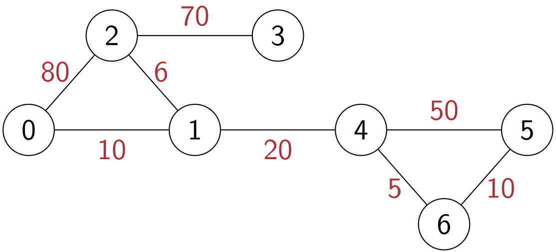

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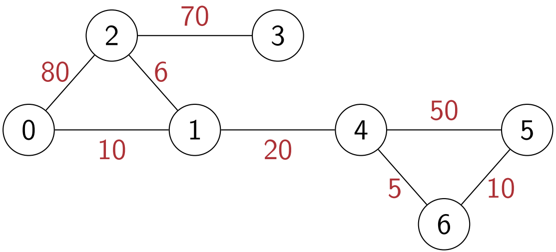

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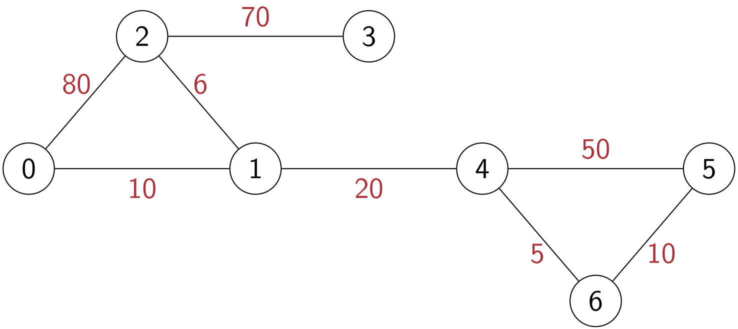

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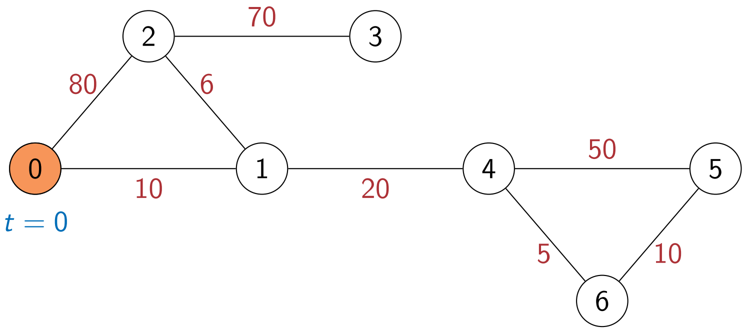

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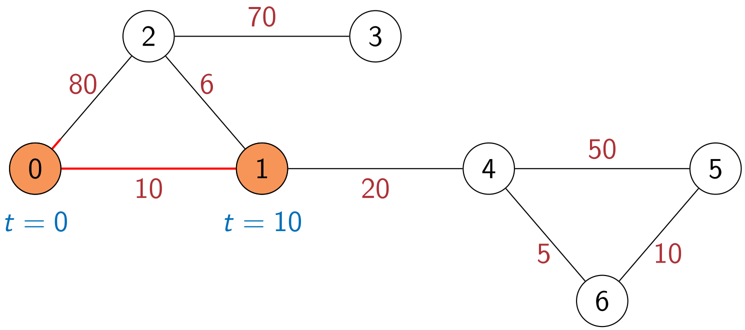

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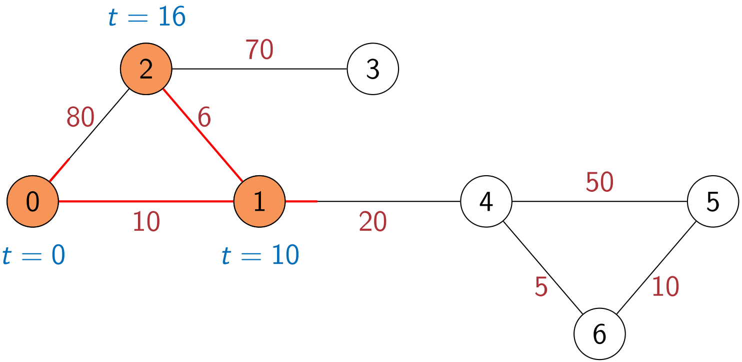

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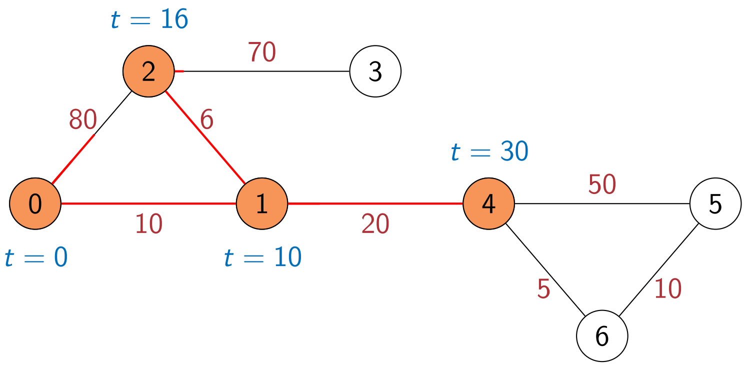

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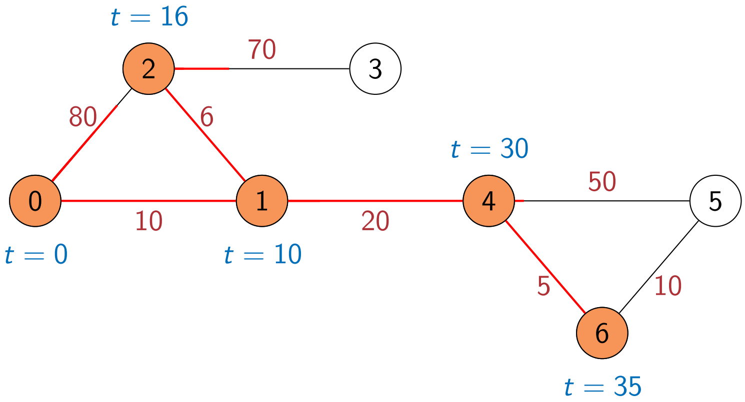

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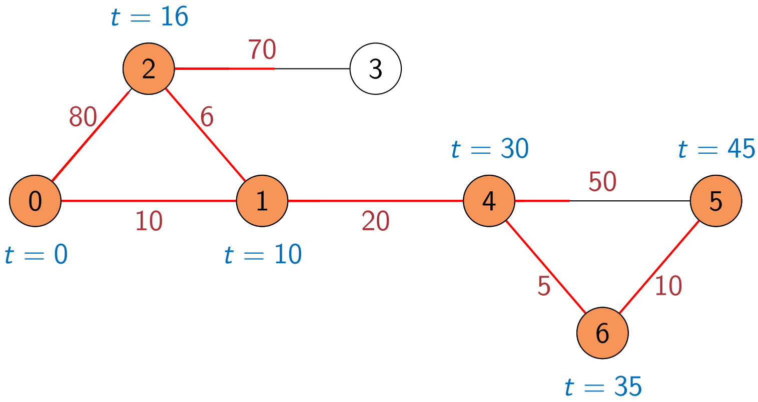

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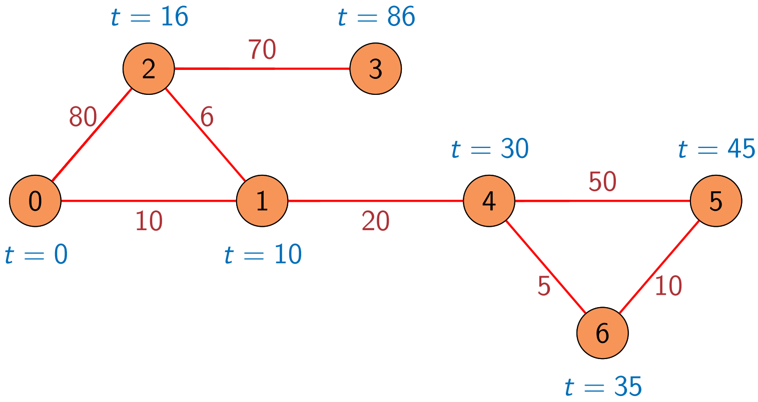
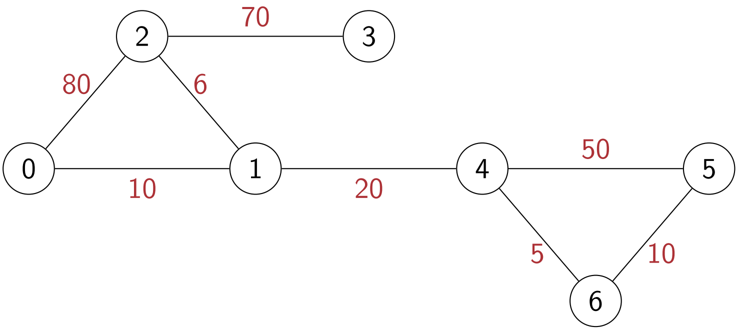


Single source shortest paths

- Compute expected burn time for each vertex
- Each time a new vertex burns, update the expected burn times of its neighbours



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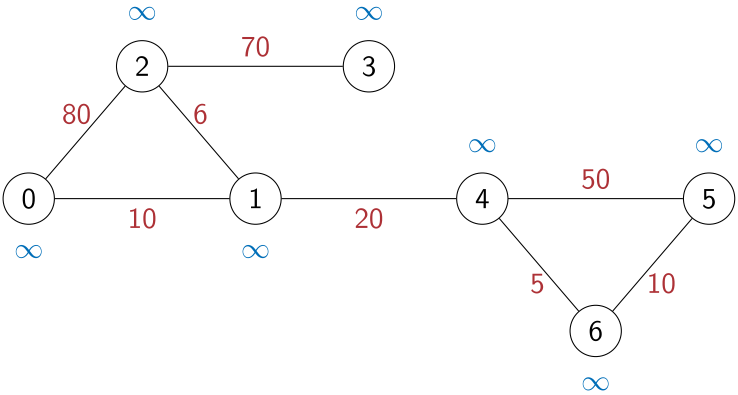

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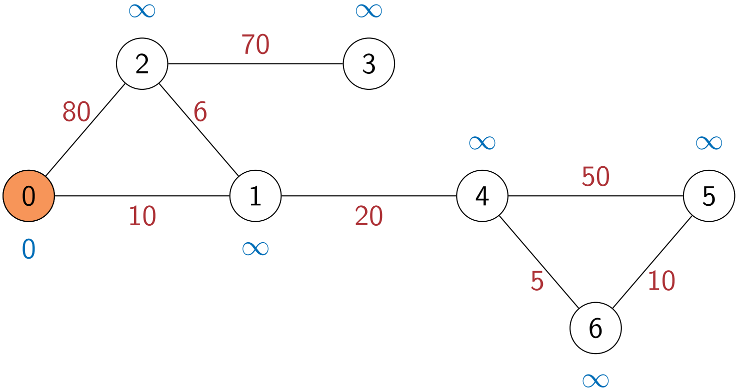

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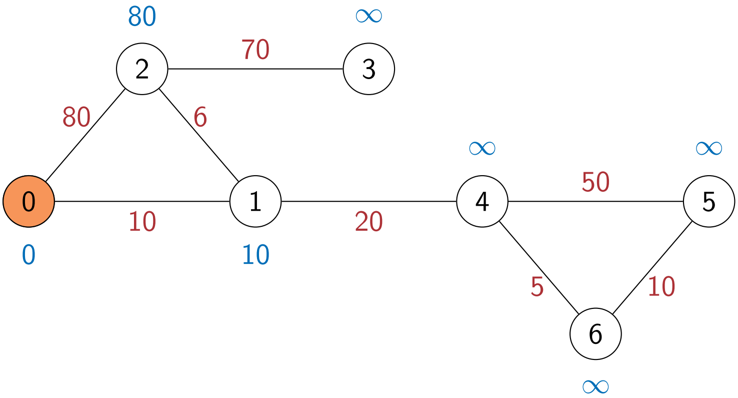

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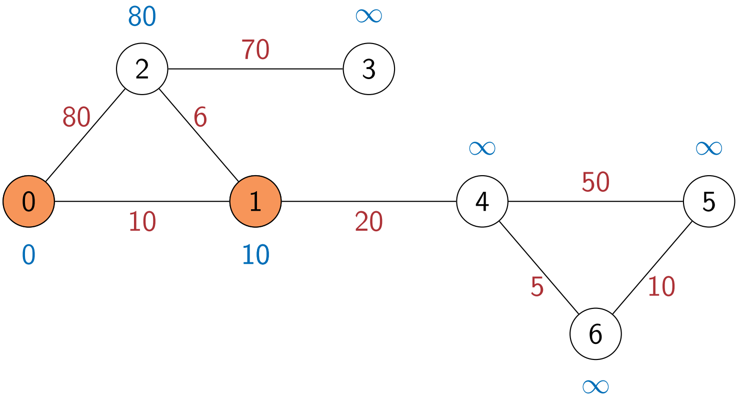

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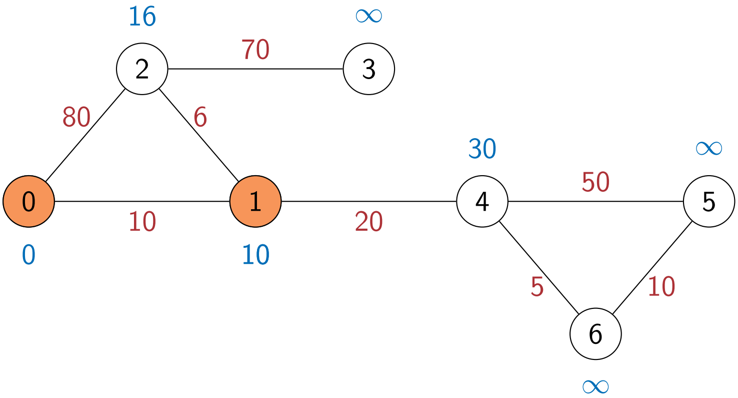

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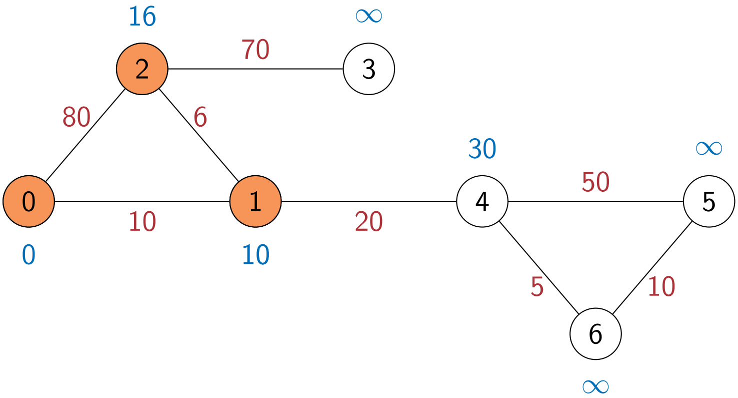

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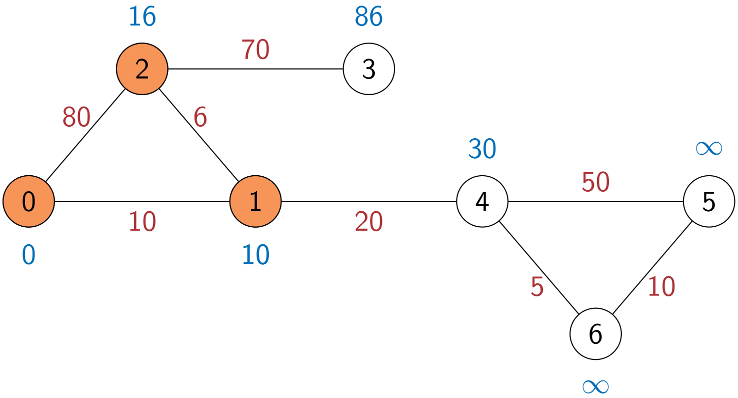

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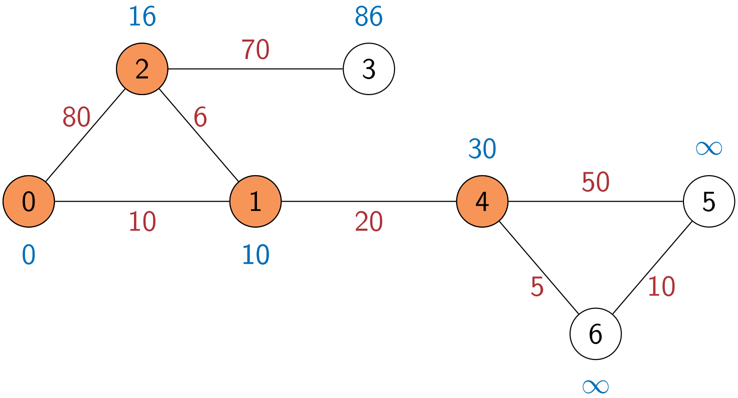

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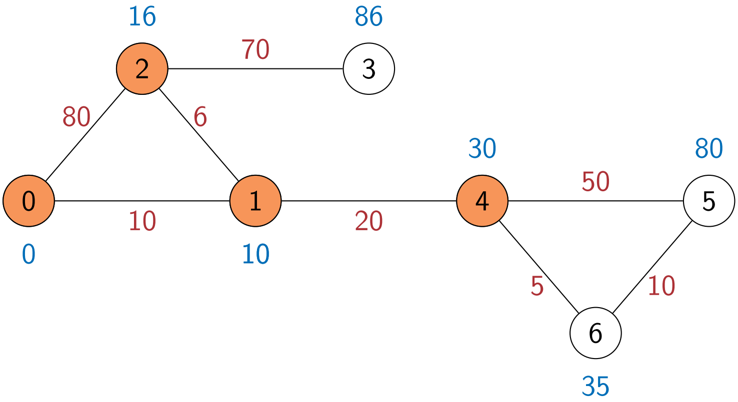

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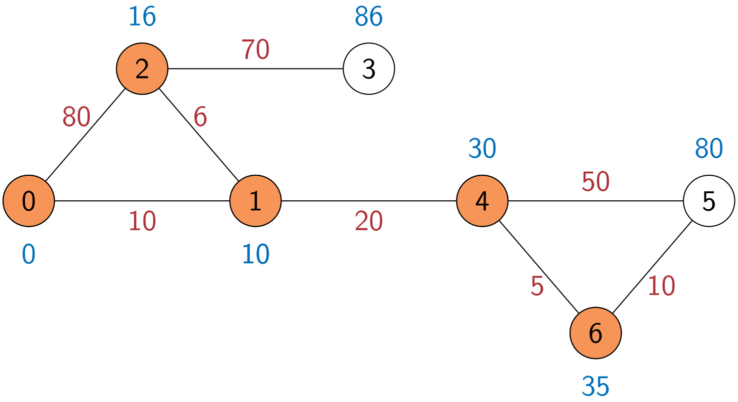

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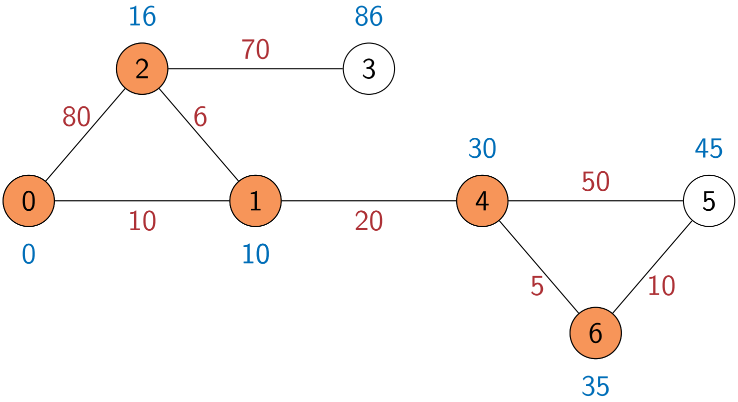

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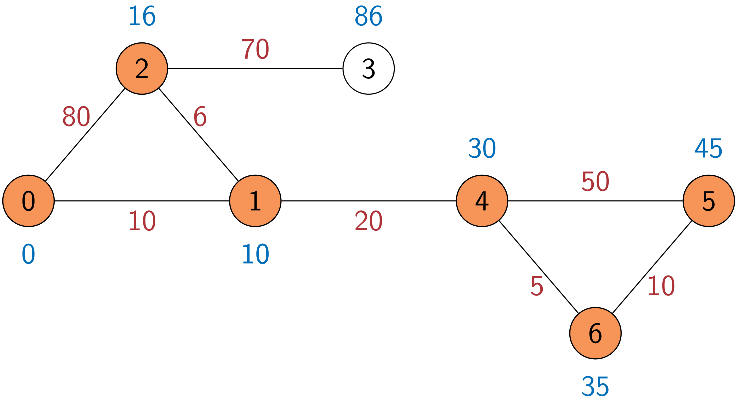

Single source shortest paths

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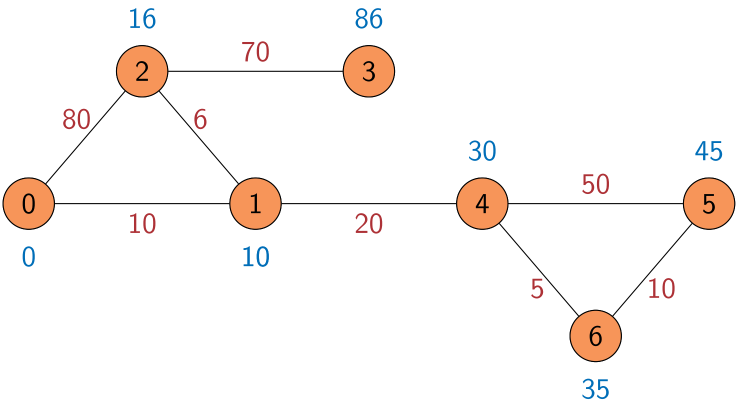
- Algorithm due to Edsger W Dikjstra



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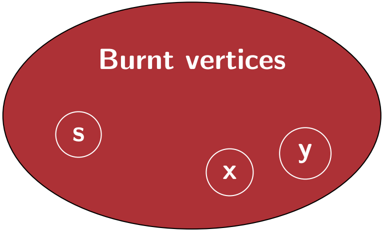

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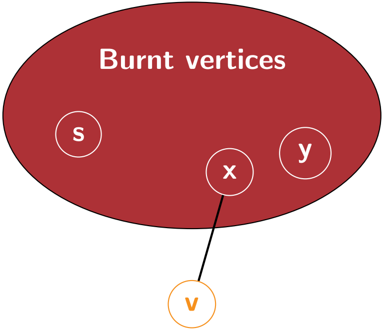

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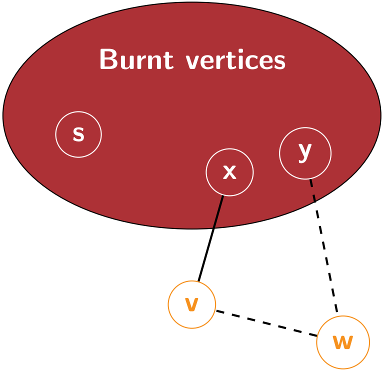
  - Burn time of **w** $\geq$ burn time of **v**
  - Edge from **w** to **v** has weight ≥ 0



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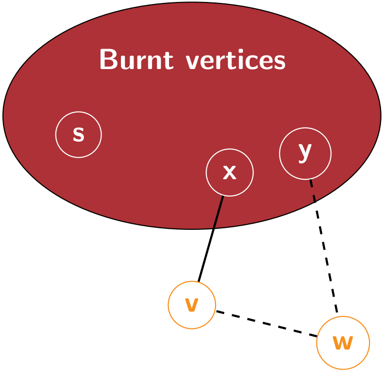
- This argument breaks down if edge (**w**,**v**) can have negative weight
  - Can't use Dijkstra's algorithm with negative edge weights



Implementation

- Maintain two dictionaries with vertices as keys
  - `visited`, initially `False` for all `v` (burnt vertices)
  - `distance`, initially `infinity` for all `v` (expected burn time)

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def dijkstra(WMat,s):
  (rows,cols,x) = WMat.shape
  infinity = np.max(WMat)*rows+1
  (visited,distance) = ({},{})
  for v in range(rows):
    (visited[v],distance[v]) = (False,infinity)
  distance[s] = 0
  for u in range(rows):
    nextd = min([distance[v] for v in range(rows)
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    nextvlist = [v for v in range(rows)
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    nextv = min(nextvlist)
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- Set `distance[s]` to `0`
- Repeat, until all reachable vertices are visited
  - Find unvisited vertex `nextv` with minimum distance
  - Set `visited[nextv]` to `True`
  - Recompute `distance[v]` for every neighbour `v` of `nextv`

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- Overall $O(n^2)$
- If we use an adjacency list
  - Setting `infinity` and updating distances both $O(m)$, amortised
  - $O(n)$ bottleneck remains to find next vertex to visit
  - Better data structure? Later ...

```
def dijkstralist(WList,s):
  infinity = 1 + len(WList.keys())*
                    max([d for u in WList.keys()
                            for (v,d) in WList[u]])
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      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for (v,d) in WList[nextv]:
      if not visited[v]:
        distance[v] = min(distance[v],distance[nextv]+d)
  return(distance)
```

Summary

- Dijkstra's algorithm computes single source shortest paths
- Use a greedy strategy to identify vertices to visit
  - Next vertex to visit is based on shortest distance computed so far
  - Need to prove that such a strategy is correct
  - Correctness requires edge weights to be non-negative
- Complexity is $O(n^2)$
  - Even with adjacency lists
  - Bottleneck is identifying unvisited vertex with minimum distance
  - Need a better data structure to identify and remove minimum (or maximum) from a collection