Least Squares

Simple case : One dimension

Dataset: (x_1, b_1) ----- (x_m, b_m)

$$b_i = \theta' x_i + \theta'' \quad , \quad i = 1, \ldots, m \quad \text{---} \quad (1)$$

offset

linear fit

System of equations (1) is equivalent to

$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \begin{bmatrix} \theta' \\ \theta'' \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

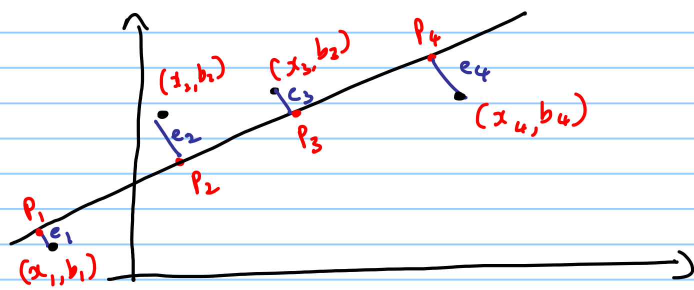
$A\theta = b$, where $\theta = \begin{pmatrix} \theta' \\ \theta'' \end{pmatrix}$

$A\theta = b$ may be inconsistent

Least squares approach: minimize $E^2 = \|b - A\theta\|^2$

$$= (b_1 - \theta' x_1 - \theta'')^2 + \cdots\cdots + (b_m - \theta' x_m - \theta'')^2$$

$$(\hat{\theta}', \hat{\theta}'') = \arg\min_{\theta} \|b - A\theta\|^2$$



If the points b_1, b_2, b_3, b_4 lie on a line, then $P_1 = b_1$, $P_2 = b_2$, $P_3 = b_3$, $P_4 = b_4$ & $E^2 = 0$ & $Ax = b$ can be solved

If not, then minimize $E^2 = \|A\theta - b\|^2$

Example:

$$A\theta = b$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \theta' \\ \theta'' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Is $A\theta = b$ consistent? Or does $b \in C(A)$?

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 3 & 5 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right] \longleftarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{array}\right]$$

inconsistent $A\theta = b$ or $b \notin C(A)$

Note Title 4/19/2021

Least-squares:

$$A^T A\, \hat{\theta} = A^T b \quad , \text{ where } \quad \hat{\theta} = \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} , \quad A^T b = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$A^T A\, \hat{\theta} = A^T b \iff \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$6\hat{\theta}' + 2\hat{\theta}'' = 6$$

$$2\hat{\theta}' + 3\hat{\theta}'' = 5$$

leads to $\hat{\theta}'' = \dfrac{9}{7}$ and $\hat{\theta}' = \dfrac{4}{7} \Rightarrow \hat{\theta} = \begin{bmatrix} 4/7 \\ 9/7 \end{bmatrix}$

Note Title 4/19/2021

Best line (in the "least squares" sense) through the given data is

$$\boxed{\frac{4}{7}x + \frac{9}{7}}$$

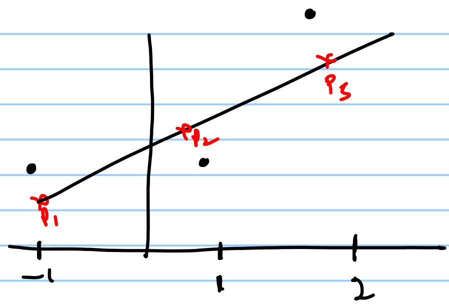
Projections: $p_1 = \frac{4}{7}(-1) + \frac{9}{7} = \frac{5}{7}$, $p_2 = \frac{13}{7}$, $p_3 = \frac{17}{7}$

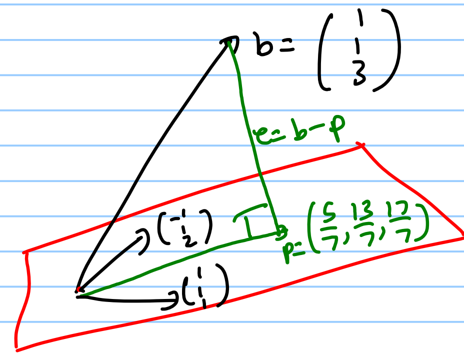
The original data is not on a line, so $E^2 > 0$

$$E^2 = \|b - A\hat{\theta}\|^2 = \|e\|^2$$

$$e = \left[1 - \left(-\frac{4}{7} + \frac{9}{7}\right)\right], \left[1 - \left(\frac{4}{7} + \frac{9}{7}\right)\right], \left[3 - \left(\frac{8}{7} + \frac{9}{7}\right)\right]$$

$$= \left(+\frac{2}{7}, -\frac{6}{7}, \frac{4}{7}\right)$$





$$e = \left(\frac{+2}{7},\ \frac{-6}{7},\ \frac{4}{7} \right)$$

$$e \perp (-1, 1, 2) \quad \text{and} \quad e \perp (1, 1, 1)$$