Greedy Algorithms: Interval Scheduling

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Programming, Data Structures and Algorithms using Python

Week 7

Greedy Algorithms

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Examples

- Dijkstra's algorithm
  - Local rule: freeze the distance to nearest unvisited vertex
  - Global optimum: distance assigned to each vertex is shortest distance from source

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Examples

- Prim's algorithm
  - Local rule: add to the spanning tree nearest non-tree vertex
  - Global optimum: final spanning tree is minimum cost spanning tree

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Examples

- Kruskal's algorithm
  - Local rule: add to the current set of edges the smallest edge that does not form a cycle
  - Global optimum: final spanning tree is minimum cost spanning tree

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- What is a sound local strategy?

Greedy strategies for interval scheduling

- Strategy 1
  Choose the booking whose starting time is earliest

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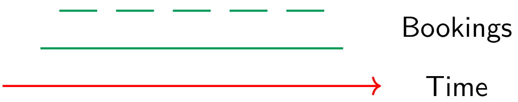
Counterexample



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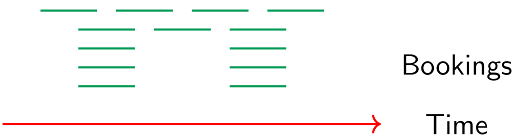

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  - Counterexample? Proof of correctness?

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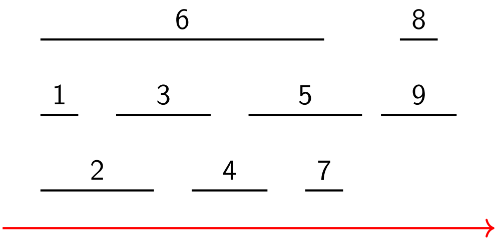

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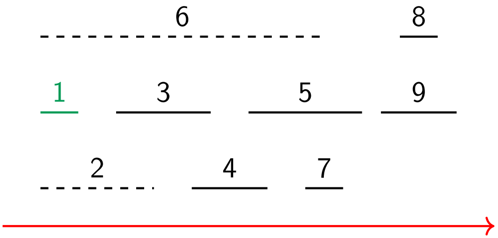

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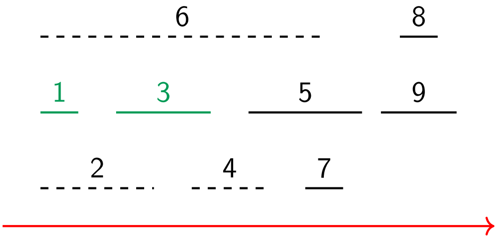

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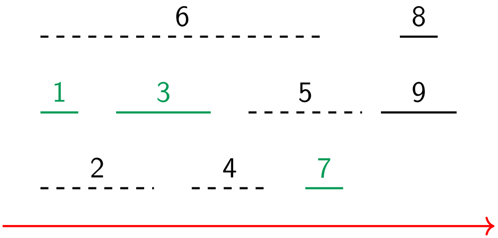

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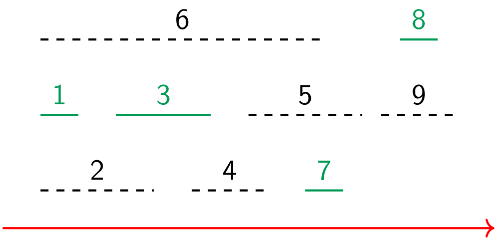

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- Our goal is to show that $k = m$

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  - We must have $f(i_\ell) \leq f(j_\ell)$

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- Greedy strategy stops when B is empty

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  - B is not empty after choosing A, contradiction!

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- In general, after adding booking `j` to A, Find the smallest `r` with `S[r] > F[j]`
  - Single scan, $O(n)$ overall

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- Correct strategy needs a proof
- One way is to show that greedy solution "stays ahead", step by step, of any optimal solutions