Q1) $f(x,y) = x^2+y^2$

$f_x = \frac{\partial(f(x,y))}{\partial x} = 2x$

↳ keeping y constant

$f_y = \frac{\partial(f(x,y))}{\partial y} = 2y$

↳ keeping x constant

$f_x = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

$\therefore$ Stationary point $= (0,0)$

Option (B)

Q2) $A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Here $a = 4 > 0$

$ac - b^2 = 4(2) - 2(2) = 4 > 0$

$\therefore$ A is positive-definite.

$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Here $a = 1 > 0$

$ac - b^2 = 1(2) - 1(1) = 1 > 0$

$\therefore$ B is positive-definite

$A+B = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a = 5 > 0$

$ac - b^2 = (5)(4) - (3)(3) = 20 - 9 = 11 > 0$

$\therefore$ A+B is ALSO positive definite.

Option (A)

Q3) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic polynomial:

$$\lambda^3 - 6\lambda^2 + \{3+3+3\}\lambda - 4 = 0$$

[**Formula:** $\lambda^3 - [\text{trace}(A)]\lambda^2 + \left(\sum \text{Minors of diagonal elements of } A\right)\lambda - \det(A) = 0$]

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Upon solving: $\lambda = 4, 1, 1$

$\lambda_1 = 4, \ \lambda_2 = 1, \ \lambda_3 = 1$

Since $\lambda_1, \lambda_2, \lambda_3 > 0$

A is positive definite

Option (A).

Q4) $f(x,y) = 2x^2 + 2xy + 2y^2 - 6x$

$f_x = 4x + 2y - 6$

$f_y = 2x + 4y$

For finding stationary point:

$f_x = 0 \ ; \ f_y = 0$

$f_x = 4x + 2y - 6 = 0 \Rightarrow 2x + y = 3 \quad -①$

$f_y = 2x + 4y = 0 \Rightarrow x + 2y = 0 \quad -②$

Upon solving ① & ②;

We get $\boxed{x = 2, y = -1}$

$\therefore$ Stationary point $= (2, -1)$

Option (D)

$2x + y = 3$

$x + 2y = 0 \ (\times 2)$

$\Downarrow$

$\cancel{2x} + y = 3$

$(-) \ \cancel{2x} + 4y = 0$

$-3y = 3$

$\boxed{y = -1}$

$2x - 1 = 3$

$2x = 4$

$\boxed{x = 2}$

Q5)

$$\begin{bmatrix} x & y & z \end{bmatrix}_{1\times3} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times1} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} ax+dy+gz & bx+ey+hz & cx+fy+iz \end{bmatrix}_{1\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$$

$$x(ax+dy+gz) + y(bx+ey+hz) + z(cx+fy+iz)$$

$$= ax^2+dxy+gxz + bxy+ey^2+hyz + cxz+fyz+iz^2$$

$$\Rightarrow ax^2+ey^2+iz^2 + (b+d)xy + (g+c)xz + (f+h)yz$$

$$\Updownarrow$$

Compare with

$$x^2+y^2-z^2-xy+yz+xz$$

$a=1$; $e=1$; $i=-1$;

$(b+d)=-1$; $(c+g)=1$; $(f+h)=1$

By elimination of Options:

Only in (A) & (C) ; diagonal elements are $\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$

as $\begin{bmatrix} a & & \\ & e & \\ & & i \end{bmatrix}$

lets check in Option (C) : $b+d = -1+0 = -1$ ✓

$c+g = 0+1 = 1$ ✓

$f+h = 0+(-1) = -1$ (✗)

lets check in option (A) : $b+d = -1+0 = -1$ ✓

$c+g = 0+1 = 1$ ✓

$f+h = 0+1 = 1$ ✓

$\therefore$ Answer is Option (A)

Q6) $f(x,y) = 3x^2 + 4xy + 2y^2$

$f_x = 6x+4y$ $\quad$ $f_y = 4x+4y$

$f_{xx} = 6$ $\quad$ $f_{yy} = 4$

Since $f_{xx} > 0$ & $f_{yy} > 0$

$\therefore$ The point $(0,0)$ is Minima. $\boxed{\text{Option (B)}}$

Q7.) $A = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a = 4 > 0$

$ac - b^2 = (4)(3) - (2)(2) = 12 - 4 = 8 > 0$

Since $a > 0$ and $ac - b^2 > 0$

$\therefore$ A is positive definite. $\boxed{\text{Option (A)}}$

Q8) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a = 1 > 0$

$ac - b^2 = (1)(1) - (2)(2) = 1 - 4 = -3 < 0$

Since $a > 0$ but $ac - b^2 < 0$

$\therefore$ A is NOT Positive definite

$\boxed{\text{Option (B)}}$

Q9) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ $= (3-\lambda)(5-\lambda)(7-\lambda) = 0$

$\Downarrow$

{derived from $\det(A - \lambda I) = 0$}

$\lambda_1 = 3$; $\lambda_2 = 5$; $\lambda_3 = 7$

Since $\lambda_1, \lambda_2, \lambda_3 > 0$,

Hence A is positive definite

$\boxed{\text{Option (A)}}$

Q16) $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{4\times 3} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}_{3\times 3}$$

Characteristic polynomial of AA^T (3×3 matrix):

$$\lambda^3 - [\text{trace}(AA^T)]\lambda^2 + \left(\begin{array}{c}\sum \text{Minors of} \\ \text{diagonal elements} \\ \text{of } AA^T\end{array}\right)\lambda - \det(AA^T) = 0$$

$$\lambda^3 - 6\lambda^2 + \left\{ \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \right\}\lambda - [3(2) - 1(2) + 2(-2)] = 0$$

$$\lambda^3 - 6\lambda^2 + 6\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 6) = 0$$

$$\boxed{\lambda = 3 \pm \sqrt{3}, 0}$$

$(\sigma = \sqrt{\lambda}) \Rightarrow \boxed{\sigma_1 = \sqrt{3+\sqrt{3}} \; ; \; \sigma_2 = \sqrt{3-\sqrt{3}}}$ are the non-zero singular values

Option (B)

Q21) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$ $\qquad A_{2\times 2} = Q_{1\,2\times 2} \Sigma_{2\times 4} Q_{2\,4\times 4}^T$

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{4\times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

Characteristic polynomial of AA^T:

$$\lambda^2 - 4\lambda + 4 = 0$$

Roots : $\lambda = 2, 2$ (repeated).

$\sigma = \sqrt{\lambda} = \sqrt{2}$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

for $\lambda = 2$: $[AA^T - 2I]X = 0$

$$\left\{ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v = k\begin{bmatrix} x \\ y \end{bmatrix}$$

for $x = 1, y = 0$ $\quad v_1 = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

for $x = 0, y = 1$ $\quad v_2 = k\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Q_1 = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ x_1 \quad x_2 \end{array}$$

$$y_1 = \frac{A^T x_1}{\sigma_1} = \frac{1}{\sqrt{2}} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{4\times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1} \right\}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$y_2 = \frac{A^T x_2}{\sigma_2} = \frac{1}{\sqrt{2}}\left\{\begin{bmatrix}1 & 0\\0 & 1\\1 & 0\\0 & 1\end{bmatrix}_{4\times2}\begin{bmatrix}0\\1\end{bmatrix}_{2\times1}\right\} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

For other two :

$$\left\{\begin{array}{l}\begin{bmatrix}a & b & c & d\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = 0 \quad \& \\ \begin{bmatrix}a & b & c & d\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = 0\end{array}\right\}$$

$a+c = 0$

$b+d = 0$

Let:

$c = k_1,\ d = k_2$

$$k_1\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + k_2\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} \xrightarrow{\text{Normalizing}} \frac{1}{\sqrt{2}}k_1\begin{bmatrix}-1\\0\\1\\0\end{bmatrix};\ \frac{1}{\sqrt{2}}k_2\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$$

$$y_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$y_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

$$y_3 = \frac{1}{\sqrt{2}}k_1\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} \quad \text{for } k_1 = -1 \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$$

$$y_4 = \frac{1}{\sqrt{2}}k_2\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} \quad \text{for } k_2 = -1 \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

$$Q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}$$

$$\therefore Q_2^T = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}$$

{Option (C)}