Machine Learning Foundations

Tutorial - Week5

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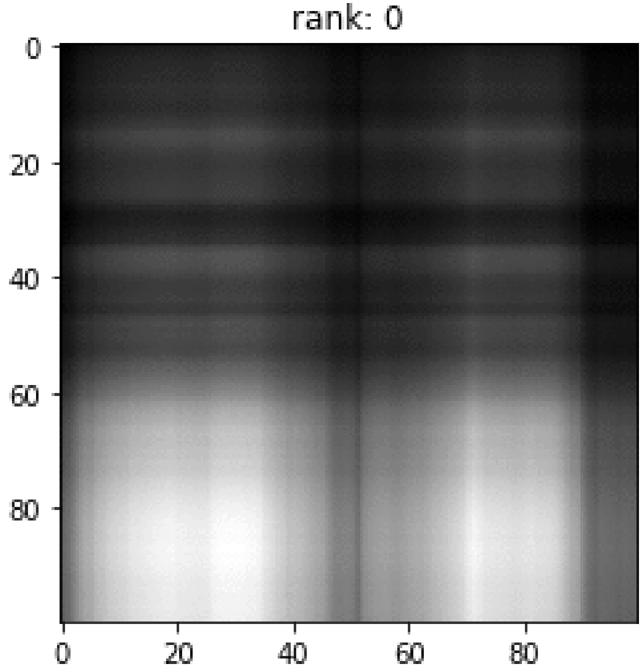
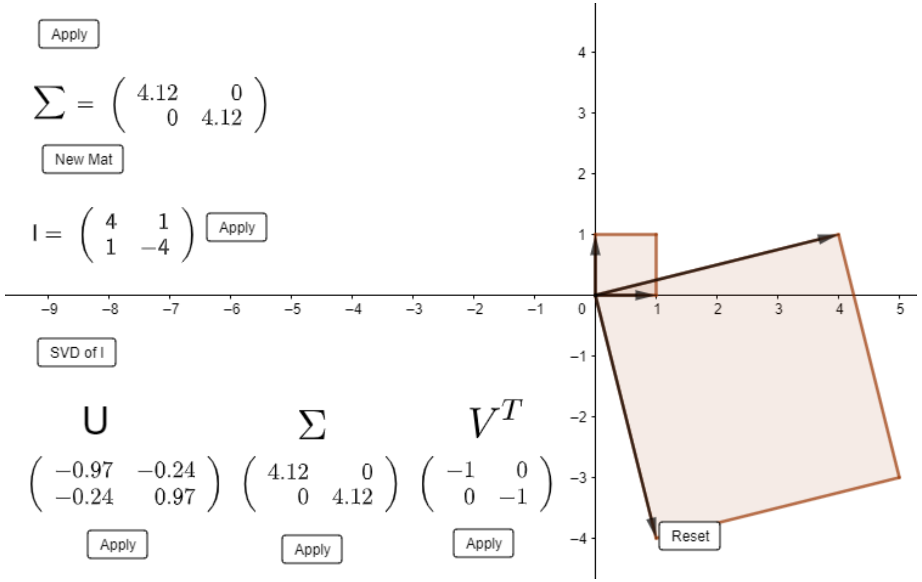
Our Mind

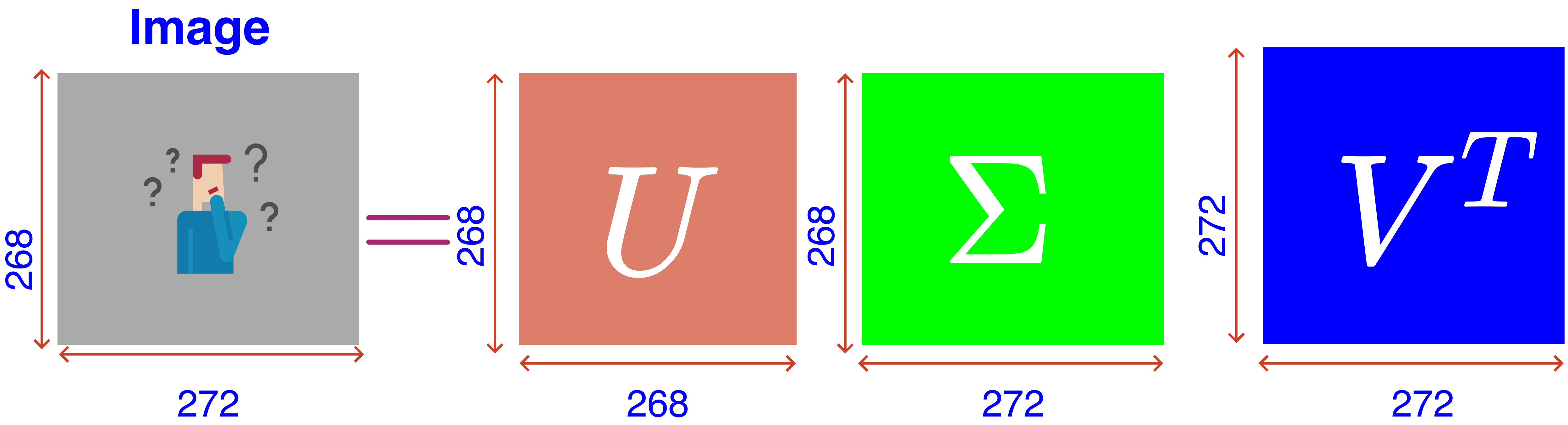
$$I = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

$$I = U \Sigma V^T$$



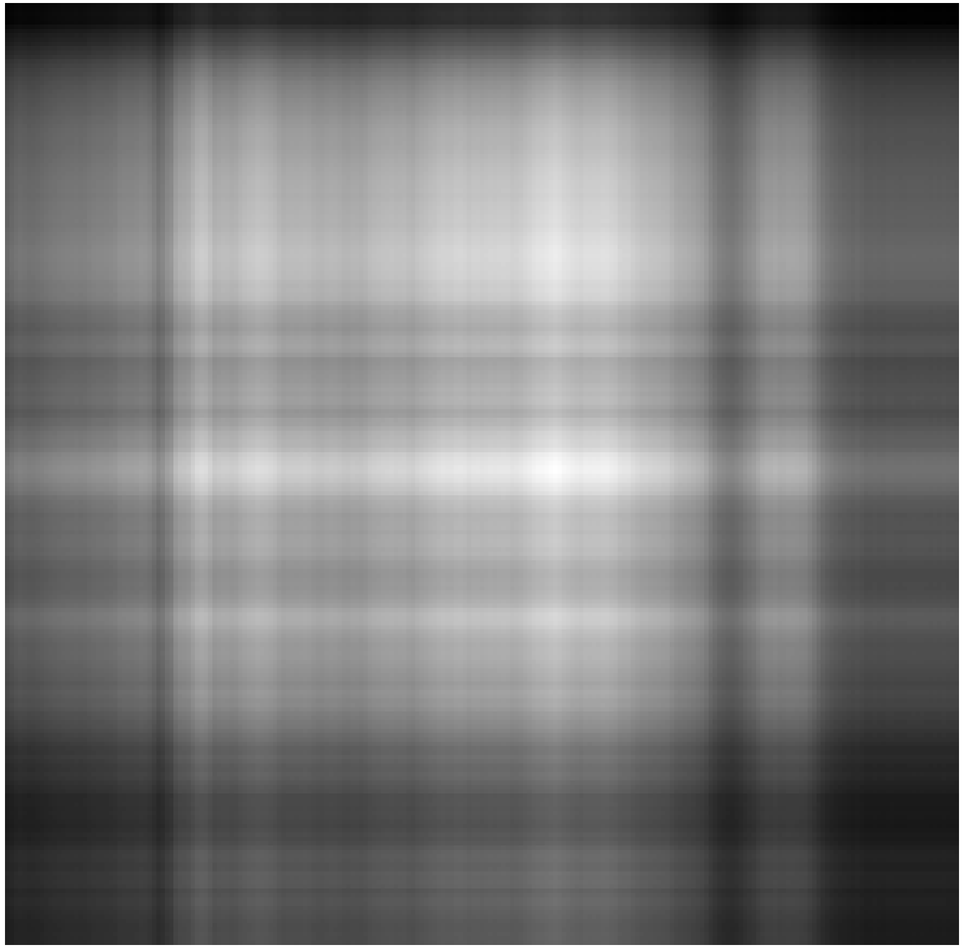


Let's play a game



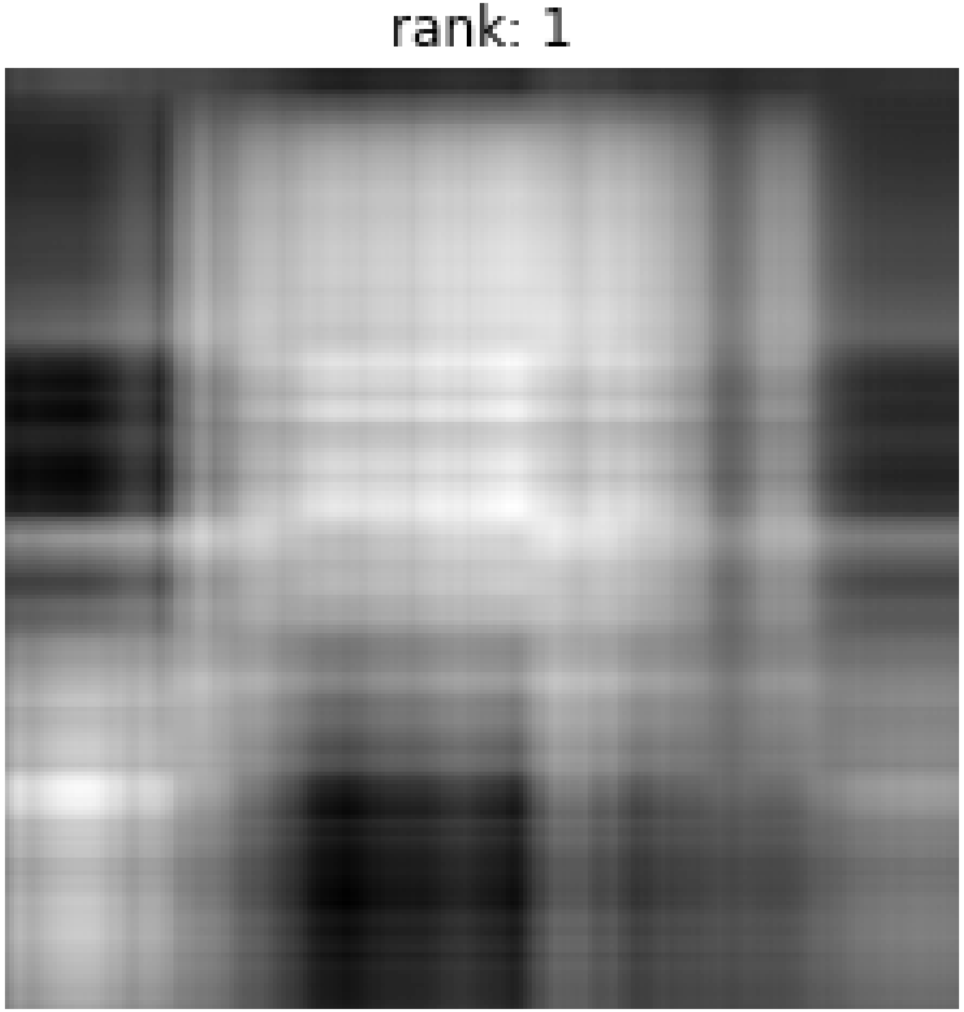
- I am going to show you a sequence of images, one after another, that contains something.
- Task : Recognise the "thing" in the images. (Note down the sequence number)
- Let's go

rank: 0

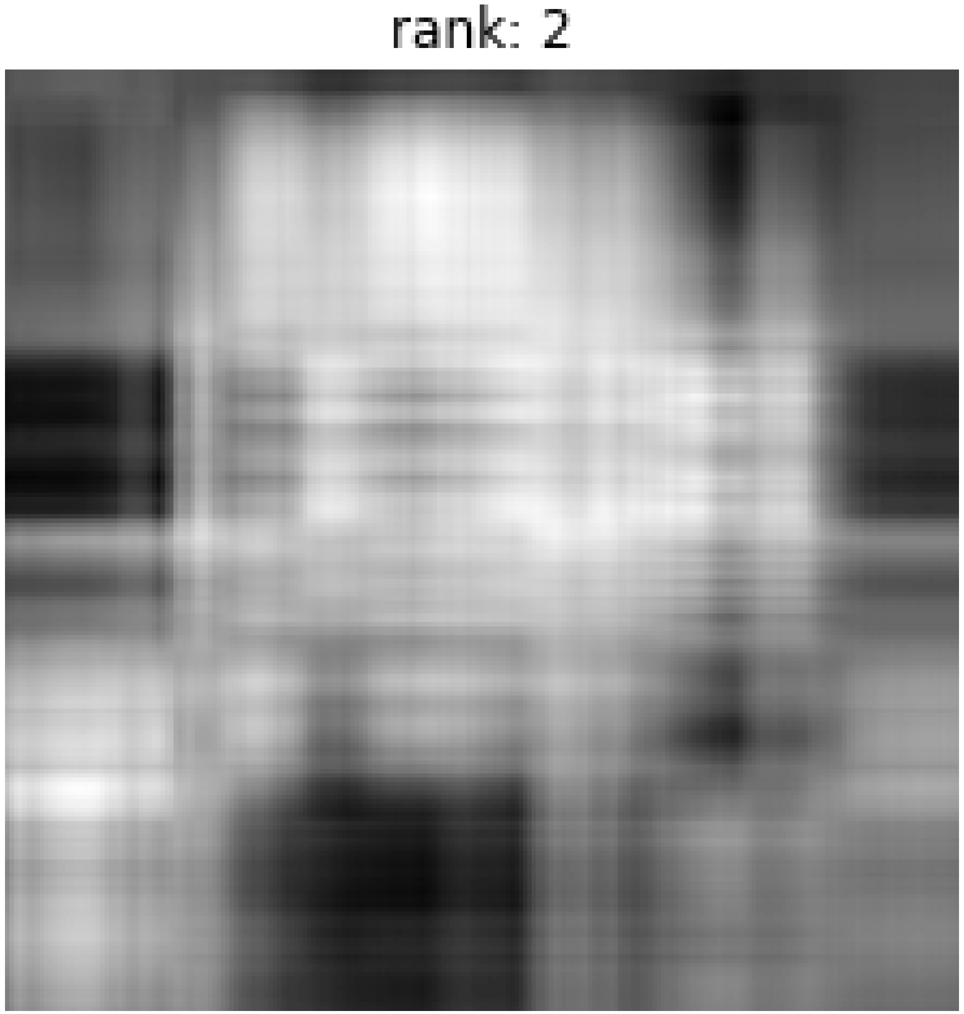


What do you perceive?

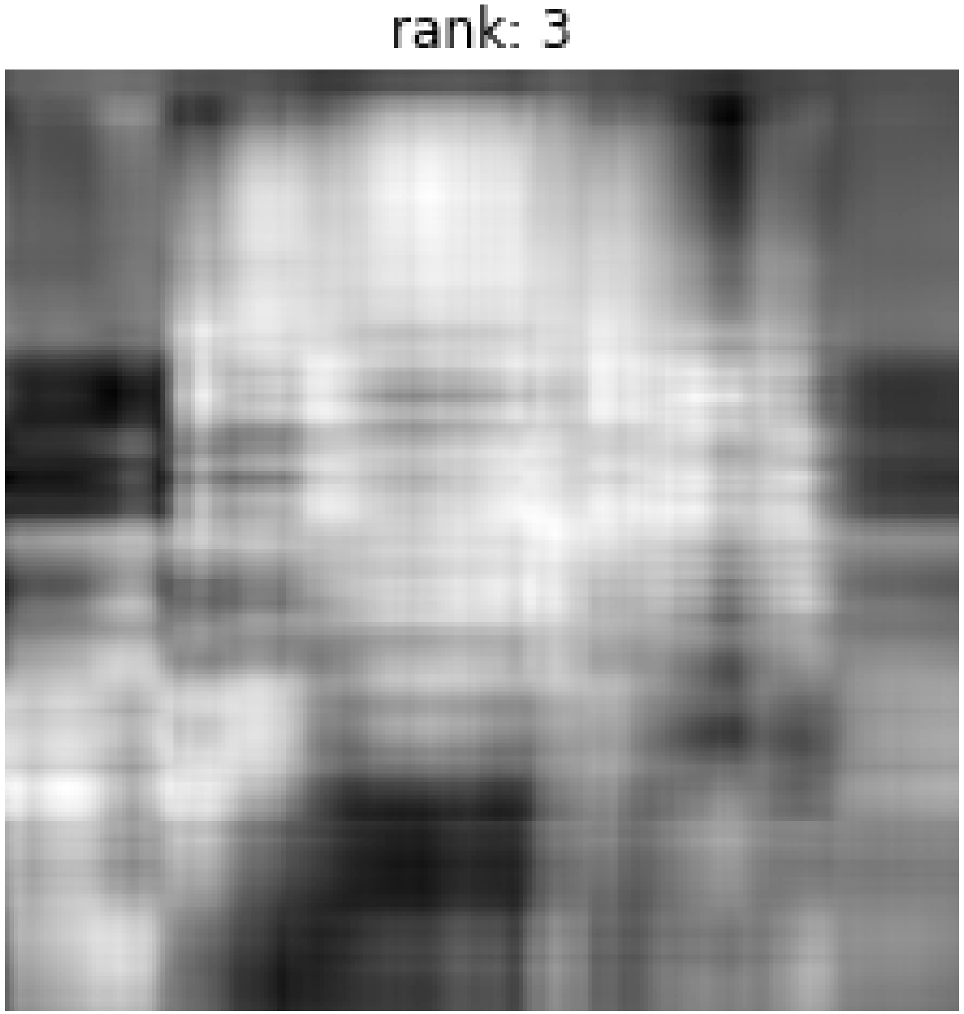
My bad: Add +1 to the rank



What do you perceive?

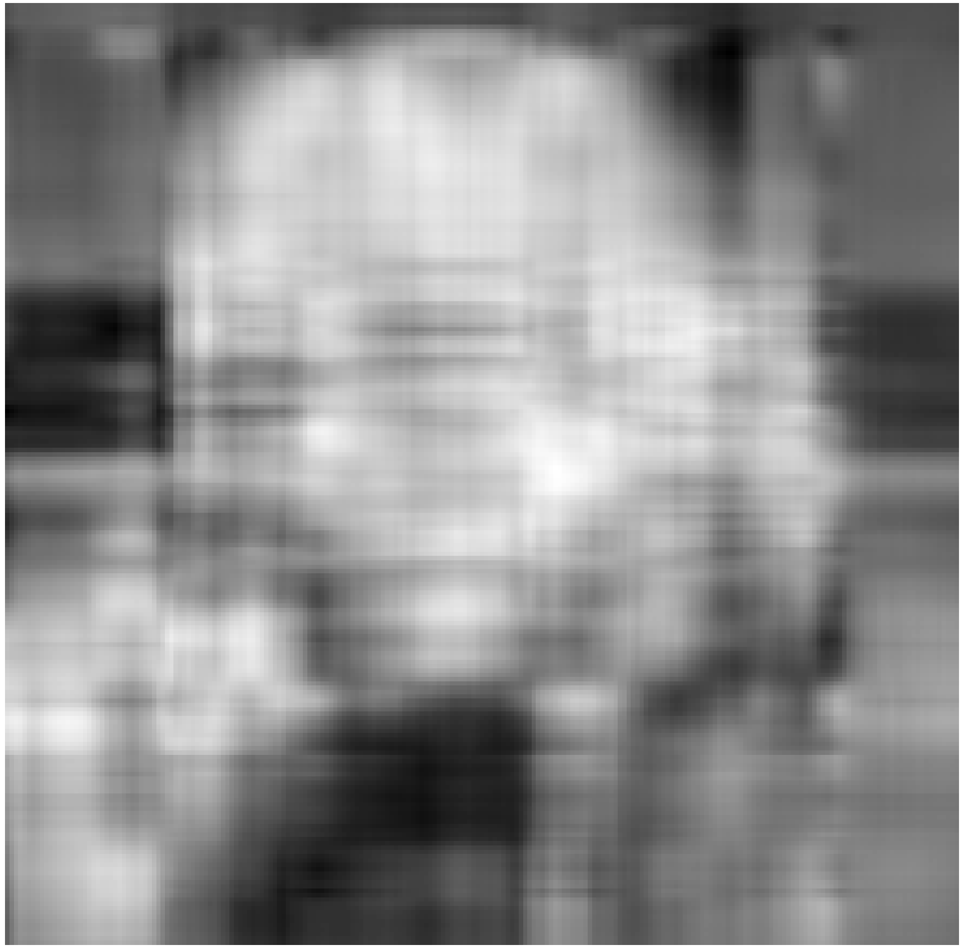


What do you perceive?

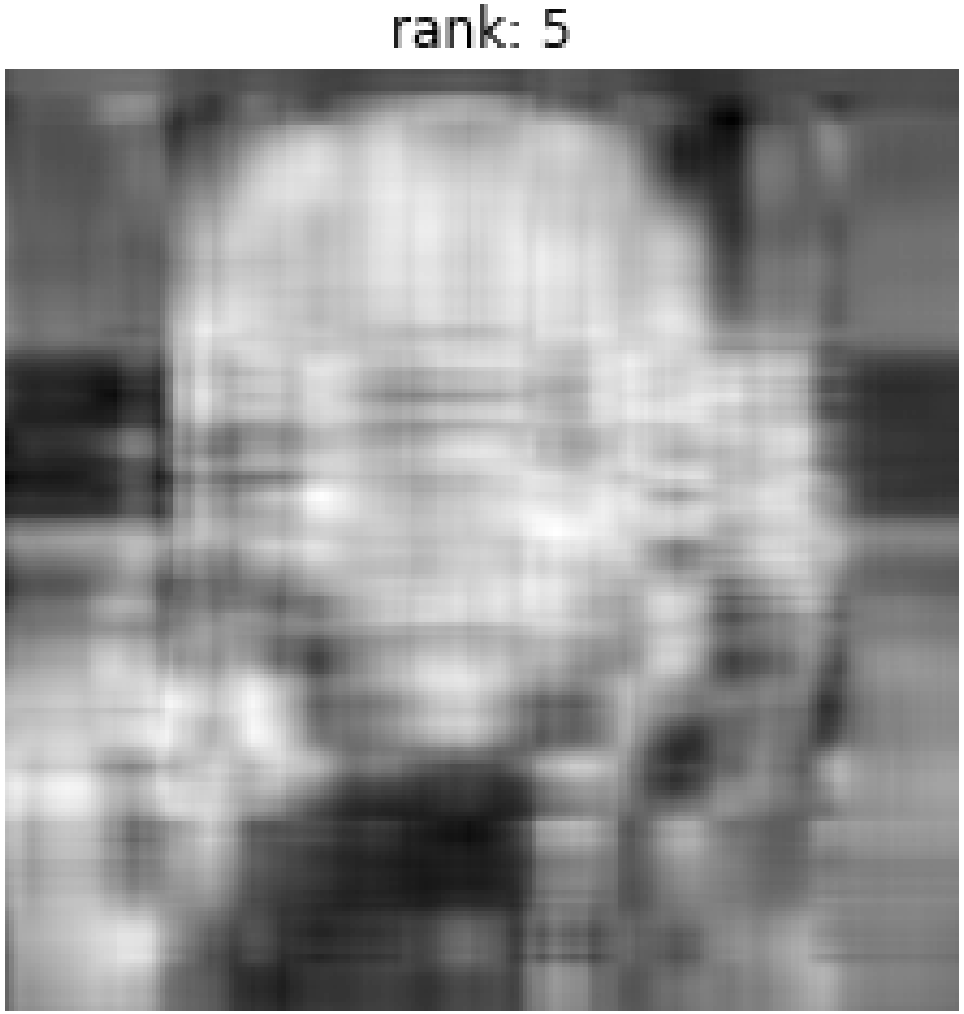


What do you perceive?

rank: 4



What do you perceive?



What do you perceive?

rank: 6

What do you perceive?

What do you perceive?

What do you perceive?

What do you perceive?

Recognise the man in the picture

rank: 10

Recognise the man in the picture

Recognise the man in the picture

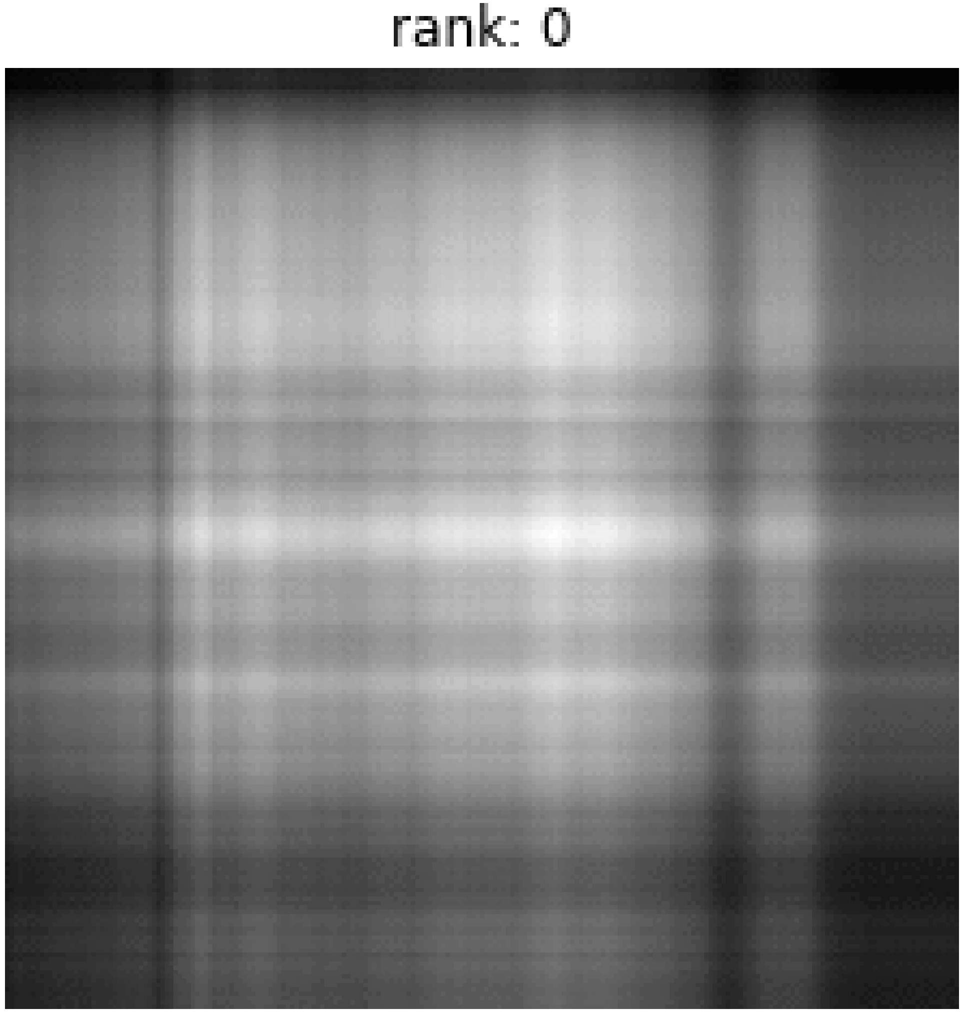
Recognise the man in the picture

Recognise the man in the picture

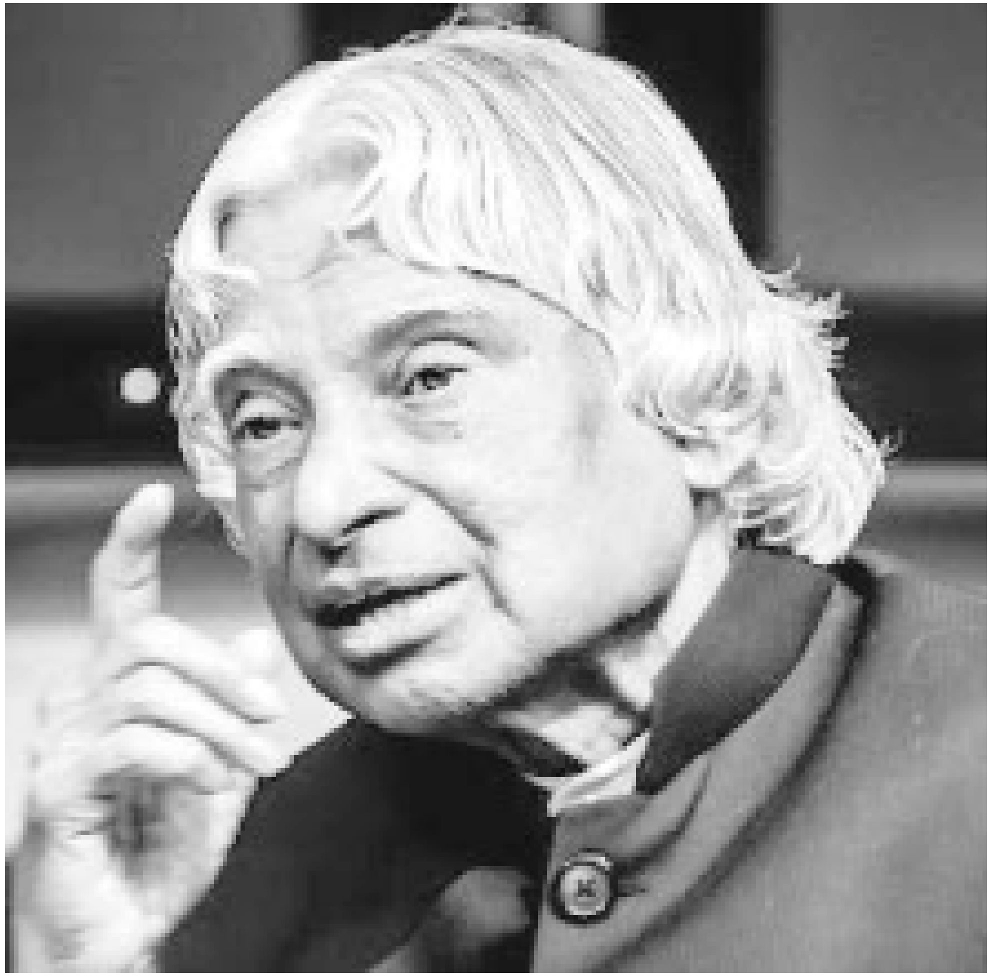
rank: 14

Recognise the man in the picture

Rank (k =50) Approximation

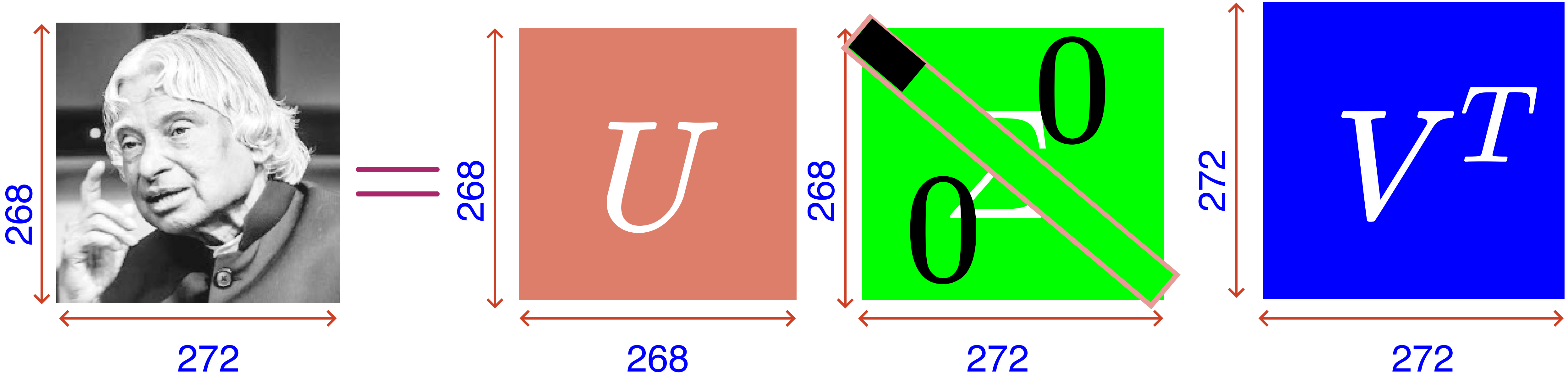


Of course, we lost some resolution (details)!



But not the gross information!

Image compression



$$I = \sum_{i=1}^{k=15} \sigma_i u_i v_i^T \qquad \sigma_i = 0, i > 15$$

Original Image : (Without Compression)

Number of elements to be stored:

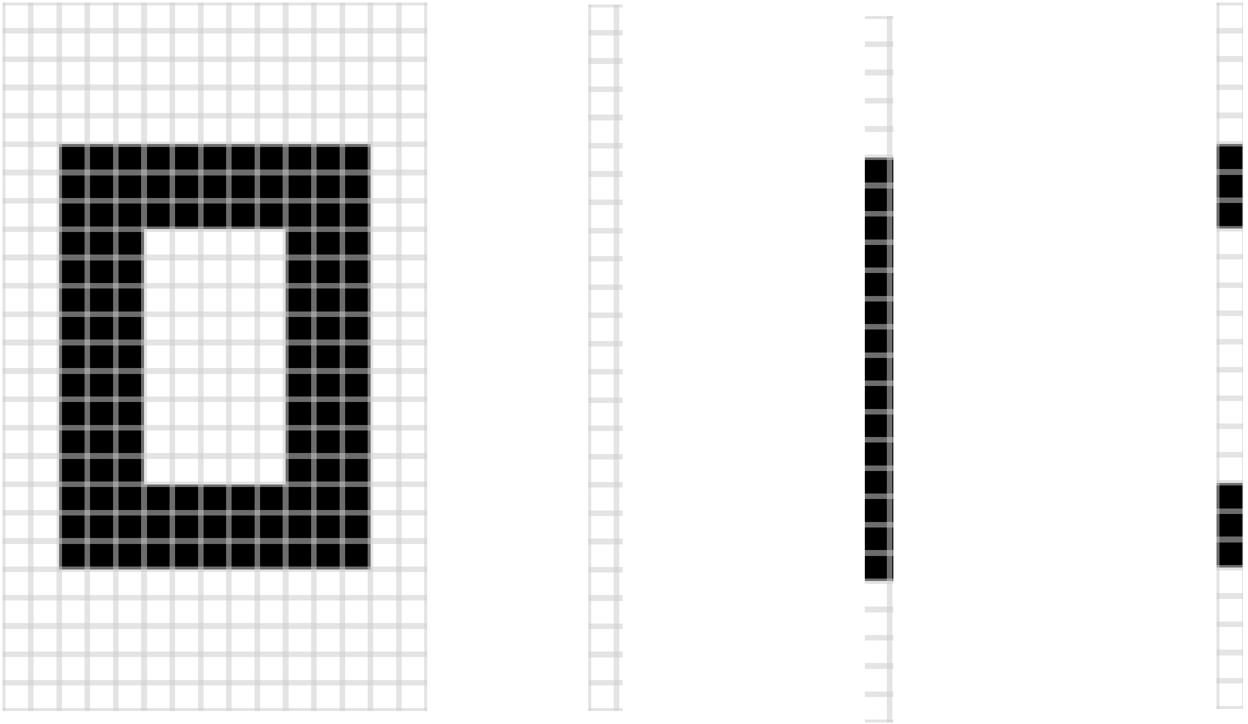
268*272 = 72,896

Approx. Image :(or Compressed)

- For U : 268*15 = 4020
- For Σ: k =15
- For V^T : 15*272 = 4080
- Total = 8115
- ~= 9 times reduction in required memory for storage

Well, but, hmm..How does it work?

The image is a linear combination of



$$\sigma_1 = 14.7$$

$$\sigma_2 = 5.22$$

$$\sigma_3 = 3.31$$

$$\sigma_i = 0, i > 3$$

It is zero by deafult!

Removes redundancy!

Image Source: http://www.ams.org/publicoutreach/feature-column/fcarc-svd

Tutorial - Week5

Geometric Interpretation of SVD

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SVD

$$I = U\Sigma V^T$$

Σ → **Diagonal Matrix**

- What happens if diagonal matrices act on a set of vectors in the canonical (standard) basis?
- Let us **see** it in $\mathbb{R}^2$ with help of Geogebra applet :-)

Diagonal Matrices

https://www.geogebra.org/material/iframe/id/nhksajgq/width/700/height/625/border/888888/sfsb/true/smb/false/stb/false/stbh/false/ai/false/asb/false/sri/false/rc/false/ld/false/sdz/true/ctl/false

Diagonal matrices preserves the direction of orthogonal vectors!

Why?

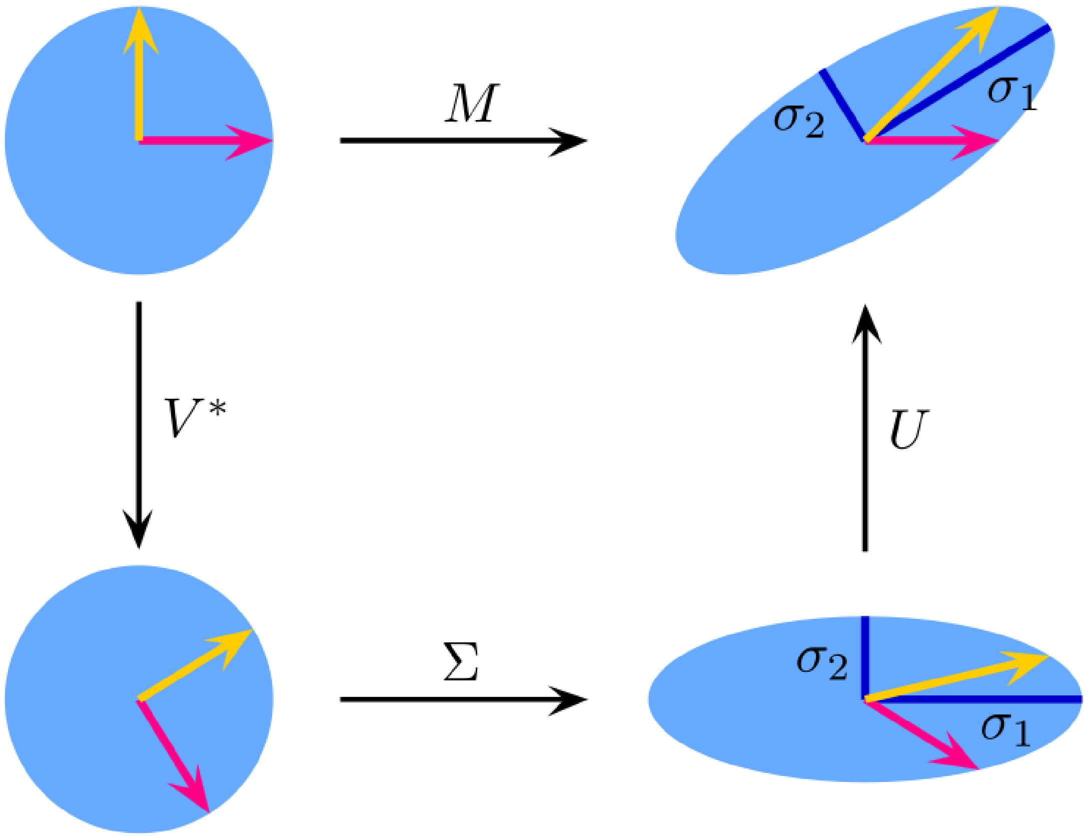
Similar Matrices

https://www.geogebra.org/material/iframe/id/dgwbf7db/width/1020/height/500/border/888888/sfsb/true/smb/false/stb/fa
lse/stbh/false/ai/false/asb/false/sri/false/rc/false/ld/false/sdz/true/ctl/false

Geometry of SVD

https://www.geogebra.org/material/iframe/id/dgwbf7db/width/1020/height/500/border/888888/sfsb/true/smb/false/stb/false/stbh/false/ai/false/asb/false/sri/false/rc/false/ld/false/sdz/true/ctl/false

Geometry of SVD



$$M = U \cdot \Sigma \cdot V^*$$

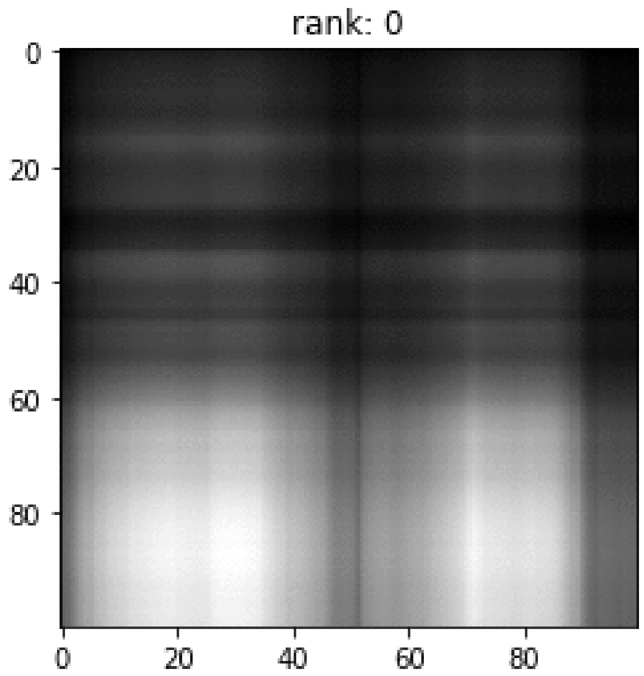
A quick summary:

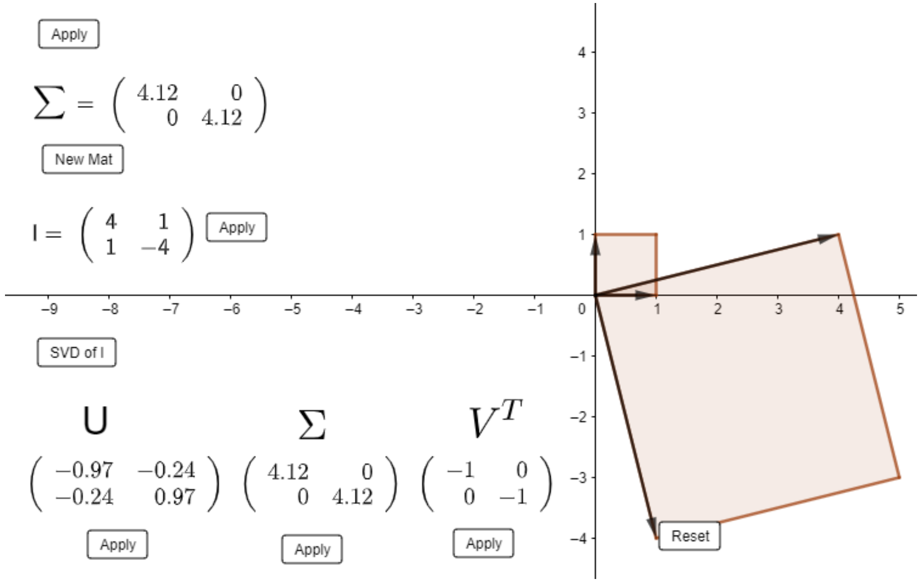
- V^T Rotates disk D and basis e_1, e_2
- Σ scales the rotated disk D and σ_1, σ_2 are semi-major and semi-minor axis of an ellipse (hyper-ellipse)
- U rotates the ellipse.

Source: Wikipedia

$$I = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

$$I = U\Sigma V^T$$





Tutorial - Week5

Some questions to think and solve

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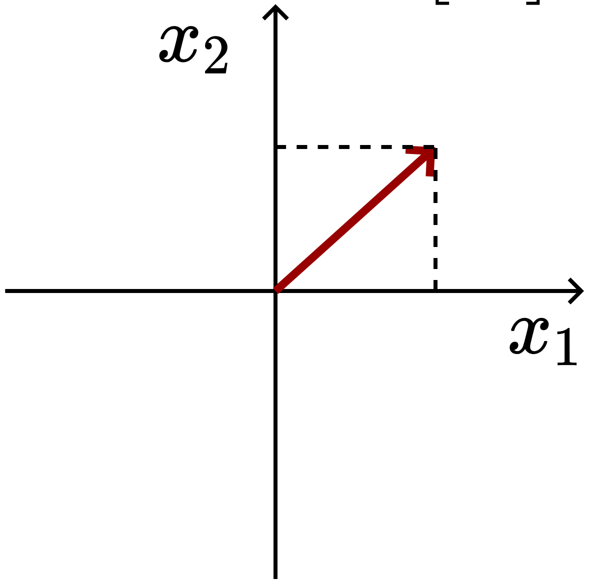
High Dimensional Visualization

"To deal with hyper-planes in a 14-dimensional space, visualize a 3-D space and say 'fourteen' to yourself very loudly. Everyone does it.

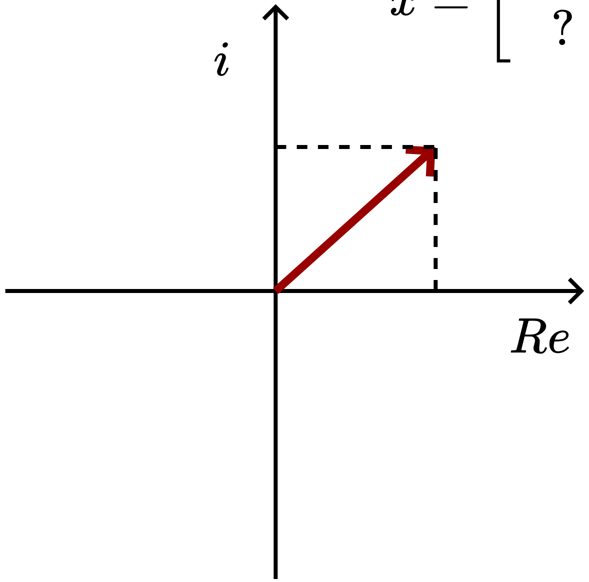
Geoffrey Hinton

1. Is it possible to visualize complex vectors $x_i \in \mathbb{C}^2$ geometrically as we do for real vectors ?. Pause the video and think about it.

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$x = \begin{bmatrix} 1+i \\ ? \end{bmatrix}$$



We need **4** dimensions to visualize a vector from $\mathbb{C}^2$

Do complex matrices find any real world applications?

Is that just an abstract mathematical stuff?

Discrete Fourier Transform
(DFT)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix}$$

😃

Countless Applications in signal processing, Digital communication, Speech processing ...

2.Compute the inner product between two vectors $x = \begin{bmatrix} 3-2i \\ -2+i \\ -4-3i \end{bmatrix}$ and $y = \begin{bmatrix} -2+4i \\ 5-i \\ -2i \end{bmatrix}$ and verify whether they are commutative (i.e. $x \cdot y = y \cdot x$)

$$x \cdot y = x^* y = \bar{x}^T y$$

$(3+2i) \times (-2+4i)$ = $-14+8i$

$(-2-i) \times (5-i)$ = $-11-3i$

$(-4+3i) \times (-2i)$ = $6+8i$

Σ

$-19+13i$

$$y \cdot x = y^* x = \bar{y}^T x$$

2.Compute the inner product between two vectors $x = [3-2i, -2+i, -4-\quad 3i]^T$ and $y = [-2+4i, 5-i, -2i]^T$ and verify whether they are commutative (i.e. $x \cdot y = y \cdot x$)

$$y \cdot x = y^* x = \bar{y}^T x$$

$(3-2i) \times (-2-4i)$ = $-14-8i$

$(-2+i) \times (5+i)$ = $-11+3i$

$(-4-3i) \times (2i)$ = $6-8i$

$$\Sigma \rightarrow -19-13i$$

$$y \cdot x = y^* x = \bar{y}^T x$$

$$\overline{x \cdot y}$$

Is it always true?

$$x \cdot y \neq y \cdot x$$

3. Prove that $x \cdot y = \overline{y \cdot x}$ where $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$

$$\overline{y \cdot x} = \overline{\bar{y}x}$$

$$= \overline{\bar{y_1}x_1} + \overline{\bar{y_2}x_2} + \cdots + \overline{\bar{y_n}x_n}$$

$$= y_1\bar{x_1} + y_2\bar{x_2} + \cdots + y_n\bar{x_n}$$

$$= \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$

$$= \ x \cdot y$$

4. Can we use inner product to compute (cosine) angle between two complex vectors, like we do for real vectors?

No!. Not Always

Let us reason why?

$$\frac{x \cdot y}{||x||\ ||y||}$$

But some authors prefers

$$\frac{Re(x \cdot y)}{||x||\ ||y||}$$

More details on it : Angles in complex vector space

5. Consider the matrix $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$. Find the complex eigenvector for the eigenvalue $\lambda = 8$

$$N[A - \lambda I] = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix}$$

$$R_2 = R_2 + \tfrac{1}{2}(1+i)R_1$$

$$= \begin{bmatrix} -6 & 3-3i \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 3-3i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-6x_1 + (3-3i)x_2 = 0$$

$$-2x_1 + (1-1i)x_2 = 0$$

$$2x_1 = (1-1i)x_2$$

$$x_1 = 1$$

$$x_2 = \tfrac{2}{1-1i}$$

$$x_2 = \tfrac{2}{1-1i} = \tfrac{2}{1-1i}\tfrac{1+1i}{1+1i}$$

$$x_2 = 1 + 1i$$

$$\therefore x = \begin{bmatrix} 1 \\ 1+1i \end{bmatrix}$$

6. Let $U = \begin{bmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{bmatrix}$, show that the matrix U is unitary.

$$U = \begin{bmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{bmatrix} \qquad U^T = \begin{bmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{bmatrix}$$

$$U * U^T = \begin{bmatrix} cos^2(t) + sin^2(t) & cos(t)sin(t) - sin(t)cos(t) \\ sin(t)cos(t) - cos(t)sin(t) & sin^2(t) + cos^2(t) \end{bmatrix}$$

$$U * U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

7. We know that $U = \begin{bmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{bmatrix}$,is unitary.Let us take a vector $x \in R^2$ and see what happens when it get transformed by the U.

https://www.geogebra.org/material/iframe/id/ynztugm7/width/700/height/500/border/888888/sfsb/true/smb/false/stb/false/stbh/false/ai/false/asb/false/sri/false/rc/false/ld/false/sdz/true/ctl/false