Background on complex matrices:

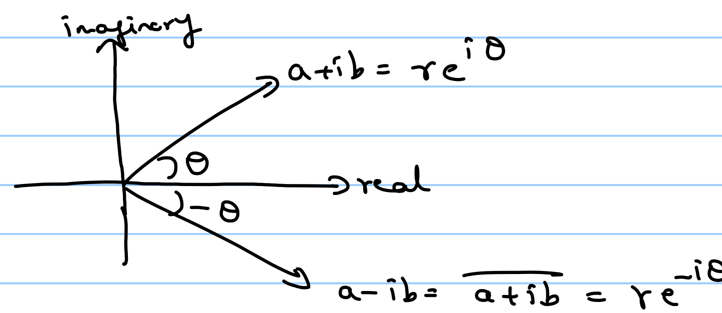
$\mathbb{C}^n$: complex counterpart of $\mathbb{R}^n$

$(x_1, \dots, x_n) \in \mathbb{C}^n$, then $x_i \to$ complex number, $i = 1 \dots n$

Addition: $(a+ib) + (c+id) = (a+c) + i(b+d)$

multiplication: $(a+ib)(c+id) = (ac-bd) + i(bc+ad)$

Complex conjugate of $(a+ib)$ is $(a-ib)$



$r^2 = a^2 + b^2 = |a+ib|$

Linear combinations: $c_1 x_1 + \dots + c_k x_k = 0$

$\in \mathbb{C}^n$

complex numbers

Inner product & length:

In $\mathbb{R}^n$, $\underbrace{\|x\|^2}_{\text{length}} = \underbrace{x^T x}_{\text{inner product}}$ ——— (*)

We cannot use the same definition in $\mathbb{C}^n$. Why?

e.g. using (*) $\quad \left\| \begin{bmatrix} 1 \\ i \end{bmatrix} \right\| = 0$

So, in $\mathbb{C}^n$, define the inner-product as $x \cdot y = \bar{x}^T y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$

Note: $\bar{x}^T y \neq \bar{y}^T x$ $\quad$ e.g. $x = \begin{bmatrix} 2-i \\ 1+i \end{bmatrix}$ $\quad y = \begin{bmatrix} 6+8i \\ 9-13i \end{bmatrix}$ $\quad$ Check $\bar{x}^T y \neq \bar{y}^T x$

Length of a complex vector: For $x \in \mathbb{C}^n$, define $\|x\|^2 = \bar{x}^T x$

$$\left\| \begin{bmatrix} 1 \\ i \end{bmatrix} \right\| = \sqrt{2} \neq 0$$

$\|x\| = 0$ if and only if $\|x\| = 0$

Check these → ① $x \cdot y = \overline{y \cdot x}$

② $x \cdot (cy) = c(x \cdot y)$

③ $(cx) \cdot y = \bar{c}(x \cdot y)$

using the definition of inner product

$x \cdot y = \bar{x}^T y = \sum_{i=1}^{n} \bar{x}_i y_i$

Conjugate transpose:

A^* = conjugate transpose of A

$A^* = \bar{A}^T = \overline{A^T}$

Example: $A = \begin{bmatrix} 1+i & 3-2i \\ 2-4i & i \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 1-i & 3+2i \\ 2+4i & -i \end{bmatrix}$ $A^* = \bar{A}^T = \begin{bmatrix} 1-i & 2+4i \\ 3+2i & -i \end{bmatrix}$

we would get the same result by transposing first & then taking the conjugate.

Remark: Real matrix A, $A^* = A^T$

Check → ① $(A^*)^* = A$

② $(AB)^* = B^* A^*$

Real case equivalents

① $(A^T)^T = A$

② $(AB)^T = B^T A^T$

Inner-product: $x \cdot y = \bar{x}^T y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$

$$= \begin{bmatrix} \bar{x}_1 & \cdots & \bar{x}_n \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\boxed{x \cdot y = x^* y}$$

Hermitian matrix: A matrix A is Hermitian if $A^* = A$

"Hermitian matrices are the equivalent of symmetric matrices in a complex vector space".