Q1.) Complex conjugate of $(a+ib) = (a-ib)$ & vice versa
Applying to every term of given matrix :
Complex conjugate of $1-1i = 1+1i$
Complex conjugate of $1-3i = 1+3i$
Complex conjugate of $6+4i = 6-4i$
Complex conjugate of $35-2i = 35+2i$

$\therefore$ Complex conjugate matrix $= \begin{bmatrix} 1+i & 1+3i \\ 6-4i & 35+2i \end{bmatrix}$

$\Rightarrow$ option (B)

Q2.) $A = \begin{bmatrix} 3-2i & 5+i \\ 1+4i & 7-2i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 3+2i & 5-i \\ 1-4i & 7+2i \end{bmatrix}$

$\bar{A}^T = \overline{A^T} = \begin{bmatrix} 3+2i & 1-4i \\ 5-i & 7+2i \end{bmatrix} \Rightarrow$ option (D)

Q3.) Inner product of x & y $= x \cdot y = \bar{x}^T y$
$= \sum_{i=1}^{n} \bar{x}_i y_i$

$x \cdot y = \boxed{x^* y}$

$x^* = \overline{x^T}$ or $\bar{x}^T = \begin{bmatrix} 1+i & -2i \end{bmatrix}$
$x^* y = \begin{bmatrix} 1+i & -2i \end{bmatrix} \begin{bmatrix} -1-i \\ i \end{bmatrix} = (1+i)(-1-i) + (-2i)(i)$

Remember, $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

Date :

$$\Rightarrow \left(\cancel{(1)(-1) - (1)(-1)}^{\,0}\right) + i\left((1)(-1) + (1)(-1)\right) + \left(-2i^2\right)$$

$$= -2i - 2(-1) \qquad \left(i^2 = -1 \text{ as } i = \sqrt{-1}\right)$$

$$= \boxed{2-2i} \Rightarrow \text{option (C)}$$

Q4) $\|x\|^2 = x^* x = \bar{x}^T x$ or $\overline{x^T} x$

$$= \begin{bmatrix} 2+i & 4+i \end{bmatrix} \begin{bmatrix} 2-i \\ 4-i \end{bmatrix}$$

$$= (2+i)(2-i) + (4+i)(4-i)$$

$$= (4+1) + i\cancel{(-2+2)}^{\,0} + (16+1) + i\cancel{(-4+4)}^{\,0}$$

$$= 5 + 17 = \boxed{22} \Rightarrow \text{option (D)}$$

Q5) A matrix is Unitary if $A^* A = I$.

$$A^* A = \begin{bmatrix} \frac{1-i}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{-2i}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{2i}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(1-i)(1+i)}{3} + \frac{1}{3} & \frac{(1-i)(1+i)}{\sqrt{3}\cdot\sqrt{6}} + \frac{2}{\sqrt{3}\cdot\sqrt{6}} \\ \frac{(1-i)(1+i)}{\sqrt{3}\cdot\sqrt{6}} + \frac{2}{\sqrt{3}\cdot\sqrt{6}} & \frac{(1-i)(1+i)}{6} + \frac{4}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} + \frac{1}{3} & \frac{2}{\sqrt{3}\cdot\sqrt{6}} + \frac{2}{\sqrt{3}\cdot\sqrt{6}} \\ \frac{2}{\sqrt{3}\cdot\sqrt{6}} + \frac{2}{\sqrt{3}\cdot\sqrt{6}} & \frac{2}{6} + \frac{4}{6} \end{bmatrix}$$

$$\left\{ (1+i)(1-i) = (1+1) + i\cancel{(-1+1)}^{\,0} = \boxed{2} \right\}$$

$$= \begin{bmatrix} 1 & \frac{4}{\sqrt{3}\cdot\sqrt{6}} \\ \frac{4}{\sqrt{3}\cdot\sqrt{6}} & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence A is NOT UNITARY → <u>Option(B)</u>

Q6) A matrix is Hermition if $A^* = A$.

$$Z^* = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = Z.$$

∴ Since $Z^* = Z$, Z is Hermitiam ⇒ <u>Option (A)</u>

Q7.) a) Diagonal elements are NOT Real

Hence, A is <u>not Hermitiam</u>.

b) $A = \begin{bmatrix} 0 & 3-2i \\ 3+2i & 4 \end{bmatrix}$; $A^* = \begin{bmatrix} 0 & 3+2i \\ 3+2i & 4 \end{bmatrix}$

$$A \neq A^*$$

Hence, A is <u>NOT</u> Hermitiam

c) $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1+i \\ 3i & 1+i & 0 \end{bmatrix}$; $A^* = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$

$$A = A^*$$

Hence, A is Hermitiam.

d) $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$; $A^* = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$

$$A = A^*$$

Hence, A is Hermitian.

Options C, D are Hermitian Matrices.

Q8.) $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$; $A^* = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$

$A = A^* \Rightarrow A$ is Hermitian

(Has real eigenvalues)

Characteristic polynomial for 3x3 matrix :

$$x^3 - \boxed{\text{trace}(A)}\, x^2 + \begin{pmatrix} \text{Minors of} \\ \text{diagonal elements} \\ \text{of } A \end{pmatrix} x - \det(A) = 0$$

$$\text{trace}(A) = 3 + 0 + 0 = \boxed{3}$$

$$M_{11} = \begin{vmatrix} 0 & 1-i \\ 1+i & 0 \end{vmatrix} = 0 - (1-i)(1+i) = [1 - (-1) + i\,\cancel{(1-1)}^{0}] = \boxed{-2}$$

$$M_{22} = \begin{vmatrix} 3 & -3i \\ 3i & 0 \end{vmatrix} = 0 - 3i(-3i) = \boxed{-9}$$

$$M_{33} = \begin{vmatrix} 3 & 2-i \\ 2+i & 0 \end{vmatrix} = 0 - (2+i)(2-i) = 0 - [(4+1) + i\,\cancel{(-2+2)}^{0}] = \boxed{-5}$$

$$\det(A) = 3(-(1-i)(1+i)) - (2-i)(-3i(1-i)) - 3i(2+i)(1+i)$$

$$= 3(-2) + 9 + \cancel{3i} + 9 - \cancel{3i}$$

$$= \boxed{12}$$

Plugging the values into eq. ① we get

$$\lambda^3 - 3\lambda^2 + (-2-9-5)\lambda - (12) = 0$$

$$\lambda^3 - 3\lambda^2 - 16\lambda - 12 = 0$$

Solutions are $\boxed{6, -1, -2}$.

$\therefore$ Eigenvalues are $-1, 6, -2 \Rightarrow$ Option (D)

Q9) $A^* = k \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$

For a matrix to be Unitary, $A^*A = I$

$$A^*A = k \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} k \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$= k^2 \begin{bmatrix} (1-i)(1+i) + (1+i)(1-i) & \overbrace{(1-i)^2 + (1+i)^2}^{0} \\ \underbrace{(1+i)^2 + (1-i)^2}_{0} & (1+i)(1-i) + (1-i)(1+i) \end{bmatrix}$$

$$= k^2 \begin{bmatrix} 2+2 & 0 \\ 0 & 2+2 \end{bmatrix}$$

$(1+i)(1+i) = (\cancel{(1)(1) - (1)(1)})^0 + i[(1)(1) + (1)(1)] = 2i$

$(1-i)(1+i) = ((1)(1) - (-1)(1)) + i[\cancel{(1)(1) + (-1)(1)}]^0 = 2$

$\boxed{(a+ib)(c+id) = (ac-bd) + i(ad+bc)}$

$(1-i)(1-i) = (\cancel{(1)(1) - (-1)(-1)})^0 + i[(1)(-1) + (-1)(1)] = -2i$

$(1+i)^2 + (1-i)^2 = 2i - 2i = \boxed{0}$

$\Rightarrow A^*A = k^2 \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ or $4k^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^*A = I$$

$$4k^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$4k^2 = 1$$

$$k^2 = \frac{1}{4}$$

$$\text{or } \boxed{k = \frac{1}{2}} \rightarrow \text{option (A)}$$

Q10) $$A^* = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{\sqrt{k}}{2} & \frac{-\sqrt{k}i}{2} \end{bmatrix}$$

$$A^*A = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ \sqrt{k} & -\sqrt{k}i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1+i & \sqrt{k} \\ 1-i & \sqrt{k}i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1-i)(1+i) + (1+i)(1-i) & (1-i)(\sqrt{k}) + (1+i)(\sqrt{k}i) \\ \sqrt{k}(1+i) - \sqrt{k}i(1-i) & \sqrt{k}\cdot\sqrt{k} + (-\sqrt{k}i)(\sqrt{k}i) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 + 2 & \cancel{\sqrt{k}} - \cancel{\sqrt{k}i} + \cancel{\sqrt{k}i} - \cancel{\sqrt{k}} \\ \cancel{\sqrt{k}} + \cancel{\sqrt{k}i} - \cancel{\sqrt{k}i} - \cancel{\sqrt{k}} & k + k \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 2k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{k}{2} \end{bmatrix}$$

$$A^* A = I$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating both sides:

$$\frac{k}{2} = 1$$

$$\therefore \boxed{k = 2} \Rightarrow \text{Option (C)}$$

Q11)

Characteristic polynomial of $A_{2\times2}$:

$$\lambda^2 - (2+3)\lambda + \left[(2)(3) - (1-i)(1+i)\right] = 0$$

$$\lambda^2 - 5\lambda + (6-2) = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda^2 - \lambda - 4\lambda + 4 = 0$$

$$\lambda(\lambda-1) - 4(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-4) = 0$$

$$\boxed{\lambda = 1, 4}$$

D = diagonal matrix of λ values

$$\therefore \boxed{D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}$$

for $\lambda = 1$,

$$\left\{\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + (1+i)x_2 = 0 \qquad \times (1-i)$

$(1-i)x_1 + 2x_2 = 0$

$$\Downarrow$$

$$\underline{(1-i)x_1 + 2x_2 = 0}$$

let $\underline{x_2 = k}$

$(1-i)x_1 = -2k$

$x_1 = \frac{-2}{(1-i)}k = \frac{-2(1+i)k}{(1-i)(1+i)} = \frac{-\cancel{2}(1+i)k}{+\cancel{2}} = \underline{(-1-i)k}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

$\|x\| = \sqrt{(-1)^2 + (-1)^2 + (1)^2} = \sqrt{1+1+1} = \sqrt{3}$

for $\lambda = 4$,

$$\left\{ \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + (1+i)x_2 = 0 \qquad \times (1-i)$

$(1-i)x_1 - x_2 = 0 \qquad \times (-2)$

$$\Downarrow$$

$$\underline{-2(1-i)x_1 + 2x_2 = 0}$$

let $\underline{x_2 = k}$

$\cancel{2}(1-i)x_1 = \cancel{2}k$

$x_1 = \frac{k}{(1-i)} \times \frac{(1+i)}{(1+i)} = \underline{\frac{(1+i)k}{2}}$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} (1+i)/2 \\ 1 \end{bmatrix}$$

$$\|x\| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + (1)^2} = \frac{\sqrt{6}}{2}$$

$$\therefore U = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$\Rightarrow$ $\boxed{\text{Option (A)}}$

Q12.) For a matrix to be Unitary, $U^*U = I$.

option A) $$U^*U = \begin{bmatrix} \frac{1}{\sqrt2} & \frac{i}{\sqrt2} \\ \frac{-i}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt2} & \frac{i}{\sqrt2} \\ \frac{-i}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{i}{2}+\frac{i}{2} \\ -\frac{i}{2}-\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Option (A) is NOT Unitary.

Option B) $$U^*U = \begin{bmatrix} \frac{-1}{\sqrt2} & \frac{-i}{\sqrt2} \\ \frac{-i}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt2} & \frac{+i}{\sqrt2} \\ \frac{i}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{i}{2}-\frac{i}{2} \\ \frac{i}{2}+\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Option (B) is NOT Unitary.

Option C) $U^*U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{i}{2}-\frac{i}{2} \\ \frac{i}{2}+\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ option(C) is NOT Unitary

Option D) $U^*U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{+i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{i}{2}+\frac{i}{2} \\ -\frac{i}{2}-\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ Option (D) is NOT Unitary

Option E) $U^*U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{i}{2}-\frac{i}{2} \\ -\frac{i}{2}+\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (I)$

$\therefore$ Option (E) is Unitary.

Correct options : option (E)

Q13.) 1) Sum of 2 Symmetric matrices is Symmetric

i.e., If A is symmetric ($A^T = A$) and B is symmetric ($B^T = B$) matrix, then $(A+B)^T = A+B$

$$(A+B)^T = A^T + B^T = A + B.$$

2) Multiplication of a scalar with a symmetric matrix is symmetric.

3) If A and B are symmetric matrices then $AB + BA$ is a symmetric matrix.

4) Any power A^n of A is symmetric.

5) Let U = symmetric matrix i.e., $U^T = U$
and V = Symmetric matrix i.e., $V^T = V$

~~Prove~~ Check if $(UT)^T = UV$

then $(UV)^T = V^T U^T = VU \neq UV$

$\therefore$ UV is NOT symmetric.

Product would be symmetric only IF $UV = VU$
(if both matrices share eigenvectors)

6) If A is an invertible symmetric matrix, then A^{-1} is also symmetric

Using 1 & 5 ; Statement 1 is FALSE
Statement 2 is TRUE.

$\therefore$ Option (C) is correct.