Outline

- Sets and Functions
  - Notations
  - Logic
  - Graphs and visualisations.
- Univariate Calculus
  - Continuity and differentiability
  - Derivatives and Linear approximations
  - Applications/Advanced rules
- **Multivariate Calculus**
  - **Lines and planes in high dimensional space.**
  - Partial derivatives
  - Gradients
  - Linear approximations and Alternate gradient interpretations
  - Applications/Advanced rules

$f : \mathbb{R} \to \mathbb{R}$

$f : \mathbb{R}^d \to \mathbb{R}$

Geometry of Lines

(i) A line in $\mathbb{R}^d \subseteq \mathbb{R}^d$

(ii) (a) A line through the point $u \in \mathbb{R}^d$ along the vector $v \in \mathbb{R}^d$

$$= \{ x \in \mathbb{R}^d : x = u + \alpha v \text{ for } \alpha \in \mathbb{R} \}$$

(b) Line through $u, u' \in \mathbb{R}^d$

$$= \{ x \in \mathbb{R}^d : x = u + \alpha (u' - u) \text{ for } \alpha \in \mathbb{R} \}$$

$$= \{ x \in \mathbb{R}^d : x = (1-\alpha) u + \alpha u' \text{ for } \alpha \in \mathbb{R} \}$$

$\Downarrow$

Line through u along $u' - u$

Line through u' along $u - u'$

Geometry of (Hyper)planes

A $(d-1)$ dimensional hyperplane $\subseteq \mathbb{R}^d$

A hyperplane normal to the vector $w \in \mathbb{R}^d$ with value $b \in \mathbb{R}$

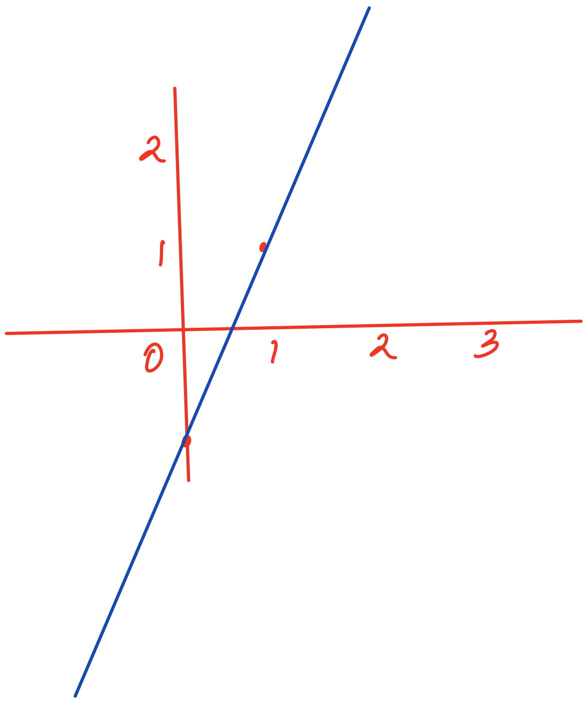
$$= \{ x \in \mathbb{R}^d : w^T x = b \}$$

$$= \{ x \in \mathbb{R}^d : \sum_{i=1}^{d} w_i x_i = b \}$$

Example Lines

Line through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ along $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$

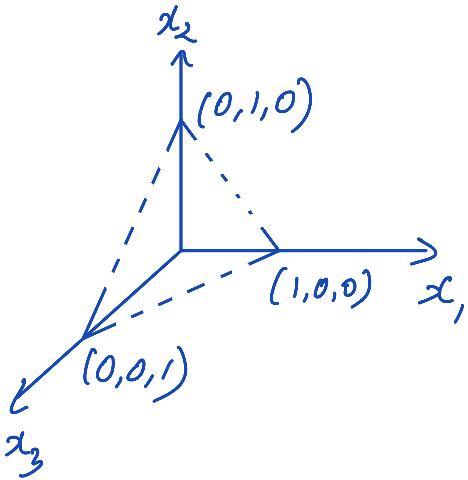


Example Planes

$d = 3$

Hyperplane normal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with value 1

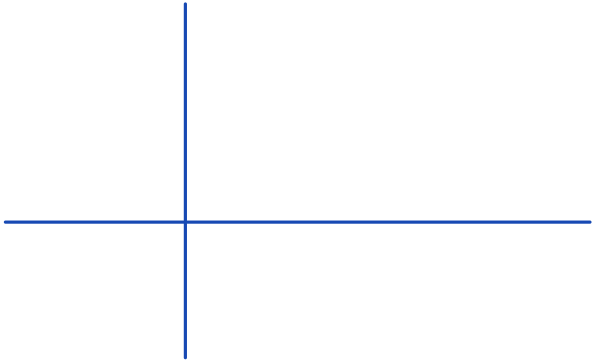
$T = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1 \}$



$(0,1,0)$ lies on T which is perpendicular to the $(1,1,1)$

Tuples vs Points vs Vectors

$\mathbb{R}^d$



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Partial Derivatives

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x_1, x_2) = x_1^2 + x_2^2$$

$$\frac{\partial f}{\partial x_1}(v) = \lim_{\alpha \to 0} \frac{f\left(v + \begin{bmatrix} \alpha \\ 0 \end{bmatrix}\right) - f(v)}{\alpha}$$

$$= \lim_{\alpha \to 0} \frac{f(v_1 + \alpha, v_2) - f(v_1, v_2)}{\alpha}$$

$$\frac{\partial f}{\partial x_2}(v) = \lim_{\alpha \to 0} \frac{f(v_1, v_2 + \alpha) - f(v_1, v_2)}{\alpha}$$

$$\frac{\partial f}{\partial x_i}(v) = \lim_{\alpha \to 0} \frac{f(v + \alpha e_i) - f(v)}{\alpha} \qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Gradients

$$f : \mathbb{R}^d \to \mathbb{R}$$

$$\frac{\partial f}{\partial x}(v) = \left[\frac{\partial f}{\partial x_1}(v), \frac{\partial f}{\partial x_2}(v) \cdots \frac{\partial f}{\partial x_d}(v) \right]$$

$$\nabla f(v) = \left[\frac{\partial f}{\partial x} \right]^T$$

Gradients

e.g 1 $\quad d=2 \qquad f(x) = x_1^2 + x_2^2$;

$$\frac{\partial f}{\partial x_1}(v) = 2v_1 \quad ; \quad \frac{\partial f}{\partial x_2}(v) = 2v_2$$

$$\nabla f(v) = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$$

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e.g 1

$f : \mathbb{R}^3 \to \mathbb{R} \qquad f(x) = x_1 + 2x_2 + 3x_3$

$$\nabla f(v) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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