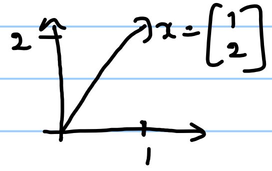
Least squares

Background on orthogonality

We will look at

1. Orthogonal vectors, length of a vector
2. Projections
3. Least-squares regression

Length of a vector



$$\|x\|^2 = x_1^2 + x_2^2$$

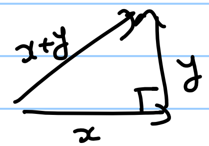
$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|^2 = 1^2 + 2^2 = 5$$

In general, for $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, $\|x\|^2 = x_1^2 + x_2^2 --- + x_n^2$

Orthogonal vectors: $x \perp y$ if $\underbrace{x^T y}_{\text{Inner product}} = 0$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \text{Then} \quad x^T y = \sum_{i=1}^{n} x_i y_i$$

From Pythagoras theorem:



$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$
$$x^T x + y^T y = (x+y)^T (x+y)$$
$$\cancel{x^T x} + \cancel{y^T y} = \cancel{x^T x} + \cancel{y^T y} + x^T y + y^T x$$
$$2\, x^T y = 0 \Rightarrow \boxed{x^T y = 0} \rightarrow x \text{ is orthogonal to } y.$$

Remark: (i) 0 is orthogonal to every x

(2) If $\{v_1 \dots v_k\}$ are mutually orthogonal "non-trivial" set of vectors $\{v_1 \dots v_k\}$ is a linearly independent set

Why? Suppose $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$

$$\Rightarrow v_1^T (c_1 v_1 + \dots + c_k v_k) = 0$$

$$\Rightarrow c_1 v_1^T v_1 = 0 \text{ , but } \|v_1\| \neq 0 \Rightarrow c_1 = 0$$

Similarly $c_i \neq 0 \ \forall i$

Examples:

(1) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(2) $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

Orthonormal vectors: $\{u, v\}$ orthonormal if $\underbrace{u^T v = 0}_{\text{orthogonal}}$ and $\underbrace{\|u\| = \|v\| = 1}_{\text{unit length}}$

Example: $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, $\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\}$

Orthogonal subspaces:

U, V are orthogonal subspaces if

$x^T y = 0 \quad \forall x \in U,\ y \in V$

Examples: ① $\{0\} \perp$ subspace

② $U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\quad V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

$U \perp V$

$$W = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$$

Then $W \perp U$, $W \perp V$

Orthogonality wrt four fundamental subspaces of a matrix A:

Claim: (1) $R(A) \perp N(A)$

Proof: $x \in N(A)$

$Ax = 0$ (A is $m \times n$)

$$\begin{bmatrix} \text{— row 1 —} \\ \text{— row 2 —} \\ \vdots \\ \text{— row } m \text{ —} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The equation above implies $\text{row } 1 \perp x$

$$\text{row } 2 \perp x$$

$$\vdots$$

$$\text{row } m \perp x$$

Thus, any linear combination $\underbrace{(c_1 \text{ row } 1 + c_2 \text{ row } 2 + \cdots + c_m \text{ row } m)}_{\in R(A)} \perp x$

So, $R(A) \perp N(A)$

$$\iff C(A^T) \perp N(A)$$

Claim 2: $C(A) \perp N(A^T)$

"follows from claim 1".

Example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

rank $(A) = \dim(C(A)) = 1$

$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in N(A)$ since "col 2 = 2 × col 1"

Row-space $R(A)$ = line through $[1 \ \ 2]$

Null space $N(A)$ = line through $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$[1 \ \ 2]\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0$ verifies $R(A) \perp N(A)$

$C(A) =$ line through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$N(A^T) = \quad y_1 + 2y_2 + 3y_3 = 0$ $\leftarrow$ is a plane

$y^T A = 0$

Connect this to the claim $C(A) \perp N(A^T)$

A dimension check:

$\dim C(A) + \dim(N(A)) = 2$

$1 + 1$ number of cols

$\dim C(A^T) + \dim(N(A^T)) = 3$

$1 + 2 =$ number of cols