Course: Machine Learning - Foundations
Test Questions - Solution
Lecture Details: Week 6

1. (1 point) The function $f(x, y) = x^2 + y^2$

   A. has no stationary point.

   B. has a stationary point at (0, 0).

   C. has a stationary point at (1, 1).

   Answer: B

   $f(x, y) = x^2 + y^2$,

   $f_x = \frac{\partial f}{\partial x} = 2x, f_y = \frac{\partial f}{\partial y} = 2y$

   Since, f_x, f_y are 0 at (0, 0). The origin is an stationary point for the function.

2. (1 point) If two matrices A and B are positive definite then $A + B$ is also positive definite.

   A. True

   B. False

   Answer: A

3. (1 point) The matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ is

   A. positive definite.

   B. positive semi-definite.

   C. negative semi-definite.

   Answer: A

   The eigenvalues are $4, 1, 1$. Since, all eigenvalues are positive for the matrix, it is positive definite.

4. (1 point) The function $f(x, y) = 2x^2 + 2xy + 2y^2 - 6x$ has a stationary point at

   A. (2, 1)

   B. (1, 2)

   C. (-1, 2)

   D. (2, -1)

Answer: D

$f(x, y) == 2x^2 + 2xy + 2y^2 - 6x,$

$f_x = \frac{\partial f}{\partial x} = 4x + 2y - 6, f_y = \frac{\partial f}{\partial y} = 2x + 4y$

Since, f_x, f_y are 0 at (2, -1). The origin is an stationary point for the function.

5. (1 point) The correct representation of $x^2 + y^2 - z^2 - xy + yz + xz$ in the matrix form is

   A. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   B. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   C. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   D. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Answer: A

Perform matrix multiplication and check which option is giving function described in the question.

6. (1 point) Given $f(x, y) = 3x^2 + 4xy + 2y^2$, the point $(0, 0)$ is a __________.

   A. maxima.

   B. minima.

   C. saddle Point.

   D. None of these

Answer: B

$f(x, y) = 3x^2 + 4xy + 2y^2,$

$f_x = \frac{\partial f}{\partial x} = 6x + 4y, f_y = \frac{\partial f}{\partial y} = 4x + 4y$

Since, f_x, f_y are 0 at (0, 0). The origin is an stationary point for the function.

$f_{xx} = \frac{\partial f^2}{\partial x^2} = 6, f_{xy} = \frac{\partial f^2}{\partial x \partial y} = 4, f_{yy} = \frac{\partial f}{\partial y^2} = 4$

Since $f_{xx} < 0, D = f_{xx}f_{yy} - f_{xy}^2 = 6 * 4 - (4)^2 = 8 > 0$, the point (0, 0) is a minima.

13. (1 point) The non-zero singular values of a matrix $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ *are*

A. $3+\sqrt{3},\ 3-\sqrt{3}$

B. $\sqrt{3+\sqrt{3}},\ \sqrt{3-\sqrt{3}}$

C. $2+\sqrt{2},\ 2-\sqrt{2}$

D. $\sqrt{2+\sqrt{2}},\ \sqrt{2-\sqrt{2}}$

Answer: B

$AA^T = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

Eigenvalues of AA^T,

$\lambda = 3+\sqrt{3}, 3-\sqrt{3}$

$\sigma = \sqrt{3+\sqrt{3}}, \sqrt{3-\sqrt{3}}, 0$

14. (1 point) The correct SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is

A. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$

B. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

C. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

D. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Answer: C

$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvalues of AA^T,

$\lambda = 2, 2$

Eigenvalues of A^TA,

$\lambda = 2, 2, 0, 0$

$\sigma = \sqrt{2}, \sqrt{2}$
Eigenvectors of A^TA,
$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
Normalizing eigen vectors,
$x_1 = v_1$
$x_2 = v_2$
$Q_1 = [x_1 \quad x_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$q_1 = \frac{1}{\sigma_1} A^T x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
$q_2 = \frac{1}{\sigma_2} A^T x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$
We can not find out q_3 and q_4 by using above formula.
q_3 and q_4 should be orthonormal basis for R^4.
$q_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$
$q_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$$Q_2 = [q_1 \quad q_2 \quad q_3 \quad q_4] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

15. (1 point) The correct SVD of the matrix $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$ is

A. $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{7}} & \frac{2}{\sqrt{7}} \\ -\frac{2}{3} & \frac{1}{\sqrt{7}} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{\sqrt{7}} \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

B. $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

C. $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{21}} & \frac{1}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} & \frac{3}{\sqrt{21}} \end{bmatrix}$

D. $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ -\frac{2}{3} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{5}} \end{bmatrix}$

Answer: B

$A^T A = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

Eigen values of $A^T A$,

$\lambda = 0, 90$

Eigen vectors of $A^T A$

$v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Normalizing,

$x_1 = \frac{1}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $x_2 = \frac{1}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$u_1 = \frac{1}{\sigma} A x_1 = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

$AA^T = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -40 & 40 & 40 \end{bmatrix}$

Null space of $AA^T = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

16. (1 point) The singular value decomposition of matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is

A. $\begin{bmatrix} 0.525 & 0.85 \\ 0.2 & 2.225 \end{bmatrix} \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix} \begin{bmatrix} 0.85 & 0.525 \\ -0.525 & 0.85 \end{bmatrix}^T$

B. $\begin{bmatrix} -0.525 & 0.85 \\ 0.2 & 2.225 \end{bmatrix} \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix} \begin{bmatrix} 0.85 & 0.525 \\ -0.525 & 0.85 \end{bmatrix}^T$

C. $\begin{bmatrix} 0.525 & 0.85 \\ 0.2 & 2.225 \end{bmatrix} \begin{bmatrix} -1.618 & 0 \\ 0 & 0.618 \end{bmatrix} \begin{bmatrix} 0.85 & 0.525 \\ -0.525 & 0.85 \end{bmatrix}^T$

D. $\begin{bmatrix} 0.525 & 0.85 \\ 0.2 & 2.225 \end{bmatrix} \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix} \begin{bmatrix} 0.85 & -0.525 \\ -0.525 & 0.85 \end{bmatrix}^T$

Answer: A
$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
Eigen values of $A^T A$,
$\lambda = 2.618, 0.382$ Eigen vectors of $A^T A$
$v_1 = \begin{bmatrix} 1.618 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -0.618 \\ 1 \end{bmatrix}$
Normalizing,
$x_1 = \begin{bmatrix} 0.85 \\ 0.525 \end{bmatrix}$ $x_2 = \begin{bmatrix} -0.525 \\ 0.85 \end{bmatrix}$
$Q_2 = [x_1 \quad x_2]$
$y_1 = \frac{1}{\sigma_1} A x_1 = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix}$
$y_2 = \frac{1}{\sigma_2} A x_2 = \begin{bmatrix} -1 \\ 0.618 \end{bmatrix}$
$Q_1 = [y_1 \quad y_2]$
$\sigma = \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix}$

17. (1 point) Find the singular values for matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

A. 1 and 9

B. 1 and 3

C. 2 and 3

D. 4 and 9

Answer: B
$A^T A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$
$|A^T A - \lambda I| = 0$

$(9-\lambda)(1-\lambda)=0$
$\lambda = 1, 9$
$\sigma = \sqrt{\lambda}$
$\sigma = 1, 3$