Intractability: P and NP

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 11

Checking algorithms

- Checking algorithm C for problem
- Takes in an input instance I and a solution "certificate" S for I
- C outputs yes if S represents a valid solution for I, no otherwise

The class NP

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- If we can solve one efficiently, we can solve them all!

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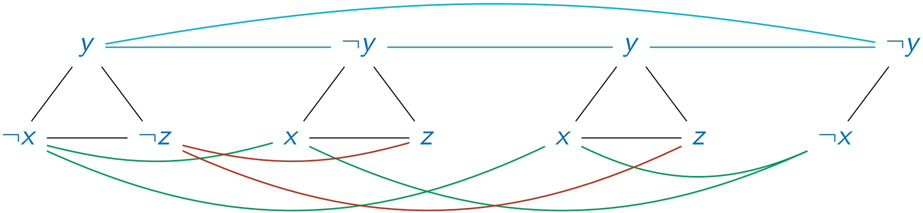
  $(v \lor \neg w \lor a) \land (\neg a \lor x \lor b) \land (\neg b \lor \neg y \lor z)$
- If SAT is hard, so is 3-SAT

3-SAT to independent set

- Construct a graph from a 3-SAT formula

$$(\neg x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (x \vee y \vee z) \wedge (\neg x \vee \neg y)$$



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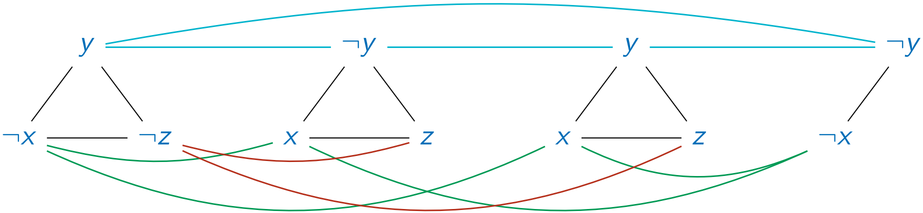

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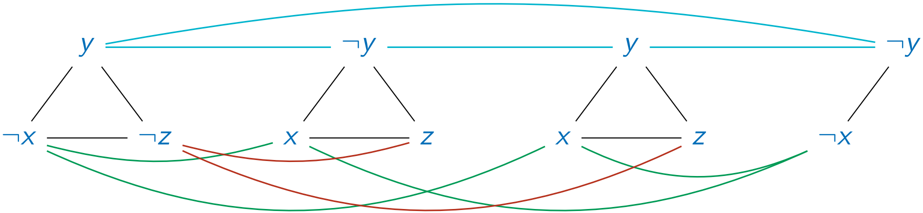

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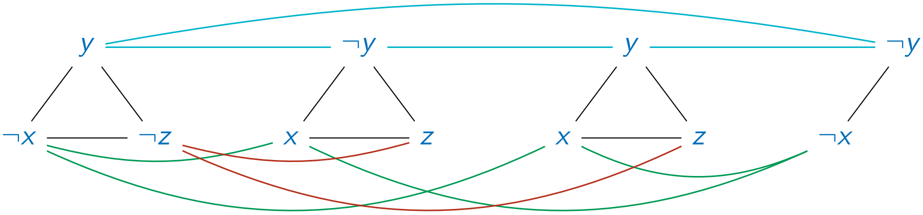

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- Edges enforce consistency across clauses
- Ask for size of independent set = number of clauses

Reductions within NP

- SAT $\rightarrow$ 3-SAT, 3-SAT $\rightarrow$ independent set, independent set $\leftrightarrow$ vertex cover
- Reduction is transitive, so SAT $\rightarrow$ vertex cover, ...
- Other inter-reducible NP problems
  - Travelling salesman, integer linear programming ... All these problems are "equally" hard

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- In general, to show P is NP-complete, reduce some existing NP-complete problem to P

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- Empirical evidence that NP is different from P
- But a formal proof is elusive, and worth $1 million!