Matrix Multiplication

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Programming, Data Structures and Algorithms using Python

Week 9

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$

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- Find an optimal order to compute the product
  - Multiply two matrices at a time
  - Bracket the expression optimally

Inductive structure

- Final step combines two subproducts
  $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$
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- $C(j, k) =$ $\min_{j < \ell \leq k} [C(j, \ell{-}1) + C(\ell, k) + r_j r_\ell c_k]$
- Base case: $C(j, j) = 0$ for $0 \leq j < n$

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$

	0	$\cdots$	i	$\cdots$	$\cdots$	j	$\cdots$	$n{-}1$
0								
$\cdots$								
i								
$\cdots$								
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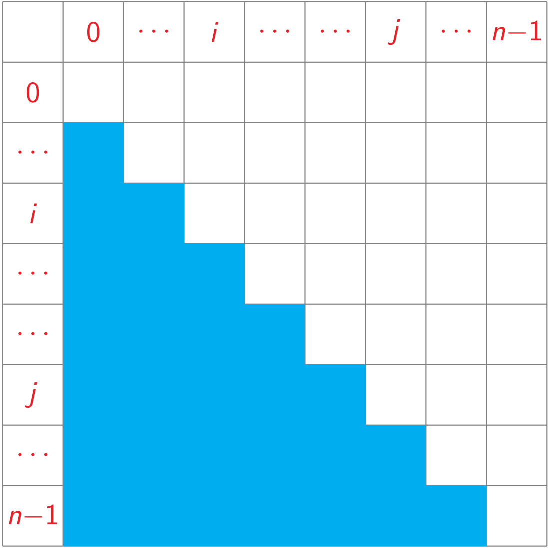
  - Entries above main diagonal



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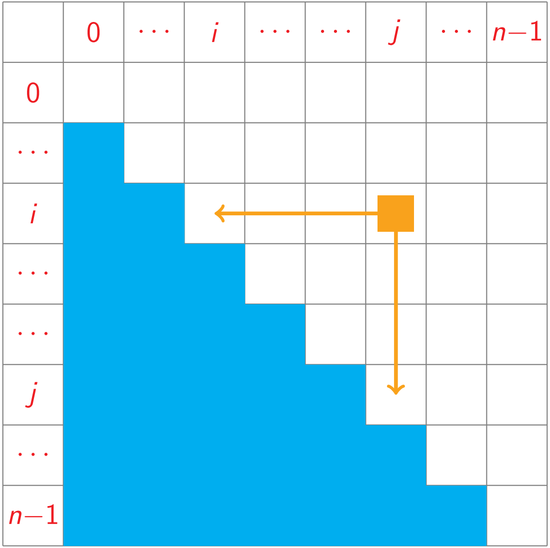

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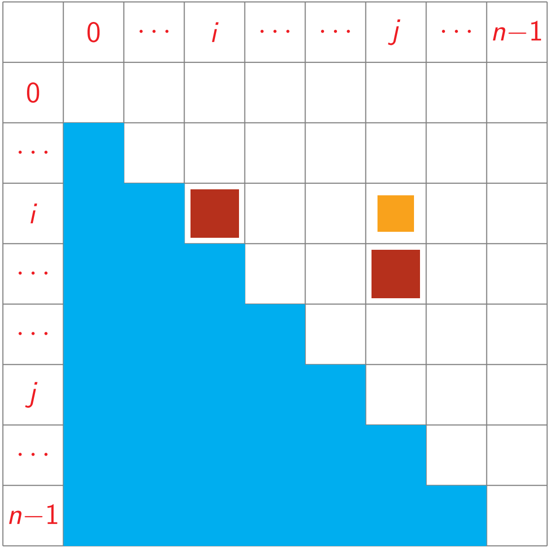

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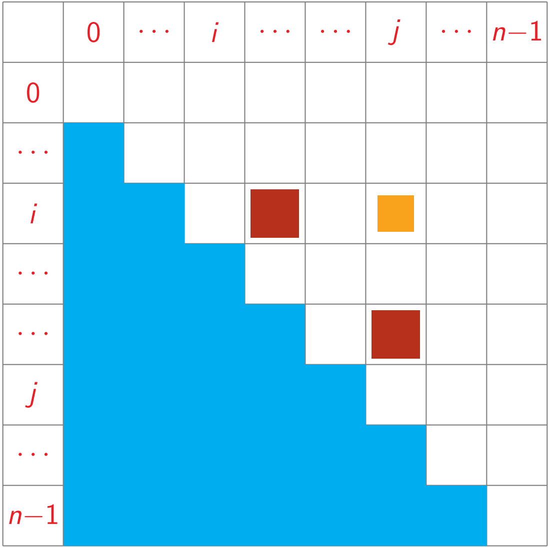

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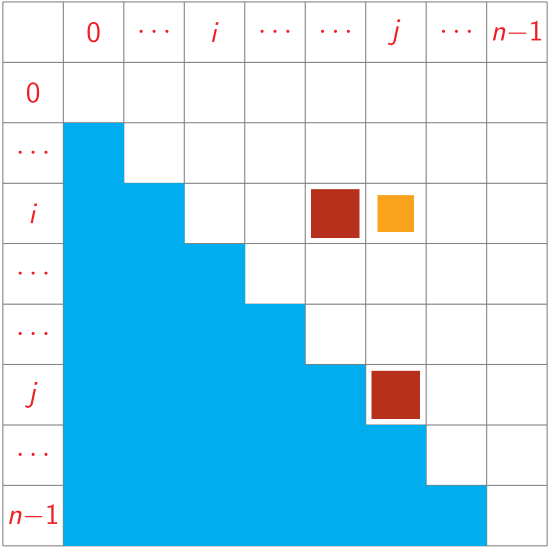

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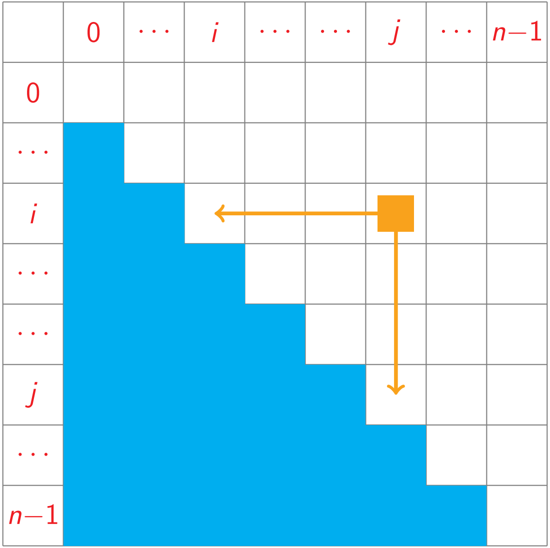
  - $O(n)$ dependencies per entry, unlike LCW, LCS and ED



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  - $O(n)$ dependencies per entry, unlike LCW, LCS and ED
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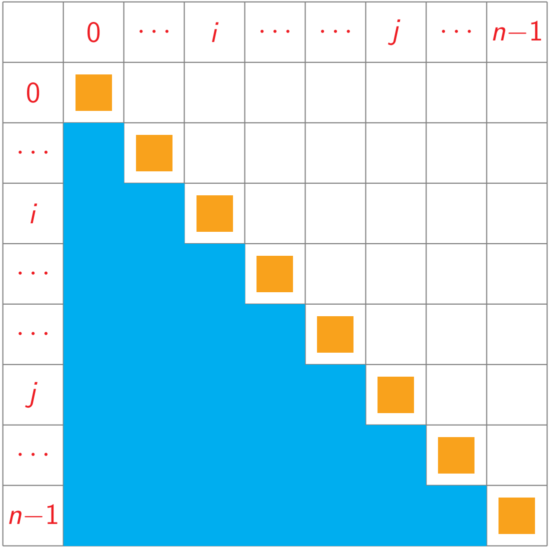

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
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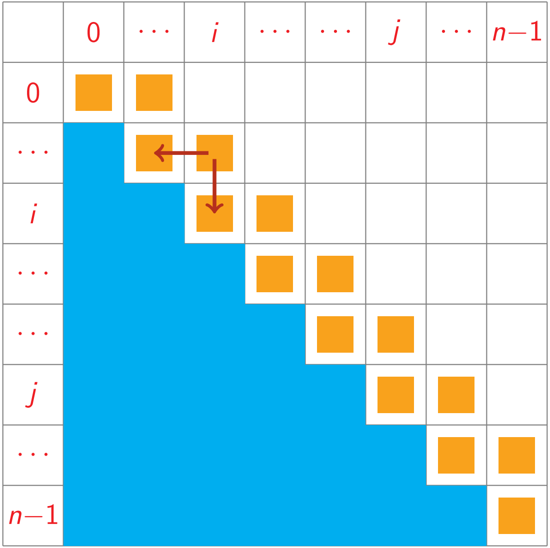

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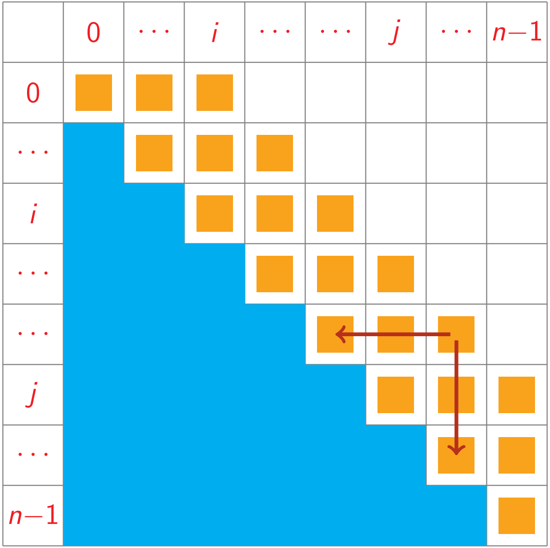

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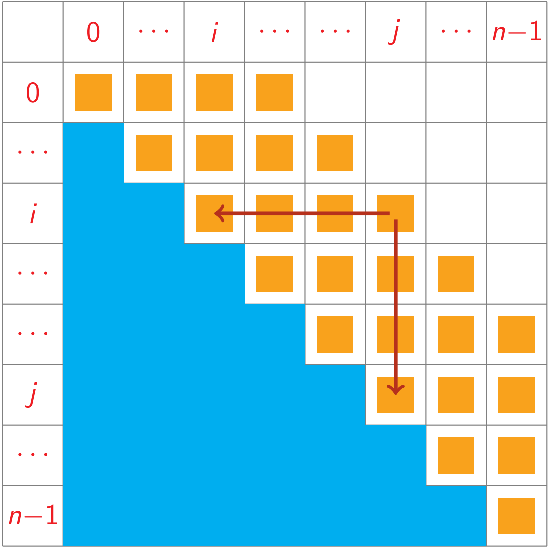

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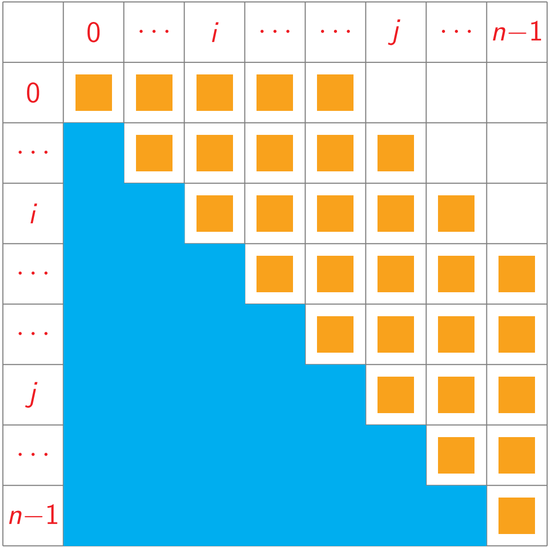

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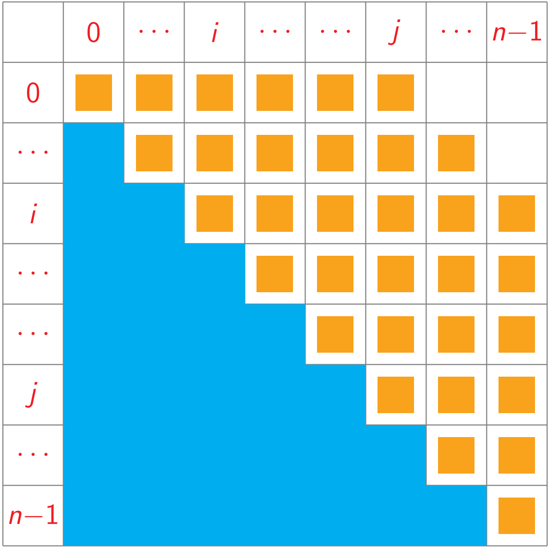

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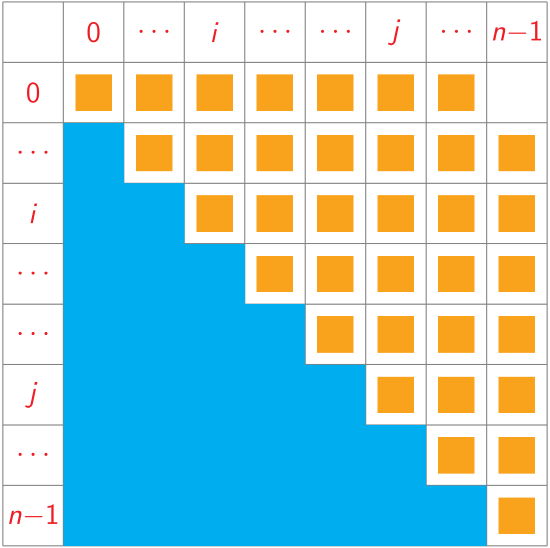

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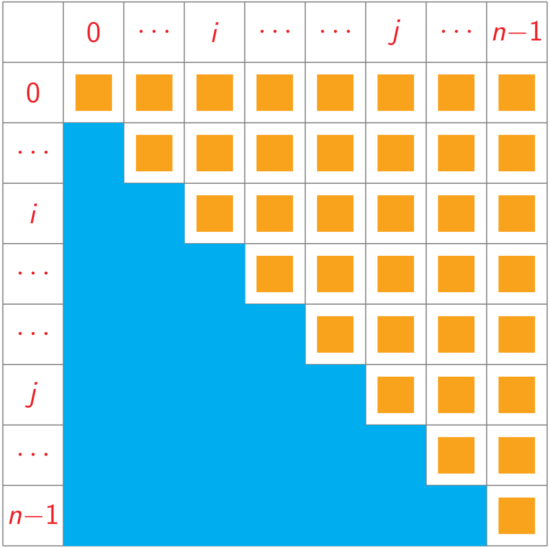

Implementation

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def C(dim):
  # dim: dimension matrix,
  #      entries are pairs (r_i,c_i)
  import numpy as np
  n = dim.shape[0]
  C = np.zeros((n,n))
  for i in range(n):
    C[i,i] = 0
  for diff in range(1,n):
    for i in range(0,n-diff):
      j = i + diff
      C[i,j] = C[i,i] +
               C[i+1,j] +
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      for k in range(i+1,j+1):
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- We have to fill a table of size $O(n^2)$

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Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$
- Overall, $O(n^3)$