1. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Which of the following vectors is not an eigenvector of this matrix ?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Solution: Option D is the correct answer. For each of the vectors given in the options you can compute the product Ax. For example, consider $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4x$$

Hence, $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. Similarly, you can show that $x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are also eigenvectors of this matrix with the corresponding eigen values being 1 and -1 respectively. You can then also check that $x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is not an eigenvector of this matrix.

2. Consider a square matrix $A \in \mathbb{R}^{3\times 3}$ such that $A^T = A$. My friend told me that the following three vectors are the eigenvectors of this matrix A:

$x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Is my friend telling the truth ?

   A. Yes

   B. No

   C. Can't say without knowing all the elements of A

   D. Yes, only if all the diagonal elements of A are 1

Solution: Note that A is a square symmetric matrix ($\because A \in \mathbb{R}^{3\times 3}$ and $A^T = A$). We know that the eigenvectors of a square symmetric matrix are orthogonal. In other words, if x, y, z are the eigenvectors of A then $x^T y = x^T z = y^T z = 0$. You can easily verify that this is not the case. Hence, my friend is not telling the truth. **Option B** is the correct answer.

A. A is positive definite.

B. A is positive semi-definite.

C. A is neither positive definite nor positive semi-definite.

D. Can not be determined

Answer: A

This is a diagonal matrix. For a diagonal matrix, the eigenvalues are elements on its principal diagonal. Since all the eigenvalues (diagonal elements) are positive, the matrix is a positive definite matrix.

Questions 10-15 are based on common data

Consider these data points to answer the following questions:

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

10. (1 point) The mean vector of the data points x_1, x_2, x_3 is

A. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 0.9 \\ 0.6 \\ 0.3 \end{bmatrix}$

D. $\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

Answer: B

$$\text{Mean vector} = \bar{X} = \Sigma_{i=1}^{n} \frac{1}{n} x_i = \frac{1}{3} \begin{bmatrix} (0+1+2) \\ (1+1+1) \\ (2+1+0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

11. (2 points) The covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ of the data points x_1, x_2, x_3 is

A. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$

C. $\begin{bmatrix} 0.7 & 0 & -0.7 \\ 0 & 0 & 0 \\ -0.7 & 0 & 0.7 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: C

$$C = \frac{1}{3}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & -0.7 \\ 0 & 0 & 0 \\ -0.7 & 0 & 0.7 \end{bmatrix}$$

12. (2 points) The eigenvalues of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are

A. 2, 0, 0

B. 1, 1, 1

C. 1.4, 0, 0

D. 0.5, 0, 0.5

Answer: C

Characteristics equation:

$$\begin{bmatrix} \frac{7}{10} - \lambda & 0 & -\frac{7}{10} \\ 0 & -\lambda & 0 \\ -\frac{7}{10} & 0 & \frac{7}{10} - \lambda \end{bmatrix} I = 0$$

The determinant of the obtained matrix is $\lambda^2(\frac{7}{5} - \lambda) = 0$

Eigenvalues:

The roots are $\lambda_1 = \frac{7}{5} = 1.4, \lambda_2 = 0, \lambda_3 = 0$

Eigenvectors:

$$\lambda_1 = 1.4, \begin{bmatrix} \frac{7}{10} - \lambda & 0 & -\frac{7}{10} \\ 0 & -\lambda & 0 \\ -\frac{7}{10} & 0 & \frac{7}{10} - \lambda \end{bmatrix} = \begin{bmatrix} -\frac{7}{10} & 0 & -\frac{7}{10} \\ 0 & -\frac{7}{5} & 0 \\ -\frac{7}{10} & 0 & -\frac{7}{10} \end{bmatrix}$$

The null space of this matrix is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, Corresponding eigenvector is, $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7 \\ 0 \\ 0.7 \end{bmatrix}$

$\lambda_2 = \lambda_3 = 0$, $\begin{bmatrix} \frac{7}{10} - \lambda & 0 & -\frac{7}{10} \\ 0 & -\lambda & 0 \\ -\frac{7}{10} & 0 & \frac{7}{10} - \lambda \end{bmatrix} = \begin{bmatrix} \frac{7}{10} & 0 & -\frac{7}{10} \\ 0 & 0 & 0 \\ -\frac{7}{10} & 0 & \frac{7}{10} \end{bmatrix}$

The null space of this matrix are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ Corresponding eigenvector are, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0 \\ 0.7 \end{bmatrix}$

13. (2 points) The eigenvectors of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are (Note: The eigenvectors should be arranged in the descending order of eigenvalues from left to right in the matrix.)

    A. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

    B. $\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$

    C. $\begin{bmatrix} -0.7 & 0 & 0.7 \\ 0 & 1 & 0 \\ 0.7 & 0 & 0.7 \end{bmatrix}$

    D. $\begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

    Answer: C

    Refer to the solution of the previous question.

14. (2 points) The data points x_1, x_2, x_3 are projected onto the one dimensional space using PCA as points z_1, z_2, z_3 respectively. (Use eigenvector with the maximum eigenvalue for this projection.)

    A. $z_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $z_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $z_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

B. $z_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $z_3 = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$

C. $z_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $z_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $z_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

D. $z_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $z_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Answer: D

$\lambda_1 = 1.4$, $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$z_1 = \frac{1}{\sqrt{2}}(\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix})\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$z_2 = \frac{1}{\sqrt{2}}(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix})\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z_3 = \frac{1}{\sqrt{2}}(\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix})\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

15. (1 point) The approximation error J on the given data set is given by $\Sigma_{i=1}^{n} ||x_i - z_i||^2$. What is the reconstruction error?

A. 3

B. 5

C. 10

D. 20

Answer: A

Approximation Error, $J = \frac{1}{n}\Sigma_{i=1}^{n} ||x_i - z_i||^2 = \frac{1}{3}[(1^2 + 1^2 + 1^2) + (1^2 + 1^2 + 1^2) + (1^2 + 1^2 + 1^2)] = 3$