Lecture-7
Algorithmic Mathematics(CSC545)

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CENTRAL DIFFERENCE FORMULA

Consider a function **f(x)** tabulated for equally spaced points $\mathbf{x_0, x_1, x_2, \ldots, x_n}$ with step length **h**. In many problems one may be interested to know the behaviour of **f(x)** in the neighbourhood of $\mathbf{x_r\ (x_0 + rh)}$. If we take the transformation $\mathbf{X = (x - (x_0 + rh)) / h}$, the data points for **X** and **f(X)** can be written as now the central difference table can be generated using the definition of central differences:

$$\mathbf{df(X) = f(X + h/2) - f(X - h/2)}$$

$$\mathbf{df_i = (E^{1/2} - E^{-1/2})f_i = (f_{i +1/2} - f_{i -1/2})}$$

$$\mathbf{d^2f_i = (E^{1/2} - E^{-1/2}) (f_{i +1/2} - f_{i -1/2})}$$

$$\mathbf{= f_1 - f_0 - f_0 + f_{-1} = f_1 - 2f_0 + f_{-1}}$$

Now the central difference table is:

X_i	f_i	df_i	d^2f_i	d^3f_i	d^4f_i
-2	f_{-2}				
		$df_{-3/2}$ ($= f_{-1} - f_{-2}$)			
-1	f_{-1}		d^2f_{-1} ($= df_{-1/2} - df_{-3/2}$)		
		$df_{-1/2}$ ($= f_0 - f_{-1}$)		$d^3f_{-1/2}$ ($= d^2f_0 - d^2f_{-1}$)	
0	f_0		d^2f_0 ($= df_{1/2} - df_{-1/2}$)		d^4f_0 ($= d^3f_{1/2} - d^3f_{-1/2}$)
		$df_{1/2}$ ($= f_1 - f_0$)		$d^3f_{1/2}$ ($= d^2f_1 - d^2f_0$)	
1	f_1		d^2f_1 ($= df_{3/2} - df_{1/2}$)		
		$df_{3/2}$ ($= f_2 - f_1$)			
2	f_2				

Types of Central Difference Interpolation Formula

1. Gauss' Interpolation(Forward / Backward).
2. Stirling's Interpolation
3. Bessel's Interpolation
4. Everett's Interpolation

Bessel' Interpolation Formula

This is a very useful formula for practical interpolation, and it uses the differences as shown in the following table, where the brackets mean that the average of the values has to be taken.

$$
\begin{array}{cccccccc}
\vdots & \vdots & & & & & & \\
x_{-1} & y_{-1} & & & & & & \\
\begin{matrix} x_0 \\ x_1 \end{matrix} & \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} & \Delta y_0 & \begin{pmatrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{pmatrix} & \Delta^3 y_{-1} & \begin{pmatrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{pmatrix} & \Delta^5 y_{-2} & \begin{pmatrix} \Delta^6 y_{-3} \\ \Delta^6 y_{-2} \end{pmatrix} \\
\vdots & \vdots & & & & & &
\end{array}
$$

Hence, Bessel's formula can be assumed in the form

$$y_p = \frac{y_0 + y_1}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1}$$

$$+ B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots$$

$$= y_0 + \left(B_1 + \frac{1}{2} \right) \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1}$$

$$+ B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \qquad (3.22)$$

Using the method outlined in Section 3.7.1, i.e. Gauss' forward formula, we obtain

$$\left.\begin{aligned} B_1 + \frac{1}{2} &= p, \\ B_2 &= \frac{p(p-1)}{2!}, \\ B_3 &= \frac{p(p-1)(p-1/2)}{3!}, \\ B_4 &= \frac{(p+1)\,p(p-1)(p-1)}{4!}, \\ &\vdots \end{aligned}\right\} \qquad (3.23)$$

Hence, Bessel's interpolation formula may be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-1/2)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\,\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \qquad (3.24)$$

3.7.4 Everett's Formula

This is an extensively used interpolation formula and uses only even order differences, as shown in the following table:

x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$	
		–		–		–		–
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$	

Hence the formula has the form

$$y_p = E_0 y_0 + E_2 \Delta^2 y_{-1} + E_4 \Delta^4 y_{-2} + \cdots + F_0 y_1 + F_2 \Delta^2 y_0 + F_4 \Delta^4 y_{-1} + \cdots \quad (3.25)$$

The coefficients $E_0, F_0, E_2, F_2, E_4, F_4, \ldots$ can be determined by the same method as in the preceding cases, and we obtain

$$\left.\begin{aligned} E_0 &= 1 - p = q, & F_0 &= p, \\ E_2 &= \frac{q(q^2 - 1^2)}{3!}, & F_2 &= \frac{p(p^2 - 1^2)}{3!}, \\ E_4 &= \frac{q(q^2 - 1^2)(q^2 - 2^2)}{5!}, & F_4 &= \frac{p(p^2 - 1^2)(p^2 - 2^2)}{5!}, \\ &\vdots & &\vdots \end{aligned}\right\} \quad (3.26)$$

Hence Everett's formula is given by

$$\left.\begin{aligned} y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots \\ + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots \end{aligned}\right\} \quad (3.27)$$

where $q = 1 - p$.

3.7.5 Relation between Bessel's and Everett's Formulae

These formulae are very closely related, and it is possible to deduce one from the other by a suitable rearrangement. To see this we start with Bessel's formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-1/2)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots$$

$$= y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-1/2)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$

$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots$$

expressing the odd order differences in terms of low even order differences. This gives on simplification

$$y_p = (1-p)y_0 + \left[\frac{p(p-1)}{4} - \frac{(p-1)\,p(p-1/2)}{6}\right]\Delta^2 y_{-1} + \cdots$$

$$+ py_1 + \left[\frac{p(p-1)}{4} + \frac{p(p-1)(p-1/2)}{6}\right]\Delta^2 y_0 + \cdots$$

$$= qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \cdots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \cdots$$

which is *Everett's formula* truncated after second differences. Hence we have a result of practical importance that Everett's formula truncated after second differences is equivalent to Bessel's formula truncated after third differences. In a similar way, Bessel's formula may be deduced from Everett's.

Bessel's Interpolation

Q Find $f(x)$ at $x = 7$ from following data.

x	2	4	6	8	10
$f(x)$	5	49	181	449	901

Solution:- Lets construct Bessel's Interpolation Table

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 2$	$y_{-2} = 5$	$\Delta y_{-2} = 44$	$\Delta^2 y_{-2} = 88$	$\Delta^3 y_{-2} = 48$	$\Delta^4 y_{-2} = 0$
$x_{-1} = 4$	$y_{-1} = 49$	$\Delta y_{-1} = 132$	$\Delta^2 y_{-1} = 136$	$\Delta^3 y_{-1} = 48$	
$x_0 = 6$	$y_0 = 181$	$\Delta y_0 = 268$	$\Delta^2 y_0 = 184$		
$x_1 = 8$	$y_1 = 449$	$\Delta y_1 = 452$			
$x_2 = 10$	$y_2 = 901$				

Bessel's Formula:-

$$f(x) = \left\{\frac{y_0 + y_1}{2} + v.\Delta y_0\right\} + \left\{\frac{v^2 - \frac{1}{4}}{2!} \times \frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right\} + \left\{\frac{v.(v^2 - \frac{1}{4})}{3!} \times \Delta^3 y_{-1}\right\}$$

$$+ \left\{\frac{(v^2 - \frac{1}{4})(v^2 - \frac{9}{4})}{4!} \times \frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right\} \ldots$$

where,

$$v = u - \frac{1}{2} = \frac{x - x_0}{h} - \frac{1}{2} = \frac{7-6}{2} - \frac{1}{2} = 0$$

x = Value of x, for which $f(x)$ needs to be found

x_0 = Value above/below x in x column

h = Difference between each value of x (interval gap)

Putting the values,

$$f(x) = \left\{ \frac{181 + 449}{2} + (0 \times 268) \right\} + \left\{ \frac{0^2 - \frac{1}{4}}{2!} \times \frac{184 + 136}{2} \right\} + \left\{ \frac{0(0^2 - \frac{1}{4})}{3!} \times 48 \right\}$$

$$= \{315 + 0\} + \left\{ -\frac{1}{8} \times 160 \right\} + \{0\}$$

$$= \{315\} + \{-20\} + \{0\}$$

$$= 315 - 20 + 0$$

$$= 295$$

$$\therefore y = f(7) = 295 \quad \underline{\text{Ans}}$$

Note:- Bessel's Interpolation Formula is similar to Newton's Forward & Backward Interpolation formulas, where the difference between each value of x is same, i.e interval gap or 'h' is fixed.

When interval gap is not same, we use Newton's Divided Difference, Lagrange Interpolation Formula.

Thank You

Any Query??