## Lecture-7 Algorithmic Mathematics(CSC545)

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#### **CENTRAL DIFFERENCE FORMULA**

Consider a function f(x) tabulated for equally spaced points  $x_0, x_1, x_2, \ldots, x_n$  with step length h. In many problems one may be interested to know the behaviour of f(x) in the neighbourhood of  $x_r(x_0 + rh)$ . If we take the transformation  $X = (x - (x_0 + rh)) / h$ , the data points for X and f(X) can be written as now the central difference table can be generated using the definition of central differences:

$$df(X) = f(X + h/2) - f(X - h/2)$$

$$df_{i} = (E^{1/2} - E^{-1/2})f_{i} = (f_{i+1/2} - f_{i-1/2})$$

$$d^{2}f_{i} = (E^{1/2} - E^{-1/2}) (f_{i+1/2} - f_{i-1/2})$$

$$= f_{1} - f_{0} - f_{0} + f_{-1} = f_{1} - 2f_{0} + f_{-1}$$

#### Now the central difference table is:

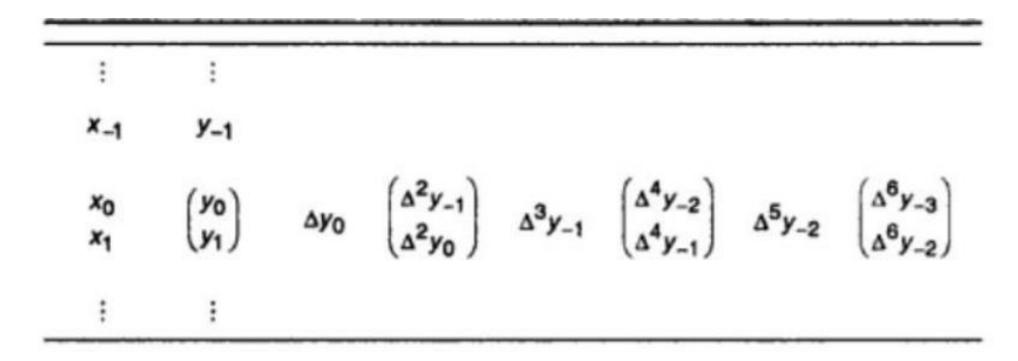
X <sub>i</sub>	f <sub>i</sub>	df <sub>i</sub>	d²f <sub>i</sub>	d³f <sub>i</sub>	d⁴f <sub>i</sub>
-2	f <sub>-2</sub>				
		df <sub>-3/2</sub> ( = f <sub>-1</sub> - f <sub>-2</sub> )			
-1	f <sub>-1</sub>		$d^{2}f_{-1}$ $( = df_{-1/2} - df_{-3/2})$		
		$df_{-1/2}  ( = f_0 - f_{-1} )$		$d^{3}f_{-1/2}$ $( = d^{2}f_{0} - d^{2}f_{-1})$	
0	$f_0$		$d^{2}f_{0}$ $( = df_{1/2} - df_{-1/2})$		$d^{4}f_{0}$ $( = d^{3}f_{1/2} - d^{3}f_{-1/2})$
		$df_{1/2}$ ( = $f_1$ - $f_0$ )		$d^{3}f_{1/2}$ ( = $d^{2}f_{1} - d^{2}f_{0}$ )	
1	$f_1$		$d^{2}f_{1}$ $( = df_{3/2} - df_{1/2})$		
		$df_{3/2}$ ( = $f_2$ - $f_1$ )			
2	f <sub>2</sub>				

### Types of Central Difference Interpolation Formula

- 1. Gauss' Interpolation(Forward / Backward).
- 2. Stirling's Interpolation
- 3. Bessel's Interpolation
- 4. Everett's Interpolation

### Bessel' Interpolation Formula

This is a very useful formula for practical interpolation, and it uses the differences as shown in the following table, where the brackets mean that the average of the values has to be taken.



#### Hence, Bessel's formula can be assumed in the form

$$y_{p} = \frac{y_{0} + y_{1}}{2} + B_{1}\Delta y_{0} + B_{2}\frac{\Delta^{2}y_{-1} + \Delta^{2}y_{0}}{2} + B_{3}\Delta^{3}y_{-1}$$

$$+ B_{4}\frac{\Delta^{4}y_{-2} + \Delta^{4}y_{-1}}{2} + \cdots$$

$$= y_{0} + \left(B_{1} + \frac{1}{2}\right)\Delta y_{0} + B_{2}\frac{\Delta^{2}y_{-1} + \Delta^{2}y_{0}}{2} + B_{3}\Delta^{3}y_{-1}$$

$$+ B_{4}\frac{\Delta^{4}y_{-2} + \Delta^{4}y_{-1}}{2} + \cdots$$
(3.22)

Using the method outlined in Section 3.7.1, i.e. Gauss' forward formula, we obtain

$$B_{1} + \frac{1}{2} = p,$$

$$B_{2} = \frac{p(p-1)}{2!},$$

$$B_{3} = \frac{p(p-1)(p-1/2)}{3!},$$

$$B_{4} = \frac{(p+1) p(p-1)(p-1)}{4!},$$

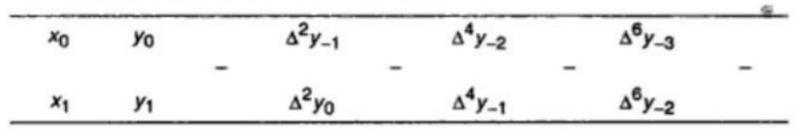
$$\vdots$$
(3.23)

Hence, Bessel's interpolation formula may be written as

$$y_{p} = y_{0} + p\Delta y_{0} + \frac{p(p-1)}{2!} \frac{\Delta^{2} y_{-1} + \Delta^{2} y_{0}}{2} + \frac{p(p-1)(p-1/2)}{3!} \Delta^{3} y_{-1} + \frac{(p+1) p(p-1)(p-2)}{4!} \frac{\Delta^{4} y_{-2} + \Delta^{4} y_{-1}}{2} + \dots$$
(3.24)

#### 3.7.4 Everett's Formula

This is an extensively used interpolation formula and uses only even order differences, as shown in the following table:



Hence the formula has the form

$$y_p = E_0 y_0 + E_2 \Delta^2 y_{-1} + E_4 \Delta^4 y_{-2} + \dots + F_0 y_1 + F_2 \Delta^2 y_0 + F_4 \Delta^4 y_{-1} + \dots$$
 (3.25)

The coefficients  $E_0$ ,  $F_0$ ,  $E_2$ ,  $F_2$ ,  $E_4$ ,  $F_4$ ,... can be determined by the same method as in the preceding cases, and we obtain

#### Hence Everett's formula is given by

$$y_{p} = qy_{0} + \frac{q(q^{2} - 1^{2})}{3!} \Delta^{2} y_{-1} + \frac{q(q^{2} - 1^{2})(q^{2} - 2^{2})}{5!} \Delta^{4} y_{-2} + \cdots$$

$$+ py_{1} + \frac{p(p^{2} - 1^{2})}{3!} \Delta^{2} y_{0} + \frac{p(p^{2} - 1^{2})(p^{2} - 2^{2})}{5!} \Delta^{4} y_{-1} + \cdots$$
(3.27)

where q = 1 - p.

#### 3.7.5 Relation between Bessel's and Everett's Formulae

These formulae are very closely related, and it is possible to deduce one from the other by a suitable rearrangement. To see this we start with Bessel's formula

$$y_{p} = y_{0} + p\Delta y_{0} + \frac{p(p-1)}{2!} \frac{\Delta^{2} y_{-1} + \Delta^{2} y_{0}}{2} + \frac{p(p-1)(p-1/2)}{3!} \Delta^{3} y_{-1}$$

$$+ \frac{(p+1) p(p-1)(p-2)}{4!} \frac{\Delta^{4} y_{-2} + \Delta^{4} y_{-1}}{2} + \cdots$$

$$= y_{0} + p(y_{1} - y_{0}) + \frac{p(p-1)}{2!} \frac{\Delta^{2} y_{-1} + \Delta^{2} y_{0}}{2} + \frac{p(p-1)(p-1/2)}{3!} (\Delta^{2} y_{0} - \Delta^{2} y_{-1})$$

$$+ \frac{(p+1) p(p-1)(p-2)}{4!} \frac{\Delta^{4} y_{-2} + \Delta^{4} y_{-1}}{2} + \cdots$$

expressing the odd order differences in terms of low even order differences.

This gives on simplification

$$y_{p} = (1-p)y_{0} + \left[\frac{p(p-1)}{4} - \frac{(p-1)p(p-1/2)}{6}\right]\Delta^{2}y_{-1} + \cdots$$

$$+ py_{1} + \left[\frac{p(p-1)}{4} + \frac{p(p-1)(p-1/2)}{6}\right]\Delta^{2}y_{0} + \cdots$$

$$= qy_{0} + \frac{q(q^{2}-1^{2})}{3!}\Delta^{2}y_{-1} + \cdots + py_{1} + \frac{p(p^{2}-1^{2})}{3!}\Delta^{2}y_{0} + \cdots$$

which is *Everett's formula* truncated after second differences. Hence we have a result of practical importance that Everett's formula truncated after second differences is equivalent to Bessel's formula truncated after third differences. In a similar way, Bessel's formula may be deduced from Everett's.

Bessel's Interpolation  (3) Find $f(x)$ at $x=7$ from following data. $\begin{array}{c ccccccccccccccccccccccccccccccccccc$					
	:- Lets c == f(x)	construct	Bessel's	9nterpol 534	ation Table
$x_0 = 6$ $x_1 = 8$	성-1= 49	△성-2= 44 △성-1= 132 △성 <sub>0</sub> = 268 △성 <sub>1</sub> = 452	Δ2 y = 136	$\triangle^3 4_{-2} = 48$ $\triangle^3 4_{-1} = 48$	

Bessel's For	mula:
	-+ v. Δ40]+ { - 1/2 + x Δ30+241]+ { v. (v2-1) x Δ21]
+ { (	12-41)(v2-27) x 14-1 + 14-27
= x-x0-1/2	x = value of x, for which f(x) needs to be found to = Value above/below x in x calumn in = Difference between each value of x (internal 34)
$=\frac{7-6}{2}-\frac{1}{2}$	putting the values,

$$f(x) = \left\{ \frac{181 + 449}{2} + (0 \times 268) \right\} + \left\{ \frac{0^2 - \frac{1}{4}}{2!} \times \frac{184 + 136}{2} \right\} + \left\{ \frac{0(0^2 - \frac{1}{4})}{3!} \times 48 \right\}$$

$$= \left\{ 315 + 0 \right\} + \left\{ -\frac{1}{8} \times 160 \right\} + \left\{ 0 \right\}$$

$$= \left\{ 315 \right\} + \left\{ -20 \right\} + \left\{ 0 \right\}$$

$$= 315 - 20 + 0$$

$$= 235$$

$$4 = \left\{ (7) = 235 \right\}$$
Ans

Note: Bessel's Interpolation Formula is similar to Newton's
Forward & Backward Interpolation formulas, where the difference between each value of x is some, ine interval gap or "h' is fixed. When interval gap is not some, we use: i'. Newton's Divided Difference, Lagrange Interpolation Formula,

# Thank You Any Query??