Lecture-5
Algorithmic Mathematics(CSC545)

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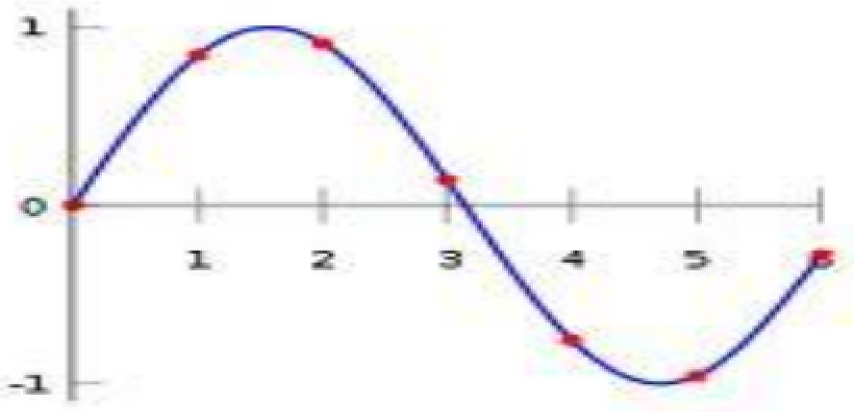
Interpolation

- **Definition**: The process of fitting a function through given data is called interpolation.



- Usually when we have data, we don't know the function f(x) that generated the data. So we fit a certain class of functions.
- The most usual class of functions fitted through data are polynomials. We will see why polynomials are fitted through data when we don't know f(x).

Figure 9.1 shows an approximate linear function and an interpolation polynomial for a set of data. Note that although the interpolation poly-

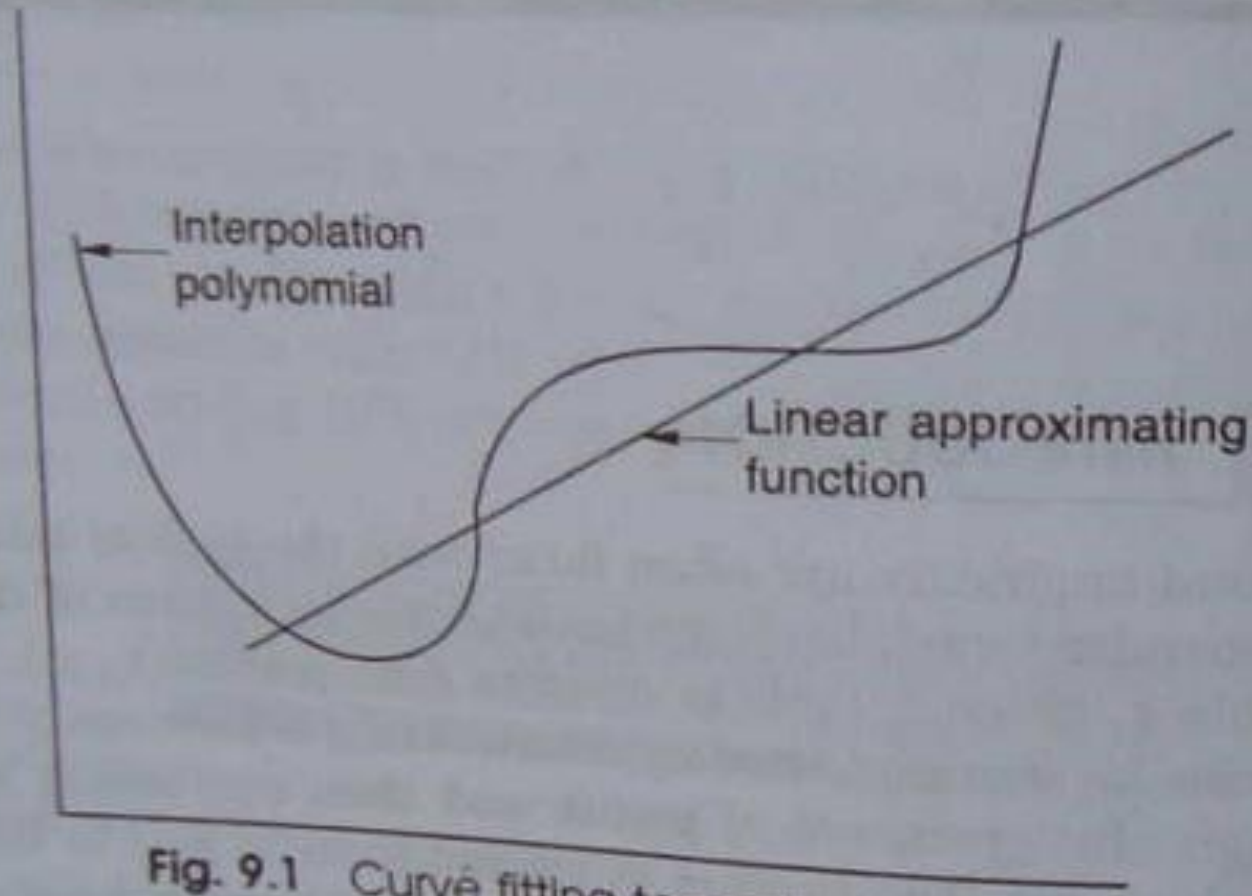


Fig. 9.1 Curve fitting to a set of points

9.2 POLYNOMIAL FORMS

The most common form of an nth order polynomial is

$$\boxed{p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n} \quad (9.1)$$

This form, known as the *power form*, is very convenient for differentiating and integrating the polynomial function and, therefore, are most widely used in mathematical analysis. However, there are situations where this form has been found inadequate, as illustrated by Example 9.1.

$$\boxed{P_n(x) = \sum_{i=0}^{n} b_i \prod_{j=0,\, j \neq i}^{n} (x - x_j)}$$

Errors in polynomial interpolation

- Let $\mathbf{P_N(x)}$ be the $\mathbf{N}^{th}$ degree polynomial through the **(N+1)** points $\mathbf{x_0}$**,** $\mathbf{x_1}$**, . . .,** $\mathbf{x_N}$ and $\mathbf{E_N(x)}$ is the error in the approximation of **f(x)** then :
- $\mathbf{E_N(x) = f(x) - P_N(x)}$
- Since both **f(x)** and $\mathbf{P_N(x)}$ have same value at the $\mathbf{x_i}$**, i = 0, 1, . . . N,** the error **E(x)** can be written as
- $\mathbf{E_N(x) = f(x) - P_N(x) = (x - x_0)(x - x_1). . .(x - x_N)\ g(x)}$
- where **g(x)** represents the $\mathbf{E_N(x)}$ at non tabulated points **x**. Obviously $\mathbf{f(x) - P_N(x) - E_N(x) = 0}$

$\mathbf{\Rightarrow f(x) - P_n(x) - (x - x_0)(x - x_1) . . . (x - x_n)\ g(x) = 0}$

Types of Interpolation

1. Linear Interpolation
2. Lagrange Interpolation
3. Newton’s Interpolation.

9.3 LINEAR INTERPOLATION

The simplest form of interpolation is to approximate two data points by a straight line. Suppose we are given two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. These two points can be connected linearly as shown in Fig. 9.2. Using the concept of similar triangles, we can show that

$$\frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

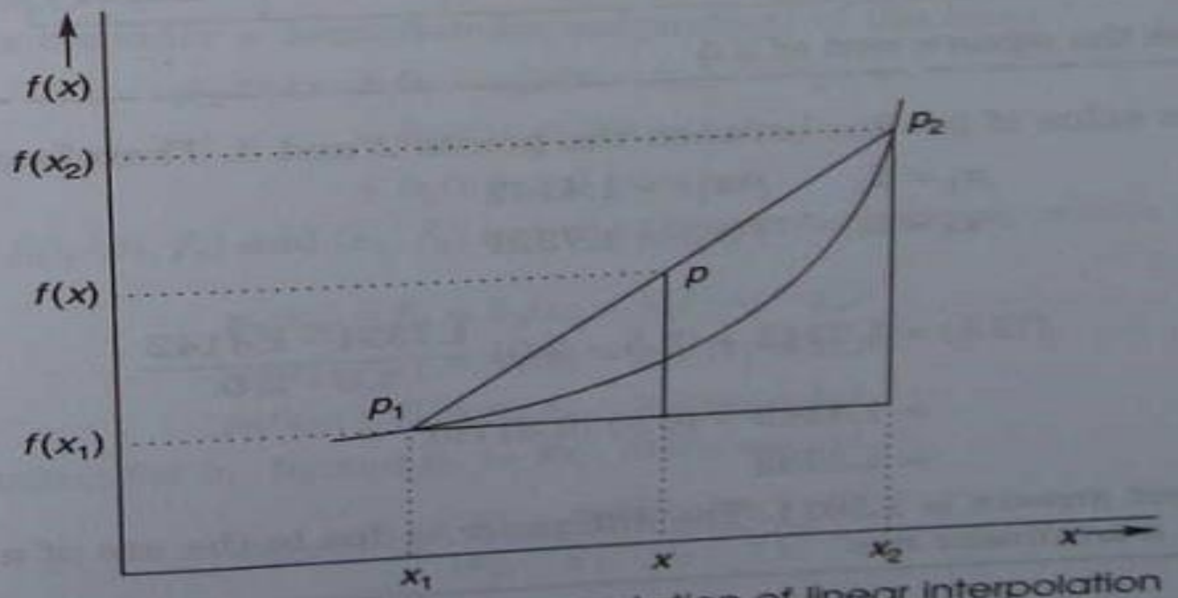


Fig. 9.2 Graphical representation of linear interpolation

Solving for $f(x)$, we get

$$\boxed{f(x) = f(x_1) + (x - x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1}} \qquad (9.5)$$

Equation (9.5) is known as *linear interpolation formula*. Note that the term

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

represents the slope of the line. Further, note the similarity of equation (9.5) with the *Newton form* of polynomial of first-order.

$$C_1 = x_1$$

$$a_0 = f(x_1)$$

$$a_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The coefficient a_1 represents the first derivative of the function.

The table below gives square roots for integers.

x	1	2	3	4	5
$f(x)$	1	1.4142	1.7321	2	2.2361

Determine the square root of 2.5

The given value of 2.5 lies between the points 2 and 3. Therefore,

Then $x_1 = 2, \quad f(x_1) = 1.4142$

$x_2 = 3, \quad f(x_2) = 1.7321$

$$f(2.5) = 1.4142 + (2.5 - 2.0)\,\frac{1.7321 - 1.4142}{3.0 - 2.0}$$

$$= 1.4142 + (0.5)\,(0.3179)$$

$$= 1.5732$$

The correct answer is 1.5811. The difference is due to the use of a linear model to a nonlinear one.

Now, let us repeat the procedure assuming $x_1 = 2$ and $x_2 = 4$.

$$f(x_1) = 1.4142$$

$$f(x_2) = 2.0$$

Then,

$$f(2.5) = 1.4142 + (2.5 - 2.0)\,\frac{2.0 - 1.4142}{4.0 - 2.0}$$

$$= 1.4142 + (0.5)\,(0.2929)$$

$$= 1.5607$$

Notice that the error has increased from 0.0079 to 0.0204. In general, the smaller the interval between the interpolating data points, the better will be the approximation.

9.4 LAGRANGE INTERPOLATION POLYNOMIAL

In this section, we derive a formula for the polynomial of degree n which takes specified values at a given set of $n + 1$ points.

Let $x_0, x_1, \ldots x_n$ denote n distinct real numbers and let $f_0, f_1, \ldots, f_n$ be arbitrary real numbers. The points $(x_0, f_0), (x_1, f_1), \ldots (x_n, f_n)$ can be imagined to be data values connected by a curve. Any function $p(x)$ satisfying the conditions

$$p(x_k) = f_k \qquad \text{for} \qquad k = 0, 1, \ldots n$$

is called an *interpolation function*. An interpolation function is, therefore, a curve that passes through the data points as pointed out in Section 9.1.

Let us consider a second-order polynomial of the form

$$\begin{aligned} p_2(x) = b_1(x - x_0)(x - x_1) \\ + b_2(x - x_1)(x - x_2) \\ + b_3(x - x_2)(x - x_0) \end{aligned} \tag{9.6}$$

If (x_0, f_0), (x_1, f_1) and (x_2, f_2) are the three interpolating points, then we have

$$p_2(x_0) = f_0 = b_2(x_0 - x_1)(x_0 - x_2)$$
$$p_2(x_1) = f_1 = b_3(x_1 - x_2)(x_1 - x_0)$$
$$p_2(x_2) = f_2 = b_1(x_2 - x_0)(x_2 - x_1)$$

Substituting for b_1, b_2 and b_3 in Eq. (9.6), we get

$$p_2(x) = f_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f_1 \frac{(x - x_2)(x - x_0)}{(x_1 - x_2)(x_1 - x_0)} + f_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \quad (9.7)$$

Equation (9.7) may be represented as

$$p_2(x) = f_0\, l_0(x) + f_1\, l_1(x) + f_2\, l_2(x) = \sum_{i=0}^{2} f_i\, l_i(x)$$

where

$$l_i(x) = \prod_{j=0,\, j \neq i}^{2} \frac{(x - x_j)}{(x_i - x_j)}$$

In general, for $n+1$ points we have nth degree polynomial as

$$\boxed{p_n(x) = \sum_{i=0}^{n} f_i l_i(x)} \tag{9.8}$$

where

$$\boxed{l_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}} \tag{9.9}$$

Equation (9.8) is called the *Lagrange interpolation polynomial*. The polynomials $l_i(x)$ are known as *Lagrange basis polynomials*. Observe that

$$l_i(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Now, consider the case $n = 1$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

Therefore,

$$p_1(x) = f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}$$

$$= \frac{f_0(x - x_1) - f_1(x - x_0)}{x_0 - x_1}$$

$$= f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0)$$

This is the *linear interpolation formula.*

Let us consider the following three points:

$$x_0 = 2, \quad x_1 = 3, \quad \text{and} \quad x_2 = 4$$

Then

$$f_0 = 1.4142, \quad f_1 = 1.7321, \quad \text{and} \quad f_2 = 2$$

For $x = 2.5$, we have

$$l_0(2.5) = \frac{(2.5 - 3.0)(2.5 - 4.0)}{(2.0 - 3.0)(2.0 - 4.0)} = 0.3750$$

$$l_1(2.5) = \frac{(2.5 - 2.0)(2.5 - 4.0)}{(3.0 - 4.0)(3.0 - 2.0)} = 0.7500$$

$$l_2(2.5) = \frac{(2.5 - 2.0)(2.5 - 3.0)}{(4.0 - 2.0)(4.0 - 3.0)} = -0.125$$

$$p_2(2.5) = (1.4142)(0.3750) + (1.7321)(0.7500) + (2.0)(-0.125)$$

$$= 0.5303 + 1.2991 - 0.250 = 1.5794$$

The error is 0.0017 which is much less than the error obtained in Example 9.3

Example 9.5

Find the Lagrange interpolation polynomial to fit the following data.

i	0	1	2	3
x_i	0	1	2	3
$e^{x_i} - 1$	0	1.7183	6.3891	19.0855

Use the polynomial to estimate the value of $e^{1.5}$.

Lagrange basis polynomials are

$$l_0(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-1)(0-3)}$$

$$= \frac{x^3 - 6x^2 + 11x - 6}{-3}$$

$$l_1(x) = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$

$$= \frac{x^3 - 5x^2 + 6x}{2}$$

$$l_2(x) = \frac{(x-0)(x-2)(x-3)}{(2-0)(2-1)(2-3)}$$

$$= \frac{x^3 - 4x^2 + 3x}{-2}$$

$$l_3(x) = \frac{(x-0)(x-2)(x-3)}{(3-0)(3-1)(3-2)}$$

$$= \frac{x^3 - 3x^2 + 2x}{6}$$

The interpolation polynomial is

$$p(x) = f_0\, l_0(x) + f_1\, l_1(x) + f_2\, l_2(x) + f_3\, l_3(x)$$

$$= 0 + \frac{1.7183\,(x^3 - 5x^2 + 6x)}{2}$$

$$= + \frac{6.3891\,(x^3 - 4x^2 + 3x)}{-2}$$

$$= + \frac{19.0856\,(x^3 - 3x^2 + 2x)}{6}$$

$$= \frac{5.0732\,x^3 - 6.3621\,x^2 + 11.5987x}{6}$$

$$= 0.8455x^3 - 1.0604\,x^2 + 1.9331x$$

$$p(1.5) = 3.3677$$

$$e^{1.5} = p(1.5) + 1 = 4.3677$$

Assignment#4

Prepare a technical report on “Uses of Interpolation and Extrapolation in mathematical Computation ”

Thank You

Any Query??