Lecture-11
Algorithmic Mathematics(CSC545)

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Simpson’s 1/3 Rule

Another popular method is Simpson's 1/3 rule. Here, the function $f(x)$ is approximated by a second-order polynomial $p_2(x)$ which passes through three sampling points as shown in Fig. 12.4. The three points include the end points a and b and a midpoint between them, i.e., $x_0 = a$, $x_2 = b$ and $x_1 = (a + b)/2$. The width of the segments h is given by

$$h = \frac{b-a}{2}$$

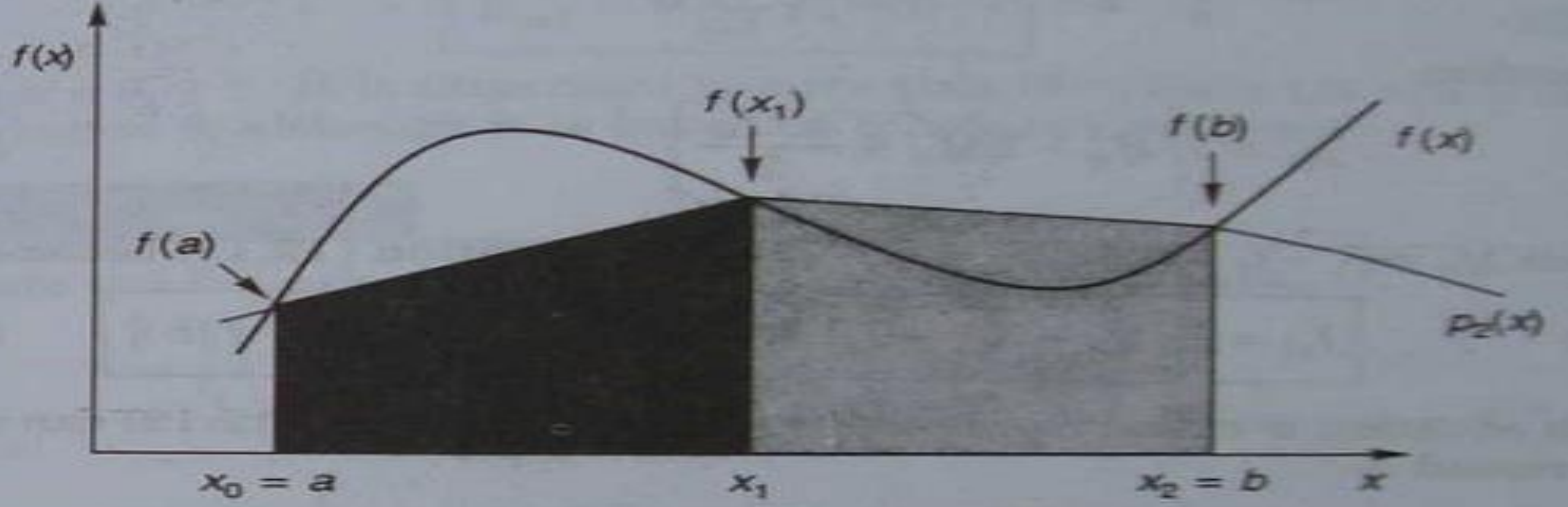

Fig. 12.4 Representation of Simpson's Three-point rule

The integral for Simpson's 1/3 rule is obtained by integrating the first three terms of equation (12.5), i.e.,

$$I_{s1} = \int_a^b p_2(x)\,dx = \int_a^b (T_0 + T_1 + T_2)\,dx$$

$$= \int_a^b T_0\,dx + \int_a^b T_1\,dx + \int_a^b T_2\,dx$$

$$= I_{s11} + I_{s12} + I_{s13}$$

where

$$I_{s11} = \int_a^b f_0\,dx$$

$$I_{s12} = \int_a^b \Delta f_0\, s\,dx$$

$$I_{s13} = \int_a^b \frac{\Delta^2 f_0}{2}\, s(s-1)\,dx$$

We know that $dx = h \times ds$ and s varies from 0 to 2 (when x varies from a to b). Thus,

$$I_{s11} = \int_0^2 f_0 h\, ds = 2hf_0$$

$$I_{s12} = \int_0^2 \Delta f_0\, sh\, ds = 2h\Delta f_0$$

$$I_{s13} = \int_0^2 \frac{\Delta^2 f_0}{2} s(s-1)h\, ds = \frac{h}{3}\Delta^2 f_0$$

Therefore,

$$I_{s1} = h\left[sf_0 + 2\Delta f_0 + \frac{\Delta^2 f_0}{3}\right] \tag{12.11}$$

Since $\Delta f_0 = f_1 - f_0$ and $\Delta^2 f_0 = f_2 - 2f_1 + f_0$, equation (12.11) becomes

$$\boxed{I_{s1} = \frac{h}{3}[f_0 + 4f_1 + f_2] = \frac{h}{3}[f(a) + 4f(x_1) + f(b)]} \tag{12.12}$$

This equation is called *Simpson's 1/3 rule*. Equation (12.12) can also be expressed as

$$I_{s1} = (b-a)\frac{f(a) + 4f(x_1) + f(b)}{6}$$

This shows that the area is given by *the product of total width of the segments and weighted average of heights $f(a)$, $f(x_1)$ and $f(b)$.*

Evaluate the following integrals using Simpson's 1/3 rule

(a) $\int_{-1}^{1} e^{x}\, dx$ (b) $\int_{0}^{\pi} \sqrt{\sin x}\, dx$

Case (a)

$$I = \int_{-1}^{1} e^{x}\, dx$$

$$I_{s1} = \frac{h}{3}\,[f(a) + f(b) + 4f(x_1)]$$

$$h = \frac{b-a}{2} = 1$$

$$f(x_1) = f(a + b)$$

Therefore,

$$I_{s1} = \frac{e^{-1} + 4e^{0} + e^{+1}}{3} = 2.36205$$

(Note that I_{s1} gives better estimate than I_{ct} when $n = 2$. This is because I_{s1} uses quadratic equation while I_{ct} uses a linear one)

Case (b)

$$I = \int_{0}^{\pi/2} \sqrt{\sin(x)}\,dx = \pi/4$$

$$I_{s1} = \frac{\pi}{12}\,[f(0) + 4f(\pi/4) + f(\pi/2)]$$

$$= 0.2617993(0 + 3.3635857 + 1)$$

$$= 1.1423841$$

Composite Simpson's 1/3 rule

Similar to the composite trapezoidal rule, we can construct a composite Simpson's 1/3 rule to improve the accuracy of the estimate of the area. Here again, the integration interval is divided into n number of segments of equal width, where n is an even number. Then the step size is

$$h = \frac{b-a}{n}$$

As usual, $x_i = a + ih$, $i = 0, 1, \ldots n$. Now, we can apply Eq. (12.12) to each of the $n/2$ pairs of segments or subintervals (x_{2i-2}, x_{2i-1}), (x_{2i-1}, x_{2i}). This gives

$$I_{cs1} = \frac{h}{3}\sum_{i=1}^{n/2}[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})]$$

$$= \frac{h}{3}[f(a) + 4f_1 + 2f_2 + 4f_3 + \ldots 2f_{n-2} + 4f_{n-1} + f(b)]$$

On regrouping terms, we get

$$\boxed{I_{cs1} = \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i}) + f(b)\right]} \qquad (12.14)$$

Compute the integral

$$\int_0^{\pi/2} \sqrt{\sin(x)}\, dx$$

applying Simpson's 1/3 rule for $n = 4$ and $n = 6$ with an accuracy to five decimal places.

The composite Simpson's 1/3 rule is given by

$$I_{cs1} = \frac{h}{3}\left[(f(x_0) + f(x_n)) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i})\right]$$

For $n = 4$, $h = \pi/8$

There are five sampling points given by $x_k = k\pi/8$, $k = 0, 1, \ldots, 4$. Substituting the values of x_k in the composite rule, we get,

$$I_{cs1} = \frac{\pi}{24}\,[f(0) + f(\pi/2) + 4f(\pi/8) + 4f(3\pi/8) + 2f(\pi/4)]$$

$$= \frac{\pi}{24}\,[0 + 1.0 + 4(0.61861 + 0.96119) + 2\,(0.84090)]$$

$$= 1.17823$$

For $n = 6,\ h = \pi/12$

There are seven sampling points given by $x_k = k\pi/12$, $k = 0, 1, \ldots, 6$.
Substituting these values in the above equation, we get

$$I_{cs1} = \frac{\pi}{36}\,[0 + 1.0 + 4(0.50874 + 0.84090 + 0.98282) + 2(0.70711 + 0.93060)]$$

$$= 1.18728$$

12.5 SIMPSON'S 3/8 RULE

Simpson's 1/3 rule was derived using three sampling points that fit a quadratic equation. We can extend this approach to incorporate four sampling points so that the rule can be exact for $f(x)$ of degree 3. Remember, even Simpson's 1/3 rule, although it is based on three points, is third-order accurate. However, a formula based on four points can be used even when the number of segments is odd.

By using the first four terms of Eq. (12.5) and applying the same procedure followed in the previous case, we can show that

$$\boxed{I_{s2} = \frac{3h}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right]} \tag{12.16}$$

where $h = (b - a)/3$. This equation is known as *Simpson's 3/8 rule*. This is also known as *Newton's three-eighths rule*.

Use Simpson's 3/8 rule to evaluate

$$\text{(a)} \int_1^2 (x^3 + 1)\, dx \qquad \text{(b)} \int_0^{\pi/2} \sqrt{\sin(x)\, dx}$$

Case (a)

Basic Simpson's 3/8 rule is based on four sampling points and, therefore, $n = 3$.

$$I_{s2} = \frac{3h}{8}[f(a) + 3f(x_1) + 3f(x_2) + f(b)]$$

$$h = \frac{b-a}{a} = \frac{1}{3}$$

$$x_1 = a + h = 1 + 1/3 = 4/3$$

$$x_2 = a + 2h = 1 + 2/3 = 5/3$$

on substitution of these values, we obtain

$$I_{s2} = \frac{1}{8}[f(1) + f(2) + 3f(4/3) + 3f(5/3)]$$

$$= 4.75$$

$= 4.75$

Note that the answer is exact. This is expected because Simpson's 3 rule is supposed to be exact for cubic polynomials.

Case (b)

$$I = \int_0^{\pi/2} \sqrt{\sin(x)}\, dx$$

Here again, $n = 3$ and the integral is given by

$$I_{s2} = \frac{3h}{8} [f(a) + 3f(x_1) + 3f(x_2) + f(b)]$$

$$h = \frac{b-a}{3} = \frac{\pi}{6}$$

$$x_1 = a + h = \frac{\pi}{6}$$

$$x_2 = a + 2h = \frac{\pi}{3}$$

on substitution of these values, we obtain

$$I_{s2} = \frac{\pi}{16} [f(0) + 3f(\pi/6) + f(\pi/3) + f(\pi/2)]$$

$$= \frac{\pi}{16} [0 + 2.12132 + 2.79181 + 1.0]$$

$$= 1.16104$$

Assignment#11

Estimate the following integrals by (a) trapezoidal method and (b) Simpson's 1/3 method using the given n:

(a) $\int_1^3 \frac{dx}{x}$, $n = 2, 4, 8$

(b) $\int_1^2 \frac{e^x dx}{x}$, $n = 4$

Thank You

Any Query??