Lecture-2
Algorithmic Mathematics(CSC545)

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6.8 NEWTON-RAPHSON METHOD

Consider a graph of $f(x)$ as shown in Fig. 6.5. Let us assume that x_1 is an approximate root of $f(x) = 0$. Draw a tangent at the curve $f(x)$ at $x = x_1$ as shown in the figure. The point of intersection of this tangent with the x-axis gives the second approximation to the root. Let the point of intersection be x_2. The slope of the tangent is given by

$$\tan \alpha = \frac{f(x_1)}{x_1 - x_2} = f'(x_1) \tag{6.19}$$

where $f'(x_1)$ is the slope of $f(x)$ at $x = x_1$. Solving for x_2 we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{6.20}$$

This is called the *Newton-Raphson formula*.

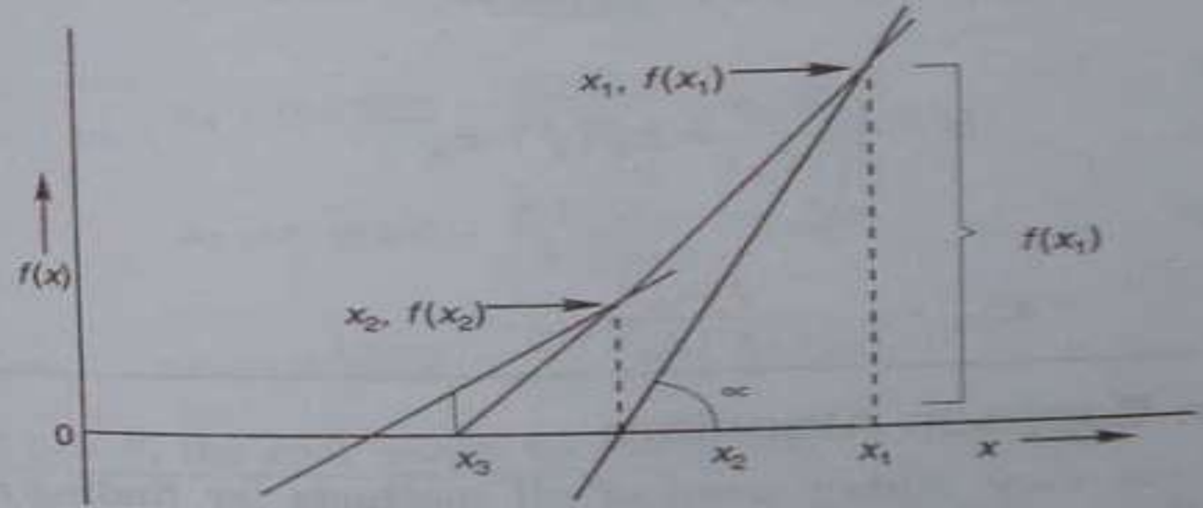

Fig. 6.5 Newton-Raphson method

The next approximation would be

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \tag{6.21}$$

This method of successive approximation is called the *Newton-Raphson method*. The process will be terminated when the difference between two successive values is within a prescribed limit.

The Newton-Raphson method approximates the curve of $f(x)$ by tangents. Complications will arise if the derivative $f'(x_n)$ is zero. In such cases, a new initial value for x must be chosen to continue the procedure.

Newton-Raphson Algorithm

Perhaps the most widely used of all methods for finding roots is the Newton-Raphson method. Algorithm 6.4 describes the steps for implementing Newton-Raphson method iteratively.

Newton-Raphson Method

1. Assign an initial value to x, say x_0.
2. Evaluate $f(x_0)$ and $f'(x_0)$
3. Find the improved estimate of x_0

   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

4. Check for accuracy of the latest estimate.

   Compare relative error to a predefined value E. If $\left|\frac{x_1 - x_0}{x_1}\right| \le E$ stop; Otherwise continue.
5. Replace x_0 by x_1 and repeat steps 3 and 4.

Algorithm 6.4

Example 6.7

Find the root of the equation

$$f(x) = x^2 - 3x + 2$$

in the vicinity of $x = 0$ using Newton-Raphson method.

$$f'(x) = 2x - 3$$

Let $x_1 = 0$ (first approximation)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0 - \frac{2}{-3} = \frac{2}{3} = 0.6667$$

Similarly,

$$x_3 = 0.6667 - \frac{0.4444}{-1.6667} = 0.9333$$

$$x_4 = 0.9333 - \frac{0.071}{-1.334} = 0.9959$$

$$x_5 = 0.9959 - \frac{0.0041}{-1.0082} = 0.9999$$

$$x_6 = 0.9999 - \frac{0.0001}{-1.0002} = 1.0000$$

Since $f(1.0) = 0$, the root closer to the point $x = 0$ is 1.000.

Convergence of Newton-Raphson Method

Let x_n be an estimate of a root of the function $f(x)$. If x_n and x_{n+1} are close to each other, then, using Taylor's series expansion, we can state

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(R)}{2}(x_{n+1} - x_n)^2 \qquad (6.22)$$

where R lies somewhere in the interval x_n to x_{n+1} and third and higher order have been dropped.

Let us assume that the exact root of $f(x)$ is x_r. Then $x_{n+1} = x_r$. Therefore $f(x_{n+1}) = 0$ and substituting these values in equation (6.22), we get

$$0 = f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(R)}{2}(x_r - x_n)^2 \qquad (6.23)$$

We know that the Newton's iterative formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Rearranging the terms, we get

$$f(x_n) = f'(x_n)\,(x_n - x_{n+1})$$

Substituting this for $f(x_n)$ in Eq. (6.23) yields

$$0 = f'(x_n)(x_r - x_{n+1}) + \frac{f''(R)}{2}(x_r - x_n)^2 \tag{6.24}$$

We know that the error in the estimate x_{n+1} is given by

$$e_{n+1} = x_r - x_{n+1}$$

Similarly,

$$e_n = x_r - x_n$$

Now, equation (6.24) can be expressed in terms of these errors as

$$0 = f'(x_n)\,e_{n+1} + \frac{f''(R)}{2}e_n^2$$

Rearranging the terms we get,

$$\boxed{e_{n+1} = -\frac{f''(R)}{2f'(x_n)}e_n^2} \tag{6.25}$$

Equation (6.25) shows that the error is roughly proportional to the square of the error in the previous iteration. Therefore, the Newton-Raphson method is said to have *quadratic convergence*.

Thanks You

Any Query??