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Unit : 1
Introduction to fuzzy set theory:

* Fuzzy logic:

- Fuzzy logic is an approach to computing based on "degree of truth" rather than the usual "true or false" (1 or 0) Boolean logic on which the modern computer is based.
- The idea of fuzzy logic was first advanced by Lotfi Zadeh in 1960s at University of california.
- The term fuzzy refers to things that are not clear or are vague. In the real world many times we encounter a situation when we can't determine whether the state is true or false, their fuzzy logic provides very valuable flexibility for reasoning.
- Fuzzy logic is a form of multi-valued logic in which the truth values of variables may be any real number between 0 & 1.
- Fuzzy logic is based on the idea that in many cases, the concept of true or false is too restrictive, so it allows for partial truths where a statement can be partially true or false, rather than fully true or false.
- The fundamental concept of fuzzy logic is the membership function, which defines the degree of membership of an input value to a certain set or category. The membership function is a mapping from an input value to a membership degree between 0 & 1, where 0 represents non-membership & 1 represent full membership.

- Example:

In the boolean system truth value, 1.0 represents the absolute truth value and 0.0 represents the absolute false value. But in fuzzy system, there is no logic for the absolute truth & absolute false value. But in fuzzy logic, there is an intermediate value to present which is partially true and partially false.

Is it cold? → Yes / 1
Is it cold? → No / 0

Boolean logic

Is it cold? → Very much / 0.9
Is it cold? → little / 0.25
Is it cold? → Very less / 0.1

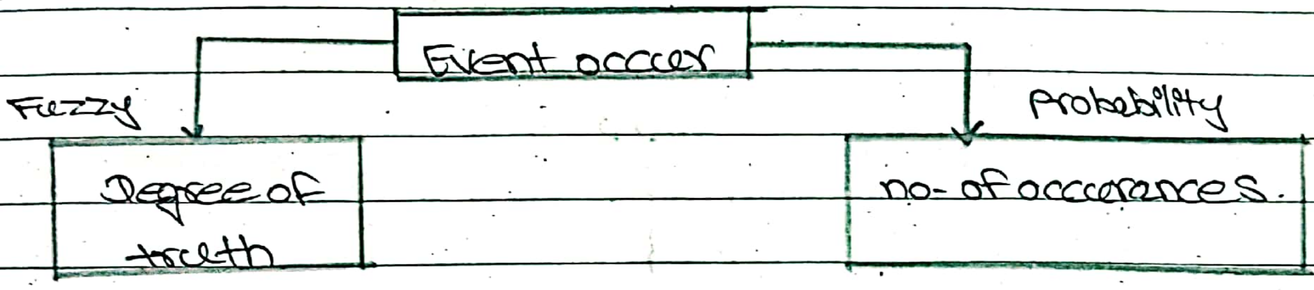
Fuzzy logic

Fuzzy { Imprecise, Vague }

Crisp { precise, clear }

* Fuzzy vs Probability:

- The probability theory is based on perception and has only two outcomes (true or false).
- Fuzzy theory is based on linguistic ~~function~~ information & is extended to handle the concept of partial truth. Fuzzy values are determined between true or false.
- e.g:-



[0,1] → multi-valued.

$x \in \begin{smallmatrix} t \\ f \end{smallmatrix}$

⇒ partial degree of truth

* Linguistic variables or hedges:

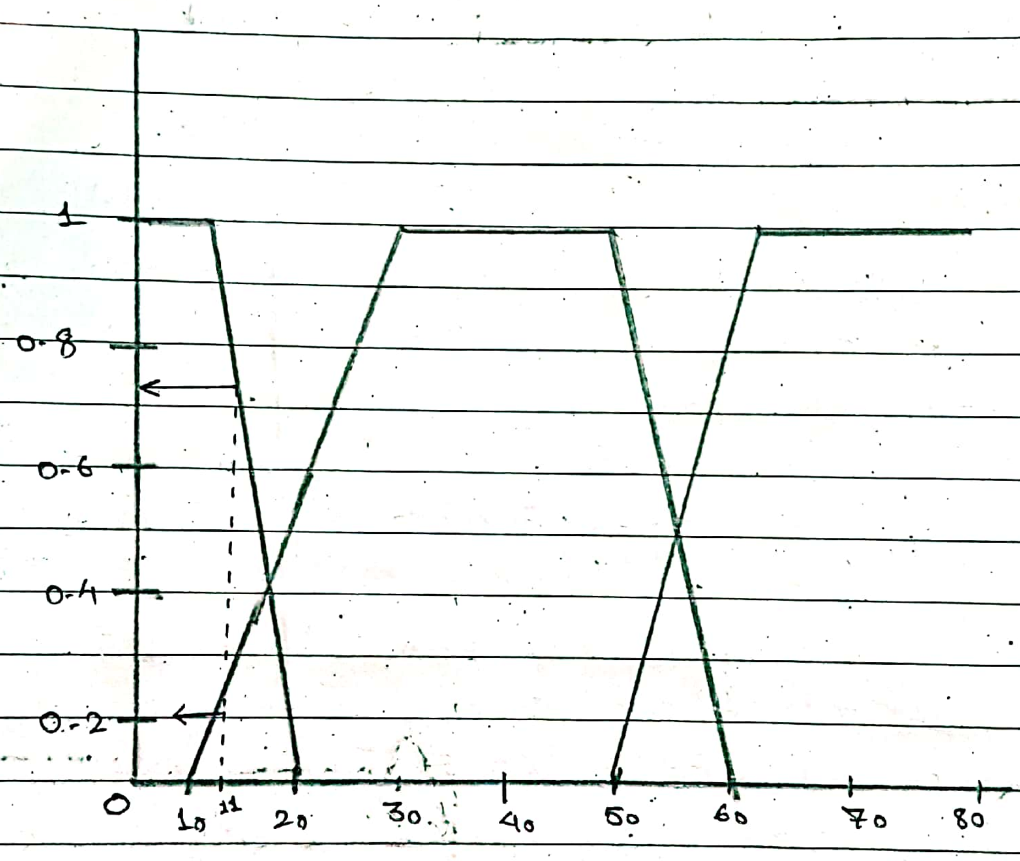
Variables in mathematics normally take numeric values, although non-numeric linguistic variables are frequently employed in fuzzy logic to make the expression of rules & facts easier. For instance, the term 'Age' can be used to indicate a linguistic variable with a value such as child, young, old & so on.

Linguistic variables are variables with a value made up of linguistic concepts (also known as linguistic words) rather than numbers, such as child, young, & so on.

Age = {child, young, old}

* for e.g:-

Each Age linguistic phrase has a membership function for a specific age range. The same age value is mapped to multiple membership values in the range of 0 to 1 by each function. These membership values can be used to identify whether a person is a child, a young person, or an elderly person.



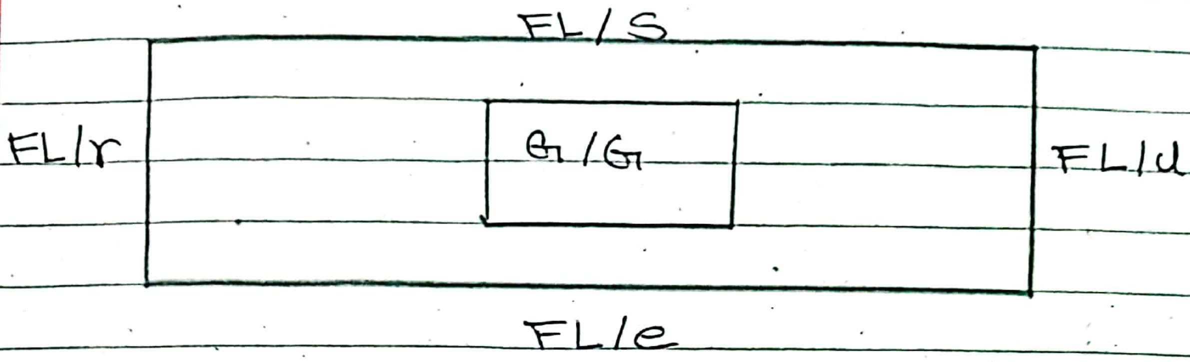
For Age = 11, we will get membership value of 0.75 (roughly) in the child set, 0.2 (approx) in the Young set, & 0 in the old set, as shown in diagram. So, if a person's age is 11, it's safe to assume that he or she is a child, perhaps little young but certainly not old.

Similarly,

eg:- If x is young then x is very talent

Membership function $= \left\{ \frac{0.9}{40}, \frac{0.7}{50} \right\}$

* Principle faces of fuzzy logic:



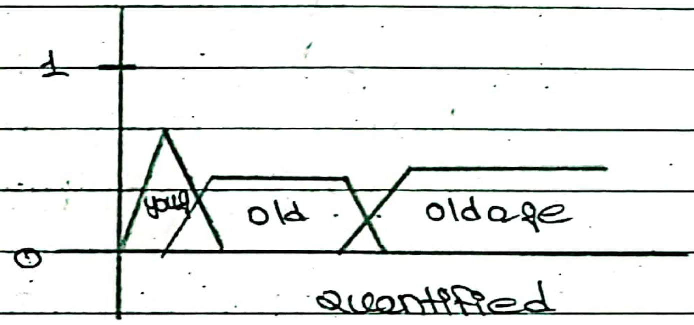
where

FL/S → set theoretic Facet
FL/l → Logical Facet
FL/e → epistemic Facet
FL/r → relational facet
G/G → Graduation / Granulation.

eg:-

Age: { young, old, very old }

Granulation



Graduation.

let Age = {0, 1, 2, ... 100}
then,

① Granulation generally assign the given set of elements to the linguistic variable i.e. young = {0 – 20}
old = {50 – 70}
very old = {70 – 100}

② Graduation is the mapping of set of elements to the memberfunction e.g:- $\{\frac{0}{0}, \frac{1}{1}, \frac{0.3}{3}, \frac{0.6}{8}, \ldots\}$

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① FL/S → Set theoretic Facet = Fuzzy set theories,
② FL/l → Logical Facet = degree of membership / multivalued logic.
③ FL/e → epistemic Facet = Natural language / linguistic variables
Fuzzy propositions / Knowledge
④ FL/r → relational Facet = Fuzzy relations over Fuzzy sets / Mappings.

* Example:-

Let, $A = \{1, 2, 3, ---, 10\}$
odd upto 5 $= \{1, 3, 5\}$
even upto 6 $= \{2, 4, 6\}$

Find, if 7 belongs to odd upto 5
$=$ $7 \notin$ odd upto 5

$X - \{$ Domain of Discourse / universe $\}$
$x \in X$
$x \in A$ where, $A \subseteq X$
$x \notin A$

$X = \{a, b, c\}$
$p(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ possible sub-sets.
then,
$A_1 = \{\{c\}, \{a, b\}, \{a, b, c\}\}$
$Y = \{\{0, 0, 1\}, \{1, 1, 0\}, \{1, 1, 1\}\}$ f : mapping.

$A_1 \rightarrow Y$

* Characteristic Function (χ):

Even upto 6 $= \{2, 4, 6\}$ ($\because$ from before example)

$8 \notin eut6$ (even upto)

$$\chi_S(x_i) \begin{cases} 0 & \text{iff } x_i \notin S \\ 1 & \text{iff } x_i \in S \end{cases}$$

So, $\chi(8) = 0$ ($\because 8 \notin eut6$)

another example: Vowel $= \{a, e, i, o, u\}$

$\chi_{Vowel}(a) = 1$ ($\because a \in$ Vowel)

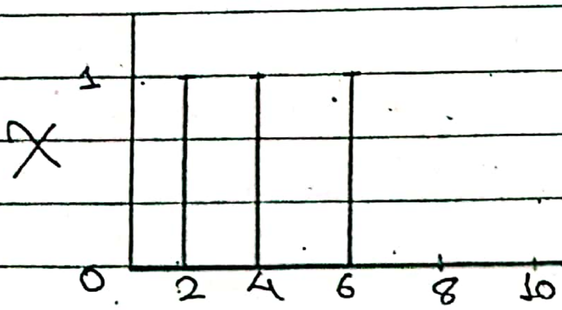
$\chi_{Vowel}(b) = 0$ ($\because b \in$ Vowel)

similarly,

$X = \{2, 4, 6, 8, 10\}$

$eut6 = \{2, 4, 6\}$

$$\chi_{\text{of any set}}(x_i) \begin{cases} 1 & \text{If element exists} \\ 0 & \text{doesn't exist.} \end{cases}$$



Graphical Notation

$$\boxed{\begin{aligned} &\exists A \subseteq X \\ &A \cup \bar{A} = X \\ &A \cap \bar{A} = \phi \end{aligned}}$$

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* Set theory operation:

i) Union
ii) Intersection,
iii) Complement.

Let, $X = \{2, 4, 6, 8, 10, 12\}$

$A = \{2, 4, 6\}$
$B = \{4, 6, 8, 10\}$

$A \cup B = \{2, 4, 6, 8, 10\}$
$A \cap B = \{4, 6\}$
$\overline{A} = \{8, 10, 12\}$

For Any two Crisp sets:

i.e A & B;

$$A \cup B = \chi_{A \cup B}(x_i) = \{ \max (\chi_A(x_i), \chi_B(x_i) \}$$

$$A \cap B = \chi_{A \cap B}(x_i) = \{ \min (\chi_A(x_i), \chi_B(x_i) \}$$

$$\overline{A} = \chi_{\overline{A}}(x_i) = 1 - \chi_A(x_i)$$

$$\chi_{A\cup B}^{(2)} = \max\{\chi_A^{(2)}, \chi_B^{(2)}\}$$
$$= \max(1, 0)$$
$$= 1$$

$$\chi_{A\cup B}^{(4)} = \max\{\chi_A^{(4)}, \chi_B^{(4)}\}$$
$$= \max(1, 1)$$
$$= 1$$

similarly,

$$\chi_{A\cup B}^{(6)} = \max(1, 1) = 1$$

$$\chi_{A\cup B}^{(8)} = \max(0, 1) = 1$$

$$\chi_{A\cup B}^{(10)} = \max(0, 1) = 1$$

Also,

$$A \subseteq B \text{ iff } \chi_A^{(x_i)} \leq \chi_B^{(x_i)}$$

e.g.:- $A = \{2, 4, 6\}$
$B = \{2, 4, 6, 8\}$

for $x_i = 2$	for $x_i = 8$
$\chi_A^{(2)} = 1$, $\chi_B^{(2)} = 1$ $\Big\}$ $1 \leq 1$ (True)	$\chi_A^{(8)} = 0$, $\chi_B^{(8)} = 1$; $0 \leq 1$ (true)

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Example:

$A = \{x_1, x_2\}$
$A_1 = \{\phi\}$
$A_2 = \{x_1\}$
$A_3 = \{x_2\}$
$A_4 = \{x_1, x_2\}$

$X \cdot A_1 = \{0, 0\}$
$X A_2 = \{1, 0\}$
$X A_3 = \{0, 1\}$
$X A_4 = \{1, 1\}$

* Fuzzy sets:

$X = \{x_1, x_2, \ldots, x_n\}$
$\underset{\sim}{A} \subseteq X$ (∵ $\underset{\sim}{A} \rightarrow$ zadeh notation for fuzzy set)

These are different ways of notations only for fuzzy sets.

A or $\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1}, \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2}, \frac{\mu_{\underset{\sim}{A}}(x_3)}{x_3}, \ldots \right\}$

A or $\underset{\sim}{A} = \left\{ \frac{\underset{\sim}{A}(x_1)}{x_1}, \frac{\underset{\sim}{A}(x_2)}{x_2}, \ldots \right\}$

A or $\underset{\sim}{A} = \{ (x_1, \mu_{\underset{\sim}{A}}(x_1)), (x_2, \mu_{\underset{\sim}{A}}(x_2)), \ldots \}$

or

$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \ldots \right\}$

where, μ is a membership function that maps every element in fuzzy set to membership value in [0,1]. which determines partial truthness.

$$\underset{\sim}{A} = \sum \frac{\mu_A(x_i)}{x_i} \quad \Big\} \text{ discrete set / Finite set}$$

$$\underset{\sim}{A} = \int \frac{\mu_A(x_i)}{x_i} \quad \Big\} \text{ Continuous set / Infinite set}$$

* Example:

$X = \{1, 2, \ldots 10\}$

Even up to 4 $= \left\{ \frac{0.001}{1}, \frac{1}{2}, \frac{0.5}{3}, \frac{1}{4} \right\}$

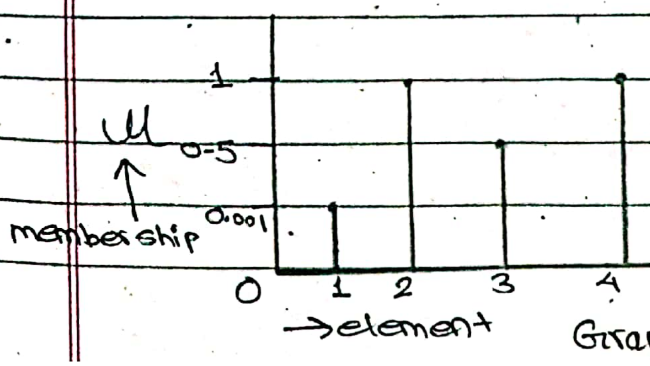
odd up to 4 $= \left\{ \frac{1}{1}, \frac{0.002}{2}, \frac{1}{3}, \frac{0.2}{4} \right\}$

$$\chi : x \rightarrow y \begin{cases} 0 \\ 1 \end{cases}$$

$$\mu : x \rightarrow y \in [0, 1]$$

$$\underset{\sim}{A} = \left\{ \frac{0.5}{1} + \frac{0.6}{2} + \frac{0.8}{3} + \frac{0.5}{4} \right\}$$

$$\underset{\sim}{A} = \{(1, 0.5), (2, 0.6), (3, 0.8), (4, 0.5)\}$$



Graph for Even upto 4

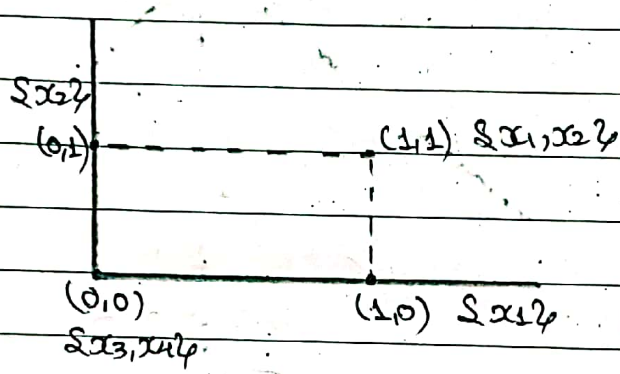
Kosko cube:

It helps to represent the given elements of a set to a hypercube to easy generalization.

Consider, $X = \{x_1, x_2\}$

$pow(x) = \{(x_1), (x_2), (x_1, x_2), (x_3, x_4)\}$

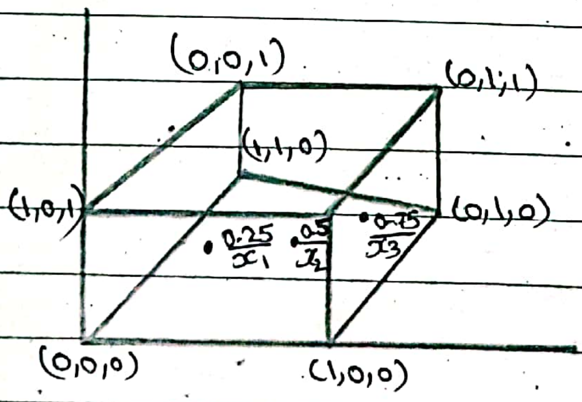
$pow(x) = \{(1,0), (0,1), (1,1), (0,0)\}$



(For crisp set)

Also, $X = \{x_1, x_2, x_3\}$.

$$\mu_x = \left\{ \frac{0.25}{x_1}, \frac{0.5}{x_2}, \frac{0.75}{x_3} \right\}$$



(For fuzzy set)

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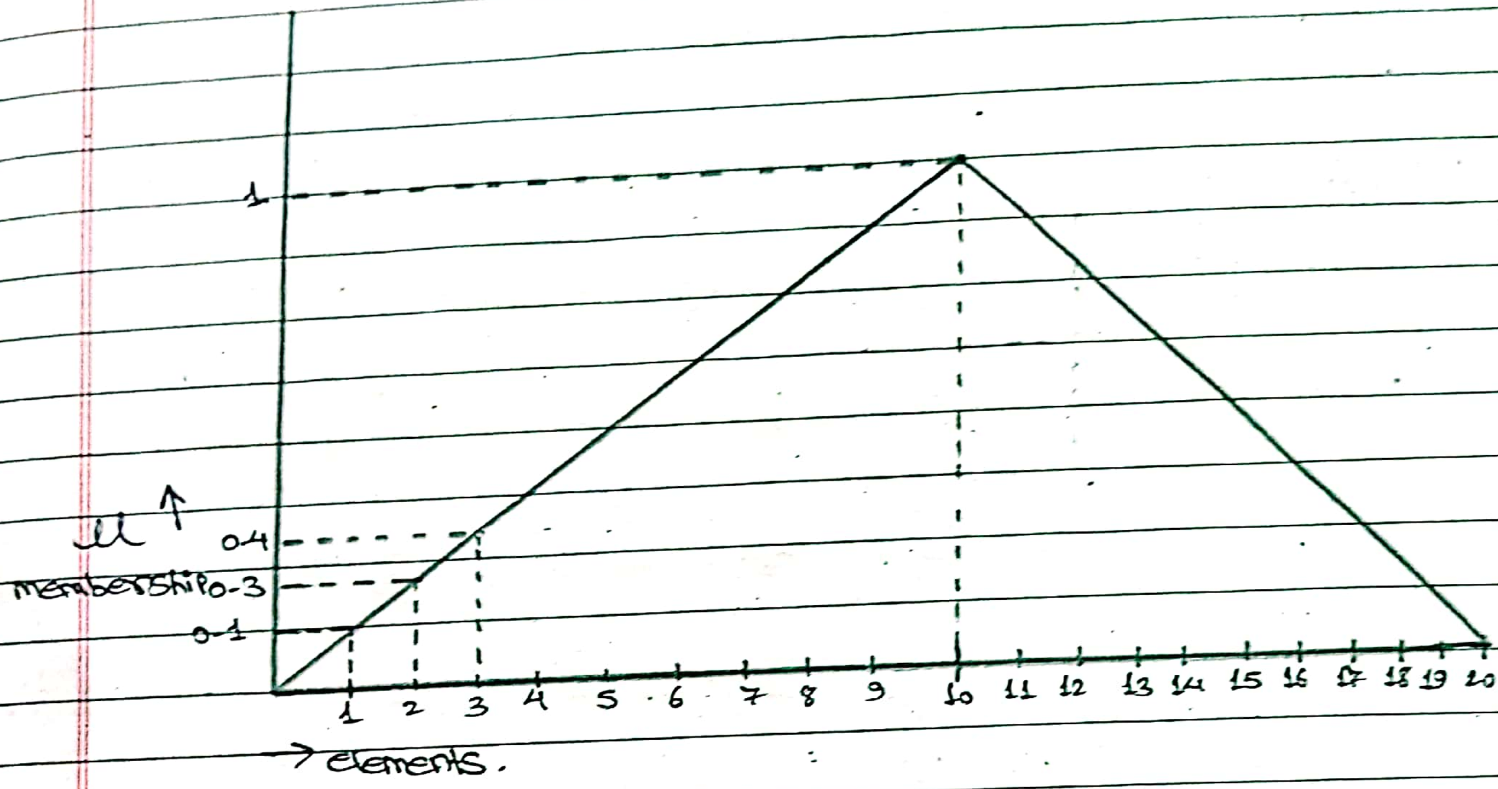
Eg:-
$\underset{\sim}{A} = \{(1,0.1), (2,0.3), (3,0.4), \ldots (10,1), (11,0.9),$
$(12,0.8), \ldots (20,0)\}$



$\text{number } 5 = \left\{ \frac{0.01}{1}, \frac{0.3}{2}, \frac{0.7}{3}, \frac{0.9}{4}, \frac{1}{5}, \frac{0.8}{6}, \frac{0.4}{7} \right\}$

* Fuzzy Set operations:

① Union: $\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x)$

$= MAX(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x))$

⑪ Intersection: $\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x)$

$= MIN(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x))$

③ Complement: $\mu_{\bar{A}}(x) = 1 - \mu_{A}(x)$

④ Subset: $A \subset X \Rightarrow \mu_{A}(x) \subset \mu_{X}(x)$

$$\forall x, x \in X, \mu_{\phi}(x) = 0$$

$$\forall x, x \in X, \mu_{X}(x) = 1$$

$$A \subseteq B \Rightarrow \mu_{A}(x) \subset \mu_{B}(x)$$

Example ①: $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$A = \left\{ \frac{0.5}{2}, \frac{0.4}{3}; \frac{0.3}{4}, \frac{0.9}{6}, \frac{1}{9} \right\}$$

$$B = \left\{ \frac{0.6}{1}, \frac{0.2}{2}, \frac{0.02}{3}, \frac{1}{10} \right\}$$

$$C = \left\{ \frac{0.7}{1}, \frac{0.01}{2}, \frac{0.4}{3} \right\}$$

(i) For $A \cup B$

$$\mu_{A \cup B}(x) = \max\left(\mu_{A}(x), \mu_{B}(x)\right)$$

$$\mu_{A \cup B}(1) = \max(0, 0.6)$$

$$= 0.6$$

$$\mu_{A \cup B}(2) = \max(0.5, 0.2)$$

$$= 0.5$$

$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(3) = \text{MAX}(0.4, 0.02)$
$= 0.4$

$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(4) = \text{MAX}(0.3, 0)$
$= 0.3$

$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(6) = \text{MAX}(0.9, 0)$
$= 0.9$

$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(9) = \text{MAX}(1, 0)$
$= 1$

$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(10) = \text{MAX}(0, 1)$
$= 1$

$\therefore \mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \left\{ \frac{0.6}{1}, \frac{0.5}{2}, \frac{0.4}{3}, \frac{0.3}{4}, \frac{0.9}{6}, \frac{1}{9}, \frac{1}{10} \right\}$

(ii) $\underset{\sim}{A} \cap \underset{\sim}{B}$

$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \text{MIN}\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x) \right)$

$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(2) = \text{MIN}\left(\mu_{\underset{\sim}{A}}(2), \mu_{\underset{\sim}{B}}(2) \right)$
$= \text{MIN}(0.5, 0.2)$
$= 0.2$

$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(3) = \text{MIN}\left(\mu_{\underset{\sim}{A}}(3), \mu_{\underset{\sim}{B}}(3) \right)$
$= \text{MIN}(0.4, 0.02)$
$= 0.02$

$$\therefore \mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \left\{ \frac{0.2}{2}, \frac{0.02}{3} \right\}$$

(iii) $\overline{A}$

$$\overline{A} = 1 - \mu_{\underset{\sim}{A}}(x)$$

$$= \left\{ \frac{1-0.5}{2}, \frac{1-0.4}{3}, \frac{1-0.3}{4}, \frac{1-0.9}{6}, \frac{1-1}{9} \right\}$$

$$= \left\{ \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.7}{4}, \frac{0.1}{6}, \frac{0}{9} \right\}$$

* Relative Complement:

Relative complement of A with respect to some Set X, i.e if given set is B then,

R.C. of A w.r.t B.

$$\overline{A} = B - A$$

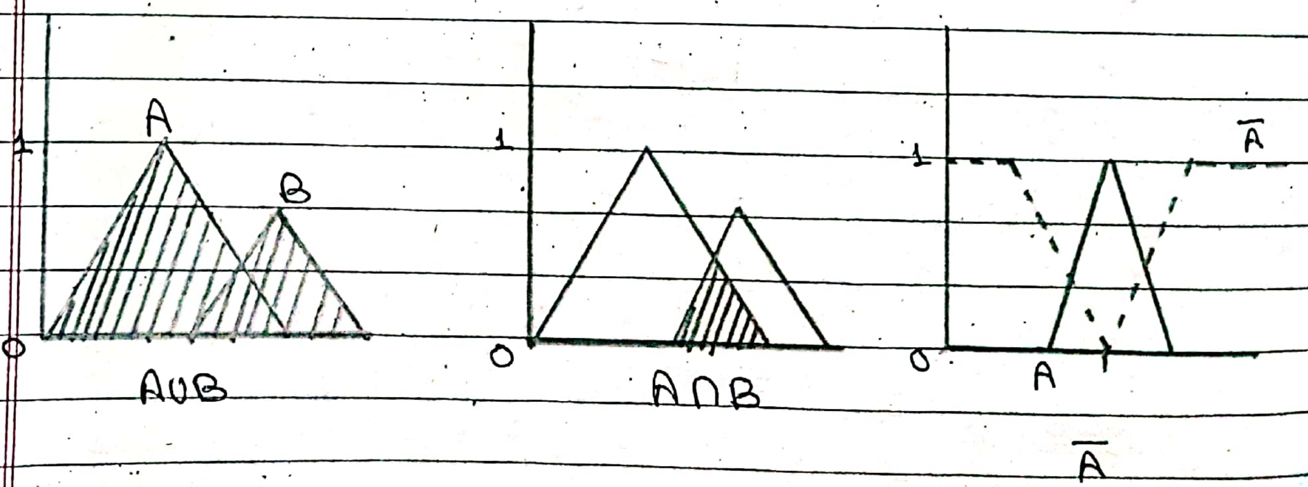
* Diagram:

$A = \{a, b, c, d, e\}$

$B = \{a, e\}$

$C = \{a, e, i, o, u\}$

then,



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* Exercise:

$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\underset{\sim}{A} = \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$

$\underset{\sim}{B} = \{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4} + \frac{0.4}{5} \}$

Compute: ① $\overline{\underset{\sim}{A}}$, ② $\overline{\underset{\sim}{B}}$, ③ $\underset{\sim}{A} \cup \underset{\sim}{B}$ ④ $\underset{\sim}{A} \cap \underset{\sim}{B}$

⑤ $A \cup \overline{A}$, ⑥ $B \cap \overline{B}$, ⑦ A difference B

i.e $A | B$

i.e $A \cap \overline{B}$

⑧ Show Demorgans rule over $\underset{\sim}{A}$ & $\underset{\sim}{B}$

i.e $(A \cup B)' = A' \cap B'$

⑨ Relative complement of B w.r.t A.

⇒ Sol

① $\mu_{\overline{\underset{\sim}{A}}}^{(x)} = 1 - \mu_{\underset{\sim}{A}}^{(x)}$

$= 1 - \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$

$= \{ \frac{1-1}{2}, \frac{1-0.5}{3}, \frac{1-0.3}{4}, \frac{1-0.2}{5} \}$

$= \{ \frac{0}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5} \}$

② $\overline{B} = 1 - \mu_{B}^{(x)}$

$= 1 - \{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \}$

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$$= \left\{ \frac{1-0.5}{2}, \frac{1-0.7}{3}, \frac{1-0.2}{4}, \frac{1-0.4}{5} \right\}$$

$$= \left\{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5} \right\}$$

③ $\underset{\sim}{A} \cup \underset{\sim}{B}$

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = MAX\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right)$$

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(2) = MAX(1, 0.5) = 1$$

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(3) = MAX(0.5, 0.7) = 0.7$$

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(4) = MAX(0.3, 0.2) = 0.3$$

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(5) = MAX(0.2, 0.4) = 0.4$$

$$\therefore \underset{\sim}{A} \cup \underset{\sim}{B} = \left\{ \frac{1}{2}, \frac{0.7}{3}, \frac{0.3}{4}, \frac{0.4}{5} \right\}$$

④ $\underset{\sim}{A} \cap \underset{\sim}{B}$

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = MIN\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right)$$

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(2) = MIN(1, 0.5) = 0.5$$

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(3) = MIN(0.5, 0.7) = 0.5$$

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$\mu^{(4)}_{\underset{\sim}{A} \cap \underset{\sim}{B}} = MIN(0.3, 0.2) = 0.2$

$\mu^{(5)}_{\underset{\sim}{A} \cap \underset{\sim}{B}} = MIN(0.2, 0.4) = 0.2$

$\therefore \underset{\sim}{A} \cap \underset{\sim}{B} = \left\{ \frac{0.5}{2}, \frac{0.5}{3}, \frac{0.2}{4}, \frac{0.2}{5} \right\}$

⑤ $A \cup \bar{A}$

$A = \left\{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \right\}$

$\bar{A} = \left\{ \frac{0}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5} \right\}$

$\mu^{(2)}_{\underset{\sim}{A} \cup \bar{A}} = MAX(1, 0) = 1$

$\mu^{(3)}_{A \cup \bar{A}} = MAX(0.5, 0.5) = 0.5$

$\mu^{(4)}_{A \cup \bar{A}} = MAX(0.3, 0.7) = 0.7$

$\mu^{(5)}_{A \cup \bar{A}} = MAX(0.2, 0.8) = 0.8$

$\therefore A \cup \bar{A} = \left\{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5} \right\}$

⑥ $B \cap \bar{B}$

$B = \left\{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \right\}$

$\bar{B} = \left\{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5} \right\}$

$\mu^{(2)}_{B \cap \bar{B}} = MIN(0.5, 0.5) = 0.5$

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$\mu^{(3)}_{B\cap\bar{B}} = MIN(0.7, 0.3) = 0.3$

$\mu^{(4)}_{B\cap\bar{B}} = MIN(0.2, 0.8) = 0.2$

$\mu^{(5)}_{B\cap\bar{B}} = MIN(0.4, 0.6) = 0.4$

$\therefore B\cap\bar{B} = \left\{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.2}{4}, \frac{0.4}{5} \right\}$

⑦ $A\cap\bar{B}$

$A = \left\{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}; \frac{0.2}{5} \right\}$

$\bar{B} = \left\{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5} \right\}$

$\mu^{(2)}_{A\cap\bar{B}} = MIN(1, 0.5) = 0.5$

$\mu^{(3)}_{A\cap\bar{B}} = MIN(0.5, 0.3) = 0.3$

$\mu^{(4)}_{A\cap\bar{B}} = MIN(0.3, 0.8) = 0.3$

$\mu^{(5)}_{A\cap\bar{B}} = MIN(0.2, 0.6) = 0.2$

$\therefore A\cap\bar{B} = \left\{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.3}{4}, \frac{0.2}{5} \right\}$

⑧ De-Morgan's Law : $(A\cup B)' = A'\cap B'$

now,

$A\cup B = \left\{ \frac{1}{2}, \frac{0.7}{3}, \frac{0.3}{4}, \frac{0.4}{5} \right\}$

So,

$(A \cup B)' = 1 - (A \cup B)$

$= 1 - \left\{ \frac{1}{2}, \frac{0.7}{3}, \frac{0.3}{4}, \frac{0.4}{5} \right\}$

$= \left\{ \frac{1-1}{2}, \frac{1-0.7}{3}, \frac{1-0.3}{4}, \frac{1-0.4}{5} \right\}$

$= \left\{ \frac{0}{2}, \frac{0.3}{3}, \frac{0.7}{4}, \frac{0.6}{5} \right\}$

we have,

$A' = \left\{ \frac{0}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5} \right\}$

$B' = \left\{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5} \right\}$

So,

$A' \cap B'$

$\mu^{(2)}_{\bar{A} \cap \bar{B}} = MIN(0, 0.5) = 0$

$\mu^{(3)}_{\bar{A} \cap \bar{B}} = MIN(0.5, 0.3) = 0.3$

$\mu^{(4)}_{\bar{A} \cap \bar{B}} = MIN(0.7, 0.8) = 0.7$

$\mu^{(5)}_{\bar{A} \cap \bar{B}} = MIN(0.8, 0.6) = 0.6$

$\therefore \bar{A} \cap \bar{B} = \left\{ \frac{0}{2}, \frac{0.3}{3}, \frac{0.7}{4}, \frac{0.6}{5} \right\}$

$\therefore (A \cup B)' = A' \cap B'$ proved.

(9) R-C of B w.r.t A

~~B-A~~ A−B

i.e, $= \left\{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \right\} - \left\{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \right\}$

$= \left\{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \right\} - \left\{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \right\}$

$= \left\{ \frac{1-0.5}{2}, \frac{0.5-0.7}{3}, \frac{0.3-0.2}{4}, \frac{0.2-0.4}{5} \right\}$

$= \left\{ \frac{0.5}{2}, \frac{-0.2}{3}, \frac{0.1}{4}, \frac{-0.2}{5} \right\}$

$= \left\{ \frac{0.5}{2}, \frac{0}{3}, \frac{0.1}{4}, \frac{0}{5} \right\}$

$\therefore A-B = \left\{ \frac{0.5}{2}, \frac{0.1}{4} \right\}$

* Fuzzy numbers:

Fuzzy numbers is a fuzzy subset of the universe of numerical numbers.

* Fuzzy integers:

Fuzzy integers is a fuzzy subset of the universe of integers.

* Fuzzy real numbers:

Fuzzy real numbers is a fuzzy subset of the universe of real numbers.

e.g:- Fuzzy number 2 defined by a fuzzy set $\underset{\sim}{A}$ over domain of integers as $\underset{\sim}{A} = \left\{ \frac{0.4}{1}, \frac{1}{2}, \frac{0.5}{3}, \frac{0.02}{4} \right\}$

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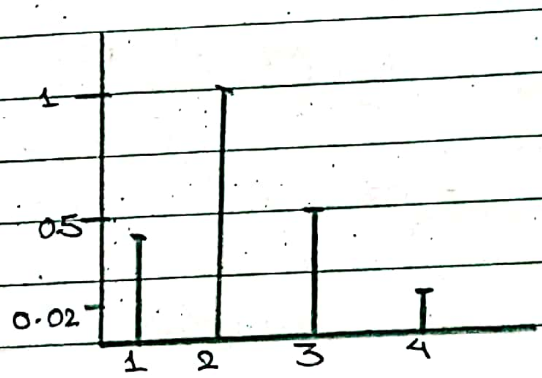
* Fuzzy Arithmetic:

Given fuzzy sets $\underset{\sim}{A}$ & $\underset{\sim}{B}$ representing fuzzy numbers x & y, the fuzzy arithmetic of fuzzy numbers x & y is defined as a fuzzy set $\underset{\sim}{C}$ representing the result z of

$$z = x \text{ op } y$$

where, op is arithmetic of operation

The membership of each z in $\underset{\sim}{C}$ is determined by extension principle.



Fuzzy number 2

* Properties of fuzzy sets:

① Normal fuzzy sets:

Given, a fuzzy set $\underset{\sim}{A}$ over X, $\underset{\sim}{A}$ is normal iff $\exists x \in A$ & $x \in X$, such that $\mu_{\underset{\sim}{A}}(x) = 1$

eg:-

$$\underset{\sim}{A} = \left\{ \frac{0.2}{3}, \frac{1}{4}, \frac{0.6}{5} \right\}$$

$$\underset{\sim}{C} = \left\{ \frac{1}{3}, \frac{1}{4}, \frac{0.5}{5} \right\}$$

$\underset{\sim}{A}$ & $\underset{\sim}{C}$ both are normal.

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A fuzzy set which is not normal is sub-normal.

e.g:-

$$B = \left\{ \frac{0.2}{A}, \frac{0.6}{5}, \frac{0.9}{8} \right\}$$

$\tilde{B}$ is sub-normal.

Note: All crisp sets except empty set are normal.

* Height of fuzzy sets:

Height of a fuzzy set $\tilde{A}$ is

$$\text{Height}(\tilde{A}) = \text{Max}\left(\mu_{\tilde{A}}(x)\right)$$

normal fuzzy set has height 1

$$\text{Height}(\tilde{A}) = 1 \quad \text{(From above example)}$$

$$\text{Height}(\tilde{B}) = 0.9$$

Since, Normal fuzzy set has atleast one element with membership 1 so height 1.

② Support of fuzzy set:

For a fuzzy set $\tilde{A}$ over X

support of $(\tilde{A})$ = crisp subset of X with elements from $\tilde{A}$ having non-zero membership.

i.e, $\text{Supp}(\tilde{A}) = \{ x / \mu_{\tilde{A}}(x) > 0 \ \forall x \in X \ \& \ x \in \tilde{A} \}$

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eg:- $\underset{\sim}{A} = \{ \frac{0.2}{4}, \frac{0.6}{5}, \frac{1}{6} \}$

$supp(\underset{\sim}{A}) = \{4, 5, 6\}$

* Core of fuzzy set:

For a fuzzy set $\underset{\sim}{A}$ over X

$core\ of(\underset{\sim}{A}) = \{ x \mid \mu_A(x) = 1\ \forall x \in X\ \&\ x \in \underset{\sim}{A} \}$

Core of $\underset{\sim}{A}$ is also crisp subset of X.

$\underset{\sim}{B} = \{ \frac{0.4}{2}, \frac{0.6}{5}, \frac{0.8}{9} \}$

$\therefore core(\underset{\sim}{B}) = \phi$

$\underset{\sim}{A} = \{ \frac{0.4}{2}, \frac{1}{6}, \frac{1}{7}, \frac{0.5}{8} \}$

$Core(\underset{\sim}{A}) = \{6, 7\}$

* Boundary of fuzzy sets:

Boundary of fuzzy sets is a crisp subset over X.

$Boundary(\underset{\sim}{A}) = \{ x \mid 0 < \mu_{\underset{\sim}{A}}(x) < 1\ \forall x \in X\ \&\ x \in \underset{\sim}{A} \}$

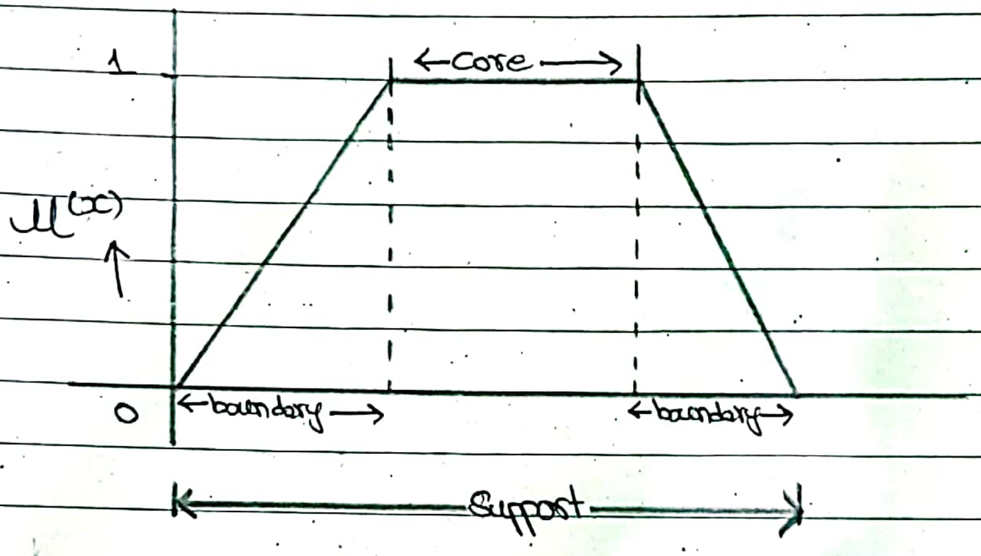
eg:- $\underset{\sim}{A} = \{ \frac{0.5}{2}, \frac{0.6}{4}, \frac{1}{8}, \frac{1}{10}, \frac{0.4}{11}, \frac{0}{12} \}$

$Boundary(\underset{\sim}{A}) = \{2, 4, 11\}$

$\underset{\sim}{B} = \{ \frac{0}{1}, \frac{1}{3}, \frac{1}{4}, \frac{0}{5} \}$

$\text{Boundary}(\underset{\sim}{B}) = \phi$

$\text{Boundary (crisp set)} = \phi$



Note: Subnormal fuzzy set $= \underset{\sim}{A}$

$\text{boundary}(\underset{\sim}{A}) = \text{support}(\underset{\sim}{A})$

* Convex & non-convex fuzzy sets:

For a fuzzy set $\underset{\sim}{A}$ over X such that for each $x, y, z \in \underset{\sim}{A}$ and they have monotonically increasing order (i.e. $x < y < z$) in $\underset{\sim}{A}$ then $\underset{\sim}{A}$ is convex iff;

$$\mu_{\underset{\sim}{A}}(y) \geq \text{Min}\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{A}}(z)\right)$$

i.e, membership of elements are all either in increasing or decreasing order or first increasing & then decreasing order.

eg:-

$$\underset{\sim}{A} = \left\{\frac{0.4}{1}, \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.9}{4}, \frac{1}{5}, \frac{0.9}{6}, \frac{0.6}{7}, \frac{0.4}{8}, \frac{0.1}{9}\right\}$$

$$\left.\begin{array}{l} \underset{\sim}{B} = \left\{ \frac{0.4}{1}, \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.7}{4} \right\} \\[2ex] \underset{\sim}{C} = \left\{ \frac{0.2}{1}, \frac{0.8}{2}, \frac{0.5}{3}, \frac{0.2}{4} \right\} \end{array}\right\} \text{Both are convex}$$

$$\underset{\sim}{P} = \left\{ \frac{0.2}{1}, \frac{0.8}{2}, \frac{0.4}{4}, \frac{0.8}{5}, \frac{0.2}{6}, \frac{0.4}{7}, \frac{0.9}{8} \right\}$$

$\underset{\sim}{P}$ is non-convex

eg:- (i) convex

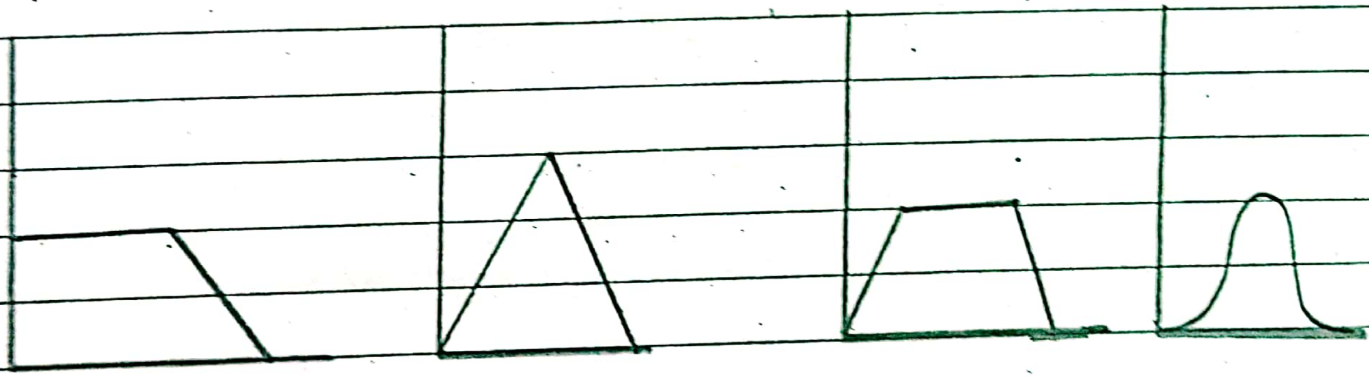


Fig of :- All are convex

(ii) Non-convex:

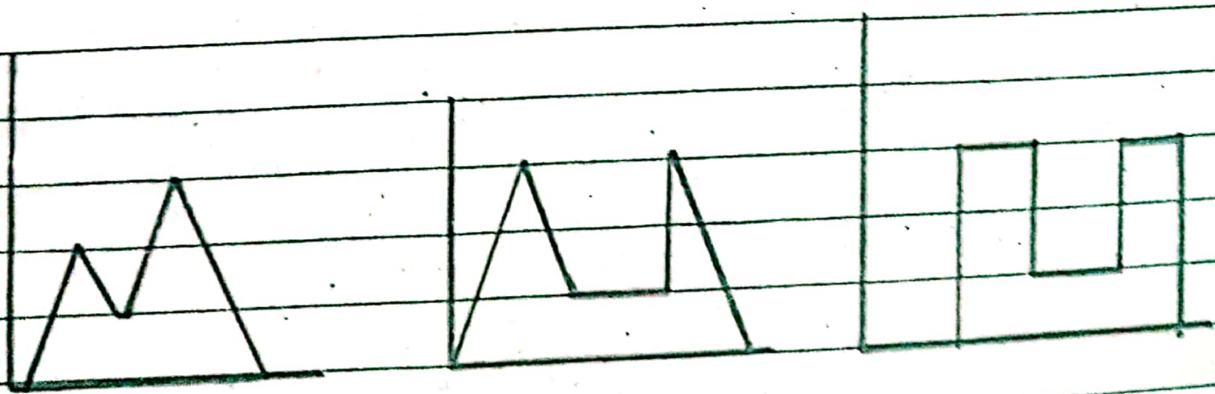


Fig: All are non-convex.

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* Prototype of fuzzy set:

For $\underset{\sim}{A}$ over X

$$\text{Prototype}(\underset{\sim}{A}) = x \text{ such that } \exists \text{ single } x \in \underset{\sim}{A}$$

$$\& \ \mu_{\underset{\sim}{A}}^{(x)} = 1$$

x is prototype or prototype element

(i) $\underset{\sim}{A} = \left\{ \frac{0.4}{2}, \frac{1}{3}, \frac{0.6}{5} \right\}$

prototype of $(\underset{\sim}{A}) = 3$

(ii) $\underset{\sim}{B} = \left\{ \frac{0.4}{2}, \frac{0.6}{3}, \frac{0.4}{5} \right\}$

prototype of $(\underset{\sim}{B}) = \phi$

(iii) $\underset{\sim}{C} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{0.2}{4} \right\}$

prototype of $(\underset{\sim}{C}) = \phi$

* Theorem:

Assume $\underset{\sim}{A}$ & $\underset{\sim}{B}$ are any two fuzzy subsets over X. If $\underset{\sim}{C} = \underset{\sim}{A} \cup \underset{\sim}{B}$ and $\underset{\sim}{D} = \underset{\sim}{A} \cap \underset{\sim}{B}$ then

(1) $\underset{\sim}{D} \subset \underset{\sim}{C}$

(2) $\underset{\sim}{A} \subset \underset{\sim}{C}$ and $\underset{\sim}{B} \subset \underset{\sim}{C}$

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③ $\underset{\sim}{D} \subset \underset{\sim}{A}$ and $\underset{\sim}{D} \subset \underset{\sim}{B}$

① Proof $\underset{\sim}{D} \subset \underset{\sim}{C}$

Given,

$$\underset{\sim}{C} = \underset{\sim}{A} \cup \underset{\sim}{B} \quad \& \quad \underset{\sim}{D} = \underset{\sim}{A} \cap \underset{\sim}{B}$$

E.g:- $\mu_A(x) = \frac{0.4}{2}$
$\mu_B(x) = \frac{0.6}{2}$

now,

for each $x \in X$, $\mu_{\underset{\sim}{D}}(x) = \text{MIN}\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right) \to \frac{0.4}{2}$

$\& \ \mu_{\underset{\sim}{C}}(x) = \text{MAX}\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right) \to \frac{0.6}{2}$

here, for each $x \in X$, it can be inferred that

$$\mu_{\underset{\sim}{D}}(x) \leq \mu_{\underset{\sim}{C}}(x) \Rightarrow \underset{\sim}{D} \subset \underset{\sim}{C}$$

$\frac{0.4}{2} \leq \frac{0.6}{2}$ True

② Proof $\underset{\sim}{A} \subset \underset{\sim}{C}$ and $\underset{\sim}{B} \subset \underset{\sim}{C}$

since,

$$\mu_{\underset{\sim}{C}}(x) = \text{MAX}\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right)$$

then,

it is clear that

$$\mu_{\underset{\sim}{C}}(x) \geq \mu_{\underset{\sim}{A}}(x) \quad \text{i.e.} \quad \mu_{\underset{\sim}{A}}(x) \leq \mu_{\underset{\sim}{C}}(x) \Rightarrow \underset{\sim}{A} \subset \underset{\sim}{C}$$

and,

$$\mu_{\underset{\sim}{C}}(x) \geq \mu_{\underset{\sim}{B}}(x) \quad \text{i.e.,} \quad \mu_{\underset{\sim}{B}}(x) \leq \mu_{\underset{\sim}{C}}(x) \Rightarrow \underset{\sim}{B} \subset \underset{\sim}{C}$$

③ Proof $\underset{\sim}{D} \subset \underset{\sim}{A}$ and $\underset{\sim}{D} \subset \underset{\sim}{B}$

Given,

$$\underset{\sim}{D} = \underset{\sim}{A} \cap \underset{\sim}{B}$$

$$\mu_{\underset{\sim}{D}}^{(x)} = \text{MIN}\left(\mu_{\underset{\sim}{A}}^{(x)}, \mu_{\underset{\sim}{B}}^{(x)}\right)$$

Thus,

$$\mu_{D}^{(x)} \leq \mu_{\underset{\sim}{A}}^{(x)} \Rightarrow \underset{\sim}{D} \subset \underset{\sim}{A}$$

Also,

$$\mu_{\underset{\sim}{D}}^{(x)} \leq \mu_{\underset{\sim}{B}}^{(x)} \Rightarrow \underset{\sim}{D} \subset \underset{\sim}{B}$$

* **Concentration & Dilation of fuzzy set:**

Assume a fuzzy set $\underset{\sim}{A}$ over X. consider a non-negative number α then A^{α} is a fuzzy set, say $\underset{\sim}{B}$ containing all elements x from $\underset{\sim}{A}$ such that for each x

$$\mu_{\underset{\sim}{B}}^{(x)} = \left(\mu_{\underset{\sim}{A}}^{(x)}\right)^{\alpha}$$

eg:- $\underset{\sim}{A} = \left\{\frac{1}{a}, \frac{0.6}{b}, \frac{0.3}{c}, \frac{0}{d}, \frac{0.5}{e}\right\}$

(i) for $\alpha = 2$ (Concentration op):

$$(\underset{\sim}{A})^2 = \left\{\frac{1}{a}, \frac{0.36}{b}, \frac{0.09}{c}, \frac{0}{d}, \frac{0.25}{e}\right\}$$

(ii) for $\alpha = \frac{1}{2} = 0.5$ (Dilation op)

$$(\underset{\sim}{A})^{0.5} = \left\{\frac{1}{a}, \frac{0.77}{b}, \frac{0.55}{c}, \frac{0}{d}, \frac{0.7}{e}\right\}$$

(i) If $\alpha > 1$ then $\underset{\sim}{A}^{\alpha} \subset \underset{\sim}{A}$, then it is known as concentration operation which decreases the fuzziness of Elements.

(ii) If $\alpha < 1$ then $\underset{\sim}{A} \subset \underset{\sim}{A}^{\alpha}$, then it is known as Dilation operation which increases fuzziness of elements.

* Bounded Sum & Difference:

Given, any two fuzzy subsets $\underset{\sim}{A}$ & $\underset{\sim}{B}$ over X, then bounded sum of $\underset{\sim}{A}$ & $\underset{\sim}{B}$ is a fuzzy set $\underset{\sim}{D}$

where, for each $x \in \underset{\sim}{D}$.

$$\mu_{\underset{\sim}{D}}^{(x)} = \text{MIN}\left(1, \left(\mu_{\underset{\sim}{A}}^{(x)} + \mu_{\underset{\sim}{B}}^{(x)}\right)\right)$$

and $\underset{\sim}{D} = \underset{\sim}{A} \oplus \underset{\sim}{D}$

& bounded difference of $\underset{\sim}{A}$ & $\underset{\sim}{B}$ is a fuzzy set $\underset{\sim}{C}$ where, for each $x \in \underset{\sim}{C}$

$$\mu_{\underset{\sim}{C}}^{(x)} = \text{MAX}\left(0, \left(\mu_{\underset{\sim}{A}}^{(x)} - \mu_{\underset{\sim}{B}}^{(x)}\right)\right)$$

and $\underset{\sim}{D} = \underset{\sim}{A} \ominus \underset{\sim}{B}$ ($\because$ Bounded difference of B w.r.t A)

Example:

$$\underset{\sim}{A} = \left\{ \frac{1}{a} + \frac{0.7}{b} + \frac{0.3}{c} + \frac{0}{d} + \frac{0.6}{e} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.2}{a} + \frac{0.1}{b} + \frac{0.4}{c} + \frac{1}{d} + \frac{0.2}{e} + \frac{0.6}{f} \right\}$$

now,

bounded sum $\underset{\sim}{D} = \underset{\sim}{A} \oplus \underset{\sim}{B}$ is

$$\mu_{\underset{\sim}{D}}(x) = \text{MIN}\left(1, \left(\mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)\right)\right)$$

for $a \in \underset{\sim}{D}$, $\mu_{\underset{\sim}{D}}(a) = \text{MIN}\left(1, \left(\mu_{\underset{\sim}{A}}(a) + \mu_{\underset{\sim}{B}}(a)\right)\right)$

$$= \text{MIN}(1, (1+0.2))$$

$$= 1$$

for $b \in \underset{\sim}{D}$, $\mu_{\underset{\sim}{D}}(b) = \text{MIN}(1, (0.7+0.1))$

$$= 0.8$$

for $c \in \underset{\sim}{D}$, $\mu_{\underset{\sim}{D}}(c) = \text{MIN}(1, (0.3+0.4))$

$$= 0.7$$

for $d \in \underset{\sim}{D}$, $\mu_{\underset{\sim}{D}}(d) = \text{MIN}(1, (0+1))$

$$= 1$$

for $e \in \underset{\sim}{D}$, $\mu_{\underset{\sim}{D}}(e) = \text{MIN}(1, (0.6+0.2))$

$$= 0.8$$

for $f \in \underset{\sim}{D}$, $\mu_{\underset{\sim}{D}}(f) = \text{MIN}(1, (0+0.6))$

$$= 0.6$$

$\therefore$ Bounded sum $\underset{\sim}{D} = \left\{ \frac{1}{a}, \frac{0.8}{b}, \frac{0.7}{c}, \frac{1}{d}, \frac{0.8}{e}, \frac{0.6}{f} \right\}$

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now, for Bounded difference of $\underset{\sim}{A}$ & $\underset{\sim}{B}$ i.e. $\underset{\sim}{D} = \underset{\sim}{A} - \underset{\sim}{B}$

$$\mu_{\underset{\sim}{D}}(x) = MAX\left(0, \left(\mu_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{B}}(x)\right)\right)$$

for $a \in \underset{\sim}{D} = \mu_{\underset{\sim}{D}}(a) = MAX(0, (1-0.2))$
$= 0.8$

for $b \in \underset{\sim}{D} = \mu_{\underset{\sim}{D}}(b) = MAX(0, (0.7-0.1))$
$= 0.6$

for $c \in \underset{\sim}{D} = \mu_{\underset{\sim}{D}}(c) = MAX(0, (0.3-0.4))$

$= 0$

for $d \in \underset{\sim}{D} = \mu_{\underset{\sim}{D}}(d) = MAX(0, (0-1))$
$= 0$

for $e \in \underset{\sim}{D} = \mu_{\underset{\sim}{D}}(e) = MAX(0, (0.6-0.2))$
$= 0.4$

for $f \in \underset{\sim}{D} = \mu_{\underset{\sim}{D}}(f) = MAX(0, (0-0.6))$

$= 0$

$\therefore$ Bounded difference $\underset{\sim}{A} \ominus \underset{\sim}{B}$.

$$= \left\{ \frac{0.8}{a}, \frac{0.6}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0.4}{e}, \frac{0}{f} \right\}$$

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* α-Level or α-cut:

Assume a fuzzy set $\underset{\sim}{A}$ over X, then the α-level or α-cut of $\underset{\sim}{A}$ is denoted by A_α and is a crisp set ~~onto~~ consisting all elements x from X for which

$$\mu_{\underset{\sim}{A}}(x) \geq \alpha$$

$$\text{i.e., } A_\alpha = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \alpha \text{ for each } x \in X\}$$

(i) Strict α-cut:

$$A_\alpha^+ = \{x \mid \mu_{\underset{\sim}{A}}(x) > \alpha \text{ for each } x \in X\}$$

eg:- $\underset{\sim}{A} = \left\{\frac{0.4}{a}, \frac{1}{b}, \frac{0.6}{c}, \frac{0.7}{d}, \frac{0.8}{e}\right\}$

For $\alpha = 0.6$,

$$A_{0.6} = \{b, c, d, e\}$$

$$A_{0.6}^+ = \{b, d, e\}$$

Case (i) If $\alpha = 1$, then for subnormal fuzzy set
$\alpha\text{-cut} = \phi$

Case (ii) If $\alpha = 0$, $\forall x \in \underset{\sim}{A}$

$\alpha\text{-cut} = x \in \underset{\sim}{A}$

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Unit : 2
Fuzzy Mapping

* Relation/Mapping:

→ Mapping between two sets

→ Presence or absence of connection or association between the elements of two sets.

* Classical Relation | Cartesian product:

For crisp sets $A_1, A_2, \ldots A_r$ the cartesian product is $A_1 \times A_2 \times \ldots \times A_r$ and contains tuples $(a_1, a_2, a_3 \ldots a_r)$ where, $a_1 \in A_1, a_2 \in A_2, \ldots, a_r \in A_r$.

Consider two crisp sets X and Y then the cartesian product of two crisp sets X & Y is denoted by;

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$ (For domain X, Y)

Some Properties:

a) $X \times Y \neq Y \times X$

b) $|X \times Y| = |X| \times |Y|$ (Cardinality i.e. no. of elements)

c) Cartesian product of 2 sets is not same as the arithmetic product of two or more sets.

Also,

$$\chi_{X \times Y}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in X \times Y \\ 0 & \text{if } (x, y) \notin X \times Y \end{cases}$$

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Example: consider the example A and B.

$A = \{2, 4, 6, 8\}$

$B = \{3, 7, 8, 9\}$

Find the cartesian product of 2 sets.

= Sol

$$A \times B = \{(2,3), (2,7), (2,8), (2,9), (4,3), (4,7), (4,8), (4,9), (6,3), (6,7), (6,8), (6,9), (8,3), (8,7), (8,8), (8,9)\}$$

Therefore, A particular mapping is done from $a \in A$ to $b \in B$ which is denoted by R (relation).

* Crisp Relation:

Crisp relation is a subset of cartesian product

$R(x, y) \subset X \times Y$

Thus,

$$\chi_R(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R \\ 0 & \text{if } (x,y) \notin R \end{cases}$$

We can represent R in a matrix form.

* Example:

Consider sets, $X = \{1, 2, 3\}$

$Y = \{a, b\}$

Then, the relation $R_1 \subseteq X \times Y$ can be

$$R_1 = \{(1,a), (2,b)\} \subseteq X \times Y$$

i.e,

$$R1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

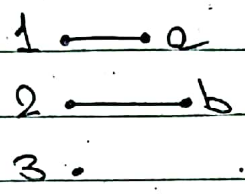

Fig:- Saggital Diagram of R1

* Operations on crisp relations:

For any two relations R and S;

(i) Union:

$$R \cup S \Rightarrow \chi_{R\cup S}(x,y) = \max\left(\chi_R(x,y), \chi_S(x,y)\right)$$

(ii) Intersection:

$$R \cap S = \chi_{R\cap S}(x,y) = \min\left(\chi_R(x,y), \chi_S(x,y)\right)$$

(iii) complement:

$$\bar{R} = \chi_{\bar{R}}(x,y) = 1 - \chi_R(x,y)$$

(iv) Containment/subset:

$$R \subset S \Rightarrow \chi_R(x,y) \leq \chi_S(x,y)$$

(V) Identity:

ϕ is identity of $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

X is identity of $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

→ Distributivity,
→ Commutativity,
→ Associativity,
→ Demorgan's law,
→ Excluded middle axioms → $R \cup \overline{R} = E$
$R \cap \overline{R} = O$

* Example:

Given, $X = \{x_1, x_2\}$
$Y = \{y_1, y_2\}$

$R_1 = \{(x_2, y_1), (x_1, y_2)\}$
$R_2 = \{(x_1, y_1), (x_1, y_2)\}$

Compute, (i) $R_1 \cup R_2$ (ii) $R_1 \cap R_2$
(iii) $\overline{R_1}$ (iv) $\overline{R_2}$

= Sol

(i) $R_1 \cup R_2$

for (x_1, y_1)

$$\chi_{R_1 \cup R_2}(x_1, y_1) = \max\left(\chi_{R_1}(x_1, y_1), \chi_{R_2}(x_1, y_1)\right)$$
$$= \max(0, 1)$$
$$= 1$$

for (x_1, y_2)

$$\chi_{R_1 \cup R_2}(x_1, y_2) = \max\left(\chi_{R_1}(x_1, y_2), \chi_{R_2}(x_1, y_2)\right)$$
$$= \max(1, 1)$$
$$= 1$$

for (x_2, y_1)

$$\chi^{(x_2,y_1)}_{R_1 \cup R_2} = \max\left(\chi^{(x_2,y_1)}_{R_1}, \chi^{(x_2,y_1)}_{R_2}\right)$$

$$= \max(1, 0)$$

$$= 1$$

(ii) <u>for $R_1 \cap R_2$</u>

for (x_1, y_1)

$$\chi^{(x_1,y_1)}_{R_1 \cap R_2} = \min\left(\chi^{(x_1,y_1)}_{R_1}, \chi^{(x_1,y_1)}_{R_2}\right)$$

$$\cancel{= \min(0, 1}$$

$$= \min(0, 1)$$

$$= 0$$

for (x_1, y_2)

$$\chi^{(x_1,y_2)}_{R_1 \cap R_2} = \min\left(\chi^{(x_1,y_2)}_{R_1}, \chi^{(x_1,y_2)}_{R_2}\right)$$

$$= \min(1, 1)$$

$$= 1$$

for (x_2, y_1)

$$\chi^{(x_2,y_1)}_{R_1 \cap R_2} = \min\left(\chi^{(x_2,y_1)}_{R_1}, \chi^{(x_2,y_1)}_{R_2}\right)$$

$$= \min(0, 1)$$

$$= 0$$

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(iii) $\overline{R_1}$

For (x_2, y_1)

$\overline{R}_1 = 1 - (\chi_{R_1}^{(x_2,y_1)})$

$= 1 - 1$

$= 0$

For (x_1, y_2)

$\overline{R}_1 = 1 - (\chi_{R_1}^{(x_1,y_2)})$

$= 1 - 1$

$= 0$

For (x_1, y_1)

$\overline{R}_1 = 1 - (\chi_{R_1}^{(x_1,y_1)})$

$= 1 - 0$

$= 1$

(iv) $\overline{R_2}$

For (x_1, y_1)

$\overline{R}_2 = 1 - (\chi_{R_2}^{(x_1,y_1)})$

$= 1 - 1$

$= 0$

For (x_1, y_2)

$\overline{R}_2 = 1 - (\chi_{R_2}^{(x_1,y_2)})$

$= 1 - 1$

$= 0$

For (x_2, y_1)

$\overline{R}_2 = 1 - (\chi_{R_2}^{(x_2,y_1)})$

$= 1 - 0$

$= 1$

another way for $R_1 \cup R_2$:

$$R_1 = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$R_2 = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$R_1 \cup R_2 = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} \max(0,1) & \max(1,1) \\ \max(1,0) & \max(0,0) \end{bmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \;//$$

* Composition operations:

Consider two relations,

$R \subseteq X \times Y$

$S \subseteq Y \times Z$

Then, the relation $T \subseteq X \times Z$ can be generated from R and S performing composition and is denoted as;

composition operator

$T = R \circ S$ where, ~~$T = \{(x,z)\} \in R$~~

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$T = \{(x,z) \mid \exists (x,y) \in R \;\&\; \exists (y,z) \in S\}$

$\mu_T^{(x,z)}$ can be determined by rules of composition.

* Rules of Composition:

① Maxmin
② Max product
③ Max Max
④ Min max
⑤ Min Min.

① Maxmin composition:

given,

$R \subseteq X \times Y$
$S \subseteq Y \times Z$

$T = R \circ S$

where,

OR (max) (Union) AND (min)

$$\chi_T^{(x,z)} = \bigvee_{y \in Y} \left(\chi_R^{(x,y)} \wedge \chi_S^{(y,z)} \right)$$

$$T(x,z) = \max\{\min\{\chi_R^{(x,y)}, \chi_S^{(y,z)}\}\}$$

② Max-Product Composition:

$T = R \cdot S$

where,

$$\chi_T^{(x,z)} = \bigvee_{y \in Y} \{\chi_R^{(x,y)} \cdot \chi_S^{(y,z)}\}$$

$$T(x,z) = \max\{\chi_R^{(x,y)} \cdot \chi_S^{(y,z)}\}$$

where, "·" is arithmetic product | Multiplication for crisp relations.

Note: max-product = max-min

since, product is same as min as;

$$\left.\begin{array}{l} 1*0=0 \\ 0*1=0 \\ 0*0=0 \\ 1*1=1 \end{array}\right\} \text{same as MIN.}$$

③ Max-Max Composition:

$$T = R \cdot S$$

where,

$$\chi_T^{(x,z)} = \bigvee_{y \in Y} \left(\chi_R^{(x,y)} \vee \chi_S^{(y,z)} \right)$$

$$T(x,z) = \max \left(\max \{ \chi_R^{(x,y)}, \chi_S^{(y,z)} \} \right)$$

④ Min max composition:

$$T = R \cdot S$$

where,

$$\chi_T^{(x,z)} = \bigwedge_{y \in Y} \{ \chi_R^{(x,y)} \vee \chi_S^{(y,z)} \}$$

$$T(x,z) = \min \{ \max \{ \chi_R^{(x,y)}, \chi_S^{(y,z)} \} \}$$

⑤ Min-min composition:

$T = R \cdot S$ where,

$$\chi_T^{(x,z)} = \bigwedge_{y \in Y} \{ \chi_R^{(x,y)} \wedge \chi_S^{(y,z)} \}$$

$$T(x,z) = \min \{ \min \{ \chi_R(x,y), \chi_S^{(y,z)} \} \}$$

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* Example:

consider, $R = \{(x_1, y_1), (x_1, y_3), (x_2, y_4)\}$

$S = \{(y_1, z_2), (y_3, z_2)\}$

For, $X = \{x_1, x_2, x_3\}$

$Y = \{y_1, y_2, y_3, y_4\}$

$Z = \{z_1, z_2\}$

Compete all rules of composition:

= Sol

here,

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$T = R \cdot S = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{cc} Z_1 & Z_2 \end{array} \\ \begin{bmatrix} \; & \; \\ \; & \; \\ \; & \; \end{bmatrix} \end{array} = ?$$

① max min Composition ?

for (x_1, z_1)

$$\chi_T^{(x_1,z_1)} = \max\{\min\{\chi_R^{(x,y)}, \chi_S^{(y,z)}\}\}$$

$$= \max\{\min\{\chi_R^{(x_1,y_1)}, \chi_S^{(y_1,z_1)}, \min(\chi_R^{(x_1,y_2)}, \chi_S^{(y_2,z_1)}), \min(\chi_R^{(x_1,y_3)}, \chi_S^{(y_3,z_1)}), \min(\chi_R^{(x_1,y_4)}, \chi_S^{(y_4,z_1)})\}\}$$

$$= \max\{\min(1,0), \min(0,0), \min(1,0), \min(0,0)\}$$

$$= \max(0,0,0,0)$$

$$= 0$$

for (x_1, z_2),

$$\chi_T^{(x_1,z_2)} = \max\{\min\{\chi_R^{(x_1,y_1)}, \chi_S^{(y_1,z_2)}\}, \min(\chi_R^{(x_1,y_2)}, \chi_S^{(y_2,z_2)}), \min(\chi_R^{(x_1,y_3)}, \chi_S^{(y_3,z_2)}), \min(\chi_R^{(x_1,y_4)}, \chi_S^{(y_4,z_2)})$$

$$= \max(\min(1,1), \min(0,0), \min(1,1), \min(0,0))$$

$$= \max(1,0,1,0)$$

$$= 1$$

Similar for, (x_2, z_1)

$$\chi_T^{(x_2,z_1)} = max[min(0,0), min(0,0), min(1,0), min(0,0)]$$

$$= max[0,0,0,0]$$

$$= 0$$

for (x_2, z_2)

$$\chi_T^{(x_2,z_2)} = max[min(0,1), min(0,0), min(0,1), min(1,0)]$$

$$= max[0,0,0,0]$$

$$= 0$$

for, (x_3, z_1)

$$\chi_T^{(x_3,z_1)} = max[min(0,0), min(0,0), min(0,0), min(0,0)]$$

$$= max[0,0,0,0,0]$$

$$= 0$$

for, (x_3, z_2)

$$\chi_T^{(x_3,z_2)} = max[min(0,1), min(0,0), min(0,1), min(0,0)]$$

$$= max[0,0,0,0]$$

$$= 0$$

$\therefore T =$ R.S using max-min composition $=$ $\begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 0 & 1 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{array}$

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b. For min-min composition :

$$T(x,z) = \min\{\min\{\mu_R(x,y), \chi_S(x,y)\}\}$$

for, (x_1, z_1)

$= \min[0,0,0,0]$

$= 0$

for (x_1, z_2)

$= \min[1,0,1,0]$

$= 0$

for, (x_2, z_1)

$= \min[0,0,0,0]$

$= 0$

for (x_2, z_2)

$= \min[0,0,0,0]$

$= 0$

for (x_3, z_1)

$= \min[0,0,0,0]$

$= 0$

for (x_3, z_2)

$= \min[0,0,0,0]$

$= 0$

$\therefore T = R \cdot S$ using min-min composition $=$
$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$
//

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* Fuzzy relations:

Consider fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ over X and Y, then fuzzy relation $\underset{\sim}{R}$ is subset of $\underset{\sim}{A} \times \underset{\sim}{B}$ i.e. $\forall x \in \underset{\sim}{A}$ and $y \in \underset{\sim}{B}$

$$R = \left\{ \frac{\mu_R(x,y)}{(x,y)} \middle| (x,y) \in X \times Y \right\}$$

where,

$$\mu_{\underset{\sim}{R}}(x,y) = MIN\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right)$$

* Example:

Consider,

$$\underset{\sim}{A} = \left\{ \frac{0.2}{2}, \frac{0.4}{3} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.1}{1}, \frac{0.6}{2} \right\}$$

now,

$$\underset{\sim}{R} \subseteq \underset{\sim}{A} \times \underset{\sim}{B}$$

For (2,1)

$$\mu_{\underset{\sim}{R}}(2,1) = MIN\left(\mu_{\underset{\sim}{A}}(2), \mu_{\underset{\sim}{B}}(1)\right)$$

$$= Min(0.2, 0.1)$$

$$= 0.1$$

For (2,2)

$$\mu_{\underset{\sim}{R}}(2,2) = MIN\left(\mu_{\underset{\sim}{A}}(2), \mu_{\underset{\sim}{B}}(2)\right)$$

$= MIN(0.2, 0.6)$
$= 0.2$

for $(3,1)$
$$\mu_{\underset{\sim}{R}}^{(3,1)} = MIN\left(\mu_{\underset{\sim}{A}}^{(3)}, \mu_{\underset{\sim}{B}}^{(1)}\right)$$
$$= MIN(0.4, 0.1)$$
$$= 0.1$$

for $(3,2)$
$$\mu_{\underset{\sim}{R}}^{(3,2)} = MIN\left(\mu_{\underset{\sim}{A}}^{(3)}, \mu_{\underset{\sim}{B}}^{(2)}\right)$$
$$= MIN(0.4, 0.6)$$
$$= 0.4$$

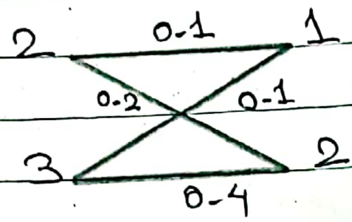
Thus,
$$\underset{\sim}{R} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \end{array}$$

i.e, $\underset{\sim}{R} = \left\{ \frac{0.1}{(2,1)}, \frac{0.2}{(2,2)}, \frac{0.1}{(3,1)}, \frac{0.4}{(3,2)} \right\}$

* Fuzzy graph:

Fuzzy graph is denoted as G and given by,
$$G = (\sigma, \mu)$$
where,
$$\sigma : S \to [0,1]$$
where, S is set of vertices, which are pair of elements from fuzzy sets.

Then, the fuzzy graph for $\underset{\sim}{R}$ in above example is given by:



which is also known as fuzzy saggital diagram.

Example:

$A = \left\{ \frac{0.2}{x_1}, \frac{0.5}{x_2}, \frac{1}{x_3} \right\}$

$B = \left\{ \frac{0.3}{y_1}, \frac{0.9}{y_2} \right\}$

$\underset{\sim}{R} \subseteq \underset{\sim}{A} \times \underset{\sim}{B} = ?$, Also draw fuzzy graph for $\underset{\sim}{R}$.

= Sol

for (x_1, y_1)

$\mu_{\underset{\sim}{R}}(x_1, y_1) = \mathrm{MIN}\left(\mu_{\underset{\sim}{A}}^{(x_1)}, \mu_{\underset{\sim}{B}}^{(y_1)}\right)$

$= \mathrm{MIN}(0.2, 0.3)$

$= 0.2$

for (x_1, y_2)

$\mu_{\underset{\sim}{R}}^{(x_1, y_1)} = \mathrm{MIN}(0.2, 0.9)$

$= 0.2$

for (x_2, y_1)

$$\mu_{\underset{\sim}{R}}(x_2, y_1) = MIN(0.5, 0.3)$$
$$= 0.3$$

for (x_2, y_2)

$$\mu_{\underset{\sim}{R}}(x_2, y_2) = MIN(0.5, 0.9)$$
$$= 0.5$$

for (x_3, y_1)

$$\mu_{\underset{\sim}{R}}(x_3, y_1) = MIN(1, 0.3)$$
$$= 0.3$$

for (x_3, y_2)

$$\mu_{\underset{\sim}{R}}(x_3, y_2) = MIN(1, 0.9)$$
$$= 0.9$$

Thus,

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

i.e. $\underset{\sim}{R} = \left\{ \frac{0.2}{(x_1, y_1)}, \frac{0.2}{(x_1, y_2)}, \frac{0.3}{(x_2, y_1)}, \frac{0.5}{(x_2, y_2)}, \frac{0.3}{(x_3, y_1)}, \frac{0.9}{(x_3, y_2)} \right\}$

Fuzzy graph:

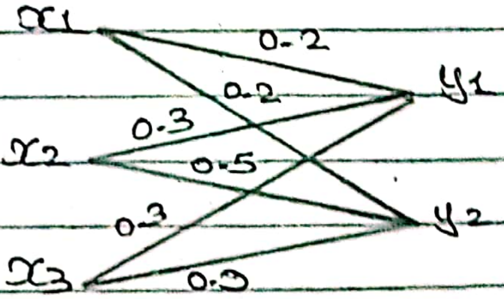


Fig :- Fuzzy Saggittal diagram.

* Fuzzy Composition:

Given,

$$\underset{\sim}{R} : X \rightarrow Y$$

$$\underset{\sim}{S} : Y \rightarrow Z$$

Then, fuzzy relation, $\underset{\sim}{T}$ can be derived from $\underset{\sim}{R}$ & $\underset{\sim}{S}$ such that $T : X \rightarrow Z$ using fuzzy composition operation.

* Fuzzy Composition operations:

1) Max-MIN
2) Max-Product
3) Max-Max
4) MIN-Max
5) MIN-MIN
6) Max-average, MIN-average

Example:

Consider $X = \{x_1, x_2\}$
$Y = \{y_1, y_2\}$
$Z = \{z_1, z_2, z_3\}$

given,

$$\underset{\sim}{R} = \left\{ \frac{0.7}{(x_1, y_1)} + \frac{0.5}{(x_1, y_2)} + \frac{0.8}{(x_2, y_1)} + \frac{0.4}{(x_2, y_2)} \right\}$$

$$\underset{\sim}{S} = \left\{ \frac{0.9}{(y_1, z_1)} + \frac{0.6}{(y_1, z_2)} + \frac{0.2}{(y_1, z_3)} + \frac{0.1}{(y_2, z_1)} + \frac{0.7}{(y_2, z_2)} + \frac{0.5}{(y_2, z_3)} \right\}$$

Compute $T = R \circ S$ using max-min composition.

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Sol

For (x_1, z_1)

$$\mu_{T}(x_1, z_1) = \max\{\min\{\mu_{R}(x_1), \mu_{S}(z_1)\}\}$$

$$= \max\{\min\{\mu_{R}(x_1, y_1), \mu_{S}(y_1, z_1)\},$$
$$\min\{\mu_{R}(x_1, y_2), \mu_{S}(y_2, z_1)\}\}$$

$$= \max\{\min(0.7, 0.9), \min(0.5, 0.1)\}$$

$$= \max(0.7, 0.1)$$
$$= 0.7$$

For (x_1, z_2)

$$= \max\{\min\{\mu_{R}(x_1, y_1), \mu_{S}(y_1, z_2)\}, \min\{\mu_{R}(x_1, y_2), \mu_{S}(y_2, z_2)$$

$$= \max\{\min(0.7, 0.6), \min(0.5, 0.7)\}$$
$$= \max(0.6, 0.5)$$
$$= 0.6$$

For (x_1, z_3)

$$= \max\{\min\{\mu_{R}(x_1, y_1), \mu_{S}(y_1, z_3)\}, \min\{\mu_{R}(x_1, y_2), \mu_{S}(y_2, z_3)$$

$$= \max\{\min(0.7, 0.2), \min(0.5, 0.5)\}$$
$$= \max\{0.2, 0.5\}$$
$$= 0.5$$

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For (x_2, z_1)

$$\mu_T^{(x_2,z_1)} = \max\{\min\{\mu_R^{(x_2,y_1)}, \mu_S^{(y_1,z_1)}\}, \{\min\{\mu_R^{(x_2,y_2)}, \mu_S^{(y_2,z_1)}\}\}$$

$$= \max\{\min(0.8, 0.9), \min(0.4, 0.1)\}$$
$$= \max(0.8, 0.1)$$
$$= 0.8$$

For (x_2, z_2)

$$\mu_T^{(x_2,z_2)} = \max\{\min\{\mu_R^{(x_2,y_1)}, \mu_S^{(y_1,z_2)}\}, \min\{\mu_R^{(x_2,y_2)}, \mu_S^{(y_2,z_2)}\}\}$$

$$= \max\{\min(0.8, 0.6), \min(0.4, 0.7)\}$$
$$= \max(0.6, 0.4)$$
$$= 0.6$$

For (x_2, z_3)

$$\mu_T^{(x_2,z_3)} = \max\{\min\{\mu_R^{(x_2,y_1)}, \mu_S^{(y_1,z_3)}\}, \min\{\mu_R^{(x_2,y_2)}, \mu_S^{(y_2,z_3)}\}\}$$

$$= \max\{\min(0.8, 0.2), \min(0.4, 0.5)\}$$

$$= \max\{0.2, 0.4\}$$
$$= 0.4$$

$\therefore$ $T = R \circ S$ under max-min composition is

$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

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* For max-product:

$$\mu_{T_2}^{(x,z)} = \max\{\mu_{R_2}^{(x,y)} \cdot \mu_{S_2}^{(y,z)}\}$$

for (x_1, z_1)

$$\mu_{T_2}^{(x_1,z_1)} = \max\{\mu_{R_2}^{(x_1,y_1)} \cdot \mu_{S_2}^{(y_1,z_1)}\}, \{\mu_{R_2}^{(x_1,y_2)} \cdot \mu_{S_2}^{(y_2,z_1)}\}$$

$= \max\{(0.7 \times 0.9); (0.5 \times 0.1)\}$
$= \max(0.63, 0.05)$
$= 0.63$

for $(x_1, z_2) = \max\{(0.7 \times 0.6), (0.5 \times 0.7)\}$.
$= \max(0.42, 0.35)$
$= 0.42$

for $(x_1, z_3) = \max\{(0.7 \times 0.2), (0.5, 0.5)\}$
$= \max(0.14, 0.25)$
$= 0.25$

for $(x_2, z_1) = \max\{(0.8 \times 0.9); (0.4 \times 0.1)\}$
$= \max(0.72, 0.04)$
$= 0.72$

for $(x_2, z_2) = \max\{(0.8 \times 0.6), (0.4 \times 0.7)\}$
$= \max(0.48, 0.28)$
$= 0.48$

for $(x_2, z_3) = \max\{(0.8 \times 0.2), (0.4 \times 0.5)\}$
$= \max(0.16, 0.2)$
$= 0.2$

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There,

$T = R \cdot S$ under max-product PS.

$$\begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.63 & 0.42 & 0.25 \\ x_2 & 0.72 & 0.48 & 0.2 \end{array}$$

* Properties of Max-min Composition:

① Associativity:

$$P \circ (Q \circ R) = (P \circ Q) \circ R$$

② Distributive over union:

$$P \circ (Q \cup R) = (P \circ Q) \cup (P \circ R)$$

③ Monotonic:

$$Q \subseteq R \Rightarrow P \circ Q \subseteq P \circ R$$

④ Commutativity:

$$P \circ Q \neq Q \circ P$$

* Example:

$$\underset{\sim}{A} = \left\{ \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.5}{y_1} + \frac{1}{y_2} \right\}$$

$$\underset{\sim}{C} = \left\{ \frac{0.2}{z_1} + \frac{0.3}{z_2} + \frac{0.9}{z_3} \right\}$$

Now, Formulate,

$$\underset{\sim}{R} \subseteq A \times B$$

$$\underset{\sim}{S} \subseteq B \times C$$

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Determine $\underset{\sim}{R} \circ \underset{\sim}{S} = \underset{\sim}{S} \circ \underset{\sim}{R}$ using max-min composition.

① We have to formulate:

$\underset{\sim}{R} \subseteq \underset{\sim}{A} \times \underset{\sim}{B}$

For (x_1, y_1)

$= \min\left(\mu_{\underset{\sim}{A}}(x_1), \mu_{\underset{\sim}{B}}(y_1)\right)$

$= \min(0.4, 0.5)$

$= 0.4$

For (x_1, y_2)

$= \min\left(\mu_{\underset{\sim}{A}}(x_1), \mu_{\underset{\sim}{B}}(y_2)\right)$

$= \min(0.4, 1)$

$= 0.4$

For (x_2, y_1)

$= \min\left(\mu_{\underset{\sim}{A}}(x_2), \mu_{\underset{\sim}{B}}(y_1)\right)$

$= \min(0.2, 0.5)$

$= 0.2$

For (x_2, y_2)

$= \min\left(\mu_{\underset{\sim}{A}}(x_2), \mu_{\underset{\sim}{B}}(y_2)\right)$

$= \min(0.2, 1)$

$= 0.2$

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for (x_3, y_1)
$= \min(\mu_{\underset{\sim}{A}}(x_3), \mu_{\underset{\sim}{B}}(y_1))$

$= \min(0.3, 0.5)$
$= 0.3$

for (x_3, y_2)
$= \min(\mu_{\underset{\sim}{A}}(x_3), \mu_{\underset{\sim}{B}}(y_2))$

$= \min(0.3, 1)$
$= 0.3$

$$\underset{\sim}{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix} \end{array}$$

(ii) $\underset{\sim}{S} \subseteq \underset{\sim}{B} \times \underset{\sim}{C}$

for (y_1, z_1);

$\mu_{\underset{\sim}{S}}(y_1, z_1) = \min(\mu_{\underset{\sim}{B}}(y_1), \mu_{\underset{\sim}{C}}(z_1))$

$= \min(0.5, 0.2)$
$= 0.2$

$\mu_{\underset{\sim}{S}}(y_1, z_2) = \min(\mu_{\underset{\sim}{B}}(y_1), \mu_{\underset{\sim}{C}}(z_2))$
$= \min(0.5, 0.3)$
$= 0.3$

$$\mu_{\tilde{S}}^{(y_1,z_3)} = \min\left(\mu_{\tilde{B}}^{(y_1)}, \mu_{\tilde{C}}^{(z_3)}\right)$$

$$= \min(0.5, 0.9)$$

$$= 0.5$$

$$\mu_{\tilde{S}}^{(y_2,z_1)} = \min\left(\mu_{\tilde{B}}^{(y_2)}, \mu_{\tilde{C}}^{(z_1)}\right)$$

$$= \min(1, 0.2)$$

$$= 0.2$$

$$\mu_{\tilde{S}}^{(y_2,z_2)} = \min\left(\mu_{\tilde{B}}^{(y_2)}, \mu_{\tilde{C}}^{(z_2)}\right)$$

$$= \min(1, 0.3)$$

$$= 0.3$$

$$\mu_{\tilde{S}}^{(y_2,z_3)} = \min\left(\mu_{\tilde{B}}^{(y_1)}, \mu_{\tilde{C}}^{(z_3)}\right)$$

$$= \min(1, 0.9)$$

$$= 0.9$$

$$\therefore \tilde{S} = \tilde{B} \times \tilde{C}$$

$$\begin{array}{c} \\ y_1 \\ y_2 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.2 & 0.3 & 0.9 \end{bmatrix} \end{array}$$

(iii) $\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S}$

$$\underset{\sim}{R} = \left\{ \frac{0.4}{(x_1,y_1)}, \frac{0.4}{(x_1,y_2)}, \frac{0.2}{(x_2,y_1)}, \frac{0.2}{(x_2,y_2)}, \frac{0.3}{(x_3,y_1)}, \frac{0.3}{(x_3,y_2)} \right\}$$

$$\underset{\sim}{S} = \left\{ \frac{0.2}{(y_1,z_1)}, \frac{0.3}{(y_1,z_2)}, \frac{0.5}{(y_1,z_3)}, \frac{0.2}{(y_2,z_1)}, \frac{0.3}{(y_2,z_2)}, \frac{0.9}{(y_2,z_3)} \right\}$$

Then, using Max-Min Composition,

For $(x_1, z_1) = \max\left\{ \min\left\{ \mu_{\underset{\sim}{R}}^{(x_1,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,z_1)} \right\}, \min\left\{ \mu_{\underset{\sim}{R}}^{(x_1,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,z_1)} \right\} \right\}$

$= \max\{ \min(0.4, 0.2), \min(0.4, 0.2) \}$

$= \max(0.2, 0.2)$

$= 0.2$

For $(x_1, z_2) =$

$\mu_T^{(x_1,z_2)} = \max\left\{ \min\left\{ \mu_{\underset{\sim}{R}}^{(x_1,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,z_2)} \right\}, \min\left\{ \mu_{\underset{\sim}{R}}^{(x_1,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,z_2)} \right\} \right\}$

$= \max\{ \min(0.4, 0.3), \min(0.4, 0.3) \}$

$= \max(0.3, 0.3)$

$= 0.3$

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for (x_1, z_3)

$$\mu_T^{(x_1,z_3)} = \max\{\min\{\mu_{R_2}^{(x_1,y_1)}, \mu_{S_2}^{(y_1,z_3)}\}, \min\{\mu_{R_2}^{(x_1,y_2)}, \mu_{S_2}^{(y_2,z_3)}$$

$= \max\{\min(0.4, 0.5), \min(0.4, 0.9)\}$

$= \max(0.4, 0.4)$

$= 0.4$

now,

for (x_2, z_1)

$$\mu_T^{(x_2,z_1)} = \max\{\min\{\mu_{R_2}^{(x_2,y_1)}, \mu_{S_2}^{(y_1,z_1)}\}, \min\{\mu_{R_2}^{(x_2,y_2)}, \mu_{S_2}^{(y_2,z_1)}\}$$

$= \max\{\min(0.2, 0.2), \min(0.2, 0.3)\}$

$= \max\{0.2, 0.2\}$

$= 0.2$

for (x_2, z_2)

$$\mu_T^{(x_2,z_2)} = \max\{\min\{\mu_{R_2}^{(x_2,y_1)}, \mu_{S_2}^{(y_1,z_2)}\}\min\{\mu_{R_2}^{(x_2,y_2)}, \mu_{S_2}^{(y_2,z_2)}$$

$= \max\{\min(0.2, 0.3), \min(0.2, 0.3)\}$

$= \max(0.2, 0.2)$

$= 0.2$

for (x_2, z_3)

$\mu_T^{(x_2,z_3)} = \max\{\min(0.2, 0.5), \min(0.2, 0.9)\}$

$= \max(0.2, 0.2)$

$= 0.2$

for (x_3, z_1)

$$\mu_T^{(x_3,z_1)} = \max\{\min\{\mu_{\underset{\sim}{R}}^{(x_3,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,z_1)}\}, \min\{\mu_{\underset{\sim}{R}}^{(x_3,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,z_1)}\}\}$$

$$= \max\{\min(0.3, 0.2), \min(0.3, 0.2)\}$$

$$= \max\{0.2, 0.2\}$$

$$= 0.2$$

for (x_3, z_2)

$$\mu_T^{(x_3,z_2)} = \max\{\min\{\mu_{\underset{\sim}{R}}^{(x_3,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,z_2)}\}, \min\{\mu_{\underset{\sim}{R}}^{(x_3,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,z_2)}\}\}$$

$$= \max\{\min(0.3, 0.3), \min(0.3, 0.3)\}$$

$$= \max(0.3, 0.3)$$

$$= 0.3$$

for (x_3, z_3)

$$\mu_T^{(x_3,z_3)} = \max\{\min\{\mu_{\underset{\sim}{R}}^{(x_3,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,z_3)}\}, \min\{\mu_{\underset{\sim}{R}}^{(x_3,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,z_3)}\}\}$$

$$= \max\{\min(0.3, 0.5), \min(0.3, 0.9)\}$$

$$= \max(0.3, 0.3)$$

$$= 0.3$$

Therefore,

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.3 \end{bmatrix} \end{array} \, //$$

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(1) $\underset{\sim}{T} = \underset{\sim}{S} \circ \underset{\sim}{R}$

$$\underset{\sim}{S} = \left\{ \frac{0.2}{(y_1,z_1)}, \frac{0.3}{(y_1,z_2)}, \frac{0.5}{(y_1,z_3)}, \frac{0.2}{(y_2,z_1)}, \frac{0.3}{(y_2,z_2)}, \frac{0.9}{(y_2,z_3)} \right\}$$

$$\underset{\sim}{R} = \left\{ \frac{0.4}{(x_1,y_1)}, \frac{0.4}{(x_1,y_2)}, \frac{0.2}{(x_2,y_1)}, \frac{0.2}{(x_2,y_2)}, \frac{0.3}{(x_3,y_1)}, \frac{0.3}{(x_3,y_2)} \right\}$$

So,

for (z_1,x_1)

$$\mu_T^{(z_1,x_1)} = \max\left\{\min\left\{\mu_{\underset{\sim}{R}}^{(z_1,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,x_1)}\right\}, \min\left\{\mu_{\underset{\sim}{R}}^{(z_1,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,x_1)}\right\}\right\}$$

$= \max\{\min(0.2, 0.4), \min(0.2, 0.4)\}$

$= \max(0.2, 0.2)$

$= 0.2$

for (z_1,x_2)

$$\mu_T^{(z_1,x_2)} = \max\left\{\min\left\{\mu_{\underset{\sim}{R}}^{(z_1,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,x_2)}\right\}, \min\left\{\mu_{\underset{\sim}{R}}^{(z_1,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,x_2)}\right\}\right\}$$

$= \max\{\min(0.2, 0.2), \min(0.2, 0.2)\}$

$= \max(0.2, 0.2)$

$= 0.2$

for (z_1,x_3)

$$\mu_T^{(z_1,x_3)} = \max\left\{\min\left\{\mu_{\underset{\sim}{A}}^{(z_1,y_1)}, \mu_{\underset{\sim}{S}}^{(y_1,x_3)}\right\}, \min\left\{\mu_{\underset{\sim}{A}}^{(z_1,y_2)}, \mu_{\underset{\sim}{S}}^{(y_2,x_3)}\right\}\right\}$$

$= \max\{\min(0.2, 0.3), \min(0.2, 0.3)\}$

$= \max(0.2, 0.2)$

$= 0.2$

for (z_2, x_1)

$$\mu_T^{(z_2,x_1)} = \max\left\{\min\left\{\mu_{R_2}^{(z_2,y_1)}, \mu_{S_2}^{(y_1,x_1)}\right\}, \min\left\{\mu_{R_2}^{(z_2,y_2)}, \mu_{S_2}^{(y_2,x_1)}\right\}\right\}$$

$$= \max\{\min(0.3, 0.4), \min(0.3, 0.2)\}$$
$$= \max\{0.3, 0.2\}$$
$$= 0.3$$

for (z_2, x_2)

$$\mu_T^{(z_2,x_2)} = \max\left\{\min\left\{\mu_{R_2}^{(z_2,y_1)}, \mu_{S_2}^{(y_1,x_2)}\right\}, \min\left\{\mu_{R_2}^{(z_2,y_2)}, \mu_{S_2}^{(y_2,x_2)}\right\}\right\}$$

$$= \max\{\min(0.3, 0.2), \min(0.3, 0.2)\}$$
$$= \max(0.2, 0.2)$$
$$= 0.2$$

for (z_2, x_3)

$$\mu_T^{(z_2,x_3)} = \max\left\{\min\left\{\mu_{R_2}^{(z_2,y_1)}, \mu_{S_2}^{(y_1,x_3)}\right\}, \min\left\{\mu_{R_2}^{(z_2,y_2)}, \mu_{S_2}^{(y_2,x_3)}\right\}\right\}$$

$$= \max\{\min(0.3, 0.3), \min(0.3, 0.3)\}$$
$$= \max\{0.3, 0.3)$$
$$= 0.3$$

for (z_3, x_1)

$$\mu_T^{(z_3,x_1)} = \max\left\{\min\left\{\mu_{R_2}^{(z_3,y_1)}, \mu_{S_2}^{(y_1,x_1)}\right\}, \min\left\{\mu_{R_2}^{(z_3,y_2)}, \mu_{S_2}^{(y_2,x_1)}\right\}\right\}$$

$$= \max\{\min(0.5, 0.4), \min(0.9, 0.4)\}$$

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$= \max(0.4, 0.4)$
$= 0.4$

for (z_3, x_2)

$$\mu_T^{(z_3,x_2)} = \max\{\min\{\mu_{R_2}^{(z_3,y_1)}, \mu_{S_2}^{(y_1,x_2)}\}, \min\{\mu_{R_2}^{(z_3,y_2)}, \mu_{S_2}^{(y_2,x_2)}\}\}$$

$= \max\{\min(0.5, 0.2), \min(0.9, 0.2)\}$
$= \max\{0.2, 0.2\}$
$= 0.2$

for (z_3, x_3)

$$\mu_T^{(z_3,x_3)} = \max\{\min\{\mu_{R_2}^{(z_3,y_1)}, \mu_{S_2}^{(y_1,x_3)}\}, \min\{\mu_{R_2}^{(z_3,y_2)}, \mu_{S_2}^{(y_2,x_3)}\}\}$$

$= \max\{\min(0.5, 0.3), \min(0.9, 0.3)\}$
$= \max(0.3, 0.3)$
$= 0.3$

Therefore, $T = S \circ R =$

$$\begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} \end{array}$$

Thus, $R \circ S \neq S \circ R$

$$\begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.3 \end{bmatrix} \neq \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

* Consider a model for Predicting score in cricket:

Suppose the fuzzy sets are,

Speed of bowling = $\left\{ \frac{0.6}{\text{Fast}}, \frac{0.8}{\text{medium}}, \frac{0.9}{\text{slow}} \right\}$

Condition on pitch = $\left\{ \frac{0.9}{\text{good wicket}}, \frac{0.5}{\text{fair wicket}}, \frac{0.2}{\text{rough wicket}} \right\}$

Condition of runs = $\left\{ \frac{0.9}{\text{Low run}}, \frac{1}{\text{average run}}, \frac{0.7}{\text{high run}} \right\}$

now,

i) Determine impact of pitch condition on bowling (I_1)

ii) Determine impact of bowling on runs (I_2)

iii) Determine impact of pitch conditions on runs using I_1 & I_2

= Sol

consider

$X = \{x_1, x_2, x_3\}$ = {Fast, medium, slow}

$Y = \{y_1, y_2, y_3\}$ = {good wicket, fair wicket, rough wicket}

$Z = \{z_1, z_2, z_3\}$ = {low run, average run, high run}

so,

$$B = \left\{ \frac{0.6}{x_1}, \frac{0.8}{x_2}, \frac{0.9}{x_3} \right\}$$

$$P = \left\{ \frac{0.9}{y_1}, \frac{0.5}{y_2}, \frac{0.2}{y_3} \right\}$$

$$R = \left\{ \frac{0.9}{z_1}, \frac{1}{z_2}, \frac{0.7}{z_3} \right\}$$

① We need determine impact of pitch condition on bowling (I_1):

$$I_1 = \underset{\sim}{R} \subseteq \underset{\sim}{P} \times \underset{\sim}{B}$$

So, for (y_1, x_1)

$$\mu_{\underset{\sim}{R}}^{(y_1,x_1)} = \min\left(\mu_{\underset{\sim}{P}}^{(y_1)}, \mu_{\underset{\sim}{B}}^{(x_1)}\right)$$

$$= \min(0.9, 0.6)$$

$$= 0.6$$

for (y_1, x_2)

$$\mu_{\underset{\sim}{R}}^{(y_1,x_2)} = \min(0.9, 0.8)$$

$$= 0.8$$

for (y_1, x_3)

$$\mu_{\underset{\sim}{R}}^{(y_1,x_3)} = \min(0.9, 0.9)$$

$$= 0.9$$

for $(y_2, x_1) = \min(0.5, 0.6)$

$$= 0.5$$

for $(y_2, x_2) = \min(0.5, 0.8)$

$$= 0.5$$

for $(y_2, x_3) = \min(0.5, 0.9)$

$$= 0.5$$

for $(y_3, x_1) = \min(0.2, 0.6)$

$$= 0.2$$

for $(y_3, x_2) = \min(0.2, 0.8)$

$$= 0.2$$

for $(y_3, x_3) = \min(0.2, 0.9)$

$$= 0.2$$

$$\therefore \underset{\sim}{R} = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} 0.6 & 0.8 & 0.9 \\ 0.5 & 0.5 & 0.5 \\ 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

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(11) We need find the impact of bowling on runs (I_2):

$$\underset{\sim}{S} \subseteq \underset{\sim}{B} \times \underset{\sim}{R}$$

for (x_1, z_1)

$$\mu_{\underset{\sim}{S}}^{(x_1,z_1)} = \min\left(\mu_{\underset{\sim}{B}}^{(x_1)}, \mu_{\underset{\sim}{R}}^{(z_1)}\right)$$

$= \min(0.6, 0.9)$

$= 0.6$

for $(x_1, z_2) = \min(0.6, 1)$

$= 0.6$

for $(x_1, z_3) = \min(0.6, 0.7)$

$= 0.6$

for $(x_2, z_1) = \min(0.8, 0.9)$

$= 0.8$

for $(x_2, z_2) = \min(0.8, 1)$

$= 0.8$

for $(x_2, z_3) = \min(0.8, 0.7)$

$= 0.7$

for $(x_3, z_1) = \min(0.9, 0.9)$

$= 0.9$

for $(x_3, z_2) = \min(0.9, 1)$

$= 0.9$

for $(x_3, z_3) = \min(0.9, 0.7)$

$= 0.7$

$$\therefore \underset{\sim}{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.8 & 0.8 & 0.7 \\ 0.9 & 0.9 & 0.7 \end{bmatrix} \end{array}$$

③ To find the impact of Pitch condition (P) on runs (R) using I1 & I2.

$\underset{\sim}{T} = \underset{\sim}{R} \cdot \underset{\sim}{S} = I_1 \cdot I_2$

So,

$$\underset{\sim}{R} = \left\{ \frac{0.6}{(y_1,x_1)}, \frac{0.8}{(y_1,x_2)}, \frac{0.9}{(y_1,x_3)}, \frac{0.5}{(y_2,x_1)}, \frac{0.5}{(y_2,x_2)}, \frac{0.5}{(y_2,x_3)}, \frac{0.2}{(y_3,x_1)}, \frac{0.2}{(y_3,x_2)}, \frac{0.2}{(y_3,x_3)} \right\}$$

$$\underset{\sim}{S} = \left\{ \frac{0.6}{(x_1,z_1)}, \frac{0.6}{(x_1,z_2)}, \frac{0.6}{(x_1,z_3)}, \frac{0.8}{(x_2,z_1)}, \frac{0.8}{(x_2,z_2)}, \frac{0.7}{(x_2,z_3)}, \frac{0.9}{(x_3,z_1)}, \frac{0.9}{(x_3,z_2)}, \frac{0.7}{(x_3,z_3)} \right\}$$

for (y_1,z_1).

$$\mu_{\underset{\sim}{T}}^{(y_1,z_1)} = \max\left\{\min\left\{\mu_{\underset{\sim}{R}}^{(y_1,x_1)}, \mu_{\underset{\sim}{S}}^{(x_1,z_1)}\right\}, \min\left\{\mu_{\underset{\sim}{R}}^{(y_1,x_2)}, \mu_{\underset{\sim}{S}}^{(x_2,z_1)}\right\}, \min\left\{\mu_{\underset{\sim}{R}}^{(y_1,x_3)}, \mu_{\underset{\sim}{S}}^{(x_3,z_1)}\right\}\right\}$$

$$= \max\left(\min(0.6,0.6), \min(0.8,0.8), \min(0.9,0.9)\right\}$$

$$= \max(0.6, 0.8, 0.9)$$

$$= 0.9$$

for (y_1,z_2)

$$\mu_{\underset{\sim}{T}}^{(y_1,z_2)} = \max\left\{\min(0.6,0.6), \min(0.8,0.8), \min(0.9,0.9)\right\}$$

$$= \max(0.6, 0.8, 0.9)$$

$$= 0.9$$

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for (y_1, z_3)

$$\mu_{T_2}^{(y_1,z_3)} = \max\{\min(0.6,0.6), \min(0.8,0.7), \min(0.9,0.7)\}$$

$$= \max(0.6, 0.7, 0.7)$$
$$= 0.7$$

now, for (y_2, z_1)

$$\mu_{T_2}^{(y_2,z_1)} = \max\{\min(0.5,0.6), \min(0.5,0.8), \min(0.5,0.9)\}$$

$$= \max(0.5, 0.5, 0.5)$$
$$= 0.5$$

for (y_2, z_2)

$$\mu_{T_2}^{(y_2,z_2)} = \max\{\min(0.5,0.6), \min(0.5,0.8), \min(0.5,0.9)\}$$

$$= \max(0.5, 0.5, 0.5)$$
$$= 0.5$$

for (y_2, z_3)

$$\mu_{T_2}^{(y_2,z_3)} = \max\{\min(0.5,0.6), \min(0.5,0.7), \min(0.5,0.7)\}$$
$$= \max(0.5, 0.5, 0.5)$$
$$= 0.5$$

for (y_3, z_1)

$$\mu_{\underset{\sim}{T}}^{(y_3,z_1)} = \max\{\min(0.2, 0.6), \min(0.2, 0.8), \min(0.2, 0.9)\}$$

$$= \max(0.2, 0.2, 0.2)$$
$$= 0.2$$

for (y_3, z_2)

$$\mu_{\underset{\sim}{T}}^{(y_3,z_2)} = \max\{\min(0.2, 0.6), \min(0.2, 0.8), \min(0.2, 0.9)\}$$

$$= \max(0.2, 0.2, 0.2)$$
$$= 0.2$$

for (y_3, z_3)

$$\mu_{\underset{\sim}{T}}^{(y_3,z_3)} = \max\{\min(0.2, 0.6), \min(0.2, 0.7), \min(0.2, 0.7)\}$$

$$= \max(0.2, 0.2, 0.2)$$
$$= 0.2$$

Therefore, $I_1 \circ I_2 = \underset{\sim}{R} \circ \underset{\sim}{S}$

$$= \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.9 & 0.7 \\ 0.5 & 0.5 & 0.5 \\ 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array} \#$$

* One to one Mapping:

Let $F(\cdot)$ be a mapping from the fuzzy universal set X to another fuzzy universal set Y. Suppose $\underset{\sim}{A}$ & $\underset{\sim}{B}$ are subsets of X & Y respectively.

Consider,

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1}, \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2}, \dots, \frac{\mu_{\underset{\sim}{A}}(x_n)}{x_n} \right\}$$

If there is a one-to-one mapping from x_i to $y_i = f(x_i)$ (∵ fuzzy arithmetic function)

then,

$\underset{\sim}{B}$ is given by;

$$\underset{\sim}{B} = \left\{ \frac{\mu_{\underset{\sim}{B}}(y_1)}{y_1}, \frac{\mu_{\underset{\sim}{B}}(y_2)}{y_2}, \dots, \frac{\mu_{\underset{\sim}{B}}(y_n)}{y_n} \right\}$$

where membership of y_i in $\underset{\sim}{B} = \mu_{\underset{\sim}{B}}(y_i)$ }

and,

$$y_1 = F(x_1) \text{ \& } \mu_{\underset{\sim}{B}}(y_1) = \mu_{\underset{\sim}{A}}(x_1)$$

$$y_2 = F(x_2) \text{ \& } \mu_{\underset{\sim}{B}}(y_2) = \mu_{\underset{\sim}{A}}(x_2)$$

$$\vdots$$

$$y_n = F(x_n) \text{ \& } \mu_{\underset{\sim}{B}}(y_n) = \mu_{\underset{\sim}{A}}(x_n)$$

$$y = F(x)$$

x_1 •——• y_1
x_2 •——• y_2
x_3 •——• y_3
⋮
x_n •——• y_n

Example:

$$\underset{\sim}{A} = \left\{ \frac{0.2}{2}, \frac{0.1}{4}, \frac{1}{6} \right\}$$

now, $\underset{\sim}{B}$ is defined by $\forall y \in \underset{\sim}{B}$, $y = F(x) = x^2$ $\forall x \in \underset{\sim}{A}$

then,

$$\underset{\sim}{B} = \left\{ \frac{0.2}{4}, \frac{0.1}{16}, \frac{1}{36} \right\}$$

2 —— 4

4 —— 16

6 —— 36

* Many to one Mapping:

Let $F(\cdot)$ be a mapping from the fuzzy universal set X to another fuzzy universal set Y. Suppose $\underset{\sim}{A}$ & $\underset{\sim}{B}$ are subsets of X and Y respectively.

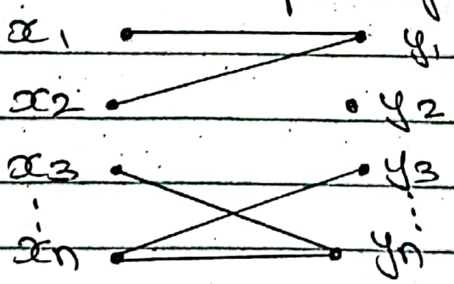
Consider,

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1}, \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2}, \ldots, \frac{\mu_{\underset{\sim}{A}}(x_n)}{x_n} \right\}$$

If there is a many to one mapping from x_i to $y_i = F(x_i)$ then $\underset{\sim}{B}$ is given by;

$$\underset{\sim}{B} = \left\{ \frac{\mu_{\underset{\sim}{B}}(y_1)}{y_1}, \frac{\mu_{\underset{\sim}{B}}(y_2)}{y_2}, \ldots, \frac{\mu_{\underset{\sim}{B}}(y_n)}{y_n} \right\}$$

where membership of y_i in $\underset{\sim}{B}$ $= \text{MAX}\left[\mu_{A}(x_i) / x_i \in F^{-1}(y_i)\right]$



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Example:

$$\underset{\sim}{A} = \left\{ \frac{0.2}{-1}, \frac{0.4}{-2}, \frac{0.6}{1}, \frac{0.8}{2}, \frac{0.9}{3} \right\}$$

now, $\underset{\sim}{B}$ is defined by $\forall y \in \underset{\sim}{B}$ $y = F(x) = x^2$ $\forall x \in \underset{\sim}{A}$.

So,

$$\underset{\sim}{B} = \left\{ \frac{MAX(\mu_{\underset{\sim}{A}}^{(-1)}, \mu_{\underset{\sim}{A}}^{(1)})}{1}, \frac{MAX(\mu_{\underset{\sim}{A}}^{(-2)}, \mu_{\underset{\sim}{A}}^{(2)})}{2}, \frac{Max(\mu_{\underset{\sim}{A}}^{(3)})}{9} \right\}$$

$$= \left\{ \frac{MAX(0.2, 0.6)}{1}, \frac{MAX(0.4, 0.8)}{2}, \frac{MAX(0.9)}{3} \right\}$$

$$\therefore \underset{\sim}{B} = \left\{ \frac{0.6}{1}, \frac{0.8}{2}, \frac{0.9}{3} \right\}$$

* Projection of a Fuzzy relation:

① X-Projection:

The Projection of $R(x,y)$ on X is defined as R_1 and given by:

$$\mu_{R_1}^{(x)} = \max_{y \in Y} \left(\mu_R^{(x,y)} \right)$$

② Y-Projection:

The Projection of $R(x,y)$ on Y is defined as R_2 & given by:

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$$\mu_{R_2}(y) = \max_{x \in X} \left(\mu_R(x,y) \right)$$

Example:

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0.43 \\ 0.43 & 1 \\ 0.16 & 0.42 \end{bmatrix} \end{array}$$

i.e, $R = \left\{ \frac{1}{(1,4)}, \frac{0.43}{(1,5)}, \frac{0.43}{(2,4)}, \frac{1}{(2,5)}, \frac{0.16}{(3,4)}, \frac{0.42}{(3,5)} \right\}$

now,

X- Projection of R is R1 where μ_{R_1} is defined as:

$\mu_{R_1}(1) = \max(\mu_R(1,4), \mu_R(1,5)) = \max(1, 0.43) = 1$

$\mu_{R_1}(2) = \max(\mu_R(2,4), \mu_R(2,5)) = \max(0.43, 1) = 1$

$\mu_{R_1}(3) = \max(\mu_R(3,4), \mu_R(3,5)) = \max(0.16, 0.42) = 0.42$

Thus, X- Projection of R is

$$R_1 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 \\ 1 \\ 0.42 \end{bmatrix}$$

i.e, $R1 = \left[\frac{1}{1}, \frac{1}{2}, \frac{0.42}{3} \right]$

or, $R_1 = \{(1,1), (2,1), (3, 0.42)\}$

now,

Y- Projection of R is R_2 where μR_2 is defined as:

$$\mu_{R_2}^{(4)} = MAX\left(\mu_{R_2}^{(1,4)}, \mu_{R_2}^{(2,4)}, \mu_{R_2}^{(3,4)}\right)$$

$$= MAX(1, 0.43, 0.16)$$

$$= 1$$

$$\mu R_2^{(5)} = MAX\left(\mu_{R_2}^{(1,5)}, \mu_{R_2}^{(2,5)}, \mu_{R_2}^{(3,5)}\right)$$

$$= MAX(0.43, 1, 0.42)$$

$$= 1$$

Thus, Y-Projection of R is,

$$R_2 = \begin{array}{cc} 4 & 5 \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

i.e. $R_2 = \left\{ \frac{1}{4}, \frac{1}{5} \right\}$

or, $R_2 = \{ (4,1), (5,1) \}$,,

* Cylindrical Extension of Fuzzy relation:

(i) Cylindrical Extension w.r.t X-Projection:

$$cyl_A(x,y) = \{ (x,y) \mid (x,y) \in R \}$$

and,

$$\mu_{cyl_A}^{(x,y)} = \mu_{X\text{-}Projection}^{(x)}.$$

(ii) Cylindrical extension w.r.t Y-Projection:

$$Cyl_A(x,y) = \{(x,y) \mid (x,y) \in R$$

and

$$\mu_{cyl_A}(x,y) = \mu_{y\text{-Projection}}(y)$$

Thus, $Cyl_A(R)$ from X-Projection:

$$\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 0.42 & 0.42 \end{array}$$

Also, $Cyl_A(R)$ from Y-Projection:

$$= \begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{array}$$

* Reflection Relation:

A relation R over X×X is said to be reflective if $\mu(x_i, x_i) = 1$.

For, $x = \{1, 2, 3\}$

$$R_1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0.9 & 0.6 \\ 2 & 0.9 & 1 & 0.5 \\ 3 & 0.6 & 0.6 & 1 \end{array}$$

i.e. $R(x_i, x_j) = 1, \forall x_i = x_j \in X$

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* Anti-reflexive:

A relation R over $X \times X$ is said to be Anti-reflexive if $\mu(x_i, x_i) = 0$

e.g:- for $X = \{1, 2, 3\}$

$$R_2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0.9 & 0.6 \\ 2 & 0.9 & 0 & 0.5 \\ 3 & 0.6 & 0.6 & 0 \end{array}$$

i.e. $R(x_i, x_j) = 0 \ \forall \ x_i = x_j \in X$

* Symmetric:

A relation R over $X \times X$ is said to be Symmetric if $\mu(x_i, x_j) = \mu(x_j, x_i)$

e.g:-

$$R_3 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.8 & 0.1^* & 0.7^* \\ 2 & 0.1^* & 1 & 0.6^\circ \\ 3 & 0.7^* & 0.6^\circ & 0.5 \end{array}$$

* Anti-Symmetric:

A relation R over $X \times X$ is said to be Anti-Symmetric if $\mu_R(x_i, x_j) > 0$ then $\mu_R(x_j, x_i) = 0$ for $x_i, x_j \in X$ $\& \ x_i \neq x_j$

e.g:-

$$R_4 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0.7 \\ 2 & 0.2 & 0 & 0 \\ 3 & 0 & 0.1 & 0 \end{array} \quad \text{or,} \quad R_5 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0.6 & 0.5 & 0.7 \\ 3 & 0.7 & 0 & 0.2 \end{array}$$

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* Transitive Relation:

A relation R over X×X is said to be Transitive if $\mu_R(x_i, x_j) = \lambda_1$ & $\mu_R(x_j, x_k) = \lambda_2$ then,

$$\mu_R^{(x_i, x_k)} = \lambda \text{ such that } \lambda \geq \min(\lambda_1, \lambda_2)$$

i.e, $\mu_R^{(x_i, x_k)} \geq \text{MIN}\left(\mu_R^{(x_i, x_j)}, \mu_R^{(x_j, x_k)}\right)$

Note: For crisp relation :

If $\mu_R^{(x_i, x_j)} = 1$

& $\mu_R^{(x_j, x_k)} = 1$

then, $\mu_R^{(x_i, x_k)} = 1$

else

0

eg:- For $X = \{x_1, x_2, x_3, x_4, x_5\}$

$$R_1 = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 1 & 0.8 & 0 & 0.1 & 0.2 \\ x_2 & 0.8 & 1 & 0.4 & 0 & 0.9 \\ x_3 & 0 & 0.4 & 1 & 0 & 0 \\ x_4 & 0.1 & 0 & 0 & 1 & 0.5 \\ x_5 & 0.2 & 0.9 & 0 & 0.5 & 1 \end{array}$$

here,

$\mu_{R_1}^{(x_1, x_2)} = 0.8$

$\mu_{R_1}^{(x_2, x_5)} = 0.9$

$\mu_{R_1}^{(x_1, x_5)} = 0.2$

i.e, $0.2 \ngeq \min(0.8, 0.9)$, Thus R_1 is not Transitive.

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$$R_2 = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 1 & 0.8 & 0.4 & 0.5 & 0.8 \\ x_2 & 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ x_3 & 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ x_4 & 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ x_5 & 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{array}$$

here,

$\mu_{R_2}(x_1, x_5) = 0.8$
$\mu_{R_2}(x_1, x_2) = 0.8$
$\mu_{R_2}(x_2, x_5) = 0.9$

i.e, $0.8 \geq \min(0.8, 0.9)$

Thus, R_2 is Transitive.

* Equivalence relation:

A relation R is said to be equivalence relation, if it satisfies reflective, symmetric and transitive relation.

* Fuzzy Tolerance Relation:

If the relation is reflective, symmetric but not transitive, then we can make it transitive by doing n-1 composition with R and its resulted composition if it transformed to transitive then it's tolerance relation.
where,

n is cardinal of domain of Discourse.

Example:

For $X = \{x_1, x_2, x_3, x_4, x_5\}$

$x_5 \rightarrow$ n i.e cardinal of domain of discourse.

$$R = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 1 & 0.8 & 0 & 0.1 & 0.2 \\ x_2 & 0.8 & 1 & 0.4 & 0 & 0.9 \\ x_3 & 0 & 0.4 & 1 & 0 & 0 \\ x_4 & 0.1 & 0 & 0 & 1 & 0.5 \\ x_5 & 0.2 & 0.9 & 0 & 0.5 & 1 \end{array}$$

Using max-min composition, find above relation is tolerance relation or not.

=> Sol. To be a Fuzzy Tolerance relation it must satisfy Reflective & symmetric property too.

(i) For Reflective:

$\mu_R(x_i, x_i) = 1$ i.e,

$\mu_R(x_1, x_1) = 1$

$\mu_R(x_2, x_2) = 1$

$\mu_R(x_3, x_3) = 1$

$\mu_R(4,4) = 1$
$\mu_R(5,5) = 1$

This satisfies reflexive property.

(2) For Symmetric:

$\mu_{\tilde{R}}(x_i,x_j) = \mu_{\tilde{R}}(x_j,x_i)$ i.e,

$\mu_R(x_1,x_2) = \mu_R(x_2,x_1) = 0.8$
$\mu_R(x_1,x_3) = \mu_R(x_3,x_1) = 0$
$\mu_R(x_1,x_4) = \mu_R(x_4,x_1) = 0.1$
$\mu_R(x_1,x_5) = \mu_R(x_5,x_1) = 0.2$
$\mu_R(x_2,x_3) = \mu_R(x_3,x_2) = 0.4$
$\mu_R(x_2,x_4) = \mu_R(x_4,x_2) = 0$
$\mu_R(x_2,x_5) = \mu_R(x_5,x_2) = 0.9$
$\mu_R(x_3,x_4) = \mu_R(x_4,x_3) = 0$
$\mu_R(x_3,x_5) = \mu_R(x_5,x_3) = 0$
$\mu_R(x_4,x_5) = \mu_R(x_5,x_4) = 0.5$

This satisfies reflexive property.

(3) For Transitive,

$\mu_R(x_1,x_5) = 0.2$
$\mu_R(x_1,x_2) = 0.8$
$\mu_R(x_2,x_5) = 0.9$

So,

$0.2 \neq \min(0.8, 0.9)$ This is not Transitive So, we need to use n-1 max-min composition i.e,

$n=5$

$n=4$ composition.

then new R_1 is tolerance relation.

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The possible composition are:

$$R_1^2 = R_1 \circ R_1$$
$$R_1^3 = R_1^2 \circ R_1^2$$
$$R_1^4 = R_1 \circ R^3$$
$$R_1^5 = R_1 \circ R^5$$

① For $R_1^2 = R_1 \circ R_1$

$$R_1 = \left\{ \frac{1}{(x_1,x_1)}, \frac{0.8}{(x_1,x_2)}, \frac{0}{(x_1,x_3)}, \frac{0.1}{(x_1,x_4)}, \frac{0.2}{(x_1,x_5)}, \dots, \frac{0.5}{(x_5,x_4)}, \frac{1}{(x_5,x_5)} \right\}$$

(i) For (x_1,x_1)

$$\mu_{R_1^2}^{(x_1,x_1)} = \text{Max}\left\{ \min\left\{\mu_{R_1}^{(x_1,x_1)}, \mu_{R_1}^{(x_1,x_1)}\right\}, \min\left\{\mu_{R_1}^{(x_1,x_2)}, \mu_{R_1}^{(x_2,x_1)}\right\}, \right.$$
$$\min\left\{\mu_{R_1}^{(x_1,x_3)}, \mu_{R_1}^{(x_3,x_1)}\right\}, \min\left\{\mu_{R}^{(x_1,x_4)}, \mu_{R}^{(x_4,x_1)}\right\},$$
$$\left. \min\left\{\mu_{R_1}^{(x_1,x_5)}, \mu_{R_1}^{(x_5,x_1)}\right\} \right\}$$

$$= \max\{\min(1,1), \min(0.8, 0.8), \min(0,0), \min(0.1, 0.1),$$
$$\min(0.2, 0.2)\}$$
$$= \max\{1, 0.8, 0, 0.1, 0.2\}$$
$$= 1$$

(ii) for (x_1,x_2)

$$\mu_{R_1^2}^{(x_1,x_2)} = \max\left\{ \min\left\{\mu_{R_1}^{(x_1,x_1)}, \mu_{R_1}^{(x_1,x_2)}\right\}, \min\left\{\mu_{R_1}^{(x_1,x_2)}, \mu_{R_1}^{(x_2,x_2)}\right\}, \right.$$
$$\min\left\{\mu_{R_1}^{(x_1,x_3)}, \mu_{R_1}^{(x_3,x_2)}\right\}, \min\left\{\mu_{R_1}^{(x_1,x_4)}, \mu_{R_1}^{(x_4,x_2)}\right\},$$
$$\left. \min\left\{\mu_{R_1}^{(x_1,x_5)}, \mu_{R_1}^{(x_5,x_2)}\right\} \right\}$$

$$= \max\{\min(1, 0.8), \min(0.8, 1), \min(0, 0.4), \min(0.1, 0),$$
$$\min(0.2, 0.9)\}$$
$$= 0.8$$

$x_1 \to$ row
$x_3 \to$ column } min } max

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(iii) for $(x_1, x_3) = 0.4$
(iv) for $(x_1, x_4) = 0.2$
(v) for $(x_1, x_5) = 0.8$

Similarly for

(x_2, x_1)
(x_2, x_2)
(x_2, x_3)
⋮
$(x_5, x_5) = ?$

(x_2, x_5)
$(x_2, x_1), (x_1, x_5) \Rightarrow (0.8, 0.2) \Rightarrow 0.2$
$(x_2, x_2), (x_2, x_5) \Rightarrow (1, 0.9) \Rightarrow 0.9$
$(x_2, x_3), (x_3, x_5) \Rightarrow (0.4, 0) \Rightarrow 0$
$(x_2, x_4), (x_4, x_5) \Rightarrow (0, 0.5) \Rightarrow 0$
$(x_2, x_5), (x_5, x_5) \Rightarrow (0.9, 1) \Rightarrow 0.9$
0.9 max

Therefore,

$$R^2_{1_2} = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 1 & 0.8 & 0.4 & 0.2 & 0.8 \\ x_2 & 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ x_3 & 0.4 & 0.4 & 1 & 0 & 0.4 \\ x_4 & 0.2 & 0.5 & 0 & 1 & 0.5 \\ x_5 & 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{array}$$

Since, $R^2_{1_2}$ is reflective & symmetric.

To check transitive,

$\mu R^2_1 (x_1, x_4) = 0.2$
$\mu R^2_1 (x_1, x_2) = 0.8$
$\mu R^2_1 (x_2, x_4) = 0.5$

i.e, $0.2 \not\geq \min(0.8, 0.5)$

So, $R^2_{1_2}$ is not transitive,

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now, we should perform next composition again, i.e

$$\underset{\sim}{R_1^3} = R_1 \circ R_1^2$$

Similarly as R_1^2
we get,

$$\underset{\sim}{R_1^3} = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 1 & 0.8 & 0.4 & 0.5 & 0.8 \\ x_2 & 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ x_3 & 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ x_4 & 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ x_5 & 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{array}$$

$\underset{\sim}{R_1^3}$ is reflective & symmetric,

for transitive,

$$\mu_{\underset{\sim}{R_1^3}}(x_1, x_5) = 0.8$$

$$\mu_{\underset{\sim}{R_1^3}}(x_1, x_2) = 0.8$$

$$\mu_{\underset{\sim}{R_1^3}}(x_2, x_5) = 0.9$$

i.e, $0.8 \geq \min(0.8, 0.9)$ (True)

Thus, R_1^3 is reflective, symmetric & transitive.
So, it is a Fuzzy Tolerance relation.

* Extension principle:

Consider domain X and Y

and $y = f(x)$ $\forall$ $y \in Y$ and $x \in X$

If for each $x \in A$

$y \in B$ and $y = f(x)$

Then,

$$\mu_B(y) = \mathrm{Max}\left[\min\left[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\right]\right]$$

$$y = f(x_1, x_2, \ldots, x_n)$$

This is called Zadeh Extension Principle.

Example:

$$\underset{\sim}{\text{Approx } 4} = \left\{\frac{0.6}{2}, \frac{1}{4}, \frac{0.3}{6}\right\}$$

$$\underset{\sim}{\text{near } 8} = \left\{\frac{0.3}{4} + \frac{0.4}{12} + \frac{1}{8} + \frac{0.01}{2}\right\}$$

For a set $\underset{\sim}{C} = \dfrac{\text{near } 8}{\text{Approx } 4}$

Then,

$$\mu_C(1) = \max\left\{\left(\min\left\{\mu_{A4}(2), \mu_{n8}(2)\right\}, \min\left\{\mu_{A4}(4), \mu_{n8}(4)\right\}\right\}$$

$$= \max\left\{\min(0.6, 0.01), \min(1, 0.3)\right\}$$

$$= \max(0.01, 0.3)$$

$$= 0.3$$

$$\mu_C(2) = \max\left\{\min\left(\mu_{A4}(2), \mu_{n8}(4)\right), \min\left(\mu_{A4}(4), \mu_{n8}(8)\right), \min\left(\mu_{A4}(6), \mu_{n8}(12)\right)\right\}$$

$= \max\{\min(0.6, 0.3), \min(1,1), \min(0.3, 0.4)\}$

$= \max(0.3, 1, 0.3)$

$= 1$

$\mu_C^{(3)} = \max\{\min(\mu_{A_4}^{(4)}, \mu_{B_8}^{(12)}\}$

$= \max\{\min(1, 0.4)\}$

$= 0.4$

$\mu_C^{(4)} = \max\{\min(\mu_{A_4}^{(2)}, \mu_{B_8}^{(8)}\}$

$= \max\{\min(0.6, 1)\}$

$= 0.6$

$\mu_C^{(6)} = \max\{\min(\mu_{A_4}^{(2)}, \mu_{B_8}^{(12)}\}$

$= \max\{\min(0.6, 0.4)\}$

$= 0.4$

Therefore,

$C = \left\{\frac{0.3}{1}, \frac{1}{2}, \frac{0.4}{3}, \frac{0.6}{4}, \frac{0.4}{6}\right\}$

$* \quad F(A, B, C) = x^2 + y + z \quad \forall x \in A, y \in B \ \& \ z \in C$

$\downarrow$

$r_i = x_i^2 + y_i + z_i$

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* Fuzzy Transform:

For any two fuzzy set, consider $f: \underset{\sim}{A} \to \underset{\sim}{B}$. Here, f determines fuzzy transform, which determines or defines a mapping from $\underset{\sim}{A}$ to $\underset{\sim}{B}$ i.e. an element x in $\underset{\sim}{A}$ to an element y in $\underset{\sim}{B}$.

If $\underset{\sim}{A} \subseteq X$ & $\underset{\sim}{B} \subseteq Y$ are finite then fuzzy mapping is:

$$\underset{\sim}{R} = \begin{array}{c|c} & \begin{array}{ccccc} y_1 & y_2 & \cdots & y_j \cdots & y_m \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} & \left[\begin{array}{ccccc} r_{11} & r_{12} & \cdots & r_{1j} \cdots & r_{1m} \\ r_{21} & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ r_{n1} & & & r_{nj} & r_{nm} \end{array}\right] \end{array}$$

Given mapping $\underset{\sim}{R}$ & fuzzy set $\underset{\sim}{A}$; $\underset{\sim}{B_i} = f(x_i)$ and

$\mu_{\underset{\sim}{B_i}}(y_j) = r_{ij}$ then,

$$\mu_{\underset{\sim}{B}}(y) = \overset{(max)}{\bigvee_{x \in X}} \left(\mu_A(x) \overset{(min)}{\wedge} \mu_R(x,y) \right)$$

$\Rightarrow$ i.e, $B = A \circ R$ (Zadeh Extension principle)

Example:

* consider:

Here, for $\underset{\sim}{A} \subset X = \{40, 50, 60, 70, 80\}$

& $\underset{\sim}{B} \subseteq Y = \{14, 15, 16, 17, 18\}$

& $R: \underset{\sim}{A} \to \underset{\sim}{B}$

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$$\underset{\sim}{A} = \left\{ \frac{0.8}{40} + \frac{1}{50} + \frac{0.6}{60} + \frac{0.2}{70} + \frac{0}{80} \right\}$$

&

$\underset{\sim}{R} =$

	14	15	16	17	18
40	1	0.8	0.2	0.1	0
50	0.8	1	0.8	0.2	0.1
60	0.2	0.8	1	0.8	0.2
70	0.1	0.2	0.8	1	0.8
80	0	0.1	0.2	0.8	1

now,

$B = ?$

$$\underset{\sim}{B} = \underset{\sim}{A} \circ \underset{\sim}{R}$$

⇒ Sol ~~for membership of 14,~~

$\cancel{\mu_{\underset{\sim}{B}}^{(14)} = \max\{\min\{\mu_A}$

for each $y \in \underset{\sim}{B}$

$$\mu_{\underset{\sim}{B}}^{(y)} = \max\{\min(\mu_A^{(x)}, \mu_R^{(x,y)})\}.$$

so, here for membership of 14,

$$\mu_{\underset{\sim}{B}}^{(14)} = \max\{\min\{\mu_{\underset{\sim}{A}}^{(40)}, \mu_{\underset{\sim}{R}}^{(40,14)}\}, \min\{\mu_{\underset{\sim}{A}}^{(50)}, \mu_{\underset{\sim}{R}}^{(50,14)}\},$$
$$\min\{\mu_{\underset{\sim}{A}}^{(60)}, \mu_{\underset{\sim}{R}}^{(60,14)}\}, \min\{\mu_{\underset{\sim}{A}}^{(70)}, \mu_{\underset{\sim}{R}}^{(70,14)}\},$$
$$\min\{\mu_{\underset{\sim}{A}}^{(80)}, \mu_{\underset{\sim}{R}}^{(80,14)}\}\}$$
$$= \max\{\min(0.8, 1), \min(1, 0.8), \min(0.6, 0.2), \min(0.2, 0.1)$$
$$\min(0, 0)\}$$

$= \max(0.8, 0.8, 0.2, 0.1, 0)$
$= 0.8$

for membership of 15,

$$\mu_{B_2}^{(15)} = \max\{\min\{\mu_{A_2}^{(40)}, \mu_{R_2}^{(40,15)}\}, \min\{\mu_{A_2}^{(50)}, \mu_{R_2}^{(50,15)}\}$$
$$\min\{\mu_{A_2}^{(60)}, \mu_{R_2}^{(60,15)}\}, \min\{\mu_{A_2}^{(70)}, \mu_{R_2}^{(70,15)}\}$$
$$\min\{\mu_{A_2}^{(80)}, \mu_{R_2}^{(80,15)}\}\}$$

$= \max\{\min(0.8, 0.8), \min(1, 1), \min(0.6, 0.8), \min(0.2, 0.2)$
$\min(0, 0.1)\}$
$= \max(0.8, 1, 0.6, 0.2, 0)$
$= 1$

for membership of 16

$$\mu_{B_2}^{(16)} = \max\{\min(0.8, 0.2), \min(1, 0.8), \min(0.6, 1), \min(0.2, 1)$$
$$\min(0, 0.2)\}$$

$= \max(0.2, 0.8, 0.6, 0.2, 0)$
$= 0.8$

for membership of 17

$$\mu_{B_2}^{(17)} = \max\{\min(0.8, 0.1), \min(1, 0.2), \min(0.6, 0.8), \min(0.2, 1), \min(0, 0.8)$$
$= \max(0.1, 0.2, 0.6, 0.2, 0)$
$= 0.6$

for membership of 18

$$\mu_{\underset{\sim}{B}}(18) = \max\{\min(0.8, 0), \min(1, 0.1), \min(0.6, 0.2), \min(0.2, 0.8), \min(0, 1)\}$$

$$= \max(0, 0.1, 0.2, 0.2, 0)$$

$$= \cancel{1}\ 0.2$$

Therefore,

$$\underset{\sim}{B} = \left\{ \frac{0.8}{14}, \frac{1}{15}, \frac{0.8}{16}, \frac{0.6}{17}, \frac{0.2}{18} \right\}$$

* Fuzzy Transform in generalized relations:

Consider we want to map ordered pairs from input universes $X_1 = \{a, b\}$ and $X_2 = \{1, 2, 3\}$ to an output universe, $Y = \{x, y, z\}$. The mapping is given by a crisp relation R as;

$$R = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} x & z & x \\ x & y & z \end{bmatrix} \end{array}$$

consider fuzzy sets $\underset{\sim}{A} \subseteq X_1$ and $\underset{\sim}{B} \subseteq X_2$ as;

$$\underset{\sim}{A} = \left\{ \frac{0.6}{a} + \frac{1}{b} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.2}{1} + \frac{0.8}{2} + \frac{0.4}{3} \right\}$$

now, if $\underset{\sim}{R}$ is defined for a relational mapping function,

$$f(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{C}$$

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Then, using max-min extension (zadeh) membership of x, y, z can be determined as;

$$\mu_{\underset{\sim}{C}}(x) = \max\{\min\{\mu_{\underset{\sim}{A}}(a), \mu_{\underset{\sim}{B}}(1)\}, \min\{\mu_{\underset{\sim}{A}}(b), \mu_{\underset{\sim}{B}}(1)\}, \min\{\mu_{\underset{\sim}{A}}(a), \mu_{\underset{\sim}{B}}(3)\}\}$$

$$= \max\{\min(0.6, 0.2), \min(1, 0.2), \min(0.6, 0.4)\}$$
$$= \max(0.2, 0.2, 0.4)$$
$$= 0.4$$

Similarly,

$$\mu_{\underset{\sim}{C}}(y) = \max\{\min\{\mu_{\underset{\sim}{A}}(b), \mu_{\underset{\sim}{B}}(2)\}\}$$
$$= \max\{\min(1, 0.8)\}$$
$$= 0.8$$

$$\mu_{\underset{\sim}{C}}(z) = \max\{\min\{\mu_{\underset{\sim}{A}}(a), \mu_{\underset{\sim}{B}}(2)\}, \min\{\mu_{\underset{\sim}{A}}(b), \mu_{\underset{\sim}{B}}(3)\}\}$$
$$= \max\{\min\{0.6, 0.8\}, \min\{1, 0.4\}\}$$
$$= \max(0.6, 0.4)$$
$$= 0.6$$

Thus, $\underset{\sim}{C} = \left\{\frac{0.4}{x} + \frac{0.8}{y} + \frac{0.6}{z}\right\}$

$$R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0.4 & 0.6 & 0.4 \\ 0.4 & 0.8 & 0.6 \end{bmatrix} \end{array}$$ #

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* α-cut in fuzzy relations:

for $\alpha \in [0,1]$

$$R_\alpha(x_i, y_j) = (x_i, y_j)$$

$$\mu_{R_\alpha}(x_i, y_j) = 1 \quad \text{if } \mu_{\underset{\sim}{R}}(x_i, y_j) \geq \alpha$$

$$0 \quad \text{if } \mu_{\underset{\sim}{R}}(x_i, y_j) < \alpha$$

* Strict α-cut:

for $\alpha \in [0,1]$

$$R_\alpha(x_i, y_j) = (x_i, y_j)$$

$$\mu_{R_\alpha}(x_i, y_j) = 1 \quad \text{if } \mu_{\underset{\sim}{R}}(x_i, y_j) > \alpha$$

$$0 \quad \text{if } \mu_{\underset{\sim}{R}}(x_i, y_j) \leq \alpha$$

* Aggregation operation on fuzzy sets:

Consider fuzzy sets $\underset{\sim}{A}$ & $\underset{\sim}{B}$ over some domain X.

Suppose: $\underset{\sim}{E} = \underset{\sim}{A} \cap \underset{\sim}{B}$

& $\underset{\sim}{F} = \underset{\sim}{A} \cup \underset{\sim}{B}$

Then, $E(x)$ & $F(x)$ can be formalized as a functions depending on $A(x)$ and $B(x)$.

Suppose,

$$E(x) = T(A(x), B(x)) \;\&\; F(x) = S(A(x), B(x))$$

where,

T and S are intersection and union operations.

$\therefore E = A \cap B = B \cap A \Rightarrow T(A(x), B(x)) = T(B(x), A(x))$
& $F = A \cup B = B \cup A \Rightarrow S(A(x), B(x)) = S(B(x), A(x))$

Similarly,
$E = A \cap (B \cap C) = (A \cap B) \cap C$ & $F = A \cup (B \cup C) = (A \cup B) \cup C$

Thus,
$$T(A(x), T(B(x), C(x))) = T(T(A(x), B(x)), C(x))$$
&
$$S(A(x), S(B(x), C(x))) = S(S(A(x), B(x)), C(x))$$

If $A(x) \geq A(y)$ & $B(x) \geq B(y)$
then,
$$T(A(x), B(x)) \geq T(A(y), B(y))$$
&
$$S(A(x), B(x)) \geq S(A(y), B(y))$$

If $A \cap 1 = A$ & $A \cup 0 = A$

then,
$$T(A(x), 1) = A(x)$$
$$S(A(x), 0) = A(x)$$

If $A \cap A = A$ & $A \cup A = A$

then,
$$T(A(x), A(x)) = A(x)$$
$$S(A(x), A(x)) = A(x)$$

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* The function T defines T-operator as:

$$T : [0,1] \times [0,1] \rightarrow [0,1]$$

and is called t-norm if

1. $T(a,b) = T(b,a)$ commutative,
2. $T(a, T(b,c)) = T(T(a,b), c)$ Associative
3. $T(a,b) \geq T(c,d)$ if $a \geq c$ & $b \geq d$ 'Monotonocity
4. $T(a,1) = a$ Identity

Example: MIN, MAX $[0, a+b-1]$, $a \cdot b$
↳ Arithmetic multiplication

* The function S defines S-operator as:

$$S : [0,1] \times [0,1] \rightarrow [0,1]$$

and is called t-conorm if

1. $S(a,b) = S(b,a)$ Commutative
2. $S(a, S(b,c)) = S(S(a,b), c)$ Associative
3. $S(a,b) \geq S(c,d)$ if $a \geq c$ & $b \geq d$ Monotonocity
4. $S(a,0) = a$ Identity

Example: Max, Min $(1, a+b)$, $a+b-ab$

* Negation operator : N

$$N : [0,1] \rightarrow [0,1] \text{ if}$$

1. $N(1) = 0$, & $N[0] = 1$ boundary rule
2. if $a > b$ then $N(a) < N(b)$ order of reversing
3. $N(N(a)) = a$ Involution.

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Example: For $\underset{\sim}{A} = \left\{ \frac{0.6}{20}, \frac{0.4}{30} \right\}$

& $\underset{\sim}{B} = \left\{ \frac{0.1}{20}, \frac{0.2}{30} \right\}$

compute t-norm operation on $\underset{\sim}{A} \times \underset{\sim}{B}$ using $\max(0, a+b-1)$ and t-conorm operation on $\underset{\sim}{A} \times \underset{\sim}{B}$ using $\min(1, a+b)$.

= Sol

for, t-norm using $\max(0, a+b-1)$

for (20)

$= \max(0, 0.6+0.1-1)$

$= \max(0, -0.3)$

$= 0$

for (30)

$= \max(0, 0.4+0.2-1)$

$= \max(0, -0.4)$

$= 0$

For t-conorm using $\min(1, a+b)$

for 20 ;

$= \min(1, 0.6+0.1)$

$= \min(1, 0.7)$

$= 0.7$

For 30

$= \min(1, 0.4+0.2)$

$= \min(1, 0.6)$

$= 0.6$,,

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Unit: 3
Membership Function:

3.1 Universe of discourse (X) } Already cover in initial chapters.
3.2 Mapping inside Fuzzy domain

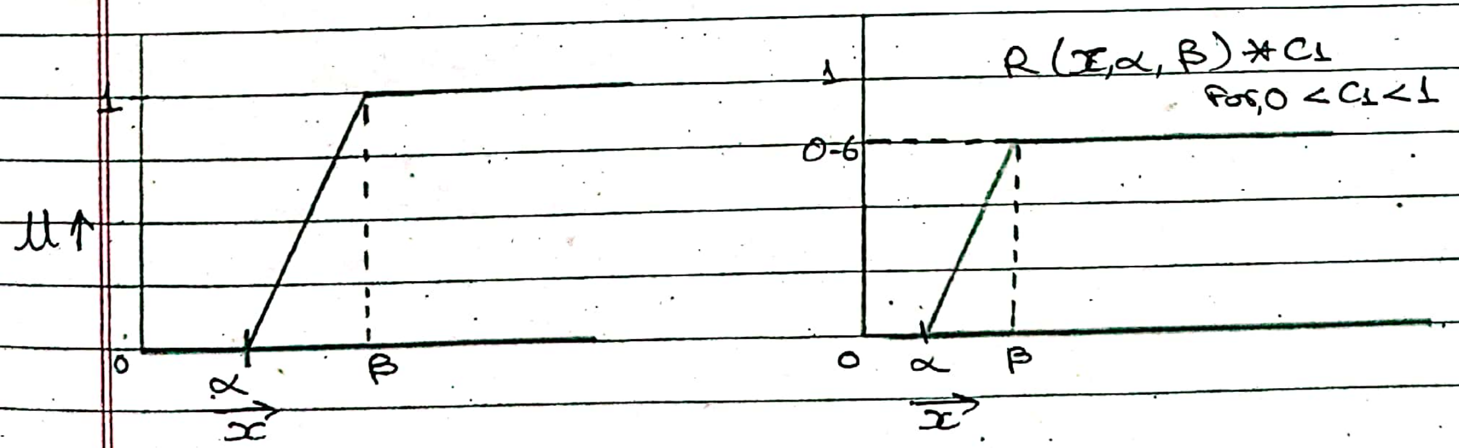
3.3 Fuzzy membership mapping methods:

1. R-Function,
2. L- Function,
3. Triangular Function,
4. Trapezoidal Function,
5. Sigmoid Function,
6. Gaussian Function,

① R-Function (Right-sided function):

- For any element x over X, the R-Function is defined as;

$$R(x, \alpha, \beta) = \begin{cases} 0 & ; x \leq \alpha \\ \dfrac{x-\alpha}{\beta-\alpha} & ; \alpha \leq x \leq \beta \\ 1 & ; x > \beta \end{cases}$$



Crisp → $\boxed{\text{Fuzzify}}$ → Reasoning → Defuzzify → crisp

↓

$\boxed{\text{Fuzzification}}$

↓

value

↓

membership function

↓ μ value

[0,1]

Example:

$X = \{20, 30, 40, 50, 60, 70, 80\}$. Compute a fuzzy set $\underset{\sim}{A} \subseteq X$ using R-function where $\alpha = 40$ & $\beta = 70$

= Sol

given, $\alpha = 40$ & $\beta = 70$

so, calculate membership for each element in X i.e,

$\mu_{\underset{\sim}{A}}(10) = 0$ $\quad$ $(\because x \leq \alpha)$

$\mu_{\underset{\sim}{A}}(20) = 0.$ $\quad$ $(\because x \leq \alpha)$

$\mu_{\underset{\sim}{A}}(30) = 0$ $\quad$ $(\because x \leq \alpha)$

$\mu_{\underset{\sim}{A}}(40) = \dfrac{x-\alpha}{\beta-\alpha} = \dfrac{40-40}{70-40} = 0$ $\quad$ $(\because \alpha \leq x \leq \beta)$.

$$\mu_{A_2}(50) = \frac{x-\alpha}{\beta-\alpha} = \frac{50-40}{70-40} = \frac{10}{30} = 0.33$$

$$\mu_{A_2}(60) = \frac{x-\alpha}{\beta-\alpha} = \frac{60-40}{70-40} = \frac{20}{30} = 0.67$$

$$\mu_{A_2}(70) = \frac{x-\alpha}{\beta-\alpha} = \frac{70-40}{70-40} = \frac{30}{30} = 1$$

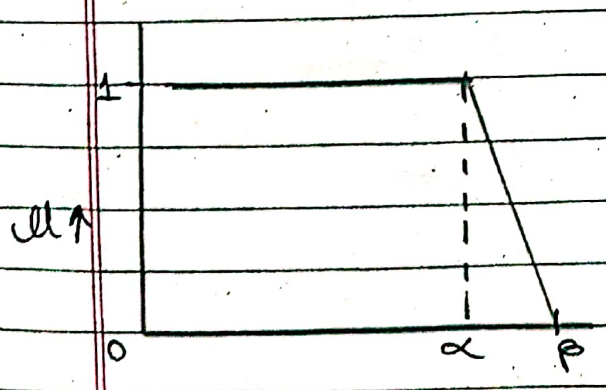
$$\mu_{A_2}(80) = 1 \quad (\because x \geq \beta)$$

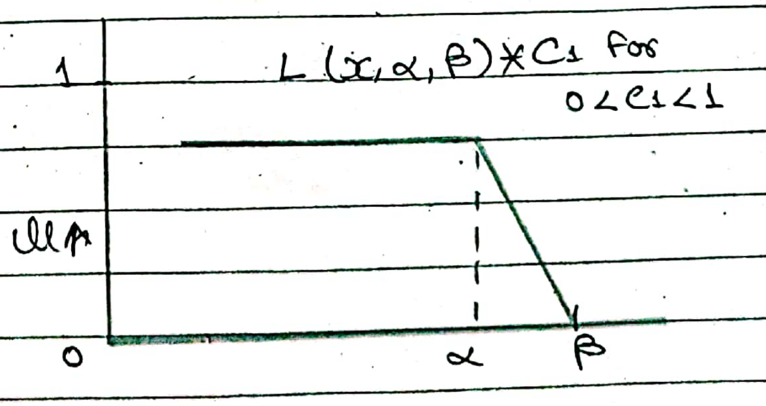
$$\therefore \mu_{A_2} = \left\{ \frac{0}{10}, \frac{0}{20}, \frac{0}{30}, \frac{0}{40}, \frac{0.33}{50}, \frac{0.67}{60}, \frac{1}{70}, \frac{1}{80} \right\}$$

② L-Function (Left-Sided Function):

For any element x over X, the L-function is defined as;

$$L(x,\alpha,\beta) = \begin{cases} 1 & \text{for } x < \alpha \\ \dfrac{\beta - x}{\beta-\alpha} & \text{for } \alpha \leq x \leq \beta \\ 0 & \text{for } x > \beta \end{cases}$$





Example:

$X = \{10, 20, 30, 40, 50, 60, 70, 80\}$. Compute a Fuzzy Set $\underset{\sim}{A} \subseteq X$ using L-Function where $\alpha = 40$ & $\beta = 70$.

= Sol

given, $\alpha = 40$ & $\beta = 70$

So,

to calculate the membership for each element in X i.e.

for 20 i.e,

$\mu_{\underset{\sim}{A}}^{(20)} = 1 \quad (\because x < \alpha)$ $\qquad \mu_{\underset{\sim}{A}}^{(10)} = 1 \quad (\because x < \alpha)$

$\mu_{\underset{\sim}{A}}^{(30)} = 1 \quad (\because x \leq \alpha)$

$\mu_{\underset{\sim}{A}}^{(40)} = \frac{\beta - x}{\beta - \alpha} \quad \cancel{(\because x \leq \alpha)}$

$= \frac{40-40}{70-40} \qquad \frac{70-40}{70-40}$

$= \frac{0}{30} \qquad = \frac{30}{30}$

$= 0 \qquad = 1$

$\mu_{\underset{\sim}{A}}^{(50)} = \frac{x-\alpha}{\beta-\alpha} \qquad \frac{\beta - x}{\beta - \alpha} = \frac{70-50}{70-40}$

$= \frac{50-40}{70-40} \qquad = \frac{20}{30}$

$= 0.33 \qquad = 0.67$

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$$\mu_{\tilde{A}}(60) = \frac{\beta - x}{\beta - \alpha}$$

$$= \frac{70-60}{70-40}$$

$$= \frac{10}{30}$$

$$= 0.33$$

$$\mu_{\tilde{A}}(70) = \frac{\beta - x}{\beta - \alpha}$$

$$= \frac{70-70}{70-40}$$

$$= 0$$

$$\mu_{\tilde{A}}(80) = 0 \quad (\because x > \beta)$$

$$\therefore \mu_{\tilde{A}} = \left\{ \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, \frac{0.67}{50}, \frac{0.33}{60}, \frac{0}{70}, \frac{0}{80} \right\}$$

③ Triangular Function:

For any element x over X, the triangular membership function is defined as:

$$F(x, \alpha, \beta, \gamma) = \begin{cases} 0 & \text{for } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{for } \alpha \leq x \leq \beta \\ \frac{\gamma - x}{\gamma - \beta} & \text{for } \beta \leq x \leq \gamma \\ 0 & \text{for } x > \gamma \end{cases}$$

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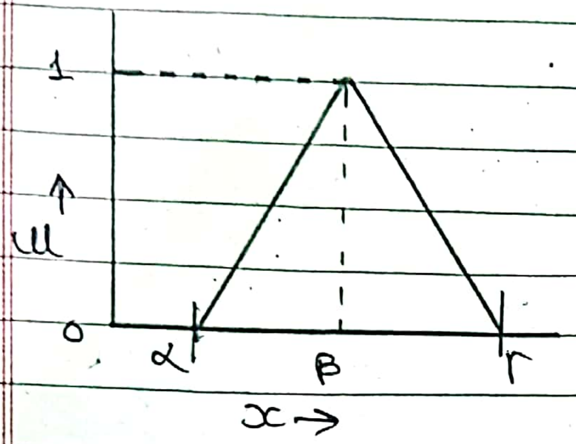
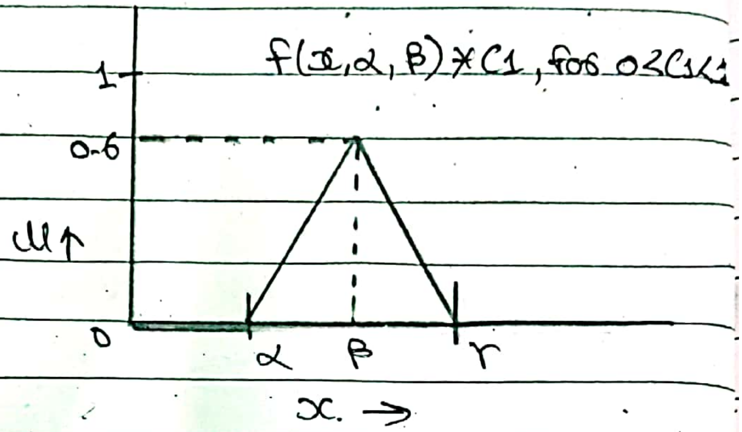




Example:

Given, $X = \{10, 20, 30, 40, 50, 60, 70, 80\}$. Compute a fuzzy set $\underset{\sim}{A} \subset X$ using Triangular function where

$\alpha = 30$; $\beta = 50$; $\gamma = 70$

= Sol

given, $\alpha = 30$, $\beta = 50$, $\gamma = 70$

To calculate the membership for each element in X i.e,

$\mu_{\underset{\sim}{A}}(10) = 0 \quad (\because x < \alpha)$

$\mu_{\underset{\sim}{A}}(20) = 0 \quad (\because x < \alpha)$

$$\mu_{\underset{\sim}{A}}(30) = \frac{x-\alpha}{\beta-\alpha}$$

$$= \frac{30-30}{50-30}$$

$$= \frac{0}{20}$$

$$= 0$$

$$\mu_{A_2}^{(40)} = \frac{x-\alpha}{\beta-\alpha}$$

$$= \frac{40-30}{50-30}$$

$$= \frac{10}{20}$$

$$= 0.5$$

$\mu_{A_2}^{(50)}$ = since $x = \beta$ we can use any formula among (ii)

$\left(\because \frac{x-\alpha}{\beta-\alpha} \text{ for } \alpha \le x \le \beta \right.$ or, $\left. \because \frac{\gamma-x}{\gamma-\beta} \text{ for } \beta \le x \le \gamma \right)$

i.e.,

$$\frac{x-\alpha}{\beta-\alpha} \quad \text{or} \quad \frac{\gamma-x}{\gamma-\beta}$$

$$= \frac{50-30}{50-30} \qquad = \frac{70-50}{70-50}$$

$$= \frac{20}{20} \qquad = \frac{20}{20}$$

$$= 1 \qquad = 1$$

$$\mu_{A_2}^{(60)} = \frac{\gamma-x}{\gamma-\beta}$$

$$= \frac{70-60}{70-50}$$

$$= \frac{10}{20}$$

$$= 0.5$$

$$\mu_{A_2}(70) = \frac{\gamma - x}{\gamma - \beta}$$

$$= \frac{70-70}{70-50}$$

$$= \frac{0}{20}$$

$$= 0$$

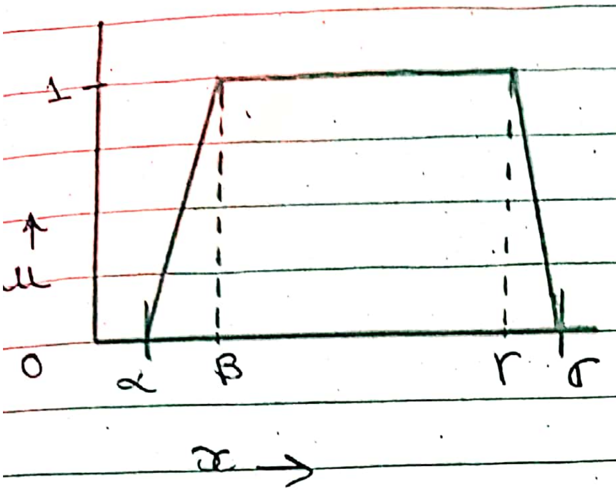
$$\mu_{A_2}(80) = 0 \quad (\because x > \gamma)$$

$$\therefore A_2 = \left\{ \frac{0}{10}, \frac{0}{20}, \frac{0}{30}, \frac{0.5}{40}, \frac{1}{50}, \frac{0.5}{60}, \frac{0}{70}, \frac{0}{80} \right\}$$

* Trapezoid membership function:

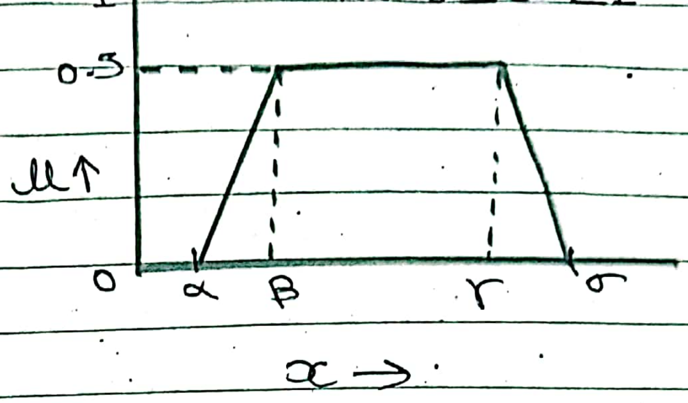
For any element x over X, the trapezoidal membership function is defined as;

$$f(x, \alpha, \beta, \gamma, \sigma) = \begin{cases} 0 & \text{if } x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha} & \text{if } \alpha \le x < \beta \\ 1 & \text{if } \beta \le x < \gamma \\ \dfrac{\sigma - x}{\sigma - \gamma} & \text{if } \gamma \le x \le \sigma \\ 0 & \text{if } x > \sigma \end{cases}$$



For $(x, \alpha, \beta, \gamma, \sigma) * C_1$

for $0 < C_1 \leq 1$



Example:

Given, $X = \{10, 20, 30, 40, 50, 60, 70, 80\}$. Compute Fuzzy set $\underset{\sim}{A} \subseteq X$ using Trapezoid membership function where,

$\alpha = 30, \quad \beta = 50, \quad \gamma = 70, \quad \sigma = 80$

calculate membership for each element in X i.e,

$\mu_{\underset{\sim}{A}}(10) = 0 \quad (\because x < \alpha)$

$\mu_{\underset{\sim}{A}}(20) = \mu_{\underset{\sim}{A}}(30) = 0 \quad (\because x < \alpha)$

$\mu_{\underset{\sim}{A}}(40) = \frac{40-80}{50-30} = 0.5$

$\mu_{\underset{\sim}{A}}(50) = 1 \quad (\because \beta \leq x \leq \gamma)$

$\mu_{\underset{\sim}{A}}(60) = 1 \quad (\because \beta \leq x \leq \gamma)$

$\mu_{\underset{\sim}{A}}(70) = \frac{80-70}{80-70}$

$= \frac{10}{10} = 1$

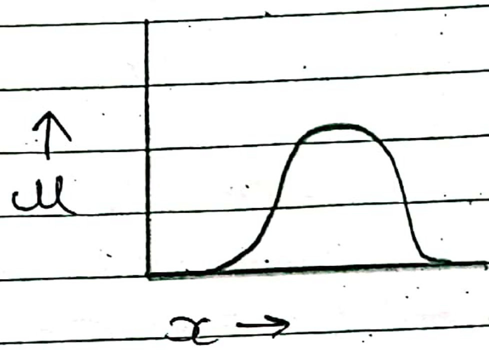
$\mu_{\underset{\sim}{A}}(80) = \frac{80-80}{80-70} = 0$

$\therefore \underset{\sim}{A} = \left\{ \frac{0}{10}, \frac{0}{20}, \frac{0}{30}, \frac{0.5}{40}, \frac{1}{50}, \frac{1}{60}, \frac{1}{70}, \frac{0}{80} \right\}$

* Gaussian Function:

For any element x over X, the Gaussian function is defined as:

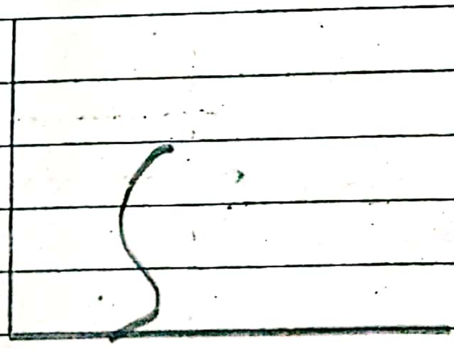
$$f(x, c, \sigma) = e^{-\frac{(x-c)^2}{\sqrt{2\sigma}}}$$



* Sigmoid function:

For any element x over X, the sigmoid function is defined as:

$$f(x, c, \sigma) = \frac{1}{1+e^{-\sigma(x-c)}}$$



Note: Besides above mentioned membership function, there can be users defined membership function where value lise $[0,1]$

e.g.:- $\frac{1}{2}\left(\frac{x+x}{x*x}\right)$

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Unit: 4
Fuzzy Knowledge based Systems:

\# Fuzzy Knowledge Based System:

In a fuzzy knowledge base system, the knowledge base contains information represented in fuzzy sets, such as linguistic variables, membership functions, and rules. The inference engine uses these fuzzy sets to reason about new data, making decisions or recommendations based on the degree of membership of the fuzzy sets.

\# Fuzzy rule based system:

The most common way to represent human knowledge is to form it into natural language expressions of the type

Condition — Action

If <antecedent>, then <consequent>

The antecedent & consequent are defined by fuzzy linguistic variables.

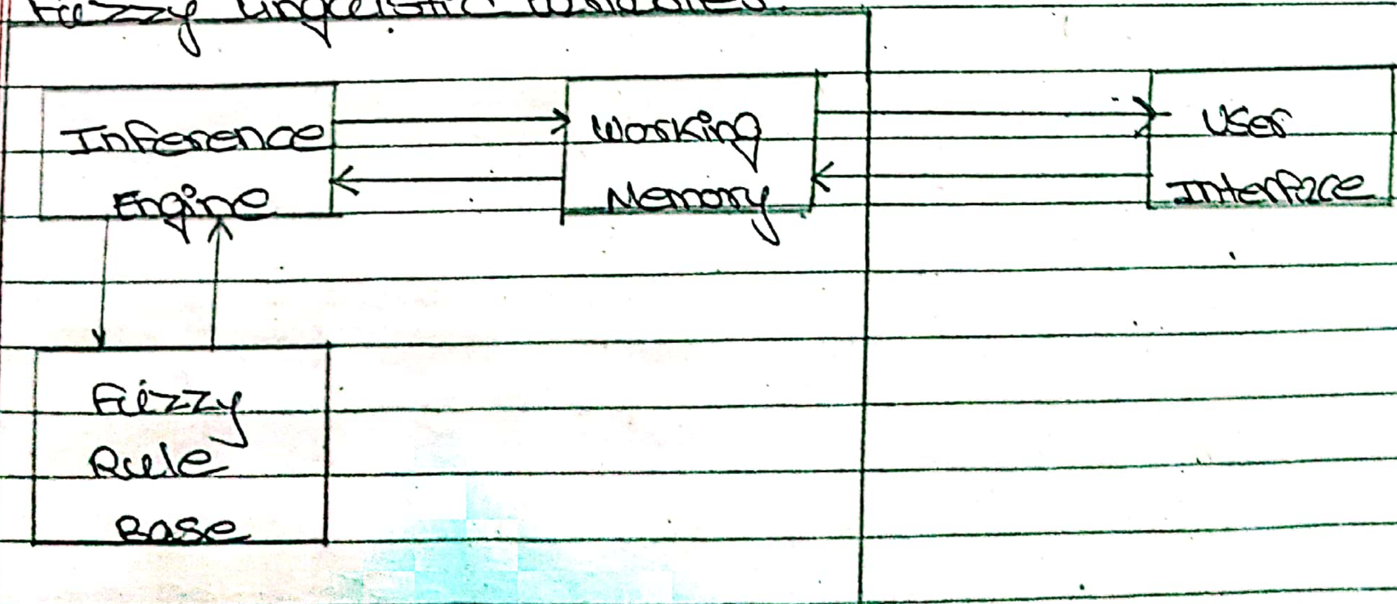

Fig: Fuzzy Rule Based System (FRBS).

The Fuzzy rules are defined as;
IF x is in A then y is in B where A & B are fuzzy sets.

Implication rules are used to resolve fuzzy rules.

* Fuzzification:

It is the process of fuzzyfying crisp values.

crisp value	→	fuzzify	→	Fuzzy reasoning	→	Defuzzify	→	crisp value

Tradition:

IF $x > 0$ then . . .

Fuzzy:

IF speed is high then power is low
Speed $\in$ high & power $\in$ Low

IF speed is moderate then power is high.

* Implication rules:

Given, a fuzzy rule:

IF x is in $\underset{\sim}{A}$ then y is in $\underset{\sim}{B}$

where,

x is in $\underset{\sim}{A}$ & y is in $\underset{\sim}{B}$ is a Fuzzy predicate $A(x)$ & $B(y)$

Then;

IF $A(x)$ then $B(y)$ can be represented as;

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$$R(x,y) : A(x) \to B(y)$$

i.e, $\mu_{\underset{\sim}{R}}(x,y) = f\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right)$

where, f is a fuzzy implication & performs task of transferring membership degree of x in $\underset{\sim}{A}$ & y in $\underset{\sim}{B}$ into those of (x,y) in $A \times B$ for implication for C : IF $\underset{\sim}{A}$ then $\underset{\sim}{B}$

① Mamdani Implication Rule :

$$R_C = \underset{\sim}{A} \times \underset{\sim}{B}$$

$$\mu(R_C) = \int_{X \times Y} \frac{\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(y)}{(x,y)}$$

Aggregation

$$= \frac{\min\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right)}{(x,y)}$$

② Larsen's Rule:

$$R_C = A \times B$$

$$\mu(R_C) = \int_{X \times Y} = \frac{\mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(y)}{(x,y)}$$

$$= \frac{\mu_{\underset{\sim}{A}}(x) * \mu_{\underset{\sim}{B}}(y)}{(x,y)}$$

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(3) Zadeh Implication Rule:

$$R_c = A \times B$$

$$\mu(R_c) = \max\left[\min\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right)\right]$$

* Example:

Consider a fuzzy rule:

If temperature is $\underset{\sim}{\text{hot}}$ then fan speed is $\underset{\sim}{\text{fast}}$.

where domain of discourse for temperature in °c is

{30, 35, 40, 45, 50}

& domain of discourse for speed in rpm is

{500, 1000, 1500, 2000}

consider,

$$\underset{\sim}{\text{hot}} = \left\{\frac{0.4}{30}, \frac{0.6}{35}, \frac{0.8}{40}, \frac{0.9}{45}\right\}$$

$$\& \ \text{fast} = \left\{\frac{0.3}{500}, \frac{0.5}{1000}, \frac{0.7}{1500}, \frac{1}{2000}\right\}$$

now, the rule R_1 using mamdani rule is;

$$\mu_R^{(30,500)} = \min\left(\mu_{\underset{\sim}{\text{hot}}}^{(30)}, \mu_{\underset{\sim}{\text{fast}}}^{(500)}\right)$$
$$= \min(0.4, 0.3)$$
$$= 0.3$$

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$$\mu_{R_2}^{(30,1000)} = \min\left(\mu_{hot_2}^{(30)}, \mu_{fast_2}^{(1000)}\right)$$

$$= \min(0.4, 0.5)$$

$$= 0.4$$

$$\mu_{R_2}^{(30,1500)} = \min\left(\mu_{hot_2}^{(30)}, \mu_{fast_2}^{(1500)}\right)$$

$$= \min(0.4, 0.7)$$

$$= 0.4$$

$$\mu_{R_2}^{(30,2000)} = \min\left(\mu_{hot_2}^{(30)}, \mu_{fast_2}^{(2000)}\right)$$

$$= \min(0.4, 1)$$

$$= 0.4$$

similarly for all, then we get,

R_1 =	500	1000	1500	2000
30	0.3	0.4	0.4	0.4
35	0.3	0.5	0.6	0.6
40	0.3	0.5	0.7	0.8
45	0.3	0.5	0.7	0.9

* Approximate Reasoning:

let,

$R: A \times B$

where, $x \in A$

$y \in B$

using, x & R we can calculate y i.e.

$B: A \circ R$

where, $y \in B: x \in A \circ R$

* Defuzzification:

- It is a process of selecting a representative element from fuzzy output.

- Defuzzification to crisp set: → Relations, → Sets — α-cut

- Defuzzification to scalar
  - → Max-membership method,
  - → Mean Max Membership method
  - → Weighted Average method
  - → Centroid method
  - ⋮

① Max-membership method (Height method):

- For $\underset{\sim}{A} = \left\{ \frac{\mu(x_1)}{x_1}, \frac{\mu(x_2)}{x_2}, \ldots, \frac{\mu(x_n)}{x_n} \right\}$

Defuzzification $(\underset{\sim}{A}) = x_i$ such that $\mu_{\underset{\sim}{A}}(x_i) \geq \mu_{\underset{\sim}{A}}(x_j)$

$$\forall\, x_i, x_j \in \underset{\sim}{A} \;\&\; x_i \neq x_j$$

- Used in descrete fuzzy sets, triangular fuzzy sets,

eg: $\underset{\sim}{A} = \left\{ \frac{0.5}{6}, \frac{0.6}{7}, \frac{0.8}{9}, \frac{0.4}{10}, \frac{0.3}{11} \right\}$

Deffuz $(\underset{\sim}{A}) = 9$

Mean-Max Membership (Middle of Maxima):

For a fuzzy set $\underset{\sim}{A}$, Defuzz$(\underset{\sim}{A}) = z^* = \frac{a+b}{2}$,

where,

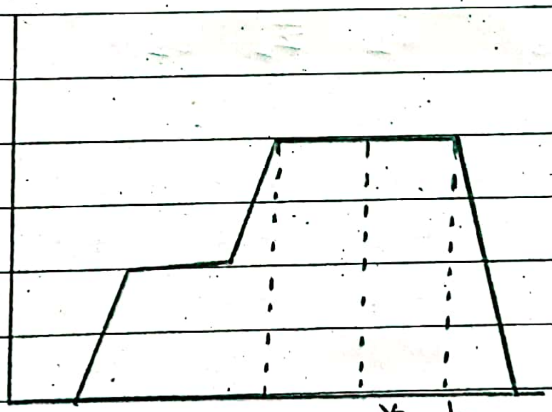
$$\mu_{\underset{\sim}{A}}(a) = \mu_{\underset{\sim}{A}}(b) \text{ and}$$

$$\mu_{\underset{\sim}{A}}(a) = \mu_{\underset{\sim}{A}}(b) \geq \mu_{\underset{\sim}{A}}(x_i) \quad \forall\, x_i \in A$$

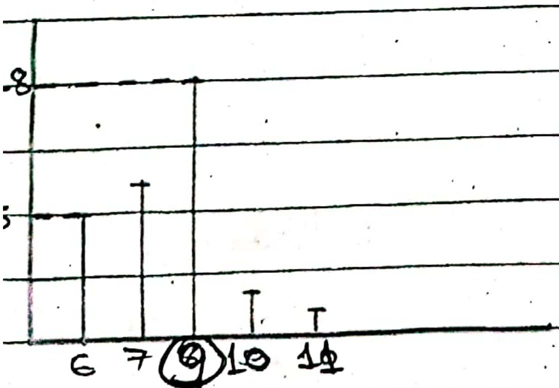
and a is first among x having max-membership & b is last among x having max-membership.

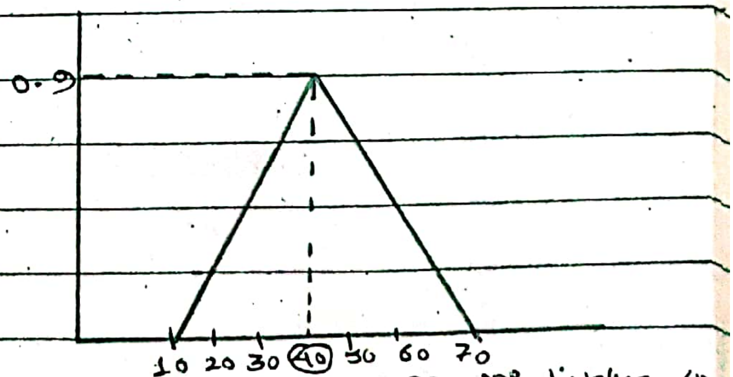
Eg:- $\underset{\sim}{A} = \left\{ \frac{0.1}{4}, \frac{0.6}{5}, \frac{0.7}{7}, \overset{a}{\frac{0.9}{8}}, \frac{0.9}{9}, \frac{0.9}{10}, \overset{b}{\frac{0.9}{12}}, \frac{0.6}{13}, \frac{0.2}{14} \right\}$

$$\text{Defuz}(\underset{\sim}{A}) = \frac{8+12}{2} = 10$$



$$z^* = \frac{a+b}{2}$$





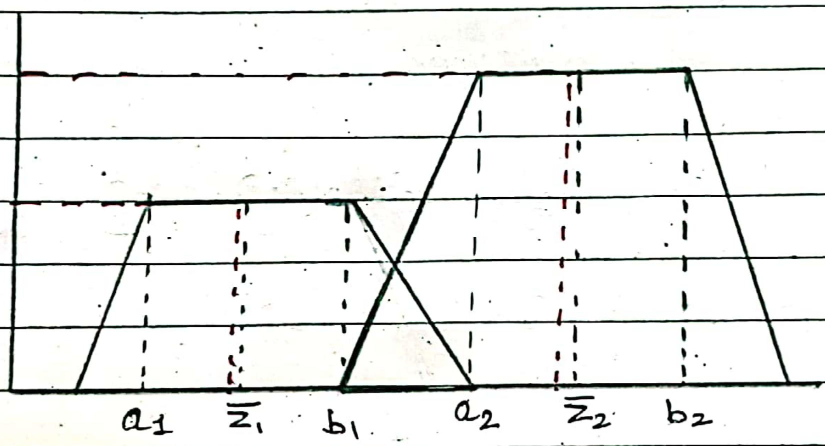
For Max-membership method

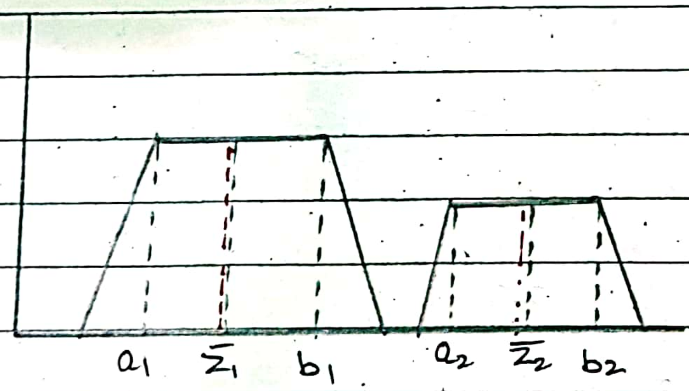
③ Weighted Average method:

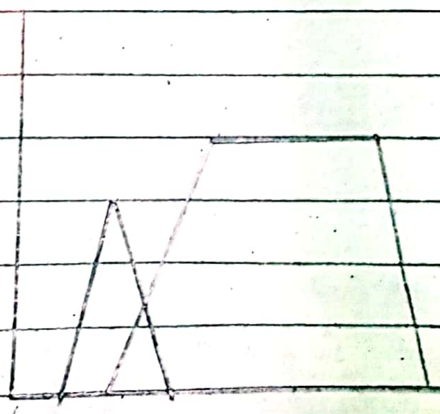
$$z^* = \frac{\sum \mu(\bar{z}) \cdot \bar{z}}{\mu(\bar{z})}$$

where,

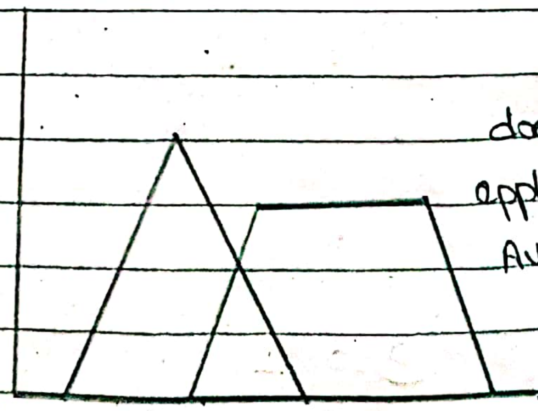
$\bar{z}$ is centroid or average of symmetric membership functions.







MAX-Mean



Max-method

doesn't need to apply weighted-Average method

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(A) Centroid Method:

- Used in Continuous fuzzy sets.

$$z^* = \frac{\Sigma \int \mu(z) \cdot z\,dz}{\int \mu(z)\,dz}$$

where, Σ = is algebric sum
$\int$ = is algebric integration

For a fuzzy set $\underset{\sim}{A}$ defined by:

$$\mu(x) = \begin{cases} 0.35x & 0 \le x < 2 \\ 0.7 & 2 \le x < 2.7 \\ x-2 & 2.7 \le x < 3 \\ 1 & 3 \le x \le 4 \end{cases}$$

Then,

$$z^* = \frac{\int_0^2 (0.35)x\,dx + \int_2^{2.7} (0.7)x\,dx + \int_{2.7}^{3} (x-2)x\,dx + \int_3^4 (1)x\,dx}{\int_0^2 (0.35x)\,dx + \int_2^{2.7} (0.7)\,dx + \int_{2.7}^{3} (x-2)\,dx + \int_3^4 (1)\,dx}$$

= ....

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* Example of Approximate Reasoning:

Given a rule R, defined as:

$$R: \text{ If } A \text{ then } B$$

Now, suppose we introduce an antecedent A' for a rule. i.e,

$$\text{If } A' \text{ then } B'$$

Then, the information derived from the rule R, it is possible to derive the consequent in the second rule. i.e. B' by:

$$B' = A' \circ R$$

Example:

$$R_1: \text{ If } x \text{ is } A \text{ then } y \text{ is } B.$$

where,

$$A = \left\{ \frac{0.2}{1}, \frac{0.5}{2}, \frac{0.7}{3} \right\}$$

$$\text{and } B = \left\{ \frac{0.6}{5}, \frac{0.8}{7}, \frac{0.4}{9} \right\}$$

now, we can evaluate B' for following rule as:

$$R_2: \text{ If } x \text{ is } A' \text{ then } y \text{ is } B'$$

where,

$$A' = \left\{ \frac{0.5}{1}, \frac{0.9}{2}, \frac{0.3}{3} \right\}$$

Also, the rule R_1 is defined using Mamdani principle i.e.

$$\mu(R) = \min_{(x,y)} \left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y) \right)$$

$$R_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 5 & 7 & 9 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.6 & 0.7 & 0.4 \end{bmatrix} \end{array}$$

$$\mu^{(1,5)} = \min\left(\mu_{A_2}^{(1)}, \mu_{B_2}^{(5)}\right)$$

$$= \min(0.2, 0.6)$$

$$= 0.2$$

$$\mu^{(1,7)} = \min\left(\mu_{A_2}^{(1)}, \mu_{B_2}^{(7)}\right)$$

$$= \min(0.2, 0.8)$$

$$= 0.2$$

$$\mu^{(1,9)} = \min\left(\mu_{A_2}^{(1)}, \mu_{B_2}^{(9)}\right)$$

$$= \min(0.2, 0.4)$$

$$= 0.2$$

$$\mu^{(2,5)} = \min\left(\mu_{A_2}^{(2)}, \mu_{B_2}^{(5)}\right)$$

$$= \min(0.5, 0.6)$$

$$= 0.5$$

$$\mu^{(2,7)} = \min\left(\mu_{A_2}^{(2)}, \mu_{B_2}^{(7)}\right)$$

$$= \min(0.5, 0.8)$$

$$= 0.5$$

$$\mu^{(2,9)} = \min\left(\mu_{A_2}^{(2)}, \mu_{B_2}^{(9)}\right)$$

$$= \min(0.5, 0.4)$$

$$= 0.4$$

$$\mu^{(3,5)} = \min(\mu_{\underset{\sim}{A}}^{(3)}, \mu_{\underset{\sim}{B}}^{(5)})$$
$$= \min(0.7; 0.6)$$
$$= 0.6$$

$$\mu^{(3,7)} = \min(\mu_{\underset{\sim}{A}}^{(3)}, \mu_{\underset{\sim}{B}}^{(7)})$$
$$= \min(0.7, 0.8)$$
$$= 0.7$$

$$\mu^{(3,9)} = \min(\mu_{\underset{\sim}{A}}^{(3)}, \mu_{\underset{\sim}{B}}^{(9)})$$
$$= \min(0.7, 0.4)$$
$$= 0.4$$

So, we have,

$$\underset{\sim}{A}' = \left\{ \frac{0.5}{1}, \frac{0.9}{2}, \frac{0.3}{3} \right\}$$

$$R = \begin{array}{c c} & \begin{array}{ccc} 5 & 7 & 9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.6 & 0.7 & 0.4 \end{bmatrix} \end{array}$$

Then,

$$B' = A' \circ R \in \{5, 7, 9\}$$

For this example we are taking Max-MIN Composition so

$$\mu_{\underset{\sim}{B}'}^{(5)} = \operatorname{Max}[\operatorname{Min}(\mu_{\underset{\sim}{A}'}^{(1)}, \mu_R^{(1,5)}), \min(\mu_{\underset{\sim}{A}'}^{(2)}, \mu_R^{(2,5)}),$$
$$\operatorname{Min}(\mu_{\underset{\sim}{A}'}^{(3)}, \mu_R^{(3,5)})]$$
$$= \operatorname{Max}[\operatorname{Min}(0.5, 0.2), \min(0.9, 0.5), \min(0.3, 0.6)]$$

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$= \max(0.2, 0.5, 0.3)$
$= 0.5$

$$\mu_{B'_2}^{(7)} = \max\left[\min\left(\mu_{A'_2}^{(1)}, \mu_R^{(1,7)}\right), \min\left(\mu_{A'_2}^{(2)}, \mu_R^{(2,7)}\right), \min\left(\mu_{A_2}^{(3)}, \mu_R^{(3,7)}\right)\right]$$

$= \max[\min(0.5, 0.2), \min(0.9, 0.5), \min(0.3, 0.7)]$
$= \max(0.2, 0.5, 0.3)$
$= 0.5$.

$$\mu_{B'_2}^{(9)} = \max\left[\min\left(\mu_{A_2}^{(1)}, \mu_R^{(1,9)}\right), \min\left(\mu_{A_2}^{(2)}, \mu_R^{(2,9)}\right), \min\left(\mu_{A_2}^{(3)}, \mu_R^{(3,9)}\right)\right]$$

$= \max[\min(0.5, 0.2), \min(0.9, 0.4), \min(0.3, 0.4)]$
$= \max(0.2, 0.4, 0.3)$
$= 0.4$.

$$\therefore B' = \left\{\frac{0.5}{5}, \frac{0.5}{7}, \frac{0.4}{9}\right\}$$

Approximate Reasoning for multiple rules:

Rule 1 : If height is tall then speed is high.
Rule 2 : If height is Medium then speed is Moderate.

The fuzzy sets for height (in foot) & speed in (m/s) are:

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$H_1 = \text{Tall} = \{ \frac{0.5}{5}, \frac{0.8}{6}, \frac{1}{7} \}$

$H_2 = \text{Medium} = \{ \frac{0.6}{5} + \frac{0.7}{6} + \frac{0.6}{7} \}$

$S_1 = \text{High} = \{ \frac{0.4}{5} + \frac{0.7}{7} + \frac{0.9}{9} \}$

$S_2 = \text{Moderate} = \{ \frac{0.6}{5} + \frac{0.8}{7} + \frac{0.7}{9} \}$

we have given, $H' = \text{Above Average}$

i.e. $H' = \{ \frac{0.5}{5} + \frac{0.9}{6} + \frac{0.8}{7} \}$

find $S' = \text{Above Average}$.

= Sol

· Evaluate R_1 & R_2 using Mamdani Implication Rule i.e,

$$\mu^{(R)}_{(x,y)} = \min \left(\mu_{A_2}^{(x)}, \mu_{B_2}^{(y)} \right)$$

we get,

$$R_1 = \begin{array}{c|ccc} & 5 & 7 & 9 \\ \hline 5 & 0.4 & 0.5 & 0.5 \\ 6 & 0.4 & 0.7 & 0.8 \\ 7 & 0.4 & 0.7 & 0.9 \end{array}$$

(Taking H_1 & S_1 similarly as before qns)

$$R_2 = \begin{array}{c|ccc} & 5 & 7 & 9 \\ \hline 5 & 0.6 & 0.6 & 0.6 \\ 6 & 0.6 & 0.7 & 0.7 \\ 7 & 0.6 & 0.6 & 0.6 \end{array}$$

(Taking H_2 & S_2)

now,

So,

let say,

now

$\mu_{S'}^{(5)}$

$\mu_{S'}$

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now,

$S' = (H' \circ R_1) \vee (H' \circ R_2)$ [∵ If two Relation is present we use (V) union as aggregation]

$= H' \circ (R_1 \vee R_2)$

So,

$$R_1 \dot{\vee} R_2 = \begin{bmatrix} \max(0.4, 0.6), \max(0.5, 0.6), \max(0.5, 0.6) \\ \max(0.4, 0.6), \max(0.7, 0.7), \max(0.8, 0.7) \\ \max(0.4, 0.6), \max(0.7, 0.6), \max(0.9, 0.6) \end{bmatrix}$$

let say, $R_3 = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.9 \end{bmatrix}$

now,

$S' = H' \circ R_3$

$$= \begin{array}{c|c} 5 & \\ 6 & \\ 7 & \end{array}\begin{bmatrix} 0.5 \\ 0.9 \\ 0.8 \end{bmatrix} \circ \begin{array}{c} \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccc} 5 & 7 & 9 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

$\mu_{S'}^{(5)} = \text{Max}[\text{Min}(\mu^{(5)}, \mu^{(5,5)}), \min(\mu^{(6)}, \mu^{(6,5)}), \min(\mu^{(7)}, \mu^{(7,5)})]$

$= \max[\min(0.5, 0.6), \min(0.9, 0.6), \min(0.8, 0.6)]$

$= \max[0.5, 0.6, 0.6)$

$= 0.6$

$\mu_{S'}^{(7)} = \max[\min(\mu^{(5)}, \mu^{(5,7)}), \min(\mu^{(6)}, \mu^{(6,7)}), \min(\mu^{(7)}, \mu^{(7,7)})]$

$= \max[\min(0.5, 0.6), \min(0.9, 0.7), \min(0.8, 0.7)]$

$= \max[0.5, 0.7, 0.7]$

$= 0.7$

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$$\mu_{S'}^{(9)} = \max[\min(\mu^{(5)}, \mu^{(5,9)}), \min(\mu^{(6)}, \mu^{(6,9)}), \min(\mu^{(7)}, \mu^{(7,9)})]$$
$$= \max[\min(0.5, 0.6), \min(0.9, 0.8), \min(0.8, 0.9)]$$
$$= \max[0.5, 0.8, 0.8]$$
$$= 0.8$$

$$\therefore S' = \left\{ \frac{0.6}{5}, \frac{0.7}{7}, \frac{0.8}{9} \right\}$$

* Fuzzy (Rule-Based) Systems:

* Fuzzy Inference:

→ Fuzzy Inference (resoning) is the actual process of mapping from a given input to an output using fuzzy logic.

→ Fuzzy inference system is also known as fuzzy rule based system or Fuzzy system.

→ The Block diagram of fuzzy Inference system is:

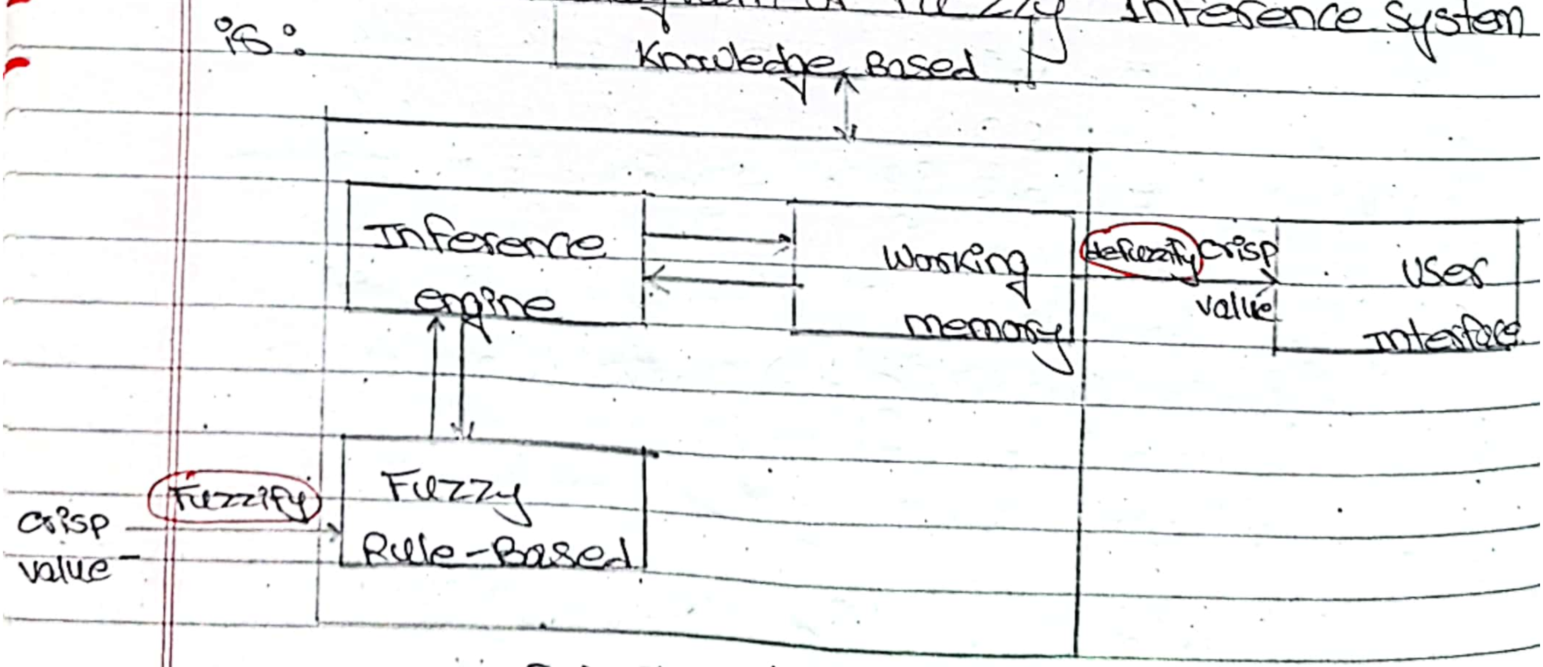

Fig:- Block diagram of fuzzy Inference system

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→ In the field of artificial Intelligence (Machine Intelligence) there are various ways to represent knowledge. Perhaps the most common way to represent human knowledge is in the form of natural language expressions of the type

IF premise (antecedent), THEN conclusion (consequent)

where, it is commonly referred as the IF-THEN rule-based form, which is generally referred to as the deductive form.

→ It typically expresses an inference such that if we know a fact (premise, hypothesis, antecedent) then we can infer, or drive another fact called a a conclusion (consequent).

→ The fuzzy-rule based system is most useful in modeling some complex systems that can be observed by humans because they make use of linguistic variables as their antecedents & consequents as described here these linguistic variables can be naturally represented by fuzzy sets.

Rule 1 : IF condition C^1, THEN restriction R^1
Rule 2 : IF condition C^2, THEN restriction R^2
⋮
Rule r : IF condition C^r, THEN restriction R^r.

Table : The canonical form for a fuzzy rule based system.

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* Multiple conjuctive antecedents:

IF x is $\tilde{A}^1$ and $\tilde{A}^2$ ... and $\tilde{A}^L$ THEN y is $\tilde{B}^s$.

Assuming a new fuzzy subset A^s as:

$$\tilde{A}^s = \tilde{A}^1 \cap \tilde{A}^2 \cap \ldots \cap \tilde{A}^L$$

expressed by means of membership function.

$$\boxed{\mu_{\tilde{A}^s}(x) = \min\left[\mu_{\tilde{A}_1}(x), \mu_{\tilde{A}^2}(x), \ldots, \mu_{\tilde{A}^L}(x)\right]}$$

Based on the definition of the standard fuzzy intersection operation, the compound rule may be rewritten as

IF $\tilde{A}^s$ THEN $\tilde{B}^s$.

* Multiple disjunctive antecedents:

IF x is $\tilde{A}^1$ OR x is $\tilde{A}^2$ ... OR x is $\tilde{A}^L$ THEN y is $\tilde{B}^s$

Could be rewritten as

IF x is $\tilde{A}^s$ THEN y is $\tilde{B}^s$,

where the fuzzy set $\tilde{A}^s$ is defined as

$$\tilde{A}^s = \tilde{A}^1 \cup \tilde{A}^2 \cup \ldots \cup \tilde{A}^L,$$

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$$\boxed{\mu_{\underset{\sim}{A}^s}(x) = \max\left[\mu_{\underset{\sim}{A}^1}(x), \mu_{\underset{\sim}{A}^2}(x), \ldots, \mu_{\underset{\sim}{A}^L}(x)\right]}$$

* Aggregation of Fuzzy rules:

→ Most rule based system involve more than one rule.

→ The process of obtaining the overall consequent (conclusion) from the individual consequents contributed by each rule in the rule-based is known as aggregation of rules.

① Conjunctive system of rules:

In case of a system of rules that must be jointly satisfied, the rules are connected by "and" connectives. In this case, the ~~o/p~~ aggregated output (consequent) y, is found by the fuzzy intersection of all individual rule consequents. y^i, where $i = 1, 2, \ldots r$ as.

$$y = y^1 \text{ and } y^2 \text{ and} \ldots \text{and } y^r$$

or,

$$y = y^1 \cap y^2 \cap \ldots \cap y^r.$$

Which is defined by the membership function as:

$$\boxed{\mu_y(y) = \min(\mu_{y^1}(y), \mu_{y^2}(y), \ldots \mu_{y^r}(y)), \text{ for } y \in Y}$$

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② Disjunctive system of rules:

For the case of a disjunctive system of rules whe the satisfaction of at least one rule is required, the rules are connected by the "or" connectives.

In this case, the aggregated output is found by the fuzzy union of all individual rule contributions

$$y = y^1 \text{ or } y^2 \text{ or} \ldots \text{ or } y^r$$

or,

$$y = y^1 \cup y^2 \cup \ldots \cup y^r,$$

which is defined by the membership function as

$$\boxed{\mu_y(y) = \max(\mu_{y^1}(y), \mu_{y^2}(y), \ldots, \mu_{y^r}(y))}$$

for $y \in Y$.

* Common Techniques of deductive inference for fuzzy systems using Graphical method:

The Most Common methods of deductive inference for fuzzy systems based on Linguistic rules are:

i) Mamdani systems,

ii) Sugeno models, (TSK)

iii) Tsukamoto ~~(TSK)~~ Models.

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i) Mamdani Inference Method:

→ Two input mamdani methods:

i) MAX-MIN inference method, (Clipping)

ii) MAX-product inference method. (Scaling)

Note: clipping doesn't preserves the membership structure but scaling preserves.

→ To begin the general illustration of this idea, we consider a simple two-rule system where each rule comprises two antecedents and one consequent. which is analogous to a dual-input & single-output fuzzy systems.

→ A fuzzy system with two non-interactive inputs x_1 & x_2 (antecedents) and a single output y (consequent) in mamdani form:

Multiple conjuctive Antecedents (MIN)

$$\text{IF } x_1 \text{ is } \underset{\sim}{A}_1^K \text{ (and) } x_2 \text{ is } \underset{\sim}{A}_2^K \text{ THEN } y^K \text{ is } \underset{\sim}{B}^K.$$

for $K = 1, 2, \ldots, n$.

Multiple disjunctive Antecedents (MAX)

$$\text{IF } x_1 \text{ is } \underset{\sim}{A}_1^K \text{ (or) } x_2 \text{ is } \underset{\sim}{A}_2^K \text{ THEN } y^K \text{ is } \underset{\sim}{B}^K.$$

Where $\underset{\sim}{A}_1^K$ and $\underset{\sim}{A}_2^K$ are the fuzzy sets representing the K^{th} antecedent pairs & $\underset{\sim}{B}K$ is the fuzzy set representing the K^{th} consequent.

→ In the following presentation, we consider two different cases of two-input Mamdani systems:

i) The inputs to the system are scalar values, and we use a max-min inference method:

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2) The inputs to the system are scalar values, and we use a max-product inference method.

① Case 1: Max-MIN Inference Method (Scaling):

If x_1 & x_2 are crisp values, that is, delta functions The rule-based system is described in Equation.

IF x_1 is $\underset{\sim}{A}_1^k$ and x_2 is $\underset{\sim}{A}_2^k$ THEN y^k is $\underset{\sim}{B}^k$.

So, Membership for the inputs x_1 & x_2 will be described as:

$$\mu(x_1) = \delta(x_1 - \text{input}(i)) = \begin{cases} 1, & x_1 = \text{input}(i); \\ 0, & \text{otherwise}. \end{cases}$$

$$\mu(x_2) = \delta(x_2 - \text{input}(j)) = \begin{cases} 1, & x_2 = \text{input}(j); \\ 0, & \text{otherwise}. \end{cases}$$

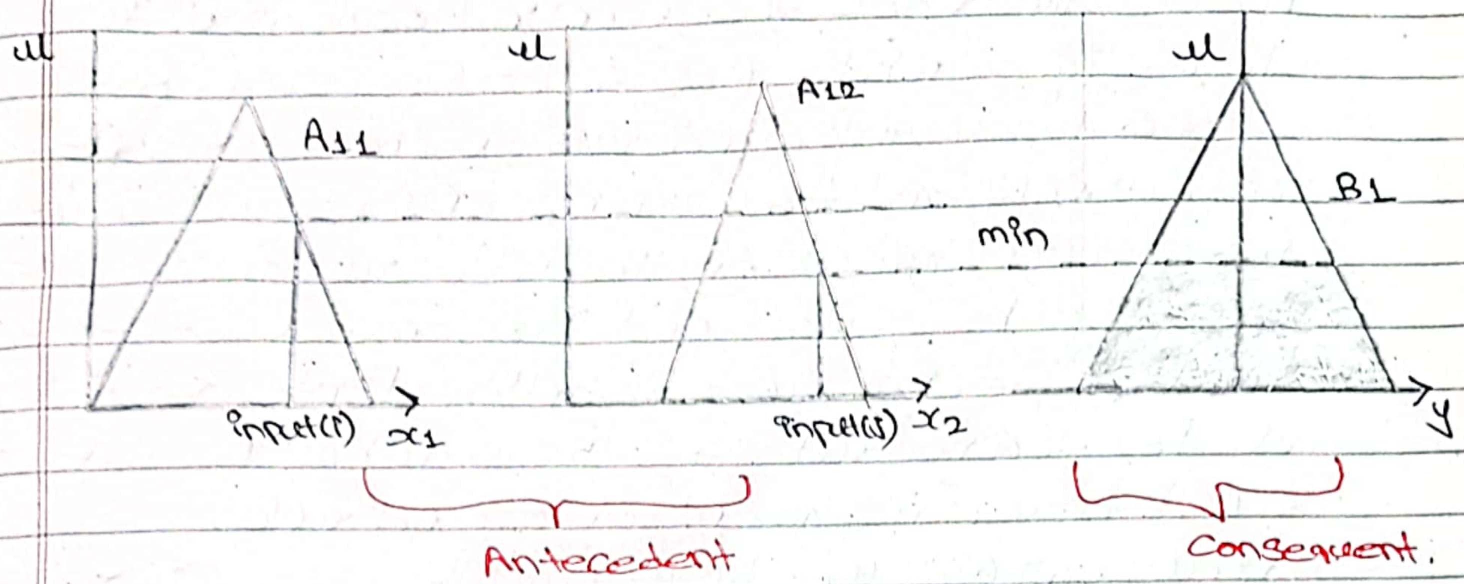
but,

Based on the Mamdani implication method of inference for a set of disjunctive (max) rule, the aggregated output for the n rules will be given as:

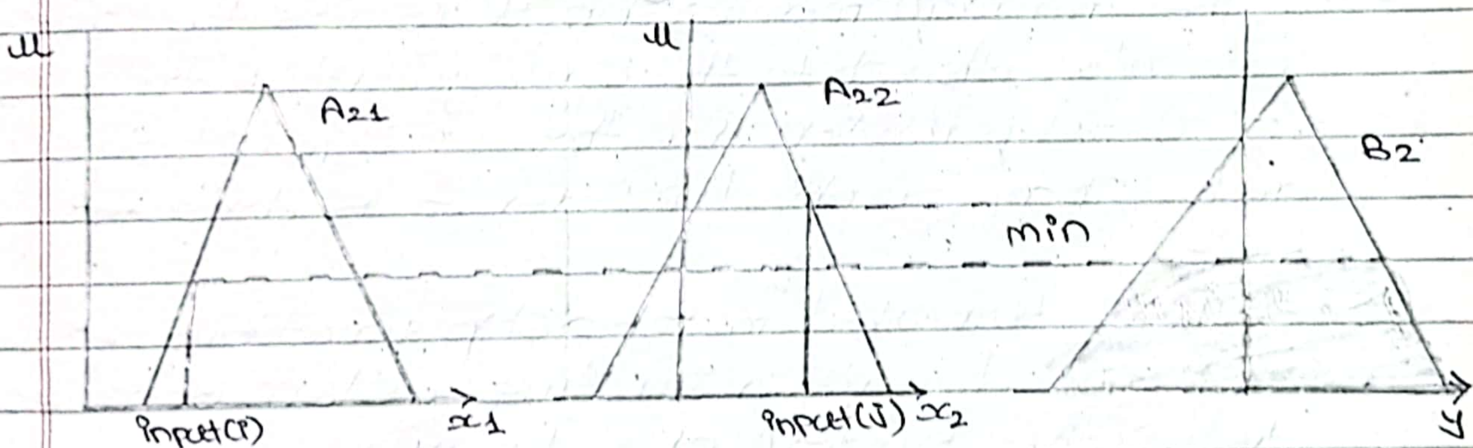
$$\mu_{\underset{\sim}{B}^k}(y) = \max_k \left[\min\left[\mu_{\underset{\sim}{A}_1^k}(\text{input}(i)), \mu_{\underset{\sim}{A}_2^k}(\text{input}(j))\right]\right],$$

$k = 1, 2, \ldots, n.$

Rule 1 :



Rule 2 :



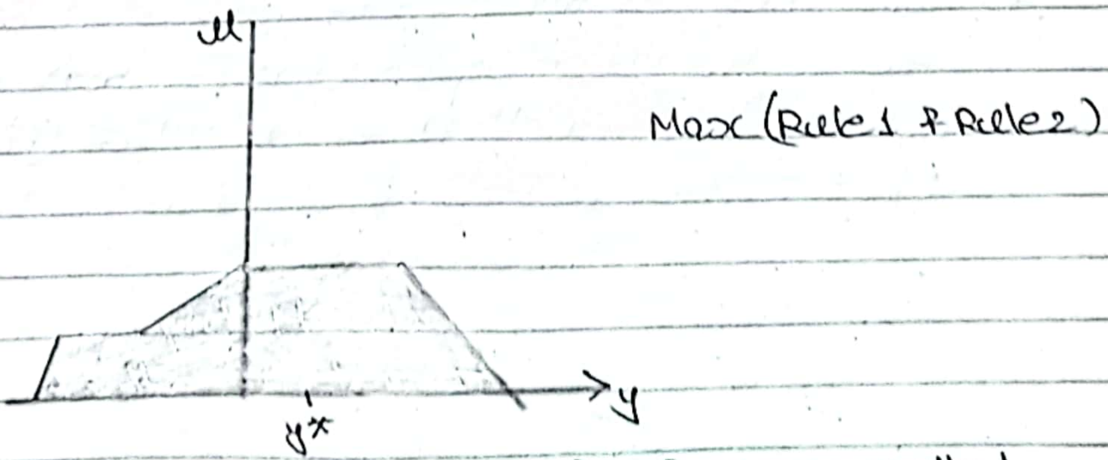


Fig:- Graphical Mamdani (Max-Min) Inference method.

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→ Here, the above figure has a very simple interpretation figure Illustrates the graphical analysis of two rules where the symbols A_{11} & A_{12} refers to first and second fuzzy antecendents of the first rule, respectively and the symbol B_1 refers to the fuzzy consequent of the first rule;

→ The symbols A_{21} and A_{22} refers to the first & second fuzzy antecedents respectively of the second rule & the symbol B_2 refers to the fuzzy consequent of second rule.

→ From eq^n ①, we know that the general rule structure for this system are connected by logical "and" connective.

→ The minimum membership value for the antecedents propagates through to the consequent & truncates the membership function for the consequent of each rule. Then, graphical inference is done for each rule.

→ Then, truncated membership functions are aggregated using conjunctive system of rules (MIN) or disjunctive system of rules. (MAX)

→ In this example, the rules are disjunctive, so the aggregation operation is max results in aggregate membership function. & a value y^* would result which is a defuzzified value.

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(2) Case (2): Max-Product Inference Method (Scaling):

If we were to use a Max-product implication technique for a set of disjunctive rule. (Max). the aggregated output for the rules would be given as:

$$\mu_{\underset{\sim}{B}^k}(y) = \max\left[\mu_{\underset{\sim}{A}_1^k}(\text{input}(i)) \cdot \mu_{\underset{\sim}{A}_2^k}(\text{input}(j))\right],$$

$$k = 1, 2, \ldots n.$$

Examples:

In Mechanics the energy of a moving body is called kinetic energy. If an object of mass m (kilograms) is moving with a velocity V (meters per second), then kinetic energy K (in joules) is given the equation $K = \frac{1}{2} mV^2$. Suppose we model the mass & velocity as inputs to system (moving body) and the energy as output, then observe the system for a while & deduce the following two disjunctive rules of inference based on our observations:

MIN

Rule 1: IF x_1 is $\underset{\sim}{A}_1^1$ (small mass) & x_2 is $\underset{\sim}{A}_2^1$ (high velocity)

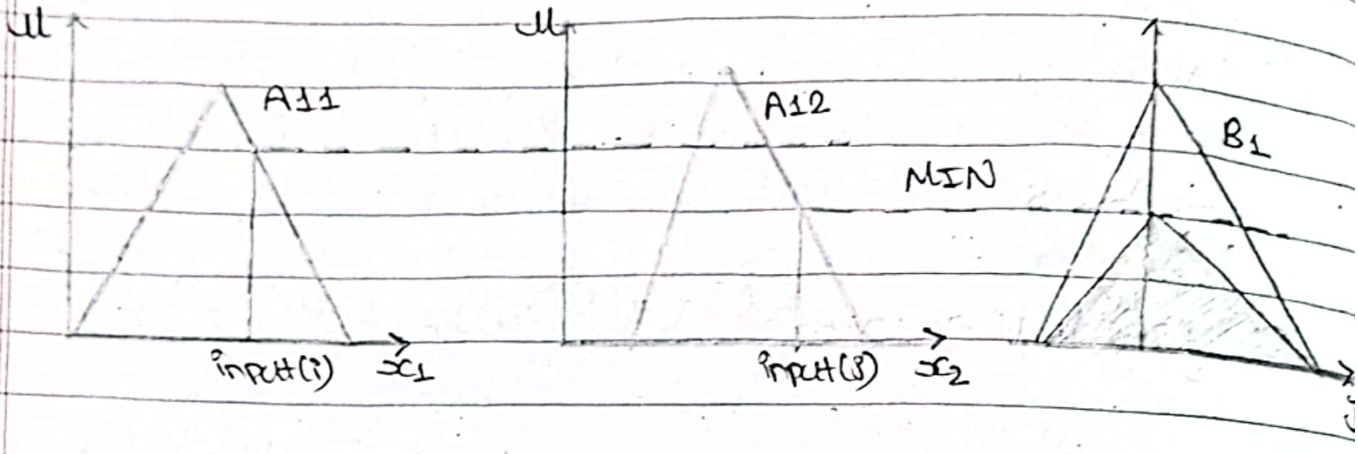
THEN y is $\underset{\sim}{B}^1$ (medium Energy).

MAX

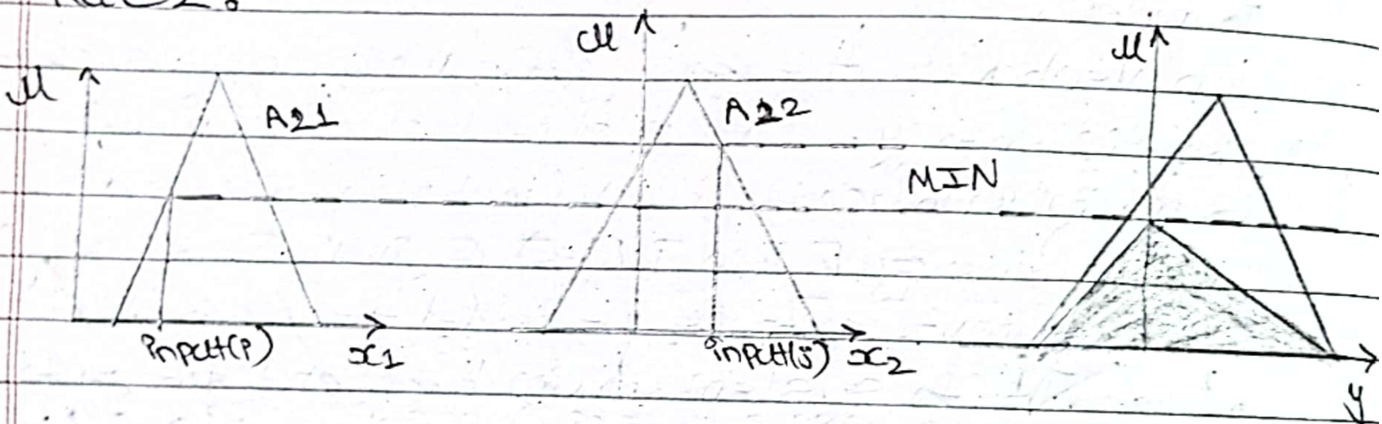
Rule 2: IF x_1 is $\underset{\sim}{A}_1^2$ (large mass) ~~and~~ OR x_2 is $\underset{\sim}{A}_2^1$ (high velocity).

THEN y is $\underset{\sim}{B}^2$ (high Energy)

Rule 1 :



Rule 2 :



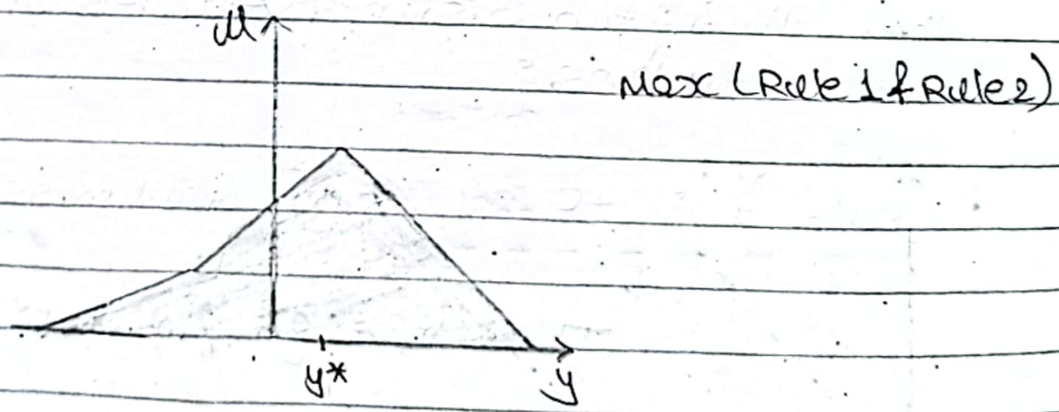


Fig :- Graphical Mamdani (Max-product) inference Method.

now, let's suppose we have made some observations of the system (moving body) & we estimate the values of two inputs, mass & velocity, as crisp values.

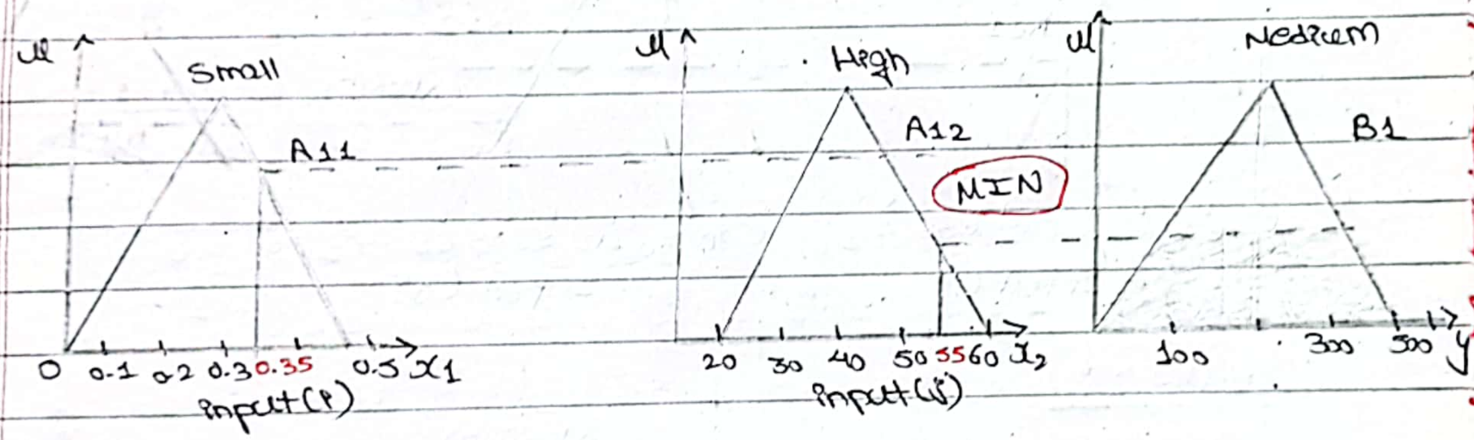
Let input $(i) = 0.35$ kg (mass)

input $(v) = 55$ m/s (velocity).

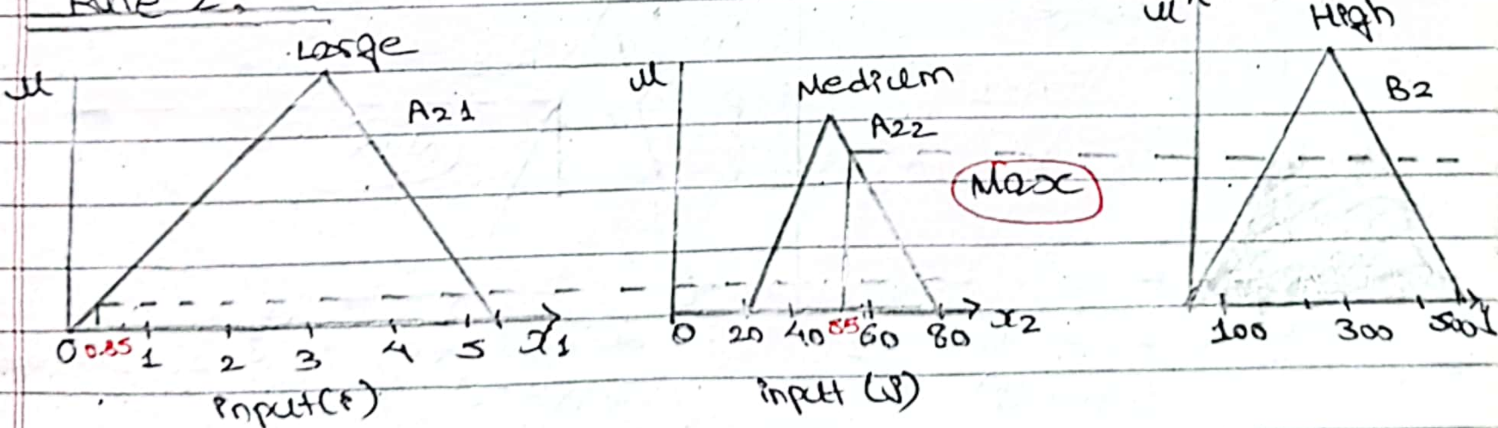
* Using case ① & ⓘ:

→ Case ① models inputs as delta functions, & uses a Mamdani (Max-Min) implication inference model. It is illustrated below, where the output fuzzy membership function is defuzzified using a centroid method.

Rule 1:



Rule 2:



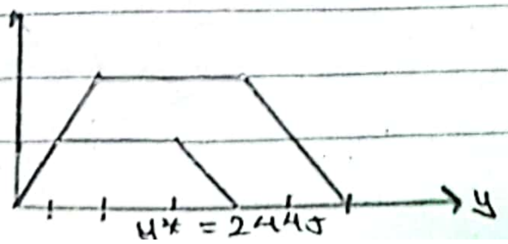
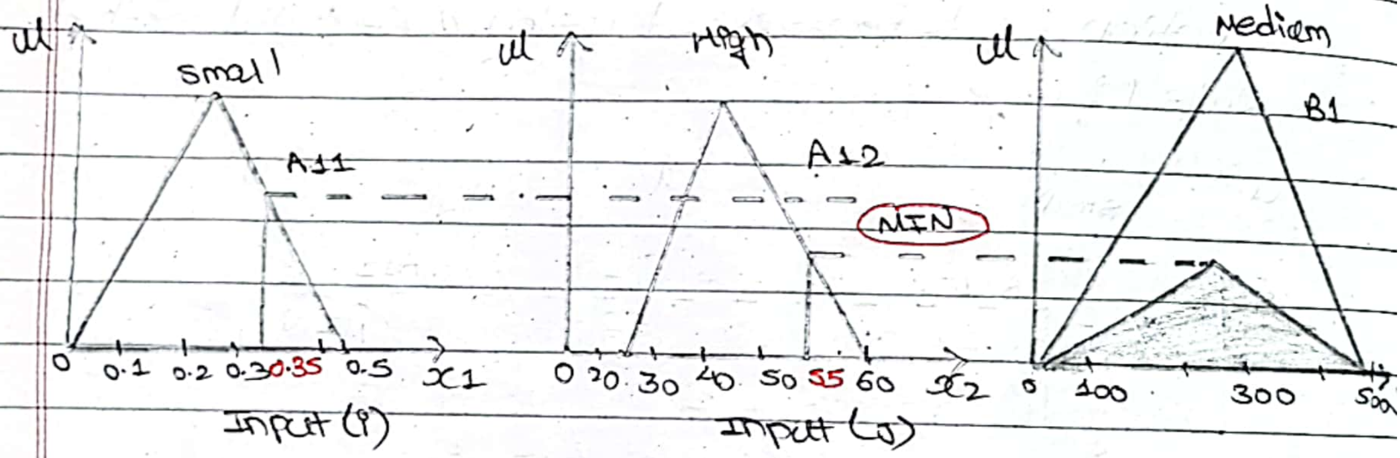


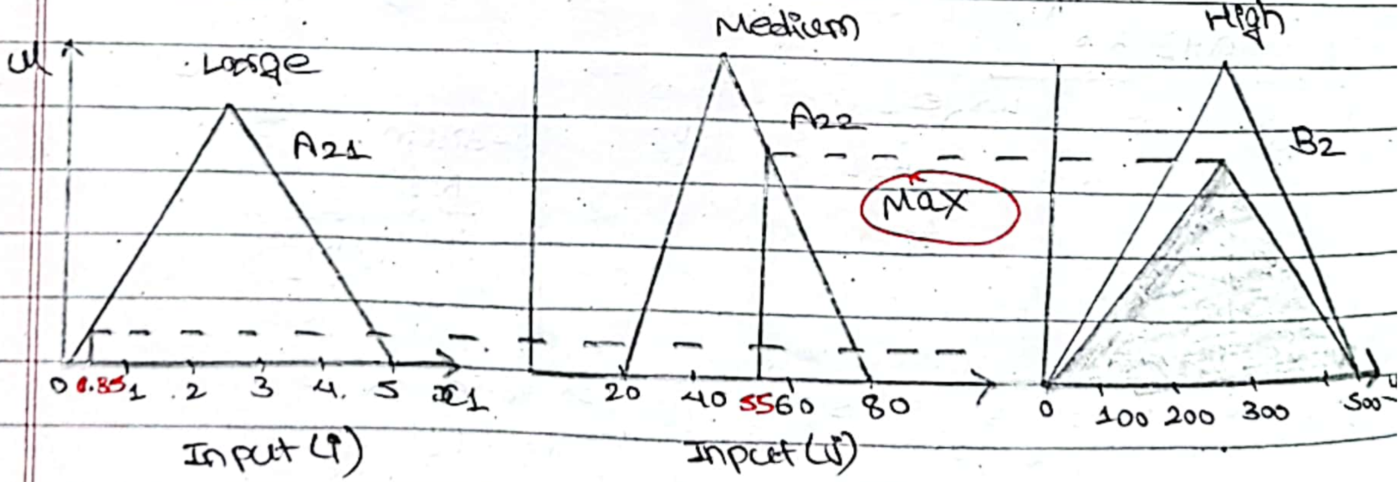
Fig: Fuzzy inference method using the case 1 graphical concept.

→ Case ②, we only change the method of implication from the first case. Now using a max-product implication method and its eqn and a centroidal defuzzification method, the graphical result can be illustrated below:

Rule 1:



Rule 2:



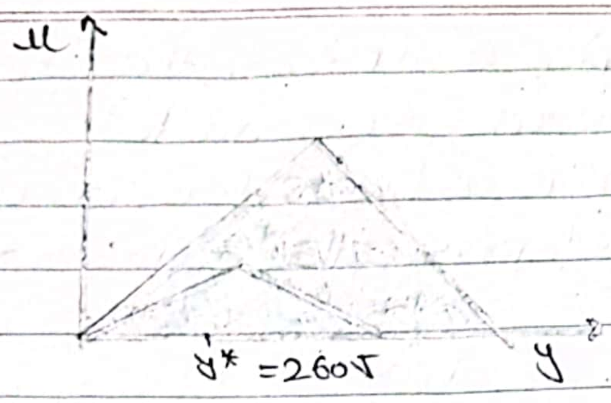


Fig :- Fuzzy inference method using case 2 graphical approach.

ii) Sugeno Method, or the TSK Method:

The defuzzied values that we seen in Mamdani inference method for the output energy are both fairly consistent. i.e 244 J & 260 J. The power of fuzzy rule-based system is their ability to yield "good" results with reasonably simple mathematical operations. # Also, we need a deffuzification method in mamdani method to get a defuzzified value but in sugeno method (TSK), the deffuzifaction is ~~done~~ done by the method itself.

→ The second inference method, generally referred to as the sugeno method, or the TSK method (Takagi, Sugeno, and Kang) was purposed in an effort to develop a systematic approach to generating fuzzy rules from a given input-output data set.

→ A typical rule in Sugeno method has two inputs x & y and output z, has the form

$$\text{IF } x \text{ is } \underset{\sim}{A} \text{ and } y \text{ is } \underset{\sim}{B}, \text{ THEN } z \text{ is } z = f(x,y)$$

where, $z = f(x,y)$ is a crisp function in the consequent.

i) When $f(x,y)$ is a constant, the inference system is called a <u>zero-order</u> Sugeno model,
ii) When $f(x,y)$ is a linear function of x and y, the inference system is called a first-order sugeno model.

* Graphical Representation:

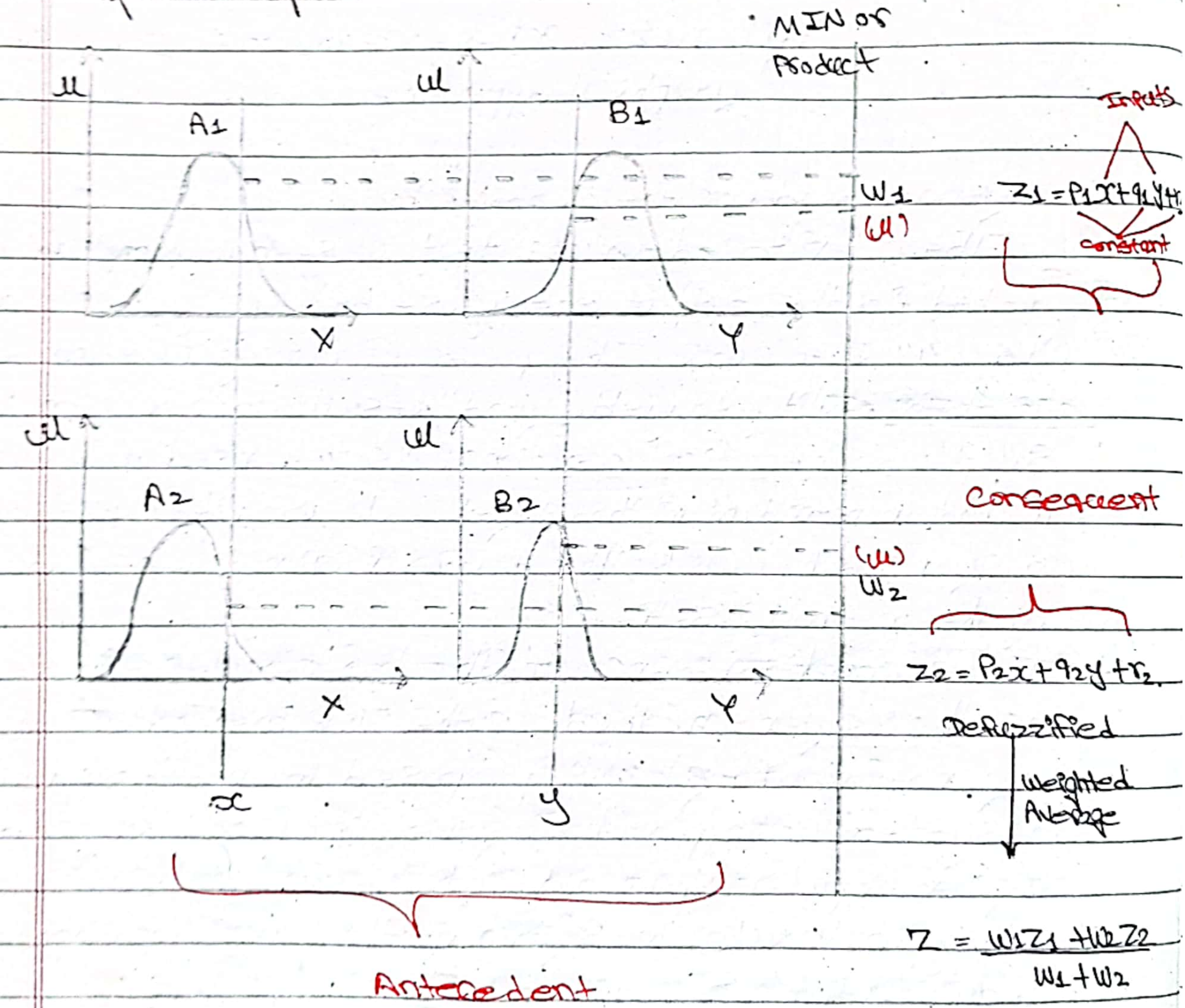


Fig:- The Sugeno fuzzy model.

→ In Sugeno model, each rule has a crisp output, given by a function because of this the overall output is obtained via a weighted Average deffuzzification as shown in above figure.
This process avoids the time-consuming methods of defuzzification necessary in the Mamdani model.

Example: An example of a two-input, single-output sugeno model with four rules are:

First order.

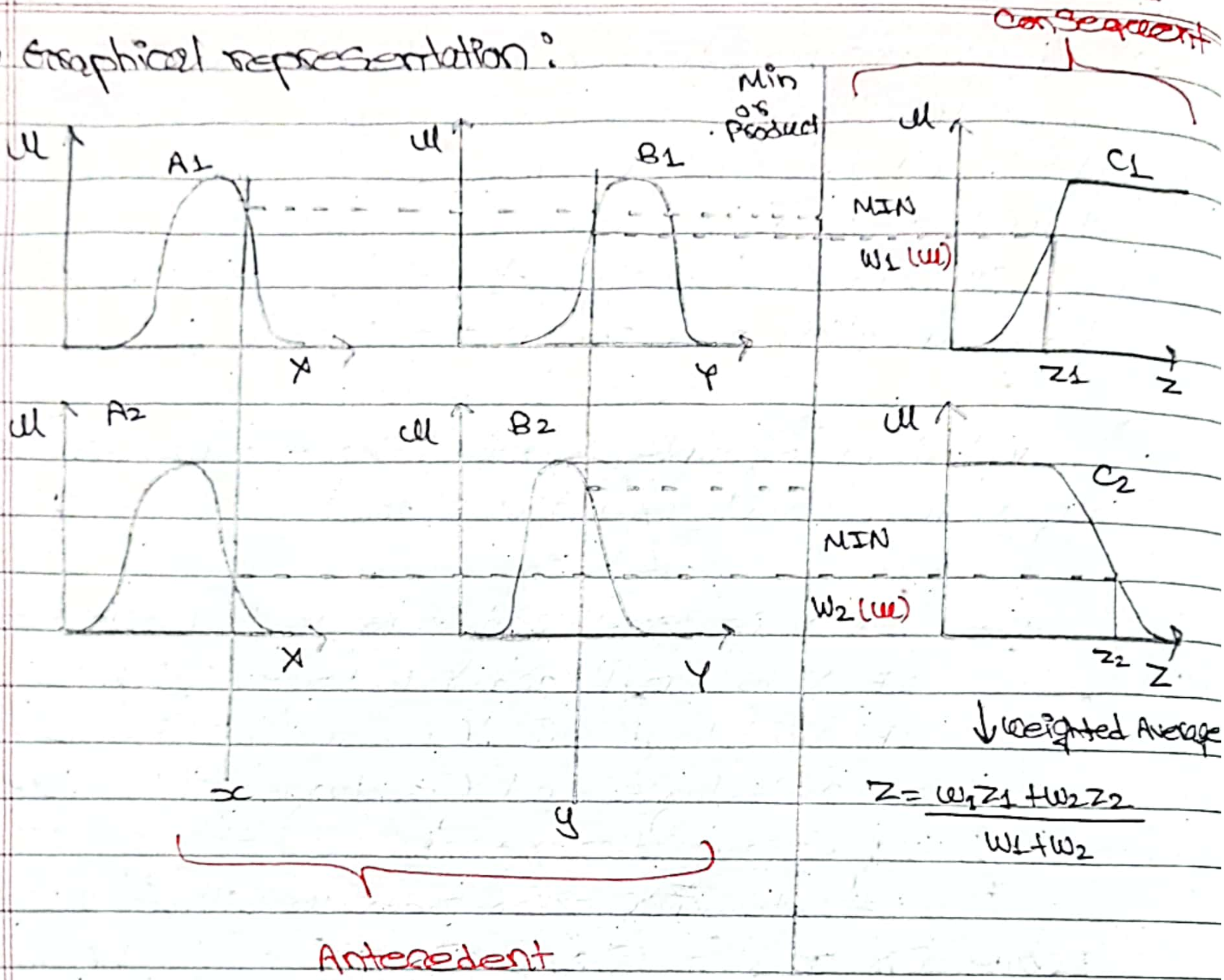
IF X is small and Y is small, THEN $Z = -x + y + 1$.
IF X is Small and Y is large, THEN $Z = -y + 3$.
IF X is large and Y is small, THEN $Z = -x + 3$.
IF X is large and Y is large, THEN $Z = x + y + 2$

Note: For zero-order we will get a constant value for Z.

iii) Tsukamoto inference method:

- The third inference method is Tsukamoto method where, the Consequent of each fuzzy rule is represented by a fuzzy set with a monotonic membership function (Shoulder function).
- The output of each rule is defined as a crisp value induced by the membership value coming from the antecedent clause of the rule. The overall output is calculated by the weighted average of each rule's output
- It also avoids the time-consuming process of defuzzification.

* Graphical representation:



* Let's see an example from our Text book for all inference methods: i) Mamdani method,

ii) Sugeno method &

iii) Tsukamoto method.

The given disjunctive rules of inferenced based on the observation of models are:

Rule 1: IF w is $\tilde{A}_1^1$ (Large flow rate) (MIN) and ΔTapp is $\tilde{A}_2^1$ (small approa

THEN ΔU is $\tilde{B}^1$ (Large heat exchange.

Rule 2: IF w is $\tilde{A}_1^2$ (small flow rate) (Max) or ΔTapp is $\tilde{A}_2^2$ (Large approa

THEN ΔU is $\tilde{B}^1$ (small heat exchange.

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Rule 3: IF W is $\underset{\sim}{A}_1^2$ (small flow rate) and ΔT_{app} is $\underset{\sim}{A}_2^1$ (small approach), THEN AU is $\underset{\sim}{B}^1$ (large heat exchanger).

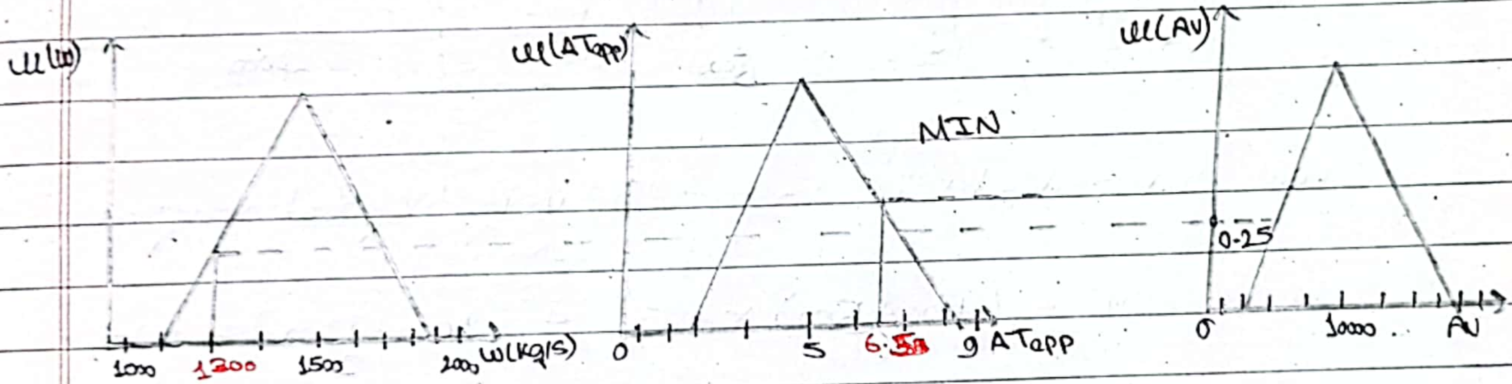
The two input values for flow rate & temperature are:

$$W = 1300\ Kg\,s^{-1}$$

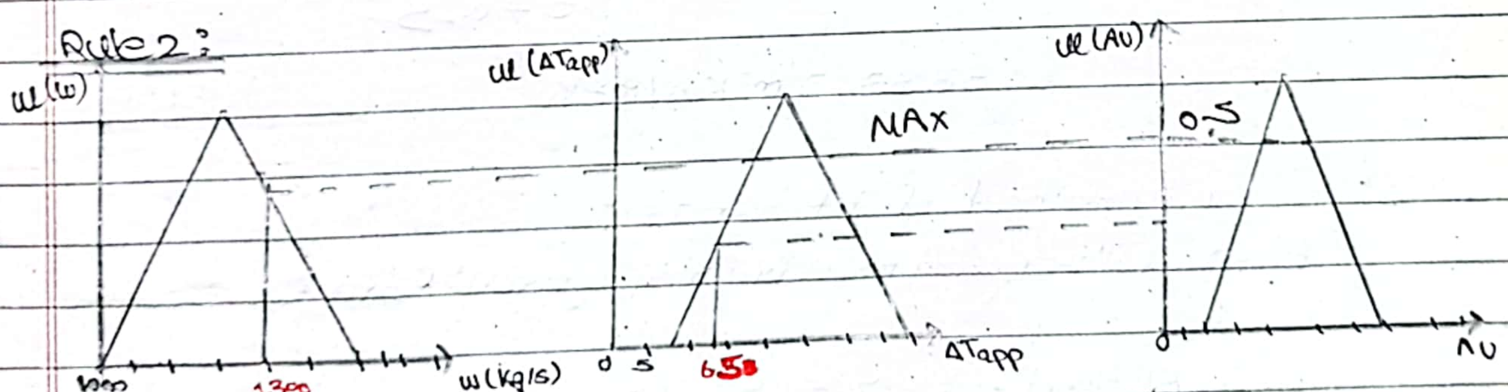
$$\Delta T_{app} = 6.5\ K$$

① Using Mamdani implication method of inference.
We are using MAX-MIN (clipping):

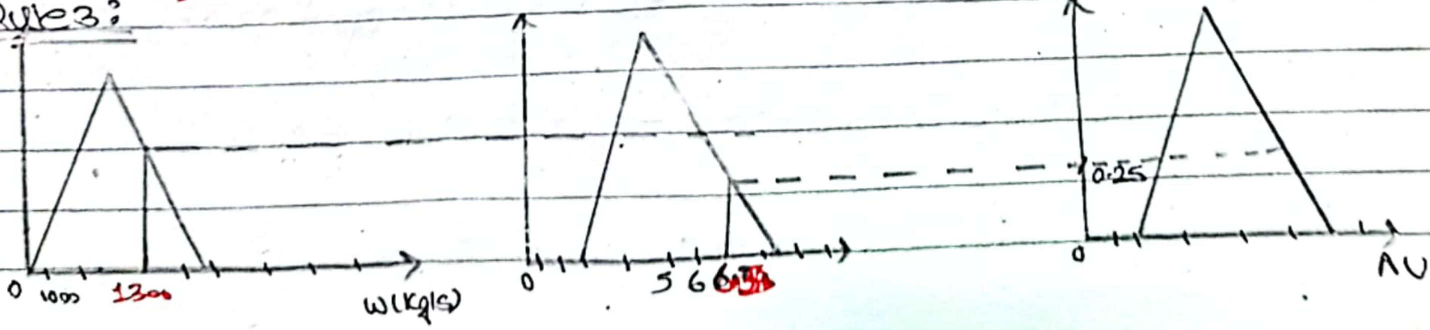
Rule 1:



Rule 2:



Rule 3:



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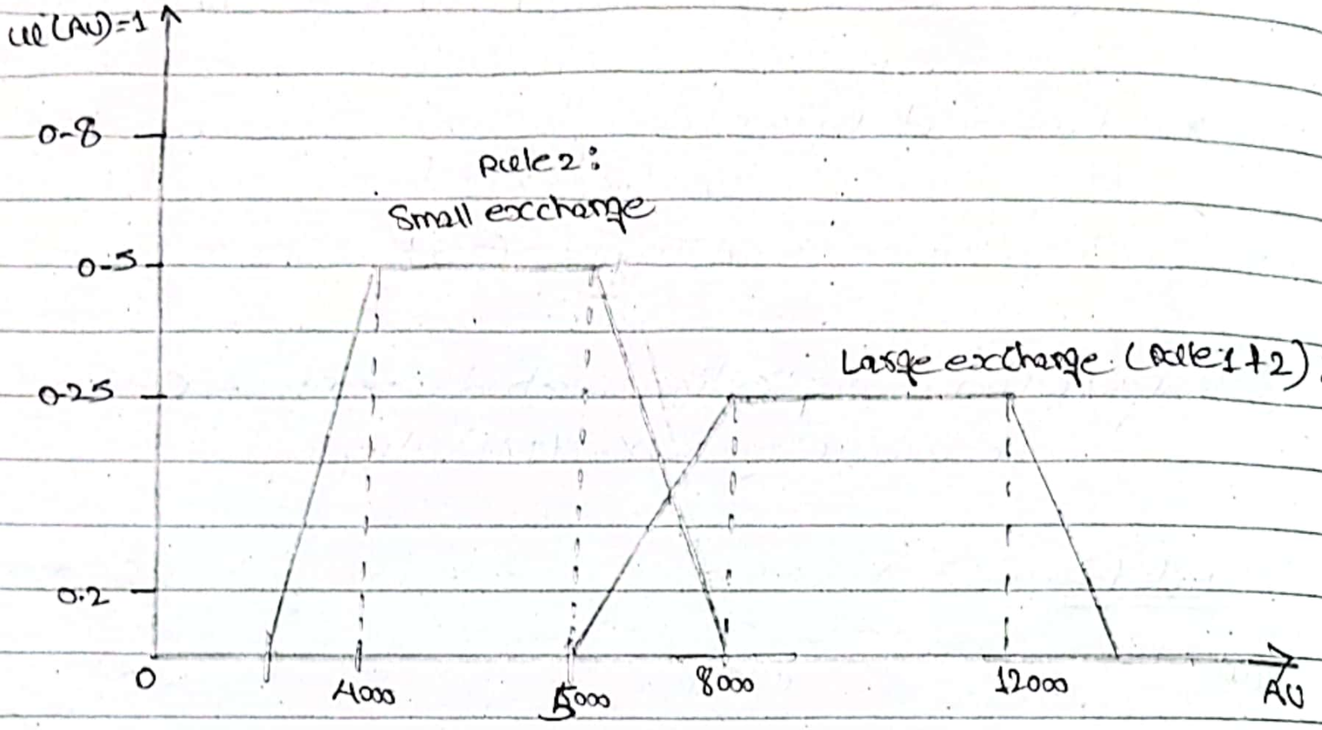
now, Aggregating all three rules we get:



now, Defuzzified value for mamdani using weighted average method is:

$$AU^* = \frac{\frac{(4000+5000)}{2} \times 0.5 + \frac{(8000+12000)}{2} \times 0.25}{0.5+0.25}$$

$= 3333.3 \ m^2 Kw/m^2K,$

2) Sugeno method of inference:
Given, following expression in polynomial form for our two consequent

$$AU_{small} = 3.4765 \omega - 210.5 \Delta T_{app} + 2103$$

$$AU_{large} = 4.6925 \omega - 526.2 \Delta T_{app} + 2031.$$

where, $\omega = 1800$ & $\Delta T_{app} = 6.33$ we get,

$AU_{small} = 3.4765 * 1300 - 210.5 * 6.5 + 2103$
$= 5256$

$AU_{large} = 4.6925 * 1300 - 526.2 * 6.5 + 2631$
$= 5311$

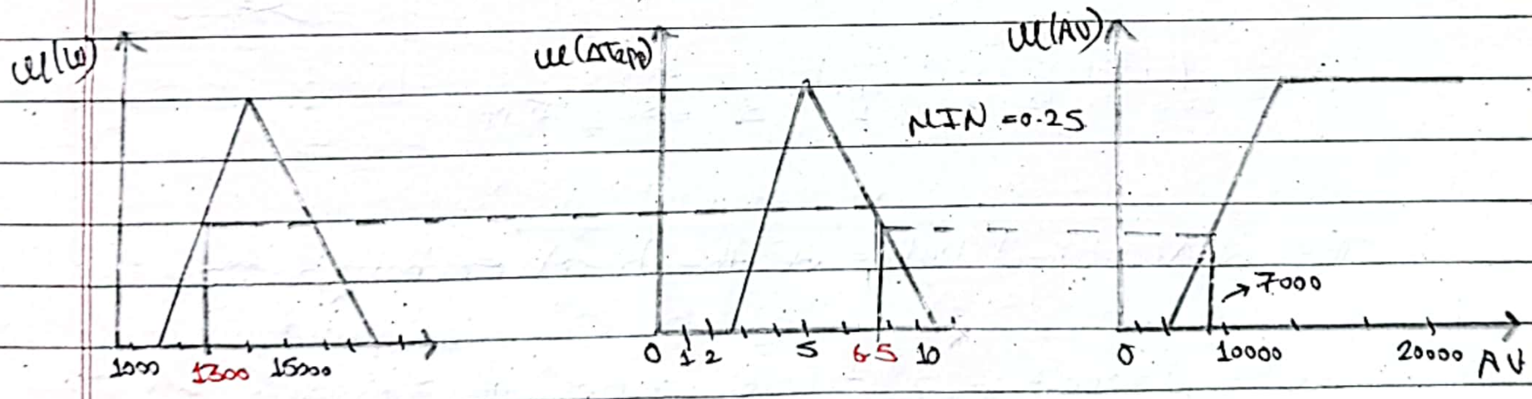
So, now,
defuzzified value using weighted Average method:

$$AU^* = \frac{(5311 \times 0.25) + (5256 * 0.5) + (5311 * 0.25)}{0.25 + 0.5 + 0.25}$$

$= 5283.5 \ m^2KW/m^2K.$

iii) Tsukamoto Inference method:
- we modify the output membership functions from the Nordame case, but we added shoulders to them for Tsukamoto.

Rule 1:



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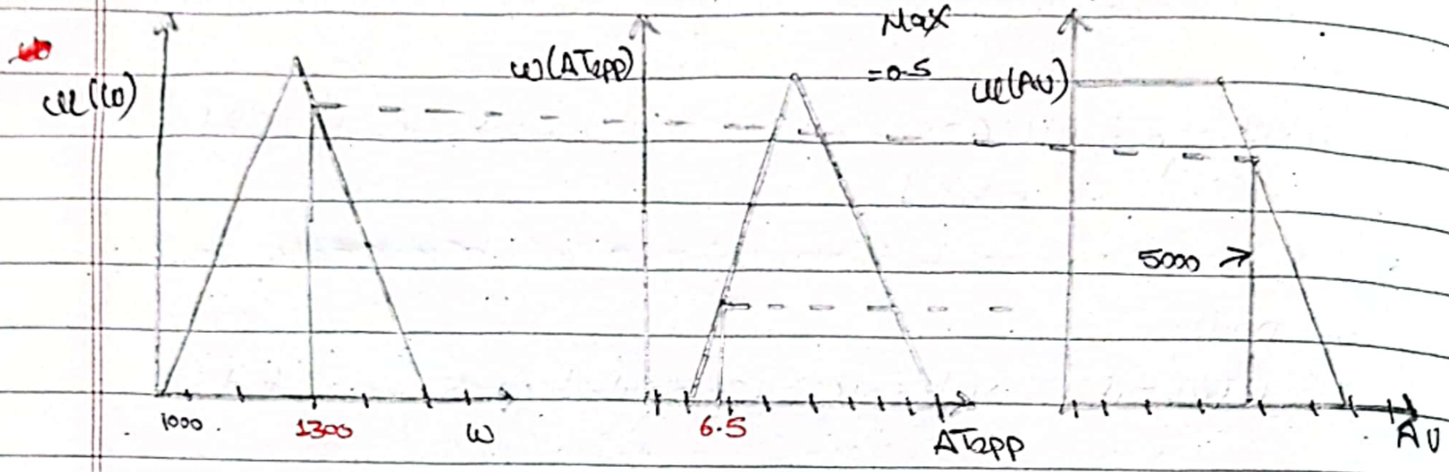
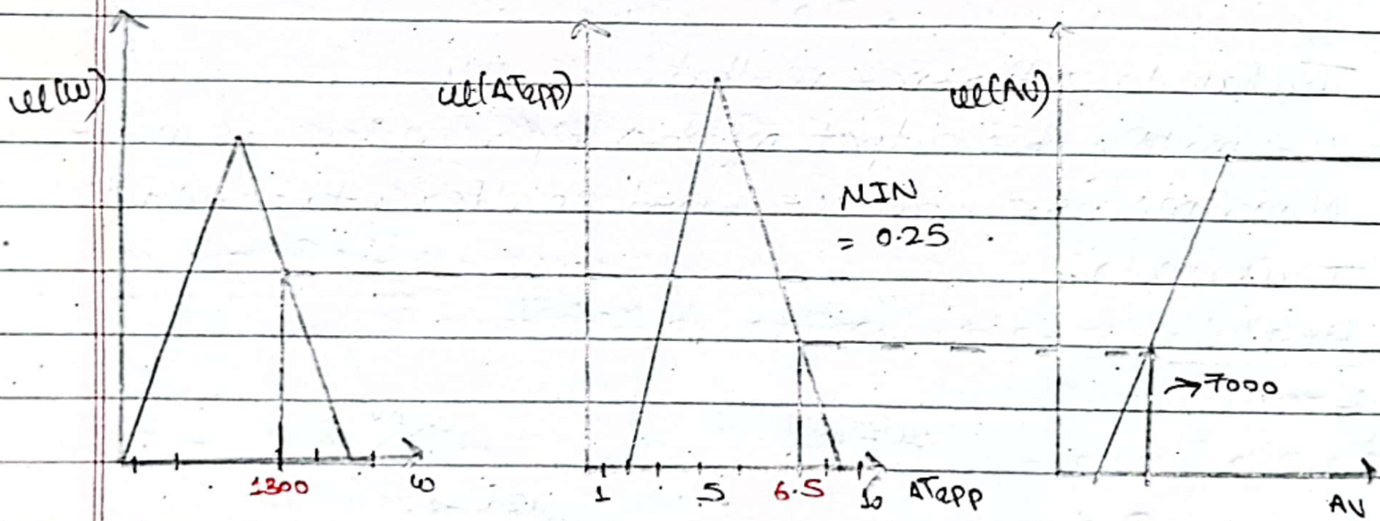
Rule2:



Rule3:



The defuzzied value of the heat exchanger size is

$$AU^* = \frac{(7000 \times 0.25) + (5000 \times 0.5) + (7000 \times 0.25)}{0.25 + 0.5 + 0.25}$$

$= 6250 \ m^2 kW/m^2K$

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* Example For inference method (Rule-based method):

Consider a Fuzzy rule based system for the study of quality of Land. The goodness of land is determined by the factors like its slopeness and Fertility. Suppose fuzzy sets defining these parameters are:

$$\text{Sloppy} = \left\{ \frac{0}{2}, \frac{0.3}{5}, \frac{0.5}{8}, \frac{1}{11}, \frac{0.6}{14}, \frac{0.3}{17}, \frac{0}{20} \right\}$$

$$\text{Slightly Sloppy} = \left\{ \frac{0}{1}, \frac{0.3}{2}, \frac{0.4}{6}, \frac{0.8}{8}, \frac{0.5}{10}, \frac{0.1}{12}, \frac{0}{13} \right\}$$

$$\text{Highly Fertile} = \left\{ \frac{0}{5}, \frac{0.3}{10}, \frac{0.7}{15}, \frac{0.9}{20}, \frac{0.8}{25}, \frac{0.3}{40}, \frac{0}{45} \right\}$$

$$\text{Low Fertile} = \left\{ \frac{0}{3}, \frac{0.2}{5}, \frac{0.4}{10}, \frac{0.7}{20}, \frac{0.5}{25}, \frac{0.1}{30}, \frac{0}{35} \right\}$$

$$\text{Excellent quality} = \left\{ \frac{0}{34}, \frac{0.3}{40}, \frac{0.5}{45}, \frac{1}{50}, \frac{0.6}{55}, \frac{0.3}{60} \right\}$$

$$\text{Good quality} = \left\{ \frac{0.4}{20}, \frac{0.6}{35}, \frac{0.8}{40}, \frac{0.5}{50}, \frac{0.3}{55}, \frac{0.2}{65}, \frac{1}{70} \right\}$$

The rules in the FRBC contains.

Max

R1 : IF Land is sloppy or highly fertile then it has excellent quality.

MIN

R2 : IF Land is slightly sloppy & Low fertile then it has good quality.

now, using mamdani model with max-min inference method of deffization & max-Product inference method of deffization determine the quality with slope value = 8
Fertility = 20

= Sol

given,

Slope value = 8

Fertility = 20

i) For Max-min inference method of defuzzification:

now,

R_1 μ sloppy (8) = 0.5

R_2 μ slightly sloppy (8) = 0.8

similarly,

R_1 μ highly fertile (20) = 0.9

R_2 μ low fertile (20) = 0.7

we can see both R_1 & R_2 will satisfy so,

① for R_1:

$$\text{Membership of antecedents} = \max(\mu \text{sloppy}(8), \mu_{HF}(20))$$
$$= \max(0.5, 0.9)$$
$$= 0.9$$

now, the consequent using min will be;

$$C_1 = \left\{ \frac{\min(0.9, 0)}{34}, \frac{\min(0.9, 0.3)}{40}, \frac{\min(0.9, 0.5)}{45}, \frac{\min(0.9, 1)}{50}, \frac{\min(0.9, 0.6)}{55}, \frac{\min(0.9, 0.3)}{60} \right\}$$

$$= \left\{ \frac{0}{34}, \frac{0.3}{40}, \frac{0.5}{45}, \frac{0.9}{50}, \frac{0.6}{55}, \frac{0.3}{60} \right\}$$

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② For R2:

Membership of Antecedents $= \min(\mu_{SS}^{(8)}, \mu_{LF}^{(20)})$
$= \min(0.8, 0.7)$
$= 0.7$

now, the consequent using min will be,

$$C_2 = \left\{ \frac{\min(0.7, 0.4)}{20}, \frac{\min(0.7, 0.6)}{35}, \frac{\min(0.7, 0.8)}{40}, \frac{\min(0.7, 0.5)}{50}, \frac{\min(0.7, 0.3)}{55}, \frac{\min(0.7, 0.9)}{65}, \frac{\min(0.7, 1)}{70} \right\}$$

$$= \left\{ \frac{0.4}{20}, \frac{0.6}{35}, \frac{0.7}{40}, \frac{0.5}{50}, \frac{0.3}{55}, \frac{0.7}{65}, \frac{0.7}{70} \right\}$$

now, Aggregating C_1 & C_2 we get,

$C = \max(C_1 \text{ and } C_2)$

$$= \left\{ \frac{0.4}{20}, \frac{0}{34}, \frac{0.6}{35}, \frac{0.7}{40}, \frac{0.9}{50}, \frac{0.6}{55}, \frac{0.3}{60}, \frac{0.7}{65}, \frac{0.7}{70} \right\}$$

using Max method the deffuzzification of C is 50.
Hence, the quality is 50.

ii) For max-product inference method of deffuzification:
For R1:
The consequent using Max-product will be;

$$C_1 = \left\{ \frac{0.9 \times 0}{34}, \frac{0.9 \times 0.3}{40}, \frac{0.9 \times 0.5}{45}, \frac{0.9 \times 1}{50}, \frac{0.9 \times 0.6}{55}, \frac{0.9 \times 0.3}{60} \right\}$$

$$= \left\{ \frac{0}{34}, \frac{0.27}{40}, \frac{0.45}{45}, \frac{0.9}{50}, \frac{0.54}{55}, \frac{0.27}{60} \right\}$$

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For R_2:

The consequent using product will be:

$$C_2 = \left\{ \frac{0.7*0.4}{20}, \frac{0.7*0.6}{35}, \frac{0.7*0.8}{40}, \frac{0.7*0.5}{50}, \frac{0.7*0.3}{55}, \frac{0.7*0.9}{65}, \frac{0.7*1}{70} \right\}$$

$$= \left\{ \frac{0.28}{20}, \frac{0.42}{35}, \frac{0.56}{40}, \frac{0.35}{50}, \frac{0.21}{55}, \frac{0.63}{65}, \frac{0.7}{70} \right\}$$

now, Aggregating C_1 & C_2 we have:

$$C = \max(C_1, C_2)$$

$$C = \left\{ \frac{0.28}{20}, \frac{0}{34}, \frac{0.42}{35}, \frac{0.56}{40}, \frac{0.45}{45}, \frac{0.9}{50}, \frac{0.54}{55}, \frac{0.27}{60}, \frac{0.63}{65}, \frac{0.7}{70} \right\}$$

using Max-method, the deffuzzification of C is 50, the quality is 50.

ii) For Sugeno or TSK method:

① using zero order TSK (quality) $= \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$

$=$ ~~[illegible]~~

where the rules in the Fuzzy rule base system are:

R_1: IF Land is sloppy or highly fertile then quality is 20.
R_2: IF land is slightly sloppy and low fertile the quality is 6

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now, using zero-order TSK model determine quality with slope value 8 & Fertility 20.

→

Slope value = 8
Fertility = 20.

① R1
now, $\mu_{slopy}(8) = 0.5$ & $\mu_{HF}(20) = 0.9$

For R1 with quality $(z_1) = 20$
Membership of antecedents (ω_1) = $\max(\mu_{slopp}(8), \mu_{HF}(20))$
$= \max(0.5, 0.9)$
$= 0.9$

② For R2 :
now,
$\mu_{SS}(8) = 0.8$ & $\mu_{LF}(20) = 0.7$
For R2 with quality $(z_2) = 60$
Membership of antecedents $(\omega_2) = \min(\mu_{SS}(8), \mu_{HF}(20))$
$= \min(0.8, 0.7)$
$= 0.7$

now, Defuzzified value is:
zero-order TSK (quality) $= \frac{\omega_1 z_1 + \omega_2 z_2}{\omega_1 + \omega_2}$

$= \frac{0.9 \times 20 + 0.7 \times 60}{0.9 + 0.7}$

$= \frac{60}{1.6}$

$= 37.5$

The quality is 37.5

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(2) For First order TSK model:

The rules in the Fuzzy rule based system contains:

R1: IF land is sloppy or highly fertile the quality is

$$= \frac{\text{sloppy} * \text{highly fertile}}{2}$$

R2: IF land is slightly sloppy and low fertile the quality = slightly sloppy + low fertile.

now, using first order TSK model determine quality with slope value 8 & fertility = 20

= Sol

We have,

slope value = 8

Fertility = 20

now,

For R1:

$\mu_{sloppy}(8) = 0.5$ & $\mu_{HF}^{(20)} = 0.9$

R1 membership of antecedents $(W_1) = \max(\mu_{sloppy}^{(8)}, \mu_{HF}^{(20)})$

$= \max(0.5, 0.9)$

$= 0.9$

For R2:

$\mu_{ss}(8) = 0.8$ & $\mu_{LF}^{(20)} = 0.7$

R2 membership of antecedent $(W_2) = \min(\mu_{ss}^{(8)}, \mu_{HF}^{(20)})$

$= \min(0.8, 0.7)$

$= 0.7$

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now,

$$quality\ (z_1) = \frac{sloppy * highly\ fertile}{2}$$

$$= \frac{8 * 20}{2}$$

$$= 80$$

$$quality\ (z_2) = slightly\ sloppy + low\ fertile$$

$$= 8 + 20$$
$$= 28$$

Therefore, using First-order TSK model, defuzzified value is

$$quality\ (z) = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

$$= \frac{0.9 * 80 + 0.7 * 28}{0.9 + 0.7}$$

$$= \frac{72 + 19.6}{1.6}$$

$$= \frac{91.6}{1.6}$$

$$= 57.25$$

$\therefore$ The quality is 57.25.

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iii) Tsukamoto Model:

All the process for R1 & R_2 are same as Mamdani model only difference is selection of Z_1 & Z_2 which represents the elements with membership values of Antecedents in R_1 & R_2 which are 0.9, 0.7 respectively so,

$$C_1 = \left\{ \frac{0}{34}, \frac{0.3}{40}, \frac{0.5}{45}, \frac{0.9}{50}, \frac{0.6}{55}, \frac{0.3}{60} \right\}$$

Z_1 = ~~bag~~ set element with membership 0.9
= 50

$$C_2 = \left\{ \frac{0.4}{20}, \frac{0.6}{35}, \frac{0.7}{40}, \frac{0.5}{50}, \frac{0.3}{50}, \frac{0.3}{55}, \frac{0.7}{65}, \frac{0.7}{70} \right\}$$

Z_2 = set element with membership 0.7
= 40. (we are consider 40 as we can see there are more elements with $\mu = 0.7$)

now,

Defuzzified value:

$$Z(\text{quality}) = \frac{w_1 Z_1 + w_2 Z_2}{w_1 + w_2}$$

$$= \frac{0.9 \times 50 + 0.7 \times 40}{0.9 + 0.7}$$

$$= \frac{73}{1.6}$$

$$= 45.625$$

Hence, the quality is 45.625