Simplex Method

Presented by : Narayan Upreti (Roll no :03)

: Shreelata Wagle Khanal (Roll no:43)

Optimization & why ?

- In general optimization means the action of making the best or most effective use of a situation or resources.

- Mathematical optimization problems deals with maximizing or minimizing some function relative to some set, often representing a range of choices available in a certain situation.

- The function allows comparison of the different choices for determining which might be “best.”

Application area of optimization

Optimization is useful in different fields:

- Manufacturing
- Inventory control
- Scheduling
- Finance
- Mechanics
- Control engineering
- Policy Modeling
- Production
- Transportation
- Networks
- Engineering
- Economics
- Marketing

Optimization technique

- It is a technique to solve the given optimization problem to find its best optimal solution (value).

- Different technique can be used to solve the optimization problem.

  Some of them are:

1. Graphical method
2. Simplex method

Slack Variable

- In any lpp , if a constraint has $\leq$ sign

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \ldots\ldots\ldots\ldots + a_{in}x_n \leq b_i$$

then in order to make it an equality, we have to add something positive to LHS,

ie. $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \ldots\ldots\ldots\ldots + a_{in}x_n + s_i = b_i$

that positive variable s_i is called as slack variable.

Surplus variable

- In any lpp , if a constraint has $\geq$ sign

  $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots\dots\dots\dots\dots + a_{in}x_n \geq b_i$

  then in order to make it an equality, we have to subtract something positive to LHS,

  ie. $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots\dots\dots\dots\dots + a_{in}x_n - s_i = b_i$

  that positive variable s_i is called as surplus variable.

Unrestricted variable

- Any variable x_i which takes either positive, negative or zero values is called as unrestricted variables.
- eg. Maximize $Z = 3x_1 + 2x_2$

$$x_1 - x_2 \geq 0$$
$$-3x_1 + x_2 \geq 3$$
$$x_2 \geq 0$$

here variable x_1 is undefined so, it can be either +ve, -ve or 0 values.

- If x_i is an unrestricted variable , we always consider $x_i = x_i' - x_i''$ where x_i' , $x_i'' \geq 0$

Canonical form of LPP

- It is said to be in canonical form if it has the following characteristics.

1. Objective function is of maximization / minimization type.
2. All constraints are of ≤ / ≥ type.
3. All decision variables are of ≥ 0.

eg. Given lpp

$$\text{Max } Z = 3x_1 + 2x_2 + 7x_3$$

$$6x_1 - 2x_2 + 5x_3 \geq 5$$

$$-x_1 + 3x_2 - 4x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

Canonical form of LPP

Solution:

The canonical form is

$$\text{Max } Z = 3x_1 + 2x_2 + 7x_3$$

$$-6x_1 + 2x_2 - 5x_3 \leq -5$$

$$-x_1 + 3x_2 - 4x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

Standard form of LPP

- A general LPP is said to be in standard form it has the following characteristics.

1. RHS of each constraint is positive .
2. All constraints are of = type.
3. All decision variable are of ≥ 0.

eg. Given lpp

$$\text{Max } Z = 2x_1 - 3x_2 + 6x_3$$
$$x_1 - 3x_2 \geq 4$$
$$2x_1 - 8x_2 + 3x_3 \leq 4$$
$$x_1 + x_2 \geq -7$$
$$x_1 , x_2 , x_3 \geq 0$$

Standard form of LPP

Solution:

Making each constraint RHS positive , we have

$$\text{Max } Z = 2x_1 - 3x_2 + 6x_3$$

$$x_1 - 3x_2 \geq 4$$

$$2x_1 - 8x_2 + 3x_3 \leq 4$$

$$-x_1 - x_2 \leq 7$$

$$x_1, x_2, x_3 \geq 0$$

Standard form of LPP

Now , The Standard form is

$$\text{Max } Z = 2x_1 - 3x_2 + 6x_3$$

$$x_1 - 3x_2 - s_1 = 4$$

$$2x_1 - 8x_2 + 3x_3 + s_2 = 4$$

$$-x_1 - x_2 + s_3 = 7$$

$$x_1 , x_2 , x_3 , s_1 , s_2 , s_3 \geq 0$$

Simplex Method Algorithm

1. Convert the given general LPP into standard LPP.
   i. Objective function of Lpp must be maximized. If it is to be minimized then we have to convert it into a problem of maximization by Max Z'= -Min Z
   ii. Check all the decision variables are greater than zero.
   iii. Express the problem in standard form by introducing slack or surplus variable to convert the inequality constraints into equation.
   iv. All the values of right hand side must be positive.
2. Write the values of initial basic feasible solution.
3. Write the standard form Lpp into matrix form.
4. Construct the initial simplex table.
5. Calculate the value of $Z_j - C_j = C_B X_j - C_j$

Simplex Method Algorithm

- I. If all $(Z_j - C_j) \geq 0$, the optimal solution will obtained.
  II. If at least one $(Z_j - C_j)$ is –ve then indicate it by an arrow and this column is called key column.
  III. If more than one $(Z_j - C_j)$ is –ve then choose the most negative of them and this
  IV. column is called key column.

6. Calculate minimum positive ratio. ie min.ratio $= \frac{X_B}{C_k}$, C_k = key column ,> 0
7. Construct the new simplex table by entering incoming vector.
8. Repeat step 5,6.

simplex method Maximization example

Q. $\text{Max } Z = 3x_1 + 2x_2 + 5x_3$

Subject to

$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420$$
$$x_1, x_2, x_3 \ge 0$$

<u>Solution:</u>

By introducing slack variables s_1, s_2, s_3, convert the problem in standard form.

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to,

$$x_1 + 2x_2 + x_3 + s_1 = 430$$
$$3x_1 + 2x_3 + s_2 = 460$$
$$x_1 + 4x_2 + s_3 = 420$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

An initial basic feasible solution is given by,

$$x_1 = x_2 = x_3 = 0, \; s_1 = 430, \; s_2 = 460, \; s_3 = 420$$

Writing in matrix form, $AX = B$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 430 \\ 460 \\ 420 \end{bmatrix}$$

An initial simplex table.

		C_j	3	2	5	0	0	0	
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min ration $= X_B/C_K$
0	S_1	430	1	2	1	1	0	0	$430/1 = 430$
0	S_2	460	3	0	2	0	1	0	$460/2 = 230$
0	S_3	420	1	4	0	0	0	1	–
$Z_j - C_j$			-3	-2	-5 ↑	0	0	0	No optimal

incoming vector

outgoing vector

Calculation for initial simplex table,

$$Z_j - C_j = C_B X_j - C_j$$

$$Z_1 - C_1 = C_B X_1 - C_1$$

$$= (0\times1 + 0\times3 + 0\times1) - 3$$

$$= 3$$

$$Z_2 - C_2 = -2$$

and same for other.

① 1st Iteration

		c_j	3	2	5	0	0	0
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	200	$-\frac{1}{2}$	2	0	1	$-\frac{1}{2}$	0
5	x_3	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0
0	s_3	420	1	4	0	0	0	1
$z_j - c_j$			$\frac{9}{2}$	$-2\uparrow$	0	0	$\frac{5}{2}$	0

No optimal

calculation for above table,

$$\frac{430\times 2 - 460\times 1}{2} = \frac{860-460}{2} = 200$$

$$\frac{420\times 2 - 460\times 0}{2} = 420$$

$$\frac{1\times 2 - 3\times 1}{2} = -\frac{1}{2}$$

$$\frac{1\times 2 - 3\times 0}{2} = 1$$

$$\frac{2\times 2 - 0\times 1}{2} = 2$$

$$\frac{4\times 2 - 0\times 0}{2} = 4$$

$$\frac{0\times 2 - 1\times 1}{2} = -\frac{1}{2}$$

$$\frac{0\times 2 - 0\times 1}{2} = 0$$

$$z_j - c_j = C_B X_j - c_j$$

$$z_1 - c_1 = C_B X_1 - c_1$$
$$= \left(0\times -\tfrac{1}{2} + 5\times \tfrac{3}{2} + 0\times 1\right) - 3$$
$$= \frac{9}{2}$$

$$z_2 - c_2 = C_B X_2 - c_2$$
$$= (0\times 2 + 5\times 0 + 0\times 4) - 2$$
$$= -2$$

And same process for other.

		c_j	3	2	5	0	0	0	
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	Min-ratio = $\frac{X_B}{X_K}$
0	S_1	200	$-\frac{1}{2}$	2	0	1	$-\frac{1}{2}$	0	$\frac{200}{2} = 100$
5	x_3	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	-
0	S_3	420	1	4	0	0	0	1	$\frac{420}{4} = 105$
$z_j - c_j$			$\frac{9}{2}$	$-2\uparrow$	0	0	$\frac{5}{2}$	0	

(2) Second Iteration.

		c_j	3	2	5	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
2	x_2	100	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0
5	x_3	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0
0	S_3	20	2	0	0	-2	1	1
$z_j - c_j$			4	0	0	1	2	0

Optimal

Calculation for above table,

$$\frac{420 \times 2 - 200 \times 4}{2} = 20$$

$$\frac{1 \times 2 - (-\frac{1}{2}) \times 4}{2} = 2$$

$$\frac{0 \times 2 - 4 \times 1}{2} = -2$$

$$\frac{0 \times 2 - 4 \times (-\frac{1}{2})}{2} = 1$$

$$\frac{1 \times 2 - 4 \times 0}{2} = 1$$

$z_j - c_j = C_B X_j - c_j$

$z_1 - c_1 = C_B X_1 - c_1$

$= (2 \times -\frac{1}{4} + 5 \cdot \frac{3}{2} + 0 \times 2) - 3$

$= 4$

$z_2 - c_2 = C_B X_2 - c_2$

$= (2 \times 1 + 5 \times 0 + 0 \times 0) - 2$

$= 0$

and same for other.

Since all $z_j - c_j \geq 0$, the solution is optimum and given by $x_2 = 100$, $x_3 = 230$, $x_1 = 0$

$$\text{Max } Z = C_B X_B$$

$$= (2 \times 100 + 5 \times 230 + 0 \times 20)$$

$$= 1350$$

simplex method Minimization Example

Q.

$$\text{Min } Z = x_1 - 3x_2 + 2x_3$$

Subject to

$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

Soln

By introducing slack variables s_1, s_2, s_3. Convert the problem in standard form.

$$\text{Max } Z' = -\text{Min } Z$$
$$= -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to,

$$3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 + 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

An initial basic feasible solution is given by,

$$x_1 = x_2 = x_3 = 0, \quad s_1 = 7, \quad s_2 = 12, \quad s_3 = 10.$$

Writing in Matrix form, $AX = B$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \end{array}$$
$$\begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}$$

An initial simplex Table

incoming vector

		C_j	-1	3	-2	0	0	0	
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	Min ratio $= X_B/x_k$
0	S_1	7	3	-1	2	1	0	0	-
0	S_2	12	-2	4	0	0	1	0	$12/4 = 3$
0	S_3	10	-4	3	8	0	0	1	$10/3 = 3.33$
$Z_j - C_j$			1	-3 ↑	2	0	0	0	No optimal

outgoing vector

calculation for initial simplex table.

$Z_j - C_j = C_B X_j - C_j$

$Z_1 - C_1 = C_B X_1 - C_1$

$= (0\times3 + 0\times-2 + 0\times-4) - (-1)$

$= 1$

$Z_2 - C_2 = C_B X_2 - C_2$

$= (0\times(-1) + 0\times4 + 0\times3) - 3$

$= 3$

and same for other.

① <u>First Iteration</u>

$R_2' \rightarrow R_2/4$

		c_j	-1	3	-2	0	0	0
c_B	B	x_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	10	$5/2$	0	2	1	$1/4$	0
3	x_2	3	$-1/2$	1	0	0	$1/4$	0
0	s_3	1	$-5/2$	0	8	0	$-3/4$	1
$z_j - c_j$			$-1/2 \uparrow$	0	2	0	$3/4$	0

No optimal

$R_1' \rightarrow R_1 + R_2'$

$R_3' \rightarrow R_3 - 3R_2'$

Calculation for above table.

$z_j - c_j = c_B x_j - c_j$

$z_1 - c_1 = c_B x_1 - c_1$

$= \left(0 \times \frac{5}{2} + 3 \times -\frac{1}{2} + 0 \times -\frac{5}{2}\right) - (-1)$

$= -\frac{1}{2}$

$z_2 - c_2 = c_B x_2 - c_2$

$= (0 \times 0 + 3 \times 1 + 0 \times 0) - 3$

$= 0$

and same process for other.

		C_j	-1	3	-2	0	0	0	
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min. Ratio $= X_B/C_k$
0	s_1	10	$5/2$	0	2	1	$1/4$	0	$10/\frac{5}{2} = 4$
3	x_2	3	$-1/2$	1	0	0	$1/4$	0	–
0	s_3	1	$-5/2$	0	8	0	$-3/4$	1	–
$z_j - c_j$			$-1/2 \uparrow$	0	2	0	$3/4$	0	

(2) Second Iteration

$R_1' \to \frac{2}{5} R_1$

		C_j	-1	3	-2	0	0	0
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	4	1	0	$4/5$	$2/5$	$1/10$	0
3	x_2	5	0	1	$2/5$	$1/5$	$3/10$	0
0	s_3	11	0	0	10	1	$-1/2$	1
$z_j - c_j$			0	0	$12/5$	$1/5$	$4/5$	0

optimal

$R_2' \to R_2 + \frac{1}{2} R_1'$

$R_3' \to R_3 + \frac{5}{2} R_1'$

calculation $z_j - c_j$ for above table.

$z_j - c_j = C_B x_j - c_j$

$z_1 - c_1 = c_B X_1 - c_1$

$= (-1\times1 + 3\times0 + 0\times0) - (-1)$

$= 0$

$z_2 - c_2 = c_B X_2 - c_2$

$= (-1\times0 + 3\times1 + 0\times0) - 3$

$= 0$

$z_3 - c_3 = (-1\times 4/5 + 3\times 2/5 + 0\times 10) - 2$

$= 12/5$

and same process for others.

Since, all $z_j - c_j \geq 0$, then solution is optimum.

The optimal solution is given by,

$\text{Max } z' = c_B X_B$

$= (-1\times4 + 3\times5 + 0\times11)$

$= 11$

$\therefore \text{Min } z = -\text{Max}(z')$

$= -11$

$x_1 = 4, x_2 = 5, x_3 = 0$

Thank you

Any Queries???