Lecture-15
Algorithmic Mathematics(CSC545)

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Boundary Values Problem

We have seen that we require m conditions to be specified in order to solve an m-order differential equation. In the previous chapter, all the m conditions were specified at one point, $x = x_0$, and, therefore, we call this problem as an *initial-value problem*. It is not always necessary to specify the conditions at one point of the independent variable. They can be specified at different points in the interval (a, b) and, therefore, such problems are called the *boundary value problems*. A large number of problems fall into this category.

In solving initial value problems, we move in steps from the given initial value of x to the point where the solution is required. In case of boundary value problems, we seek solutions at specified points within the domain of given boundaries, for instance, given

$$\frac{d^2y}{dx^2} = f(x, y, y') \qquad y(a) = y_a, \qquad y(b) = y_b \tag{14.1}$$

we are interested in finding the values of y in the range $a \le x \le b$.

There are two popular methods available for solving the boundary value problems. The first one is known as the *shooting method*. This method makes use of the techniques of solving initial value problems. The second one is called the *finite difference method* which makes use of the finite difference equivalents of derivatives.

Some boundary value

14.3 FINITE DIFFERENCE METHOD

In this method, the derivatives are replaced by their finite difference equivalents, thus converting the differential equation into a system of algebraic equations. For example, we can use the following "central difference" approximations:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \tag{14.5}$$

$$y_i'' = \frac{y_{i+1} - 2y_i - y_{i-1}}{h^2} \tag{14.5}$$

These are second-order equations and the accuracy of estimates can be improved by using higher-order equations.

The given interval (a, b) is divided into n subintervals, each of width h. Then

$$
\begin{aligned}
x_i &= x_0 + ih = a + ih \\
y_i &= y(x_i) = y(a + ih) \\
y_0 &= y(a) \\
y_n &= y(a + nh) = y(b)
\end{aligned}
$$

This is illustrated in Fig. 14.2. The difference equation is written for each of the internal points $i = 1, 2, \ldots, n - 1$. If the DE is linear, this will result with $(n - 1)$ unknowns $y_1, y_2, \ldots, y_{n-1}$. We can solve for these unknowns using any of the elimination methods.

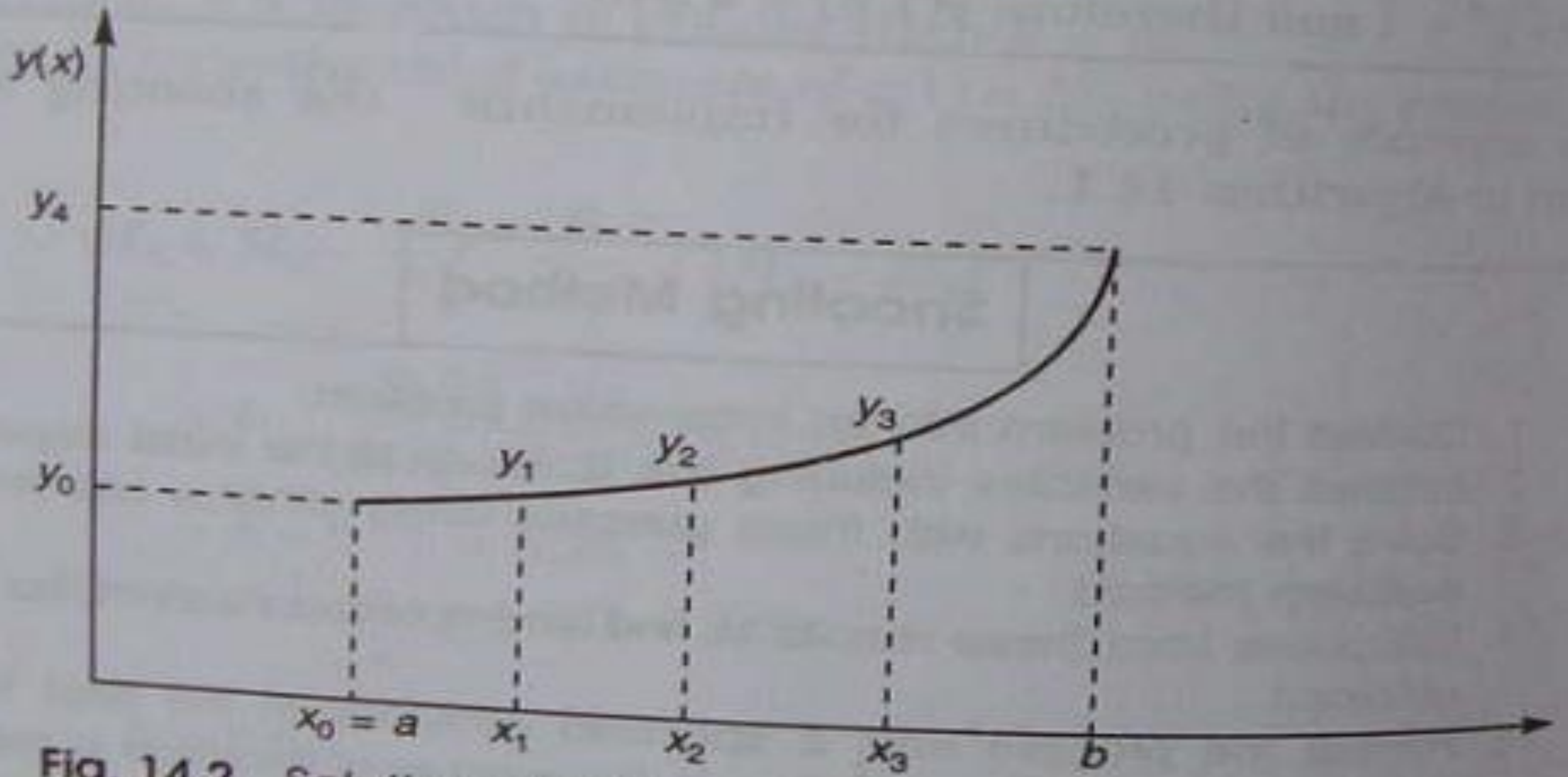


Fig. 14.2 Solution of DE by finite difference method

Note that smaller the size of h, more the subintervals and, therefore, more are the equations to be solved. However, a smaller h yields better estimates.

Example 14.2

Given the equation

$$\frac{d^2y}{dx^2} = e^{x^2} \qquad \text{with} \qquad y(0) = 0, \qquad y(1) = 0$$

estimate the values of $y(x)$ at $x = 0.25, 0.5$ and 0.75.

We know that

$$y_0 = y(0) = 0$$
$$y_1 = y(0.25)$$
$$y_2 = y(0.5)$$
$$y_3 = y(0.75)$$
$$y_4 = y(1) = 0$$
$$h = 0.25$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = e^{x^2}$$

$i = 1, x = 0.25$

$$y_1'' = \frac{y_2 - 2y_1 + y_0}{0.0625} = e^{(0.25)^2} = 1.0645$$

$$\boxed{y_2 - 2y_1 + y_0 = 0.0665} \quad (1)$$

$i = 2, x = 0.50$

$$y_2'' = \frac{y_3 - 2y_2 + y_1}{0.0625} = e^{(0.5)^2} = 1.2840$$

$$\boxed{y_3 - 2y_2 + y_1 = 0.0803} \quad (2)$$

$i = 3, x = 0.75$

$$y_3'' = \frac{y_4 - 2y_3 + y_2}{0.0625} = e^{(0.75)^2} = 1.7551$$

$$\boxed{y_4 - 2y_3 + y_2 = 0.1097} \tag{3}$$

Letting $y_0 = 0$ and $y_4 = 0$, we have the following system of three equations.

$$-2y_1 + y_2 = 0.0665$$

$$y_1 - 2y_2 + y_3 = 0.0803$$

$$y_2 - 2y_3 = 0.1097$$

Solution of these equations results in

$$y_1 = y(0.25) = -0.1175$$

$$y_2 = y(0.50) = -0.1684$$

$$y_3 = y(0.75) = -0.1391$$

Finite Difference Method

1. Divide the given interval into n subintervals.
2. At each point of x, obtain difference equation using a suitable difference formula. This will result in $(n - 1)$ equations with $(n - 1)$ unknowns, $y_1, y_2, ..., y_n - 1$.
3. Solve for y_i, $i = 1, 2, ..., n - 1$ using any of the standard elimination methods.

Algorithm 14.2

Solution of Partial Differential Equations

A **Partial Differential Equation** commonly denoted as PDE is a differential equation containing partial derivatives of the dependent variable (one or more) with more than one independent variable. A PDE for a function $u(x_1, \ldots\ldots x_n)$ is an equation of the form

$$f\left(x_1, \ldots . x_n; u, \frac{\partial u}{\partial x_1}, \ldots \ldots, \frac{\partial u}{\partial x_n}; \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}; \ldots .\right) = 0$$

The PDE is said to be linear if f is a linear function of u and its derivatives. The simple PDE is given by;

$\partial u/\partial x\ (x,y) = 0$

The above relation implies that the function u(x,y) is independent of x which is the reduced form of partial differential equation formula stated above. The order of PDE is the order of the highest derivative term of the equation.

Partial Differential Equation Classification

Each type of PDE has certain functionalities that help to determine whether a particular finite element approach is appropriate to the problem being described by the PDE. The solution depends on the equation and several variables contain partial derivatives with respect to the variables. There are three-types of second-order PDEs in mechanics. They are

- Elliptic PDE
- Parabolic PDE
- Hyperbolic PDE

Consider the example, $au_{xx}+bu_{yy}+cu_{yy}=0$, $u=u(x,y)$. For a given point (x,y), the equation is said to be **Elliptic** if $b^2-ac<0$ which are used to describe the equations of elasticity without inertial terms. **Hyperbolic** PDEs describe the phenomena of wave propagation if it satisfies the condition $b^2-ac>0$. For **parabolic** PDEs, it should satisfy the condition $b^2-ac=0$. The heat conduction equation is an example of a parabolic PDE.

8.2 FINITE-DIFFERENCE APPROXIMATIONS TO DERIVATIVES

Let the (x, y) plane be divided into a network of rectangles of sides $\Delta x = h$ and $\Delta y = k$ by drawing the sets of lines

$$x = ih, \qquad i = 0, 1, 2, \ldots$$

$$y = jk, \qquad j = 0, 1, 2, \ldots$$

The points of intersection of these families of lines are called *mesh* points, *lattice* points or *grid* points. Then, we have (see Section 7.10 of Chapter 7)

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \tag{8.10}$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \tag{8.11}$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \tag{8.12}$$

and

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \tag{8.13}$$

where

$$u_{i,j} = u(ih, jk) = u(x, y)$$

Similarly, we have the approximations

$$u_y = \frac{u_{i,\,j+1} - u_{i,\,j}}{k} + O(k) \tag{8.14}$$

$$= \frac{u_{i,\,j} - u_{i,\,j-1}}{k} + O(k) \tag{8.15}$$

$$= \frac{u_{i,\,j+1} - u_{i,\,j-1}}{2k} + O(k^2) \tag{8.16}$$

and

$$u_{yy} = \frac{u_{i,\,j-1} - 2u_{i,\,j} + u_{i,\,j+1}}{k^2} + O(k^2) \tag{8.17}$$

We can now obtain the *finite-difference analogues* of partial differential equations by replacing the derivatives in any equation by their corresponding difference approximations given above. Thus, the Laplace equation in two dimensions, namely

$$u_{xx} + u_{yy} = 0$$

has its finite-difference analogue

$$\frac{1}{h^2}(u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}) + \frac{1}{k^2}(u_{i,\,j+1} - 2u_{i,\,j} + u_{i,\,j-1}) = 0. \tag{8.18}$$

If $h = k$, this gives

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}), \tag{8.19}$$

which shows that the value of u at any point is the mean of its values at the four neighbouring points. This is called the *standard five-point formula* (*see* Fig. 8.1a), and is written

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \tag{8.20}$$

By expanding the terms on the right side of (8.19) by Taylor's series, it can be shown that

$$\begin{aligned} u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} &= h^2(u_{xx} + u_{yy}) - \frac{1}{6}h^4 u_{xxyy} + O(h^6) \\ &= -\frac{1}{6}h^4 u_{xxyy} + O(h^6) \end{aligned} \tag{8.21}$$

Instead of formula (8.19), we may also use the formula

$$u_{i,j} = \frac{1}{4}(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}) \tag{8.22}$$

Poisson's Equations

Equation (15.1), when $a = 1$, $b = 0$, $c = 1$ and $F(x, y, f, f_x, f_y) = g(x,$
becomes

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = g(x, y)$$

or (15.

$$\nabla^2 f = g(x, y)$$

Equation (15.16) is called *Poisson's equation*. Using the notation g_{ij}
$g(x_i, y_j)$, Eq. 15.10 used for Laplace's equation may be modified to sol
Eq. 15.16. The finite difference formula for solving Poisson's equati
then takes the form

$$\boxed{f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{ij} = h^2 g_{ij}} \quad (15.1$$

By applying the replacement formula to each grid point in the domain
consideration, we will get a system of linear equations in terms of f
These equations may be solved either by any of the elimination method
or by any iteration techniques as done in solving Laplace's equation.

Example 15.3

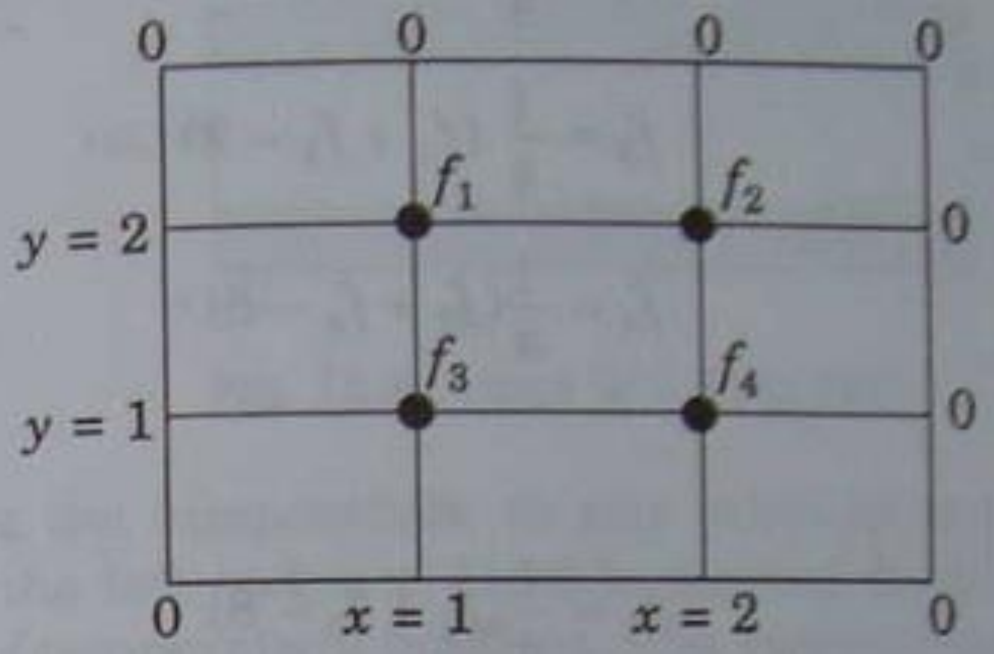
Solv

Solve the Poisson equation

$$\nabla^2 f = 2x^2y^2$$

over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$.

The domain is divided into squares of one unit size as illustrated below:



By applying Eq. (15.17) at each grid point, we get the following set of equations:

Point 1: $0 + 0 + f_2 + f_3 - 4f_1 = 2(1)^2(2)^2$

i.e. $f_2 + f_3 - 4f_1 = 8$ (a)

Point 2: $0 + 0 + f_1 + f_4 - 4f_2 = 2(2)^2(2)^2$

i.e. $f_1 - 4f_2 + f_4 = 32$ (b)

Point 3: $0 + 0 + f_1 + f_4 - 4f_3 = 2(1)^2(1)^2$

i.e. $f_1 - 4f_3 + f_4 = 2$ (c)

Point 4: $0 + 0 + f_2 + f_3 - 4f_4 = 2(2)^2(1)^2$

i.e. $f_2 + f_3 - 4f_4 = 8$ (d)

Rearranging the equations (a) to (d), we get

$$\begin{aligned}
-4f_1 + f_2 + f_3 &= 8 \\
f_1 - 4f_2 + f_4 &= 32 \\
f_1 - 4f_3 + f_4 &= 2 \\
f_2 + f_3 - 4f_4 &= 8
\end{aligned}$$

Solving these equations by elimination method, we get the answers.

$$f_1 = -\frac{22}{4}, \qquad f_2 = -\frac{43}{4}$$

$$f_3 = -\frac{13}{4}, \qquad f_4 = -\frac{22}{4}$$

Thank You

Any Query??