Lecture-14
Algorithmic Mathematics(CSC545)

Prepared by Asst. Prof. Bal Krishna Subedi

CDCSIT, TU

7.4.2 Modified Euler's Method

Instead of approximating $f(x, y)$ by $f(x_0, y_0)$ in (7.6), we now approximate the integral in (7.6) by means of trapezoidal rule to obtain

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \tag{7.13}$$

We thus obtain the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})], \quad n = 0, 1, 2, \ldots \tag{7.14}$$

where $y_1^{(n)}$ is the nth approximation to y_1. The iteration formula (7.14) can be started by choosing $y_1^{(0)}$ from Euler's formula:

$$y_1^{(0)} = y_0 + hf(x_0, y_0).$$

Example 7.7 Determine the value of y when $x = 0.1$ given that

$$y(0) = 1 \quad \text{and} \quad y' = x^2 + y$$

We take $h = 0.05$. With $x_0 = 0$ and $y_0 = 1.0$, we have $f(x_0, y_0) = 1.0$. Hence Euler's formula gives

$$y_1^{(0)} = 1 + 0.05(1) = 1.05$$

Further, $x_1 = 0.05$ and $f(x_1, y_1^{(0)}) = 1.0525$. The average of $f(x_0, y_0)$ and $f(x_1, y_1^{(0)})$ is 1.0262. The value of $y_1^{(1)}$ can therefore be computed by using (7.14) and we obtain

$$y_1^{(1)} = 1.0513.$$

Repeating the procedure, we obtain $y_1^{(2)} = 1.0513$. Hence we take $y_1 = 1.0513$, which is correct to four decimal places.

Next, with $x_1 = 0.05$, $y_1 = 1.0513$ and $h = 0.05$, we continue the procedure to obtain y_2, i.e. the value of y when $x = 0.1$. The results are

$$y_2^{(0)} = 1.1040, \quad y_2^{(1)} = 1.1055, \quad y_2^{(2)} = 1.1055.$$

Hence we conclude that the value of y when $x = 0.1$ is 1.1055.

7.5 RUNGE–KUTTA METHODS

As already mentioned, Euler's method is less efficient in practical problems since it requires h to be small for obtaining reasonable accuracy. The Runge–Kutta methods are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points on the subinterval.

If we substitute $y_1 = y_0 + hf(x_0, y_0)$ on the right side of Eq. (7.13), we obtain

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h, y_0 + hf_0)],$$

where $f_0 = f(x_0, y_0)$. If we now set

$$k_1 = hf_0 \quad \text{and} \quad k_2 = hf(x_0 + h,\ y_0 + k_1)$$

then the above equation becomes

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2), \tag{7.15}$$

which is the *second-order Runge–Kutta* formula. The error in this formula can be shown to be of order h^3 by expanding both sides by Taylor's series. Thus, the left side gives

$$y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \cdots$$

and on the right side

$$k_2 = hf\,(x_0 + h, y_0 + hf_0) = h\left[f_0 + h\frac{\partial f}{\partial x_0} + hf_0\frac{\partial f}{\partial y_0} + O(h^2)\right].$$

Since

$$\frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y},$$

we obtain

$$k_2 = h[f_0 + hf_0' + O(h^2)] = hf_0 + h^2 f_0' + O(h^3),$$

leads to the fourth-order Runge–Kutta formula, the most commonly used one in practice:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (7.21a)$$

where

where

$$\left.\begin{aligned}
k_1 &= hf(x_0, y_0)\\
k_2 &= hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)\\
k_3 &= hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)\\
k_4 &= hf(x_0 + h, y_0 + k_3)
\end{aligned}\right\} \tag{7.21b}$$

in which the error is of order h^5. Complete derivation of the formula is exceedingly complicated, and the interested reader is referred to the book by Levy and Baggot. We illustrate here the use of the fourth-order formula by means of examples.

We take $h = 0.2$. With $x_0 = y_0 = 0$, we obtain from (7.21a) and (7.21b),

$$k_1 = 0.2,$$

$$k_2 = 0.2\,(1.01) = 0.202,$$

$$k_3 = 0.2\,(1 + 0.010201) = 0.20204,$$

$$k_4 = 0.2\,(1 + 0.040820) = 0.20816,$$

and

$$y(0.2) = 0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2027,$$

which is correct to four decimal places.

To compute $y(0.4)$, we take $x_0 = 0.2$, $y_0 = 0.2027$ and $h = 0.2$. With these values (7.21a) and (7.21b) gives

$$k_1 = 0.2\,[1 + (0.2027)^2] = 0.2082,$$

$$k_2 = 0.2\,[1+(0.3068)^2] = 0.2188,$$

$$k_3 = 0.2\,[1+(0.3121)^2] = 0.2195,$$

$$k_4 = 0.2\,[1+(0.4222)^2] = 0.2356,$$

and

$$y(0.4) = 0.2027 + 0.2201 = 0.4228,$$

correct to four decimal places.

Finally, taking $x_0 = 0.4$, $y_0 = 0.4228$ and $h = 0.2$, and proceeding as above, we obtain $y(0.6) = 0.6841$.

Example 7.10 We consider the initial value problem $y' = 3x + y/2$ with the condition $y(0) = 1$.

The following table gives the values of $y(0.2)$ by different methods, the exact value being 1.16722193. It is seen that the *fourth-order* Runge–Kutta method gives the accurate value for $h = 0.05$.

Method	*h*	*Computed value*
Euler	0.2	1.100 000 00
	0.1	1.132 500 00
	0.05	1.149 567 58
Modified Euler	0.2	1.100 000 00
	0.1	1.150 000 00
	0.05	1.162 862 42
Fourth-order Runge–Kutta	0.2	1.167 220 83
	0.1	1.167 221 86
	0.05	1.167 221 93

7.6 PREDICTOR–CORRECTOR METHODS

In the methods described so far, to solve a differential equation over a single interval, say from $x = x_n$ to $x = x_{n+1}$, we required information only at the beginning of the interval, i.e. at $x = x_n$. *Predictor–corrector* methods are the ones which require function values at $x_n, x_{n-1}, x_{n-2}, \ldots$ for the computation of the function value at x_{n+1}. A *predictor* formula is used to predict the value of y at x_{n+1} and then a *corrector* formula is used to improve the value of y_{n+1}.

In Section 7.6.1 we derive Predictor–corrector formulae which use backward differences and in Section 7.6.2 we describe Milne's method which uses forward differences.

7.6.1 Adams–Moulton Method

Newton's backward difference interpolation formula can be written as

$$f(x, y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + \cdots \quad (7.22)$$

where

$$n = \frac{x - x_0}{h} \quad \text{and} \quad f_0 = f(x_0, y_0).$$

If this formula is substituted in

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\,dx, \quad (7.23)$$

we get

$$y_1 = y_0 + \int_{x_0}^{x_1} \left[f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots \right] dx$$

$$= y_0 + h \int_0^1 \left[f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots \right] dn$$

$$= y_0 + h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \cdots\right) f_0.$$

It can be seen that the right side of the above relation depends only on y_0, $y_{-1}, y_{-2}, \ldots$; all of which are known. Hence this formula can be used to compute y_1. We therefore write it as

$$y_1^p = y_0 + \left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \cdots\right) f_0 \qquad (7.24)$$

This is called *Adams–Bashforth* formula and is used as a *predictor* formula (the superscript p indicating that it is a predicted value).

A corrector formula can be derived in a similar manner by using Newton's backward difference formula at f_1:

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+1)}{6}\nabla^3 f_1 + \cdots \qquad (7.25)$$

Substituting (7.25) in (7.23), we obtain

$$y_1 = y_0 + \int_{x_0}^{x_1}\left[f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right]dx$$

$$= y_0 + h\int_{1}^{0}\left[f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right]dn$$

$$= y_0 + h\left(1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \cdots\right) f_1 \qquad (7.26)$$

The right side of (7.26) depends on $y_1, y_0, y_{-1}, \ldots$ where for y_1 we use y_1^p, the predicted value obtained from (7.24). The new value of y_1 thus obtained from (7.26) is called the *corrected* value, and hence we rewrite the formula as

$$y_1^c = y_0 + h\left(1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \cdots\right) f_1^p \qquad (7.27)$$

This is called *Adams–Moulton corrector* formula the superscript c indicates that the value obtained is the corrected value and the superscript p on the right indicates that the predicted value of y_1 should be used for computing the value of $f(x_1, y_1)$.

In practice, however, it will be convenient to use formulae (7.24) and (7.27) by ignoring the higher-order differences and expressing the lower-order differences in terms of function values. Thus, by neglecting the fourth and higher-order differences, formulae (7.24) and (7.27) can be written as

$$y_1^{p} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \tag{7.28}$$

and

$$y_1^{c} = y_0 + \frac{h}{24}(9f_1^{p} + 19f_0 - 5f_{-1} + f_{-2}) \tag{7.29}$$

in which the errors are approximately

$\frac{251}{720}h^5 f_0^{(4)}$ and $-\frac{19}{720}h^5 f_0^{(4)}$ respectively.

and

$$y^c_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}) \tag{7.34a}$$

The application of this method is illustrated by the following example.

Example 7.12 We consider again the differential equation discussed in Examples 7.9 and 7.10, viz., to solve $y' = 1 + y^2$ with $y(0) = 0$ and we wish to compute $y(0.8)$ and $y(1.0)$.

With $h = 0.2$, the values of $y(0.2)$, $y(0.4)$ and $y(0.6)$ are computed in Example 7.9 and these are given in the table below:

x	y	$y' = 1 + y^2$
0	0	1.0
0.2	0.2027	1.0411
0.4	0.4228	1.1787
0.6	0.6841	1.4681

To obtain $y(0.8)$, we use (7.32) and obtain

$$y(0.8) = 0 + \frac{0.8}{3}[2(1.0411) - 1.1787 + 2(1.4681)] = 1.0239$$

This gives

$$y'(0.8) = 2.0480.$$

To correct this value of $y(0.8)$, we use formula (7.34) and obtain

$$y(0.8) = 0.4228 + \frac{0.2}{3}[1.1787 + 4(1.4681) + 2.0480] = 1.0294.$$

Proceeding similarly, we obtain $y(1.0) = 1.5549$. The accuracy in the values of $y(0.8)$ and $y(1.0)$ can, of course, be improved by repeatedly using formula (7.34).

Example 7.13 The differential equation $y' = x^2 + y^2 - 2$ satisfies the following data:

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x	y
0.1	1.0900
0	1.0000
0.1	0.8900
0.2	0.7605

Use Milne's method to obtain the value of $y(0.3)$.

We first form the following table:

x	y	$y' = x^2 + y^2 - 2$
−0.1	1.0900	−0.80190
0	1.0	−1.0
0.1	0.8900	−1.19790
0.2	0.7605	−1.38164

Using (7.32), we obtain

$$y(0.3) = 1.09 + \frac{4(0.1)}{3}[2(-1) - (-1.19790) + 2(-1.38164)] = 0.614616.$$

In order to apply (7.34), we need to compute $y'(0.3)$. We have

$$y'(0.3) = (0.3)^2 + (0.614616)^2 - 2 = -1.532247.$$

Now, (7.34) gives the corrected value of $y(0.3)$:

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In order to apply (7.34), we need to compute $y'(0.3)$. We have

$$y'(0.3) = (0.3)^2 + (0.614616)^2 - 2 = -1.532247.$$

Now, (7.34) gives the corrected value of $y(0.3)$:

$$y(0.3) = 0.89 + \frac{0.1}{3}[-1.197900 + 4(-1.38164) + (-1.532247)] = 0.614776.$$

Thank You

Any Query??