Lecture-6
Algorithmic Mathematics(CSC545)

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The Newton Polynomial Interpolation

The **Lagrange interpolation** relies on the interpolation points , all of which need to be available to calculate each of the basis polynomials . If additional points are to be used when they become available, all basis polynomials need to be recalculated.

In comparison, in **the Newton interpolation**, when more data points are to be used, additional basis polynomials and the corresponding coefficients can be calculated, while all existing basis polynomials and their coefficients remain unchanged. Due to the additional terms, the degree of interpolation polynomial is higher and the approximation error may be reduced (e.g., when interpolating higher order polynomials).

We have seen that, in Lagrange interpolation, we cannot use the work that has already been done if we want to incorporate another data point in order to improve the accuracy of estimation. It is therefore necessary to look for some other form of representation to overcome this drawback.

Let us now consider another form of polynomial known as *Newton form* which was discussed in Section 9.2. The Newton form of polynomial is

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1}) \quad (9.10)$$

where the interpolation points $x_0, x_1, \dots x_{n-1}$ act as centres.

To construct the interpolation polynomial, we need to determine the coefficients $a_0, a_1, \dots a_n$. Let us assume that $(x_0, f_0), (x_1, f_1), \dots (x_{n-1}, f_{n-1})$ are the interpolating points. That is,

$$p_n(x_k) = f_k \qquad k = 0, 1, \dots n - 1$$

Now, at $x = x_0$, we have (using Eq. (9.10))

$$p_n(x_0) = \boxed{a_0 = f_0} \quad (9.11)$$

Similarly, at $x = x_1$,

$$p_n(x_1) = a_0 + a_1(x_1 - x_0) = f_1$$

Substituting for a_0 from Eq. (9.11), we get

$$\boxed{a_1 = \frac{f_1 - f_0}{x_1 - x_0}} \quad (9.12)$$

At $x = x_2$,

$$p_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$$

Substituting for a_0 and a_1 from Eqs. (9.11) and (9.12) and rearranging the terms, we get

$$\boxed{a_2 = \frac{[(f_2 - f_1)/(x_2 - x_1)] - [(f_1 - f_0)/(x_1 - x_0)]}{x_2 - x_0}} \quad (9.13)$$

Let us define a notation

$$f[x_k] = f_k$$

$$f[x_k, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k}$$

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}$$

$$f[x_k, x_{k+1}, \dots x_i, x_{i+1}] = \frac{f[x_{k+1} \dots x_{i+1}] - f[x_k \dots x_i]}{x_{i+1} - x_k} \quad (9.14)$$

These quantities are called *divided differences*. Now we can express the coefficients a_i in terms of these divided differences.

$$a_0 = f_0 = f[x_0]$$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$$

$$a_2 = \frac{\dfrac{f_2 - f_1}{x_2 - x_1} - \dfrac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= f[x_0, x_1, x_2]$$

Thus,

$$a_n = f[x_0, x_1, \dots x_n] \tag{9.15}$$

Note that a_1 represents the first divided d...

(9.15)

Note that a_1 represents the *first divided difference* and a_2 the *second divided difference* and so on.

Substituting for a_i coefficients in equation (9.10), we get

$$p_n(x) = f[x_0] + f[x_0, x_1]\,(x - x_0) + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1)$$
$$+ \ldots$$
$$+ f[x_0, x_1, \ldots x_n]\,(x - x_0)\,(x - x_1)\ldots(x - x_{n-1})$$

This can be written more compactly as

$$\boxed{p_n(x) = \sum_{i=1}^{n} f[x_0, \ldots x_i] \prod_{j=0}^{i-1} (x - x_j)} \qquad (9.16)$$

Equation (9.16) is called *Newton's divided difference interpolation polynomial.*

Example: Newton's Interpolation

Given below is a table of data for log x. Estimate log 2.5 using second order Newton interpolation polynomial.

i	0	1	2	3
x_i	1	2	3	4
$\log x_i$	0	0.3010	0.4771	0.6021

Second order polynomials require only three data points. We use the first three points

$$a_0 = f[x_0] = 0$$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0.3010}{2-1} = 0.3010$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0.4771 - 0.3010}{3-2} = 0.1761$$

Therefore,

$$a_2 = \frac{0.1761 - 0.3010}{3-1} = -0.06245$$

$$\begin{aligned}
p_2(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\
&= 0 + 0.3010(x - 1) + (-0.06245)(x - 1)(x - 2) \\
p_2(2.5) &= 0.3010 \times 1.5 - (0.06245)(1.5)(0.5) \\
&= 0.4515 - 0.0468 \\
&= 0.4047
\end{aligned}$$

Note that, in Example 9.6, had we used a linear polynomial, we would have obtained the result as follows:

$$p_1(x) = a_0 + a_1(x - x_0)$$

$$p_1(2.5) = 0 + 0.3010\ (1.5) = 0.4515$$

This shows that p_2 (2.5) is obtained by simply adding a correction factor due to third data point. That is

$$p_2(x) = p_1(x) + a_2(x - x_0)\ (x - x_1)$$

$$= p_1(x) + \Delta_2$$

If we want to improve the results further, we can apply further correction by adding another data point. That is

$$p_3(x) = p_2(x) + \Delta_3$$

where

$$\Delta_3 = a_3(x - x_0)(x - x_1)(x - x_2)$$

This shows that the Newton interpolation formula provides a very convenient form for interpolation at an increasing number of interpolation points. Newton formula can be expressed recursively as follows:

$$\boxed{p_{k+1}(x) = p_k(x) + f[x_0, \ldots x_{k+1}]\,\phi_k(x)\,(x - x_k)} \tag{9.17}$$

where $\quad p_k(x) = f[x_0, \ldots x_i]\,\phi_i(x) = \sum_{i=0}^{k} a_i\,\phi_i(x)$

and $\quad \phi_i(x) = (x - x_0)(x - x_1)\ldots(x - x_{i-1})$

Divided Difference Table

We have seen that the coefficients of Newton divided difference interpolation polynomial are evaluated using the divided differences at the interpolating points. We have also seen that a higher-order divided difference is obtained using the lower-order differences. Finally, the first-order divided differences use the given interpolating points (i.e., x_k and f_k values). For example, consider the second-order divided difference

$$a_2 = f[x_0, x_1, x_2]$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

where $f[x_1, x_2]$ and $f[x_0, x_1]$ are first-order divided differences and are given by

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f_1 - f_0}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f_2 - f_1}{x_2 - x_1}$$

$x_2 - x_1 \qquad x_2 - x_1$

This shows that, given the interpolating points, we can obtain recursively a higher-order divided difference, starting from the first-order differences. While this can be conveniently implemented in a computer, we can generate a *divided difference table* for manual computing. A divided difference table for five data points is shown in Fig. 9.3. A particular entry in the table is obtained as follows:

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$$

i	x_i	$f[x_i]$	First difference	Second difference	Third difference	Fourth difference
0	x_0	$f[x_0]$				
			$f[x_0, x_1]$			
1	x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
			$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
2	x_2	$f[x_2]$		$\boxed{f[x_1, x_2, x_3]}$		$f[x_0, \ldots, x_4]$
			$f[x_2, x_3]$		$\boxed{f[x_1, x_2, x_3, x_4]}$	
3	x_3	$f[x_3]$		$\boxed{f[x_2, x_3, x_4]}$		
			$f[x_3, x_4]$			
4	x_4	$f[x_4]$				

Fig. 9.3 Divided difference table

Given the following set of data points, obtain the table of divided differences. Use the table to estimate the value of $f(1.5)$.

i	0	1	2	3	4
x_i	1	2	3	4	5
$f(x_i)$	0	7	26	63	124

The divided difference table is given below:

i	x_i	$f(x_i)$	*First difference*	*Second difference*	*Third difference*	*Fourth difference*
0	1	0				
			7			
1	2	7		12		
			19		6	
2	3	26		18		0
			37		6	
3	4	63		24		
			61			
4	5	124				

The value of polynomial at $x = 1.5$ is computed as follows:

$$p_0(1.5) = 0$$
$$p_1(1.5) = 0 + 7(1.5 - 1) = 3.5$$
$$p_2(1.5) = 3.5 + 12(1.5 - 1)(1.5 - 2) = 0.5$$
$$p_3(1.5) = 0.5 + 6\,(1.5 - 1)(1.5 - 2)(1.5 - 3) = 2.25$$
$$p_4(1.5) = 2.25 + 0 = 2.25$$

The function value at $x = 1.5$ is 2.25

9.7 INTERPOLATION WITH EQUIDISTANT POINTS

In this section, we consider a particular case where the function values are given at a sequence of equally spaced points. Most of the engineering and scientific tables are available in this form. We often use such tables to estimate the value at a non-tabular point. Let us assume that

$$x_k = x_0 + kh$$

where x_0 is the reference point and h is the step size. The integer k may take either positive or negative values depending on the position of the reference point in the table. We also assume that we are going to use *simple differences* rather than *divided differences*. For this purpose, we define the following:

The *first forward difference* Δf_i is defined as

$$\Delta f_i = f_{i+1} - f_i$$

The *second forward difference* is defined as

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$

In general,

$$\boxed{\Delta^j f_i = \Delta^{j-i} f_{i+1} - \Delta^{j-1} f_i} \tag{9.18}$$

We can now express the *simple forward differences* in terms of the *divided differences*. We know that

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f_1 - f_0}{h}$$

Therefore,

$$f_1 - f_0 = h\, f[x_0, x_1]$$

Then

$$\Delta f_0 = f_1 - f_0 = h\, f[x_0, x_1]$$

Similarly,

$$\Delta f_1 = h\, f[x_1, x_2]$$

Now,

$$\begin{aligned}
\Delta^2 f_0 &= \Delta f_1 - \Delta f_0 \\
&= h\, f[x_1, x_2] - h\, f[x_0, x_1] \\
&= h\, \{f[x_1, x_2] - f[x_0, x_1]\} \\
&= h \,.\, 2h \,.\, f[x_0, x_1, x_2] \\
&= 2\, h^2 f[x_0, x_1, x_2]
\end{aligned}$$

In general, by induction,

$$\Delta^j f_i = j!\, h^j f[x_i, x_{i+1}, \ldots x_{i+j}]$$

Therefore,

$$f[x_0, x_1, \ldots x_j] = \frac{\Delta^j f_0}{j!h^j}$$

Substituting this in the Newton's divided difference interpolation polynomial (Eq. (9.16)) we get,

$$\boxed{p_n(x) = \sum_{j=0}^{n} \frac{\Delta^j f_0}{j!h^j} \prod_{k=0}^{j-1} (x - x_k)} \quad (9.19)$$

Let us set $x = x_0 + sh$ and $p_n(s) = p_n(x)$

We know that $x_k = x_0 + kh$

Thus we get $x - x_k = (s - k)h$

Substituting this in Eq. (9.19), we get

$$p_n(s) = \sum_{j=0}^{n} \frac{\Delta^j f_0}{j!\, h^j} \prod_{k=0}^{j-1} (s-k)h$$

$$= \sum_{j=0}^{n} \frac{\Delta^j f_0}{j!\, h^j} [s(s-1)\dots(s-j+1)]\, h^j$$

Thus,

$$\boxed{p_n(s) = \sum_{j=0}^{n} \binom{s}{j} \Delta^j f_0} \tag{9.20}$$

where

$$\binom{s}{j} = \frac{s(s-1)\dots(s-j+1)}{j!}$$

is the binomial coefficient. Equations (9.19) and (9.20) are known as *Gregory-Newton forward difference formula*.

Forward Difference Table

The coefficients $\Delta^j f_i$ can be conveniently obtained from the *forward difference table* shown in Fig. 9.4. According to Eq. (9.18), each entry is merely the difference between the two diagonal entries immediately on its left. That is

$$\Delta^j f_i = \Delta^{j-1} f_{i+1} - \Delta^{j-1} f_i$$

The differences which appear on the top of each column correspond to the differences of equation (9.20).

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
x_0	f_0						
		Δf_0					
x_1	f_1		$\Delta^2 f_0$				
		Δf_1		$\Delta^3 f_0$			
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$		
		Δf_2		$\Delta^3 f_1$		$\Delta^5 f_0$	
x_3	f_3		$\Delta^2 f_2$		$\Delta^4 f_1$		
		Δf_3		$\Delta^3 f_2$			
x_4	f_4		$\Delta^2 f_3$				
		Δf_4					
x_5	f_5						

Fig. 9.4 Forward difference table

Estimate the value of sin θ at $\theta = 25°$ using the Newton-Gregory forward difference formula with the help of the following table.

θ	10	20	30	40	50
sin θ	0.1736	0.3420	0.5000	0.6428	0.7660

In order to use the Newton-Gregory forward difference formula, we need the values of $\Delta^j f_0$. These coefficients can be obtained from the difference table given below. The required coefficients are boldfaced.

θ	sin θ	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
10	**0.1736**					
		0.1684				
20	0.3420		**−0.0104**			
		0.1580		**0.0048**		
30	0.5000		−0.0152		**−0.0004**	
		0.1428		0.0044		
40	0.6428		−0.0196			
		0.1232				
50	0.7660					

$$x_0 = \theta_0 = 10$$

$$h = 10$$

Therefore,

$$s = \frac{x - x_0}{h} = \frac{25 - 10}{10} = 1.5$$

Using Eq. (9.20), we have

$$p_1(s) = 0.1736 + (1.5)(0.1684) = 0.4262$$

$$p_2(s) = 0.4262 + \frac{(1.5)(0.5)(-0.0104)}{2} = 0.4223$$

$$p_3(s) = 0.4223 + \frac{(1.5)(0.5)(-0.5)(0.0048)}{6} = 0.4220$$

$$p_4(s) = 0.4220 + \frac{(1.5)(0.5)(-0.5)(-1.5)(-0.0004)}{24} = 0.4220$$

Thus,

$$\sin 25 = 0.4220$$

which is accurate to four decimal places.

Backward Difference Table

If the table is too long and if the required point is close to the end of the table, we can use another formula known as *Newton-Gregory backward difference formula*. Here, the reference point is x_n, instead of x_0. Therefore, we have

$$x = x_n + sh$$

$$x_k = x_n - kh$$

$$x - x_k = (s + k)h$$

Then, the Newton-Gregory backward difference formula is given by

$$p_n(s) = f_n + s\,\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \dots$$

$$+ \frac{s(s+1)\dots(s+n-1)}{n!}\nabla^n f_n \qquad (9.21)$$

For a given table of data, the backward difference table will be identical to the forward difference table. However, the reference point will be below the point for which the estimate is required. This implies that the value of s will be negative for backward interpolation. The coefficients $\nabla j\, f_i$ can be obtained from the backward difference table shown in Fig. 9.5.

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$	$\nabla^6 f$
x_0	f_0						
		∇f_1					
x_1	f_1		$\nabla^2 f_2$				
		∇f_2		$\nabla^3 f_3$			
x_2	f_2		$\nabla^2 f_3$		$\nabla^4 f_4$		
		∇f_3		$\nabla^3 f_4$		$\nabla^5 f_5$	
x_3	f_3		$\nabla^2 f_4$		$\nabla^4 f_5$		
		∇f_4		$\nabla^3 f_5$			
x_4	f_4		$\nabla^2 f_5$				
		∇f_5					
x_5	f_5						

Fig. 9.5 Backward difference table

Repeat the estimation of sin 25 in Example 9.8 using Newton's backward difference formula

$$s = \frac{(x - x_n)}{h} = \frac{25 - 50}{10} = -2.5$$

Using Eq. (9.21), we get

$$p_4(2.5) = 0.7660 + (-2.5)(0.1232)$$

$$+ \frac{(-2.5)(-1.5)(-0.0196)}{2}$$

$$+ \frac{(-2.5)(-1.5)(-0.5)(0.0044)}{6}$$

$$+ \frac{(-2.5)(-1.5)(-0.5)(0.5)(-0.0004)}{24}$$

$$= 0.4200$$

Assignment#5

Given the data

x	1.2	1.3	1.4	1.5
$f(x)$	1.063	1.091	1.119	1.145

(a) Calculate $f(1.35)$ using Newton's interpolating polynomial of order 1 through 3. Choose base points to attain good accuracy.

(b) Comment on the accuracy of results on the order of polynomial.

Find the divided differences $f[x_0, x_1]$, $f[x_1, x_2]$ and $f[x_0, x_1, x_2]$ for the data given below.

i	0	1	2
x_i	1.0	1.5	2.5
$f(x_i)$	3.2	3.5	4.5

Assignment#5

Estimate the value of ln(3.5) using Newton-Gregory forward difference formula and Newton- Gregory backward difference formula and compare their result by help following data:

X	1.0	2.0	3.0	4.0
ln	0.0	0.6931	1.0986	1.3863

Thank You

Any Query??