Lecture-4
Algorithmic Mathematics(CSC545)

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Newton-Raphson Method

The Newton-Raphson method, which was discussed in Section 6.8 for solving single nonlinear equations, can be extended to systems of non-linear equations. Recall that a first order Taylor series of the form

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)\, f'(x) \tag{6.48}$$

was used to derive the Newton iteration formula

$$x_{i+1} = x_i - \frac{f(x)}{f'(x)} \tag{6.49}$$

$$(6.49)$$

for solving one equation. For the sake of simplicity, let us again consider a two-equation nonlinear system

$$f(x, y) = 0 \qquad g(x, y) = 0$$

First order Taylor series of these equations can be written as

$$f(x_{i+1}, y_{i+1}) = f(x_i, y_i) + (x_{i+1} - x_i)\left|\frac{\partial f_i}{\partial x}\right| + (y_{i+1} - y_i)\left|\frac{\partial f_i}{\partial y}\right| \tag{6.50a}$$

$$g(x_{i+1}, y_{i+1}) = g(x_i, y_i) + (x_{i+1} - x_i)\left|\frac{\partial g_i}{\partial x}\right| + (y_{i+1} - y_i)\left|\frac{\partial g_i}{\partial y}\right| \tag{6.50b}$$

If the root estimates are x_{i+1} and y_{i+1}, then

$$f(x_{i+1}, y_{i+1}) = g(x_{i+1}, y_{i+1}) = 0$$

Substituting this in Eq. (6.50) we get the following two linear equations:

$$\Delta x\, f_1 + \Delta y\, f_2 + f = 0 \tag{6.51a}$$

$$\Delta x\, g_1 + \Delta y\, g_2 + g = 0 \tag{6.51b)}$$

where we denote

$$\Delta x = x_{i+1} - x_i$$

$$\Delta y = y_{i+1} - y_i$$

$$f_1 = \left|\frac{\partial f_i}{\partial x}\right|, \quad f_2 = \left|\frac{\partial f_i}{\partial y}\right|$$

$$g_1 = \left|\frac{\partial g_i}{\partial x}\right|, \quad g_2 = \left|\frac{\partial g_i}{\partial y}\right|$$

$$f = f(x_i, y_i), \quad g = g(x_i, y_i)$$

Solving for x and y, we get

$$\Delta x = -\frac{f \cdot g_2 - g \cdot f_2}{f_1 g_2 - f_2 g_1} = -\frac{Dx}{D} \tag{6.52a}$$

$$\Delta y = -\frac{g \cdot f_1 - f \cdot g_1}{f_1 g_2 - f_2 g_1} = -\frac{Dy}{D} \tag{6.52b}$$

where

$$D = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} = f_1 g_2 - g_1 f_2$$

is called the *Jacobian matrix*. From Eq. (6.52a) and (6.52b), we can establish the following recurring relations:

$$\boxed{\begin{aligned} x_{i+1} &= x_i - \frac{Dx}{D} \qquad (6.53(a)) \\ y_{i+1} &= y_i - \frac{Dy}{D} \qquad (6.53b) \end{aligned}}$$

Equations (6.53a) and (6.53b) are similar to the single-equation Newton formula and may be called the *two-equation* Newton formula. These equations can be used iteratively and simultaneously to solve for the roots of $f(x, y)$ and $g(x, y)$.

Algorithm 6.8 lists the steps involved in implementing the Newton iteration formula for a two-equation system.

Two equation Newton-Raphson method

1. Define the functions f and g
2. Define the Jacobian elements
   f_1, f_2, g_1 and g_2
3. Decide starting points x_0 and y_0 and error tolerance E.
4. Evaluate f, g, f_1, f_2, g_1, g_2 at (x_0, y_0)
   Compute Dx, Dy and D
   $x_1 = x_0 - Dx/D$
   $y_1 = y_0 - Dy/D$

(Contd.)

5. Test for accuracy.
   If $|x_1 - x_0| < E$ and
   $|y_1 - y_0| < E$, then
   solution obtained;
   go to step 7
6. Otherwise, set
   $x_0 = x_1$
   $y_0 = y_1$
   go to step 4
7. Write results
8. Stop

Example 6.14

Determine the roots of equations

$$x^2 + xy = 6$$
$$x^2 - y^2 = 3$$

using the Newton-Raphson method

Let

$$F(x, y) = x^2 + xy - 6$$
$$G(x, y) = x^2 - y^2 - 3$$
$$f_1 = \frac{\partial F}{\partial x} = 2x + y$$
$$f_2 = \frac{\partial F}{\partial y} = y$$
$$g_1 = \frac{\partial G}{\partial x} = 2x$$
$$g_2 = \frac{\partial G}{\partial y} = -2y$$

Assume the initial guesses as

Iteration 1 $\quad x_0 = 1 \quad$ and $\quad y_0 = 1$

$$f_1 = 3, f_2 = 1$$

and therefore $\quad g_1 = 2, g_2 = -2$

$$D = -6 - 2 = -8$$

The values of functions at x_0 and y_0

$$F = 1^2 + 1 \times 1 - 6 = -4$$
$$G = 1^2 - 1^2 - 3 = -3$$
$$x_1 = 1 - \frac{(-4)(-2) - (-3)(1)}{(-8)} = 2.375$$

$$y_1 = 1 - \frac{(-3)(3) - (-4)(2)}{(-8)} = 0.875$$

Iteration 2

$$f_1 = 2 \times 2.375 + 0.875 = 5.625$$
$$f_2 = 0.875$$
$$g_1 = 4.75$$
$$g_2 = -1.75$$
$$F = (2.375)^2 + (2.375)(0.875) - 6 = 1.71187$$
$$G = (2.375)^2 - (0.875)^2 = 4.8750$$
$$D = (5.625)(-1.75) - (4.75)(0.875)$$
$$= -9.8436 - 4.1563 = -14$$
$$x_2 = 2.375 - \frac{(1.7187)(-1.75) - (4.875)(0.875)}{-14}$$
$$= 2.375 - \frac{(-3.0077) - 4.2656}{-14} = 2.375 - 0.5195$$
$$= 1.8555$$
$$y_2 = 0.875 - \frac{(4.875)(5.625) - (1.7187)(4.75)}{-14}$$
$$= 0.875 - \frac{27.4218 - 8.1638}{-14} = 2.2506$$

Continue further to obtain correct answer.

Assignment#3

Use Newton's method to solve the following systems of equations:

(a) $3x^2 - 2y^2 = 1$

$x^2 - 2x + y^2 + 2y = 8$

(Assume $x_0 = -1$ and $y_0 = 1$)

(b) $x^3 - y^2 + 1 = 0$

$x^2 - 2x + y^3 - 2 = 0$

(Assume $x_0 = 1$ and $y_0 = 1$)

Thanks You

Any Query??