Lecture-12
Algorithmic Mathematics(CSC545)

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Solving system of linear equations

A system of n linear equations is represented generally as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\cdot \qquad \cdot \qquad & \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad & \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad & \cdot \qquad \cdot \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
\tag{7.3}
$$

In matrix notation, Eq. (7.3) can be expressed as

$$Ax = b \tag{7.4}$$

where A is an $n \times n$ matrix, b is an n vector, and x is a vector of n unknowns.

The techniques and methods for solving systems of linear algebraic equations belong to two fundamentally different approaches:

1. Elimination approach
2. Iterative approach

7.6 GAUSS-JORDAN METHOD

Gauss-Jordan method is another popular method used for solving a system of linear equations. Like Gauss elimination method, Gauss-Jordan method also uses the process of elimination of variables, but there is a major difference between them. In Gauss elimination method, a variable is eliminated from the rows below the pivot equation. But in Gauss-Jordan method, it is eliminated from all other rows (both below and above). This process thus eliminates all the off-diagonal terms producing a diagonal matrix rather than a triangular matrix. Further, all rows are normalised by dividing them by their pivot elements. This is illustrated in Fig. 7.4. Consequently, we can obtain the values of unknowns directly from the b vector, without employing back-substitution. Algorithm 7.3 enumerates the Gauss-Jordan elimination steps.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

Result of Gauss elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b'_1 \\ b''_2 \\ b'''_3 \end{bmatrix}$$

Result of Gauss-Jordan elimination

Fig. 7.4 Comparison of Gauss and Gauss-Jordan methods of elimination

	Gauss-Jordan elimination

Comparison of Gauss and Gauss-Jordan methods of elimination

Gauss-Jordan elimination

1. Normalise the first equation by dividing it by its pivot element.
2. Eliminate x_1 term from all the other equations.
3. Now, normalise the second equation by dividing it by its pivot element.
4. Eliminate x_2 from all the equations, above and below the normalised pivotal equation.
5. Repeat this process until x_n is eliminated from all but the last equation.
6. The resultant b vector is the solution vector.

Algorithm 7.3

The Gauss-Jordan method requires approximately 50 per cent more arithmetic operations compared to Gauss method. Therefore, this method is rarely used.

Example 7.4

Solve the system

$$2x_1 + 4x_2 - 6x_3 = -8$$
$$x_1 + 3x_2 + x_3 = 10$$
$$2x_1 - 4x_2 - 2x_3 = -12$$

using Gauss-Jordan method.

Step 1: Normalise the first equation by dividing it by 2 (pivot element). The result is:

$$x_1 + 2x_2 - 3x_3 = -4$$
$$x_1 + 3x_2 + x_3 = 10$$
$$2x_1 - 4x_2 - 2x_3 = -12$$

Step 2: Eliminate x_1 from the second equation, subtracting 1 time the first equation from it. Similarly, eliminate x_1 from the third equation by subtracting 2 times the first equation from it. The result is:

$$x_1 + 2x_2 - 3x_3 = -4$$
$$0 + x_2 + 4x_3 = 14$$
$$0 - 8x_2 + 4x_3 = -4$$

Step 3: Normalise the second equation. (Note that it is already in normalised form.)

Step 4: Following similar approach, eliminate x_2 from first and third equations. This gives

$$x_1 + 0 - 11x_3 = -32$$
$$0 + x_2 + 4x_3 = 14$$
$$0 + 0 + 36x_3 = 108$$

Step 5: Normalise the third equation

$$x_1 + 0 - 11x_3 = -32$$
$$0 + x_2 + 4x_3 = 14$$
$$0 + 0 + x_3 = 3$$

Step 6: Eliminate x_3 from the first and second equations. We get

$$x_1 + 0 + 0 = 1$$
$$0 + x_2 + 0 = 2$$
$$0 + 0 + x_3 = 3$$

Solve the following systems of equations by Gauss-Jordan method

(a) $x_1 + 2x_2 - 3x_3 = 4$

$2x_1 + 4x_2 - 6x_3 = 8$

$x_1 - 2x_2 + 5x_3 = 4$

(b) $2x_1 + x_2 + x_3 = 7$

$4x_1 + 2x_2 + 3x_3 = 4$

$x_1 - x_2 + x_3 = 0$

Solving system of linear equations using Iterative Methods

- There three iterative methods for solving system of linear equations
  - Jacobi iterative methods
  - Gauss Seidel iterative methods
  - Successive Relaxation iterative methods

of the iteration process.

8.2 JACOBI ITERATION METHOD

Jacobi method is one of the simple iterative methods. The basic idea behind this method is essentially the same as that for the fixed point method discussed in Chapter 6. Recall that an equation of the form

$$f(x) = 0$$

can be rearranged into a form

$$x = g(x)$$

The function $g(x)$ can be evaluated iteratively using an initial approximation x as follows:

$$x_{i+1} = g(x_i) \qquad \text{for } i = 0, 1, 2 \ldots$$

Jacobi method extends this idea to a system of equations. It is a direct substitution method where the values of unknowns are improved by substituting directly the previous values.

Let us consider a system of n equations in n unknowns.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned} \tag{8.1}$$

We rewrite the original system as

$$\begin{aligned}
x_1 &= \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n)}{a_{11}} \\
x_2 &= \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \dots + a_{2n}x_n)}{a_{22}} \\
&\ \ \vdots \\
x_n &= \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn-1}x_n)}{a_{nn}}
\end{aligned} \tag{8.2}$$

Now, we can compute $x_1, x_2, \ldots x_n$ by using initial guesses for these values. These new values are again used to compute the next set of x values. The process can continue till we obtain a desired level of accuracy in the x values.

In general, an iteration for x_i can be obtained from the ith equation as follows

$$x_i^{(k+1)} = \frac{b_i - \left(a_{i1}x_1^{(k)} + a_{ii-1}x_{i-1}^{(k)} + a_{ii+1}x_{i+1}^{(k)} + \ldots a_{in}x_n^{(k)}\right)}{a_{ii}} \tag{8.3}$$

The computational steps of Jacobi iteration process are given in Algorithm 8.1.

Jacobi iteration method

1. Obtain n, a_{ij} and b_i values.
2. Set $x_{0i} = b_i / a_{ii}$ for $i = 1, \ldots n$
3. Set key = 0
4. For $i = 1, 2, \ldots n$
   (i) Set sum = b_i
   (ii) For $j = 1, 2 \ldots n$ ($j \neq i$)
        Set sum = sum $- a_{ij} x_{0j}$
      Repeat j
   (iii) Set x_i = sum/a_{ii}
   (iv) if key = 0 then

   $$\text{if} \left| \frac{x_i - x_{0i}}{x_i} \right| > \text{error then}$$

      set key = 1
   Repeat i
5. If key = 1 then
      set $x_{0i} = x_i$
      go to step 3
6. Write results

Algorithm 8.1

Example 8.1

Obtain the solution of the following system using the Jacobi iteration method

$$2x_1 + x_2 + x_3 = 5$$
$$3x_1 + 5x_2 + 2x_3 = 15$$
$$2x_1 + x_2 + 4x_3 = 8$$

First, solve the equations for unknowns on the diagonal. That is

$$x_1 = \frac{5 - x_2 - x_3}{2}$$

$$x_2 = \frac{15 - 3x_1 - 2x_3}{5}$$

$$x_3 = \frac{8 - 2x_1 - x_2}{4}$$

If we assume the initial values of x_1, x_2 and x_3 to be zero, then we get

$$x_1^{(1)} = \frac{5}{2} = 2.5$$

$$x_2^{(1)} = \frac{15}{5} = 3$$

$$x_3^{(1)} = \frac{8}{4} = 2$$

(Note that these values are nothing but $x_i^1 = b_i/a_{ii}$)

For the second iteration, we have

$$x_1^{(2)} = \frac{5-3-2}{2} = 0$$

$$x_2^{(2)} = \frac{15-3\times 2.5-2\times 2}{5} = \frac{3.5}{5} = 0.7$$

$$x_3^{(2)} = \frac{8-2\times 2.5-3}{4} = 0$$

After third iteration,

$$x_1^{(3)} = \frac{5-0.7}{2} = 2.15$$

$$x_2^{(3)} = \frac{15-3\times 0-2\times 0}{5} = 3$$

$$x_3^{(3)} = \frac{8-2\times 0-0.7}{4} = 1.825$$

After fourth iteration,

$$x_1^{(4)} = \frac{5 - 3 - 1.825}{2} = 0.0875$$

$$x_2^{(4)} = \frac{15 - 3 \times 2.15 - 2 \times 1.825}{4} = 1.225$$

$$x_3^{(4)} = \frac{8 - 2 \times 2.15 - 3}{4} = 0.175$$

The process can be continued till the values of x reach a desired level of accuracy.

8.3 GAUSS-SEIDEL METHOD

Gauss-Seidel method is an improved version of Jacobi iteration method. In Jacobi method, we begin with the initial values

$$x_1^{(0)}, x_2^{(2)}, \ldots, x_n^{(0)}$$

and obtain next approximation

$$x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$$

Note that, in computing $x_2^{(1)}$, we used $x_1^{(0)}$ and not $x_1^{(1)}$ which has just been computed. Since, at this point, both $x_1^{(0)}$ and $x_1^{(1)}$ are available, we can use $x_1^{(1)}$ which is a better approximation for computing $x_2^{(1)}$. Similarly, for computing $x_3^{(1)}$, we can use $x_1^{(1)}$ and $x_2^{(1)}$ along with $x_4^{(0)}, \ldots, x_n^{(0)}$. This idea can be extended to all subsequent computations. This approach is called the *Gauss-Seidel* method.

The Gauss-Seidel method uses the most recent values of x as soon as they become available at any point of iteration process. During the $(k+1)$th iteration of Gauss-Seidel method, x_i takes the form

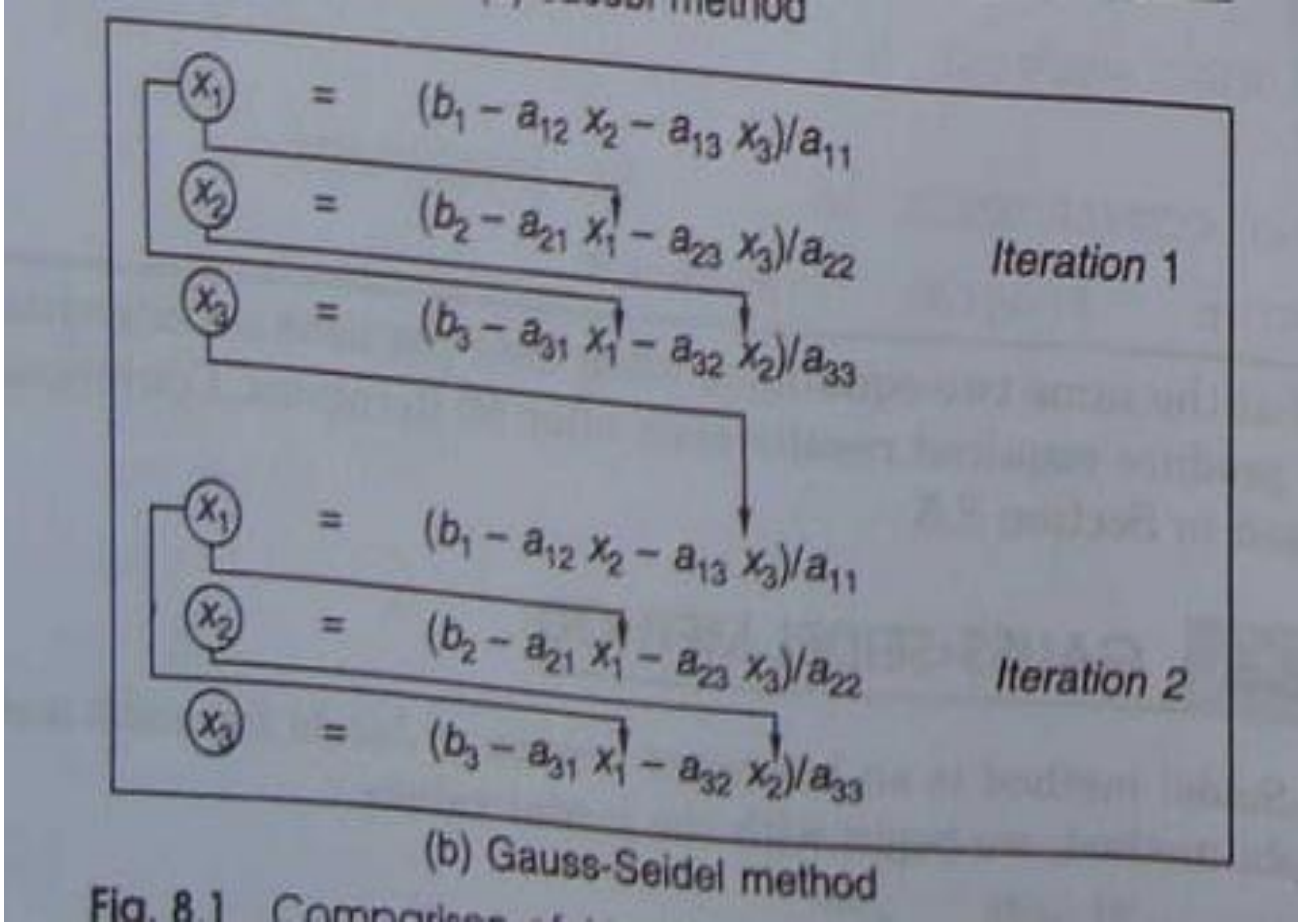
$$\boxed{x_i^{(k+1)} = \frac{b_i - \left(a_{i1}x_1^{(k+1)} + \ldots + a_{ii-1}x_{i-1}^{(k+1)} + a_{ii+1}x_{i+1}^{(k)} \ldots + a_{in}x_n^{(k)}\right)}{b_{ii}}} \qquad (8.4)$$

When $i = 1$, all superscripts in the right-hand side become (k) only. Similarly, when $i = n$, all become $(k + 1)$. Figure 8.1 illustrates pictorially the difference between the Jacobi and Gauss-Seidel method.

$$\begin{aligned}
x_1 &= (b_1 - a_{12}\,x_2 - a_{13}\,x_3)/a_{11} \\
x_2 &= (b_2 - a_{21}\,x_1 - a_{23}\,x_3)/a_{22} \qquad \textit{Iteration 1} \\
x_3 &= (b_3 - a_{31}\,x_1 - a_{32}\,x_2)/a_{33}
\end{aligned}$$

$$\begin{aligned}
x_1 &= (b_1 - a_{12}\,x_2 - a_{13}\,x_3)/a_{11} \\
x_2 &= (b_2 - a_{21}\,x_1 - a_{23}\,x_3)/a_{22} \qquad \textit{Iteration 2} \\
x_3 &= (b_3 - a_{31}\,x_1 - a_{32}\,x_2)/a_{33}
\end{aligned}$$

(a) Jacobi method



$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (b_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$
Iteration 1
$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (b_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$
Iteration 2
(b) Gauss-Seidel method

Example 8.2

Obtain the solution of the following system using Gauss-Seidel iteration method

$$2x_1 + x_2 + x_3 = 5$$
$$3x_1 + 5x_2 + 2x_3 = 15$$
$$2x_1 + x_2 + 4x_3 = 8$$

$$x_1 = (5 - x_2 - x_3)/2$$
$$x_2 = (15 - 3x_1 - 2x_3)/5$$
$$x_3 = (8 - 2x_1 - x_2)/4$$

Assuming initial value as $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$

Iteration 1 $x_1 = (5 - 0 - 0)/2 = 2.5$

$x_2 = (15 - 3 \times 2.5 - 0)/5 = 1.5$

$x_3 = (8 - 2 \times 2.5 - 1.5)/4 = 0.4$ (rounded to one decimal)

Iteration 2 $x_1 = (5 - 1.5 - 0.4)/2 = 1.6$

$x_2 = (15 - 3 \times 1.6 - 2 \times 0.4)/5 = 1.9$

$x_3 = (8 - 2 \times 1.6 - 1.9)/4 = 0.7$

We can continue this process until we get $x_1 = 1.0$, $x_2 = 2.0$ and $x_3 = 1.0$ (correct answers)

Assignment#12

- Prepare a report on “Gaussian Integration and Multiple Integration”

Thank You

Any Query??