

Unit : 1

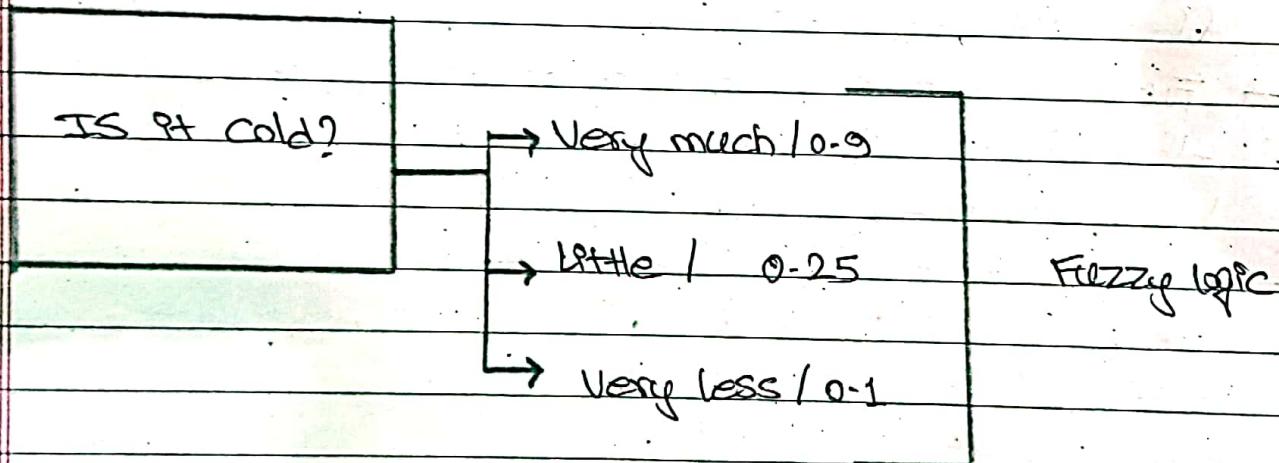
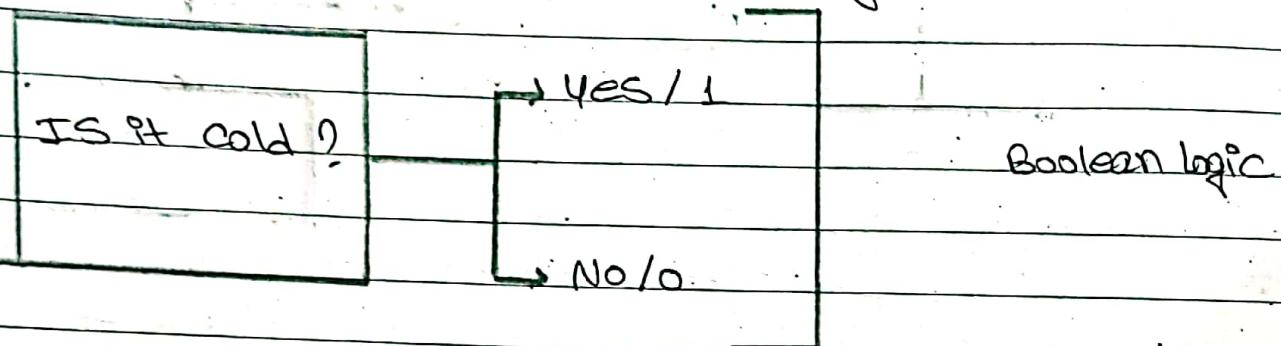
Introduction to Fuzzy Set theory:

*** Fuzzy logic:**

- Fuzzy logic is an approach to computing based on "degree of truth" rather than the usual "true or false" (1 or 0) Boolean logic on which the modern computer is based.
- The idea of fuzzy logic was first advanced by Lotfi Zadeh in 1960s at University of California.
- The term fuzzy refers to things that are not clear or are vague. In the real world many times we encounter a situation when we can't determine whether the state is true or false; their fuzzy logic provides very valuable flexibility for reasoning.
- Fuzzy logic is a form of multi-valued logic in which the truth values of variables may be any real number between 0 & 1.
- Fuzzy logic is based on the idea that in many cases the concept of true or false is too restrictive, so it allows for partial truths where a statement can be partially true or false, rather than fully true or false.
- The fundamental concept of fuzzy logic is the membership function, which defines the degree of membership of an input value to a certain set or category. The membership function is a mapping from an input value to a membership degree between 0 & 1, where 0 represents non-membership & 1 represents full membership.

- Example:

In the boolean system truth value, 1.0 represents the absolute truth value and 0.0 represents the absolute false value. But in fuzzy system, there is no logic for the absolute truth & absolute false value. But in fuzzy logic, there is an intermediate value to present which is partially true and partially false.

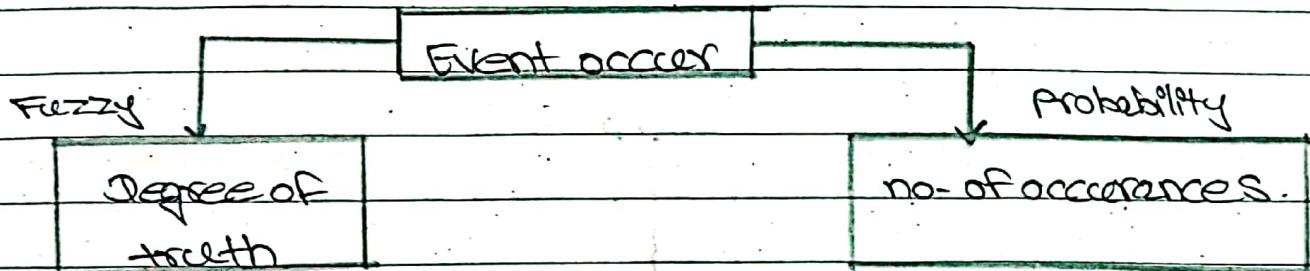


Fuzzy { Imprecise
Vague

Crisp { Precise
Clean

* Fuzzy vs Probability:

- The probability theory is based on perception and has only two outcomes (true or false).
- Fuzzy theory is based on linguistic ~~function~~ information & is extended to handle the concept of partial truth. Fuzzy Values are determined between true or false.
- e.g:-



$[0,1] \rightarrow$ multi-valued.

$$\begin{matrix} x \in t \\ f \end{matrix} \Rightarrow \text{partial degree of truth}$$

* Linguistic Variables or hedges:

Variables in mathematics normally take numeric values, although non-numeric linguistic variables are frequently employed in fuzzy logic to make the expression of rules & facts easier. For instance, the term 'Age' can be used to indicate a linguistic variable with a value such as child, young, old & so on.

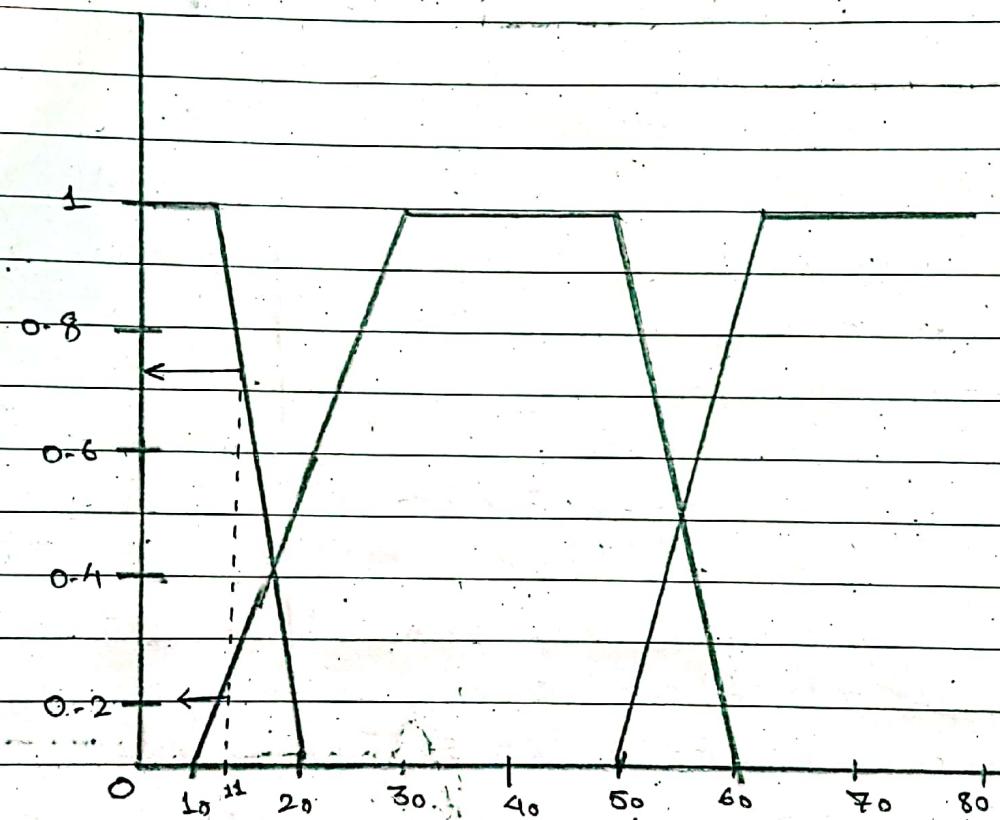
Linguistic variables are variables with a value made up of linguistic concepts (also known as linguistic words) rather than numbers, such as child, young, & so on.

$$\text{Age} = \{ \text{child}, \text{young}, \text{old} \}$$

* For e.g:-

Each Age linguistic phrase has a membership function for a specific age range. The same age value is mapped

to multiple membership values in the range of 0 to 1 by each function. These membership values can be used to identify whether a person is a child, a young person, or an elderly person.



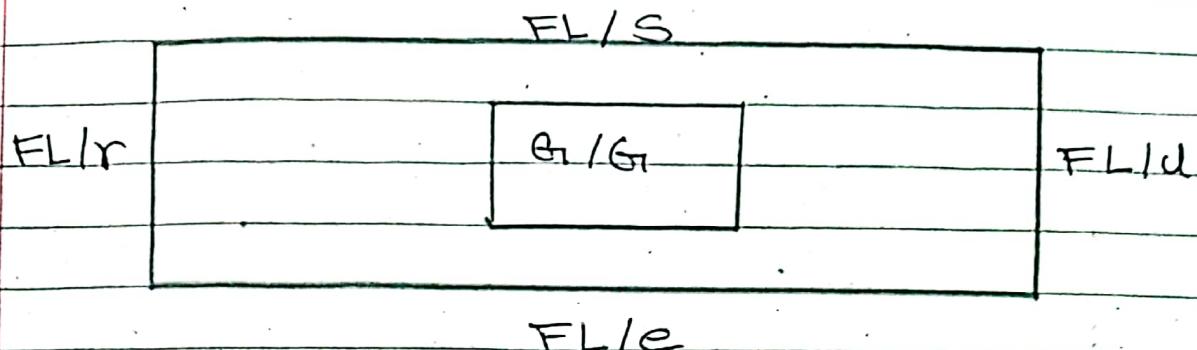
For Age = 11, we will get membership value of 0.75 (roughly) in the child set, 0.2 (approx) in the Young set, & 0 in the Old set, as shown in diagram. So, if a person's age is 11, it's safe to assume that he or she is a child, perhaps little young but certainly not old.

Similarly,

e.g:- If x is young then x is very tall

$$\text{Membership function} = \left\{ \frac{0.9}{40}, \frac{0.7}{50} \right\}$$

* Principle facets of fuzzy logic:



where

FL/S \rightarrow set theoretic Facet

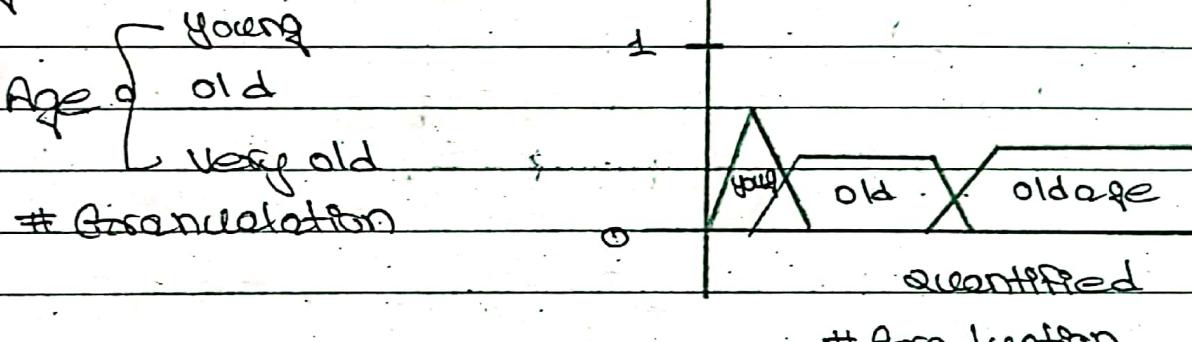
FL/d \rightarrow Logical Facet

FL/e \rightarrow epistemic Facet

FL/r \rightarrow relational Facet

G/G_i \rightarrow Granulation / Granrelation

e.g:-



let Age = 20, 1, 2, ... 100%

then,

① Granulation generally assign the given set of elements to the linguistic variable i.e. young = 20 - 20%

$$\text{old} = 50 - 70\%$$

$$\text{very old} = 70 - 100\%$$

② Graduation is the mapping of set of elements to the membership function. e.g. $\frac{2}{0}, \frac{1}{1}, \frac{0.3}{3}, \frac{0.6}{8}, \dots$

- ① FLIS \rightarrow Set theoretic Facet = Fuzzy set theories,
- ② FLIL \rightarrow Logical Facet = degree of membership / multi-valued logic.
- ③ FLIE \rightarrow epistemic Facet = Natural language / linguistic variables

④ FLIR \rightarrow relational Facet = Fuzzy relations over Fuzzy sets / Mappings.

* Example:-

Let, $A = \{1, 2, 3, \dots, 10\}$

Odd upto 5 = {1, 3, 5}

Even upto 6 = {2, 4, 6}

Find, if 7 belongs to odd upto 5

= 7 \notin odd upto 5

X - Σ Domain of discourse / universe?

$x \in X$

$x \in A$ where, $A \subset X$

$x \notin A$

$X = \{a, b, c\}$

$P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
 $\{a, b, c\}$ possible sub-sets.

then,

$A_1 = \{a, c\}, \{a, b\}, \{a, b, c\} \}$

$Y = \{0, 0, 1\}, \{1, 1, 0\}, \{1, 1, 1\} \}$ $\downarrow f: \text{mapping}$

$A_1 \rightarrow Y$

* Characteristic Function (X):

Even upto 6 = {2, 4, 6} (\because from before example)

8 \notin even 6 (even upto)

$$X_S(x_i) \begin{cases} 0 & \text{iff } x_i \notin S \\ 1 & \text{iff } x_i \in S \end{cases}$$

So, $X(8) = 0$ ($\because 8 \notin \text{even 6}$)

another example: vowel = {a, e, i, o, u}

$$X(a) = 1 \quad (\because a \in \text{vowel})$$

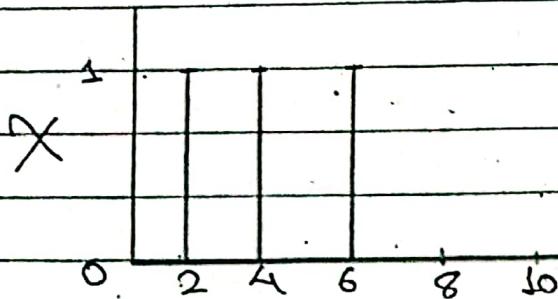
$$X(b) = 0 \quad (\because b \notin \text{vowel})$$

Similarly,

$$X = \{2, 4, 6, 8, 10\}$$

$$\text{even 6} = \{2, 4, 6\}$$

$$X(x) \begin{cases} 1 & \text{if element exists} \\ 0 & \text{doesn't exist.} \end{cases}$$



Graphical Notation

$$\begin{aligned} \exists A \subset X \\ A \cup \bar{A} = X \\ A \cap \bar{A} = \emptyset \end{aligned}$$

* Set theory operation:

- i) Union
- ii) Intersection,
- iii) Complement

Let, $X = \{2, 4, 6, 8, 10, 12\}$

$$A = \{2, 4, 6\}$$

$$B = \{4, 6, 8, 10\}$$

$$A \cup B = \{2, 4, 6, 8, 10\}$$

$$A \cap B = \{4, 6\}$$

$$\bar{A} = \{8, 10, 12\}$$

For Any two Crisp sets:

i.e $A \neq B$:

$$A \cup B = X_{A \cup B}(x_i) = \max(X_A(x_i), X_B(x_i))$$

$$A \cap B = X_{A \cap B}(x_i) = \min(X_A(x_i), X_B(x_i))$$

$$\bar{A} = X_{\bar{A}}(x_i) = 1 - X_A(x_i)$$

$$\begin{aligned} X_{A \cup B}^{(2)} &= \max \{X_A^{(2)}, X_B^{(2)}\} \\ &= \max (1, 0) \\ &= 1 \end{aligned}$$

$$\begin{aligned} X_{A \cup B}^{(4)} &= \max \{X_A^{(4)}, X_B^{(4)}\} \\ &= \max (1, 1) \\ &= 1 \end{aligned}$$

Similarly,

$$X_{A \cup B}^{(6)} = \max (1, 1) = 1$$

$$X_{A \cup B}^{(8)} = \max (0, 1) = 1$$

$$X_{A \cup B}^{(16)} = \max (0, 1) = 1$$

Also,

$$A \subset B \text{ iff. } X_A^{(xi)} \leq X_B^{(xi)}$$

$$\text{eg:- } A = \{2, 4, 6\} \\ B = \{2, 4, 6, 8\}$$

for $x_i = 2$

$$\begin{aligned} X_A^{(2)} &= 1 \\ X_B^{(2)} &= 1 \end{aligned} \quad ? \quad 1 \leq 1 \text{ (True)}$$

for $x_i = 8$

$$\begin{aligned} X_A^{(8)} &= 0 \\ X_B^{(8)} &= 1 \end{aligned} \quad 0 \leq 1 \text{ (True)}$$

Example :

$$A = \{x_1, x_2\}$$

$$A_1 = \{ \emptyset \}$$

$$A_2 = \{x_1\}$$

$$A_3 = \{x_2\}$$

$$A_4 = \{x_1, x_2\}$$

$$X \cdot A_1 = \{0, 0\}$$

$$X \cdot A_2 = \{1, 0\}$$

$$X \cdot A_3 = \{0, 1\}$$

$$X \cdot A_4 = \{1, 1\}$$

* Fuzzy sets:

$$X = \{x_1, x_2, \dots, x_n\}$$

$$A \subseteq X$$

($\because A \rightarrow$ zadeh
notation
for fuzzy set)

$$A \text{ or } \underline{A} = \left\{ \frac{\underline{u}_A(x_1)}{x_1}, \frac{\underline{u}_A(x_2)}{x_2}, \frac{\underline{u}_A(x_3)}{x_3}, \dots \right\}$$

$$A \text{ or } \underline{A} = \left\{ \frac{A(x_1)}{x_1}, \frac{A(x_2)}{x_2}, \dots \right\}$$

$$A \text{ or } \underline{A} = \{(x_1, \underline{u}_A(x_1)), (x_2, \underline{u}_A(x_2)), \dots\}$$

or

$$\underline{A} = \left\{ \frac{\underline{u}_A(x_1)}{x_1} + \frac{\underline{u}_A(x_2)}{x_2} + \dots \right\}$$

These are different ways of notations only for fuzzy sets.

where, μ_A is a membership function that maps every element in fuzzy set to membership value in $[0,1]$. which determines partial truthness.

$$A = \sum_{x_i} \mu_A(x_i) \quad \left\{ \begin{array}{l} \text{discrete Set / Finite Set} \end{array} \right.$$

$$A = \int \mu_A(x_i) \quad \left\{ \begin{array}{l} \text{Continuous Set / Infinite Set} \end{array} \right.$$

* Example:

$$X = \{1, 2, \dots, 10\}$$

Even up to 4 = $\{\frac{0.001}{1}, \frac{1}{2}, \frac{0.5}{3}, \frac{1}{4}\}$

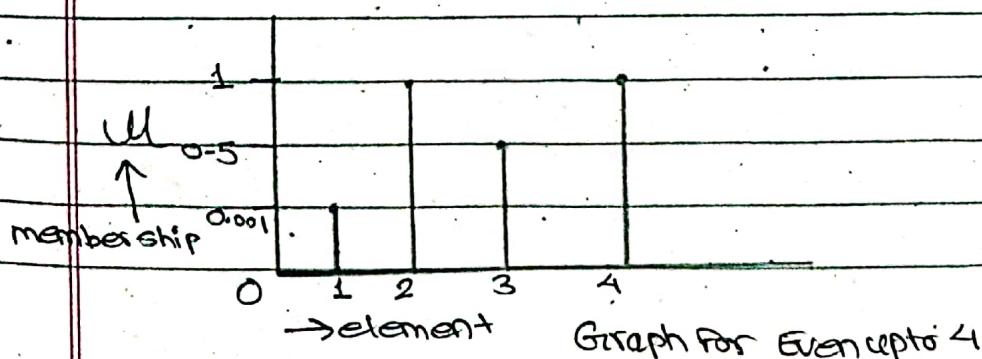
$$\text{odd up to } 4 = \{\frac{1}{1}, \frac{0.002}{2}, \frac{1}{3}, \frac{0.2}{4}\}$$

$$X : x \rightarrow Y \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right.$$

$$\mu : x \rightarrow y \in [0, 1]$$

$$A = \{\frac{0.5}{1}, \frac{0.6}{2}, \frac{0.8}{3}, \frac{0.5}{4}\}$$

$$A = \{(1, 0.5), (2, 0.6), (3, 0.8), (4, 0.5)\}$$



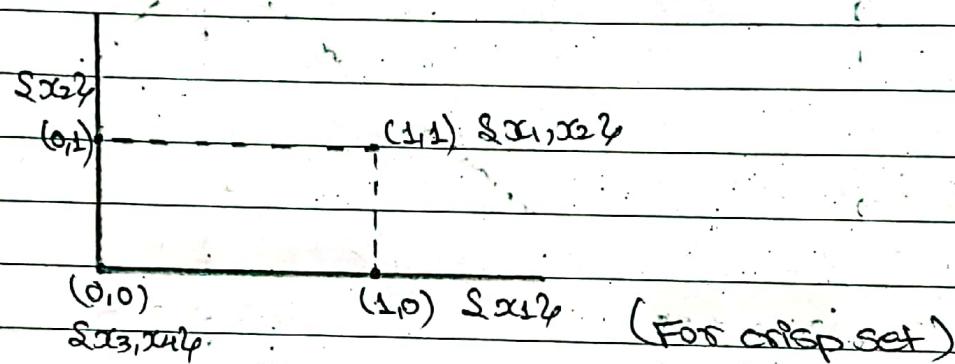
* Kosko Cube:

It helps to represent the given elements of a set to a hypercube to easy generalization.

Consider, $X = \{x_1, x_2\}$

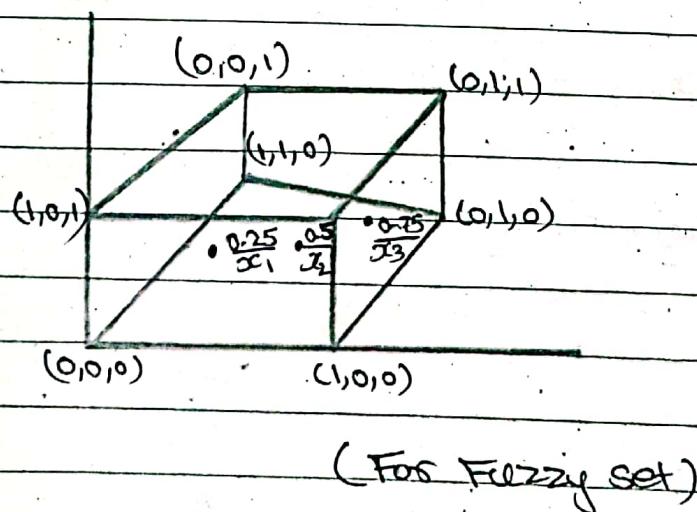
$$\text{pow}(X) = \{(x_1), (x_2), (x_1, x_2), (\emptyset)\}$$

$$\text{pow}(X) = \{(1,0), (0,1), (1,1), (0,0)\}$$

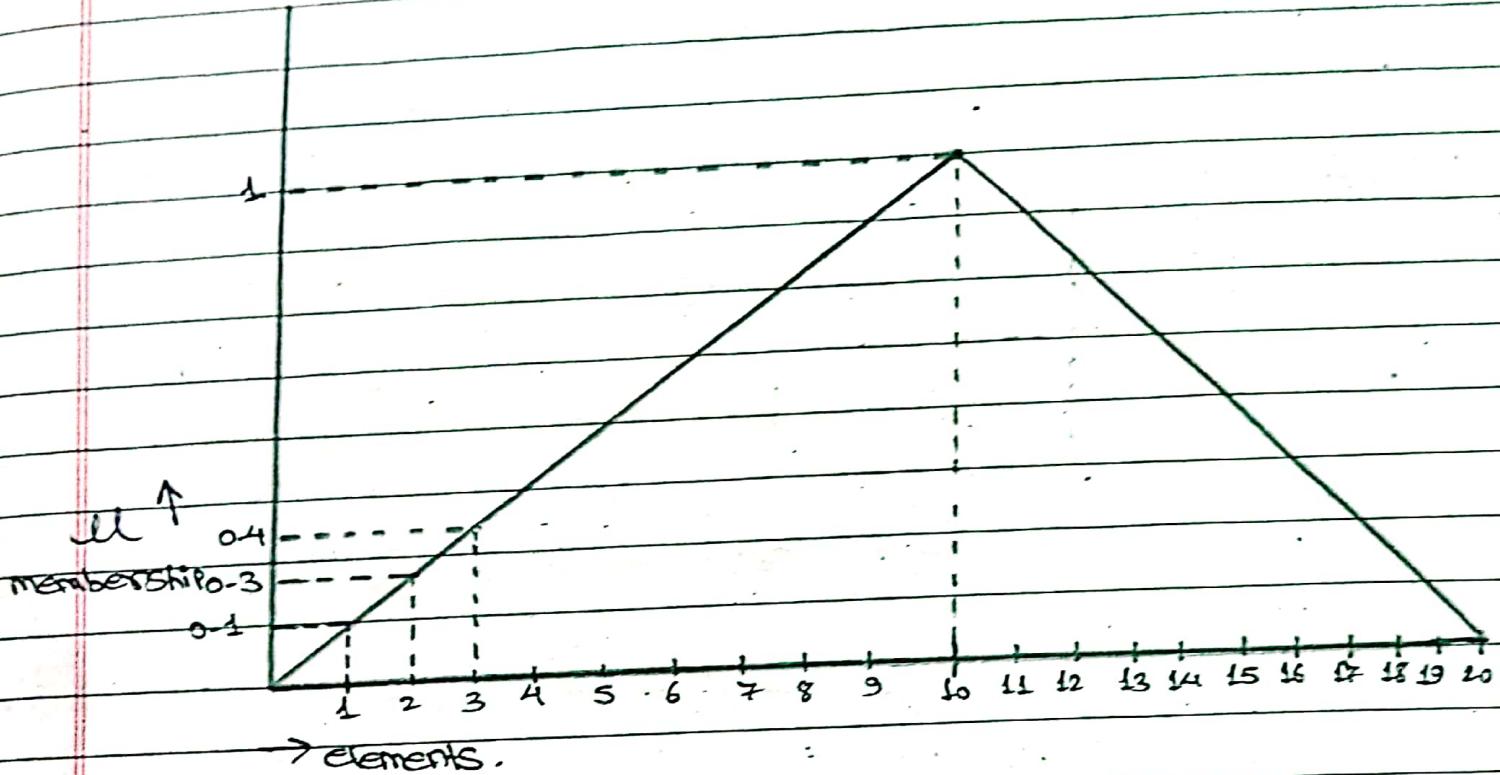


Also, $X = \{x_1, x_2, x_3\}$.

$$\text{Ull } X = \left\{ \frac{0.25}{x_1}, \frac{0.5}{x_2}, \frac{0.75}{x_3} \right\}$$



Eg:-
 $A = \{ (1, 0.1), (2, 0.3), (3, 0.4), \dots, (10, 1), (11, 0.9), (12, 0.8), \dots, (20, 0) \}$



number 5 = $\{ \frac{0.1}{1}, \frac{0.3}{2}, \frac{0.7}{3}, \frac{0.9}{4}, \frac{1}{5}, \frac{0.8}{6}, \frac{0.4}{7} \}$

* Fuzzy Set operations:

① Union: $\cup_{x \in \Omega}^{(x)} A \cup B = \cup_{x \in \Omega}^{(x)} A(x) \vee \cup_{x \in \Omega}^{(x)} B(x)$

$$= \text{MAX}(\cup_{x \in \Omega}^{(x)} A(x), \cup_{x \in \Omega}^{(x)} B(x))$$

② Intersection: $\cup_{x \in \Omega}^{(x)} A \cap B = \cup_{x \in \Omega}^{(x)} A(x) \wedge \cup_{x \in \Omega}^{(x)} B(x)$

$$= \text{MIN}(\cup_{x \in \Omega}^{(x)} A(x), \cup_{x \in \Omega}^{(x)} B(x))$$

$$\textcircled{2} \text{ Complement: } \underline{\text{ul}}_A^{(x)} = 1 - \underline{\text{ul}}_A^{(x)}$$

$$\textcircled{1} \text{ Subset: } A \subseteq X \Rightarrow \underline{\text{ul}}_A^{(x)} \subseteq \underline{\text{ul}}_X^{(x)}$$

$$\forall x, x \in X, \underline{\text{ul}}_\emptyset^{(x)} = 0$$

$$\forall x, x \in X, \underline{\text{ul}}_X^{(x)} = 1$$

$$A \subseteq B \Rightarrow \underline{\text{ul}}_A^{(x)} \subseteq \underline{\text{ul}}_B^{(x)}$$

Example \textcircled{1}: $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$A = \{0.5, 0.4, 0.3, 0.9, 1.2\}$$

$$B = \{0.6, 0.2, 0.02, 1.0\}$$

$$C = \{0.7, 0.01, 0.4\}$$

\textcircled{1}. For $A \cup B$

$$\underline{\text{ul}}_{A \cup B}^{(x)} = \max(\underline{\text{ul}}_A^{(x)}, \underline{\text{ul}}_B^{(x)})$$

$$\begin{aligned} \underline{\text{ul}}_{A \cup B}^{(1)} &= \max(0, 0.6) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} \underline{\text{ul}}^{(2)} &= \max(0.5, 0.2) \\ \underline{\text{ul}}_{A \cup B}^{(2)} &= 0.5 \end{aligned}$$

$$\underline{u}_A^{(3)} = \max_{\text{A} \cup \text{B}} (0.4, 0.02) \\ = 0.4$$

$$\underline{u}_A^{(4)} = \max_{\text{A} \cup \text{B}} (0.3, 0) \\ = 0.3$$

$$\underline{u}_A^{(5)} = \max_{\text{A} \cup \text{B}} (0.9, 0) \\ = 0.9$$

$$\underline{u}_A^{(6)} = \max_{\text{A} \cup \text{B}} (1, 0) \\ = 1$$

$$\underline{u}_A^{(7)} = \max_{\text{A} \cup \text{B}} (0, 1) \\ = 1$$

$$\therefore \underline{u}_{A \cup B}^{(x)} = \{ 0.6, \frac{0.5}{1}, \frac{0.4}{2}, \frac{0.3}{3}, \frac{0.2}{4}, \frac{0.1}{5}, \frac{0}{6}, \frac{1}{10} \}$$

(ii) $\underline{A} \cap \underline{B}$

$$\underline{u}_{A \cap B}^{(x)} = \min (\underline{u}_A^{(x)}, \underline{u}_B^{(x)})$$

$$\underline{u}_{A \cap B}^{(2)} = \min (\underline{u}_A^{(2)}, \underline{u}_B^{(2)}) \\ = \min (0.5, 0.2) \\ = 0.2$$

$$\underline{u}_{A \cap B}^{(3)} = \min (\underline{u}_A^{(3)}, \underline{u}_B^{(3)}) \\ = \min (0.4, 0.02) \\ = 0.02$$

$$\therefore \text{U}_{AB}^{(x)} = \left\{ \frac{0.2}{2}, \frac{0.02}{3} \right\}$$

$$\bar{A} = \mathbb{I} - \mathbf{u}\mathbf{u}^T$$

$$= \left\{ \frac{1-0.5}{2}, \frac{1-0.4}{3}, \frac{1-0.3}{4}, \frac{1-0.9}{6}, \frac{1-1}{9} \right\}$$

$$= \left\{ \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.7}{4}, \frac{0.1}{6}, \frac{0}{9} \right\}$$

* Relative Complement:

Relative complement of A with respect to some set X, i.e if given set is B then,

R.C. of A w.r.t B.

$$\bar{A} = B - A$$

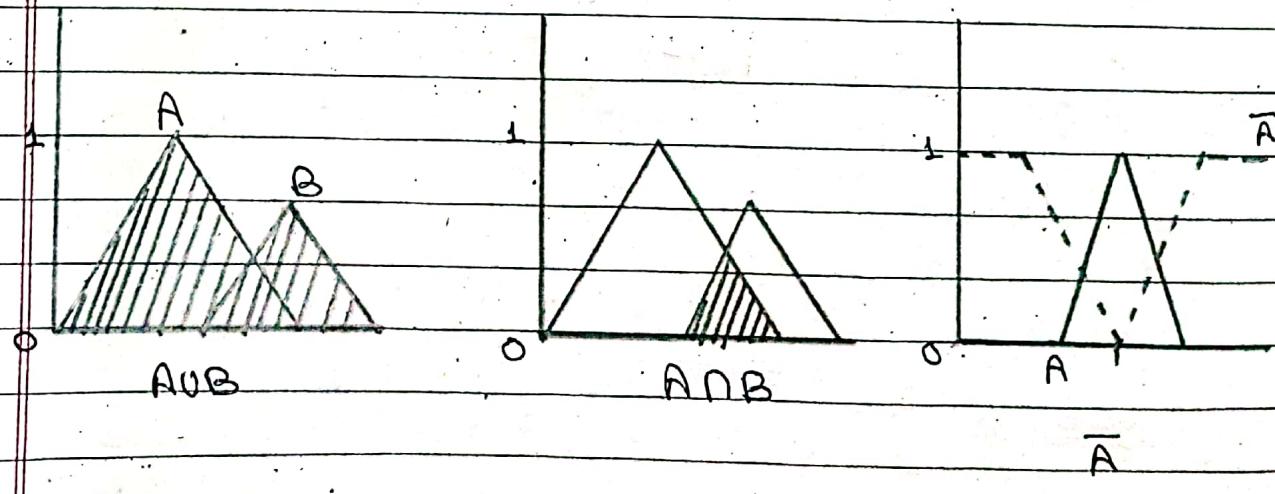
* Diagram:

$$A = \{a, b, c, d, e\}$$

$$\mathbf{B} = S \mathbf{a}_1 \mathbf{e}_2$$

$c = \{a, e, i, o, u\}$

~~theo,~~



* Exercise:

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A = \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$$

$$B = \{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \}$$

Compute: ① \bar{A} , ② \bar{B} , ③ $A \cup B$, ④ $A \cap B$

⑤ $A \cup \bar{A}$, ⑥ $B \cap \bar{B}$, ⑦ A difference B

i.e. $A \setminus B$

i.e. $A \cap \bar{B}$

⑧ Show DeMorgan's rule over $\bar{A} \cup \bar{B}$

$$\text{i.e. } (\bar{A} \cup \bar{B})' = A' \cap B'$$

⑨ Relative complement of B w.r.t. A.

\therefore Sol

$$① \text{ul}_{\bar{A}}^{(x)} = 1 - \text{ul}_A^{(x)}$$

$$= 1 - \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$$

$$= \{ \frac{1-1}{2}, \frac{1-0.5}{3}, \frac{1-0.3}{4}, \frac{1-0.2}{5} \}$$

$$= \{ \frac{0}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5} \}$$

$$② \bar{B} = 1 - \text{ul}_B^{(x)}$$

$$= 1 - \{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \}$$

$$= \left\{ \frac{1-0.5}{2}, \frac{1-0.7}{3}, \frac{1-0.2}{4}, \frac{1-0.4}{5} \right\}$$

$$= \left\{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5} \right\}$$

③ $\underset{\sim}{A} \cup \underset{\sim}{B}$

$$\underset{\sim \cup \sim}{U}^{(x)} = \max \left(\underset{\sim}{U}_A^{(x)}, \underset{\sim}{U}_B^{(x)} \right)$$

$$\underset{\sim \cup \sim}{U}^{(2)} = \max (1, 0.5) = 1$$

$$\underset{\sim \cup \sim}{U}^{(3)} = \max (0.5, 0.7) = 0.7$$

$$\underset{\sim \cup \sim}{U}^{(4)} = \max (0.3, 0.2) = 0.3$$

$$\underset{\sim \cup \sim}{U}^{(5)} = \max (0.2, 0.4) = 0.4$$

$$\therefore \underset{\sim \cup \sim}{A} \cup \underset{\sim \cup \sim}{B} = \left\{ \frac{1}{2}, \frac{0.7}{3}, \frac{0.3}{4}, \frac{0.4}{5} \right\}$$

④ $\underset{\sim \cap \sim}{A} \cap \underset{\sim \cap \sim}{B}$

$$\underset{\sim \cap \sim}{U}^{(x)} = \min \left(\underset{\sim}{U}_A^{(x)}, \underset{\sim}{U}_B^{(x)} \right)$$

$$\underset{\sim \cap \sim}{U}^{(2)} = \min (1, 0.5) = 0.5$$

$$\underset{\sim \cap \sim}{U}^{(3)} = \min (0.5, 0.7) = 0.5$$

$$u_{A \cap B}^{(4)} = \min(0.3, 0.2) = 0.2$$

$$u_{A \cap B}^{(5)} = \min(0.2, 0.4) = 0.2$$

$$\therefore A \cap B = \{ \frac{0.5}{2}, \frac{0.5}{3}, \frac{0.2}{2}, \frac{0.2}{5} \}$$

⑤ $A \cup \bar{A}$

$$A = \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$$

$$\bar{A} = \{ \frac{0}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5} \}$$

$$u_{A \cup \bar{A}}^{(2)} = \max(1, 0) = 1$$

$$u_{A \cup \bar{A}}^{(3)} = \max(0.5, 0.5) = 0.5$$

$$u_{A \cup \bar{A}}^{(4)} = \max(0.3, 0.7) = 0.7$$

$$u_{A \cup \bar{A}}^{(5)} = \max(0.2, 0.8) = 0.8$$

$$\therefore A \cup \bar{A} = \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5} \}$$

⑥ $B \cap \bar{B}$

$$B = \{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \}$$

$$\bar{B} = \{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5} \}$$

$$u_{B \cap \bar{B}}^{(2)} = \min(0.5, 0.5) = 0.5$$

$$U_{B \cap \bar{B}}^{(3)} = \min(0.7, 0.3) = 0.3$$

$$U_{B \cap \bar{B}}^{(4)} = \min(0.2, 0.8) = 0.2$$

$$U_{B \cap \bar{B}}^{(5)} = \min(0.4, 0.6) = 0.4$$

$$\therefore B \cap \bar{B} = \{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.2}{4}, \frac{0.4}{5} \}$$

⑦ $A \cap \bar{B}$

$$A = \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$$

$$\bar{B} = \{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5} \}$$

$$U_{A \cap \bar{B}}^{(2)} = \min(1, 0.5) = 0.5$$

$$U_{A \cap \bar{B}}^{(3)} = \min(0.5, 0.3) = 0.3$$

$$U_{A \cap \bar{B}}^{(4)} = \min(0.3, 0.8) = 0.3$$

$$U_{A \cap \bar{B}}^{(5)} = \min(0.2, 0.6) = 0.2$$

$$\therefore A \cap \bar{B} = \{ \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$$

⑧ De Morgan's Law : $(A \cup B)' = A' \cap B'$

now,

$$A \cup B = \{ \frac{1}{2}, \frac{0.7}{3}, \frac{0.3}{4}, \frac{0.4}{5} \}$$

So,

$$(A \cup B)' = 1 - (A \cup B)$$

$$= 1 - \{ \frac{1}{2} + 0.\overline{7} + 0.\overline{3} + 0.\overline{4} \}$$

$$= \{ 1 - \frac{1}{2}, 1 - 0.\overline{7}, 1 - 0.\overline{3}, 1 - 0.\overline{4} \}$$

$$= \{ \frac{1}{2}, 0.\overline{3}, 0.\overline{7}, 0.\overline{6} \}$$

we have,

$$A' = \{ \frac{1}{2}, 0.\overline{5}, 0.\overline{7}, 0.\overline{8} \}$$

$$B' = \{ \frac{1}{2}, 0.\overline{3}, 0.\overline{8}, 0.\overline{5} \}$$

So,

$$A' \cap B'$$

$$\underline{\mathcal{U}}_{A \cap B}^{(2)} = \min(0, 0.5) = 0$$

$$\underline{\mathcal{U}}_{A \cap B}^{(3)} = \min(0.5, 0.3) = 0.3$$

$$\underline{\mathcal{U}}_{A \cap B}^{(4)} = \min(0.7, 0.8) = 0.7$$

$$\underline{\mathcal{U}}_{A \cap B}^{(5)} = \min(0.8, 0.6) = 0.6$$

$$\therefore \bar{A} \cap \bar{B} = \{ \frac{1}{2}, 0.\overline{3}, 0.\overline{7}, 0.\overline{6} \}$$

$$\therefore (A \cup B)' = A' \cap B' \text{ proved.}$$

⑤ R-C of B w.r.t A

~~B~~ A-B

$$\text{i.e., } = \{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \} - \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \}$$

$$= \{ \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5} \} - \{ \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5} \}$$

$$= \{ \frac{1-0.5}{2}, \frac{0.5-0.7}{3}, \frac{0.3-0.2}{4}, \frac{0.2-0.4}{5} \}$$

$$= \{ \frac{0.5}{2}, -\frac{0.2}{3}, \frac{0.1}{4}, -\frac{0.2}{5} \}$$

$$= \{ \frac{0.5}{2}, \frac{0}{3}, \frac{0.1}{4}, \frac{0}{5} \}$$

$$\therefore A-B = \{ \frac{0.5}{2}, \frac{0.1}{4} \}$$

* Fuzzy numbers:

Fuzzy numbers is a fuzzy subset of the universe of numerical numbers.

* Fuzzy integers:

Fuzzy integers is a fuzzy subset of the universe of integers.

* Fuzzy real numbers:

Fuzzy real numbers is a fuzzy subset of the universe of real numbers.

e.g:- Fuzzy number 2 defined by a fuzzy set A over domain of integers as $A = \{ \frac{0.4}{1}, \frac{1}{2}, \frac{0.5}{3}, \frac{0.02}{4} \}$

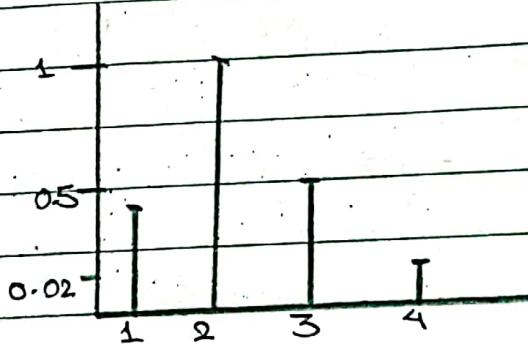
* Fuzzy Arithmetic:

Given fuzzy sets \tilde{A} & \tilde{B} representing fuzzy numbers x & y , the fuzzy arithmetic of fuzzy numbers x & y is defined as a fuzzy set \tilde{C} representing the result z of

$$z = x \text{ op } y$$

where, op is arithmetic operation

The membership of each z in \tilde{C} is determined by extension principle.



Fuzzy number 2

* Properties of fuzzy sets:

(i) Normal Fuzzy Sets:

Given, a fuzzy set \tilde{A} over X , \tilde{A} is normal iff $\exists x \in A \forall x \in X$, such that $\mu_{\tilde{A}}^{(x)} = 1$

e.g:-

$$\tilde{A} = \left\{ \frac{0.2}{3}, \frac{1}{4}, \frac{0.6}{5} \right\}$$

$$\tilde{C} = \left\{ \frac{1}{3}, \frac{1}{4}, \frac{0.5}{5} \right\}$$

\tilde{A} & \tilde{C} both are normal.

A fuzzy set which is not normal is sub-normal.

e.g:-

$$B = \{0.2, \frac{0.6}{5}, \frac{0.9}{8}\}$$

\tilde{B} is sub-normal.

Note: All crisp sets except empty set are normal.

* Height of fuzzy sets:

Height of a fuzzy set \tilde{A} is

$$\text{Height } (\tilde{A}) = \max_{\sim} (\mu_{\tilde{A}}(x))$$

normal fuzzy set has height 1

$$\text{Height } (\tilde{A}) = 1 \quad (\text{From above example.})$$

$$\text{Height } (\tilde{B}) = 0.9$$

Since, Normal fuzzy set has one element with membership 1 so height 1.

② Support of fuzzy set:

For a fuzzy set \tilde{A} over X

support of (\tilde{A}) = crisp subset of X with elements from \tilde{A} having non-zero membership.

$$\text{i.e., Supp } (\tilde{A}) = \{x / \mu_{\tilde{A}}^{(x)} > 0 \text{ for } x \in X\}$$

$$\text{eg: } A = \{0.2, 0.6, \frac{1}{6}\}$$

$$\text{supp}(A) = \{4, 5, 6\}$$

* Core of Fuzzy Set:

For a fuzzy set A over X

$$\text{core of } A = \{x \mid \mu_A^{(x)} = 1 \text{ for } x \in X \text{ & } x \in A\}$$

Core of A is also crisp subset of X .

$$B = \{0.4, \frac{1}{2}, \frac{6}{5}, \frac{8}{9}\}$$

$$\therefore \text{core } (B) = \emptyset$$

$$A = \{\frac{1}{2}, \frac{1}{6}, \frac{1}{7}, \frac{5}{8}\}$$

$$\text{core } (A) = \{6, 7\}$$

* Boundary of Fuzzy sets:

Boundary of fuzzy sets is a crisp subset over X .

$$\text{Boundary } (A) = \{x \mid 0 < \mu_A^{(x)} < 1 \text{ for } x \in X \text{ & } x \in A\}$$

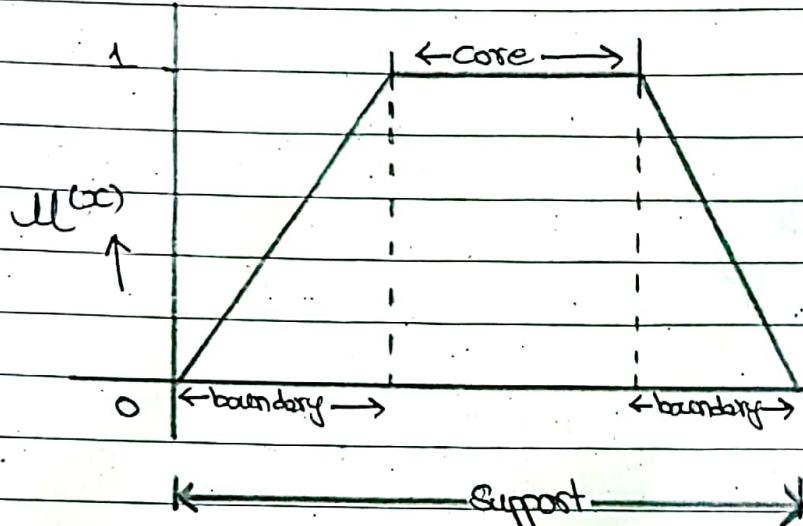
$$\text{eg: } A = \{\frac{1}{2}, \frac{6}{1}, \frac{1}{8}, \frac{1}{10}, \frac{4}{11}, \frac{2}{12}\}$$

$$\text{Boundary } (A) = \{2, 4, 11\}$$

$$B = \{\frac{1}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{5}\}$$

$\text{Boundary } (\tilde{A}) = \emptyset$

$\text{Boundary } (\text{crisp set}) = \emptyset$



Note: Subnormal fuzzy set \tilde{A}

$\text{boundary } (\tilde{A}) = \text{support } (\tilde{A})$

* Convex & non-convex fuzzy sets:

For a fuzzy set \tilde{A} over X such that for each $x, y, z \in \tilde{A}$ and they have monotonically increasing order (i.e. $x < y < z$) in \tilde{A} then \tilde{A} is convex iff;

$$\mu_{\tilde{A}}^{(y)} \geq \min(\mu_{\tilde{A}}^{(x)}, \mu_{\tilde{A}}^{(z)})$$

i.e., membership of elements are all either in increasing or decreasing order or first increasing & then decreasing order.

e.g:-

$$\tilde{A} = \left\{ \frac{0.4}{1}, \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.9}{4}, \frac{1}{5}, \frac{0.9}{6}, \frac{0.6}{7}, \frac{0.4}{8}, \frac{0.1}{9} \right\}$$

$$B = \left\{ \frac{0.4}{1}, \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.7}{4} \right\}$$

Both are convex

$$C = \left\{ \frac{0.9}{1}, \frac{0.8}{2}, \frac{0.5}{3}, \frac{0.2}{4} \right\}$$

$$P = \left\{ \frac{0.2}{1}, \frac{0.8}{2}, \frac{0.4}{3}, \frac{0.3}{4}, \frac{0.2}{5}, \frac{0.4}{6}, \frac{0.9}{7}, \frac{0.9}{8} \right\}$$

P is non-convex

e.g.: Non-convex

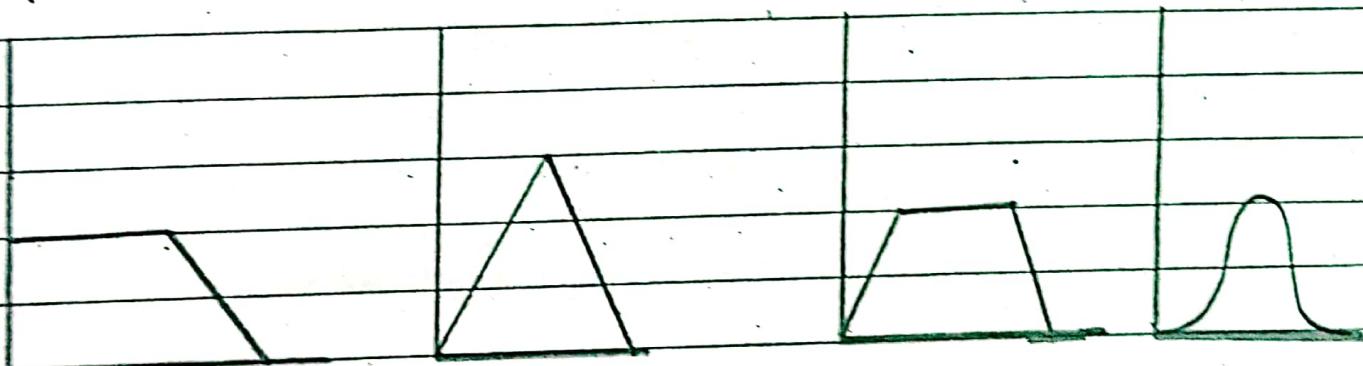


Fig :- All are convex

① Non-convex

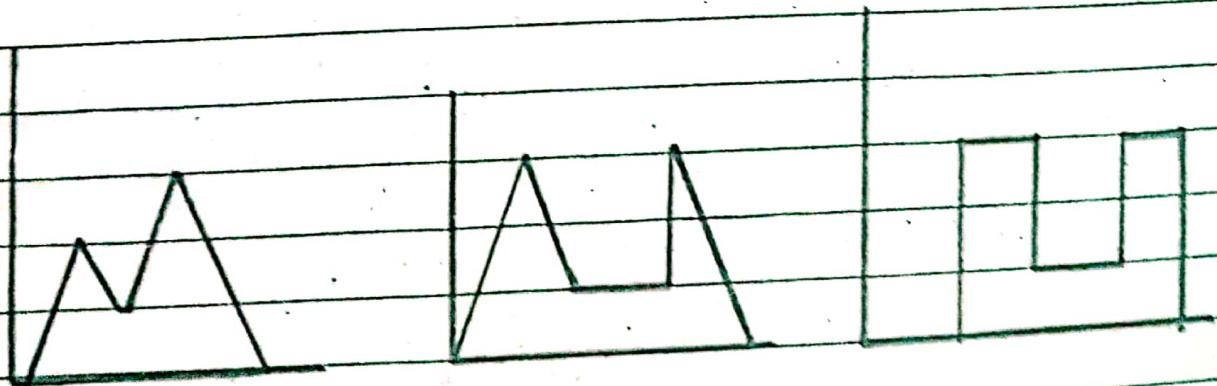


Fig: All are non-convex

* Prototype of fuzzy set:

For \tilde{A} over X

Prototype (\tilde{A}) = x such that \exists single $x \in A$

$$\text{f.u}_{\tilde{A}}^{(x)} = 1$$

x is prototype or prototype element

i) $\tilde{A} = \{0.4, \frac{1}{3}, 0.6\}$

prototype of (\tilde{A}) = 3

ii) $\tilde{B} = \{0.4, \frac{1}{3}, 0.6, 0.4\}$

prototype of (\tilde{B}) = \emptyset

iii) $\tilde{C} = \{\frac{1}{2}, \frac{1}{3}, 0.2\}$

prototype of (\tilde{C}) = \emptyset

* Theorem:

Assume \tilde{A} & \tilde{B} are any two fuzzy subsets over X . If $\tilde{C} = \tilde{A} \cup \tilde{B}$ and $\tilde{D} = \tilde{A} \cap \tilde{B}$ then

① $\tilde{D} \subset \tilde{C}$

② $\tilde{A} \subset \tilde{C}$ and $\tilde{B} \subset \tilde{C}$

③ $\frac{A}{n} \subset A$ and $\frac{B}{n} \subset B$

① Proof $\frac{A}{n} \subset C$

Given,

$$\frac{C}{n} = \frac{A}{n} \cup \frac{B}{n} \quad \text{&} \quad \frac{D}{n} = \frac{A}{n} \cap \frac{B}{n}$$

E.g:- $ul_A^{(x)} = \frac{0.4}{2}$
 $ul_B^{(x)} = \frac{0.6}{2}$

now,

$$\text{for each } x \in X, ul_{\frac{C}{n}}^{(x)} = \min \left(ul_A^{(x)}, ul_B^{(x)} \right) \rightarrow \frac{0.4}{2}$$

$$\text{&} ul_{\frac{C}{n}}^{(x)} = \max \left(ul_A^{(x)}, ul_B^{(x)} \right) \rightarrow \frac{0.6}{2}$$

here, for each $x \in X$, it can be inferred that

$$ul_{\frac{C}{n}}^{(x)} \leq ul_{\frac{C}{n}}^{(x)} \Rightarrow \frac{A}{n} \subset \frac{C}{n} \quad \frac{0.4}{2} \leq \frac{0.6}{2} \text{ TRUE}$$

② Proof $\frac{A}{n} \subset \frac{C}{n}$ and $\frac{B}{n} \subset \frac{C}{n}$

Since,

$$ul_{\frac{C}{n}}^{(x)} = \max \left(ul_A^{(x)}, ul_B^{(x)} \right)$$

then,

it is clear that

$$ul_{\frac{C}{n}}^{(x)} \geq ul_A^{(x)} \quad \text{i.e., } ul_A^{(x)} \leq ul_{\frac{C}{n}}^{(x)} \Rightarrow \frac{A}{n} \subset \frac{C}{n}$$

and,

$$ul_{\frac{C}{n}}^{(x)} \geq ul_B^{(x)} \quad \text{i.e., } ul_B^{(x)} \leq ul_{\frac{C}{n}}^{(x)} \Rightarrow \frac{B}{n} \subset \frac{C}{n}$$

③ Proof $\bigcap_{\sim} C_A$ and $\bigcap_{\sim} C_B$

Given,

$$\bigcap_{\sim} = A \cap B$$

$$\text{Ull}_{\sim}^{(x)} = \min (\text{Ull}_A^{(x)}, \text{Ull}_B^{(x)})$$

Thus,

$$\text{Ull}_{\sim}^{(x)} < \text{Ull}_A^{(x)} \Rightarrow \bigcap_{\sim} C_A$$

Also,

$$\text{Ull}_{\sim}^{(x)} \leq \text{Ull}_B^{(x)} \Rightarrow \bigcap_{\sim} C_B$$

* Concentration & dilution of fuzzy set:

Assume a fuzzy set A over X . consider a non-negative number α then A^α is a fuzzy set, say B containing all elements x from A such that for each x

$$\text{Ull}_B^{(x)} = (\text{Ull}_A^{(x)})^\alpha$$

$$\text{e.g.: } A = \left\{ \frac{1}{a}, \frac{0.6}{b}, \frac{0.3}{c}, \frac{0}{d}, \frac{0.5}{e} \right\}$$

i) for $\alpha = 2$ (Concentration op)

$$(A)^2 = \left\{ \frac{1}{a}, \frac{0.36}{b}, \frac{0.09}{c}, \frac{0}{d}, \frac{0.25}{e} \right\}$$

ii) for $\alpha = \frac{1}{2} = 0.5$ (Dilution op)

$$(A)^{0.5} = \left\{ \frac{1}{a}, \frac{0.77}{b}, \frac{0.55}{c}, \frac{0}{d}, \frac{0.7}{e} \right\}$$

① If $\alpha > 1$ then $\tilde{A}^\alpha \subset \tilde{A}$, then it is known as concentration operation which decreases the fuzziness of elements.

② If $\alpha < 1$ then $\tilde{A}^\alpha \supset \tilde{A}$, then it is known as dilation operation which increases fuzziness of elements.

* Bounded Sum & Difference:

Given, any two fuzzy subsets \tilde{A} & \tilde{B} over X , then bounded sum of \tilde{A} & \tilde{B} is a fuzzy set \tilde{D}

where, for each $x \in D$,

$$\text{ul}_{\tilde{D}}^{(x)} = \min_{\tilde{A}, \tilde{B}} \left(1, (\text{ul}_{\tilde{A}}^{(x)} + \text{ul}_{\tilde{B}}^{(x)}) \right)$$

$$\text{and } \tilde{D} = \tilde{A} \oplus \tilde{B}$$

& bounded difference of \tilde{A} & \tilde{B} is a fuzzy set \tilde{C}
whose, for each $x \in C$

$$\text{ul}_{\tilde{C}}^{(x)} = \max \left(0, (\text{ul}_{\tilde{A}}^{(x)} - \text{ul}_{\tilde{B}}^{(x)}) \right)$$

$$\text{and } \tilde{C} = \tilde{A} \ominus \tilde{B} \quad (\because \text{Bounded difference of } \tilde{B} \text{ w.r.t } \tilde{A})$$

Example:

$$\tilde{A} = S \frac{1}{a} + \frac{0.7}{b} + \frac{0.3}{c} + \frac{0}{d} + \frac{0.6}{e} \}$$

$$\tilde{B} = S \frac{0.2}{a} + \frac{0.1}{b} + \frac{0.4}{c} + \frac{1}{d} + \frac{0.2}{e} + \frac{0.6}{f} \}$$

note,

Bounded sum $\frac{D}{n} = \frac{A}{n} \oplus \frac{B}{n}$ is

$$U\ell_{\frac{D}{n}}^{(x)} = \min(1, (\ell\ell_{\frac{A}{n}}^{(x)} + \ell\ell_{\frac{B}{n}}^{(x)}))$$

$$\begin{aligned} \text{for } a \in \frac{D}{n}, \quad U\ell_{\frac{D}{n}}^{(a)} &= \min(1, (\ell\ell_{\frac{A}{n}}^{(a)} + \ell\ell_{\frac{B}{n}}^{(a)})) \\ &= \min(1, (1+0.2)) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{for } b \in \frac{D}{n}, \quad U\ell_{\frac{D}{n}}^{(b)} &= \min(1, (0.7+0.1)) \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} \text{for } c \in \frac{D}{n}, \quad U\ell_{\frac{D}{n}}^{(c)} &= \min(1, (0.3+0.4)) \\ &= 0.7 \end{aligned}$$

$$\begin{aligned} \text{for } d \in \frac{D}{n}, \quad U\ell_{\frac{D}{n}}^{(d)} &= \min(1, (0+1)) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{for } e \in \frac{D}{n}, \quad U\ell_{\frac{D}{n}}^{(e)} &= \min(1, (0.6+0.2)) \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} \text{for } f \in \frac{D}{n}, \quad U\ell_{\frac{D}{n}}^{(f)} &= \min(1, (0+0.6)) \\ &= 0.6 \end{aligned}$$

$$\therefore \text{Bounded sum } \frac{D}{n} = \sum \frac{1}{a}, \frac{0.8}{b}, \frac{0.7}{c}, \frac{1}{d}, \frac{0.8}{e}, \frac{0.6}{f}$$

Now, for Bounded difference of $\frac{A}{n} \oplus \frac{B}{n}$ i.e., $\frac{D}{n} = \frac{A}{n} - \frac{B}{n}$

$$U\ell_{\frac{D}{n}}^{(x)} = \max(0, (U\ell_{\frac{A}{n}}^{(x)} - U\ell_{\frac{B}{n}}^{(x)}))$$

$$\text{for } a \in \frac{A}{n} = U\ell_{\frac{A}{n}}^{(x)} = \max(0, (1-0.2)) \\ = 0.8$$

$$\text{for } b \in \frac{B}{n} = U\ell_{\frac{B}{n}}^{(x)} = \max(0, (0.7-0.1)) \\ = 0.6$$

$$\text{for } c \in \frac{D}{n} = U\ell_{\frac{D}{n}}^{(c)} = \max(0, (0.3-0.4)) \\ = 0$$

$$\text{for } d \in \frac{D}{n} = U\ell_{\frac{D}{n}}^{(d)} = \max(0, (0-1)) \\ = 0$$

$$\text{for } e \in \frac{D}{n} = U\ell_{\frac{D}{n}}^{(e)} = \max(0, (0.6-0.2)) \\ = 0.4$$

$$\text{for } f \in \frac{D}{n} = U\ell_{\frac{D}{n}}^{(f)} = \max(0, (0-0.6)) \\ = 0$$

\therefore Bounded difference $\frac{A}{n} \ominus \frac{B}{n}$.

$$= \{ \frac{0.8}{2}, \frac{0.6}{6}, \frac{0}{c}, \frac{0}{d}, \frac{0.4}{e}, \frac{0.2}{f} \}$$

* α -Level or α -cut:

Assume a fuzzy set A over X , then the α -level or α -cut of A is denoted by A_α and is a crisp set consisting of all elements x from X for which

$$\underline{u}_A^{(x)} > \alpha$$

i.e., $A_\alpha = \{x \mid \underline{u}_A^{(x)} \geq \alpha \text{ for each } x \in X\}$

i) Strict α -cut:

$$A_{\alpha^+} = \{x \mid \underline{u}_A^{(x)} > \alpha \text{ for each } x \in X\}$$

$$\text{eg: } A = \{a, b, c, d, e\} \text{ where } \underline{u}_A^{(a)} = \frac{0.4}{e}, \underline{u}_A^{(b)} = \frac{1}{d}, \underline{u}_A^{(c)} = 0.6, \underline{u}_A^{(d)} = 0.7, \underline{u}_A^{(e)} = 0.8$$

For $\alpha = 0.6$,

$$A_{0.6} = \{b, c, d, e\}$$

$$A_{0.6^+} = \{b, d, e\}$$

case i) If $\alpha = 1$, then for Subnormal fuzzy set
 α -cut = \emptyset

case ii) If $\alpha = 0$, $\forall x \in A$

$$\alpha\text{-cut} = \{x \in A\}$$

Unit: 2

Fuzzy Mapping

* Relation Mapping:

→ Mapping between two sets

→ Presence or absence of connection or association between the elements of two sets.

* Classical Relation | Cartesian product:

For crisp sets A_1, A_2, \dots, A_r the cartesian product is $A_1 \times A_2 \times \dots \times A_r$ and contains tuples (a_1, a_2, a_3, \dots) where, $a_1 \in A_1, a_2 \in A_2, \dots, a_r \in A_r$.

Consider two crisp sets X and Y then the cartesian product of two crisp sets $X \times Y$ is denoted by;

$$X \times Y = S(x, y) \mid x \in X, y \in Y \text{ } \& \text{ (For domain } X, Y)$$

Some Properties:

a) $X \times Y \neq Y \times X$

b) $|X \times Y| = |X| \times |Y|$ (Cardinality i.e. no. of elements)

c) Cartesian product of 2 sets is not same as the arithmetic product of two or more sets.

Also,

$$\chi_{(x,y)} = \begin{cases} 1 & \text{if } (x,y) \in X \times Y \\ 0 & \text{if } (x,y) \notin X \times Y \end{cases}$$

Example: consider the example A and B.

$$A = \{2, 4, 6, 8\}$$

$$B = \{3, 7, 8, 9\}$$

Find the Cartesian product of 2 sets.

= Sol

$$A \times B = \{(2,3), (2,7), (2,8), (2,9), (4,3), (4,7), (4,8), (4,9), (6,3), (6,7), (6,8), (6,9), (8,3), (8,7), (8,8), (8,9)\}$$

Therefore, A particular mapping is done from $a \in A$ to $b \in B$ which is denoted by R (relation).

* Crisp Relation:

Crisp relation is a subset of Cartesian product.

$$R(x,y) \subset X \times Y$$

Thus,

$$\chi_{R(x,y)} = \begin{cases} 1 & \text{if } (x,y) \in R \\ 0 & \text{if } (x,y) \notin R \end{cases}$$

We can represent R in a matrix form.

* Example:

Consider sets, $X = \{1, 2, 3\}$

$$Y = \{a, b\}$$

Then, the relation $R_1 \subseteq X \times Y$ can be

$$R_1 = \{(1,a), (2,b)\} \subseteq X \times Y$$

i.e.

$$R_1 = \begin{matrix} & a & b \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \left[\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \right] \end{matrix}$$

1 → a

2 → b

3.

Fig:- Sagittal diagram of R_1

* Operations on crisp relations:

For any two relations R and S;

i) Union :

$$R \cup S \Rightarrow X_{R \cup S}^{(x,y)} = \max (X_R^{(x,y)}, X_S^{(x,y)})$$

ii) Intersection :

$$R \cap S = X_{R \cap S}^{(x,y)} = \min (X_R^{(x,y)}, X_S^{(x,y)})$$

iii) complement :

$$\bar{R} = X_{\bar{R}}^{(x,y)} = 1 - X_R^{(x,y)}$$

iv) Containment (subset) :

$$R \subseteq S \Rightarrow X_R^{(x,y)} \leq X_S^{(x,y)}$$

v) Identity :

$$\phi \text{ is identity of } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$I \text{ is identity of } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Distributivity,
- Commutativity,
- Associativity,
- DeMorgan's Law,
- Excluded middle axioms $\rightarrow R \vee \bar{R} = E$
- $R \wedge \bar{R} = O$

Example:

$$\text{Given, } X = \{x_1, x_2\}$$

$$Y = \{y_1, y_2\}$$

$$R_1 = \{(x_2, y_1), (x_1, y_2)\}$$

$$R_2 = \{(x_1, y_1), (x_1, y_2)\}$$

Compare, (i) $R_1 \cup R_2$ (ii) $R_1 \cap R_2$
 (iii) \bar{R}_1 (iv) \bar{R}_2

= Sol

(i) $R_1 \cup R_2$

for (x_1, y_1)

$$X_{R_1 \cup R_2}(x_1, y_1) = \max(X_{R_1}(x_1, y_1), X_{R_2}(x_1, y_1))$$

$$= \max(0, 1)$$

$$= 1$$

for (x_1, y_2)

$$X_{R_1 \cup R_2}(x_1, y_2) = \max(X_{R_1}(x_1, y_2), X_{R_2}(x_1, y_2))$$

$$= \max(1, 1)$$

$$= 1$$

for (x_2, y_1)

$$X_{R_1 \cup R_2}^{(x_2, y_1)} = \max (X_{R_1}^{(x_2, y_1)}, X_{R_2}^{(x_2, y_1)})$$

$$= \max (1, 0)$$

$$= 1$$

ii) for $R_1 \cap R_2$

for (x_1, y_1)

$$X_{R_1 \cap R_2}^{(x_1, y_1)} = \min (X_{R_1}^{(x_1, y_1)}, X_{R_2}^{(x_1, y_1)})$$

$$= \min (0, 1)$$

$$= \min (0, 1)$$

$$= 0$$

for (x_1, y_2)

$$X_{R_1 \cap R_2}^{(x_1, y_2)} = \min (X_{R_1}^{(x_1, y_2)}, X_{R_2}^{(x_1, y_2)})$$

$$= \min (1, 1)$$

$$= 1$$

for (x_2, y_1)

$$X_{R_1 \cap R_2}^{(x_2, y_1)} = \min (X_{R_1}^{(x_2, y_1)}, X_{R_2}^{(x_2, y_1)})$$

$$= \min (0, 1)$$

$$= 0$$

(ii) \bar{R}_1 for (x_2, y_1)

$$\bar{R}_1 = 1 - (X_{R_1}^{(x_2, y_1)})$$

$$= 1 - 1$$

$$= 0$$

for (x_1, y_2)

$$\bar{R}_1 = 1 - (X_{R_1}^{(x_1, y_2)})$$

$$= 1 - 1$$

$$= 0$$

for (x_3, y_1)

$$\bar{R}_1 = 1 - (X_{R_1}^{(x_3, y_1)})$$

$$= 1 - 0$$

$$= 1$$

(iii) \bar{R}_2 for (x_1, y_1)

$$\bar{R}_2 = 1 - (X_{R_2}^{(x_1, y_1)})$$

$$= 1 - 1$$

$$= 0$$

for (x_1, y_2)

$$\bar{R}_2 = 1 - (X_{R_2}^{(x_1, y_2)})$$

$$= 1 - 1$$

$$= 0$$

for (x_2, y_1)

$$\bar{R}_2 = 1 - (X_{R_2}^{(x_2, y_1)})$$

$$= 1 - 0$$

$$= 1$$

another way for $R_1 \cup R_2$:

$$R_1 = \alpha_1 \left[\begin{array}{cc} y_1 & y_2 \\ 0 & 1 \\ \hline x_2 & 1 & 0 \end{array} \right]$$

$$R_2 = \alpha_1 \left[\begin{array}{cc} y_1 & y_2 \\ 1 & 1 \\ \hline x_2 & 0 & 0 \end{array} \right]$$

$R_1 \cup R_2 =$

$$\left[\begin{array}{cc} y_1 & y_2 \\ \hline x_1 & \max(0,1) & \max(1,1) \\ x_2 & \max(1,0) & \max(0,0) \end{array} \right]$$

$$= \left[\begin{array}{cc} y_1 & y_2 \\ \hline x_1 & 1 & 1 \\ x_2 & 1 & 0 \end{array} \right]$$

* Composition operations:

Consider two relations,

$R \subset X \times Y$

$S \subset Y \times Z$

Then, the relation $T \subset X \times Z$ can be generated from R and S performing composition and is denoted as;

$T = R \cdot S$ where, $T = \underbrace{S \cap T_{R \rightarrow Z}}_{\text{composition operation}}$ for R

$$T = \{ (x, z) \mid \exists (x, y) \in R \text{ & } \exists (y, z) \in S \}$$

$\chi_T^{(x,z)}$ can be determined by rules of composition.

* Rules of Composition:

- ① Maxmin
- ② Max product
- ③ Max Max
- ④ Min max
- ⑤ Min min.

① Maxmin composition:

Given,

$$R \subseteq X \times Y$$

$$S \subseteq Y \times Z$$

$$T = R \circ S$$

where,

$$\chi_T^{(x,z)} = \bigvee_{y \in Y} (\chi_R^{(x,y)} \wedge \chi_S^{(y,z)})$$

OR (max) (Union) AND (min)

$$T(x,z) = \max \left\{ \min \left\{ \chi_R^{(x,y)}, \chi_S^{(y,z)} \right\} \right\}$$

② Max-Product composition:

$$T = R \circ S$$

where,

$$\chi_T^{(x,z)} = \bigvee_{y \in Y} \{ \chi_R^{(x,y)} \cdot \chi_S^{(y,z)} \}$$

$$T(x,z) = \max \{ \chi_R^{(x,y)} \cdot \chi_S^{(y,z)} \}$$

where, " \cdot " is arithmetic product | multiplication for crisp relations.

Note: max-product = max-min

since, product is same as min as;

$$1 \times 0 = 0$$

$$0 \times 1 = 0$$

$$0 \times 0 = 0$$

$$1 \times 1 = 1$$

Same as MIN.

③ Nor-Nor Composition:

$$T = R \cdot S$$

where,

$$X_T^{(x,z)} = \bigvee_{y \in Y} (X_R^{(x,y)} \vee X_S^{(y,z)})$$

$$T(x,z) = \max \{ \max \{ X_R^{(x,y)}, X_S^{(y,z)} \} \}$$

④ Min max composition:

$$T = R \cdot S$$

where,

$$X_T^{(x,z)} = \bigwedge_{y \in Y} \{ X_R^{(x,y)} \vee X_S^{(y,z)} \}$$

$$T(x,z) = \min \{ \max \{ X_R^{(x,y)}, X_S^{(y,z)} \} \}$$

⑤ Min-min composition:

$$T = R \cdot S \text{ where,}$$

$$X_T^{(x,z)} = \bigwedge_{y \in Y} \{ X_R^{(x,y)} \wedge X_S^{(y,z)} \}$$

$$T(x,z) = \min \{ \min \{ X_R^{(x,y)}, X_S^{(y,z)} \} \}$$

* Example:

consider, $R = \{ (x_1, y_1), (x_1, y_3), (x_2, y_4) \}$

$S = \{ (y_1, z_2), (y_3, z_2) \}$

For, $X = \{ x_1, x_2, x_3 \}$

$y = \{ y_1, y_2, y_3, y_4 \}$

$Z = \{ z_1, z_2 \}$

Complete all rules of composition:

= Sol

here,

	y_1	y_2	y_3	y_4
x_1	1	0	1	0
x_2	0	0	0	1
x_3	0	0	0	0

	z_1	z_2
y_1	0	1
y_2	0	0
y_3	0	1
y_4	0	0

	z_1	z_2	=?
x_1			
x_2			
x_3			

① max min composition?

For (x_1, z_1)

$$\begin{aligned}
 X_T^{(x_1, z_1)} &= \max \left\{ \min \left\{ X_R^{(x_1, y_4)}, X_S^{(y_1, z_1)} \right\} \right\} \\
 &= \max \left\{ \min \left\{ X_R^{(x_1, y_4)}, X_S^{(y_1, z_1)}, \min(X_R^{(x_1, y_2)}, \right. \right. \\
 &\quad \left. \left. X_S^{(y_2, z_1)}, \min(X_R^{(x_1, y_3)}, X_S^{(y_3, z_1)}) \right\} \right\} \\
 &\quad \min(X_R^{(x_1, y_4)}, X_S^{(y_4, z_1)}) \Big\} \\
 &= \max \left\{ \min(1, 0), \min(0, 0), \min(1, 0), \min(0, 0) \right\} \\
 &= \max(0, 0, 0, 0) \\
 &= 0
 \end{aligned}$$

for (x_1, z_2) ,

$$\begin{aligned}
 X_T^{(x_1, z_2)} &= \max \left\{ \min \left\{ X_R^{(x_1, y_1)}, X_S^{(y_1, z_2)} \right\}, \right. \\
 &\quad \left. \min(X_R^{(x_1, y_2)}, X_S^{(y_2, z_2)}) \right\} \min(X_R^{(x_1, y_3)}, X_S^{(y_3, z_2)}) \\
 &\quad \min(X_R^{(x_1, y_4)}, X_S^{(y_4, z_2)}) \\
 &= \max(\min(1, 1), \min(0, 0), \min(1, 1), \min(0, 0)) \\
 &= \max(1, 0, 1, 0) \\
 &= 1
 \end{aligned}$$

Shifter for. (x_2, z_1)

$$\begin{aligned} X_T^{(x_2, z_1)} &= \max[\min(0, 0), \min(0, 0), \min(1, 0), \min(0, 0)] \\ &= \max[0, 0, 0, 0] \\ &= 0 \end{aligned}$$

for. (x_2, z_2)

$$\begin{aligned} X_T^{(x_2, z_2)} &= \max[\min(0, 1), \min(0, 0), \min(0, 1), \min(1, 0)] \\ &= \max[0, 0, 0, 0] \\ &= 0 \end{aligned}$$

for. (x_3, z_1)

$$\begin{aligned} X_T^{(x_3, z_1)} &= \max[\min(0, 0), \min(0, 0), \min(0, 0), \min(0, 0)] \\ &= \max[0, 0, 0, 0] \\ &= 0 \end{aligned}$$

for. (x_3, z_2)

$$\begin{aligned} X_T^{(x_3, z_2)} &= \max[\min(0, 1), \min(0, 0), \min(0, 1), \min(0, 0)] \\ &= \max[0, 0, 0, 0] \\ &= 0 \end{aligned}$$

$$\therefore T = R-S \text{ using max-min composition} = x_1 \begin{array}{|c|c|} \hline z_1 & z_2 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$x_2 \begin{array}{|c|c|} \hline z_1 & z_2 \\ \hline 0 & 0 \\ \hline \end{array}$$

$$x_3 \begin{array}{|c|c|} \hline z_1 & z_2 \\ \hline 0 & 0 \\ \hline \end{array}$$

b. For min-min composition:

$$T(x, z) = \min \{ \min \{ x_R^{(x,y)}, x_S^{(x,y)} \} \}$$

for. (x_1, z_1)

$$= \min [0, 0, 0, 0]$$

$$= 0$$

for (x_1, z_2)

$$= \min [1, 0, 1, 0]$$

$$= 0$$

for. (x_2, z_1)

$$= \min [0, 0, 0, 0]$$

$$= 0$$

for (x_2, z_2)

$$= \min [0, 0, 0, 0]$$

$$= 0$$

for (x_3, z_1)

$$= \min [0, 0, 0, 0]$$

$$= 0$$

for (x_3, z_2)

$$= \min [0, 0, 0, 0]$$

$$= 0$$

$$\therefore T = R.S \text{ using min-min composition} = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{array} \quad \boxed{1}$$

* Fuzzy relations:

Consider fuzzy sets \tilde{A} and \tilde{B} over X and Y , then fuzzy relation \tilde{R} is subset of $\tilde{A} \times \tilde{B}$ i.e. $\forall x \in \tilde{A}$ and $y \in \tilde{B}$

$$\tilde{R} = \left\{ \frac{\mu_{\tilde{R}}(x,y)}{(x,y)} \mid (x,y) \in X \times Y \right\}$$

where,

$$\mu_{\tilde{R}}^{(x,y)} = \min \left(\mu_{\tilde{A}}^{(x)}, \mu_{\tilde{B}}^{(y)} \right)$$

* Example :

Consider,

$$\tilde{A} = \left\{ \frac{0.2}{2}, \frac{0.4}{3} \right\}$$

$$\tilde{B} = \left\{ \frac{0.1}{1}, \frac{0.6}{2} \right\}$$

now,

$$\tilde{R} \subseteq \tilde{A} \times \tilde{B}$$

For $(2,1)$

$$\begin{aligned} \mu_{\tilde{R}}^{(2,1)} &= \min \left(\mu_{\tilde{A}}^{(2)}, \mu_{\tilde{B}}^{(1)} \right) \\ &= \min (0.2, 0.1) \\ &= 0.1 \end{aligned}$$

For $(2,2)$

$$\mu_{\tilde{R}}^{(2,2)} = \min \left(\mu_{\tilde{A}}^{(2)}, \mu_{\tilde{B}}^{(2)} \right)$$

$$= \text{MIN}(0.2, 0.6)$$

$$= 0.2$$

for (3,1)

$$U_{R_2}^{(3,1)} = \text{MIN}\left(U_A^{(3)}, U_B^{(1)}\right)$$

$$= \text{MIN}(0.4, 0.1)$$

$$= 0.1$$

for (3,2)

$$U_{R_2}^{(3,2)} = \text{MIN}\left(U_A^{(3)}, U_B^{(2)}\right)$$

$$= \text{MIN}(0.4, 0.6)$$

$$= 0.4$$

$$\text{Thus, } R = 2 \begin{bmatrix} 1 & 2 \\ 0.1 & 0.2 \\ 3 & 0.1 & 0.4 \end{bmatrix}$$

$$\text{i.e., } R = \{ \begin{matrix} 0.1 & ; & 0.2 & , & 0.1 & , & 0.4 \\ (2,1) & & (2,2) & & (3,1) & & (3,2) \end{matrix} \}$$

* Fuzzy graph:

Fuzzy graph is denoted as G_f and given by,

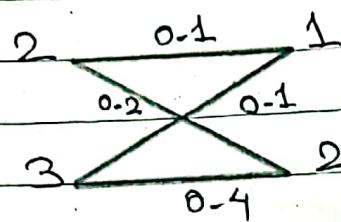
$$G_f = (\sigma, \mu)$$

where,

$$\sigma : S \rightarrow [0,1]$$

where, S is set of vertices, which are pair of elements from fuzzy sets.

Then, the fuzzy graph for R in above example is given by:



which is also known as Fuzzy Sagittal diagram.

Example:

$$A = \left\{ \frac{0.2}{x_1}, \frac{0.5}{x_2}, \frac{1}{x_3} \right\}$$

$$B = \left\{ \frac{0.3}{y_1}, \frac{0.9}{y_2} \right\}$$

$R \subseteq A \times B = ?$, Also draw fuzzy graph for R .

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for (x_1, y_1)

$$\text{ul}_R^{(x_1, y_1)} = \min \left(\text{ul}_A^{(x_1)}, \text{ul}_B^{(y_1)} \right)$$

$$= \min (0.2, 0.3)$$

$$= 0.2$$

for (x_1, y_2)

$$\text{ul}_R^{(x_1, y_2)} = \min (0.2, 0.9)$$

$$= 0.2$$

for (x_1, y_1)

$$U_R^{(x_1, y_1)} = \min(0.5, 0.3) \\ = 0.3$$

for (x_2, y_2)

$$U_R^{(x_2, y_2)} = \min(0.5, 0.9) \\ = 0.5$$

for (x_3, y_1)

$$U_R^{(x_3, y_1)} = \min(1, 0.3) \\ = 0.3$$

for (x_3, y_2)

$$U_R^{(x_3, y_2)} = \min(1, 0.9) \\ = 0.9$$

Thus,

$$R = \begin{matrix} & y_1 & y_2 \\ x_1 & 0.2 & 0.2 \\ x_2 & 0.3 & 0.5 \\ x_3 & 0.3 & 0.9 \end{matrix}$$

$$\text{i.e., } R = \{ (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2) \}$$

fuzzy graph:

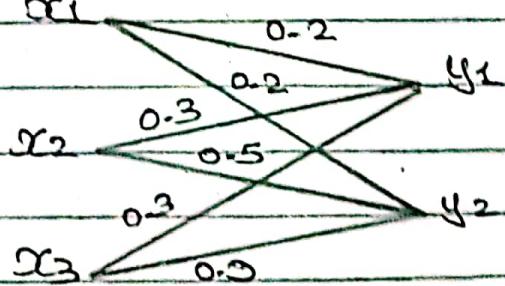


Fig: Fuzzy sagittal diagram.

* Fuzzy Composition:

Given,

$$\underset{\sim}{R} : X \rightarrow Y$$

$$\underset{\sim}{S} : Y \rightarrow Z$$

Then, fuzzy relation,

$\underset{\sim}{T}$ can be derived from $\underset{\sim}{R}$ & $\underset{\sim}{S}$ such that $\underset{\sim}{T} : X \rightarrow Z$
using fuzzy composition operation.

* Fuzzy Composition operations:

- 1) MAX-MIN
- 2) MAX-Product
- 3) MAX-MAX
- 4) MIN-MAX
- 5) MIN-MIN
- 6) MAX-average, MIN-average

Example:

$$\text{Consider } X = \{x_1, x_2\}$$

$$Y = \{y_1, y_2\}$$

$$Z = \{z_1, z_2, z_3\}$$

Given,

$$\underset{\sim}{R} = \{ \frac{0.7}{(x_1, y_1)}, \frac{0.5}{(x_1, y_2)}, \frac{0.8}{(x_2, y_1)}, \frac{0.4}{(x_2, y_2)} \}$$

$$\underset{\sim}{S} = \{ \frac{0.9}{(y_1, z_1)}, \frac{0.6}{(y_1, z_2)}, \frac{0.2}{(y_1, z_3)}, \frac{0.1}{(y_2, z_1)}, \frac{0.7}{(y_2, z_2)}, \frac{0.5}{(y_2, z_3)} \}$$

Compute $\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S}$ using max-min composition.

Solfor (x_1, z_1)

$$u_L^{(x_1, z_1)} = \max \{ \min \{ u_R^{(x_1)}, u_S^{(z_1)} \} \}$$

$$= \max \{ \min \{ u_R^{(x_1, y_1)}, u_S^{(y_1, z_1)} \} \}$$

$$\min \{ u_R^{(x_1, y_2)}, u_S^{(y_2, z_1)} \}$$

$$= \max \{ \min \{ 0.7, 0.9 \}, \min \{ 0.5, 0.1 \} \}$$

$$= \max \{ 0.7, 0.1 \}$$

$$= 0.7$$

for (x_1, z_2)

$$= \max \{ \min \{ u_R^{(x_1, y_1)}, u_S^{(y_1, z_2)} \}, \min \{ u_R^{(x_1, y_2)}, u_S^{(y_2, z_2)} \} \}$$

$$= \max \{ \min \{ 0.7, 0.6 \}, \min \{ 0.5, 0.4 \} \}$$

$$= \max \{ 0.6, 0.5 \}$$

$$= 0.6$$

for (x_1, z_3)

$$= \max \{ \min \{ u_R^{(x_1, y_1)}, u_S^{(y_1, z_3)} \}, \min \{ u_R^{(x_1, y_2)}, u_S^{(y_2, z_3)} \} \}$$

$$= \max \{ \min \{ 0.7, 0.2 \}, \min \{ 0.5, 0.5 \} \}$$

$$= \max \{ 0.2, 0.5 \}$$

$$= 0.5$$

for (x_2, z_1)

$$\begin{aligned}
 u_{LT}^{(x_2, z_1)} &= \max \{ \min_{u_R}^{(x_2, y_1)}, \min_{u_S}^{(y_1, z_1)}, \min_{u_R}^{(x_2, y_2)}, \min_{u_S}^{(y_2, z_1)} \} \\
 &= \max \{ \min(0.8, 0.9), \min(0.4, 0.1) \} \\
 &= \max(0.8, 0.1) \\
 &= 0.8
 \end{aligned}$$

for (x_2, z_2)

$$\begin{aligned}
 u_{LT}^{(x_2, z_2)} &= \max \{ \min_{u_R}^{(x_2, y_1)}, \min_{u_S}^{(y_1, z_2)}, \min_{u_R}^{(x_2, y_2)}, \min_{u_S}^{(y_2, z_2)} \} \\
 &= \max \{ \min(0.8, 0.6), \min(0.4, 0.7) \} \\
 &= \max(0.6, 0.4) \\
 &= 0.6
 \end{aligned}$$

for (x_2, z_3)

$$\begin{aligned}
 u_{LT}^{(x_2, z_3)} &= \max \{ \min_{u_R}^{(x_2, y_1)}, \min_{u_S}^{(y_1, z_3)}, \min_{u_R}^{(x_2, y_2)}, \min_{u_S}^{(y_2, z_3)} \} \\
 &= \max \{ \min(0.8, 0.2), \min(0.4, 0.5) \} \\
 &= \max(0.2, 0.4) \\
 &= 0.4
 \end{aligned}$$

$\therefore T = R \cdot S$ under max-min composition is

	z_1	z_2	z_3
x_1	0.7	0.6	0.5
x_2	0.8	0.6	0.4

* For max-product:

$$u_{I_2}^{(x_1 z)} = \max \{ u_R^{(x_1 y)} \cdot u_S^{(y, z)} \}$$

for (x_1, z_1)

$$\begin{aligned} u_{I_2}^{(x_1, z_1)} &= \max \{ u_R^{(x_1, y_1)} \cdot u_S^{(y_1, z_1)}, u_R^{(x_1, y_2)} \cdot u_S^{(y_2, z_1)} \} \\ &= \max \{ (0.7 \times 0.9), (0.5 \times 0.1) \} \\ &= \max (0.63, 0.05) \\ &= 0.63 \end{aligned}$$

$$\begin{aligned} \text{for } (x_1, z_2) &= \max \{ (0.7 \times 0.6), (0.5 \times 0.7) \} \\ &= \max (0.42, 0.35) \\ &= 0.42 \end{aligned}$$

$$\begin{aligned} \text{for } (x_1, z_3) &= \max \{ (0.7 \times 0.2), (0.5, 0.5) \} \\ &= \max (0.14, 0.25) \\ &= 0.25 \end{aligned}$$

$$\begin{aligned} \text{for } (x_2, z_1) &= \max \{ (0.8 \times 0.9), (0.4 \times 0.1) \} \\ &= \max (0.72, 0.04) \\ &= 0.72 \end{aligned}$$

$$\begin{aligned} \text{for } (x_2, z_2) &= \max \{ (0.8 \times 0.6), (0.4 \times 0.7) \} \\ &= \max (0.48, 0.28) \\ &= 0.48 \end{aligned}$$

$$\begin{aligned} \text{for } (x_2, z_3) &= \max \{ (0.8 \times 0.2), (0.4 \times 0.5) \} \\ &= \max (0.16, 0.2) \\ &= 0.2 \end{aligned}$$

These,

$T = R \cdot S$ under max-product PS.

	z_1	z_2	z_3	
x_1	0.63	0.42	0.25	
x_2	0.72	0.48	0.2	

* Properties of Max-Min Composition:

① Associativity:

$$P \circ (Q \circ R) = (P \circ Q) \circ R$$

② Distributive over union:

$$P \circ (Q \cup R) = (P \circ Q) \cup (P \circ R)$$

③ Monotonic:

$$Q \subseteq R \Rightarrow P \circ Q \subseteq P \circ R$$

④ Commutativity:

$$P \circ Q \neq Q \circ P$$

* Example:

$$A = \{ \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} \}$$

$$B = \{ \frac{0.5}{y_1} + \frac{1}{y_2} \}$$

$$C = \{ \frac{0.2}{z_1} + \frac{0.3}{z_2} + \frac{0.9}{z_3} \}$$

Now, formulate,

$$R \subseteq A \times B$$

$$S \subseteq B \times C$$

Determine $\underset{\sim}{R} \circ \underset{\sim}{S} = \underset{\sim}{S} \circ \underset{\sim}{R}$ using max-min composition.

① We have to formulate:

$$\underset{\sim}{R} \subseteq \underset{\sim}{A} \times \underset{\sim}{B}$$

for (x_1, y_1)

$$= \min \left(\underset{\sim}{\mu}_A^{(x_1)}, \underset{\sim}{\mu}_B^{(y_1)} \right)$$

$$= \min (0.4, 0.5)$$

$$= 0.4$$

For (x_1, y_2)

$$= \min \left(\underset{\sim}{\mu}_A^{(x_1)}, \underset{\sim}{\mu}_B^{(y_2)} \right)$$

$$= \min (0.4, 1)$$

$$= 0.4$$

for (x_2, y_1)

$$= \min \left(\underset{\sim}{\mu}_A^{(x_2)}, \underset{\sim}{\mu}_B^{(y_1)} \right)$$

$$= \min (0.2, 0.5)$$

$$= 0.2$$

for (x_2, y_2)

$$= \min \left(\underset{\sim}{\mu}_A^{(x_2)}, \underset{\sim}{\mu}_B^{(y_2)} \right)$$

$$= \min (0.2, 1)$$

$$= 0.2$$

$$\text{for } (x_3, y_1) \\ = \min \left(\underline{u}_A^{(x_3)}, \underline{u}_B^{(y_1)} \right)$$

$$= \min(0.3, 0.5) \\ = 0.3$$

$$\text{for } (x_3, y_2) \\ = \min \left(\underline{u}_A^{(x_3)}, \underline{u}_B^{(y_2)} \right)$$

$$= \min(0.3, 1) \\ = 0.3$$

\underline{u}_2	y_1	y_2
x_1	0.4	0.4
x_2	0.2	0.2
x_3	0.3	0.3

$$\textcircled{1} \quad S \subseteq \underbrace{B}_{n} \times \underbrace{C}_{n}$$

for (y_1, z_1) :

$$\underline{u}_S^{(y_1, z_1)} = \min \left(\underline{u}_B^{(y_1)}, \underline{u}_C^{(z_1)} \right)$$

$$= \min(0.5, 0.2) \\ = 0.2$$

$$\underline{u}_S^{(y_1, z_2)} = \min \left(\underline{u}_B^{(y_1)}, \underline{u}_C^{(z_2)} \right) \\ = \min(0.5, 0.3) \\ = 0.3$$

$$u_{ls}^{(y_1, z_3)} = \min \left(u_{lp}^{(y_1)}, u_{lc}^{(z_3)} \right)$$

$$= \min (0.5, 0.9)$$

$$= 0.5$$

$$u_{ls}^{(y_2, z_1)} = \min \left(u_{lp}^{(y_2)}, u_{lc}^{(z_1)} \right)$$

$$= \min (1, 0.2)$$

$$= 0.2$$

$$u_{ls}^{(y_2, z_2)} = \min \left(u_{lp}^{(y_2)}, u_{lc}^{(z_2)} \right)$$

$$= \min (1, 0.3)$$

$$= 0.3$$

$$u_{ls}^{(y_2, z_3)} = \min \left(u_{lp}^{(y_2)}, u_{lc}^{(z_3)} \right)$$

$$= \min (1, 0.9)$$

$$= 0.9$$

$$\therefore S = B \times C$$

	z_1	z_2	z_3
y_1	0.2	0.3	0.5
y_2	0.2	0.3	0.9

$$\textcircled{W} T = R \circ S$$

$$R = \sum_{\textcircled{W}} \begin{cases} 0.4 & (x_1, y_1) \\ 0.4 & (x_1, y_2) \\ 0.2 & (x_2, y_1) \\ 0.2 & (x_2, y_2) \\ 0.3 & (x_3, y_1) \\ 0.3 & (x_3, y_2) \end{cases}$$

$$S = \sum_{\textcircled{W}} \begin{cases} 0.2 & (y_1, z_1) \\ 0.3 & (y_1, z_2) \\ 0.5 & (y_2, z_1) \\ 0.2 & (y_2, z_2) \\ 0.3 & (y_3, z_1) \\ 0.9 & (y_3, z_2) \end{cases}$$

Then, Using Max-Min Composition,

$$\text{for } (x_1, z_1) = \max \sum_{\textcircled{W}} \min \begin{cases} \text{U}_{R_{(x_1, y_1)}} & (x_1, y_1) \\ \text{U}_{S_{(y_1, z_1)}} & (y_1, z_1) \end{cases}$$

$$= \max \sum_{\textcircled{W}} \min (0.4, 0.2), \min (0.4, 0.2)$$

$$= \max (0.2, 0.2)$$

$$= 0.2$$

for $(x_1, z_2) =$

$$\text{U}_{T_{(x_1, z_2)}} = \max \sum_{\textcircled{W}} \min \begin{cases} \text{U}_{R_{(x_1, y_1)}} & (x_1, y_1) \\ \text{U}_{S_{(y_1, z_2)}} & (y_1, z_2) \end{cases}$$

$$, \text{U}_{S_{(y_2, z_2)}} \quad \text{if}$$

$$= \max \sum_{\textcircled{W}} \min (0.4, 0.3), \min (0.4, 0.3)$$

$$= \max (0.3, 0.3)$$

$$= 0.3$$

for (x_1, z_3)

$$\begin{aligned}
 u_{l_T}^{(x_1, z_3)} &= \max \left\{ \min \left\{ u_{l_R}^{(x_1, y_1)}, u_{l_S}^{(y_1, z_3)} \right\}, \min \left\{ u_{l_R}^{(x_1, y_2)}, u_{l_S}^{(y_2, z_3)} \right\} \right\} \\
 &= \max \left\{ \min (0.4, 0.5), \min (0.4, 0.9) \right\} \\
 &= \max (0.4, 0.4) \\
 &= 0.4
 \end{aligned}$$

now,

for (x_2, z_1)

$$\begin{aligned}
 u_{l_T}^{(x_2, z_1)} &= \max \left\{ \min \left\{ u_{l_R}^{(x_2, y_1)}, u_{l_S}^{(y_1, z_1)} \right\}, \min \left\{ u_{l_R}^{(x_2, y_2)}, u_{l_S}^{(y_2, z_1)} \right\} \right\} \\
 &= \max \left\{ \min (0.2, 0.2), \min (0.2, 0.3) \right\} \\
 &= \max (0.2, 0.2) \\
 &= 0.2
 \end{aligned}$$

for (x_2, z_2)

$$\begin{aligned}
 u_{l_T}^{(x_2, z_2)} &= \max \left\{ \min \left\{ u_{l_R}^{(x_2, y_1)}, u_{l_S}^{(y_1, z_2)} \right\}, \min \left\{ u_{l_R}^{(x_2, y_2)}, u_{l_S}^{(y_2, z_2)} \right\} \right\} \\
 &= \max \left\{ \min (0.2, 0.3), \min (0.2, 0.3) \right\} \\
 &= \max (0.2, 0.2) \\
 &= 0.2
 \end{aligned}$$

for (x_2, z_3)

$$\begin{aligned}
 u_{l_T}^{(x_2, z_3)} &= \max \left\{ \min (0.2, 0.5), \min (0.2, 0.9) \right\} \\
 &= \max (0.2, 0.2) \\
 &= 0.2
 \end{aligned}$$

for (x_3, z_1)

$$\begin{aligned} u_{LT}^{(x_3, z_1)} &= \max \left\{ \min \left\{ u_{LR}^{(x_3, y_1)}, u_{LS}^{(y_1, z_1)} \right\}, \min \left\{ u_{LR}^{(x_3, y_2)}, u_{LS}^{(y_2, z_1)} \right\} \right\} \\ &= \max \left\{ \min (0.3, 0.2), \min (0.3, 0.2) \right\} \\ &= \max \{ 0.2, 0.2 \} \\ &= 0.2 \end{aligned}$$

for (x_3, z_2)

$$\begin{aligned} u_{LT}^{(x_3, z_2)} &= \max \left\{ \min \left\{ u_{LR}^{(x_3, y_1)}, u_{LS}^{(y_1, z_2)} \right\}, \min \left\{ u_{LR}^{(x_3, y_2)}, u_{LS}^{(y_2, z_2)} \right\} \right\} \\ &= \max \left\{ \min (0.3, 0.3), \min (0.3, 0.3) \right\} \\ &= \max (0.3, 0.3) \\ &= 0.3 \end{aligned}$$

for (x_3, z_3)

$$\begin{aligned} u_{LT}^{(x_3, z_3)} &= \max \left\{ \min \left\{ u_{LR}^{(x_3, y_1)}, u_{LS}^{(y_1, z_3)} \right\}, \min \left\{ u_{LR}^{(x_3, y_2)}, u_{LS}^{(y_2, z_3)} \right\} \right\} \\ &= \max \left\{ \min (0.3, 0.5), \min (0.3, 0.3) \right\} \\ &= \max (0.3, 0.3) \\ &= 0.3 \end{aligned}$$

Therefore,

$T = R \cdot S =$	x_1	x_2	x_3	
x_1	0.2	0.3	0.4	
x_2	0.2	0.2	0.2	
x_3	0.2	0.3	0.3	1

$$\textcircled{N} \quad T = S \circ R$$

$$S = S \left[\begin{array}{c} 0.2 \\ (y_1, z_1) \end{array}, \begin{array}{c} 0.3 \\ (y_1, z_2) \end{array}, \begin{array}{c} 0.5 \\ (y_1, z_3) \end{array}, \begin{array}{c} 0.2 \\ (y_2, z_1) \end{array}, \begin{array}{c} 0.3 \\ (y_2, z_2) \end{array}, \begin{array}{c} 0.9 \\ (y_2, z_3) \end{array} \right] \quad \text{f}$$

$$R = R \left[\begin{array}{c} 0.4 \\ (x_1, y_1) \end{array}, \begin{array}{c} 0.4 \\ (x_1, y_2) \end{array}, \begin{array}{c} 0.2 \\ (x_2, y_1) \end{array}, \begin{array}{c} 0.2 \\ (x_2, y_2) \end{array}, \begin{array}{c} 0.3 \\ (x_3, y_1) \end{array}, \begin{array}{c} 0.3 \\ (x_3, y_2) \end{array} \right] \quad \text{f}$$

So,

for (z_1, x_1)

$$\begin{aligned} \text{Ull}_T^{(z_1, x_1)} &= \max \left\{ \min \left\{ \text{Ull}_S^{(z_1, y_1)}, \text{Ull}_S^{(y_1, x_1)} \right\}, \min \left\{ \text{Ull}_R^{(z_1, y_1)}, \text{Ull}_R^{(y_2, x_1)} \right\} \right\} \quad \text{f} \\ &= \max \{ \min(0.2, 0.4), \min(0.2, 0.4) \} \\ &= \max(0.2, 0.2) \\ &= 0.2 \end{aligned}$$

for (z_1, x_2)

$$\begin{aligned} \text{Ull}_T^{(z_1, x_2)} &= \max \left\{ \min \left\{ \text{Ull}_R^{(z_1, y_1)}, \text{Ull}_S^{(y_1, x_2)} \right\}, \min \left\{ \text{Ull}_R^{(z_1, y_2)}, \text{Ull}_S^{(y_2, x_2)} \right\} \right\} \quad \text{f} \\ &= \max \{ \min(0.2, 0.2), \min(0.2, 0.2) \} \\ &= \max(0.2, 0.2) \\ &= 0.2 \end{aligned}$$

for (z_1, x_3)

$$\begin{aligned} \text{Ull}_T^{(z_1, x_3)} &= \max \left\{ \min \left\{ \text{Ull}_R^{(z_1, y_1)}, \text{Ull}_S^{(y_1, x_3)} \right\}, \min \left\{ \text{Ull}_R^{(z_1, y_2)}, \text{Ull}_S^{(y_2, x_3)} \right\} \right\} \quad \text{f} \\ &= \max \{ \min(0.2, 0.3), \min(0.2, 0.3) \} \\ &= \max(0.2, 0.2) \\ &= 0.2 \end{aligned}$$

for (z_2, x_1)

$$\begin{aligned}
 U_{LR}^{(z_2, x_1)} &= \max \left\{ \min \left\{ U_R^{(z_2, y_1)}, U_S^{(y_1, x_1)} \right\}, \min \left\{ U_R^{(z_2, y_2)}, U_S^{(y_2, x_1)} \right\} \right\} \\
 &= \max \{ \min(0.3, 0.4), \min(0.3, 0.2) \} \\
 &= \max \{ 0.3, 0.2 \} \\
 &= 0.3
 \end{aligned}$$

for (z_2, x_2)

$$\begin{aligned}
 U_{LR}^{(z_2, x_2)} &= \max \left\{ \min \left\{ U_R^{(z_2, y_1)}, U_S^{(y_1, x_2)} \right\}, \min \left\{ U_R^{(z_2, y_2)}, U_S^{(y_2, x_2)} \right\} \right\} \\
 &= \max \{ \min(0.3, 0.2), \min(0.3, 0.2) \} \\
 &= \max(0.2, 0.2) \\
 &= 0.2
 \end{aligned}$$

for (z_2, x_3)

$$\begin{aligned}
 U_{LR}^{(z_2, x_3)} &= \max \left\{ \min \left\{ U_R^{(z_2, y_1)}, U_S^{(y_1, x_3)} \right\}, \min \left\{ U_R^{(z_2, y_2)}, U_S^{(y_2, x_3)} \right\} \right\} \\
 &= \max \{ \min(0.3, 0.3), \min(0.3, 0.3) \} \\
 &= \max \{ 0.3, 0.3 \} \\
 &= 0.3
 \end{aligned}$$

for (z_3, x_1)

$$\begin{aligned}
 U_{LR}^{(z_3, x_1)} &= \max \left\{ \min \left\{ U_R^{(z_3, y_1)}, U_S^{(y_1, x_1)} \right\}, \min \left\{ U_R^{(z_3, y_2)}, U_S^{(y_2, x_1)} \right\} \right\} \\
 &= \max \{ \min(0.5, 0.4), \min(0.3, 0.4) \}
 \end{aligned}$$

$$= \max(0.4, 0.4)$$

$$= 0.4$$

for (z_3, x_2)

$$U_{LT}^{(z_3, x_2)} = \max \left\{ \min_{i=1}^{(z_3, y_1)} U_{R_i}^{(y_1, x_2)}, \min_{i=2}^{(z_3, y_2)} U_{R_i}^{(y_2, x_2)} \right\}$$

$$= \max \{ \min(0.5, 0.2), \min(0.9, 0.2) \}$$

$$= \max \{ 0.2, 0.2 \}$$

$$= 0.2$$

for (z_3, x_3)

$$U_{LT}^{(z_3, x_3)} = \max \left\{ \min_{i=1}^{(z_3, y_1)} U_{R_i}^{(y_1, x_3)}, \min_{i=2}^{(z_3, y_2)} U_{R_i}^{(y_2, x_3)} \right\}$$

$$= \max \{ \min(0.5, 0.3), \min(0.9, 0.3) \}$$

$$= \max \{ 0.3, 0.3 \}$$

$$= 0.3$$

	x_1	x_2	x_3
z_1	0.2	0.2	0.2
z_2	0.3	0.2	0.3
z_3	0.4	0.2	0.3

Thus, $R \circ S \neq S \circ R$

$$\begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.3 \end{bmatrix} \neq \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

* Consider a model for Predicting score in cricket.

Suppose the Fuzzy Sets are,

$$\text{Speed of bowling} = \{ \frac{0.6}{\text{Fast}}, \frac{0.8}{\text{medium}}, \frac{0.9}{\text{Slow}} \}$$

$$\text{Condition on pitch} = \{ \frac{0.9}{\text{good wicket}}, \frac{0.5}{\text{fair wicket}}, \frac{0.2}{\text{rough wicket}} \}$$

$$\text{Condition of runs} = \{ \frac{0.9}{\text{low run}}, \frac{1}{\text{average run}}, \frac{0.7}{\text{high run}} \}$$

now,

- i) determine impact of Pitch Condition on bowling (I_1)
- ii) determine impact of bowling on runs (I_2)
- iii) determine impact of pitch conditions on runs using I_1 & I_2

= Sol

Consider

$$X = \{x_1, x_2, x_3\} = \{\text{Fast, medium, Slow}\}$$

$$Y = \{y_1, y_2, y_3\} = \{\text{good wicket, fair wicket, rough wicket}\}$$

$$Z = \{z_1, z_2, z_3\} = \{\text{low run, average run, high run}\}$$

So,

$$B = \{ \frac{0.6}{x_1}, \frac{0.8}{x_2}, \frac{0.9}{x_3} \}$$

$$P = \{ \frac{0.9}{y_1}, \frac{0.5}{y_2}, \frac{0.2}{y_3} \}$$

$$R = \{ \frac{0.9}{z_1}, \frac{1}{z_2}, \frac{0.7}{z_3} \}$$

① We need determine impact of pitch condition on housing (I_1):

$$I_1 = \frac{R}{N} \leq \frac{P \times B}{N}$$

so, For (y_1, x_1)

$$UL_R^{(y_1, x_1)} = \min (UL_P^{(y_1)}, UL_B^{(x_1)})$$

$$= \min (0.9, 0.6)$$

$$= 0.6$$

for (y_1, x_2)

$$UL_R^{(y_1, x_2)} = \min (0.9, 0.8) \\ = 0.8$$

for (y_1, x_3)

$$UL_R^{(y_1, x_3)} = \min (0.9, 0.9) \\ = 0.9$$

$$\text{for } (y_2, x_1) = \min (0.5, 0.6) \\ = 0.5$$

$$\text{for } (y_2, x_2) = \min (0.5, 0.8) \\ = 0.5$$

$$\text{for } (y_2, x_3) = \min (0.5, 0.9) \\ = 0.5$$

$$\text{for } (y_3, x_1) = \min (0.2, 0.6) \\ = 0.2$$

$$\text{for } (y_3, x_2) = \min (0.2, 0.8) \\ = 0.2$$

$$\text{for } (y_3, x_3) = \min (0.2, 0.9) \\ = 0.2$$

	x_1	x_2	x_3
y_1	0.6	0.8	0.9
y_2	0.5	0.5	0.5
y_3	0.2	0.2	0.2

(1) We need find the impact of booking on rents (I_2):

$$\underline{s} \subseteq \underbrace{B}_{\sim} \times \underbrace{R}_{\sim}$$

for (x_1, z_1)

$$\begin{aligned}\underline{u}_S^{(x_1, z_1)}_{\sim} &= \min \left(\underline{u}_B^{(x_1)}_{\sim}, \underline{u}_R^{(z_1)}_{\sim} \right) \\ &= \min (0.6, 0.9) \\ &= 0.6\end{aligned}$$

$$\begin{aligned}\text{for } (x_1, z_2) &= \min (0.6, 1) \\ &= 0.6\end{aligned}$$

$$\begin{aligned}\text{for } (x_1, z_3) &= \min (0.6, 0.7) \\ &= 0.6\end{aligned}$$

$$\begin{aligned}\text{for } (x_2, z_1) &= \min (0.8, 0.9) \\ &= 0.8\end{aligned}$$

$$\begin{aligned}\text{for } (x_2, z_2) &= \min (0.8, 1) \\ &= 0.8\end{aligned}$$

$$\begin{aligned}\text{for } (x_2, z_3) &= \min (0.8, 0.7) \\ &= 0.7\end{aligned}$$

$$\begin{aligned}\text{for } (x_3, z_1) &= \min (0.9, 0.9) \\ &= 0.9\end{aligned}$$

$$\begin{aligned}\text{for } (x_3, z_2) &= \min (0.9, 1) \\ &= 0.9\end{aligned}$$

$$\begin{aligned}\text{for } (x_3, z_3) &= \min (0.9, 0.7) \\ &= 0.7\end{aligned}$$

	z_1	z_2	z_3
x_1	0.6	0.6	0.6
x_2	0.8	0.8	0.7
x_3	0.9	0.9	0.7

③ To find the impact of pitch condition (P) on runs (R) using I_1 & I_2

$$\bar{I} = \frac{R \cdot S}{n} = I_1 \cdot I_2$$

So,

$$R = \frac{0.6}{(y_1, x_1)}, \frac{0.8}{(y_1, x_2)}, \frac{0.9}{(y_1, x_3)}, \frac{0.5}{(y_2, x_1)}, \frac{0.5}{(y_2, x_2)}, \frac{0.5}{(y_2, x_3)}, \frac{0.2}{(y_3, x_1)}, \frac{0.2}{(y_3, x_2)}, \\ \frac{0.2}{(y_3, x_3)}$$

~~$S = \frac{0.6}{(x_1, z_1)}, \frac{0.6}{(x_1, z_2)}, \frac{0.6}{(x_1, z_3)}, \frac{0.8}{(x_2, z_1)}, \frac{0.8}{(x_2, z_2)}, \frac{0.7}{(x_2, z_3)}, \frac{0.9}{(x_3, z_1)}, \frac{0.7}{(x_3, z_2)}$~~

for (y_1, z_1)

$$U_{\bar{I}}^{(y_1, z_1)} = \max \left\{ \min \left\{ U_R^{(y_1, x_1)}, U_S^{(y_1, x_1)} \right\}, \min \left\{ U_R^{(y_1, x_2)}, U_S^{(y_1, x_2)} \right\}, \right. \\ \left. \min \left\{ U_R^{(y_1, x_3)}, U_S^{(x_3, z_1)} \right\} \right\}$$

$$= \max (\min (0.6, 0.6), \min (0.8, 0.8), \min (0.9, 0.9))$$

$$= \max (0.6, 0.8, 0.9)$$

$$= 0.9$$

for (y_1, z_2)

$$U_{\bar{I}}^{(y_1, z_2)} = \max \left\{ \min (0.6, 0.6), \min (0.8, 0.8), \min (0.9, 0.9) \right\}$$

$$= \max (0.6, 0.8, 0.9)$$

$$= 0.9$$

for (y_1, z_3)

$$\text{U}_{\frac{1}{2}}^{(y_1, z_3)} = \max \{ \min(0.6, 0.6), \min(0.8, 0.7), \min(0.9, 0.7) \}$$

$$= \max(0.6, 0.7, 0.7)$$

$$= 0.7$$

now, for (y_2, z_1)

$$\text{U}_{\frac{1}{2}}^{(y_2, z_1)} = \max \{ \min(0.5, 0.6), \min(0.5, 0.8), \min(0.5, 0.9) \}$$

$$= \max(0.5, 0.5, 0.5)$$

$$= 0.5$$

for (y_2, z_2)

$$\text{U}_{\frac{1}{2}}^{(y_2, z_2)} = \max \{ \min(0.5, 0.6), \min(0.5, 0.8), \min(0.5, 0.9) \}$$

$$= \max(0.5, 0.5, 0.5)$$

$$= 0.5$$

for (y_2, z_3)

$$\text{U}_{\frac{1}{2}}^{(y_2, z_3)} = \max \{ \min(0.5, 0.6), \min(0.5, 0.7), \min(0.5, 0.7) \}$$

$$= \max(0.5, 0.5, 0.5)$$

$$= 0.5$$

for (y_3, z_1)

$$\text{Ull}_T^{(y_3, z_1)} = \max \left\{ \min \{0.2, 0.6\}, \min \{0.2, 0.8\}, \min \{0.2, 0.9\} \right\}$$

$$= \max (0.2, 0.2, 0.2)$$

$$= 0.2$$

for (y_3, z_2)

$$\text{Ull}_T^{(y_3, z_2)} = \max \left\{ \min \{0.2, 0.6\}, \min \{0.2, 0.8\}, \min \{0.2, 0.9\} \right\}$$

$$= \max (0.2, 0.2, 0.2)$$

$$= 0.2$$

for (y_3, z_3)

$$\text{Ull}_T^{(y_3, z_3)} = \max \left\{ \min \{0.2, 0.6\}, \min \{0.2, 0.7\}, \min \{0.2, 0.7\} \right\}$$

$$= \max (0.2, 0.2, 0.2)$$

$$= 0.2$$

Therefore, $I_1 \circ I_2 = R \circ S$

	z_1	z_2	z_3	
y_1	0.9	0.9	0.7	
y_2	0.5	0.5	0.5	
y_3	0.2	0.2	0.2	*

* One to one Mapping:

Let $f(\cdot)$ be a mapping from the fuzzy universal set X to another fuzzy universal set Y . Suppose $A \sim f B$ are subsets of $X \sim Y$ respectively.

Consider,

$$A = \{ \underline{\text{ul}_A^{(x_1)}}, \underline{\text{ul}_A^{(x_2)}}, \dots, \underline{\text{ul}_A^{(x_n)}} \}$$

If there is a one-to-one mapping from x_i to $y_i = f(x_i)$ (\because Fuzzy arithmetic function)

then,

B is given by;

$$B = \{ \underline{\text{ul}_B^{(y_1)}}, \underline{\text{ul}_B^{(y_2)}}, \dots, \underline{\text{ul}_B^{(y_n)}} \}$$

where membership of y_i in B = $\text{ul}_B^{(y_i)}$

and,

$$y_1 = f(x_1) \text{ & } \text{ul}_B^{(y_1)} = \text{ul}_A^{(x_1)}$$

$$y_2 = f(x_2) \text{ & } \text{ul}_B^{(y_2)} = \text{ul}_A^{(x_2)}$$

$$y_n = f(x_n) \text{ & } \text{ul}_B^{(y_n)} = \text{ul}_A^{(x_n)}$$

$$y = f(x)$$

$$x_1 \rightarrow y_1$$

$$x_2 \rightarrow y_2$$

$$x_3 \rightarrow y_3$$

$$\vdots \quad \vdots \\ x_n \rightarrow y_n$$

Example:

$$\tilde{A} = \{ \frac{0.2}{2}, \frac{0.1}{4}, \frac{1}{6} \}$$

now, \tilde{B} is defined by $\forall y \in \tilde{B}, y = F(x) = x^2$
 $\forall x \in \tilde{A}$

then,

$$\tilde{B} = \{ \frac{0.2}{4}, \frac{0.1}{16}, \frac{1}{36} \}$$

$$\begin{array}{ccc} 2 & \longrightarrow & 4 \\ 4 & \longrightarrow & 16 \\ 6 & \longrightarrow & 36 \end{array}$$

* Many to one Mapping:

Let $F(\cdot)$ be a mapping from the fuzzy universal

Set X to another fuzzy universal set Y . Suppose \tilde{A} & \tilde{B} are subsets of X and Y respectively.

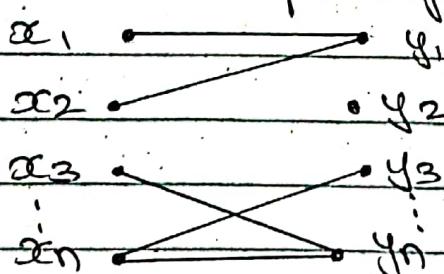
Consider,

$$\tilde{A} = \{ \frac{\text{ul } \tilde{A}_1(x_1)}{x_1}, \frac{\text{ul } \tilde{A}_1(x_2)}{x_2}, \dots, \frac{\text{ul } \tilde{A}_1(x_n)}{x_n} \}$$

If there is a many to one mapping from x_i to y_i
 $= F(x_i)$ then \tilde{B} is given by;

$$\tilde{B} = \{ \frac{\text{ul } \tilde{B}_1(y_1)}{y_1}, \frac{\text{ul } \tilde{B}_1(y_2)}{y_2}, \dots, \frac{\text{ul } \tilde{B}_1(y_n)}{y_n} \}$$

where membership of y_i in $\tilde{B} = \text{MAX} \left[\frac{\text{ul } \tilde{A}_j(x_i)}{x_i} \mid x_i \in F^{-1}(y_i) \right]$



Example:

$$A = \{ \frac{0.2}{-1}, \frac{0.4}{-2}, \frac{0.6}{-1}, \frac{0.8}{2}, \frac{0.9}{3} \}$$

now, B is defined by $\forall y \in B \quad y = F(x) = x^2$

So,

$$B = \{ \max(\underline{\mu}_A^{(-1)}, \underline{\mu}_A^{(1)}), \max(\underline{\mu}_A^{(-2)}, \underline{\mu}_A^{(2)}) \}$$

$$\max(\underline{\mu}_A^{(0)}) \}$$

$$= \{ \max(0.2, 0.6), \max(0.4, 0.8), \max(0.9) \}$$

$$\therefore B = \{ \frac{0.6}{1}, \frac{0.8}{2}, \frac{0.9}{3} \}$$

* **Projection of a Fuzzy relation:**

① **X-Projection:**

The projection of $R(x,y)$ on X is defined as R_1 and given by:

$$\underline{\mu}_{R_1}^{(x)} = \max_{y \in Y} (\underline{\mu}_R^{(x,y)})$$

② **Y-Projection:**

The projection of $R(x,y)$ on Y is defined as R_2 and given by:

$$\text{Ull}_{R_2}^{(y)} = \max_{x \in X} (\text{Ull}_R^{(x,y)})$$

Example:

	4	5
1	1	0.43
2	0.43	1
3	0.16	0.42

$$\text{i.e. } R = \{ \underset{(1,4)}{1}, \underset{(1,5)}{0.43}, \underset{(2,4)}{0.43}, \underset{(2,5)}{1}, \underset{(3,4)}{0.16}, \underset{(3,5)}{0.42} \}$$

now,

X-Projection of R is R_1 where Ull_{R_1} is defined as

$$\text{Ull}_{R_1}^{(1)} = \max_{x \in X} (\text{Ull}_R^{(1,x)}), \text{Ull}_{R_1}^{(2)} = \max_{x \in X} (\text{Ull}_R^{(2,x)}) = \max(1, 0.43) = 1$$

$$\text{Ull}_{R_1}^{(2)} = \max_{x \in X} (\text{Ull}_R^{(2,x)}, \text{Ull}_R^{(3,x)}) = \max(0.43, 1) = 1$$

$$\text{Ull}_{R_1}^{(3)} = \max_{x \in X} (\text{Ull}_R^{(3,x)}, \text{Ull}_R^{(4,x)}) = \max(0.16, 0.42) = 0.42$$

Thus, X-Projection of R is

$R_1 = 1$	1
2	1
3	0.42

$$\text{i.e. } R_1 = \{ \underset{1}{1}, \underset{2}{1}, \underset{3}{0.42} \}$$

$$\text{or, } R_1 = \{ (1,1), (2,1), (3,0.42) \}$$

Now,

Y-Projection of R is R_2 where U_{R_2} is defined as:

$$U_{R_2}^{(4)} = \text{MAX}(U_{R_2}^{(1,4)}, U_{R_2}^{(2,4)}, U_{R_2}^{(3,4)}) \\ = \text{MAX}(1, 0.48, 0.16) \\ = 1$$

$$U_{R_2}^{(5)} = \text{MAX}(U_{R_2}^{(1,5)}, U_{R_2}^{(2,5)}, U_{R_2}^{(3,5)}) \\ = \text{MAX}(0.43, 1, 0.42) \\ = 1$$

Thus, Y-Projection of R is,

$$R_2 = 4 \ 5$$

$$\boxed{1 \ 1}$$

$$\text{i.e., } R_2 = \{ \frac{1}{4}, \frac{1}{5} \}$$

$$\text{or, } R_2 = \{ (4,1), (5,1) \}$$

* Cylindrical Extension of Fuzzy relation:

① Cylindrical Extension w.r.t X-Projection:

$$\text{cyl}_A(x,y) = \{ (x,y) \mid (x,y) \in R \}$$

and,

$$U_{\text{cyl}_A}^{(x,y)} = U_A^{(x)}$$

X-Projection..

(ii) cylindrical extension w.r.t Y-Projection:

$$\text{Cyl}_A(x,y) = S(x,y) \mid (x,y) \in R$$

and

$$\text{llcyl}_A(x,y) = \text{ll}_y(y)$$

y-Projection

Thus, $\text{Cyl}_A^{(R)}$ from X-Projection:

	4	5	
1	1	1	
2	1	1	
3	0.42	0.42	

Also, $\text{Cyl}_A^{(R)}$ from Y-Projection:

	4	5	
1	1	1	
2	1	1	
3	1	1	

* Reflection Relation:

A relation R over $X \times X$ is said to be reflexive if $\text{ll}(x_i, x_i) = 1$.

For, $x = \{1, 2, 3\}$

	1	2	3
1	1	0.9	0.6
2	0.9	1	0.5
3	0.6	0.6	1

i.e., $R(x_i, x_i) = 1 \forall x_i \in X$

* Anti-reflexive:

A relation R over $X \times X$ is said to be Anti-reflexive if $ll(x_i, x_i) = 0$

e.g:- for $x = \{1, 2, 3\}$

	1	2	3
1	0	0.9	0.6
2	0.9	0	0.5
3	0.6	0.6	0

i.e., $R(x_i, x_i) = 0 \forall x_i \in X$

* Symmetric:

A relation R over $X \times X$ is said to be Symmetric if $ll(x_i, x_j) = ll(x_j, x_i)$

e.g:-

	1	2	3
1	0.8	0.1*	0.7*
2	0.1*	1	0.6°
3	0.7*	0.6°	0.5

* Anti-Symmetric:

A relation R over $X \times X$ is said to be Anti-Symmetric if $ll_R(x_i, x_j) > 0$ then $ll_R(x_j, x_i) = 0$ for $x_i, x_j \in X$

e.g:-

for $x_i \neq x_j$

	1	2	3
1	0	0	0.7
2	0.2	0	0
3	0	0.1	0

	1	2	3
1	1	0	0
2	0.6	0.5	0.7
3	0.7	0	0.2

* Transitive Relation:

A relation R over $X \times X$ is said to be Transitive if $u_{LR}(x_i, x_j) = \lambda_1$ & $u_{LR}(x_i, x_k) = \lambda_2$ then

$$u_{LR}^{(x_i, x_k)} = \lambda \text{ such that } \lambda \geq \min(\lambda_1, \lambda_2)$$

$$\text{i.e., } u_{LR}^{(x_i, x_k)} > \min(u_{LR}^{(x_i, x_j)}, u_{LR}^{(x_j, x_k)})$$

Note: For Crisp relation :

$$\text{IF } u_{LR}^{(x_i, x_j)} = 1$$

$$\text{& } u_{LR}^{(x_j, x_k)} = 1$$

$$\text{then, } u_{LR}^{(x_i, x_k)} = 1$$

else

0

e.g:- For $X = \{x_1, x_2, x_3, x_4, x_5\}$

	x_1	x_2	x_3	x_4	x_5
x_1	1	0.8	0	0.1	0.2
x_2	0.8	1	0.4	0	0.9
x_3	0	0.4	1	0	0
x_4	0.1	0	0	1	0.5
x_5	0.2	0.9	0	0.5	1

here,

$$u_{LR}^{(x_1, x_2)} = 0.8$$

$$u_{LR}^{(x_2, x_5)} = 0.9$$

$$u_{LR}^{(x_1, x_5)} = 0.2$$

i.e., $0.2 \geq \min(0.8, 0.9)$, Thus R_1 is not Transitive.

	x_1	x_2	x_3	x_4	x_5
x_1	1	0.8	0.4	0.5	0.8
x_2	0.8	1	0.4	0.5	0.9
x_3	0.4	0.4	1	0.4	0.4
x_4	0.5	0.5	0.4	1	0.5
x_5	0.8	0.9	0.4	0.5	1

here,

$$\text{U}(R_2)(x_1, x_5) = 0.8$$

$$\text{U}(R_2)(x_4, x_5) = 0.8$$

$$\text{U}(R_2)(x_2, x_5) = 0.9$$

$$\text{i.e., } 0.8 \geq \min(0.8, 0.9)$$

Thus, R_2 is Transitive

* Equivalence Relation:

A relation R is said to be equivalence relation, if it satisfies reflective, symmetric and transitive relation.

* Fuzzy Tolerance Relation:

If the relation is reflective, symmetric but not transitive, then we can make it transitive by doing $n-1$ composition with R and its resulted composition if it transformed to transitive then it's tolerance relation where,

n is cardinal of domain of discourse.

Example:

For $X = \{x_1, x_2, x_3, x_4, x_5\}$,

	x_1	x_2	x_3	x_4	x_5	\rightarrow n i.e. cardinal of domain of discourse.
x_1	1	0.8	0	0.1	0.2	
x_2	0.8	1	0.4	0	0.9	
x_3	0	0.4	1	0	0	
x_4	0.1	0	0	1	0.5	
x_5	0.2	0.9	0	0.5	1	

Using max-min composition, find above relation is tolerance relation or not.

= To be a Fuzzy Tolerance relation it must satisfy Reflective & Symmetric property too.

① For Reflective:

$$\text{ll}_R(x_i, x_i) = 1 \text{ i.e.,}$$

$$\text{ll}_R(x_1, x_1) = 1$$

$$\text{ll}_R(x_2, x_2) = 1$$

$$\text{ll}_R(x_3, x_3) = 1$$

$$\text{Ull}_R(4,4) = 1$$

$$\text{Ull}_R(5,5) = 1$$

This satisfies reflexive property.

② For Symmetric:

$$\text{Ull}_R(x_i, x_i) = \text{Ull}_R^{(x_i, x_i)} \text{ i.e.,}$$

$$\text{Ull}_R(x_1, x_2) = \text{Ull}_R^{(x_1, x_2)} = 0.8$$

$$\text{Ull}_R^{(x_1, x_3)} = \text{Ull}_R^{(x_3, x_1)} = 0$$

$$\text{Ull}_R^{(x_1, x_4)} = \text{Ull}_R^{(x_4, x_1)} = 0.1$$

$$\text{Ull}_R^{(x_1, x_5)} = \text{Ull}_R^{(x_5, x_1)} = 0.2$$

$$\text{Ull}_R^{(x_2, x_3)} = \text{Ull}_R^{(x_3, x_2)} = 0.4$$

$$\text{Ull}_R^{(x_2, x_4)} = \text{Ull}_R^{(x_4, x_2)} = 0$$

$$\text{Ull}_R^{(x_2, x_5)} = \text{Ull}_R^{(x_5, x_2)} = 0.9$$

$$\text{Ull}_R^{(x_3, x_4)} = \text{Ull}_R^{(x_4, x_3)} = 0$$

$$\text{Ull}_R^{(x_3, x_5)} = \text{Ull}_R^{(x_5, x_3)} = 0$$

$$\text{Ull}_R^{(x_4, x_5)} = \text{Ull}_R^{(x_5, x_4)} = 0.5$$

This satisfies reflexive property.

③ For Transitive,

$$\text{Ull}_R^{(x_1, x_5)} = 0.2$$

$$\text{Ull}_R^{(x_1, x_2)} = 0.8$$

$$\text{Ull}_R^{(x_2, x_5)} = 0.9$$

So,

$0.2 \neq \min(0.8, 0.9)$ This is not transitive so,

we need to use $n-1$ max-min composition i.e.,

$$n=5$$

$n=4$ composition.

then new R_1 is tolerance relation.

The possible composition case:

$$R_1^2 = R_1 \circ R_1$$

$$R_1^3 = R_1 \circ R_1^2$$

$$R_1^4 = R_1 \circ R_1^3$$

$$R_1^5 = R_1 \circ R_1^4$$

① For $R_1^2 = R_1 \circ R_1$

$$R_1 = \{ \begin{array}{llllll} 1 & 0.8 & 0 & 0.1 & 0.2 & 0.5 & 1 \\ (x_1, x_1) & (x_1, x_2) & (x_1, x_3) & (x_1, x_4) & (x_1, x_5) & (x_1, x_6) & (x_1, x_7) \end{array}$$

For (x_1, x_1)

$$\text{LL}_{R_1^2} = \max \{ \min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \dots \}$$

$$\min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \dots$$

$$\min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \dots$$

$$= \max \{ \min(1, 1), \min(0.8, 0.8), \min(0, 0), \min(0.1, 0.1) \}$$

$$\min(0.2, 0.2) \}$$

$$= \max \{ 1, 0.8, 0, 0.1, 0.2 \}$$

$$= 1$$

② For (x_1, x_2)

$$\text{LL}_{R_1^2} = \max \{ \min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \dots \}$$

$$\min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \dots$$

$$\min \{ \text{LL}_{R_1}, \text{LR}_1 \}, \dots$$

$$= \max \{ \min(1, 0.8), \min(0.8, 1), \min(0, 0.4), \min(0.1, 0) \}$$

$$\min(0.2, 0.3) \}$$

$$= 0.8,$$

⑧ For $(x_1, x_3) = 0.4$

⑨ For $(x_1, x_4) = 0.2$

⑩ For $(x_1, x_5) = 0.8$

Similarly for

(x_2, x_1)

(x_2, x_2)

(x_2, x_3)

:

$(x_5, x_5) = ?$

(x_2, x_5)

$(x_2, x_1), (x_1, x_5) \Rightarrow (0.8, 0.2) \geq 0.2$

$(x_2, x_2), (x_2, x_5) \Rightarrow (1, 0.2) \geq 0.2$

$(x_2, x_3), (x_3, x_5) \Rightarrow (0.4, 0) \geq 0$

$(x_2, x_4), (x_4, x_5) \Rightarrow (0, 0.5) \geq 0$

$(x_2, x_5), (x_5, x_5) \Rightarrow (0.9, 1) \geq 0.9$

Therefore,

	x_1	x_2	x_3	x_4	x_5
R_1^2	x_1	1	0.8	0.4	0.2
	x_2	0.8	1	0.4	0.5
	x_3	0.4	0.4	1	0
	x_4	0.2	0.5	0	1
	x_5	0.8	0.9	0.4	0.5

since, R_1^2 is reflexive & symmetric.

To check transitive,

$$llR_1^2(x_1, x_4) = 0.2$$

$$llR_1^2(x_1, x_2) = 0.8$$

$$llR_1^2(x_2, x_4) = 0.5$$

i.e., $0.2 \neq \min(0.8, 0.5)$

So, R_1^2 is not transitive,

now, we should perform next composition again, i.e

$$\underset{\sim}{R_1^3} = R_1 \circ R_1^2$$

Similarly as R_1^2
we get,

	x_1	x_2	x_3	x_4	x_5
x_1	1	0.8	0.4	0.5	0.8
x_2	0.8	1	0.4	0.5	0.9
x_3	0.4	0.4	1	0.4	0.4
x_4	0.5	0.5	0.4	1	0.5
x_5	0.8	0.9	0.4	0.5	1

$\underset{\sim}{R_1^3}$ is reflexive & symmetric.

for transitive,

$$\text{ll } \underset{\sim}{R_1^3}^{(x_1, x_3)} = 0.8$$

$$\text{ll } \underset{\sim}{R_1^3}^{(x_1, x_2)} = 0.8$$

$$\text{ll } \underset{\sim}{R_1^3}^{(x_2, x_3)} = 0.9$$

$$\text{i.e., } 0.8 \geq \min(0.8, 0.9) \text{ (True)}$$

Thus, $\underset{\sim}{R_1^3}$ is reflexive, symmetric & transitive.

so, it is a Fuzzy Tolerance relation.

* Extension principle:

Consider domain X and Y

and $y = f(x)$ & $y \in Y$ and $x \in X$

If for each $x \in A$

$y \in B$ and $y = f(x)$

Then,

$$U_{f(A)}^{(Y)} = \max \left\{ \min \left[U_{A_1}^{(x_1)}, U_{A_2}^{(x_2)}, \dots, U_{A_n}^{(x_n)} \right] \right\}$$

$$y = f(x_1, x_2, \dots, x_n)$$

This is called Zadeh Extension Principle.

Example:

$$\text{Approx } 4 = \{ \frac{0.6}{2}, \frac{1}{4}, \frac{0.3}{6} \}$$

$$\text{near } 8 = \{ \frac{0.3}{4}, \frac{0.1}{12}, \frac{1}{8}, \frac{0.01}{2} \}$$

$$\text{for a set } C = \underbrace{\text{near } 8}_{\text{Approx } 4}$$

Then,

$$U_C^{(1)} = \max \left\{ \min \left(U_{A_4}^{(2)}, U_{B_8}^{(2)} \right), \min \left(U_{A_4}^{(4)}, U_{B_8}^{(4)} \right) \right\}$$

$$= \max \left\{ \min (0.6, 0.01), \min (1, 0.3) \right\}$$

$$= \max (0.01, 0.3)$$

$$= 0.3$$

$$U_C^{(2)} = \max \left\{ \min \left(U_{A_4}^{(2)}, U_{B_8}^{(4)} \right), \min \left(U_{A_4}^{(4)}, U_{B_8}^{(8)} \right), \min \left(U_{A_4}^{(6)}, U_{B_8}^{(12)} \right) \right\}$$

$$\begin{aligned}
 &= \max \{ \min(0.6, 0.3), \min(1, 1), \min(0.3, 0.4) \} \\
 &= \max (0.3, 1, 0.3) \\
 &= 1
 \end{aligned}$$

$$U_{lC}^{(3)} = \max \{ \min(U_{lA4}^{(4)}, U_{lB8}^{(12)}) \}$$

$$\begin{aligned}
 &= \max \{ \min(1, 0.4) \} \\
 &= 0.4
 \end{aligned}$$

$$U_{lC}^{(4)} = \max \{ \min(U_{lA4}^{(2)}, U_{lB8}^{(8)}) \}$$

$$\begin{aligned}
 &= \max \{ \min(0.6, 1) \} \\
 &= 0.6
 \end{aligned}$$

$$U_{lC}^{(5)} = \max \{ \min(U_{lA4}^{(2)}, U_{lB8}^{(10)}) \}$$

$$\begin{aligned}
 &= \max \{ \min(0.6, 0.4) \} \\
 &= 0.4
 \end{aligned}$$

Therefore,

$$c = \{ \frac{0.3}{1}, \frac{1}{2}, \frac{0.4}{3}, \frac{0.6}{4}, \frac{0.4}{6} \}$$

$$* F(A, B, C) = x^2 + y + z \quad \forall x \in A, y \in B \text{ & } z \in C$$



$$r_i = x_i^2 + y_i + z_i$$

* Fuzzy Transform:

For any two fuzzy set, consider $f: \tilde{A} \rightarrow \tilde{B}$. Here, f determines fuzzy transform, which determines or defines a mapping from \tilde{A} to \tilde{B} i.e. an element x_i in \tilde{A} to an element y_j in \tilde{B} .

If $\tilde{A} \subseteq X$ & $\tilde{B} \subseteq Y$ are finite then fuzzy mapping is:

$\tilde{R} = x_1$	y_1	y_2	\dots	y_j	\dots	y_m
x_2	r_{11}	r_{12}		r_{1j}	\dots	r_{1m}
\vdots	\vdots	\vdots		\vdots	\vdots	\vdots
x_n	r_{n1}			r_{nj}	\dots	r_{nm}

Given mapping \tilde{R} & fuzzy set $\tilde{A}; \tilde{B}_i = f(x_i)$ and

$\text{U}_{\tilde{B}_i}(y_i) = r_{ij}$ then;

$$\text{U}_{\tilde{B}}(y) = \bigvee_{x \in X} (\text{U}_{\tilde{A}}(x) \wedge \text{U}_{\tilde{R}}(x, y))$$

\Rightarrow i.e., $B = A \circ R$ (Zadeh Extension principle)

Example:

* Consider:

Here, for $\tilde{A} \subseteq X = \{40, 50, 60, 70, 80\}$

& $\tilde{B} \subseteq Y = \{14, 15, 16, 17, 18\}$
& $R: \tilde{A} \rightarrow \tilde{B}$

$$\tilde{A} = \{ \frac{0.8}{40}, \frac{1}{50}, \frac{0.6}{60}, \frac{0.2}{70}, \frac{0}{80} \}$$

2

	14	15	16	17	18
40	1	0.8	0.2	0.1	0
50	0.8	1	0.8	0.2	0.1
60	0.2	0.8	1	0.8	0.2
70	0.1	0.2	0.8	1	0.8
80	0	0.1	0.2	0.8	1

now,

$$\tilde{B} = ?$$

$$\tilde{B} = \tilde{A} \circ R$$

\Rightarrow so! for membership of 14,

$$\text{ul}_{\tilde{B}}^{(14)} = \max \{ \min \{ \text{ul}_A^{(x)}, \text{ul}_R^{(x,y)} \} \}$$

for each $y \in B$

$$\text{ul}_{\tilde{B}}^{(y)} = \max \{ \min (\text{ul}_A^{(x)}, \text{ul}_R^{(x,y)}) \}$$

so, here for membership of 14,

$$\text{ul}_{\tilde{B}}^{(14)} = \max \{ \min \{ \text{ul}_A^{(40)}, \text{ul}_R^{(40,14)} \}, \min \{ \text{ul}_A^{(50)}, \text{ul}_R^{(50,14)} \} \}$$

$$\min \{ \text{ul}_A^{(60)}, \text{ul}_R^{(60,14)} \}, \min \{ \text{ul}_A^{(70)}, \text{ul}_R^{(70,14)} \}$$

$$\min \{ \text{ul}_A^{(80)}, \text{ul}_R^{(80,14)} \}$$

$$= \max \{ \min (0.8, 1), \min (1, 0.8), \min (0.6, 0.2), \min (0.2, 0.1) \\ \min (0, 0) \}$$

$$= \max(0.8, 0.8, 0.2, 0.1, 0)$$

$$= 0.8$$

for membership of 15,

$$\text{U}_{\text{L}}^{(15)} = \max \left\{ \min \left\{ \text{U}_{\text{L}}^{(40)}, \text{U}_{\text{R}}^{(40, 15)} \right\}, \min \left\{ \text{U}_{\text{L}}^{(50)}, \text{U}_{\text{R}}^{(50, 15)} \right\} \right. \\ \left. \min \left\{ \text{U}_{\text{L}}^{(60)}, \text{U}_{\text{R}}^{(60, 15)} \right\}, \min \left\{ \text{U}_{\text{L}}^{(70)}, \text{U}_{\text{R}}^{(70, 15)} \right\} \right. \\ \left. \min \left\{ \text{U}_{\text{L}}^{(80)}, \text{U}_{\text{R}}^{(80, 15)} \right\} \right\}$$

$$= \max \left\{ \min(0.8, 0.8), \min(1, 1), \min(0.6, 0.8), \min(0.2, 0) \right. \\ \left. \min(0, 0.1) \right\}$$

$$= \max(0.8, 1, 0.6, 0.2, 0)$$

$$= \cancel{0.8} \quad 1$$

for membership of 16

$$\text{U}_{\text{L}}^{(16)} = \max \left\{ \min(0.8, 0.2), \min(1, 0.8), \min(0.6, 1), \min(0.2, 0) \right. \\ \left. \min(0, 0.2) \right\}$$

$$= \max(0.2, 0.8, 0.6, 0.2, 0)$$

$$= 0.8$$

for membership of 17

$$\text{U}_{\text{L}}^{(17)} = \max \left\{ \min(0.8, 0.1), \min(1, 0.2), \min(0.6, 0.8), \min(0.2, 1), \min(0, 0.1) \right\}$$

$$= \max(0.1, 0.2, 0.6, 0.2, 0)$$

$$= 0.6$$

for membership of 18

$$\begin{aligned} m_B^{(18)} &= \max \left\{ \min(0.8, 0), \min(1, 0.1), \min(0.6, 0.2), \min(0.2, 0) \right. \\ &\quad \left. \min(0, 1) \right\} \\ &= \max(0, 0.1, 0.2, 0.2, 0) \\ &= 0.2 \end{aligned}$$

Therefore,

$$B = \{ \frac{0.8}{14}, \frac{1}{15}, \frac{0.8}{16}, \frac{0.6}{17}, \frac{0.2}{18} \}$$

* Fuzzy Transform in generalized relations:

Consider we want to map ordered pairs from input universes $X_1 = S_{0,1,6,7}$ and $X_2 = S_{1,2,3,7}$ to an output universe $Y = S_{x,y,z}$. The mapping is given by a crisp relation R_{ab} :

$$R = a \begin{bmatrix} 1 & 2 & 3 \\ x & z & x \\ b & x & y & z \end{bmatrix}$$

consider fuzzy sets $\tilde{A} \subseteq X_1$ and $\tilde{B} \subseteq X_2$ as;

$$\tilde{A} = \{ \frac{0.6}{a} + \frac{1}{b} \}$$

$$\tilde{B} = \{ \frac{0.2}{1} + \frac{0.8}{2} + \frac{0.4}{3} \}$$

now, if \tilde{R}_{ab} is defined for a relational mapping function,

$$f(\tilde{A}_2, \tilde{B}) = \tilde{C}$$

Then, using max-min extension (zadeh) membership of x, y, z can be determined as;

$$\begin{aligned} \text{U}_{\text{C}}^{(x)} &= \max \left\{ \min \left\{ \text{U}_{\text{A}}^{(1)}, \text{U}_{\text{B}}^{(1)} \right\}, \min \left\{ \text{U}_{\text{A}}^{(2)}, \text{U}_{\text{B}}^{(2)} \right\} \right\} \\ &= \min \left\{ \text{U}_{\text{A}}^{(2)}, \text{U}_{\text{B}}^{(3)} \right\} \\ &= \max \left\{ \min (0.6, 0.2), \min (1, 0.2), \min (0.6, 0.4) \right\} \\ &= \max (0.2, 0.2, 0.4) \\ &= 0.4 \end{aligned}$$

Similarly,

$$\begin{aligned} \text{U}_{\text{C}}^{(y)} &= \max \left\{ \min \left\{ \text{U}_{\text{A}}^{(1)}, \text{U}_{\text{B}}^{(2)} \right\}, \text{U}_{\text{B}}^{(3)} \right\} \\ &= \max \left\{ \min (1, 0.8) \right\} \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} \text{U}_{\text{C}}^{(z)} &= \max \left\{ \min \left\{ \text{U}_{\text{A}}^{(1)}, \text{U}_{\text{B}}^{(2)} \right\}, \min \left\{ \text{U}_{\text{A}}^{(2)}, \text{U}_{\text{B}}^{(3)} \right\} \right\} \\ &= \max \left\{ \min (0.6, 0.8), \min (1, 0.4) \right\} \\ &= \max (0.6, 0.4) \\ &= 0.6 \end{aligned}$$

Thus, $\text{C} = \frac{0.4}{x} + \frac{0.8}{y} + \frac{0.6}{z}$

$$R = \begin{bmatrix} 1 & 2 & 3 \\ a & \begin{bmatrix} 0.4 & 0.6 & 0.4 \end{bmatrix} \\ b & \begin{bmatrix} 0.4 & 0.8 & 0.6 \end{bmatrix} \end{bmatrix} *$$

* α -Cut in fuzzy relations:

for $\alpha \in [0, 1]$

$$R_\alpha(x_i, y_j) = (x_i, y_j)$$

$$\text{ul}_{R_\alpha}(x_i, y_j) = 1 \text{ if } \text{ul}_R(x_i, y_j) \geq \alpha$$

$$0 \text{ if } \text{ul}_R(x_i, y_j) < \alpha$$

* Strict α -cut:

for $\alpha \in [0, 1]$

$$R_\alpha(x_i, y_j) = (x_i, y_j)$$

$$\text{ul}_{R_\alpha}(x_i, y_j) = 1 \text{ if } \text{ul}_R(x_i, y_j) > \alpha$$

$$0 \text{ if } \text{ul}_R(x_i, y_j) \leq \alpha$$

* Aggregation operation on fuzzy sets:

Consider fuzzy sets A & B over some domain X .

$$\text{Suppose: } E = A \cap B$$

$$\text{f } E = A \cup B$$

Then, $E(x)$ & $F(x)$ can be formalized as a function depending on $A(x)$ and $B(x)$.

Suppose,

$$E(x) = T(A(x), B(x)) \quad \text{f } F(x) = S(A(x), B(x))$$

Where,

T and S are intersection and union operations

$$\therefore E = A \cap B = B \cap A \Rightarrow T(A(x), B(x)) = T(B(x), A(x))$$

$$\& F = A \cup B = B \cup A \Rightarrow S(A(x), B(x)) = S(B(x), A(x))$$

Similarly,

$$E = A \cap (B \cap C) = (A \cap B) \cap C \& F = A \cup (B \cup C) = (A \cup B) \cup C$$

Thus,

$$T(A(x), T(B(x)), C(x)) = T(T(A(x), B(x)), C(x))$$

$$\& S(A(x), S(B(x), C(x))) = S(S(A(x), B(x)), C(x))$$

$$\text{If } A(x) \geq A(y) \& B(x) \geq B(y)$$

then,

$$T(A(x), B(x)) \geq T(A(y), B(y))$$

&

$$S(A(x), B(x)) \geq S(A(y), B(y))$$

$$\text{If } A \cap 1 = A \& A \cup 0 = A$$

then,

$$T(A(x), 1) = A(x)$$

$$S(A(x), 0) = A(x)$$

$$\text{If } A \cap A = A \& A \cup A = A$$

then,

$$T(A(x), A(x)) = A(x)$$

$$S(A(x), A(x)) = A(x)$$

* The function T defines T -operator as:

$$T: [0,1] \times [0,1] \rightarrow [0,1]$$

and is called t-norm if

- ① $T(a,b) = T(b,a)$ commutative,
- ② $T(a, T(b,c)) = T(T(a,b), c)$ associative
- ③ $T(a,b) \geq T(c,d)$ if $a \geq c$ & $b \geq d$ Monotonicity
- ④ $T(a,1) = a$ Identity

Example: MIN, MAX, $[0, a+b-1], a \cdot b$

↳ Arithmetic multiplication

* The function S defines S -operator as:

$$S: [0,1] \times [0,1] \rightarrow [0,1]$$

and is called t-conorm if

- ① $S(a,b) = S(b,a)$ commutative
- ② $S(a, S(b,c)) = S(S(a,b), c)$ associative
- ③ $S(a,b) \geq S(c,d)$ if $a \geq c$ & $b \geq d$ Monotonicity
- ④ $S(a,0) = a$ Identity

Example: Max, Min, $(1, a+b), a+b - ab$

* Negation operator: N

$$N: [0,1] \rightarrow [0,1] \text{ if}$$

- ① $N(1) = 0$, & $N[0] = 1$ boundary rule
- ② if $a > b$ then $N(a) < N(b)$ order of reversing
- ③ $N(N(a)) = a$ Involution.

Example: For $A = \left\{ \frac{0.6}{20}, \frac{0.4}{30} \right\}$

$$\& B = \left\{ \frac{0.1}{20}, \frac{0.2}{30} \right\}$$

Compute t-norm operation on $A \times B$ using $\max(0, a \cdot b - 1)$ and t-conorm operation on $A \times B$ using $\min(1, a + b)$.

= Sol

for, t-norm using $\max(0, a \cdot b - 1)$

for 20)

$$\begin{aligned} &= \max(0, 0.6 \cdot 0.1 - 1) \\ &= \max(0, -0.3) \\ &= 0 \end{aligned}$$

for (30)

$$\begin{aligned} &= \max(0, 0.4 \cdot 0.2 - 1) \\ &= \max(0, -0.4) \\ &= 0 \end{aligned}$$

For t-conorm using $\min(1, a + b)$

for 20 :

$$\begin{aligned} &= \min(1, 0.6 + 0.1) \\ &= \min(1, 0.7) \\ &= 0.7 \end{aligned}$$

For 30

$$\begin{aligned} &= \min(1, 0.4 + 0.2) \\ &= \min(1, 0.6) \\ &= 0.6 \end{aligned}$$

Unit: 3

Membership Function :

3.1 Universe of discourse (X)

3.2 Mapping inside Fuzzy domain

{ Already cover in initial chapters.

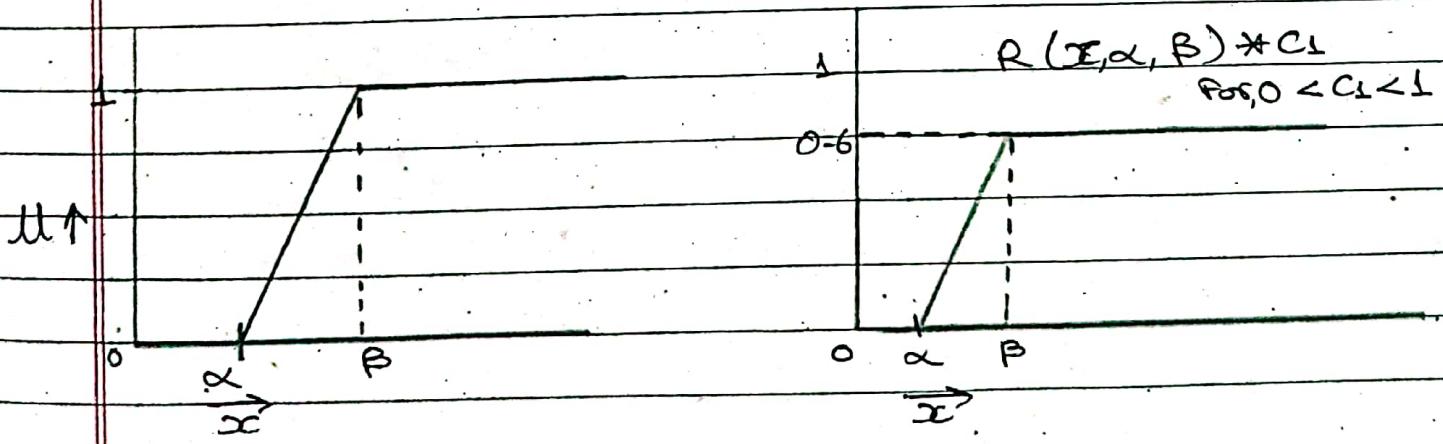
3.3 Fuzzy membership mapping methods:

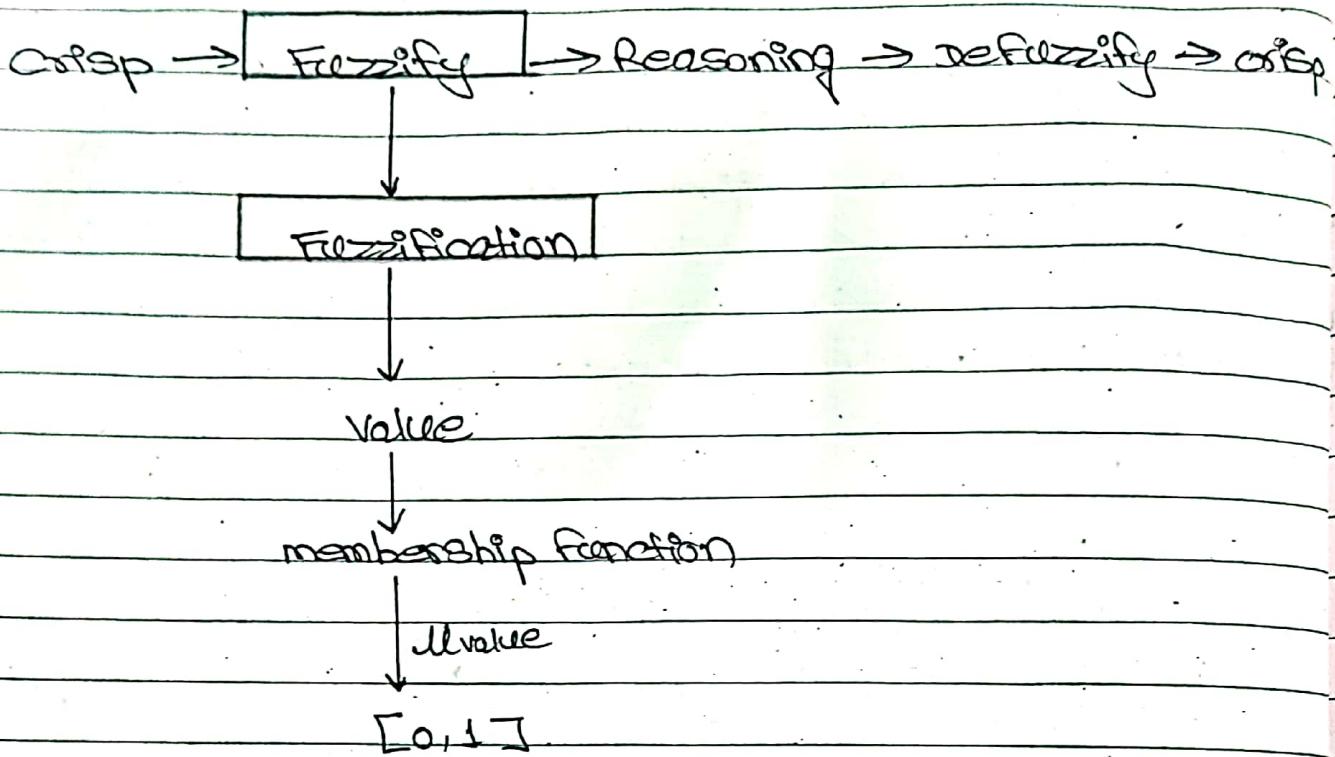
- ① R-function,
- ② L-function,
- ③ Triangular Function,
- ④ Trapezoidal Function,
- ⑤ Sigmoid Function,
- ⑥ Gaussian Function,

① R-Function (Right-sided function):

For any element or over X , the R-Function is defined as;

$$R(x, \alpha, \beta) = \begin{cases} 0 & ; x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & ; \alpha \leq x \leq \beta \\ 1 & ; x > \beta \end{cases}$$



**Example:**

$X = \{20, 30, 40, 50, 60, 70, 80\}$. Complete a fuzzy set $A \subseteq X$ using R-function where $\alpha = 40$ & $\beta = 70$

= Sol

given, $\alpha = 40$ & $\beta = 70$

so. calculate membership for each element in X i.e.,

$$\text{ul}_A^{(40)} = 0 \quad (\because 20 < \alpha \leq \beta)$$

$$\text{ul}_A^{(20)} = 0. \quad (\because \alpha \leq \alpha)$$

$$\text{ul}_A^{(30)} = 0 \quad (\because \alpha \leq \alpha)$$

$$\text{ul}_A^{(40)} = \frac{x-\alpha}{\beta-\alpha} = \frac{40-40}{70-40} = 0 \quad (\because \alpha < x \leq \beta)$$

$$U_{\alpha}^{(50)} = \frac{x-\alpha}{\beta-\alpha} = \frac{50-40}{70-40} = \frac{10}{30} = 0.33$$

$$U_{\alpha}^{(60)} = \frac{x-\alpha}{\beta-\alpha} = \frac{60-40}{70-40} = \frac{20}{30} = 0.67$$

$$U_{\alpha}^{(70)} = \frac{x-\alpha}{\beta-\alpha} = \frac{70-40}{70-40} = \frac{30}{30} = 1$$

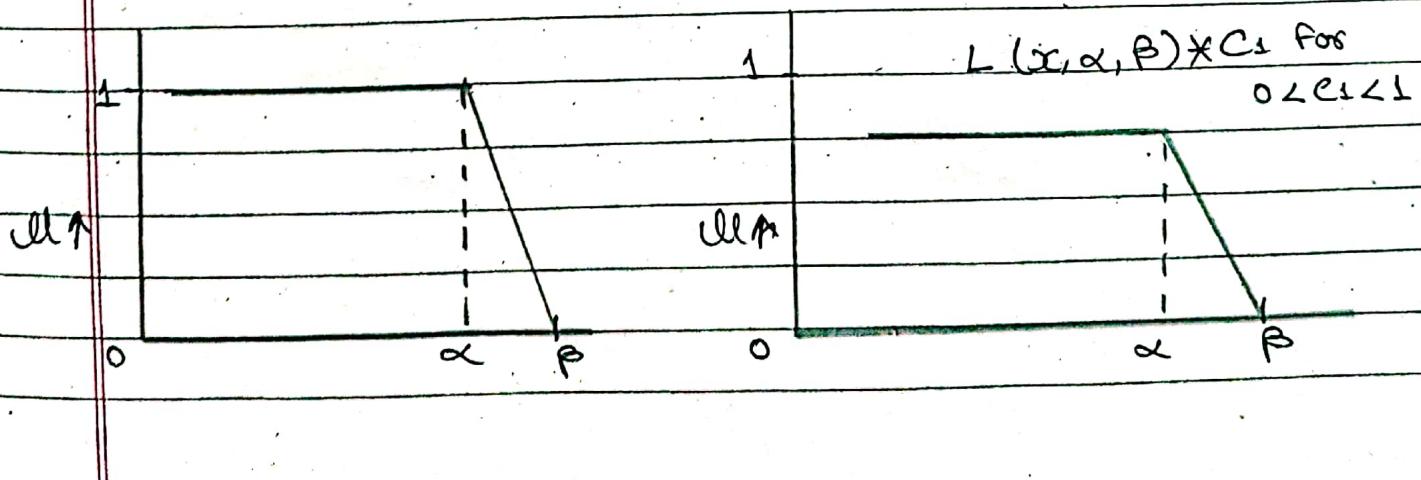
$$U_{\alpha}^{(80)} = 1 \quad (\because x \geq \beta)$$

$$\therefore U_{\alpha} = \left\{ \begin{array}{l} 0 \quad \text{for } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} \quad \text{for } \alpha \leq x \leq \beta \\ 1 \quad \text{for } x > \beta \end{array} \right.$$

② L-Function (Left-sided Function):

For any element x over X , the L-function is defined as:

$$L(x, \alpha, \beta) = \begin{cases} 1 & \text{for } x < \alpha \\ \frac{\beta-x}{\beta-\alpha} & \text{for } \alpha \leq x \leq \beta \\ 0 & \text{for } x > \beta \end{cases}$$



Example:

$X = \{10, 20, 30, 40, 50, 60, 70, 80\}$. compute a Fuzzy Set $A \subseteq X$ using L-Funcion where $\alpha = 40$ & $\beta = 70$.

= Sol

given, $\alpha = 40$ & $\beta = 70$

So,

to calculate the membership for each element in X i.e.

for 20 i.e.,

$$m_A^{(20)} = 1$$

($\because x \leq \alpha$)

$$m_A^{(60)} = 1$$

($\because x \geq \alpha$)

$$m_B^{(30)} = 1$$

($\because x \leq \beta$)

$$m_A^{(40)} = \frac{0}{70-40}$$

($\because x \leq \alpha$)

$$= \frac{40-40}{70-40}$$

$$\frac{70-40}{70-40}$$

$$= \frac{0}{30}$$

$$= \frac{20}{20}$$

$$= 0$$

$$= 1$$

$$m_A^{(50)} = \frac{x-\alpha}{\beta-\alpha}$$

$$\frac{\beta-x}{\beta-\alpha} = \frac{70-50}{70-40}$$

$$= \frac{20}{30}$$

$$= \frac{20}{30}$$

$$= 0.67$$

$$= 0.67$$

$$\text{U}_{\text{A}_2}^{(60)} = \frac{\beta - x}{\beta - \alpha}$$

$$= \frac{70 - 60}{70 - 40}$$

$$= \frac{10}{30}$$

$$= 0.33$$

$$\text{U}_{\text{A}_2}^{(70)} = \frac{\beta - x}{\beta - \alpha}$$

$$= \frac{70 - 70}{70 - 40}$$

$$= 0$$

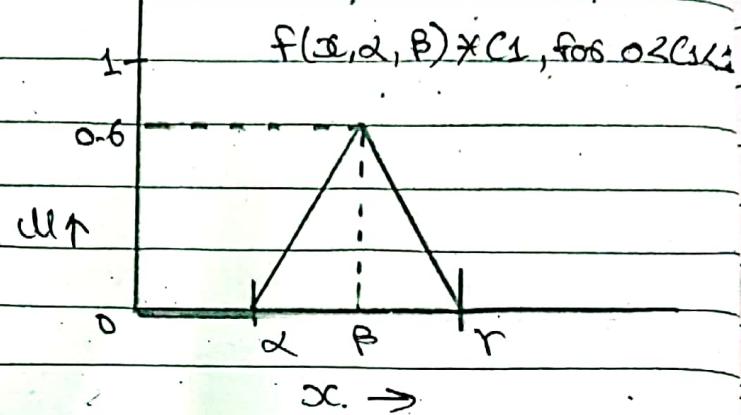
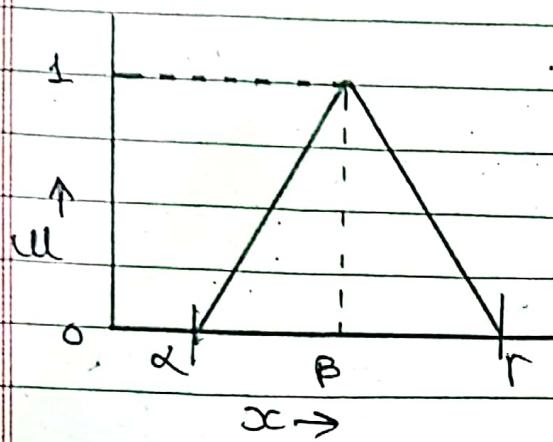
$$\text{U}_{\text{A}_2}^{(80)} = 0 \quad (\because x > \beta)$$

$$\therefore \text{U}_{\text{A}_2} = S \left[\frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, 0.67, 0.33, \frac{0}{70}, \frac{0}{80} \right]$$

⑤ Triangular Function:

For any element x over X , the triangular membership function is defined as:

$$f(x, \alpha, \beta, r) = \begin{cases} 0 & \text{for } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{for } \alpha \leq x \leq \beta \\ \frac{r-x}{r-\beta} & \text{for } \beta < x \leq r \\ 0 & \text{for } x > r \end{cases}$$



Example:

Given, $X = \{10, 20, 30, 40, 50, 60, 70, 80\}$. Compute a fuzzy set $A \subset X$ using Triangular function where $\alpha = 30, \beta = 50, \gamma = 70$

= Sol

given, $\alpha = 30, \beta = 50, \gamma = 70$

To calculate the membership for each element in X i.e,

$$u_{A_1}^{(10)} = 0 \quad (\because x < \alpha)$$

$$u_{A_2}^{(20)} = 0 \quad (\because x < \alpha)$$

$$u_{A_3}^{(30)} = \frac{\gamma - x}{\beta - \alpha}$$

$$= \frac{30 - 30}{50 - 30}$$

$$= \frac{0}{20}$$

$$= 0$$

$$\text{Ull}_2^{(50)} = \frac{x-\alpha}{B-\alpha}$$

$$= \frac{40-30}{50-30}$$

$$= \frac{10}{20}$$

$$= 0.5$$

$\text{Ull}_2^{(50)} = \text{Since } x = B \text{ we } (\because \frac{x-\alpha}{B-\alpha} \text{ for } \alpha < x < B)$
 can use any formula
 among (i) or,

i.e.

$$\frac{x-\alpha}{B-\alpha} \text{ or } \frac{r-x}{r-B}$$

$$= \frac{50-30}{50-50} = \frac{20}{20}$$

$$= \frac{20}{20} = 1$$

$$\text{Ull}_2^{(60)} = \frac{r-x}{r-B}$$

$$= \frac{70-60}{70-50}$$

$$= \frac{10}{20}$$

$$= 0.5$$

$$\mu_A^{(70)} = \frac{r-x}{r-b}$$

$$= \frac{70-70}{70-50}$$

$$= \frac{0}{20}$$

$$= 0$$

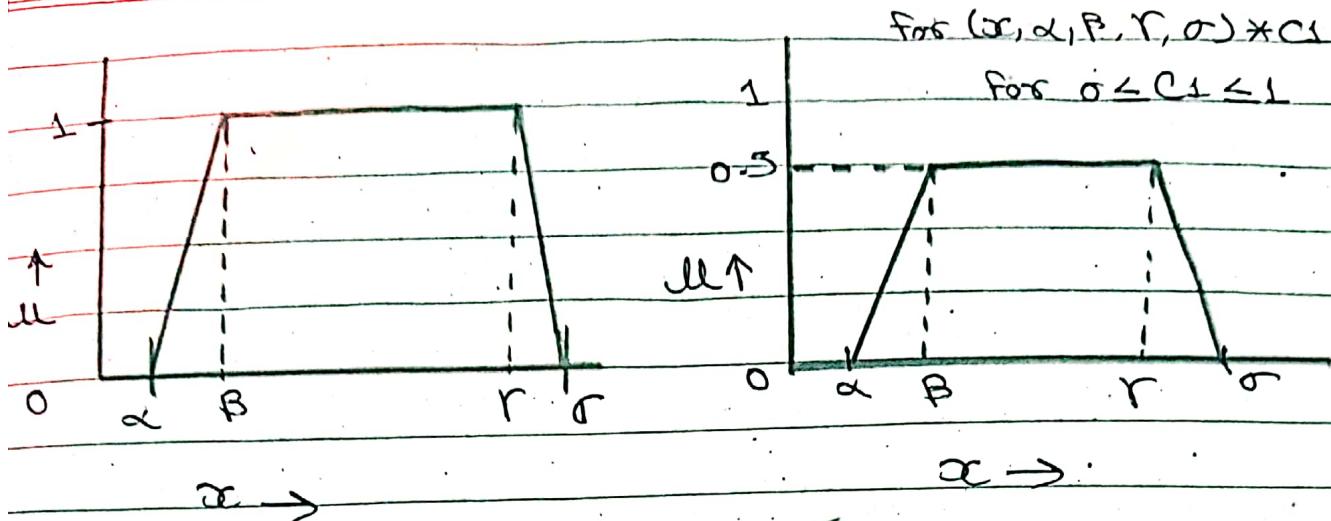
$$\mu_A^{(80)} = 0 \quad (\because x > r)$$

$$\therefore A = \{ \frac{0}{10}, \frac{0}{20}, \frac{0}{30}, \frac{0.25}{40}, \frac{1}{50}, \frac{0.5}{60}, \frac{0}{70}, \frac{0}{80} \}$$

* Trapezoidal membership function:

For any element x over X , the trapezoidal membership function is defined as;

$$f(x, \alpha, \beta, r, \sigma) = \begin{cases} 0 & \text{if } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{if } \alpha \leq x < \beta \\ 1 & \text{if } \beta \leq x < r \\ \frac{\sigma-x}{\sigma-r} & \text{if } r \leq x < \sigma \\ 0 & \text{if } x > \sigma \end{cases}$$



Example:

Given, $X = \$10, 20, 30, 40, 50, 60, 70, 80\}$. Complete Fuzzy Set $A \subseteq X$ using Trapezoid membership function where, $\alpha = 30, \beta = 50, r = 70, \sigma = 80$ calculate membership for each element in X i.e.,

$$\underline{\mu}_A^{(30)} = 0 \quad (\because x < \alpha)$$

$$\underline{\mu}_A^{(20)} = 0 \quad \underline{\mu}_A^{(30)} = 0 \quad (\because x < \alpha)$$

$$\underline{\mu}_A^{(40)} = \frac{40 - 30}{50 - 30} = 0.5$$

$$\underline{\mu}_A^{(50)} = 1 \quad (\because \beta \leq x \leq r)$$

$$\underline{\mu}_A^{(60)} = 1 \quad (\because \beta \leq x \leq r)$$

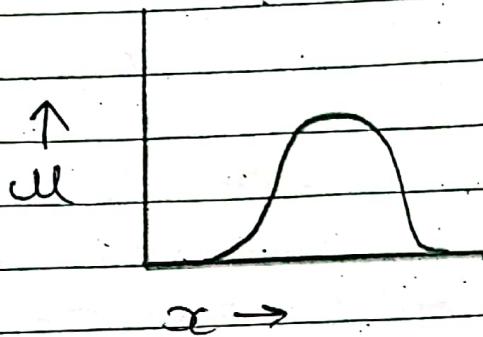
$$\begin{aligned} \underline{\mu}_A^{(70)} &= \frac{80 - 70}{80 - 70} \\ &= \frac{10}{10} = 1 \end{aligned}$$

$$\begin{aligned} \underline{\mu}_A^{(80)} &= \frac{80 - 80}{80 - 70} = 0 \\ &\therefore A = \{ \frac{0}{10}, \frac{0}{20}, \frac{0}{30}, \frac{0.5}{40}, \frac{1}{50}, \frac{1}{60}, \frac{1}{70}, \frac{0}{80} \} \end{aligned}$$

* Gaussian Function:

For any element x over X , the Gaussian function is defined as;

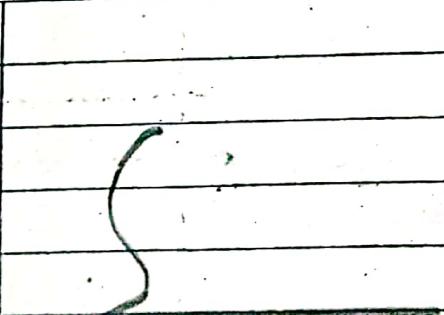
$$f(x, c, \sigma) = e^{-\frac{(x-c)^2}{2\sigma^2}}$$



* Sigmoid function:

For any element x over X , the sigmoid function is defined as;

$$f(x, c, \sigma) = \frac{1}{1 + e^{-(x-c)/\sigma}}$$



Note: Besides above mentioned membership function, there can be user defined membership function where value lies [0, 1]

$$\text{e.g:- } \frac{1}{2} \cdot \left(\frac{x+x}{x*x} \right)$$

Unit: 4

Fuzzy Knowledge based Systems

* Fuzzy Knowledge Based System:

In a fuzzy knowledge base system, the knowledge base contains information represented in fuzzy sets such as linguistic variables, membership functions, and rules. The inference engine uses these fuzzy sets to reason about new data, making decisions or recommendations based on the degree of membership of the fuzzy sets.

* Fuzzy rule based system:

The most common way to represent human knowledge is to form it into natural language expressions of the type

^{Condition}
IF <antecedent>, then <consequent>
^{Action}

The antecedent & consequent are defined by fuzzy linguistic variables.

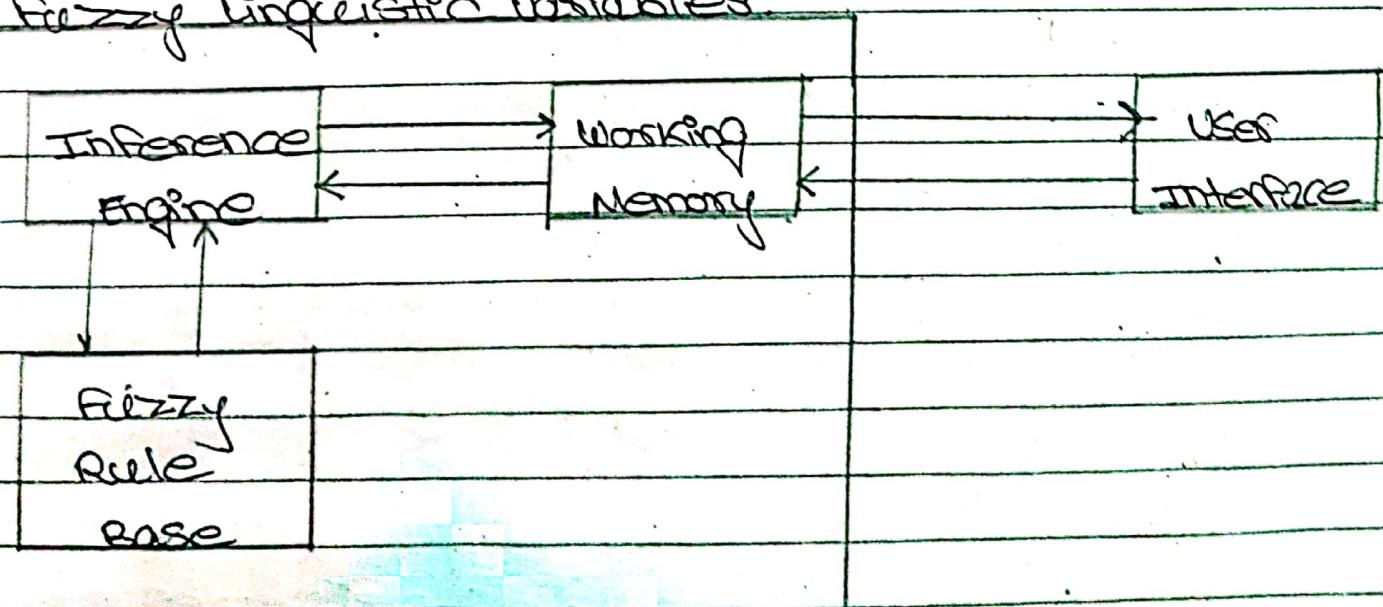


Fig : Fuzzy Rule Based System (FRBS).

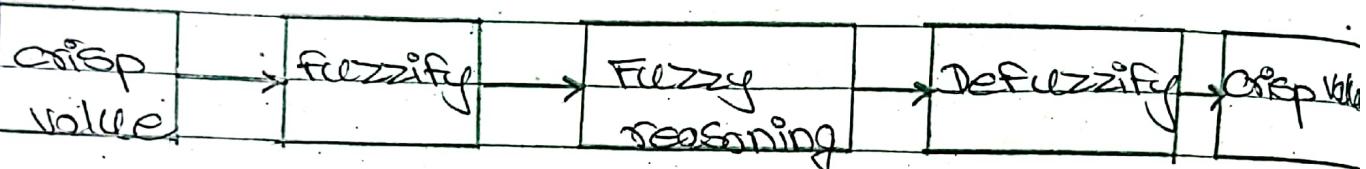
The Fuzzy rules are defined as;

IF x is in A then y is in B where A & B are fuzzy sets.

Implication rules are used to resolve fuzzy rules.

* **Fuzzification:**

It is the process of fuzzyfying crisp values.



Tradition:

IF $x > 0$ then ...

Fuzzy:

IF speed is high then power is low

Speed e high & power e low.

IF speed is moderate then power is high.

* **Implication rules:**

Given, a fuzzy rule:

IF x is in A then y is in B

where,

x is in A & y is in B is a Fuzzy predicate
 $A(x) \wedge B(y)$

Then;

IF $(A(x))$ then $B(y)$ can be represented as;

$$R(x,y) : A \underset{\sim}{\underset{\sim}{\sim}} \rightarrow B \underset{\sim}{\underset{\sim}{\sim}}$$

i.e. $\text{ll}_R^{(x,y)} = f(\text{ll}_A^{(x)}, \text{ll}_B^{(y)})$

where, f is a fuzzy implication & performs task of transferring membership degree of x in A & y in B into those of (x,y) in $A \times B$ for implication

for C : IF $A \underset{\sim}{\underset{\sim}{\sim}}$ then $B \underset{\sim}{\underset{\sim}{\sim}}$

① Mamdani Implication Rule:

$$R_C = A \underset{\sim}{\underset{\sim}{\sim}} \times B \underset{\sim}{\underset{\sim}{\sim}}$$

$$\text{ll}(R_C) = \int_{X \times Y} \text{ll}_A^{(x)} \cap \text{ll}_B^{(y)}$$

Aggregation → (x,y)

$$= \min (\text{ll}_A^{(x)}, \text{ll}_B^{(y)})$$

(x,y)

② Lotfi's Rule:

$$R_C = A \underset{\sim}{\underset{\sim}{\sim}} \times B \underset{\sim}{\underset{\sim}{\sim}}$$

$$\text{ll}(R_C) = \int_{X \times Y} \text{ll}_A^{(x)} \cdot \text{ll}_B^{(y)}$$

(x,y)

$$= \text{ll}_A^{(x)} * \text{ll}_B^{(y)}$$

(x,y)

(2) Zadeh Implication Rule:

$$R_C = A \times B$$

$$u_l(R_C) = \max [\min (u_A^{(x)}, u_B^{(y)})]$$

* Example:

Consider a fuzzy rule:

IF temperature is hot then fan speed is fast.

where domain of discourse for temperature in °C
is

{30, 35, 40, 45, 50}

& domain of discourse for speed in rpm is

{500, 1000, 1500, 2000}

Consider,

$$\text{hot} = \left\{ \frac{0.4}{30}, \frac{0.6}{35}, \frac{0.8}{40}, \frac{0.9}{45} \right\}$$

$$\text{fast} = \left\{ \frac{0.3}{500}, \frac{0.5}{1000}, \frac{0.7}{1500}, \frac{1}{2000} \right\}$$

now, the rule R, using min-doni rule is;

$$\begin{aligned} u_R^{(30, 500)} &= \min (u_{\text{hot}}^{(30)}, u_{\text{fast}}^{(500)}) \\ &= \min (0.4, 0.3) \\ &= 0.3 \end{aligned}$$

$$\text{ULR}_{\frac{1}{2}}^{(30, 1000)} = \min \left(\text{UL}_{\frac{1}{2}}^{\text{hot}}^{(30)}, \text{UL}_{\frac{1}{2}}^{\text{fast}}^{(1000)} \right)$$

$$= \min (0.4, 0.5)$$

$$= 0.4$$

$$\text{ULR}_{\frac{1}{2}}^{(30, 1500)} = \min \left(\text{UL}_{\frac{1}{2}}^{\text{hot}}^{(30)}, \text{UL}_{\frac{1}{2}}^{\text{fast}}^{(1500)} \right)$$

$$= \min (0.4, 0.7)$$

$$= 0.4$$

$$\text{ULR}_{\frac{1}{2}}^{(30, 2000)} = \min \left(\text{UL}_{\frac{1}{2}}^{\text{hot}}^{(30)}, \text{UL}_{\frac{1}{2}}^{\text{fast}}^{(2000)} \right)$$

$$= \min (0.4, 1)$$

$$= 0.4$$

similarly for all, then we get,

		500	1000	1500	2000
R ₁ = 30		0.3	0.4	0.4	0.4
35		0.3	0.5	0.6	0.6
40		0.3	0.5	0.7	0.8
45		0.3	0.5	0.7	0.9

* Approximate Reasoning:

Let,

$$R : A \times B$$

where, $x \in A$

$$y \in B$$

using, $x \in R$ we can estimate y i.e.

$$B : A \circ R$$

where, $y \in B : x \in A \circ R$

* Defuzzification:

- It is a process of selecting a representative element from fuzzy output.
- Defuzzification to crisp set: \rightarrow Relations \rightarrow Sets \rightarrow α -cut
- Defuzzification to scalar
 - \rightarrow Max-membership method
 - \rightarrow Mean Max Membership method
 - \rightarrow Weighted Average method
 - \rightarrow Centroid method
 - ⋮
 - ⋮

① Max-membership method (Height method):

- For $A = \{ \frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \dots, \frac{\mu_A(x_n)}{x_n} \}$
- Defuzzification (A) = x_i such that $\mu_A^{(x_i)} > \mu_A^{(x_j)}$
 $\forall x_i, x_j \in A, \text{ if } x_i \neq x_j$
- Used in discrete fuzzy sets, triangular fuzzy sets,

e.g.: $A = \{ \frac{0.5}{1}, \frac{0.6}{4}, \frac{0.8}{3}, \frac{0.4}{10}, \frac{0.3}{11} \}$

Defuzz (A) = 9

Mean-Max Membership (Middle of Maxima):

For a fuzzy set \tilde{A} , defuzz $(\tilde{A}) = z^* = \frac{a+b}{2}$,
where,

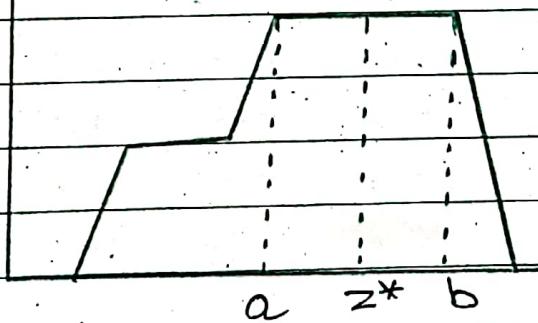
$$\text{ll}_{\tilde{A}}^{(a)} = \text{ll}_{\tilde{A}}^{(b)} \text{ and}$$

$$\text{ll}_{\tilde{A}}^{(a)} = \text{ll}_{\tilde{A}}^{(b)} > \text{ll}_{\tilde{A}}^{(x_i)} \quad \forall x_i \in A$$

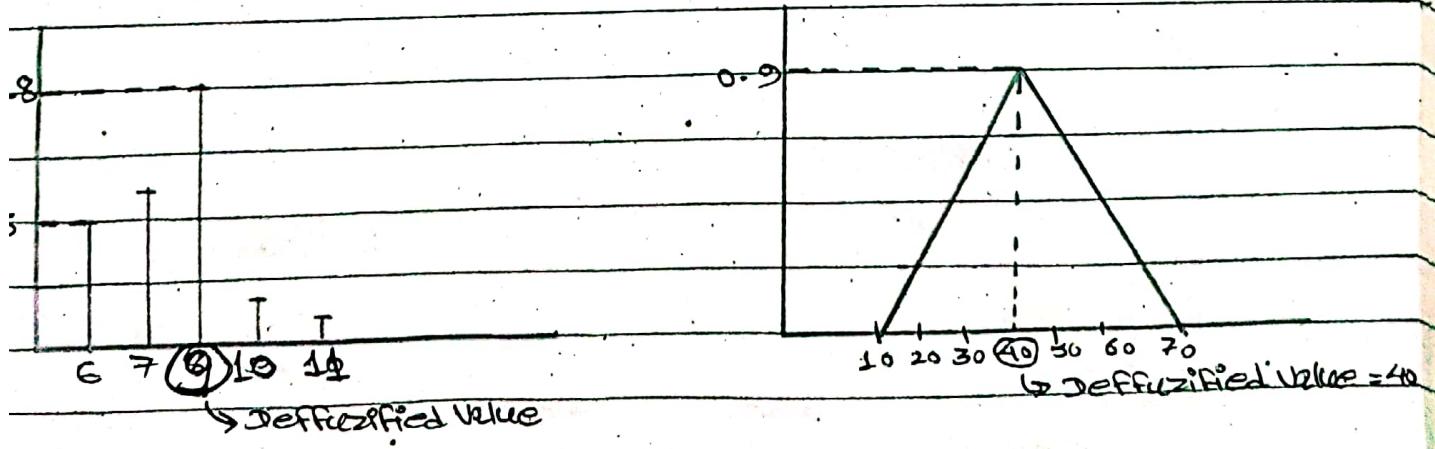
and a is first among x having max-membership
& b is last among x having max-membership.

e.g:- $\tilde{A} = \left\{ \frac{0.1}{1}, \frac{0.6}{5}, \frac{0.7}{7}, \frac{0.9}{8}, \frac{0.9}{9}, \frac{0.9}{10}, \frac{0.9}{12}, \frac{0.6}{13}, \frac{0.2}{14} \right\}$

$$\text{defuz } (\tilde{A}) = \frac{8+12}{2} = 10$$



$$z^* = \frac{a+b}{2}$$



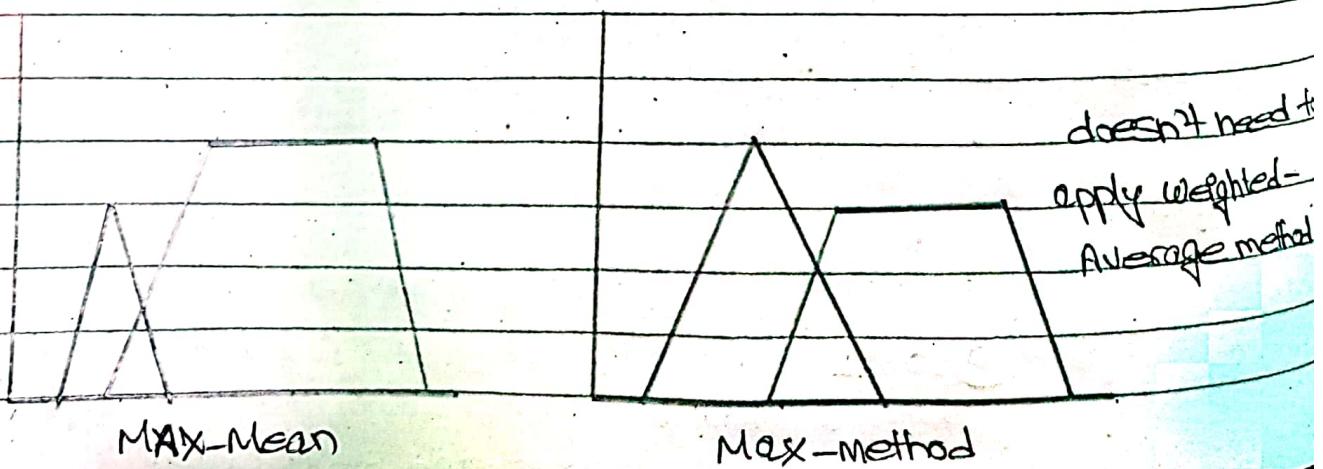
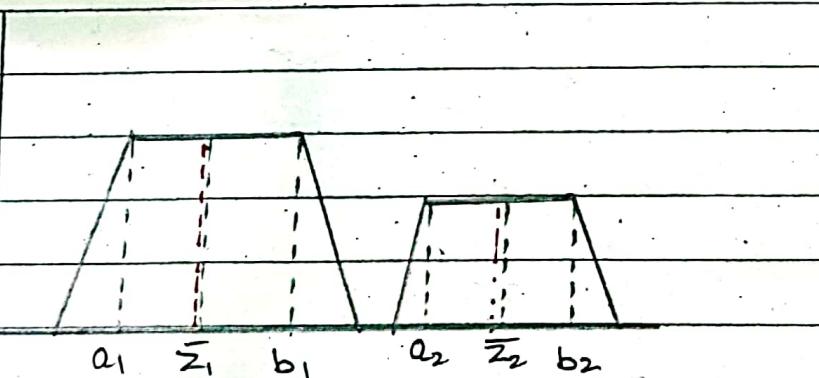
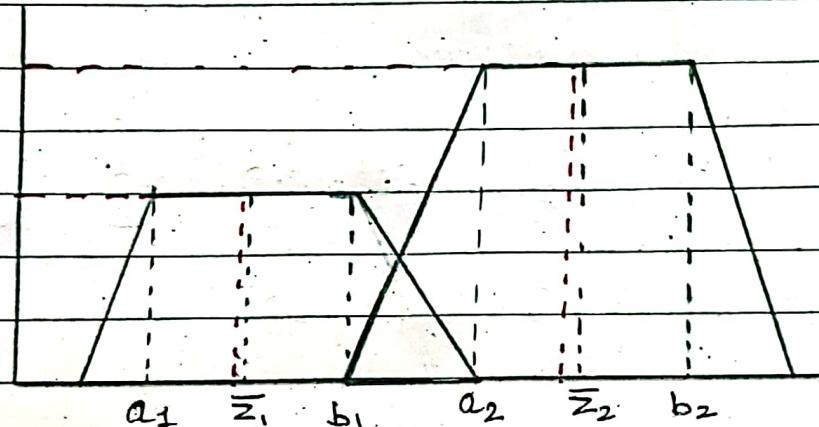
For Max-membership method

③ Weighted Average method:

$$z^* = \frac{\sum w_i(z) \cdot z}{\sum w_i(z)}$$

where,

\bar{z} is centroid or average of symmetric membership functions.



A) Centroid Method:

- Used in Continuous fuzzy sets.

$$z^* = \Sigma \int u(z) \cdot z dz$$

$$\int u(z) dz$$

where, Σ = is algebraic sum

\int = is algebraic integration

for a fuzzy set A defined by:

$$u(x) = \begin{cases} 0.35x & 0 \leq x < 2 \\ 0.7 & 2 \leq x < 2.7 \\ x-2 & 2.7 \leq x < 3 \\ 1 & 3 \leq x \leq 4 \end{cases}$$

Then,

$$z^* = \int_0^2 (0.35x) dx + \int_2^{2.7} (0.7)x dx + \int_{2.7}^3 (x-2)x dx + \int_3^4 (1)x dx$$

$$\int_0^2 (0.35x) dx + \int_2^{2.7} (0.7)x dx + \int_{2.7}^3 (x-2)x dx + \int_3^4 (1)x dx$$

=

* Example of Approximate Reasoning:

Given a rule R, defined as:

R : IF A then B

Now, Suppose we introduce an antecedent A' for a rule, i.e.,

IF A' then B'

Then, the information derived from the rule R, it's possible to derive the consequent in the second rule, i.e., B' by:

$$B' = A' \circ R$$

Example:

R₁: IF x is A then y is B.

where,

$$A = \{ \frac{0.2}{1}, \frac{0.5}{2}, \frac{0.7}{3} \}$$

$$\text{and } B = \{ \frac{0.6}{5}, \frac{0.8}{7}, \frac{0.4}{9} \}$$

now, we can elaborate B' for following rule as:

R₂: IF x is A' then y is B'

where,

$$A' = \{ \frac{0.5}{1}, \frac{0.9}{2}, \frac{0.3}{3} \}$$

Also, the rule R₁ is defined using Mamdani principle
i.e.

$$M(R) = \min_{(x,y)} \left(M_A^{(x)}, M_B^{(y)} \right)$$

	5	7	9
1	0.2	0.2	0.2
2	0.5	0.5	0.4
3	0.6	0.7	0.4

$$\text{ul}^{(1,5)} = \min \left(\text{ul}_A^{(1)}, \text{ul}_B^{(5)} \right)$$

$$= \min (0.2, 0.6)$$

$$= 0.2$$

$$\text{ul}^{(1,7)} = \min \left(\text{ul}_A^{(1)}, \text{ul}_B^{(7)} \right)$$

$$= \min (0.2, 0.8)$$

$$= 0.2$$

$$\text{ul}^{(1,9)} = \min \left(\text{ul}_A^{(1)}, \text{ul}_B^{(9)} \right)$$

$$= \min (0.2, 0.4)$$

$$= 0.2$$

$$\text{ul}^{(2,5)} = \min \left(\text{ul}_A^{(2)}, \text{ul}_B^{(5)} \right)$$

$$= \min (0.5, 0.6)$$

$$= 0.5$$

$$\text{ul}^{(2,7)} = \min \left(\text{ul}_A^{(2)}, \text{ul}_B^{(7)} \right)$$

$$= \min (0.5, 0.8)$$

$$= 0.5$$

$$\text{ul}^{(2,9)} = \min \left(\text{ul}_A^{(2)}, \text{ul}_B^{(9)} \right)$$

$$= \min (0.5, 0.4)$$

$$= 0.4$$

$$\begin{aligned} u_{ll}^{(3,5)} &= \min(u_{llA}^{(3)}, u_{llB}^{(5)}) \\ &= \min(0.7, 0.6) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} u_{ll}^{(3,7)} &= \min(u_{llA}^{(3)}, u_{llB}^{(7)}) \\ &= \min(0.7, 0.8) \\ &= 0.7 \end{aligned}$$

$$\begin{aligned} u_{ll}^{(3,9)} &= \min(u_{llA}^{(3)}, u_{llB}^{(9)}) \\ &= \min(0.7, 0.4) \\ &= 0.4 \end{aligned}$$

So, we have,

$$\begin{aligned} A' &= \begin{bmatrix} 0.5 & 0.9 & 0.3 \\ 1 & 2 & 3 \end{bmatrix} \\ R &= \begin{bmatrix} 5 & 7 & 9 \\ 1 & [0.2 \cdot 0.2 & 0.2] \\ 2 & 0.5 & 0.5 & 0.4 \\ 3 & 0.6 & 0.7 & 0.4 \end{bmatrix} \end{aligned}$$

Then,

$$B' = A' \circ R : \in \{5, 7, 9\}$$

For this example we are taking Max-MIN composition.

$$\begin{aligned} u_{llB}^{(5)} &= \text{Max} [\min(u_{llA'}^{(1)}, u_{llR}^{(1,5)}), \min(u_{llA'}^{(2)}, u_{llR}^{(2,5)})] \\ &\quad \min(u_{llA'}^{(3)}, u_{llR}^{(3,5)}) \\ &= \text{Max} [\min(0.5, 0.2), \min(0.9, 0.5), \min(0.3, 0.6)] \end{aligned}$$

$$= \max(0.2, 0.5, 0.3)$$

E 0-5

$$\underline{uL}_B^{(7)} = \max \left[\min \left(\underline{uL}_A^{(1)}, \underline{uL}_R^{(1,7)} \right), \min \left(\underline{uL}_A^{(2)}, \underline{uL}_R^{(2,7)} \right), \min \left(\underline{uL}_A^{(3)}, \underline{uL}_R^{(3,7)} \right) \right]$$

$$= \max [\min(0.5, 0.2), \min(0.9, 0.5), \min(0.3, 0.7)]$$

$$= \text{Max}(0.2, 0.5, 0.3)$$

$$= 0.5$$

$$M_{S'}^{(0)} = \max \left[\min \left(M_A^{(1)}, M_R^{(1,0)} \right), \min \left(M_A^{(2)}, M_R^{(2,0)} \right), \min \left(M_A^{(3)}, M_R^{(3,0)} \right) \right]$$

$$= \max [\min(0.5, 0.2), \min(0.9, 0.4), \min(0.3, 0.4)]$$

$$= \max(0.2, 0.4, 0.3)$$

- 0-4 -

$$\therefore B' = \left\{ \frac{0.5}{5}, \frac{0.5}{7}, \frac{0.4}{9} \right\}$$

Approximate Reasoning for multiple roles:

Rule 1: If height is tall then speed is high.

Rule 1: If height is tall then speed is high
Rule 2: If height is Medium then speed is Moderate.

The fuzzy sets for height (in foot) & speed in (mls)
are:

$$H_1 = \text{Tall} = \{ \frac{0.5}{5}, \frac{0.8}{6}, \frac{1}{7} \}$$

$$H_2 = \text{Medium} = \{ \frac{0.6}{5}, \frac{0.7}{6}, \frac{0.6}{7} \}$$

$$S_1 = \text{High} = \{ \frac{0.4}{5}, \frac{0.7}{6}, \frac{0.9}{7} \}$$

$$S_2 = \text{Moderate} = \{ \frac{0.6}{5}, \frac{0.8}{6}, \frac{0.7}{7} \}$$

We have given, $H' = \text{Above Average}$

$$\text{i.e. } H' = \{ \frac{0.5}{5}, \frac{0.9}{6}, \frac{0.8}{7} \}$$

find $S' = \text{Above Average}$.

Let say,

= Sol

Evaluate R_1 & R_2 using Mamdani Implication Rule

i.e.,

$$U_I^{(R)} = \min \left(U_A^{(x)}, U_B^{(y)} \right)_{(x,y)}$$

We get,

$$R_1 = 5 \begin{bmatrix} 5 & 7 & 9 \\ 0.4 & 0.5 & 0.5 \\ 6 & 0.4 & 0.7 & 0.8 \\ 7 & 0.4 & 0.7 & 0.9 \end{bmatrix} \quad \begin{array}{l} (\text{Taking } H_1 \text{ & } S_1) \\ (\text{similarly as before qns}) \end{array}$$

$$R_2 = 5 \begin{bmatrix} 5 & 7 & 9 \\ 0.6 & 0.6 & 0.6 \\ 6 & 0.6 & 0.7 & 0.7 \\ 7 & 0.6 & 0.6 & 0.6 \end{bmatrix} \quad (\text{Taking } H_2 \text{ & } S_2)$$

now,

$$\begin{aligned}
 S' &= (H' \circ R_1) \cup (H' \circ R_2) \quad [\because \text{if two relation} \\
 &= H' \circ (R_1 \cup R_2) \quad \text{is present we use} \\
 &\quad (\cup) \text{ Union as aggregation}]
 \end{aligned}$$

So,

$$R_1 \cup R_2 = \left[\begin{array}{lll} \max(0.4, 0.6), \max(0.5, 0.6), \max(0.5, 0.6) \\ \max(0.4, 0.6), \max(0.7, 0.7), \max(0.8, 0.7) \\ \max(0.4, 0.6), \max(0.7, 0.6), \max(0.9, 0.6) \end{array} \right]$$

$$\text{let say, } R_3 = \left[\begin{array}{ccc} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.9 \end{array} \right]$$

now,

$$\begin{aligned}
 S' &= H' \circ R_3 \\
 &= 5 \begin{bmatrix} 0.5 \\ 0.3 \\ 0.8 \end{bmatrix} \circ 5 \begin{bmatrix} 5 & 7 & 9 \\ 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.9 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 U_{S'}^{(5)} &= \max[\min(U_U^{(5)}, U_U^{(5,5)}), \min(U_U^{(6)}, U_U^{(6,5)}), \min(U_U^{(7)}, U_U^{(7,5)})] \\
 &= \max[\min(0.5, 0.6), \min(0.3, 0.6), \min(0.8, 0.6)] \\
 &= \max[0.5, 0.6, 0.6] \\
 &= 0.6
 \end{aligned}$$

$$\begin{aligned}
 U_{S'}^{(7)} &= \max[\min(U_U^{(5)}, U_U^{(5,7)}), \min(U_U^{(6)}, U_U^{(6,7)}), \min(U_U^{(7)}, U_U^{(7,7)})] \\
 &= \max[\min(0.5, 0.6), \min(0.3, 0.7), \min(0.8, 0.7)] \\
 &= \max[0.5, 0.7, 0.7]
 \end{aligned}$$

$$\begin{aligned}
 U_{f_1}^{(5)} &= \max [\min(U_1^{(5)}, U_2^{(5)}), \min(U_1^{(6)}, U_2^{(6)}), \min(U_1^{(7)}, U_2^{(7)})] \\
 &= \max [\min(0.5, 0.6), \min(0.9, 0.8), \min(0.8, 0.9)] \\
 &= \max [0.5, 0.8, 0.8] \\
 &= 0.8
 \end{aligned}$$

$$S' = \left\{ \frac{0.6}{5}, \frac{0.7}{7}, \frac{0.8}{9} \right\}$$

* Fuzzy (Rule-Based) Systems:

* Fuzzy Inference:

- Fuzzy inference (Reasoning) is the actual process of mapping from a given input to an output using fuzzy logic.
- Fuzzy inference system is also known as fuzzy rule based system or Fuzzy System.
- The Block diagram of fuzzy Inference system is:

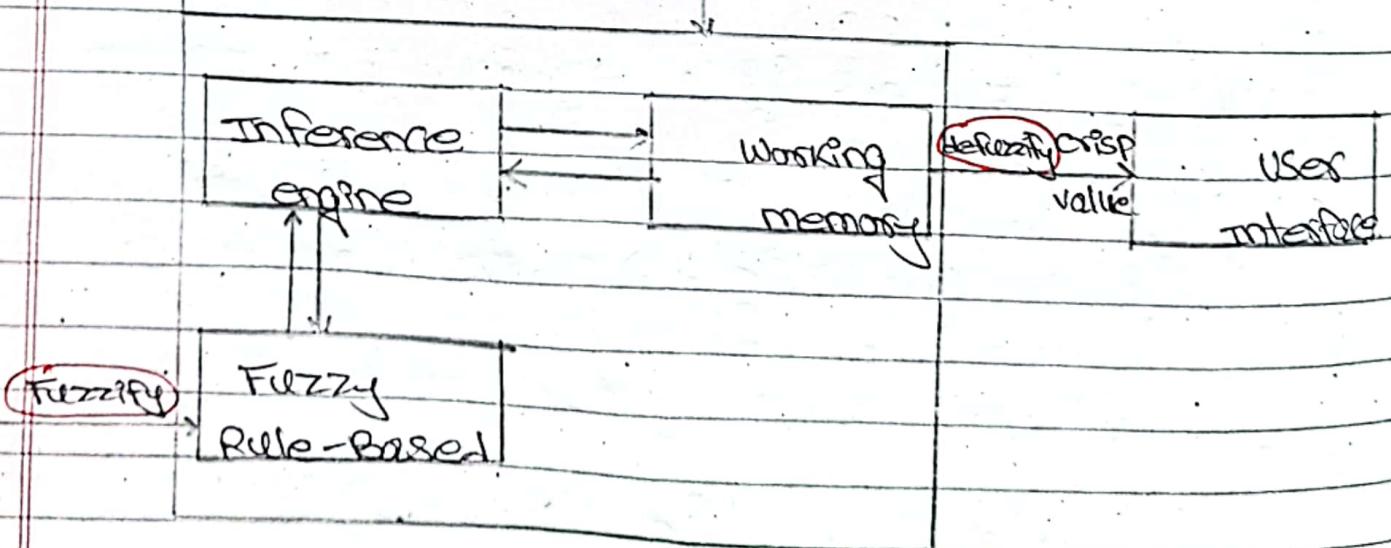


Fig:- Block diagram of fuzzy Inference System

- In the field of artificial intelligence (Machine Intelligence) there are various ways to represent knowledge - Perhaps the most common way to represent human knowledge is in the form of natural language Expressions of the type

IF Premise (antecedent), THEN Conclusion (consequent)

where, it is commonly referred as the IF-THEN rule-based form, which is generally referred to as the deductive Form.

- It typically expresses an inference such that if we know a fact (premise, hypothesis, antecedent) then we can infer, or derive another fact called a conclusion (consequent).
 - The fuzzy-rule based system is most useful in modeling some complex systems that can be observed by humans because they make use of linguistic variables as their antecedents & consequents as described here these linguistic variables can be naturally represented by fuzzy sets.

Rule 1: IF Condition C¹; THEN restriction R¹

Rule 2 : IF condition C², THEN restriction R².

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Rule r: IF Condition C^r, THEN restriction R^r.

Table 8: The Canonical Form for a fuzzy relational system.

* Multiple conjunctive antecedents:

IF x is \tilde{A}_1^1 and \tilde{A}_2^2 ... and \tilde{A}_L^L THEN y is \tilde{B}_S^S .

Assuming a new fuzzy subset \tilde{A}^S as:

$$\tilde{A}^S = \tilde{A}_1^1 \cap \tilde{A}_2^2 \cap \dots \cap \tilde{A}_L^L$$

expressed by means of membership function.

$$\mu_{\tilde{A}^S}(x) = \min [\mu_{\tilde{A}_1^1}(x), \mu_{\tilde{A}_2^2}(x), \dots, \mu_{\tilde{A}_L^L}(x)]$$

Based on the definition of the standard fuzzy intersection operation, the compound rule may be rewritten as

IF \tilde{A}^S THEN \tilde{B}^S

* Multiple disjunctive antecedents:

IF x is \tilde{A}_1^1 OR x is \tilde{A}_2^2 ... OR x is \tilde{A}_L^L THEN y is \tilde{B}_S^S

Could be rewritten as

IF x is \tilde{A}^S THEN y is \tilde{B}^S ,

where the fuzzy set \tilde{A}^S is defined as

$$\tilde{A}^S = \tilde{A}_1^1 \cup \tilde{A}_2^2 \cup \dots \cup \tilde{A}_L^L,$$

$$M_{A_2}(x) = \max [M_{A_2^1}(x), M_{A_2^2}(x), \dots, M_{A_2^L}(x)].$$

* Aggregation of Fuzzy rules:

- Most rule based system involve more than one rule.
- The process of obtaining the overall consequent (conclusion) from the individual consequents contributed by each rule in the rule-based is known as aggregation of rules.

① Conjunctive system of rules:

In case of a system of rules that must be jointly satisfied, the rules are connected by "and" connectives. In this case, the aggregated output (consequent) y , is found by the fuzzy intersection of all individual rule consequents y_i , where $i = 1, 2, \dots, r$ as,

$$y = y^1 \text{ and } y^2 \text{ and } \dots \text{ and } y^r$$

or

$$y = y^1 \cap y^2 \cap \dots \cap y^r,$$

which is defined by the membership function as:

$$M_y(y) = \min (M_{y^1}(y), M_{y^2}(y), \dots, M_{y^r}(y)), \text{ for } y \in Y$$

② Disjunctive system of rules:

For the case of a disjunctive system of rules, the satisfaction of at least one rule is required, the rules are connected by the "or" connectives.

In this case, the aggregated output is found by the fuzzy union of all individual rule contributions.

$$y = y^1 \text{ or } y^2 \text{ or } \dots \text{ or } y^n$$

or,

$$y = y^1 \cup y^2 \cup \dots \cup y^n$$

which is defined by the membership function as

$$u_{\{y\}}(y) = \max(u_{y^1}(y), u_{y^2}(y), \dots, u_{y^n}(y))$$

for $y \in Y$.

* Common Techniques of deductive inference for fuzzy systems using Graphical method:

The Most Common methods of deductive inference for fuzzy systems based on linguistic rules are:

- i) Mamdani Systems,
- ii) Sugeno models, (TSK)
- iii) Tsukamoto (TSK) Models.

1) Mamdani Inference Method:

→ Two input mamdani methods:

i) Max-MIN inference method, (clipping)

ii) Max-product inference method. (Scaling)

Note: clipping preserves the membership structure but scaling preserves

→ To begin the general illustration of this idea, we consider a simple two-rule system where each rule comprises two antecedents and one consequent which is analogous to a dual-input & single-output fuzzy systems.

→ A fuzzy system with two non-interactive inputs x_1 & x_2 (antecedents) and a single output y (consequent) in mamdani form:

Multiple Consecutive Antecedents
(MIN)

IF x_1 is \tilde{A}_1^k and x_2 is \tilde{A}_2^k THEN y^k is \tilde{B}^k

Multiple Disjunctive Antecedents
(Max) For $k = 1, 2, \dots, n$.

IF x_1 is \tilde{A}_1^k or x_2 is \tilde{A}_2^k THEN y^k is \tilde{B}^k

Where, \tilde{A}_1^k and \tilde{A}_2^k are the fuzzy sets representing the k^{th} antecedent pairs & \tilde{B}^k is the fuzzy set representing the k^{th} consequent.

→ In the following presentation, we consider two different cases of two-input Mamdani systems:

i) The inputs to the system are scalar values, and we use a max-min inference method.

- 2) The inputs to the system are scalar values, and we use a max-product inference method.

① Case 1: Max-NIN Inference Method (Scaling):

If x_1 & x_2 are crisp values, that is, delta functions
The rule-based System is described in Equation.

IF x_1 is A_1^k and x_2 is A_2^k THEN y^k is B_k

So, Membership for the inputs x_1 & x_2 will be described as:

$$\mu(x_1) = \delta(x_1 - \text{input}(i)) = \begin{cases} 1, & x_1 = \text{input}(i), \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu(x_2) = \delta(x_2 - \text{input}(j)) = \begin{cases} 1, & x_2 = \text{input}(j); \\ 0, & \text{otherwise.} \end{cases}$$

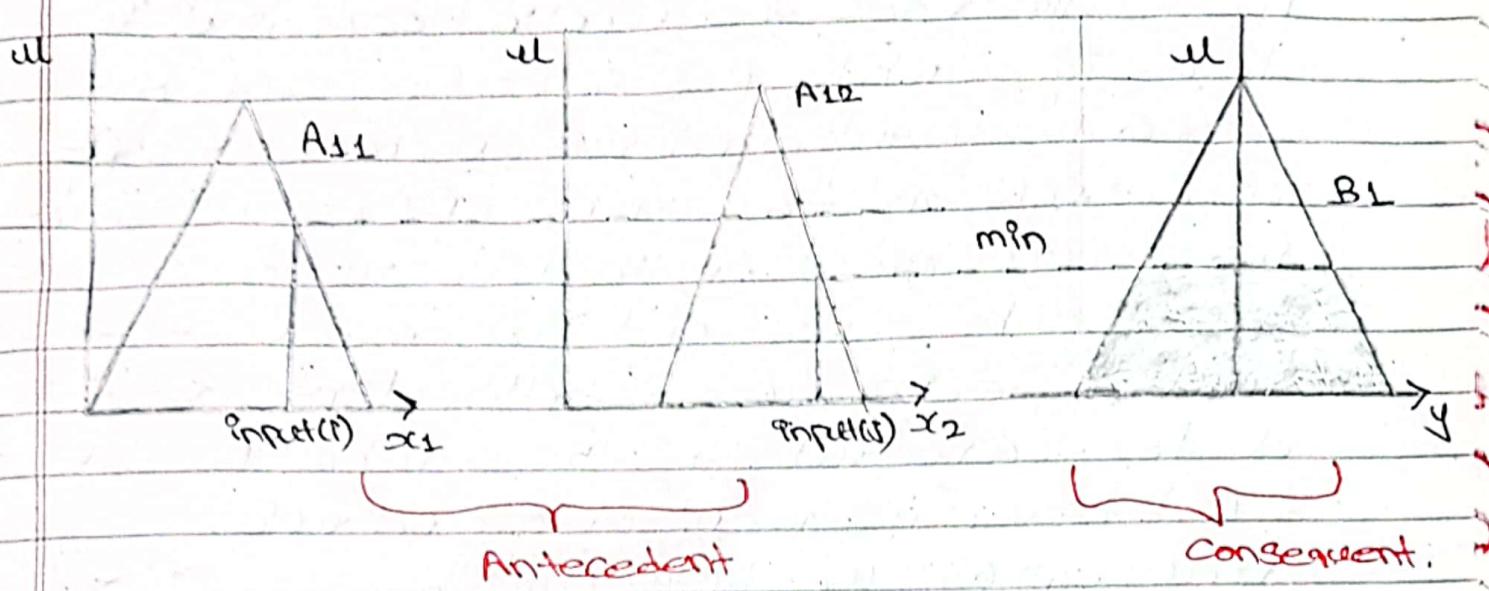
but,

Based on the Namdani implication method of inference for a set of disjunctive (max) rule, the aggregated output for the n rules will be given as:

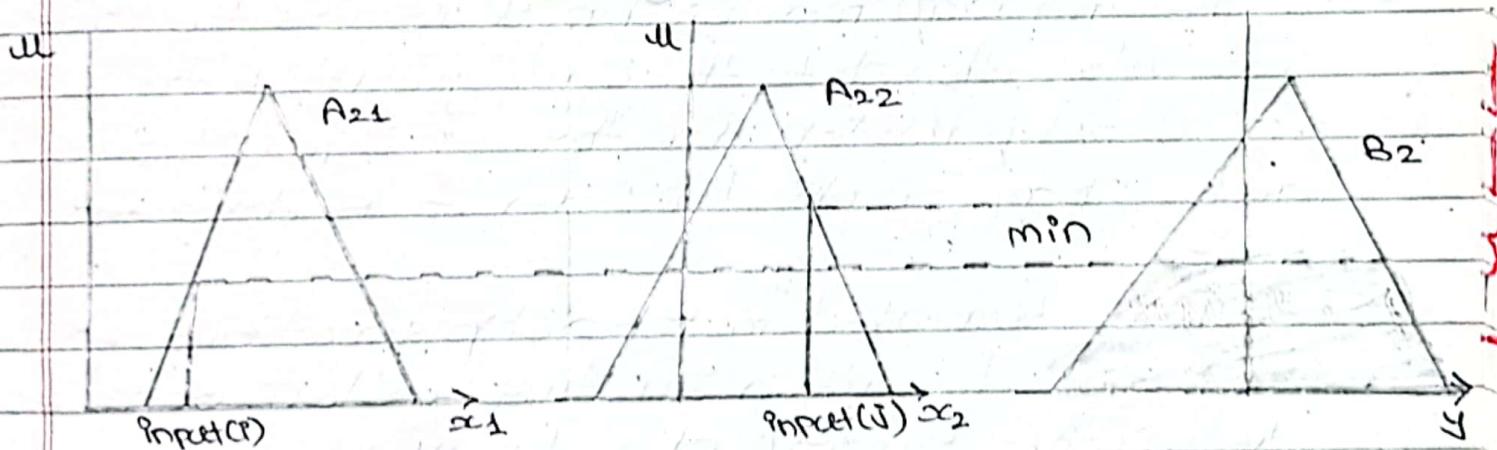
$$\mu_{B^k}^{(i)} = \max_k [\min [\mu_{A_1^k}(\text{input}(i)), \mu_{A_2^k}(\text{input}(i))]],$$

$$k = 1, 2, \dots, n.$$

Rule 1:



Rule 2:



Max (Rule 1 & Rule 2)

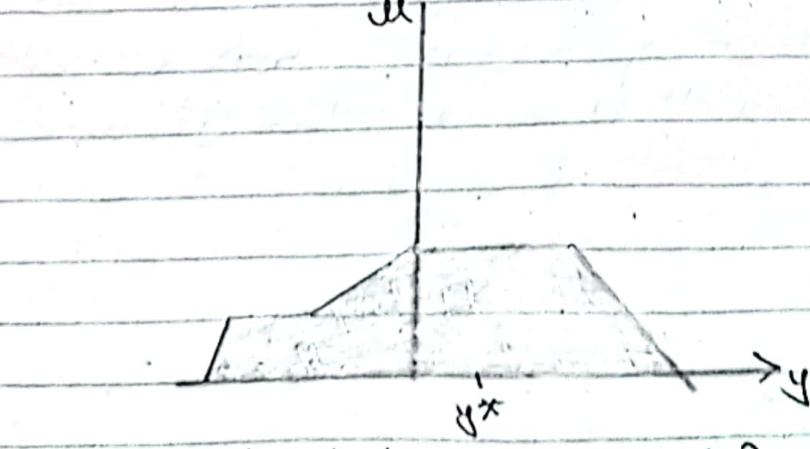


Fig:- Graphical Mamdani (Max-Min) Inference method.

- Here, the above figure has a very simple interpretation. The figure illustrates the graphical analysis of two rules, where the symbols A_{11} & A_{12} refers to first and second fuzzy antecedents of the first rule, respectively and the symbol B_1 refers to the fuzzy consequent of the first rule;
- The symbols A_{21} and A_{22} refers to the first & second fuzzy antecedents respectively of the second rule & the symbol B_2 refers to the fuzzy consequent of second rule.
- From eqn ①, we know that the general rule structure for this system are connected by logical "and" connective.
- The minimum membership value for the antecedent propagates through to the consequent & truncates the membership function for the consequent of each rule. Then, graphical inference is done for each rule.
- Then, truncated membership functions are aggregated using conjunctive system of rules or disjunctive system of rules. (MAX)
- In this example, the rules are disjunctive, so the aggregation operation is max results in aggregate membership function. If a value y^* would result which is a defuzzified value.

② Case ②: Max-Product Inference Method (Scaling):

If we were to use a Max-product implication technique for a set of disjunctive rule (Max), the aggregated output for the rules would be given as:

$$\text{ulB}^k(y) = \max_{\tilde{x}_1} [\text{ulA}_{\tilde{x}_1}^k(\text{Input}(y)) \cdot \text{ulA}_{\tilde{x}_2}^k(\text{Input}(y))],$$

$$k = 1, 2, \dots, n.$$

Examples:

In Mechanics the energy of a moving body is called kinetic energy. If an object of mass m (kilograms) is moving with a velocity v (meters per second), then kinetic energy K (in Joules) is given the equation

$K = \frac{1}{2} mv^2$. Suppose we model the mass & velocity as inputs to system (moving body) and the energy as output, then observe the system for a while & deduce the following two disjunctive rules of inference based on our observations:

MIN

Rule 1: IF x_1 is \tilde{A}_1^1 (small mass) & x_2 is \tilde{A}_2^1 (high velocity)

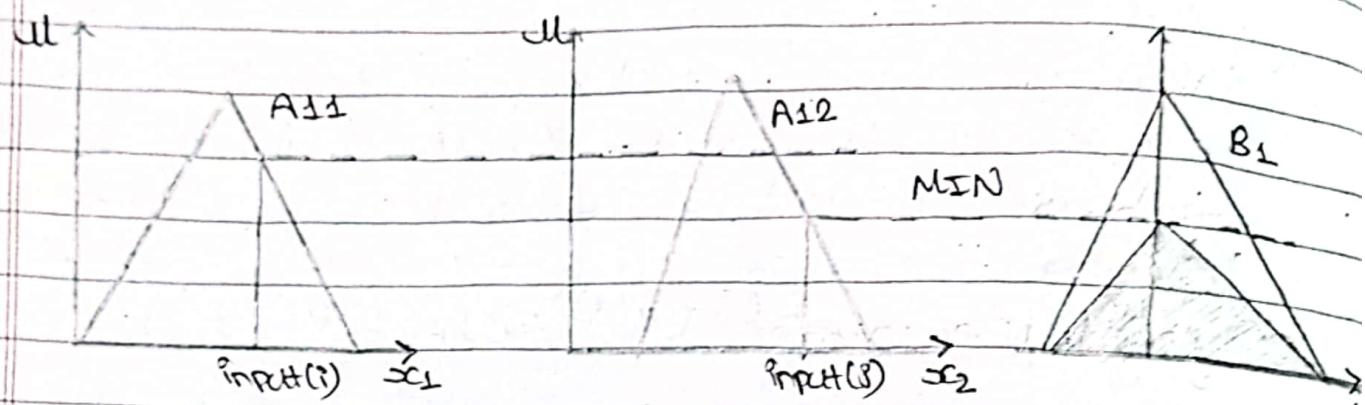
THEN y is \tilde{B}^1 (medium Energy).

MAX

Rule 2: IF x_1 is \tilde{A}_1^2 (large mass) and x_2 is \tilde{A}_2^1 (high velocity)

THEN y is \tilde{B}^2 (high Energy)

Rule 1:



Rule 2:

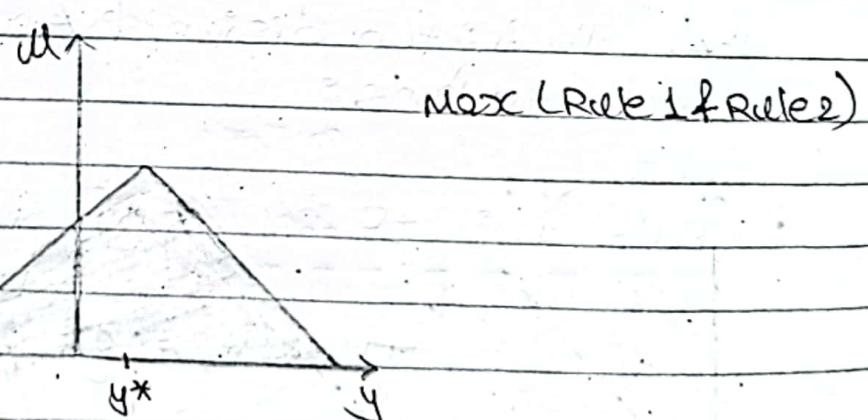
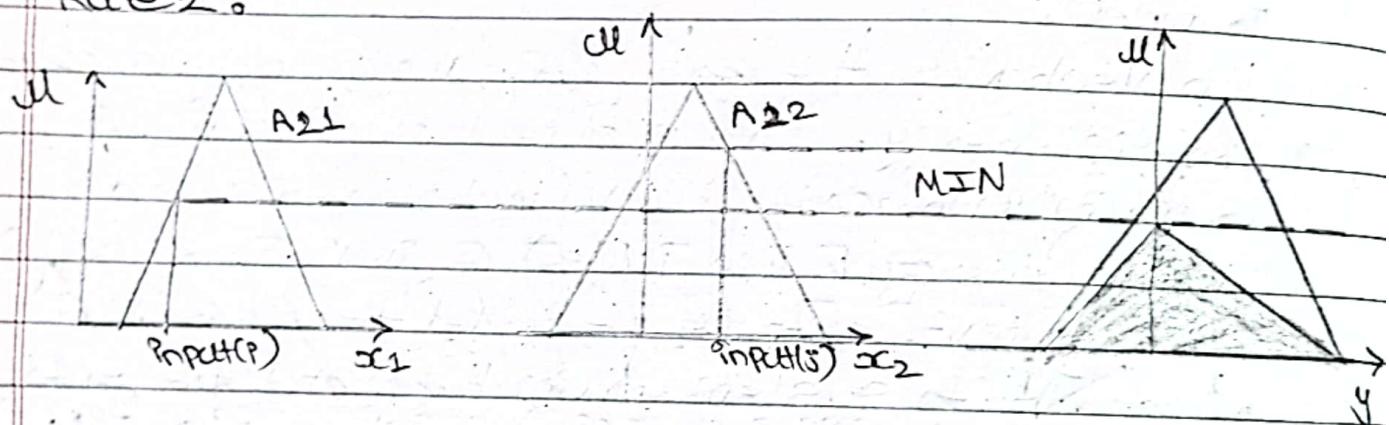


Fig:- Graphical Mamdani (Max-product) inference Method.

now, let's suppose we have made some observations of the system (moving body) & we estimate the values of two inputs, mass & velocity, as crisp values.

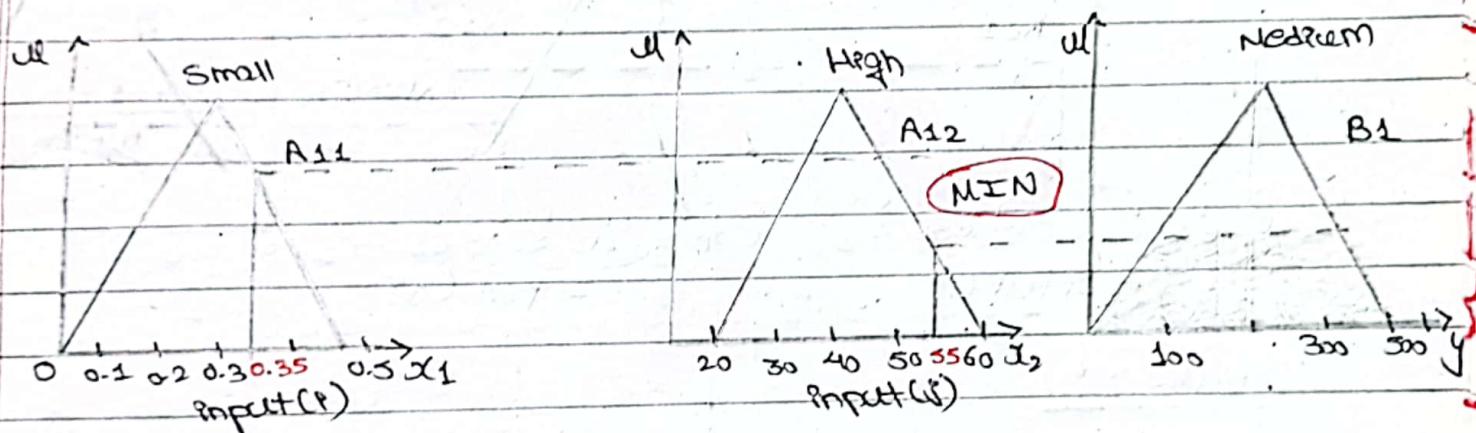
let input (i) = 0.35 kg (mass)

input (ii) = 55 m/s (velocity)

* Using case ① & ②:

→ Case ① models inputs as delta functions, fuses a Mamdani (Max-Min) implication inference model. It is illustrated below, where the output fuzzy membership function is defuzzified using a Centroid method.

Rule 1:



Rule 2:

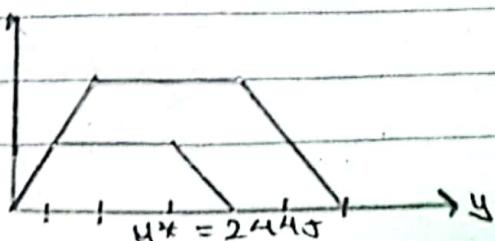
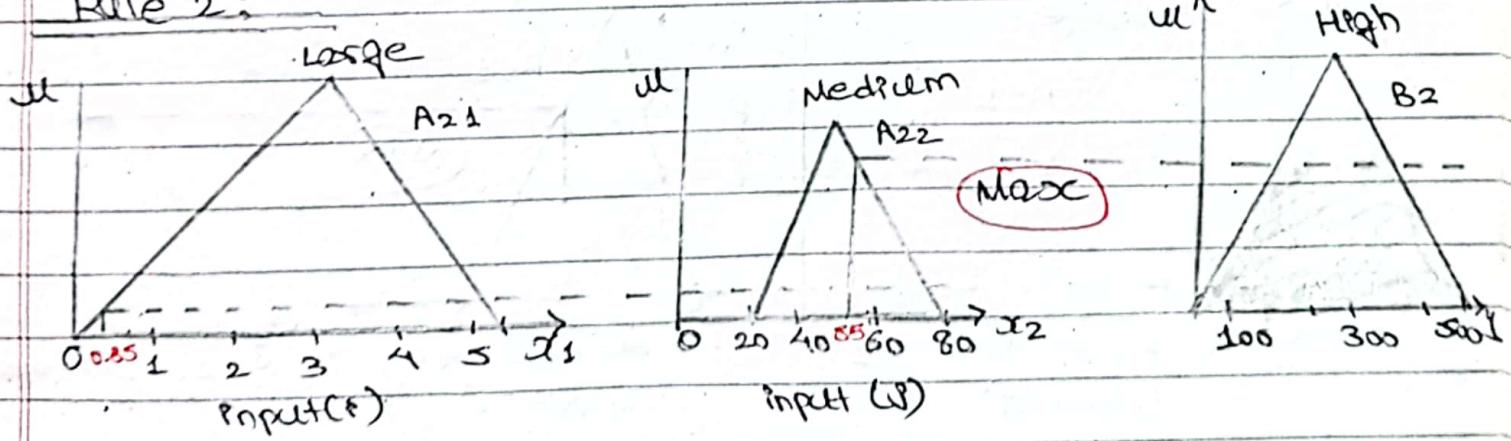
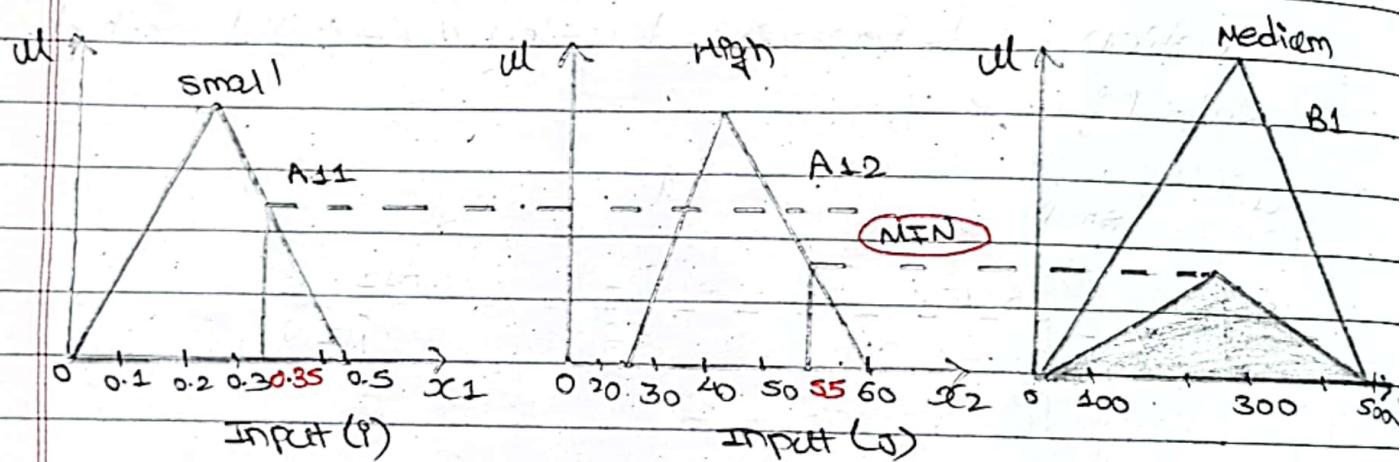


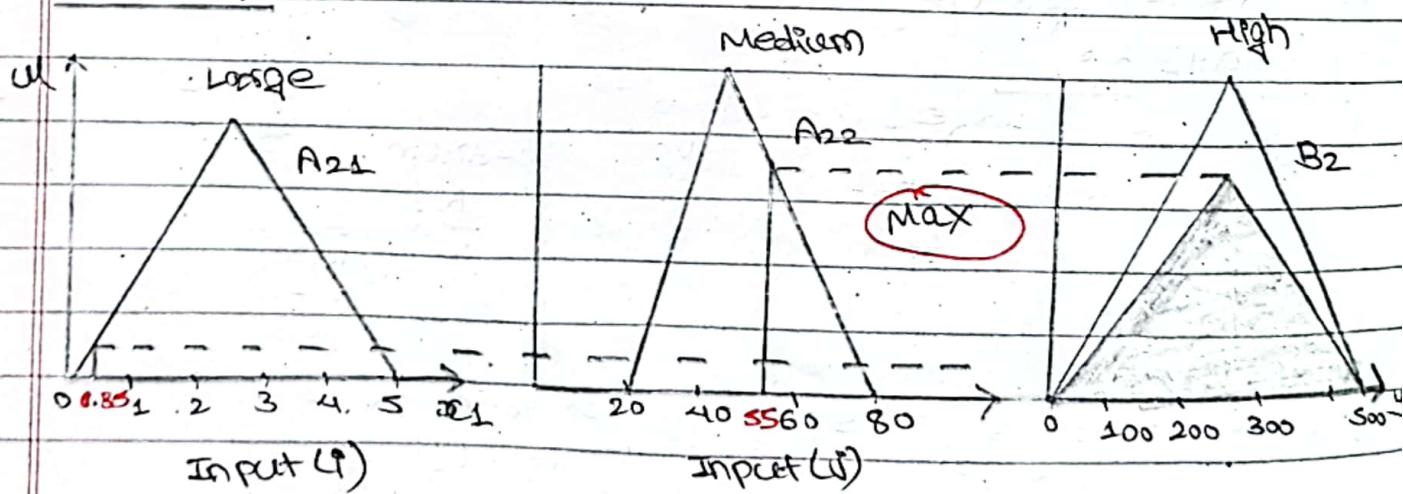
Fig: Fuzzy inference method using the case 1 graphic concept.

→ Case ②: We only change the method of implication from the first case. Now using a max-product implication method and its eqn and a centroidal defuzzification method, the graphical result can be illustrated below:

Rule 1:



Rule 2:



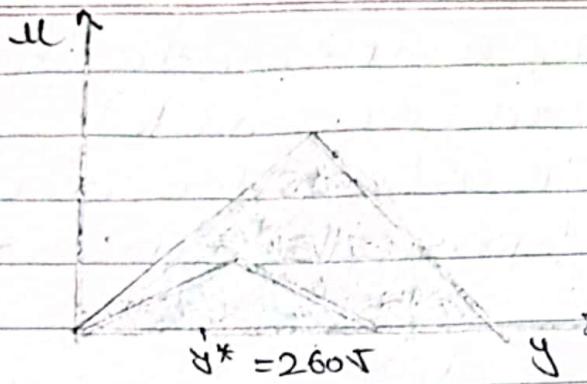


Fig:- Fuzzy inference method using case 2 graphical approach.

ii) Sugeno Method, or the TSK Method:

The defuzzified values that we seen in Mamdani inference method for the output energy are both fairly consistent, i.e 244 J & 260 J. The power of fuzzy rule-based system is their ability to yield "good" results with reasonably simple mathematical operations. [#] Also, we need a defuzzification method in mamdani method to get a defuzzified value but in Sugeno method (TSK), the defuzzification is done by the method itself.

→ The second inference method, generally referred to as the Sugeno method, or the TSK method (Takagi, Sugeno, and Kang) was proposed in an effort to develop a systematic approach to generating fuzzy rules from a given input-output dataset.

→ A typical rule in Sugeno method has two inputs x and y and output z , has the form

$$\text{IF } x \text{ is } A \text{ and } y \text{ is } B, \text{ THEN } z \text{ is } z = f(x,y)$$

where, $z = f(x,y)$ is a crisp function in the consequent.

- i) When $f(x,y)$ is a constant, the inference system is called a zero-order Sugeno model,
- ii) When $f(x,y)$ is a linear function of x and y , the inference system is called a first-order Sugeno model.

* Graphical Representation:

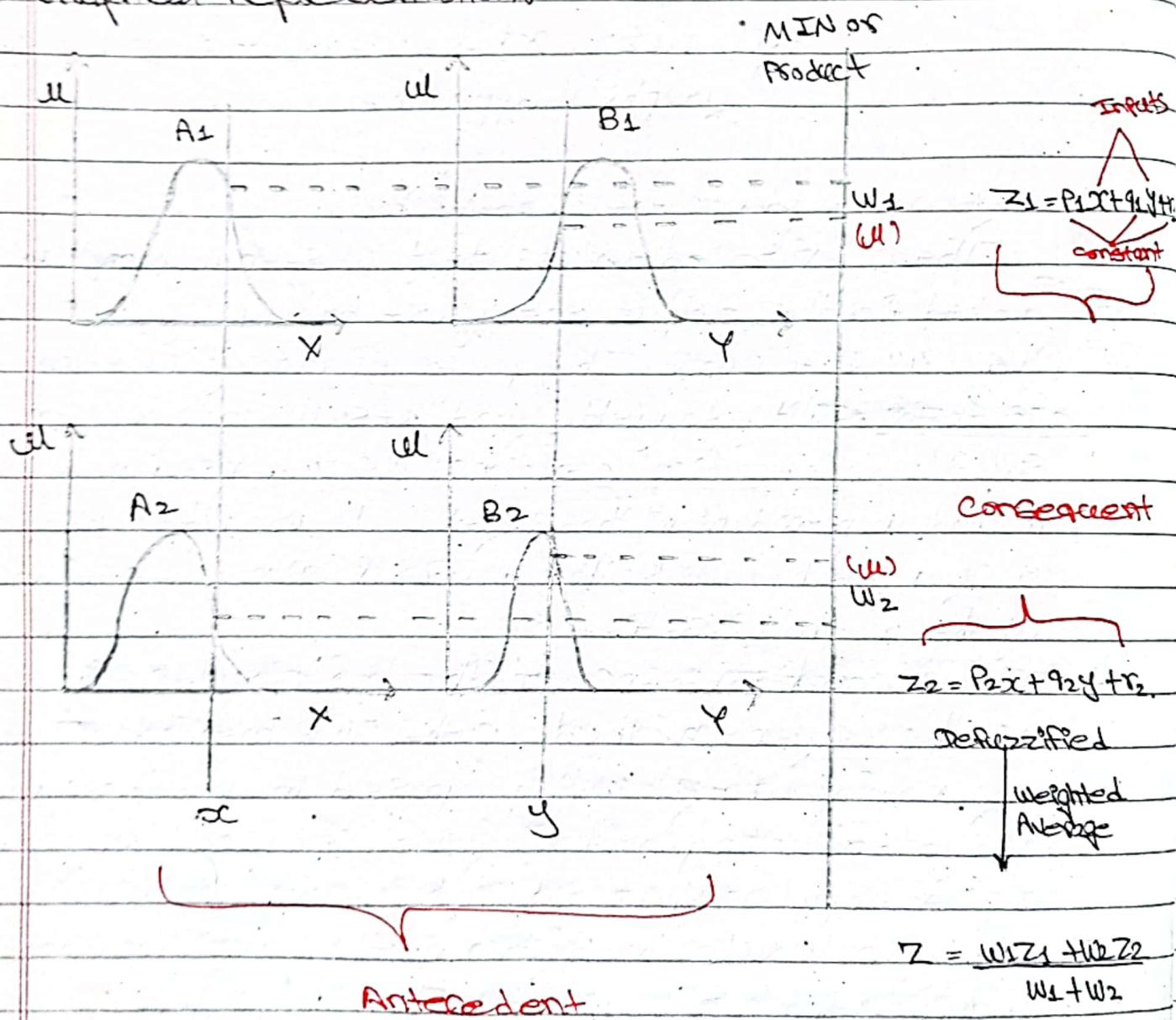


Fig:- The Sugeno fuzzy model.

→ In Sugeno model, each rule has a crisp output, given by a function because of this the overall output is obtained via a weighted Average defuzzification as shown in above figure.

This process avoids the time-consuming methods of defuzzification necessary in the Mamdani model.

Example: An example of a two-input, single-output Sugeno model with four rules are:

~~Pristorder~~

IF X is small and Y is small, THEN $Z = -x + y + 1$.

IF X is small and Y is large, THEN $Z = -y + 3$.

IF X is large and Y is small, THEN $Z = -x + 3$.

IF X is large and Y is large, THEN $Z = x + y + 2$.

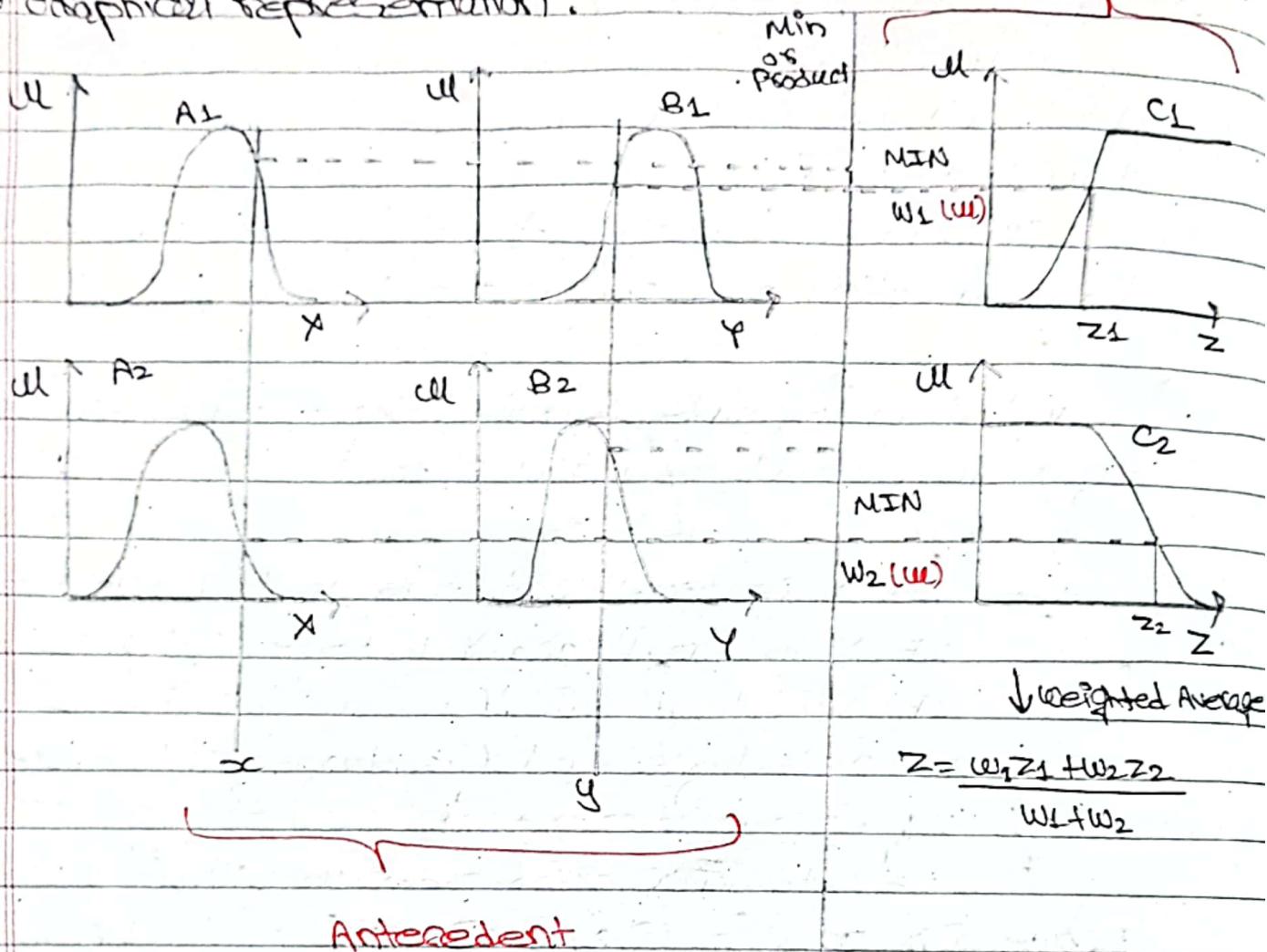
Note: For zero-order we will get a constant value for Z .

iii) Tsukamoto inference method:

- The third inference method is Tsukamoto method where, the consequent of each fuzzy rule is represented by a fuzzy set with a monotonic membership function (Bouider function).
- The output of each rule is defined as a crisp value induced by the membership value coming from the antecedent clause of the rule. The overall output is calculated by the weighted average of each rule's output.
- It also avoids the time-consuming process of defuzzification.

* Graphical representation:

consequent



- * Let's see an example from our Text book for all inference methods:
 - i) Mamdani method,
 - ii) Sugeno method &
 - iii) Tsukamoto method.

The given disjunctive rules of inferred based on the observation of models are:

Rule 1: IF w is \tilde{A}_1^1 (large flow rate) and ΔT_{app} is \tilde{A}_2^1 (small app)
 THEN AU is \tilde{B}^1 (large heat exchange).

Rule 2: IF w is \tilde{A}_1^2 (small flow rate) or ΔT_{app} is \tilde{A}_2^2 (large app)
 THEN AU is \tilde{B}^2 (small heat exchange).

Rule 3: IF w is A_1^1 (small flow rate) and ΔT_{app} is A_2^1 (small approach), THEN A_U is B^1 (large heat exchange). (MIN)

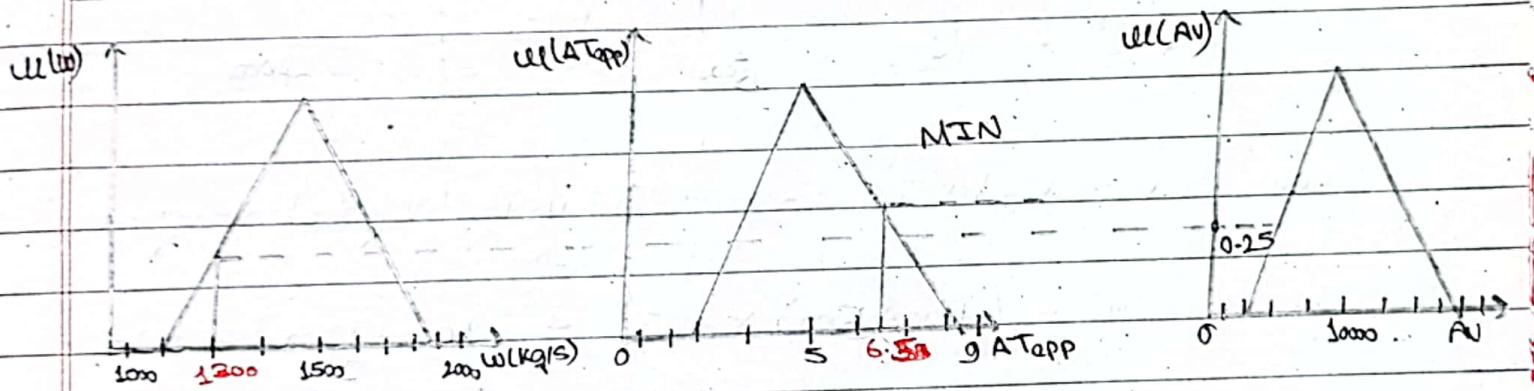
The two input values for flow rate & temperature are:

$$w = 1300 \text{ kg s}^{-1}$$

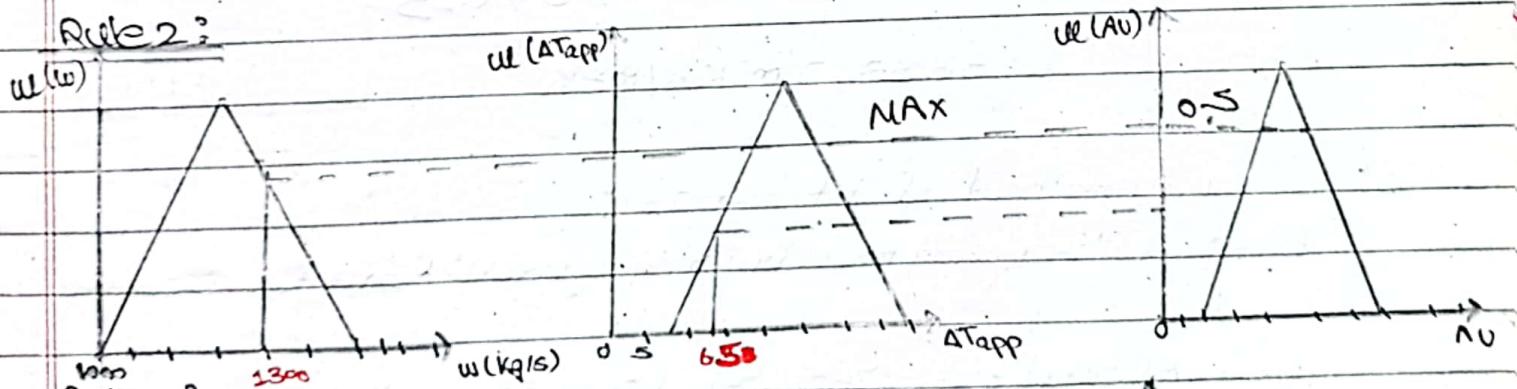
$$\Delta T_{app} = 6.5 \text{ K}$$

① Using Mamdani implication method of inference.
We are using MAX-MIN (clipping):

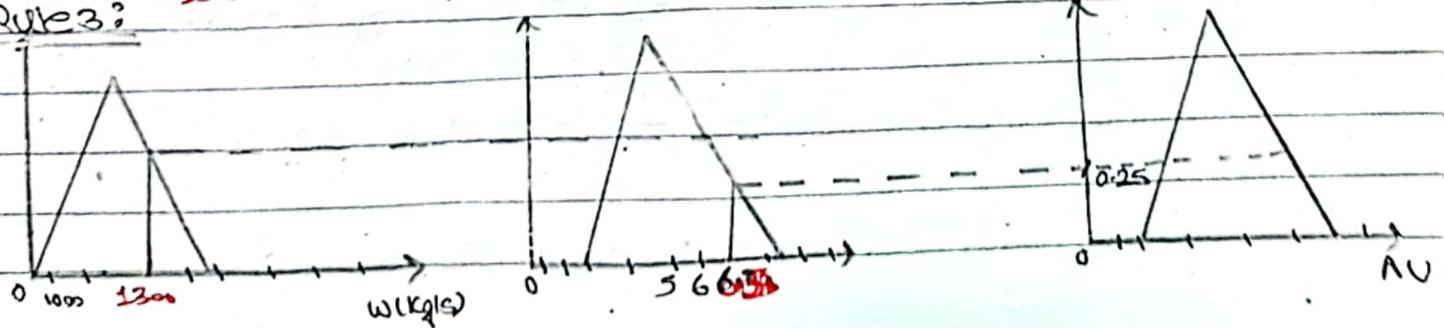
Rule 1:



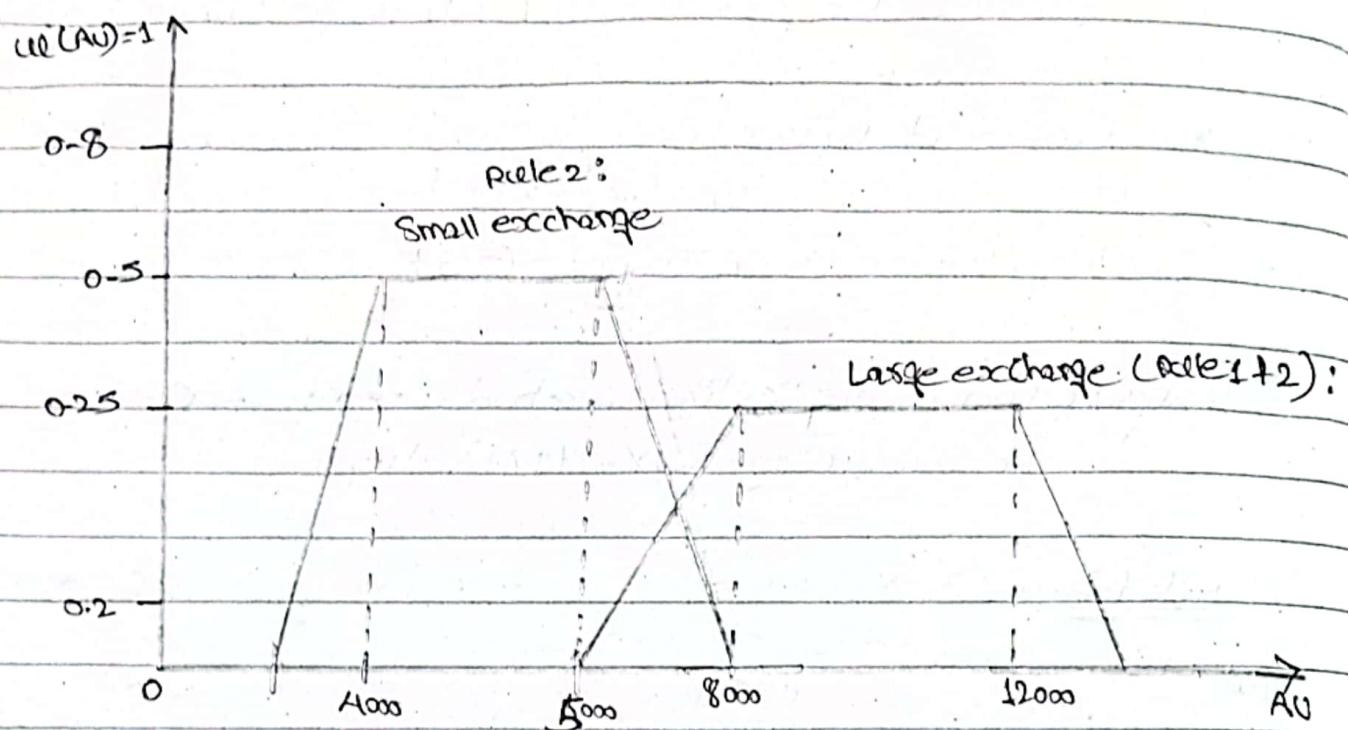
Rule 2:



Rule 3:



now, Aggregating all three rules we get:



now, Defuzzified value for mandani using weighted average method is

$$\begin{aligned} AV^* &= \frac{(4000 + 5000) \times 0.5}{2} + \frac{(8000 + 12000) \times 0.25}{2} \\ &= \frac{0.5 + 0.25}{0.5 + 0.25} \\ &= 3333.3 \text{ m}^2 \text{Kw/m}^2 \text{K}, \end{aligned}$$

2) Sugeno method of inference:

Given, following expression in polynomial form for over two concept

$$AV_{\text{small}} = 3.4765w - 210.5AT_{\text{app}} + 2103$$

$$AV_{\text{large}} = 4.6925w - 526.2AT_{\text{app}} + 2031$$

where, $w = 1800$ & $AT_{\text{app}} = 6.33$ we get,

$$AU_{small} = 3.4765 \times 1300 - 210.5 \times 6.5 + 2103 \\ = 5256$$

$$AU_{large} = 4.6925 \times 1300 - 526.2 \times 6.5 + 2631 \\ = 5311$$

So, now,

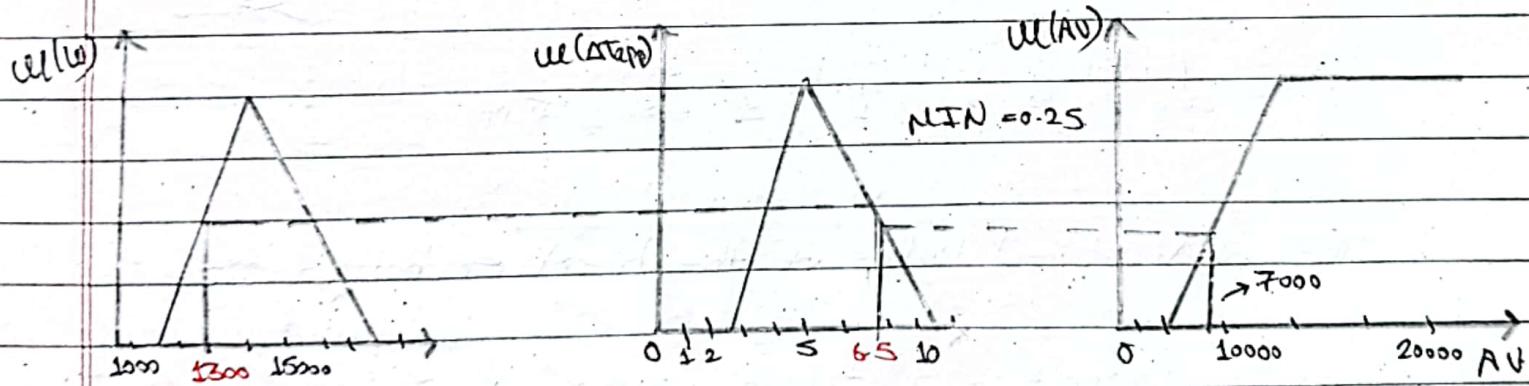
defuzzified value using weighted Average method:

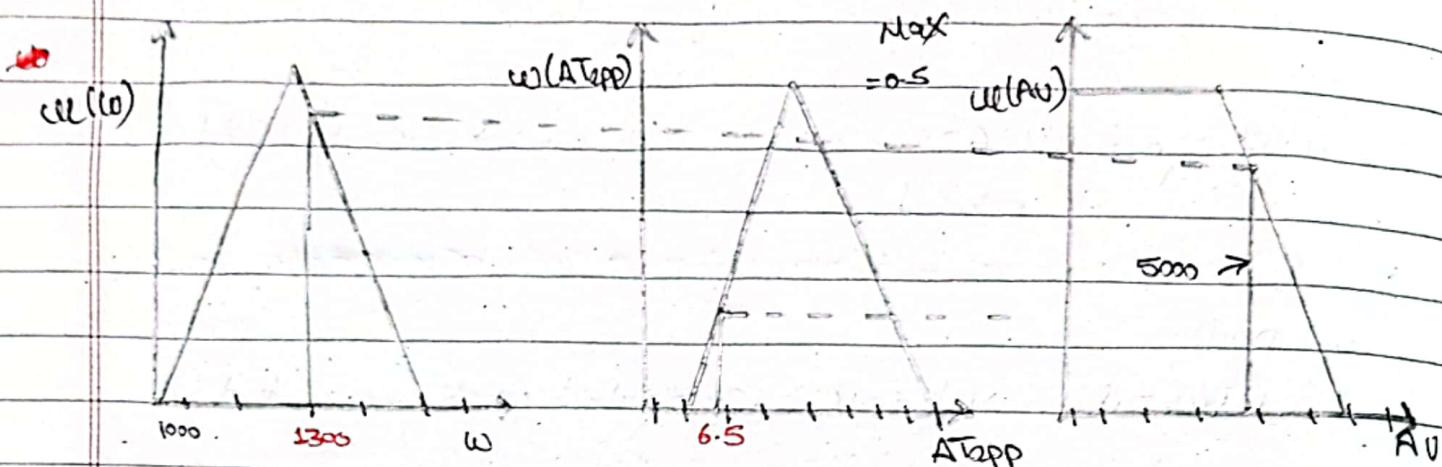
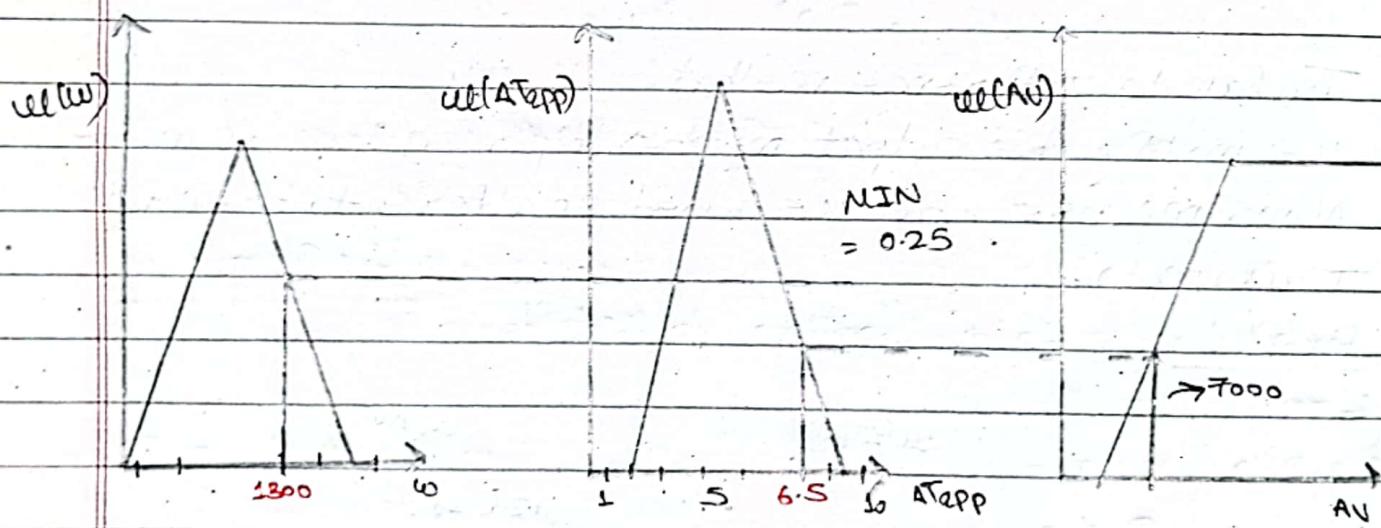
$$AU^* = \frac{(5311 \times 0.25) + (5256 \times 0.5) + (5311 \times 0.25)}{0.25 + 0.5 + 0.25} \\ = 5283.5 \text{ m}^2\text{KW/m}^2\text{K}$$

iii) Tsukamoto Inference method:

- we modify the output membership functions from the Normdane case , but we added shoulders to them for Tsukamoto.

Rule 1:



Rule 2:Rule 3:

The defuzzified value of the heat exchanger size is

$$AU^* = \frac{(7000 \times 0.25) + (5000 \times 0.5) + (7000 \times 0.25)}{0.25 + 0.5 + 0.25}$$

$$= 6250 \text{ m}^2 \text{Kw/m}^2 \text{K}$$

* Example For inference method (Rule-based method):

Consider a Fuzzy rule based system for the study of quality of land. The goodness of land is determined by the factors like its slope and fertility. Suppose fuzzy sets defining these parameters are:

$$\text{Sloppy} = \left\{ \frac{0}{2}, \frac{0.3}{5}, \frac{0.5}{8}, \frac{1}{11}, \frac{0.6}{14}, \frac{0.3}{7}, \frac{0}{20} \right\}$$

$$\text{Slightly Sloppy} = \left\{ \frac{0}{1}, \frac{0.3}{2}, \frac{0.4}{6}, \frac{0.8}{8}, \frac{0.5}{10}, \frac{0.1}{12}, \frac{0}{13} \right\}$$

$$\text{Highly Fertile} = \left\{ \frac{0}{5}, \frac{0.3}{10}, \frac{0.7}{15}, \frac{0.9}{20}, \frac{0.8}{25}, \frac{0.3}{40}, \frac{0}{45} \right\}$$

$$\text{Low Fertile} = \left\{ \frac{0}{3}, \frac{0.2}{5}, \frac{0.4}{10}, \frac{0.7}{20}, \frac{0.5}{25}, \frac{0.1}{30}, \frac{0}{35} \right\}$$

$$\text{Excellent quality} = \left\{ \frac{0}{34}, \frac{0.3}{40}, \frac{0.5}{45}, \frac{1}{50}, \frac{0.6}{55}, \frac{0.3}{60} \right\}$$

$$\text{Good quality} = \left\{ \frac{0.4}{20}, \frac{0.6}{35}, \frac{0.8}{40}, \frac{0.5}{50}, \frac{0.3}{55}, \frac{0.9}{65}, \frac{1}{70} \right\}$$

The rules in the FRBC Contains:

R₁: IF Land is sloppy or highly fertile then it has excellent quality.

R₂: IF Land is slightly sloppy & low fertile then it has good quality.

now, using mamdani model with max-min inference method of defuzzification & max-product inference method of defuzzification determine the quality with slope value = 8
Fertility = 20

= Sol

given,

$$\text{Slope value} = 8$$

$$\text{Fertility} = 20$$

Q) For Max-min inference method of classification:

now,

$$R_1 \quad \text{Ul sloopy}(8) = 0.5$$

$$R_2 \quad \text{Ul slightly sloopy}(8) = 0.8$$

Similarly,

$$R_1 \quad \text{Ul highly fertile}(20) = 0.9$$

$$R_2 \quad \text{Ul low fertile}(20) = 0.7$$

We can see both R_1 & R_2 will satisfy so,① for R_1 :

$$\begin{aligned} \text{Membership of antecedents} &= \max(\text{Ul sloopy}(8), \\ &\quad \text{Ul HF}(20)) \\ &= \max(0.5, 0.9) \\ &= 0.9 \end{aligned}$$

now, the consequent using min will be;

$$C_1 = \left\{ \min\left(\frac{0.9}{34}, 0\right), \min\left(\frac{0.9}{40}, 0.3\right), \min\left(\frac{0.9}{45}, 0.5\right), \min\left(\frac{0.9}{50}, 0.7\right) \right\}$$

$$= \left\{ \min\left(\frac{0.9}{34}, 0.6\right), \min\left(\frac{0.9}{40}, 0.3\right) \right\}$$

$$= \left\{ \frac{0}{34}, \frac{0.3}{40}, \frac{0.5}{45}, \frac{0.9}{50}, \frac{0.6}{55}, \frac{0.3}{60} \right\}$$

② For R2:

$$\text{Membership of Antecedents} = \min (\text{Uss}^{(8)}, \text{ULF}^{(20)})$$

$$= \min (0.8, 0.7)$$

$$= 0.7$$

now, the consequent using min will be,

$$C_2 = \sum_{20} \min (0.7, 0.4), \min_{35} (0.7, 0.6), \min_{40} (0.7, 0.8), \min_{50} (0.7, 0.9)$$

$$= \sum_{55} \min (0.7, 0.3), \min_{65} (0.7, 0.9), \min_{70} (0.7, 1) \}$$

$$= \sum_{20} \frac{0.4}{0.7}, \frac{0.6}{0.7}, \frac{0.7}{0.7}, \frac{0.5}{0.7}, \frac{0.3}{0.7}, \frac{0.4}{0.7}, \frac{0.7}{0.7} \}$$

now, Aggregating C₁ & C₂ we get,

$$C = \max (C_1 \text{ and } C_2)$$

$$= \sum_{20} \frac{0.4}{0.7}, \frac{0}{0.7}, \frac{0.6}{0.7}, \frac{0.7}{0.7}, \frac{0.9}{0.7}, \frac{0.6}{0.7}, \frac{0.3}{0.7}, \frac{0.7}{0.7} \}$$

$$= \frac{0.7}{0.7} \}$$

using Max method the defuzzification of C is 50.
Hence, the quality is 50.

i) For max-product inference method of defuzzification:

For R1:

The consequent using Max-product will be;

$$C_1 = \sum_{34} \frac{0.9 \times 0}{0.9}, \frac{0.9 \times 0.3}{0.9}, \frac{0.9 \times 0.5}{0.9}, \frac{0.9 \times 1}{0.9}, \frac{0.9 \times 0.6}{0.9}, \frac{0.9 \times 0.3}{0.9} \}$$

$$= \sum_{34} \frac{0}{0.9}, \frac{0.27}{0.9}, \frac{0.45}{0.9}, \frac{0.9}{0.9}, \frac{0.54}{0.9}, \frac{0.27}{0.9} \}$$

For R₁:

The consequent using product will be:

$$\begin{aligned}
 C_2 = & \{ \frac{0.7 \times 0.4}{20}, \frac{0.7 \times 0.6}{35}, \frac{0.7 \times 0.8}{40}, \frac{0.7 \times 0.5}{50}, \frac{0.7 \times 0.1}{55} \\
 & \quad \frac{0.7 \times 0.9}{65}, \frac{0.7 \times 1.2}{70} \} \\
 = & \{ \frac{0.28}{20}, \frac{0.42}{35}, \frac{0.56}{40}, \frac{0.35}{50}, \frac{0.21}{55}, \frac{0.63}{65}, \frac{0.7}{70} \}
 \end{aligned}$$

now, Aggregating C₁ & C₂ we have:

$$C = \max(C_1, C_2)$$

$$\begin{aligned}
 C = & \{ \frac{0.28}{20}, \frac{0}{34}, \frac{0.42}{35}, \frac{0.56}{40}, \frac{0.45}{45}, \frac{0.9}{50}, \frac{0.8}{55} \\
 & \quad \frac{0.27}{60}, \frac{0.63}{65}, \frac{0.7}{70} \}
 \end{aligned}$$

Using Max-method, the defuzzification of C is 50,
the quality is 50.

ii) For Sugeno or TSK method:

$$\text{① Using zero order TSK (Quality)} = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

$$= \cancel{\max}$$

where the rules in the Fuzzy rule base system are:

R₁: IF land is sloppy or highly fertile then quality is 20.

R₂: IF land is slightly sloppy and low fertile the quality is 6.

now, using zero-order TSK model determine quality with slope value 8 & Fuzzify 20.

$$\text{Slope value} = 8$$

$$\text{Fuzzify} = 20$$

① R1

$$\text{now, } \text{Ullsppg}(s) = 0.5 \quad \& \quad \text{UllHF}^{(20)} = 0.9$$

For R1 with quality (z_1) = 20

$$\begin{aligned}\text{Membership of antecedents } (w_1) &= \max (\text{Ullsppg}(s), \text{UllHF}^{(20)}) \\ &= \max (0.5, 0.9) \\ &= 0.9\end{aligned}$$

② For R2 :

now,

$$\text{Ulls}^{(8)} = 0.8 \quad \& \quad \text{UllF}^{(20)} = 0.7$$

For R2 with quality (z_2) = 60

$$\begin{aligned}\text{Membership of antecedents } (w_2) &= \min (\text{Ulls}^{(8)}, \text{UllF}^{(20)}) \\ &= \min (0.8, 0.7) \\ &= 0.7\end{aligned}$$

now, defuzzified value is:

$$\text{zero-order TSK (quality)} = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

$$= \frac{0.9 \times 20 + 0.7 \times 60}{0.9 + 0.7}$$

$$= \frac{60}{1.6}$$

$$= 37.5$$

The quality is 37.5

② For First order TSK model:

The rules in the Fuzzy rule based system contains:

R₁: IF land is sloppy or highly fertile the quality is

$$= \frac{\text{sloppy} + \text{highly fertile}}{2}$$

R₂: IF land is slightly sloppy and low fertile the quality = slightly sloppy + low fertile.

now, using first order TSK model determine quality with slope value 8 & fertility = 20

= Sol

We have,

$$\text{Slope value} = 8$$

$$\text{Fertility} = 20$$

now,

For R₁:

$$\mu_{\text{sloppy}}(8) = 0.5 \text{ & } \mu_{\text{HF}}^{(20)} = 0.9$$

$$\begin{aligned} R_1 \text{ membership of antecedents } (w_1) &= \max(\mu_{\text{sloppy}}^{(8)}, \mu_{\text{HF}}^{(20)}) \\ &= \max(0.5, 0.9) \\ &= 0.9 \end{aligned}$$

For R₂:

$$\mu_{\text{ss}}(8) = 0.8 \text{ & } \mu_{\text{LF}}^{(20)} = 0.7$$

$$\begin{aligned} R_2 \text{ membership of antecedent } (w_2) &= \min(\mu_{\text{ss}}^{(8)}, \mu_{\text{LF}}^{(20)}) \\ &= \min(0.8, 0.7) \\ &= 0.7 \end{aligned}$$

now,

quality (z_1) = ~~sloppy + highly festive~~

$$= \frac{8 * 20}{2}$$

$$= 80$$

quality (z_2) = slightly sloppy + low festive

$$= 8 + 20$$

$$= 28$$

Therefore, using First-order TSK model, defuzzified value
is

$$\text{quality } (z) = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

$$= \frac{0.9 * 80 + 0.7 * 28}{0.9 + 0.7}$$

$$= \frac{72 + 19.6}{1.6}$$

$$= \frac{91.6}{1.6}$$

$$= 57.25$$

\therefore The quality is 57.25.

iii) Tsukamoto Model:

All the process for R_1 & R_2 are same as Mamdani model only difference is selection of z_1 & z_2 which represent the elements with membership values of Antecedents in R_1 & R_2 which are 0.9, 0.7 respectively so,

$$C_1 = \{ \frac{0}{34}, \frac{0.3}{40}, \frac{0.5}{45}, \frac{0.9}{50}, \frac{0.6}{55}, \frac{0.3}{60} \}$$

$$z_1 = \text{largest element with membership } 0.9 \\ = 50$$

$$C_2 = \{ \frac{0.4}{20}, \frac{0.6}{35}, \frac{0.7}{40}, \frac{0.5}{50}, \frac{0.3}{55}, \frac{0.3}{65}, \frac{0.7}{70}, \frac{0.7}{70} \}$$

$$z_2 = \text{set element with membership } 0.7 \\ = 40 \quad (\text{we are consider 40 as we can see there are more elements with } M=0.7)$$

now,

$$\text{Defuzzified value:} \\ z(\text{quality}) = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

$$= \frac{0.9 \times 50 + 0.7 \times 40}{0.9 + 0.7}$$

$$= \frac{73}{1.6}$$

$$= 45.625$$

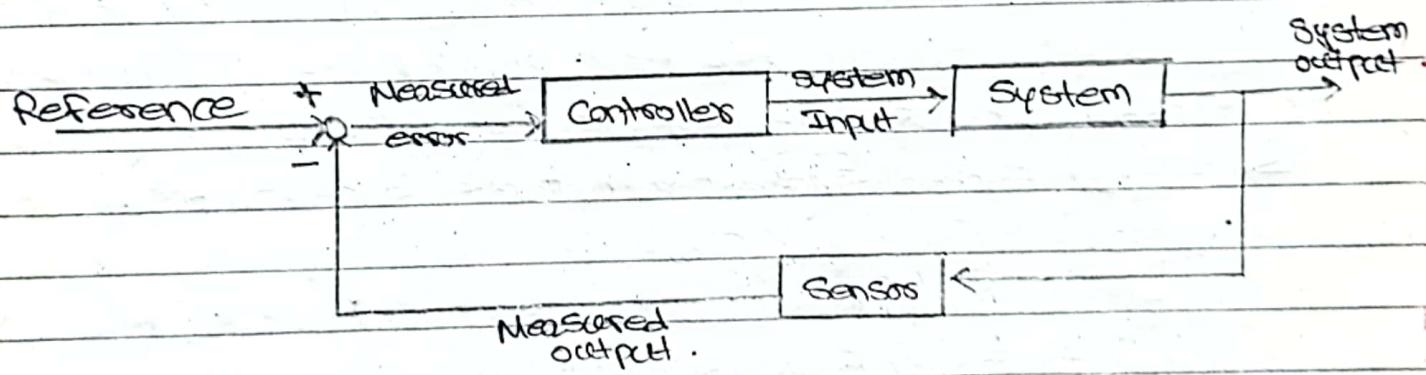
Hence, the quality is 45.625.

Unit 5 & 6

Fuzzy Controller:

* Control theory:

- Control theory is a branch of engineering and mathematics that deals with the behaviors of dynamic systems with inputs. (probable output)
- The external input of a system is called **Reference**.
- Control theory explains about the controller which manipulates the inputs to a system to obtain the desired effect as an output from the system.



- The error signal is calculated by subtracting the sensed value by the system from the desired value.

* Functions of control systems:

- i) Measure,
- ii) Compare
- iii) Compute
- iv) Correct

The main objective of a control theory is to calculate or find solutions for the corrective action from the controller

which helps to provide a stable system.

The System will hold at the **Set point** (desired value) & don't oscillate around it.

* Terminology:

- i) **Setpoint:** The desired value from the system.
 - ii) **Process variable:** The measure value by system.
 - iii) **Manipulated Variable (MV) or Control Variable (CV):** The output of the controller.
 - iv) **Closed-loop system:** Feedback system where output from system is used as input in the system to meet the setpoint.

* Fuzzy control system:

→ Fuzzy Control System is a control system based on fuzzy logic.

i.e. it is a mathematical system that analyzes input to the system in terms of logical variables ~~that~~ which take on continuous values between [0, 1].

→ Fuzzy Controllers consists of an input stage, a processing stage, an output stage where,

The core processing stage is based on a collection of logic rules in the form of IF-THEN statements.

* Process Gain:

$$\rightarrow K = \frac{\Delta \text{Output}}{\Delta \text{Input}}$$

The process Gain (K) is the ratio of change of the output variable to the change of the input variable.

\rightarrow It specifically defines the sensitivity of the output variable to given change in the input variable.

* PID Controller:

\rightarrow PID Controllers are a type of continuous controller because they continually adjust the output.

\rightarrow Basically, Proportional - Integral - Derivative controller is a Control loop feedback controller widely used in industrial areas.

\rightarrow A PID Controller calculates an error value as the difference between a measured process variable & a desired set point.

\rightarrow Consists of 3 separate control variables that can be interpreted in terms of time:

i) P → depends on the present errors,

ii) I → Accumulation of past errors,

iii) D → Prediction of future errors, based on current rate of change

\rightarrow The weighted sum of these three actions is used to adjust the process via a control elements (control valve, a damper, pump, supply)

$$u(t) = K_p e(t) + K_i \int_0^t e(t) dt + K_d \frac{de(t)}{dt}$$

where,

$$\left\{ \begin{array}{l} K_p = \text{proportional gain,} \\ K_i = \text{integral gain,} \\ K_d = \text{derivative gain} \end{array} \right.$$

$e(t)$ = $s_p - p_v$ (difference of set point & process variable)

t = variable of integration ranging from 0 to present t .

Above K_p , K_i , K_d are usually picked to optimize the controller performance & ensure that the system remains stable for all possible actions.

- i) $K_p e(t)$: is proportion term & produces output proportional to current errors.
- ii) $K_i \int_0^t e(t) dt$: is integral term that accelerates the movement of process towards set point. So it responds to accumulated error from the past.
- iii) $K_d \frac{de(t)}{dt}$: is derivative term that determines errors or system behaviors over time. i.e. rate of change.

* Adaptive Controllers:

- Adaptive Controllers follows a control method which must adapt to a controlled system with parameters that vary, or are initially uncertain.
For example, as an aircraft flies, its mass will slowly decrease as a result of fuel consumption; so, a control method is needed that adapts itself to such changing conditions.
- Mostly used for uncertainty where there is no prior information.
- It can be feed forward Adaptive control where no feedback is feed to the system, and feed back Adaptive Control where feedbacks are feeded to the system.
- It can be direct & indirect methods:
 - i) **direct**: Estimated parameters are used directly in Adaptive controller.
 - ii) **Indirect**: Estimated parameters are used to calculate required controller parameter.

* Components of Adaptive controllers:

- i) **Process monitor**:
- ii) **Adaptive Mechanism**:

i) Process monitor:

A Process monitor detects the changes in the process characteristics. It is usually in one of two forms:

- 1) A performance measure that assesses how well the controller is controlling.
- 2) A parameter estimator that constantly updates a model of the pro-

i) Adaptation Mechanism:

It uses the information passed to it by the process monitor to update the controller parameters & so adapts the controller to the changing process characteristics.

Adaptive controllers can be classified as:

- i) performance Adaptive,
- ii) parameter Adaptive.

→ There is also non-Adaptive controller where the parameter do not change once the controller is being used on line.

* Model-Based Controller:

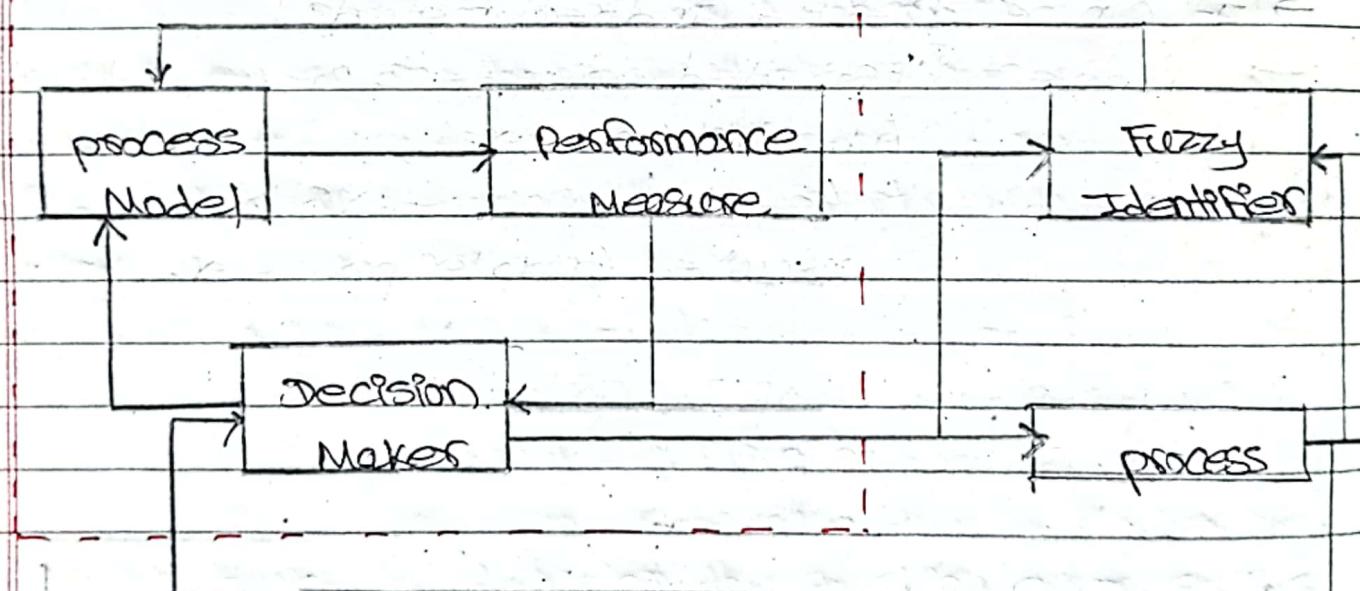


Fig :- Model-Based adaptive fuzzy controller.

→ The controller was originally developed by Hendry (96) and consists of 3 parts:

- ① a fuzzy process model,
- ② a controller performance measure,

② Fuzzy controller.

→ The fuzzy process model consists of a set of if-then rules.

* Sequence of operations:

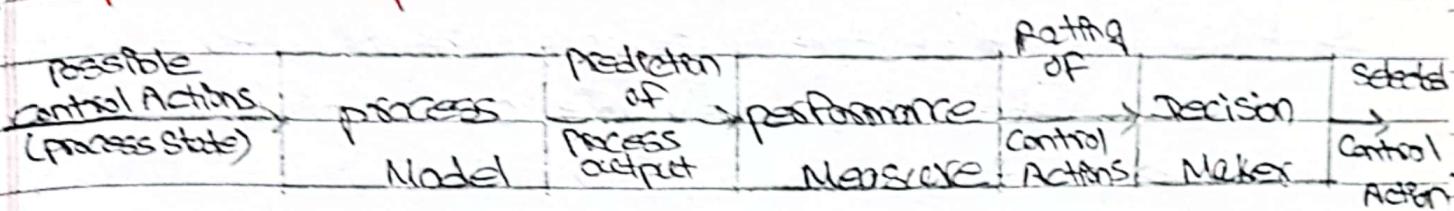


Fig of :- Flowchart of model based controller.

* Model Predictive controllers:

- Future values of actuator variables are predicted using a dynamic p model of process and current measurement
- Compute the predictive value of error in input for in output.

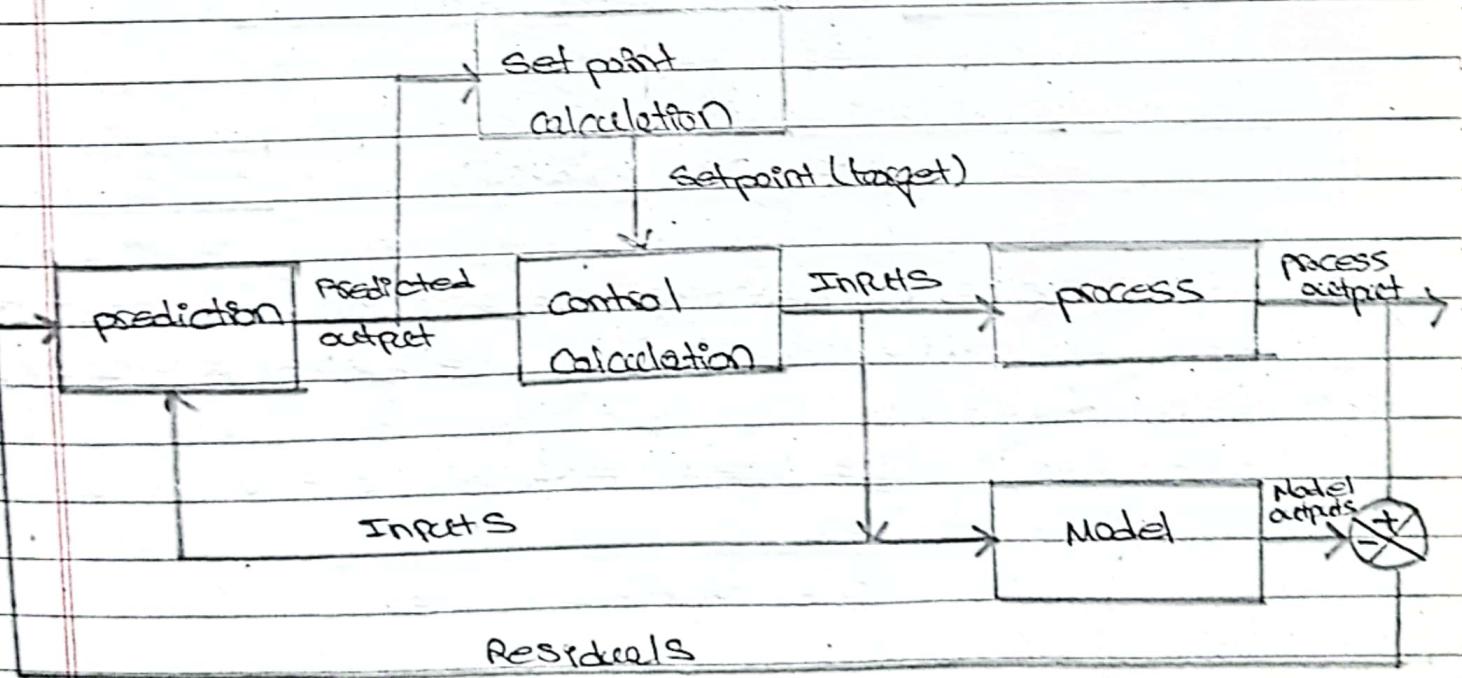


Fig of :- Model Predictive Controller.

* Model Reference Adaptive Controllers (Model+Adaptive):

- The general idea behind Model Reference Adaptive Controllers is to create a closed loop (feed back) controller with parameters that can be updated to change the response of the system.
- The output of the system is compared to a desired response from a reference model. The control parameters are updated based on this error.
- The main goal for updating these parameters is to converge to ideal values.

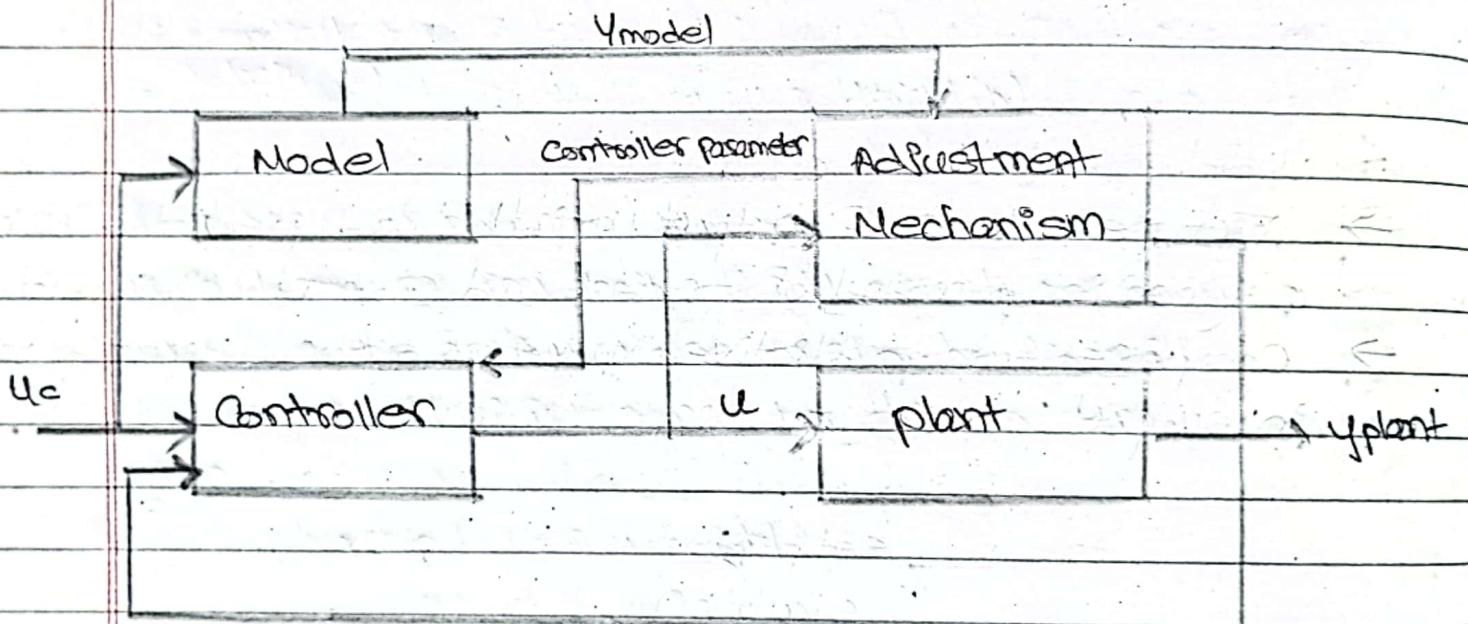


Fig of i:- MRAC

- MRAC begins by defining the tracking error, e . This is simply the difference between the plant output & the model output
i.e., $e = y_{model} - y_{plant}$.

$$e = y_{plant} - y_{model}$$

* Linear & Non-linear System:

→ The system which follows the principle of superposition is known as linear system, where the principle of superposition is the combination of two law i.e.:

i) Law of Additivity, : $f(x+y) = f(x) + f(y)$

ii) Law of Homogeneity. $f(\alpha x) = \alpha f(x)$
(Scaling Property)

→ The system which doesn't follow the principle of superposition is known as non-linear system i.e. non-linear system fails to satisfy either of laws of additivity or homogeneity or both.

→ The condition of additivity & homogeneity are often combined to give superposition principle i.e.

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

① Additivity property or Superposition property:

Let $y_1 = f(x)$, $y_2 = f(y+z)$. Then for the additivity property to be obeyed it is required that,

that, $y_1 + y_2 = f(x+z)$. Hence,

$$f(x) + f(z) = f(x+z).$$

② Scaling property or homogeneity property:

let $y = f(x)$. Then for the scaling or homogeneity property to be obeyed it is required that

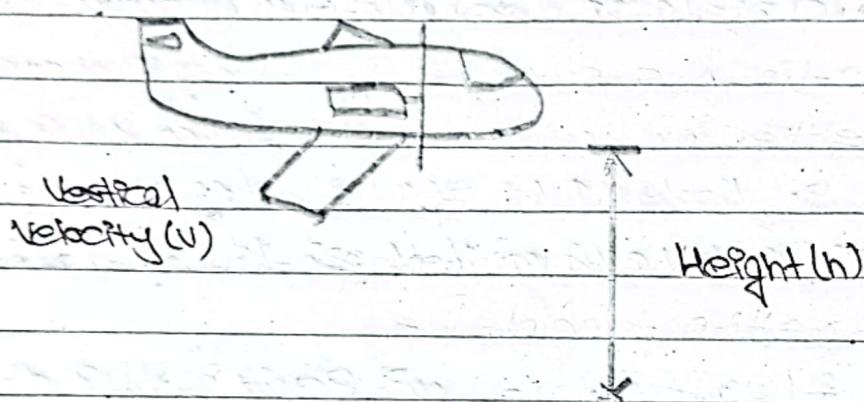
$$\alpha \cdot y = f(\alpha \cdot x)$$

$$\alpha \cdot f(x) = f(\alpha \cdot x).$$

* Sliding Mode controller:

- type of non-linear controller,
- Control variable is used to calculate error which is Input - disturbance.

* Aircraft Landing control problem:



With decrease in velocity Height also decrease, we need to evaluate the applied force to stop the Aircraft.

We use Mamdani inference for this problem.

Given,

① Membership Values of height:

	Height (ft)										
	0	100	200	300	400	500	600	700	800	900	1000
Large(L)	0	0	0	0	0	0	0.2	0.4	0.6	0.8	1*
Medium(M)	0	0	0	0	0.2	0.4	0.6	0.8	1	0.8	0.6*
Small(s)	0.4	0.6	0.8	1	0.8	0.6	0.4	0.2	0	0	0
Near zero(Nz)	1	0.8	0.6	0.4	0.2	0	0	0	0	0	0

Membership values for velocity

Vertical Velocity (PI/S)

-30 -25 -20 ⁻¹⁵
-10 -5 0 5 10 15 20 25 30

plage(VL)	0	0	0	0	0	0	0	0	0.5	1	1
sp small(US)	0	0	0	0	0	0	0.5	1	0.5	0	0
zero(z)	0	0	0	0	0	0.5	1	0.5	0	0	0
zero small(DS)	0	0	0	0.5	1	0.5	0	0	0	0	0
down large(DL)	1	1	[*] 1	0.5	0	0	0	0	0	0	0

Membership values for control force

actpt force (lb)

-30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30

plage(VL)	0	0	0	0	0	0	0	0	0.5	1	1
sp small(US)	0	0	0	0	0	0	0.5	1	0.5	0	0
zero(z)	0	0	0	0	0.5	1	0.5	0	0	0	0
zero small(DS)	0	0	0	0.5	1	0.5	0	0	0	0	0
down large(DL)	1	1	1	0.5	0	0	0	0	0	0	0

Rules:

Velocity

Height	DL	DS	zero	US	VL
L	[*] z	z	DL	DL	DL
M	US	z	DS	DL	DL
S	UL	US	z	DS	DL
JZ	UL	UL	z	DS	DS

e.g.: IF Height is Large and velocity is Downlarge then actpt force is zero.

Initial height h_0 : 1000 ft

Initial velocity, v_0 : -20 ft s⁻¹

control f_0 : to be computed.

① Height h (100) fires large(1) with $u_l = 1$

Medium(0.6) with $u_l = 0.6$ (\because See height membership table)

② When velocity v (-20) fires downlarge(0) with $u_l = 1$

So, Rules are:

R₁: IF Height is large AND Velocity is downlarge then zero

R₂: IF height is Medium AND Velocity is downlarge then upper small

Both Rules fires so,

R₁:

$$\begin{aligned} &= \text{MIN} (\text{large}, \text{downlarge}) \\ &= \text{MIN} (u_l^{(100)}, u_l^{(-20)}) \\ &= \text{MIN} (1, 1) \\ &= 1 \end{aligned}$$

R₂:

$$\begin{aligned} &= \text{MIN} (\text{medium}, \text{downlarge}) \\ &= \text{MIN} (u_l^{(100)}, u_l^{(-20)}) \\ &= \text{MIN} (0.6, 1) \\ &= 0.6 \end{aligned}$$

$$C_1 = \left\{ \min\left(1, 0\right), \min\left(1, 0\right), \min\left(1, 0\right), \min\left(1, 0\right), \min\left(1, 0\right), \min\left(1, 0\right), \right. \\ \left. \min\left(1, 0.5\right), \min\left(1, 1\right), \min\left(1, 0.5\right), \min\left(1, 0\right), \min\left(1, 0\right), \right. \\ \left. \min\left(1, 0\right), \min\left(1, 0\right), \min\left(1, 0\right) \right\} \\ = \left\{ \frac{0}{-30}, \frac{0}{-25}, \frac{0}{-20}, \frac{0}{-15}, \frac{0}{-10}, \frac{0.5}{-5}, \frac{1}{0}, \frac{0.5}{5}, \frac{0}{10}, \frac{0}{15}, \frac{0}{20}, \frac{0}{25}, \frac{0}{30} \right\}$$

$$C_2 = \left\{ \min\left(0.6, 0\right), \min\left(0.6, 0\right), \min\left(0.6, 0\right), \min\left(0.6, 0\right), \min\left(0.6, 0\right), \right. \\ \left. \min\left(0.6, 0\right), \min\left(0.6, 0\right), \min\left(0.6, 0.5\right), \min\left(0.6, 1\right), \min\left(0.6, 0.5\right), \right. \\ \left. \min\left(0.6, 0\right), \min\left(0.6, 0\right), \min\left(0.6, 0\right) \right\} \\ = \left\{ \frac{0}{-30}, \frac{0}{-25}, \frac{0}{-20}, \frac{0}{-15}, \frac{0}{-10}, \frac{0}{-5}, \frac{0}{0}, \frac{0.5}{5}, \frac{0.6}{10}, \frac{0.5}{15}, \frac{0}{20}, \frac{0}{25}, \frac{0}{30} \right\}$$

now, Aggregation $C_1 \& C_2$

$$C = \max(C_1, C_2)$$

$$= \left\{ \frac{0}{-30}, \frac{0}{-25}, \frac{0}{-20}, \frac{0}{-15}, \frac{0}{-10}, \frac{0}{-5}, \frac{0}{0}, \frac{0.5}{5}, \frac{0.6}{10}, \frac{0.5}{15}, \frac{0}{20}, \frac{0}{25}, \frac{0}{30} \right\}$$

$$\left. \frac{0}{20}, \frac{0}{25}, \frac{0}{30} \right\}$$

~~for defuzzification let us go max-mean value~~

$$a_{fb} = \frac{0+10}{2}$$

For defuzzification the authors used the centroid method
to get $f_0 = 5.8 \text{ lb}$.

now, we need to update height & velocity i.e.

$$h_1 = h_0 + v_0 = 1000 + (-20) = 980 \text{ ft}$$

$$v_1 = v_0 + g t = -20 + 5 \cdot 8 = -14.2 \text{ ft s}^{-1}$$

now, with these new values repeat the same process until desired outcome is achieved.

$$\text{Height} = 980 \text{ ft}$$

$$\text{Velocity} = -14.2 \text{ ft s}^{-1}$$

observe how many times are fixed & apply random inference.