[Introduction]
Computational Geometry (CSc 635)

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Elementary Geometric Objects

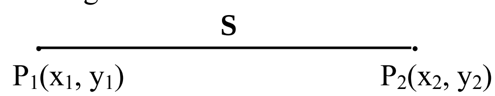
Point: - Point is a pair of numbers, in general integers, in 2D and a triplet of integers in 3D.

e.g.

. $P(x_1, y_1)$.$Q(x_1, y_1, z_1)$; where, x_1, y_1, z_1 are integers;

- Object with d dimension and zero extent, location in d-space.

Line segment: - A line segment is a pair of points, say P_1 & P_2, L (P_1, P_2), where P_1 & P_2 being the end points of the segment.

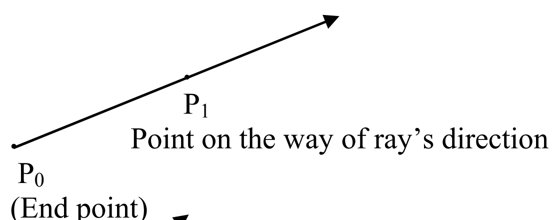


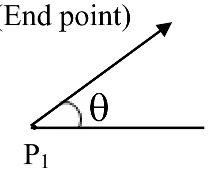
Here, S (P_1, P_2) is a line segment.

Thus, line segment is a closed subset of a line contained between two points. The subset is closed in the sense that it includes the end points.

Ray: - A ray is an infinite one dimensional subset of a line determined by two points: say P_0, P_1, where one point is denoted as the endpoint.

Thus, a ray consists of a bounded point & is extended to infinitely along a line segment.





Source point & direction

Line: - Line is represented by a pair of points P_0 and P_1 say, which is extended in both way to infinity along the segment represented by the pair of points P_0 & P_1.

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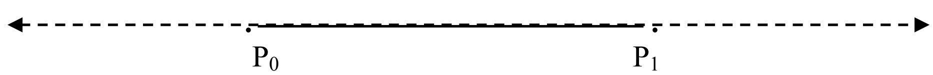
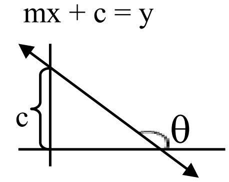


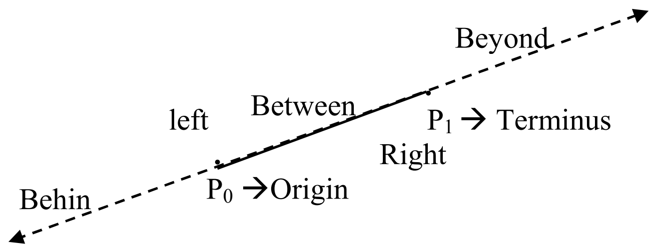
$Ax + By + C = 0$

$mx + c = y$



Point line classification:-

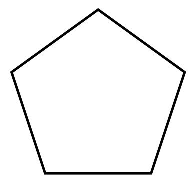
- A directed line segment partitions the plane into seven non overlapping regions. The possibilities are shown below,



Plane: - Infinite 2D subset of space.

Polygon (Jordan's Curve): - Camille Jordan → (Coursd' Analysis)

- Simply polygon is a homeomorphic image of a circle, i.e. it is a certain deformation of circle.



Simple Polygon:

A simple polygon is a region of plane bounded by a finite collection of line segments to form a simple closed curve.

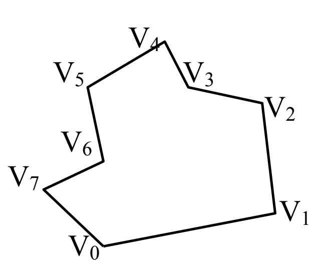
Mathematically, let V_0, V_1, V_2, ---------, V_{n-1} are n ordered vertices in the plane, then the line segments e_0 (V_0, V_1), e_1 (V_1, V_2), ……., e_{n-1} (V_{n-1}, V_0) form a simple polygon if and only if;

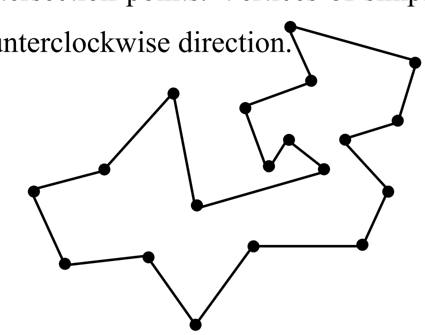
- the intersection of each pair of segments adjacent in cyclic ordering is a simple single point shared by them;

  $e_i \cap e_{i+1} = V_{i+1}$ &
- non-adjacent segments do not intersect;

  $e_i \cap e_j = \Phi$

Thus, a polygon is simple if there are no points between non-consecutive line-segments, i.e. vertices are only intersection points. Vertices of simple polygon are assumed to be ordered into counterclockwise direction.

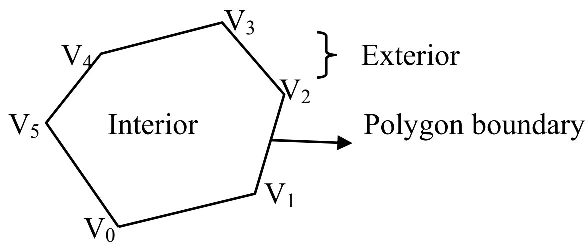




Jordan Curve Theorem:

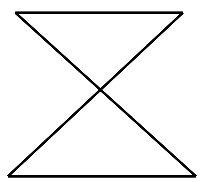
Every simple polygon partitions a plane into three (two*) connected components;

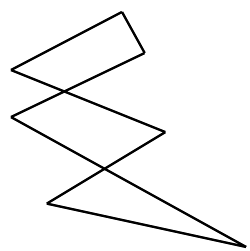
- unbounded connected exterior
- bounded connected interior
- Polygon (boundary) itself.



Non-Simple polygon (Self Intersecting)

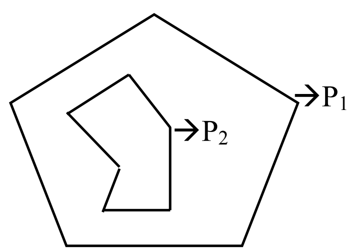
- A polygon is non-simple if there is no single interior region, i.e. non-adjacent edges intersect each other.



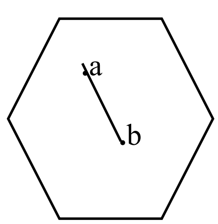


Polygon with hole:-

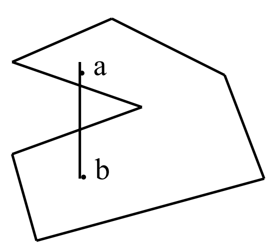
- A simple polygon with another simple polygon in its interior is called a polygon with hole.



Convex Polygon: - A simple polygon is said to be convex if any line segment connecting two interior points lies completely inside the polygon. Otherwise, it is non-convex or concave.



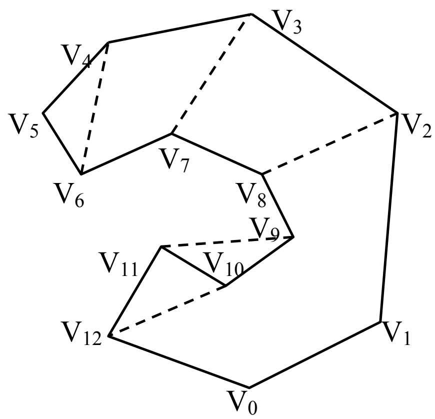
Convex Polygon



Non Convex (Concave)

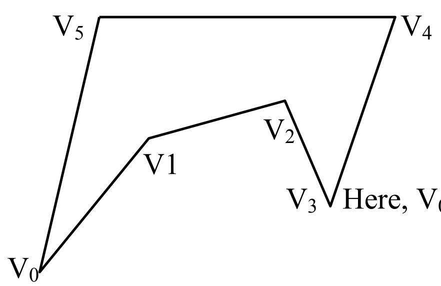
(Thus, a convex polygon has the points inside it such that all pairs of points are visible.)

Diagonal of a simple polygon: - A diagonal of a simple polygon is a line segments connecting two non-adjacent vertices and lies completely inside the polygon.



Here all (V_2, V_8), (V_3, V_7), (V_4, V_6) & (V_{10}, V_{12}) are diagonals of the polygon but (V_9, V_{11}) is not a diagonal

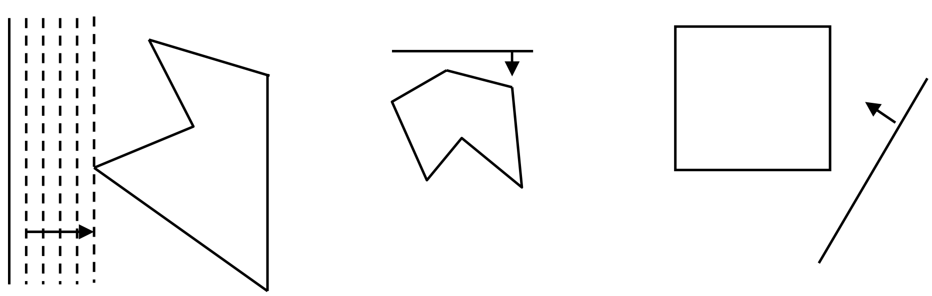
Convex vertex: - A vertex of a polygon is convex if the interior angle in the vertex is strictly smaller than π.



Here, V_0, V_3, V_4, V_5 are convex vertices of the polygon.

Lemma 1: -

Every polygon must have at least one strictly convex vertex.

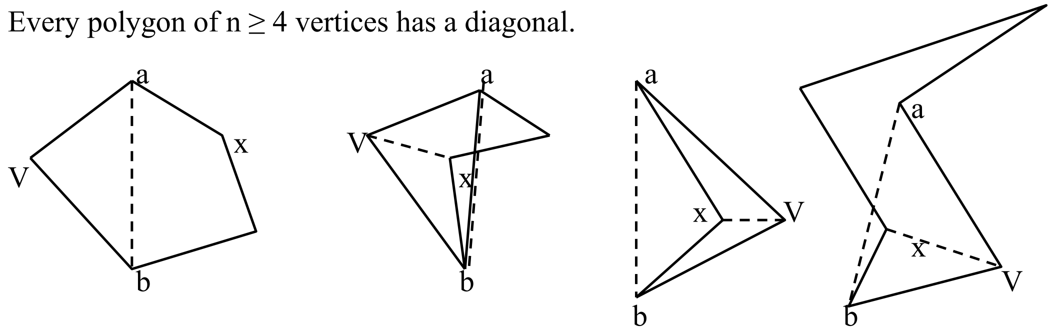


Proof:

Let us consider a vertical line moving closer to the polygon, keeping its position parallel to itself from infinity. This line must touch a vertex if it touches single point of the polygon. If the line is not touching into a single point, we can assume horizontal line or inclined line moving towards the polygon. If any line touches at a single point of polygon, that must be the convex vertex.

***Lemma 2:* -** (Meister's Lemma) (Digital Existence Theorem)

Every polygon of $n \geq 4$ vertices has a diagonal.



Proof: - Consider a polygon with a convex vertex V. The existence of V is guaranteed by the lemma 1.

Let a & b be vertices adjacent to V. If the line segment joining a & b lies completely inside the polygon then ab is a diagonal. So we are done. Otherwise, ab is either exterior to the polygon or intersects the polygon. For both cases ΔabV contains at least one vertex other than a, b or V, since n>3.

Let x be the vertex inside the ΔabV that is closest to V. joining x with V, it results xV which completely lies inside the polygon. Since x & V are non-adjacent vertices and line segment xV lies inside the polygon, clearly xV is a diagonal.

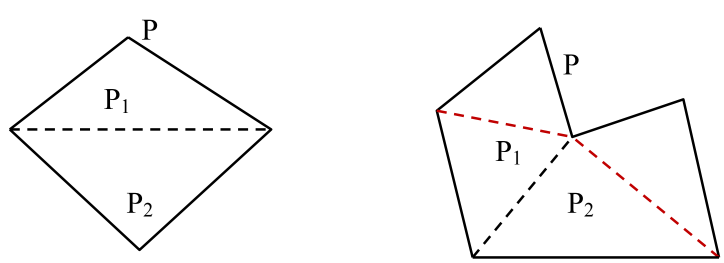
This completes the proof.

***Lemma 3:* -** Every polygon P of n vertices can be portioned into triangles by adding zero or more diagonals.

Proof: - If n = 3, the polygon itself is a triangle with zero diagonals. Hence, theorem holds trivially.

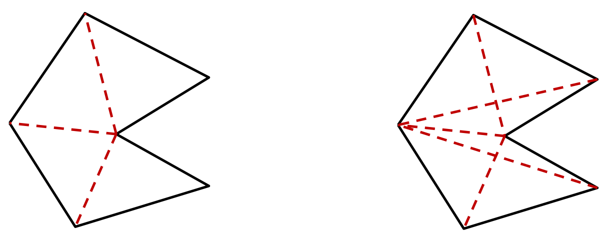
If $n \geq 4$, let d = (a, b) be a diagonal of the polygon say P, as guaranteed by Meister's Lemma. Hence d partitions the polygon P into two polygons, say P_1 and P_2, each sharing d as one of their edge and have no. of vertices fewer than P. Applying the induction hypothesis to the both sub-polygons P_1 & P_2 of P, it completes the proof.

Hence, the polygon P can be partitioned into triangles.



Polygon Triangulation

- Triangulation of a polygon P is its partition into non-overlapping triangles whose union is P. Geometrically, partitioning the polygon is done by adding the non-intersecting diagonals. In strictest sense, the triangles may have vertices only at the vertices of P.
- Triangulation reduces complex shapes to a collection of simpler ones.
- Thus, polygon partitioning, in general triangulation, is an important preprocessing step for many geometric algorithms, because most geometric problems are simpler & faster on convex objects than on non-convex ones. Triangles are always convex and have only three sides so any geometric operation performed on triangle is destinated to be as simple as it can be. Further, it is easier to work with the small convex pieces independently than with complex original object. (So flexibility & simplicity)
- Applications area may be in visibility problems, mesh generations & so on.



Triangulation Not a triangulation

- Generally, a polygon can be partitioned (triangulated) in more than one ways.

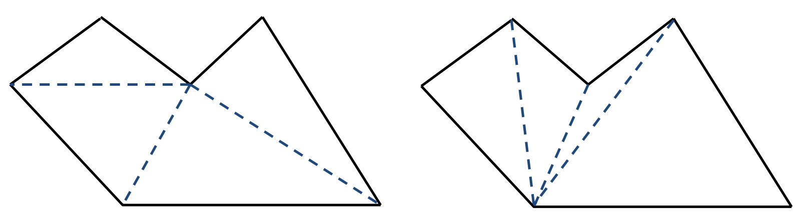


Fig: - Same polygon with two different triangulations

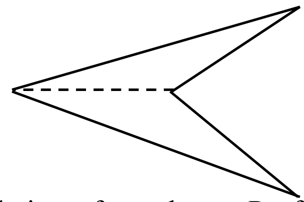
Thus, triangulations are usually not unique.

- There is a simple naïve polynomial time algorithm based on adding successive diagonals. But it can not complete with practical algorithms like monotone partitioning that takes O (nlogn) time.

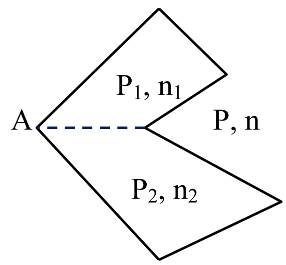
  # Chazelle (1991) – O(n) time algorithm for triangulation

  - but, is so difficult that it can not compete with others.

 # Construct a polygon with n = 4 vertices that can be triangulated in only one way.
Sol^n: -



Theorem: - Every triangulation of a polygon P of n-vertices uses n-3 diagonals and consists of n-2 triangles.



Proof: - If n=3, polygon itself is a triangle with zero diagonal and one triangle.

Therefore, it has 3 – 3 = 0 diagonals and 3 – 2 = 1 triangle.

This holds the theorem true for n = 3.

If $n \geq 4$, let d = (a, b) be the diagonal, which is guaranteed by the diagonal existence theorem. Clearly diagonal d partitions the polygon P into sub-polygon P_1 & P_2, say sharing d as common edge. Let n_1 & n_2 be the no. of vertices in P_1 & P_2 respectively.

So, $n_1 + n_2 = n + 2$

Now, applying the induction hypothesis the no. of diagonals in P_1 is $n_1 – 3$ & no. of diagonals in P_2 is $n_2 – 3$.

$\therefore$ No. of diagonals in P will be:

No. of diagonals in P_1 + No. of diagonals in P_2 + 1

$= n_1 - 3 + n_2 - 3 + 1$

$= n_1 + n_2 - 5$

$= n + 2 - 5$

$= n - 3$

Similarly, no. of triangles in P will be:

No. of triangles in P_1 + No. of triangles in P_2

$= n_1 - 2 + n_2 - 2$

$= n_1 + n_2 - 4$

$= n + 2 - 4$

$= n - 2$

Hence P uses n–3 diagonals and consists of n–2 triangles.

Corollary 1: The sum of interior angles of a simple polygon with n vertices is (n–2)π.

Proof: As there can be (n-2) triangles in a polygon, through triangulation and each triangle contributes π to the internal angles. Hence, sum of interior angle of the polygon is (n-2) π.

Corollary 2: The sum of exterior angles of a simple polygon with n vertices is (n+2)π.

Proof: Total angle at one vertex = 2 π

As there are n vertices, so the altogether total angle for n vertices is 2πn.

∴ Sum of exterior angles = 2πn – sum of interior angles

$$= 2\pi n - (n-2)\pi$$

$$= 2\pi n - n\pi + 2\pi$$

$$= \pi n + 2\pi$$

$$= (n+2)\pi$$

Triangulation Dual:

The dual T of a triangulation of a polygon is a graph with a node associated with each triangle and an edge between two nodes if and only if their triangles share an edge (diagonal).

Given, we have a polygon P, then triangulation dual of P can be obtained as:

- Triangulate P by adding diagonals.
- Consider each triangle as a node.
- Two nodes are connected by an edge/arc if their corresponding triangles share a diagonal.
- The resulting graph G (V, E) is the dual of triangulation of P.

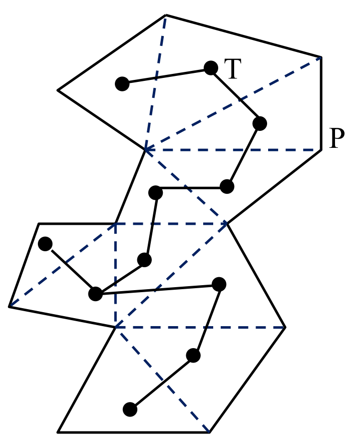


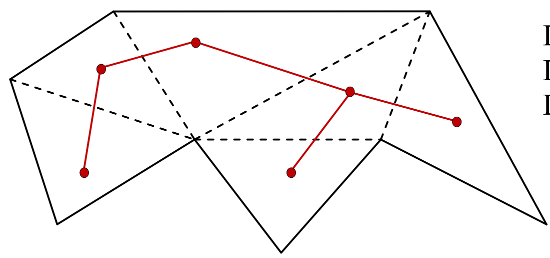
Fig: - Dual T of triangulation of polygon P.

Lemma: - *The dual T of a triangulation of a polygon is a tree, with each node of degree at most three.*

Proof:

(1) Dual in Tree: Suppose the dual T of the polygon P is not a tree, then it must have a cycle say C in it. If this cycle is drawn as a path π in the plane, connecting the midpoints of the diagonals shared by the triangles whose nodes comprise C. Then, the path π must also enclose the point's exterior to the polygon. This implies the polygon P is not a simple polygon. This is a contradiction. Hence, T is a tree.

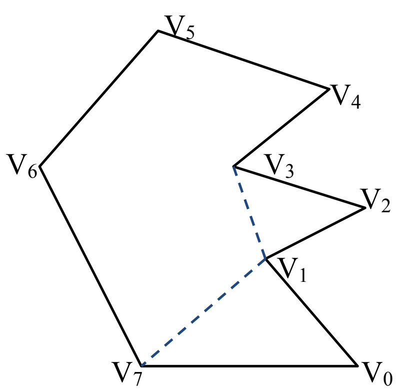
(2) Degree of each node of T is at most three: As each triangle have three sides. Hence, any node can have at most three edges incident to it. Hence, the degree of any node in the dual of triangulation can have at most three.



Degree one – leaf node
Degree 2 – lie on the path
Degree 3 – branch point.

Ear of Polygon

Three consecutive vertices V_i, V_{i+1}, V_{i+2} of a polygon form an ear if (V_i, V_{i+2}) is a diagonal, V_{i+1} is the tip of the ear.



(V_1, V_2, V_3) is an ear.
But, (V_0, V_1, V_2) is not an ear.
(V_7, V_0, V_1) & (V_1, V_2, V_3) are non-overlapping ears
(V_1, V_2, V_3) & (V_3, V_4, V_5) are non-overlapping ears
(V_3, V_4, V_5) & (V_4, V_5, V_6) are overlapping ears.

Two ears are said to be non-overlapping if their triangle interior are disjoint otherwise they are overlapping.

Mouth

Three consecutive vertices V_i, V_{i+1}, V_{i+2} of a polygon form a mouth if (V_i, V_{i+2}) is an external diagonal. In above figure, (V_0, V_1, V_2) & (V_2, V_3, V_4) are mouths of the polygon.

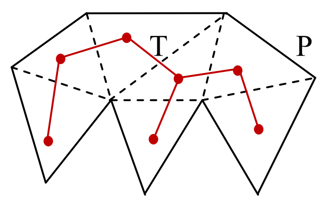
Meister's Two Ear Theorem:

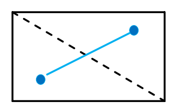
Every polygon of $n \geq 4$ vertices has at least two non-overlapping ears.

Proof:

Consider a dual T of triangulation of a polygon P with $n \geq 4$ vertices. Clearly, the no. of nodes in T is equal to n-2, since the no. of triangles is n-2.

A tree with two or more nodes, $n-2 \geq 2$ must have at two leaves. Any leaf in a dual tree corresponds to the ear of the polygon (which is non-overlapping to the ear with another leaf).





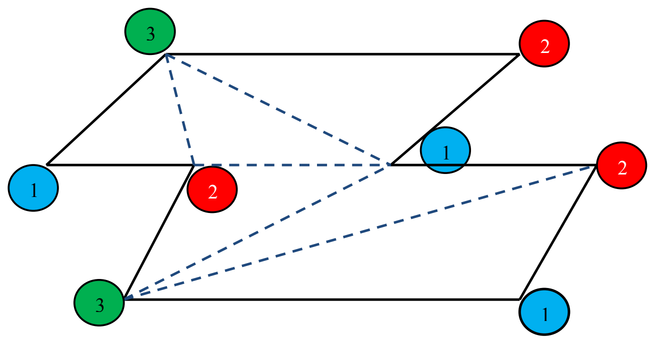
Hence, the dual shows that polygon has at least two non-overlapping ears.

Three Coloring of a polygon:

Problem: Color the vertices of a polygon with three colors such that adjacent vertices must have different colors.

⇒ The solution to this problem may be as follows:

- Triangulate the polygon.
- Clearly, vertices of each triangle correspond to the vertices of polygon.
- Assign different colors to the vertices of triangle.

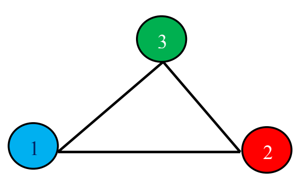


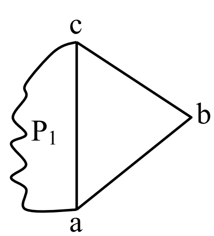
Theorem: The triangulation graph of a polygon P may be three colored.

Proof: The proof is by induction on no. of vertices. If n = 3, polygon itself is a triangle. So, to color it, assign there different color to each of vertices of the polygon. Hence, it can be three colored.

If $n \geq 4$, then by Meister's two ear theorem, there is an ear in the polygon. Let Δabc represents an ear with tip b in the polygon, say P. Now, cutting off the ear, Δabc, from the polygon P, it forms a sub-polygon $P1 = P - \Delta abc$ having vertices n-1.

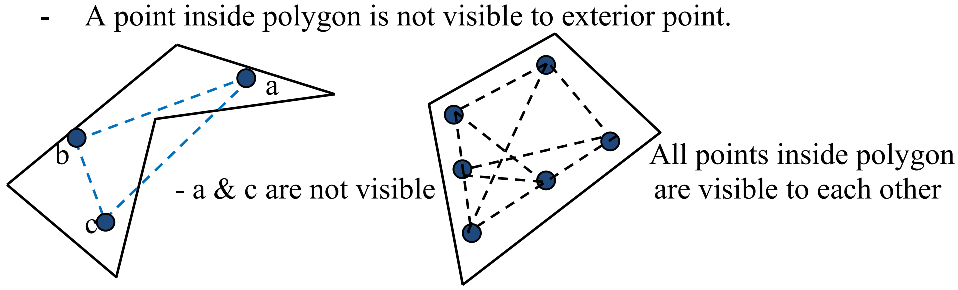
By induction hypothesis, P1 can be three colored. So, to color P just assign a color to be that is not used by a & c. Hence, P can be three colored.





Visibility inside Polygon:

- Two points inside a polygon are visible to each other if line segment joining these two points lies completely inside the polygon.
- In convex polygon each pair of interior points are visible to each other.
- Convexity to visibility refers same concept in polygon.
- A point inside polygon is not visible to exterior point.



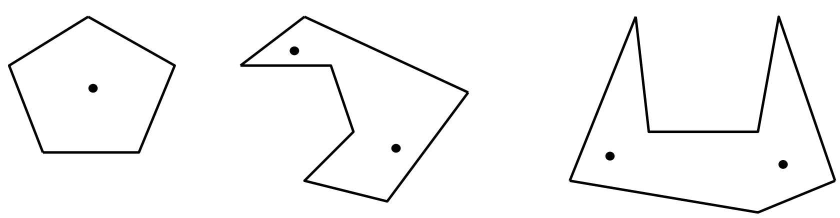
Art Gallery Problem:

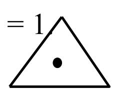
This art gallery problem was posed by Klee (1973). This problem leads us into the issues of triangulation. The problem is that;

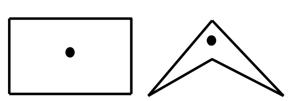
- Imagine an art gallery whose floor plan can be modeled by a simple polygon with n-vertices; the question is; how many stationary guards are needed to guard (illuminate) the gallery.

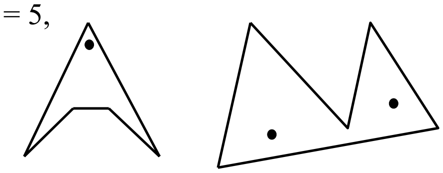
  # When a polygon is convex, one guard is sufficient to guard completely.

  # When a polygon is non-convex, one guard may not be sufficient to guard it completely.
- Generally, guards are points & have 360^0 range of visibility.



If n = 3, it is a triangle; no. of guard = 1 

$\Rightarrow$ If n = 4, 

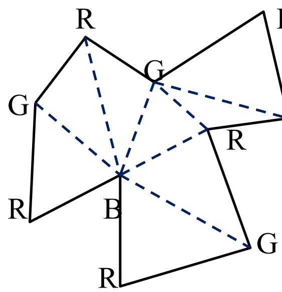
$\Rightarrow$ If n = 5, 

in 1975, Chvatal gave a proof for G (n) = $\lfloor n/3 \rfloor$ by induction.

Fisk's Proof for Art Gallery Theorem

Art Gallery Theorem: Any polygon with n-vertices can be guarded sufficiently with $\lfloor n/3 \rfloor$ guards i.e. G (n) = $\lfloor n/3 \rfloor$

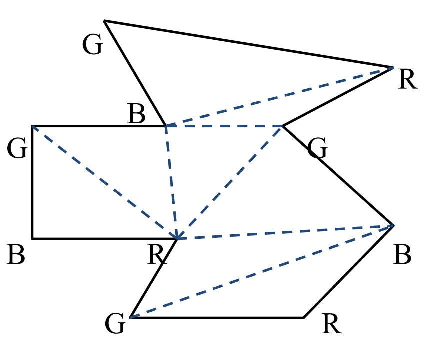
Proof: - Let us consider an arbitrary polygon P with n vertices. The polygon can be triangulated by adding diagonals and it can be three colored.



Now, placing guards at all the vertices, assigned one color guarantees visibility coverage of the polygon. Let R, G, B be the colors used in three coloring. Each triangle must have each of three colors at its three corners. Thus, every triangle has a red color (R) at one of its corner. So, placing guard at the corner of red color, the triangle will be illuminated or covered. Since, the collection of each triangle is the polygon P, hence collection f the triangles in triangulation covers the polygon. Now the final step of Fisk's proof applies the pigeon-hole principle. If n objects are placed in k holes, then at least one hole must contain no more than n/k objects. Here, n-objects are the n-nodes of the triangulation graph and k-holes are 3-colors, i.e. n-nodes of triangulated graph are to be assigned three colors i.e. k = 3.

So, by this principle, one color must be used no more than n/3 times. Since n is integer, one color is used no more than $\lfloor n/3 \rfloor$.

Hence, $\lfloor n/3 \rfloor$ guards are sufficient to guard the art gallery.

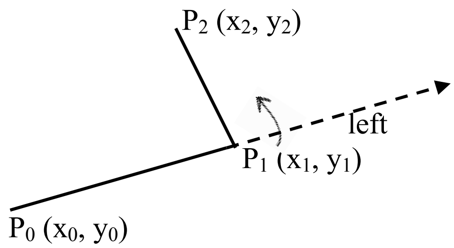


G – 4	n = 10
R – 3	So, $\lfloor n/3 \rfloor = \lfloor 10/3 \rfloor = 3$
B – 3	

Notion of Left Turn & Right Turns:

Left Turn:

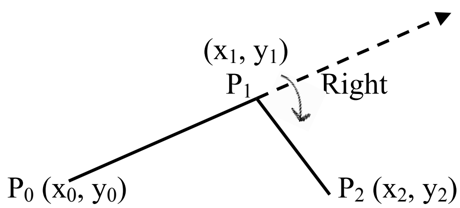
For three points P_0, P_1, P_2 in a place, P_0, P_1, P_2 is said to be left turn if line segment (P_1, P_2) lies to the left of line segment (P_0, P_1).



So, left turn $P_0P_1P_2$.

Right Turn:

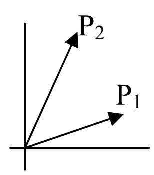
If P line segment (P_1, P_2) lies to the right of (P_0, P_1) then P_0, P_1, P_2 is a right turn.



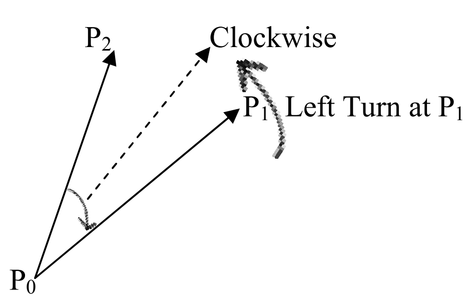
So, right turn $P_0P_1P_2$

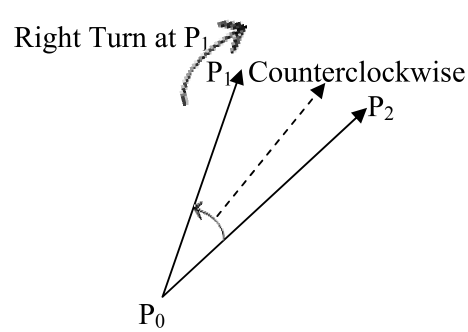
Computation of Left Turn & Right Turn:

Given the points $P_0(x_0, y_0)$, $P_1(x_1, y_1)$ & $P_2(x_2, y_2)$; to find whether P0P1P2 makes left or right turn we check whether the vector P_0P_1 in clockwise or counterclockwise with respect to vector P_0P_2. (For two vectors P_1 & P_2, P_1 is clockwise from P_2 With respect to (0, 0) is $P_1 \times P_2$ is +ve, if it is –ve, P1 & counter clockwise from P2.)



First we compute cross product of vectors; i.e. $(P_1\text{-}P_0) \times (P_2, P_0)$ (as origin is as P_0).





The cross product is:

$$(P_1-P_0) \times (P_2-P_0)$$

$$= \begin{vmatrix} x_1-x_0 & y_1-y_0 \\ x_2-x_0 & y_2-y_0 \end{vmatrix}$$

$$= (x_1-x_0)(y_2-y_0) - (x_2-x_0)(y_1-y_0)$$

(Since, geometrical interpretation of cross product gives area of parallelogram having the vectors as adjacent sides).

Thus, we can represent above equation as;

$$\Delta P_0P_1P_2 = 0.5 * \begin{vmatrix} x_0 & y_o & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = (x_1-x_0)(y_2-y_0) - (x_2-x_0)(y_1-y_0)$$

So, if the cross product; i.e. the area of triangle $P_0P_1P_2$ is positive;

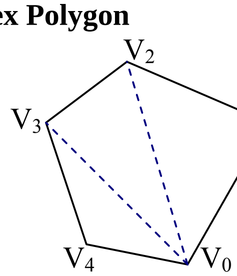
Area of Δ = +ve, $P_0P_1P_2$ is left turn at P_1 as cross product is +ve so, P_0P_1 is clockwise with respect to P_0P_2.

If Area of Δ = -ve, there is right turn at P_1, as cross product is -ve so, P_0P_1 is counter clockwise w. r. t. P_0P_2.

If Area of $\Delta = 0$, then P_0, P_1, P_2 are collinear.

Area of polygon

For Convex Polygon



Consider a convex polygon P with vertices V_0, V_1, V_2, ….. , V_{n-1} ordered in counter-clockwise direction. Let us assume the triangulation of P with every diagonal originated from the vertex V_0.

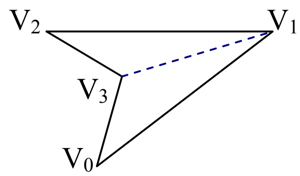
Now, the sum of area of triangles gives the area of polygon. So, area of convex polygon with n-vertices can be calculated as;

$$A(P) = A(V_0, V_1, V_2) + A(V_0, V_2, V_3) + A(V_0, V_3, V_4) + \text{-------------------} + A(V_0, V_{n-2}, V_{n-1})$$

$$= \sum_{i=1}^{n-2} A(V_0, V_i, V_{i+1})$$

For Non-convex polygon:

Let us consider a non-convex polygon with n = 4, i.e. non-convex quadrilateral.



Then, A (P) = A (V_0, V_1, V_3) + A (V_1, V_2, V_3)

Here, both areas are positive since $V_0V_1V_3$ & $V_1V_2V_3$ are left turns. So sum of areas gives the total area of the non-convex polygon.

If we consider $\Delta V_0V_1V_2$ & $V_0V_2V_3$ as in convex polygon, then,

A (P) = A (V_0, V_1, V_2) + A (V_0, V_2, V_3)

Since, $V_0V_2V_3$ is right turn which implies A (V_0, V_2, V_3) is negative. But $V_0V_1V_2$ is left turn so its area remains to be positive. So,

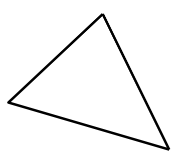
A (P) = A (V_0, V_1, V_2) + A (V_0, V_2, V_3)

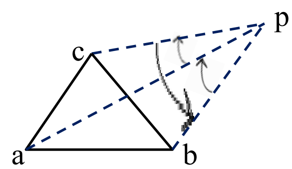
Hence, for any non-convex polygon with n-vertices this rule for obtaining area of polygon holds equally.

Area of Polygon from an arbitrary center:

Lemma: - If T = Δabc is a triangle with vertices ordered in counter clockwise and p is any point in the plane, then

A (T) = A (p, a, b) + A (p, b, c) + A (p, c, a)





⇒ Here, p is an arbitrary point in the plane. So, let us join p with a, b & c, the vertices of triangle T, as pa, pb, pc respectively.

Then, we claim that, A (T) = A (p, a, b) + A (p, b, c) + A (p, c, a)

Here, A (p, a, b) is positive since pab is a left turn.

A (p, b, c) is negative since pbc is a right turn.

A (p, c, a) is positive since pca is a left turn.

Now, A (p, b, c) subtracts exactly that portion of quadrilateral (a, b, p, c) that lies outside T, leaving the total sum precisely A (T) as claimed.

∴ A (T) = A (p, a, b) + A (p, b, c) + A (p, c, a)

= Δpab - Δpbc + pca

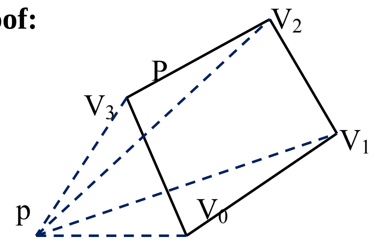
= □ abpc - Δpbc

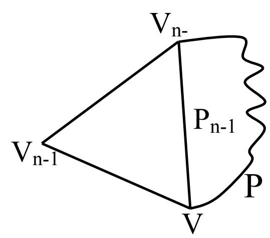
= Δabc, which is true.

Theorem: let a polygon (convex or non-convex) P having vertices V_0, V_1, ---------, V_{n-1} labeled in counter-clockwise direction & let p be any point in the plane. Then,

$A(P) = A(p, V_0, V_1) + A(p, V_1, V_2) + A(p, V_2, V_3) + \text{-------} + A(p, V_{n-2}, V_{n-1}) + A(p, V_{n-1}, V_0)$.

Proof:





Base Case:

If n = 3, polygon is a triangle, so by the lemma for area of triangle form any arbitrary point in plane, the theorem holds true.

$\therefore A(P) = A(p, V_0, V_1) + A(p, V_1, V_2) + A(p, V_2, V_0)$

Suppose a polygon P with n-vertices. Consider, the theorem is true for all the polygons with vertex size n-1. Now according to the ear existence theorem, p has an ear. Let E = (V_{n-2}, V_{n-1}, V_0) be the ear of the polygon with V_{n-1} as its tip.

Let P_{n-1} be a polygon with n-1 vertices, obtained by removing the ear E form P. Thus, by induction hypothesis,

$A(P_{n-1}) = A(p, V_0, V_1) + A(p, V_1, V_2) + \text{----------} + A(p, V_{n-2}, V_0)$ …… (i)

As, V_{n-2}, V_{n-1}, V_0 forms a triangle so its area is;

$A(E) = A(p, V_0, V_{n-2}) + A(p, V_{n-2}, V_{n-1}) + A(p, V_{n-1}, V_0)$ …………… (ii)

Clearly, area of the polygon P is

$A(P) = A(P_{n-1}) + A(E)$

$= A(p, V_0, V_1) + A(p, V_1, V_2) + \text{---------} + A(p, V_{n-2}, V_0) + A(p, V_0, V_{n-2}) + A(p, V_{n-2}, V_{n-1}) + A(p, V_{n-1}, V_0)$

But,

$A(p, V_0, V_{n-2}) = - A(p, V_{n-2}, V_0)$

So, $A(P) = A(p, V_0, V_1) + A(p, V_1, V_2) + \text{----------} + A(p, V_{n-2}, V_{n-1}) + A(p, V_{n-1}, V_0)$

Proved