Lecture-16
Algorithmic Mathematics(CSC545)

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Solution of Partial Differential Equations

A **Partial Differential Equation** commonly denoted as PDE is a differential equation containing partial derivatives of the dependent variable (one or more) with more than one independent variable. A PDE for a function $u(x_1, \ldots x_n)$ is an equation of the form

$$f\left(x_1, \ldots . x_n; u, \frac{\partial u}{\partial x_1}, \ldots \ldots, \frac{\partial u}{\partial x_n}; \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}; \ldots .\right) = 0$$

The PDE is said to be linear if f is a linear function of u and its derivatives. The simple PDE is given by;

$$\partial u/\partial x\,(x,y) = 0$$

The above relation implies that the function u(x,y) is independent of x which is the reduced form of partial differential equation formula stated above. The order of PDE is the order of the highest derivative term of the equation.

Partial Differential Equation Classification

Each type of PDE has certain functionalities that help to determine whether a particular finite element approach is appropriate to the problem being described by the PDE. The solution depends on the equation and several variables contain partial derivatives with respect to the variables. There are three-types of second-order PDEs in mechanics. They are

- Elliptic PDE
- Parabolic PDE
- Hyperbolic PDE

Consider the example, $au_{xx}+bu_{yy}+cu_{yy}=0$, $u=u(x,y)$. For a given point (x,y), the equation is said to be **Elliptic** if $b^2-ac<0$ which are used to describe the equations of elasticity without inertial terms. **Hyperbolic** PDEs describe the phenomena of wave propagation if it satisfies the condition $b^2-ac>0$. For **parabolic** PDEs, it should satisfy the condition $b^2-ac=0$. The heat conduction equation is an example of a parabolic PDE.

$4, \quad f_3 = -\frac{[illegible]}{4}$

15.4 PARABOLIC EQUATIONS

Elliptic equations studied previously describe problems that are time-independent. Such problems are known as steady-state problems. But we come across problems that are not steady-state. This means that the function is dependent on both space and time. Parabolic equations, for which

$$b^2 - 4ac = 0$$

describe the problems that depend on space and time variables.

A popular case for parabolic type of equation is the study of heat flow in one-dimensional direction in an insulated rod. Such problems are governed by both boundary and initial conditions.

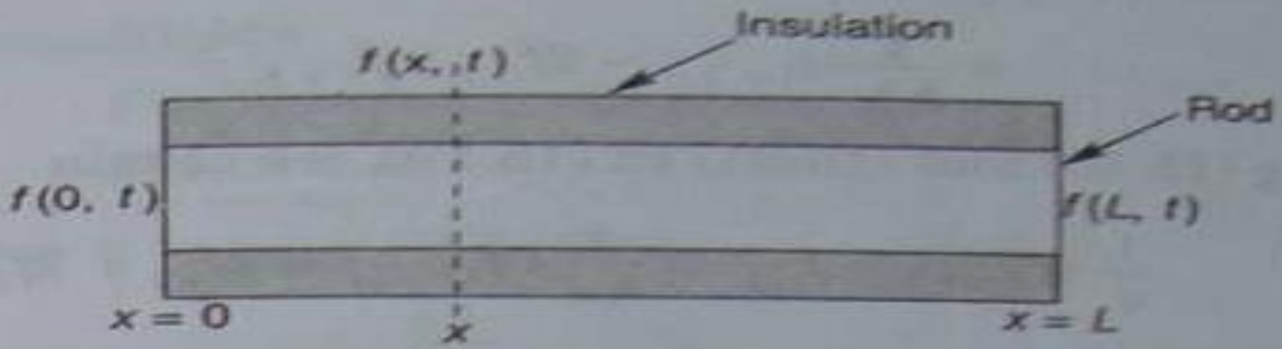


Fig. 15.3 Heat flow in a rod

Let f represent the temperature at any point in rod (Fig. 15.3) whose distance from the left end is x. Heat is flowing from left to right under the influence of temperature gradient. The temperature $f(x, t)$ in the rod at the position x and time t, is governed by the *heat equation*

$$k_1 \frac{\partial^2 f}{\partial x^2} = k_2 k_3 \frac{\partial f}{\partial t} \tag{15.18}$$

where k_1 = Coefficient of thermal conductivity; k_2 = Specific heat; and k_3 = Density of the material.

Equation (15.18) may be simplified as

$$\boxed{k f_{xx}(x, t) = f_t(x, t)} \tag{15.19}$$

where

$$k = \frac{k_1}{k_2 k_3}$$

The initial condition will be the initial temperatures at all points along the rod.

$$f(x, 0) = f(x), \qquad 0 \le x \le L$$

The boundary conditions $f(0, t)$ and $f(L, t)$ describe the temperature at each end of the rod as functions of time. If they are held constant, then

$$f(0, t) = c_1, \qquad 0 \le t < \infty$$
$$f(L, t) = c_2, \qquad 0 \le t < \infty$$

Solution of Heat Equation

We can solve the heat equation given by Eq. (15.19) using the finite difference formulae given below:

$$\begin{aligned} f_t(x, t) &= \frac{f(x, t+\tau) - f(x, t)}{\tau} \\ &= \frac{1}{\tau}(f_{i,j+1} - f_{i,j}) \end{aligned} \qquad (15.20)$$

$$f_{xx}(x, t) = \frac{f(x-h, t) - 2f(x, t) + f(x+h, t)}{h^2}$$

$$= \frac{1}{h^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) \quad (15.2$$

Substituting (15.20) and (15.21) in (15.19), we obtain

$$\frac{1}{\tau}(f_{i,j+1} - f_{i,j}) = \frac{k}{h^2}(f_{i-1,j} - 2f_{i,j} + 2f_{i} +_{1,j}) \quad (15.22)$$

Solving for $f_{i,j+1}$

$$f_{i,j+1} = \left(1 - \frac{2\tau k}{h^2}\right) f_{ij} + \frac{\tau k}{h^2}(f_{i-1,j} + f_{i+1,j})$$

$$= (1 - 2r) f_{ij} + r(f_{i-1,j} + f_{i+1,j})$$

where

$$r = \frac{\tau k}{h^2} \quad (15.23)$$

Bender-Schmidt Method

The recurrence Eq. (15.23) allows us to evaluate f at each point x and at any time t. If we choose step sizes Δt and Δx such that

$$1 - 2r = 1 - \frac{2\tau k}{h^2} = 0 \tag{15.24}$$

Equation 15.23 simplifies to

$$\boxed{f_{i,j+1} = \frac{1}{2}(f_{i+1,j} + f_{i-1,j})} \tag{15.25}$$

Equation 15.25 is known as the *Bender-Schmidt recurrence equation*. This equation determines the value of f at $x = x_i$, at time $t = t_j + \tau$, as the average of the values right and left of x_i at time t_j.

Note that the step size in time Δt obtained from Eq. (15.24)

$$\tau = \frac{h^2}{2k}$$

gives the Eq. (15.25). Equation (15.23) is stable, if and only if the step size τ satisfies the condition $\tau \leq \dfrac{h^2}{2k}$.

$2k$

Example 15.5

Solve the equation

$$2f_{xx}(x, t) = f_t(x, t), \qquad 0 < t < 1.5 \quad \text{and} \quad 0 < x < 4$$

given the initial condition

$$f(x, 0) = 50(4 - x), \qquad 0 \le x \le 4$$

and the boundary conditions

$$f(0, t) = 0, \qquad 0 \le t \le 1.5$$

$$f(4, t) = 0, \qquad 0 \le t \le 1.5$$

If we assume $\Delta x = h = 1$, $\Delta t = \tau$ must be

$$\tau \le \frac{1^2}{2 \times 2} = 0.25$$

Taking $\tau = 0.25$, we have

$$f_{i,j+1} = \frac{1}{2}(f_{i-1,j} + f_{i+1,j})$$

Using this formula, we can generate successfully $f(x, t)$. The estimates are recorded in Table 15.2. At each interior point, the temperature at any single point is just average of the values at the adjacent points of the previous time value.

Table 15.2

t \ x	0.0	1.0	2.0	3.0	4.0
0.00	0.0	150.0	100.0	50.0	0.0
0.25	0.0	50.0	100.0	50.0	0.0
0.50	0.0	50.0	50.0	50.0	0.0
0.75	0.0	25.0	25.0	25.0	0.0
1.00	0.0	12.5	25.0	12.5	0.0
1.25	0.0	12.5	12.5	12.5	0.0
1.50	0.0	6.25	12.5	6.25	0.0

$f(x,0) = 50(4-x)$

15.5 HYPERBOLIC EQUATIONS

Hyperbolic equations model the vibration of structures such as buildings, beams and machines. We consider here the case of a vibrating string that is fixed at both the ends as shown in Fig. 15.5.

The lateral displacement of string f varies with time t and distance x along the string. The displacement $f(x, t)$ is governed by the *wave equation*

$$T\frac{\partial^2 f}{\partial x^2} = \rho\frac{\partial^2 f}{\partial t^2}$$

where T is the tension in the string and ρ is the mass per unit length.

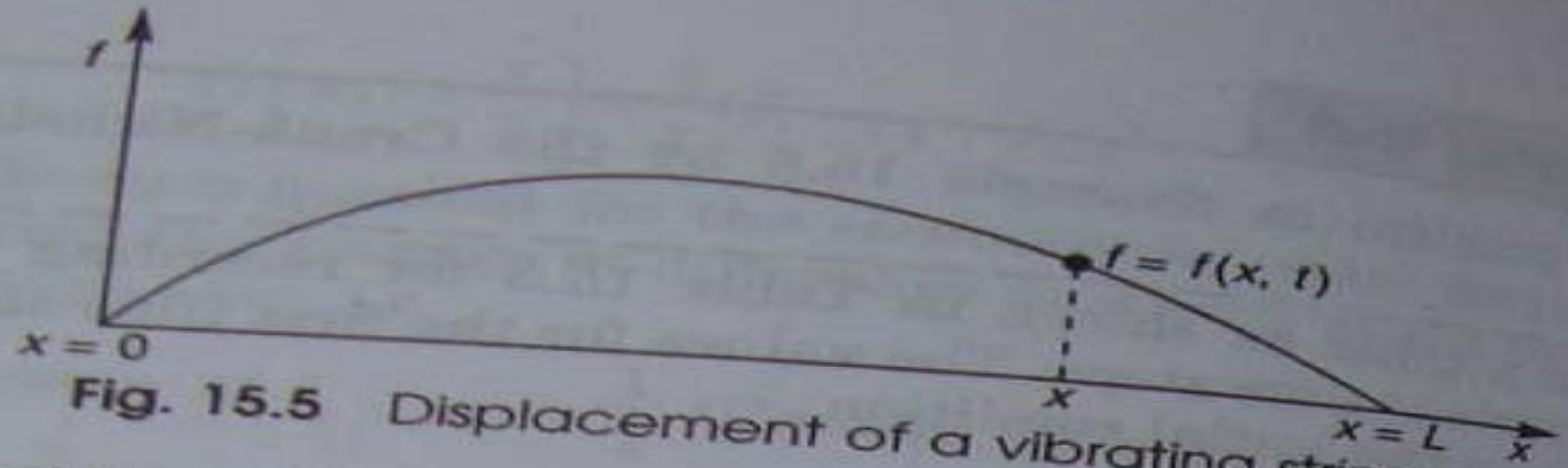


Fig. 15.5 Displacement of a vibrating string

Hyperbolic problems are also governed by both boundary and initial conditions, if time is one of the independent variables. Two boundary conditions for the vibrating string problem under consideration are

$$f(0, t) = 0 \qquad 0 \le t \le b$$
$$f(L, t) = 0 \qquad 0 \le t \le b$$

Two initial conditions are

$$f(x, 0) = f(x) \qquad 0 \le x \le a$$
$$f_t(x, 0) = g(x) \qquad 0 \le x \le a$$

Solution of Hyperbolic Equations

The domain of interest, $0 \le x \le a$ and $0 \le t \le b$, is partitioned as shown Fig. 15.6. The rectangles of size $\Delta x = h$ and $\Delta t = \tau$

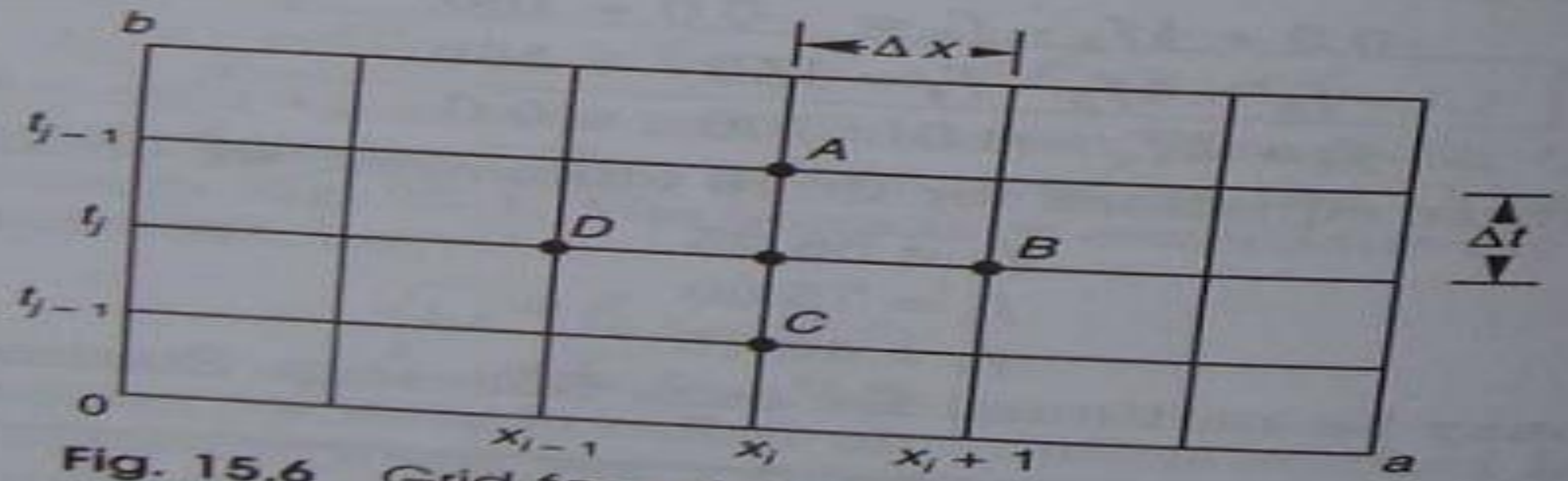


Fig. 15.6 Grid for solving hyperbolic equation

The difference equations for $f_{xx}(x, t)$ and $f_{tt}(x, t)$ are:

$$f_{xx}(x, t) = \frac{f(x-h, t) - 2f(x, t) + f(x+h, t)}{h^2}$$

$$f_{tt}(x, t) = \frac{f(x, t-\tau) - 2f(x, t) + f(x, t+\tau)}{\tau^2}$$

This implies that,

$$T\frac{f_{i-1,j}-2f_{i,j}+f_{i+1,j}}{h^2}=\rho\frac{f_{i,j-1}-2f_{i,j}+f_{i,j+1}}{\tau^2}$$

Solving this for $f_{i,j+1}$, we obtain

$$f_{i,j+1}=-f_{i,j-1}+2\left(1-\frac{T\tau^2}{\rho h^2}\right)f_{ij}+\frac{T\tau^2}{\rho h^2}(f_{i+1,j}+f_{i-1,j})$$

If we can make

$$1-\frac{T\tau^2}{\rho h^2}=0$$

then, we have

$$\boxed{f_{i,j+1}=-f_{i,j-1}+f_{i+1,j}+f_{i-1,j}} \tag{15.30}$$

The value of f at $x = x_i$ and $t = t_j + \tau$ is equal to the sum of the values of f at the point $x = x_i - h$ and $x = x_i + h$ at the time $t = t_j$ (previous time) minus the value of f at $x = x_i$ at time $t = t_j - \tau$. From Fig. 15.6, we can say that,

$$f_A = f_B + f_D - f_C$$

Starting Values

We need two rows of starting values, corresponding to $j = 1$ and $j = 2$ in order to compute the values at the third row. First row is obtained using the condition

$$f(x, 0) = f(x)$$

The second row can be obtained using the second initial condition as follows:

$$f_t(x, 0) = g(x)$$

We know that

$$f_t(x, 0) = \frac{f_{i,0+1} - f_{i,0-1}}{2\tau} = g_i$$

$$f_{i,-1} = f_{i,1} - 2\,\tau g_i \qquad \text{for } t = 0 \text{ only}$$

Substituting this in Eq. (15.30), we get for $t = t_1$

$$\boxed{f_{i,1} = \frac{1}{2}(f_{i+1,0} + f_{i-1,0}) + \tau g_i} \tag{15.31}$$

In many cases, $g(x_i) = 0$. Then, we have

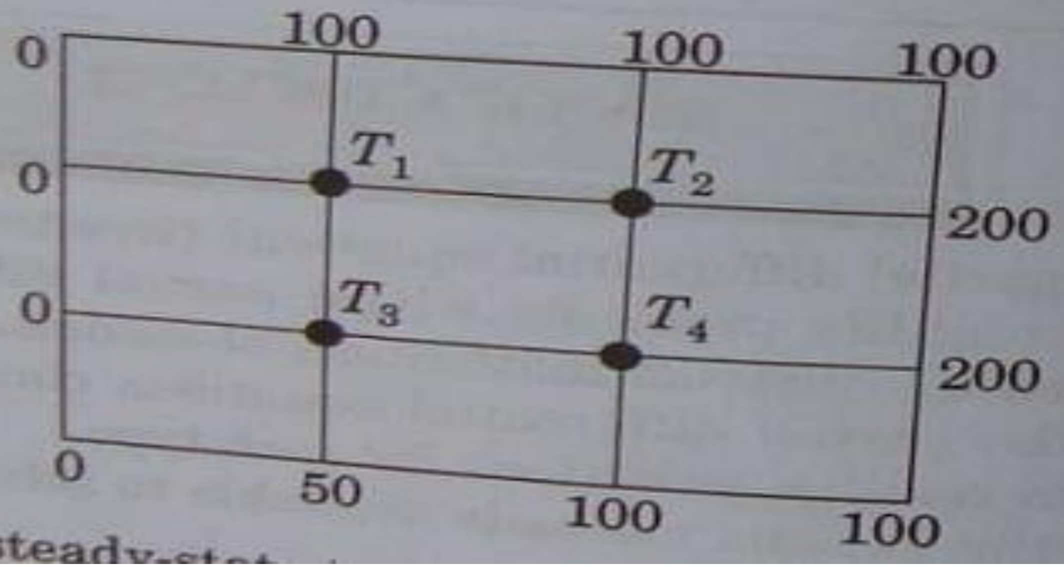
$$f_{i,1} = \frac{1}{2}(f_{i+1,0} + f_{i-1,0})$$

Assignment#16

The steady-state two-dimensional heat-flow in a metal plate is by

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

Given the boundary conditions as shown in the figure below, find the temperatures T_1, T_2, T_3, and T_4.



Solve for the steady-

Thank You

Any Query??