Lecture-3
Algorithmic Mathematics(CSC545)

Prepared by Asst. Prof. Bal Krishna Subedi

CDCSIT, TU

Secant Methods for Solving Nonlinear Equations

The Newton-Raphson method of solving a nonlinear equation $f(x)=0$ is given by the iterative formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1}$$

One of the drawbacks of the Newton-Raphson method is that you have to evaluate the derivative of the function.

To overcome these drawbacks, the derivative of the function,

$f(x)$ is approximated as

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \tag{2}$$

Substituting Equation (2) in Equation (1) gives

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \tag{3}$$

The above equation is called the secant method. This method now requires two initial guesses, but unlike the bisection method, the two initial guesses do not need to bracket the root of the equation. The secant method is an open method and may or may not converge. However, when secant method converges, it will typically converge faster than the bisection method. However, since the derivative is approximated as given by Equation (2), it typically converges slower than the Newton-Raphson method.

The secant method can also be derived from geometry, as shown in Figure 1. Taking two initial guesses, x_{i-1} and x_i, one draws a straight line between $f(x_i)$ and $f(x_{i-1})$ passing through the x-axis at x_{i+1}. ABE and DCE are similar triangles.

Hence

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

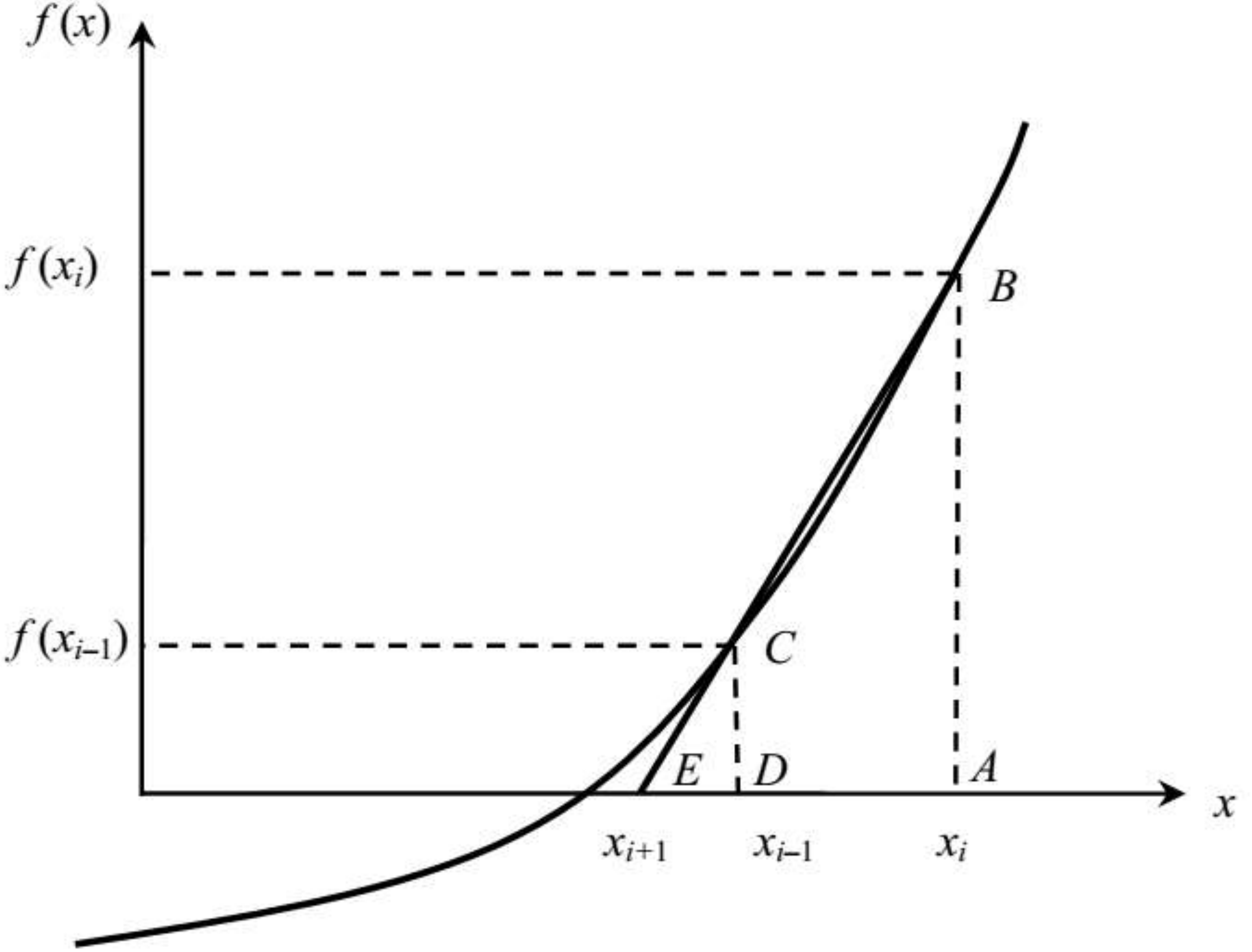

Figure 1 Geometrical representation of the secant method.

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats (Figure 2) for ABC commodes. The floating ball has a specific gravity of 0.6 and a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.
The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the secant method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error and the number of significant digits at least correct at the end of each iteration.

Solution

$$f(x)=x^3-0.165x^2+3.993\times10^{-4}$$

Let us assume the initial guesses of the root of $f(x)=0$ as $x_{-1}=0.02$ and $x_0=0.05$.

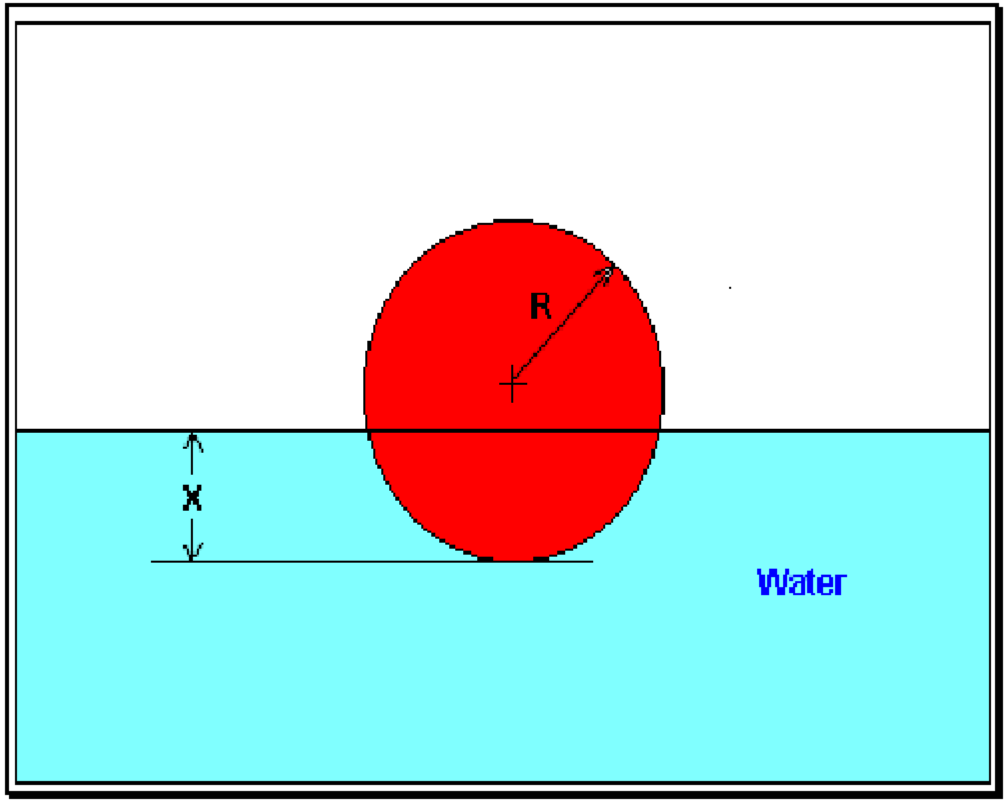

Figure 2 Floating ball problem.

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= x_0 - \frac{\left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right)\times\left(x_0 - x_{-1}\right)}{\left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right) - \left(x_{-1}^3 - 0.165x_{-1}^2 + 3.993\times10^{-4}\right)}$$

$$= 0.05 - \frac{\left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right]\times\left[0.05 - 0.02\right]}{\left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right] - \left[0.02^3 - 0.165(0.02)^2 + 3.993\times10^{-4}\right]}$$

$$= 0.06461$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

$$= \left|\frac{0.06461 - 0.05}{0.06461}\right| \times 100$$

$$= 22.62\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for one significant digit to be correct in your result.

Iteration 2

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= x_1 - \frac{\left(x_1^3 - 0.165x_1^2 + 3.993\times10^{-4}\right)\times\left(x_1 - x_0\right)}{\left(x_1^3 - 0.165x_1^2 + 3.993\times10^{-4}\right) - \left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06461 - \frac{\left[0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right]\times(0.06461 - 0.05)}{\left[0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right] - \left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right]}$$

$$= 0.06241$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{x_2 - x_1}{x_2}\right|\times100$$

$$= \left|\frac{0.06241 - 0.06461}{0.06241}\right|\times100$$

$$= 3.525\%$$

The number of significant digits at least correct is 1, as you need an absolute relative approximate error of 5% or less.

Iteration 3

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= x_2 - \frac{\left(x_2^3 - 0.165x_2^2 + 3.993\times10^{-4}\right)\times\left(x_2 - x_1\right)}{\left(x_2^3 - 0.165x_2^2 + 3.993\times10^{-4}\right) - \left(x_1^3 - 0.165x_1^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06241 - \frac{\left[0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right]\times(0.06241 - 0.06461)}{\left[0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right] - \left[0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right]}$$

$$= 0.06238$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left|\frac{x_3 - x_2}{x_3}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06241}{0.06238}\right| \times 100$$

$$= 0.0595\%$$

The number of significant digits at least correct is 2, as you need an absolute relative approximate error of 0.5% or less. Table 1 shows the secant method calculations for the results from the above problem.

Table 1 Secant method results as a function of iterations.

Iteration Number, i	x_{i-1}	x_i	x_{i+1}	$\lvert \in_a \rvert \%$	$f(x_{i+1})$
1	0.02	0.05	0.06461	22.62	-1.9812×10^{-5}
2	0.05	0.06461	0.06241	3.525	-3.2852×10^{-7}
3	0.06461	0.06241	0.06238	0.0595	2.0252×10^{-9}
4	0.06241	0.06238	0.06238	-3.64×10^{-4}	-1.8576×10^{-13}

Secant Method

1. Decide two initial points x_1 and x_2, accuracy level required, E.
2. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$
3. Compute $x_3 = \dfrac{f_2 x_1 - f_1 x_2}{f_2 - f_1}$
4. Test for accuracy of x_3.

   If $\left|\dfrac{x_3 - x_2}{x_3}\right| > E$, then

   set $x_1 = x_2$ and $f_1 = f_2$
   set $x_2 = x_3$ and $f_2 = f(x_3)$
   go to step 3

   otherwise,

   set root $= x_3$

   print results
5. Stop

Use the secant method to estimate the root of the equation

$$x^2 - 4x - 10 = 0$$

with the initial estimates of $x_1 = 4$ and $x_2 = 2$

Given $x_1 = 4$ and $x_2 = 2$

$$f(x_1) = f(4) = -10$$

$$f(x_2) = f(2) = -14$$

(Note that these points do not bracket a root)

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 2 - \frac{-14(2-4)}{-14-(-10)} = 9$$

For second iteration,

$$x = x_2 = 2$$

$$x_2 = x_3 = 9$$

$$f(x_1) = f(2) = -14$$

$$f(x_2) = f(9) = 95$$

$$x_3 = 9 - \frac{35(9-2)}{35+14} = 4$$

For third iteration,

$$x_1 = 9$$

$$x_2 = 4$$

$$f(x_1) = f(9) = 95$$

$$f(x_2) = f(4) = -10$$

$$x_3 = 4 - \frac{-10(4-9)}{-10-35} = 5.1111$$

For fourth iteration,

$$x_1 = 4$$

$$x_2 = 5.1111$$

$$f(x_1) = f(4) = -10$$

$$f(x_2) = f(5.1111) = -4.3207$$

$$x_3 = 5.1111 - \frac{-4.3207(5.1111-4)}{-4.3207-10} = 5.9563$$

For fifth iteration,

$$x_1 = 5.1111$$

$$x_2 = 5.9563$$

$$f(x_1) = f(5.1111) = -4.3207$$

$$f(x_2) = f(5.9563) = 5.0331$$

$$x_3 = 5.9563 - \frac{5.0331(5.9563 - 5.1111)}{5.0331 + 4.3207} = 5.5014$$

For sixth iteration,

$$x_1 = 5.9563$$

$$x_2 = 5.5014$$

$$f(x_1) = f(5.9539) = 5.0331$$

$$f(x_2) = f(5.5014) = -1.7392$$

$$x_3 = 5.5014 - \frac{-1.7392(5.5014 - 5.9563)}{-1.7392 + 5.0331} = 5.6182$$

The value can be further refined by continuing the process, if necessary.

Convergence of Secant Method

The secant formula of iteration is

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \qquad (6.29)$$

Let x_r be actual root of $f(x)$ and e_i the error in the estimate of x_i. Then,

$$x_{i+1} = e_{i+1} + x_r$$
$$x_i = e_i + x_r$$
$$x_{i-1} = e_{i-1} + x_r$$

Substituting these in Eq. (6.29) and simplifying, we get the error equation as

$$e_{i+1} = \frac{e_{i-1} f(x_i) - e_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \qquad (6.30)$$

According to the Mean Value Theorem, there exists at least one point, say $x = R_i$, in the interval x_i and x_r such that

$$f'(R_i) = \frac{f(x_i) - f(x_r)}{x_i - x_r}$$

We know that

$$f(x_r) = 0$$
$$x_i - x_r = e_i$$

and therefore

$$f'(R_i) = \frac{f(x_i)}{e_i}$$

or

$$f(x_i) = e_i f'(R_i)$$

Similarly,

$$f(x_{i-1}) = e_{i-1} f'(R_{i-1})$$

Substituting these in the numerator of Eq. (6.30), we get

$$e_{i+1} = e_i e_{i-1} \frac{f'(R_i) - f'(R_{i-1})}{f(x_i) - f(x_{i-1})}$$

That is, we can say

$$\boxed{e_{i+1} \propto e_i \, e_{i-1}} \tag{6.31}$$

We know that the order of convergence of an iteration process is p, if

$$\boxed{e_i \propto e_{i-1}^p} \tag{6.32}$$

or

$$e_{i+1} \propto e_i^p \tag{6.33}$$

Substituting for e_{i+1} and e_i in Eq. (6.31), we get

$$e_i^p \propto e_{i-1}^p \, e_{i-1}$$

or

$$\boxed{e_i \propto e_{i-1}^{(p+1)/p}} \tag{6.34}$$

Comparing the relations (6.32) and (6.31), we observe that

$$p = (p + 1)/p$$

That is,

$$p^2 - p - 1 = 0$$

which has the solutions

$$p = \frac{1 \pm \sqrt{5}}{2}$$

Since p is always positive, we have

$$p = 1.618$$

It follows that the order of convergence of the secant method is 1.618 and the convergence is referred to as *superlinear convergence*.

Advantages of Secant Methods

Advantages of secant method:

1. It converges at faster than a linear rate, so that it is more rapidly
convergent than the bisection method.
2. It does not require use of the derivative of the function, something that is not available in a number of applications.
3. It requires only one function evaluation per iteration, as compared with Newton's method which requires two.

Disadvantages of Secant Methods

Disadvantages of secant method:

1. It may not converge.
2. There is no guaranteed error bound for the computed iterates.
3. It is likely to have difficulty if $f\ 0(\alpha) = 0$. This means the x-axis is tangent to the graph of $y = f(x)$ at $x = \alpha$.
4. Newton's method generalizes more easily to new methods for solving simultaneous systems of nonlinear equations.

Assignment#2

1. Use the secant Method to compute a root of the following equations:
   - $4x^2-2x-6=0$
   - $x \sin x - 1=0$
   - $e^x-3x=0$

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Submission to: balkrishna@cdcsit.edu.np

Thanks You

Any Query??