Lecture-13
Algorithmic Mathematics(CSC545)

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A **matrix eigenvalue problem** considers the vector equation

(1) $$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

Here $\mathbf{A}$ is a given square matrix, λ an unknown scalar, and $\mathbf{x}$ an unknown vector. In a matrix eigenvalue problem, the task is to determine λ's and $\mathbf{x}$'s that satisfy (1).

Since $\mathbf{x} = \mathbf{0}$ is always a solution for any and thus not interesting, we only admit solutions with $\mathbf{x} \neq \mathbf{0}$.

The solutions to (1) are given the following names: The λ's that satisfy (1) are called **eigenvalues of A** and the corresponding nonzero $\mathbf{x}$'s that also satisfy (1) are called **eigenvectors of A**.

We formalize our observation. Let $\mathbf{A} = [a_{jk}]$ be a given nonzero square matrix of dimension $n \times n$. Consider the following vector equation:

(1) $$\mathbf{Ax} = \lambda \mathbf{x}.$$

The problem of finding nonzero **x**'s and λ's that satisfy equation (1) is called **an eigenvalue problem**.

A value of λ for which (1) has a solution $\mathbf{x} \neq \mathbf{0}$ is called an **eigenvalue** or *characteristic value* of the matrix **A**.

The corresponding solutions $\mathbf{x} \neq \mathbf{0}$ of (1) are called the **eigenvectors** or *characteristic vectors* of **A** corresponding to that eigenvalue λ.

The set of all the eigenvalues of **A** is called the **spectrum** of **A**. We shall see that the spectrum consists of at least one eigenvalue and at most of n numerically different eigenvalues.

How to Find Eigenvalues and Eigenvectors

EXAMPLE 1

Determination of Eigenvalues and Eigenvectors

We illustrate all the steps in terms of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}.$$

EXAMPLE 1 (continued 1)
Determination of Eigenvalues and Eigenvectors

Solution.

(a) ***Eigenvalues.*** These must be determined ***first***.

Equation (1) is

$$\mathbf{Ax} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

in components

$$\begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2 . \end{aligned}$$

Solution. ***(continued 1)***

(a) ***Eigenvalues.*** ***(continued 1)***

Transferring the terms on the right to the left, we get

$$(2^*)\qquad \begin{aligned}(-5-\lambda)x_1 + \qquad\qquad 2x_2 &= 0\\ 2x_1 + (-2-\lambda)x_2 &= 0\end{aligned}$$

This can be written in matrix notation

$$(3^*)\qquad (\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0$$

Because (1) is $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{Ax} - \lambda\mathbf{Ix} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, which gives (3*).

Solution. *(continued 2)*

(a) ***Eigenvalues.*** *(continued 2)*

We see that this is a ***homogeneous*** linear system. It has a nontrivial solution (an eigenvector of **A** we are looking for) if and only if its coefficient determinant is zero, that is,

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$(4^*) \qquad = (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0.$$

Solution. ***(continued 3)***

(a) ***Eigenvalues.*** ***(continued 3)***

We call $D(\lambda)$ the **characteristic determinant** or, if expanded, the **characteristic polynomial**, and $D(\lambda) = 0$ the **characteristic equation** of **A**. The solutions of this quadratic equation are $\lambda_1 = -1$ and $\lambda_2 = -6$. These are the eigenvalues of **A**.

(b$_1$) ***Eigenvector of* A *corresponding to* λ_1.** This vector is obtained from (2*) with $\lambda = \lambda_1 = -1$, that is,

$$-4x_1 + 2x_2 = 0$$
$$2x_1 - x_2 = 0.$$

Solution. *(continued 4)*

(b_1) ***Eigenvector of* A *corresponding to* λ_1.** *(continued)*

A solution is $x_2 = 2x_1$, as we see from either of the two equations, so that we need only one of them. This determines an eigenvector corresponding to $\lambda_1 = -1$ up to a scalar multiple. If we choose $x_1 = 1$, we obtain the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{Check:} \quad \mathbf{Ax}_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)x_1 = \lambda_1 x_1.$$

Solution. *(continued 5)*

(b$_2$) ***Eigenvector of A corresponding to λ_2.***

For $\lambda = \lambda_2 = -6$, equation (2*) becomes

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0.$$

A solution is $x_2 = -x_1/2$ with arbitrary x_1. If we choose $x_1 = 2$, we get $x_2 = -1$. Thus an eigenvector of **A** corresponding to $\lambda_2 = -6$ is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{Check:} \quad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = (-6)x_2 = \lambda_2 x_2$$

This example illustrates the general case as follows. Equation (1) written in components is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n . \end{aligned}$$

Transferring the terms on the right side to the left side, we have

$$\begin{aligned} (a_{11} - \lambda)x_1 + \quad a_{12}x_2 + \cdots + \quad a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + \quad a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + \quad a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n &= 0. \end{aligned} \tag{2}$$

In matrix notation,

(3) $$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

By **Cramer's theorem** in Sec. 7.7, this homogeneous linear system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero:

(4) $$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

$\mathbf{A} - \lambda\mathbf{I}$ is called the **characteristic matrix** and $D(\lambda)$ the **characteristic determinant** of **A**. Equation (4) is called the **characteristic equation** of **A**. By developing $D(\lambda)$ we obtain a polynomial of nth degree in λ. This is called the **characteristic polynomial** of **A**.

Solution of ordinary Differential Equations

Initial Value Problems

In order to obtain the values of the integration constants, we need additional information. For example, consider the solution $y = ae^x$ to the equation $y' = y$. If we are given a value of y for some x, the constant can be determined. Suppose $y = 1$ at $x = 0$, then,

$$y(0) = ae^0 = 1$$

Therefore,

$$a = 1$$

and the particular solution is

$$y = e^x$$

If the order of the equation is n, we will have to obtain n constants and therefore, we need n conditions in order to obtain a unique solution. When all the conditions are specified at a particular value of the independent variable x, then the problem is called an *initial-value problem*.

It is also possible to specify the conditions at different values of the independent variable. Such problems are called the *boundary-value problems*. For example, if, instead of specifying only $y(0) = 1$, we also specify $y(0) + y(1) = 2$, then the problem will be a boundary-value problem. In this case,

$$y(0) + y(1) = a(1 + e) = 2$$

giving

$$a = 2/(1 + e)$$

13.2 TAYLOR SERIES METHOD

We can expand a function $y(x)$ about a point $x = x_0$ using Taylor's theorem of expansion

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + (x - x_0)^2 \frac{y''(x_0)}{2!} + \ldots + (x - x_0)^n \frac{y^n(x_0)}{n!} \quad (13.11)$$

where $y^i(x_0)$ is the ith derivative of $y(x)$, evaluated at $x = x_0$. The value of $y(x)$ can be obtained if we know the values of its derivatives. This implies that if we are given the equation

$$y' = f(x, y) \quad (13.12)$$

we must then repeatedly differentiate $f(x, y)$ implicitly with respect to x and evaluate them at x_0.

For example, if $y' = f(x, y)$ then

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f(x, y)]$$

$$= \frac{\partial}{\partial x}[f(x, y)] + \frac{\partial}{\partial y}[f(x, y)]\frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f = f_x + f \times f_y \quad (13.13)$$

where f denotes the function $f(x, y)$ and f_x and f_y denote the partial derivatives of the function $f(x, y)$ with respect to x and y, respectively. Similarly, we can obtain

$$y''' = f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2 \quad (13.14)$$

Consider the equation

$$y' = x^2 + y^2$$

under the condition $y(x) = 1$ when $x = 0$,

$$y' = x^2 + y^2$$

$$y'' = 2x + 2yy'$$

$$y''' = 2 + 2yy'' + 2(y')^2$$

at $x = 0$, $y(0) = 1$ and, therefore,

$$y'(0) = 1$$

$$y''(0) = 2$$

$$y'''(0) = 2 + (2)(1)(2) + (2)(1)^2 = 8$$

Substituting these values, the Taylor series becomes

$$y(x) = 1 + x + x^2 + \frac{8}{3!}x^3 + \dots \quad (13.15)$$

The number of terms to be used depends on the accuracy of the solution needed.

Example 13.1

Use the Taylor method to solve the equation

$$y' = x^2 + y^2$$

for $x = 0.25$ and $x = 0.5$ given $y(0) = 1$

The solution of this equation is given by Eq. (13.15). That is,

$$y(x) = 1 + x + x^2 + 8\frac{x^3}{3!} + \ldots$$

Therefore,

$$y(0.25) = 1 + 0.25 + (0.25)^2 + \frac{8}{6}(0.25)^3 + \ldots$$

$$= 1.33333$$

Similarly,

$$y(0.5) = 1 + 0.5 + 0.5^2 + \frac{8}{6}(0.5)^3 + \ldots$$

$$= 1.81667$$

Impr

13.3 EULER'S METHOD

Euler's method is the simplest one-step method and has a limited application because of its low accuracy. However, it is discussed here as it serves as a starting point for all other advanced methods.

Consider the first two terms of the expansion (13.11)

$$y(x) = y(x_0) + y'(x_0)(x - x_0)$$

Given the differential equation

$$y'(x) = f(x, y) \text{ with } y(x_0) = y_0$$

we have

$$y'(x_0) = f(x_0, y_0)$$

and therefore

$$y(x) = y(x_0) + (x - x_0) f(x_0, y_0)$$

Then, the value of $y(x)$ at $x = x_1$ is given by

$$y(x_1) = y(x_0) + (x_1 - x_0) f(x_0, y_0)$$

Letting $h = x_1 - x_0$, we obtain

$$y_1 = y_0 + h f(x_0, y_0)$$

Similarly, $y(x)$ at $x = x_2$ is given by

$$y_2 = y_1 + h f(x_1, y_1)$$

In general, we obtain a recursive relation as

$$\boxed{y_{i+1} = y_i + h f(x_i, y_i)} \quad (13.19)$$

... used recursively

This formula is known as *Euler's method* and can be used recursively to evaluate $y_1, y_2, \ldots$ of $y(x_1), y(x_2), \ldots$, starting from the initial condition $y_0 = y(x_0)$. Note that this does not involve any derivatives.

A new value of y is estimated using the previous value of y as the initial condition. Note that the term $h\, f(x_i, y_i)$ represents the incremental value of y and $f(x_i, y_i)$ is the slope of $y(x)$ at (x_i, y_i), i.e. the new value is obtained by extrapolating linearly over the step size h using the slope at its previous value. That is

New value = old value + slope × step size

This is illustrated in Fig. 13.1. Remember that y_1 approximates $y(x_1)$ and y_2 approximates $y(x_2)$. The difference between them is the error introduced by the method.

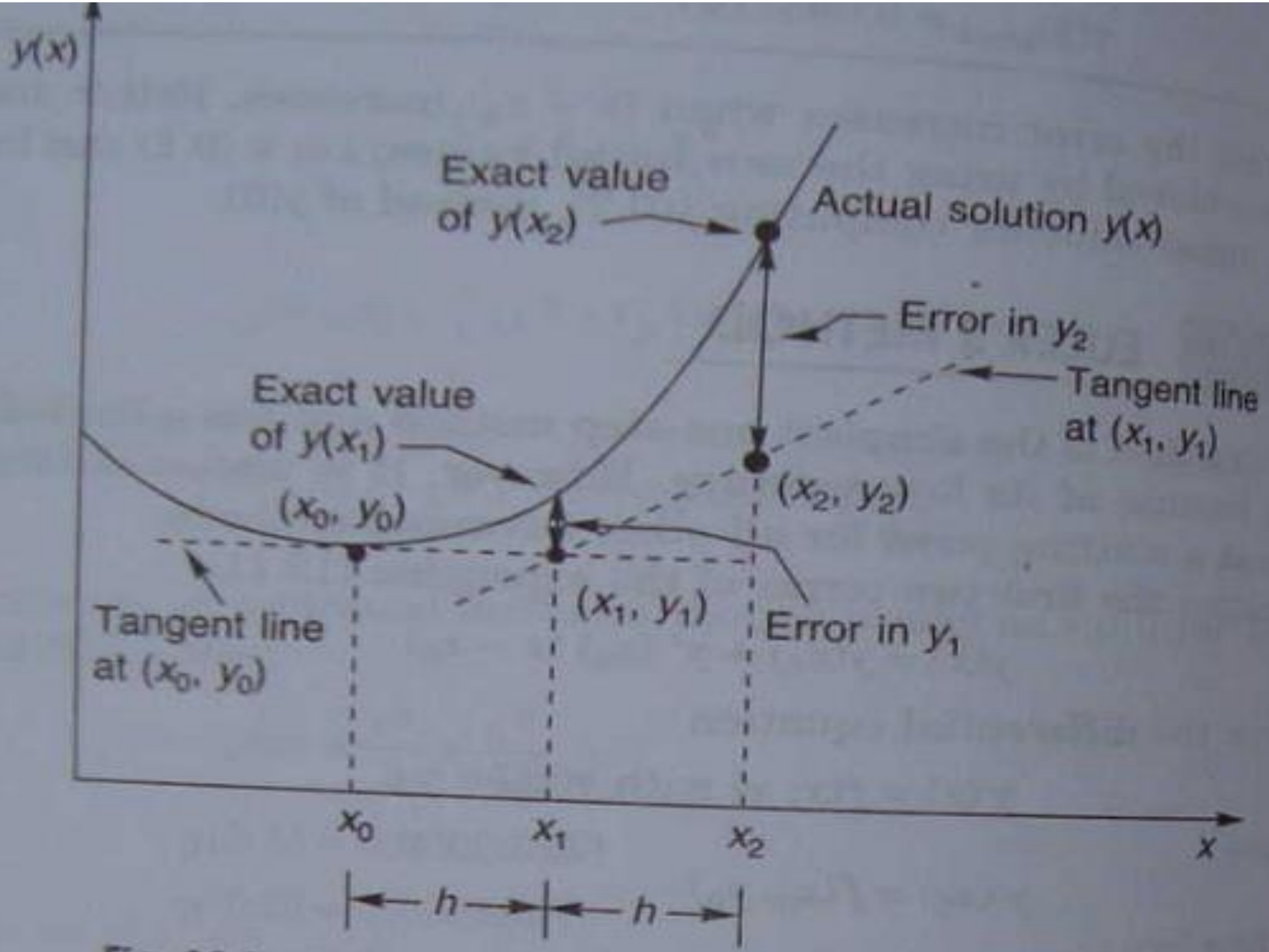

Fig. 13.1 Illustration of Euler's method for two steps

Example 13.4

Given the equation

$$\frac{dy}{dx} = 3x^2 + 1 \qquad \text{with } y(1) = 2$$

estimate $y(2)$ by Euler's method using (i) $h = 0.5$ and (ii) $h = 0.25$.

(i) $h = 0.5$

$$y(1) = 2$$
$$y(1.5) = 2 + 0.5[3(1.0)^2 + 1] = 4.0$$
$$y(2.0) = 4.0 + 0.5[3(1.5)^2 + 1] = 7.875$$

(ii) $h = 0.25$

$$y(1) = 2$$
$$y(1.25) = 2 + 0.25[3(1)^2 + 1] = 3.0$$
$$y(1.5) = 3 + 0.25[3(1.25)^2 + 1] = 5.42188$$
$$y(1.75) = 5.42188 + 0.25[3(1.5)^2 + 1] = 7.35938$$
$$y(2.0) = 7.35938 + 0.25[3(1.75)^2 + 1] = 9.90626$$

Thank You

Any Query??