Figure-8 Trajectory Generation in Robotics

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1 Introduction

A smooth and well-defined trajectory is crucial for testing and evaluating state estimation models, measurement systems, and prediction algorithms in robotics. This document outlines the design and implementation of a extbfFigure-8 Trajectory Generator, which provides a robust motion profile for dynamic testing scenarios.

2 Trajectory Configuration

The Figure-8 trajectory is configured using a set of well-defined parameters:

- **Maximum Velocity** ($v_{\max}$): Controls the peak velocity along the trajectory.
- **Length** (a): Specifies the amplitude of the X-axis motion.
- **Width** (b): Specifies the amplitude of the Y-axis motion.
- **Width Slope** (c): Determines the inclination in the Z-direction.
- **Angular Scale** (α): Scales the angular motion for orientation dynamics.

3 Mathematical Formulation

The position in the Figure-8 trajectory is parameterized by time as follows:

$$\mathbf{p}(t) = \begin{bmatrix} a\cos(w_2 t) \\ b\sin(w_1 t) \\ c\sin(w_1 t) \end{bmatrix}$$

Where:

- $a = \frac{\text{length}}{2}$ (X amplitude)
- $b = \frac{\text{width}}{2}$ (Y amplitude)
- $c = \text{width} \times \tan(\text{width slope})$ (Z amplitude)
- $w_1 = \frac{2\pi}{T}$ (frequency for Y and Z motion)
- $w_2 = w_1 \times 0.5$ (frequency for X motion)
- $T = \frac{\pi\sqrt{a^2+4(b^2+c^2)}}{v_{\max}}$ (Trajectory period)

4 Orientation and Angular Dynamics

To ensure the body frame's X-axis aligns with the velocity direction, the orientation is defined using a quaternion. The axis of rotation is computed as the cross product of the inertial velocity direction and the X-axis in the world frame:

$$\mathbf{r} = \hat{x} \times \frac{\mathbf{v}}{||\mathbf{v}||}$$

The angle of rotation is the angle between these two vectors:

$$\theta = \cos^{-1}\left(\hat{x} \cdot \frac{\mathbf{v}}{||\mathbf{v}||}\right)$$

The corresponding quaternion representation is constructed as:

$$\mathbf{q} = \left[\cos\left(\frac{\theta}{2}\right), \mathbf{r}\sin\left(\frac{\theta}{2}\right)\right]$$

This quaternion is normalized to ensure numerical stability:

$$\mathbf{q} = \frac{\mathbf{q}}{||\mathbf{q}||}$$

The velocity in the body frame is calculated by transforming the inertial velocity vector:

$$\mathbf{v}_{\text{body}} = \mathbf{q}^{-1}\mathbf{v}_{\text{inertial}}$$

The angular velocity in the world frame is calculated as:

$$\boldsymbol{\omega}_{\text{world}} = \frac{\mathbf{v} \times \mathbf{a}}{||\mathbf{v}||^2}$$

The angular velocity is then converted to the body frame:

$$\boldsymbol{\omega}_{\text{body}} = \mathbf{q}^{-1}\boldsymbol{\omega}_{\text{world}}$$

The body-frame acceleration is adjusted to incorporate full Coriolis compensation:

$$\mathbf{a}_{\text{body}} = \mathbf{q}^{-1}\mathbf{a} - 2\boldsymbol{\omega}_{\text{body}} \times \mathbf{v}_{\text{body}}$$

The angular acceleration is derived as:

$$\boldsymbol{\alpha}_{\text{world}} = \frac{\mathbf{v} \times \mathbf{j} \cdot ||\mathbf{v}||^2 - 2\mathbf{v} \times \mathbf{a} \cdot \mathbf{v} \cdot \mathbf{a}}{||\mathbf{v}||^4}$$

Finally, the angular acceleration is transformed to the body frame:

$$\boldsymbol{\alpha}_{\text{body}} = \mathbf{q}^{-1}\boldsymbol{\alpha}_{\text{world}}$$

5 Conclusion

This Figure-8 trajectory generator offers a robust and versatile tool for testing state estimation, measurement models, and prediction systems. By combining smooth position dynamics with controlled angular motion, it effectively replicates challenging real-world motion patterns for robotics testing environments.