1 Trajectory Definition

$$a = 0.5 \times \text{length}, \quad b = 0.5 \times \text{width}, \quad c = \text{width} \tan(\text{width\_slope})$$

$$\omega_1 = \frac{2\pi}{T}, \quad \omega_2 = \frac{\omega_1}{2}, \quad T = \frac{\pi\sqrt{a^2 + 4(b^2 + c^2)}}{\text{max\_velocity}}$$

2 Core Kinematics

2.1 Position

$$\boldsymbol{r}(t) = \begin{bmatrix} a\cos(\omega_2 t) \\ b\sin(\omega_1 t) \\ c\sin(\omega_1 t) \end{bmatrix}$$

2.2 Velocity

$$\boldsymbol{v}(t) = \begin{bmatrix} -a\omega_2 \sin(\omega_2 t) \\ b\omega_1 \cos(\omega_1 t) \\ c\omega_1 \cos(\omega_1 t) \end{bmatrix}$$

2.3 Acceleration

$$\boldsymbol{a}(t) = \begin{bmatrix} -a\omega_2^2 \cos(\omega_2 t) \\ -b\omega_1^2 \sin(\omega_1 t) \\ -c\omega_1^2 \sin(\omega_1 t) \end{bmatrix}$$

2.4 Jerk

$$\boldsymbol{j}(t) = \begin{bmatrix} a\omega_2^3 \sin(\omega_2 t) \\ -b\omega_1^3 \cos(\omega_1 t) \\ -c\omega_1^3 \cos(\omega_1 t) \end{bmatrix}$$

3 Angular Motion

3.1 Orientation Angles

$$\phi = \text{(Roll - Future Extension)}, \quad \theta = -\arcsin\left(\frac{v_z}{\|\boldsymbol{v}\|}\right), \quad \psi = \arctan\left(\frac{v_y}{v_x}\right)$$

3.2 Pitch and Yaw Rates

$$\text{Pitch Rate:} \quad \dot{\theta} = -\frac{\|\boldsymbol{v}\| a_z - v_z \boldsymbol{v}^\top \boldsymbol{a}/\|\boldsymbol{v}\|}{\boldsymbol{v}^\top \boldsymbol{v}\sqrt{1 - v_z^2/\boldsymbol{v}^\top \boldsymbol{v}}}$$

$$\text{Yaw Rate:} \quad \dot{\psi} = \frac{\boldsymbol{v}_{xy}^\top \boldsymbol{J} \boldsymbol{a}_{xy}}{\|\boldsymbol{v}_{xy}\|^2}$$

3.3 Pitch and Yaw Accelerations

Pitch Acceleration: $$\ddot{\theta} = -\frac{-v_z^2\|\boldsymbol{v}\|^2 j_z + v_z\|\boldsymbol{v}\|^2(a_z^2 - \|\boldsymbol{v}\|j_z + 2a_z a_{||}) + \|\boldsymbol{v}\|^3(\|\boldsymbol{v}\|j_z - 2a_z a_{||}) + v_z^3(\|\boldsymbol{v}\|a_{||} - a$$
$$\|\boldsymbol{v}\|^3\sqrt{1 - v_z^2/\|\boldsymbol{v}\|^2}(\|\boldsymbol{v}\|^2 - v_z^2)$$

Yaw Acceleration: $$\ddot{\psi} = \frac{(\boldsymbol{v}_{xy}^\top \boldsymbol{J}\boldsymbol{j}_{xy})\|\boldsymbol{v}_{xy}\|^2 - 2(\boldsymbol{v}_{xy}^\top \boldsymbol{a}_{xy})(\boldsymbol{v}_{xy}^\top \boldsymbol{J}\boldsymbol{a}_{xy})}{\|\boldsymbol{v}_{xy}\|^4}$$

Where:

- $\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (90°counterclockwise rotation matrix)
- $\boldsymbol{v}_{xy} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$
- $\boldsymbol{a}_{xy} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$
- $\boldsymbol{j}_{xy} = \begin{bmatrix} j_x \\ j_y \end{bmatrix}$

4 Key Properties

- **Generalized Rotation**: Full 321-sequence tensor accommodates future roll extensions
- **Singularity Aware**: Explicit $\tan\theta$ terms highlight $\theta = \pm\pi/2$ singularities
- **Frame Consistency**: Proper treatment of inertial-body transformations
- **Computational Preservation**: Maintains original velocity-based angular rate derivations