Mathematical Derivation and Justification of the Rigid Body State Model with Quaternion Orientation

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1 Introduction

This document provides a comprehensive mathematical explanation of the rigid body state model implemented in the provided code. The model incorporates:

- 3D position - Quaternion orientation - 3D linear and angular velocity - 3D linear and angular acceleration

Quaternion kinematics are employed for orientation representation to avoid gimbal lock and maintain robust rotation modeling.

2 State Representation

The state vector is defined as follows:

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{v}_l \\ \mathbf{v}_a \\ \mathbf{a}_l \\ \mathbf{a}_a \end{bmatrix} \tag{1}$$

Where: - $\mathbf{p}$: 3D position vector - $\mathbf{q}$: Quaternion orientation (4D vector) - $\mathbf{v}_l$: 3D linear velocity - $\mathbf{v}_a$: 3D angular velocity - $\mathbf{a}_l$: 3D linear acceleration - $\mathbf{a}_a$: 3D angular acceleration

3 State Prediction Model

The state evolution model is defined as follows:

3.1 Position Model

The position update is given by:

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{R}(\mathbf{q}_k)\left(\mathbf{v}_{l,k} + \frac{1}{2}\mathbf{a}_{l,k}\Delta t\right)\Delta t \tag{2}$$

Where $\mathbf{R}(\mathbf{q})$ is the rotation matrix derived from the quaternion $\mathbf{q}$.

3.2 Quaternion Update via Exponential Map

Quaternion kinematics are defined by:

$$\mathbf{q}_{k+1} = \exp\left(\frac{1}{2}\boldsymbol{\Omega}\Delta t + \frac{1}{4}\alpha\Delta t^2\right)\mathbf{q}_k \tag{3}$$

Where: - $\boldsymbol{\Omega}$ is the angular velocity vector - α is the angular acceleration vector

The exponential map is calculated using the Rodrigues rotation formula:

$$\exp(\boldsymbol{\theta}) = \cos\left(\frac{\|\boldsymbol{\theta}\|}{2}\right) + \frac{\sin\left(\frac{\|\boldsymbol{\theta}\|}{2}\right)}{\|\boldsymbol{\theta}\|}\boldsymbol{\theta} \tag{4}$$

For small angular motion (when $\|\boldsymbol{\theta}\| \to 0$), the exponential map simplifies to the identity quaternion.

3.3 Velocity Model

The linear and angular velocity updates are given by:

$$\mathbf{v}_{l,k+1} = \mathbf{v}_{l,k} + \mathbf{a}_{l,k}\Delta t$$
$$\mathbf{v}_{a,k+1} = \mathbf{v}_{a,k} + \mathbf{a}_{a,k}\Delta t$$

3.4 Acceleration Model

The exponential decay model for acceleration is:

$$\mathbf{a}_{l,k+1} = \mathbf{a}_{l,k}e^{-\lambda\Delta t}$$
$$\mathbf{a}_{a,k+1} = \mathbf{a}_{a,k}e^{-\lambda\Delta t}$$

Where $\lambda = 46.05$ controls the exponential decay of acceleration terms.

4 Jacobian Derivation

The Jacobian matrix linearizes the system's state transition model:

$$\mathbf{A} = \left. \frac{\partial f}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{k-1}^{+}, u_k} \tag{5}$$

Key components include:

4.1 Position Jacobian Terms

The position Jacobian terms are derived from the position update equation:

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{R}(\mathbf{q}) \left(\mathbf{v}_{l,k} + \frac{1}{2}\mathbf{a}_{l,k}\Delta t \right) \Delta t \tag{6}$$

Taking the partial derivatives:

$$\begin{aligned} \frac{\partial \mathbf{p}}{\partial \mathbf{v}_l} &= \mathbf{R}(\mathbf{q})\Delta t \\ \frac{\partial \mathbf{p}}{\partial \mathbf{a}_l} &= \mathbf{R}(\mathbf{q})\frac{\Delta t^2}{2} \\ \frac{\partial \mathbf{p}}{\partial \mathbf{q}} &= 0 \end{aligned}$$

4.2 Quaternion Jacobian Terms

Starting from the quaternion update equation:

$$\mathbf{q}_{k+1} = \exp\left(\frac{1}{2}\mathbf{\Omega}\Delta t + \frac{1}{4}\alpha\Delta t^2 \right) \mathbf{q}_k \tag{7}$$

4.3 Angular Motion Vector Definition

We define the combined angular motion vector as:

$$\mathbf{v} = \frac{1}{2}\mathbf{\Omega}\Delta t + \frac{1}{4}\alpha\Delta t^2 \tag{8}$$

This vector represents the net angular displacement when both angular velocity and angular acceleration are considered.

4.4 Exponential Map and Its Derivative

The quaternion update uses the exponential map:

$$\exp(\boldsymbol{v}) = \cos\left(\frac{\|\mathbf{v}\|}{2} \right) + \frac{\sin\left(\frac{\|\mathbf{v}\|}{2} \right)}{\|\mathbf{v}\|}\mathbf{v} \tag{9}$$

For small motion, this reduces to approximately:

$$\exp(\boldsymbol{v}) \approx 1 + \frac{1}{2}\boldsymbol{v} \tag{10}$$

4.5 Jacobian Terms for Angular Velocity

Differentiating the quaternion update with respect to angular velocity gives:

$$\frac{\partial \mathbf{q}}{\partial \boldsymbol{\Omega}} \approx \frac{\Delta t}{2}\mathbf{I} - \frac{\Delta t^3}{16}\frac{\boldsymbol{\Omega}\boldsymbol{\Omega}^T}{\|\boldsymbol{\Omega}\|^2} \tag{11}$$

4.6 Jacobian Terms for Angular Acceleration

Similarly, for angular acceleration:

$$\frac{\partial \mathbf{q}}{\partial \alpha} \approx \frac{\Delta t^2}{4}\mathbf{I} - \frac{\Delta t^4}{32}\frac{\alpha\alpha^T}{\|\alpha\|^2} \tag{12}$$

4.7 Correction Terms for Stability

When motion is non-zero, higher-order corrections improve accuracy. The correction terms scale the quaternion update:

$$\mathbf{C}_{\boldsymbol{\Omega}} = \frac{\Delta t^3\|\mathbf{v}\|}{16}\frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2}$$

$$\mathbf{C}_{\alpha} = \frac{\Delta t^4\|\mathbf{v}\|}{32}\frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2}$$

The correction terms ensure the Jacobian better approximates the non-linear rotation behavior for larger rotations.

4.8 Velocity and Acceleration Jacobians

Velocity Jacobians are identity matrices with appropriate scaling for integration over time:

$$\frac{\partial \mathbf{v}l, k+1}{\partial \mathbf{v}l, k} = \mathbf{I} \quad \frac{\partial \mathbf{v}l, k+1}{\partial \mathbf{a}l, k} = \Delta t\mathbf{I} \quad \frac{\partial \mathbf{v}a, k+1}{\partial \mathbf{v}a, k} = \mathbf{I} \quad \frac{\partial \mathbf{v}a, k+1}{\partial \mathbf{a}a, k} = \Delta t\mathbf{I}$$

Acceleration Jacobians reflect the exponential decay model:

$$\frac{\partial \mathbf{a}l, k+1}{\partial \mathbf{a}l, k} = e^{-\lambda \Delta t}\mathbf{I} \tag{13}$$

5 Conclusion

This document provides a detailed derivation and mathematical justification for the provided rigid body state model. The combination of quaternion kinematics and exponential decay for acceleration provides a numerically stable and physically accurate prediction model for 3D rigid body dynamics in state estimation systems.