Directional Velocity Constraint in Sensor-Model Kalman Filter

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1 Problem Definition

Estimate a 19D state vector containing position, orientation (quaternion), velocities, and accelerations using:

- Asynchronous position measurements $\mathbf{z}_p \in \mathbb{R}^3$
- Scalar velocity magnitude $z_v \in \mathbb{R}$ aligned with heading

2 State Representation

Define the state vector:

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{v} \\ \boldsymbol{\omega} \\ \mathbf{a} \\ \boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} x & y & z & q_w & q_x & q_y & q_z & v_x & v_y & v_z \\ \cdots \end{bmatrix}^T$$

where $\mathbf{q} = (q_w, q_x, q_y, q_z)$ is a unit quaternion.

3 Measurement Model

3.1 Position Measurement

$$\mathbf{z}_p = \mathbf{p} + \boldsymbol{\nu}_p, \quad \boldsymbol{\nu}_p \sim \mathcal{N}(0, \mathbf{R}_p)$$

Jacobian: $\mathbf{H}_p = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_{3\times16} \end{bmatrix}$

3.2 Velocity-Direction Constraint

Define heading vector from quaternion:

$$\mathbf{h}(\mathbf{q}) = \mathbf{R}(\mathbf{q}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - 2(q_y^2 + q_z^2) \\ 2(q_x q_y + q_w q_z) \\ 2(q_x q_z - q_w q_y) \end{bmatrix}$$

Velocity measurement model:

$$z_v = \mathbf{v}^T \mathbf{h}(\mathbf{q}) + \nu_v, \quad \nu_v \sim \mathcal{N}(0, R_v)$$

4 Constraint Jacobian

Compute $\mathbf{H}_v = \frac{\partial}{\partial \mathbf{x}}(\mathbf{v}^T \mathbf{h}(\mathbf{q}))$:

4.1 Velocity Components

$$\frac{\partial z_v}{\partial \mathbf{v}} = \mathbf{h}(\mathbf{q})^T = \begin{bmatrix} 1 - 2(q_y^2 + q_z^2) & 2(q_x q_y + q_w q_z) & 2(q_x q_z - q_w q_y) \end{bmatrix}$$

4.2 Quaternion Components

$$\frac{\partial z_v}{\partial \mathbf{q}} = \mathbf{v}^T \frac{\partial \mathbf{h}}{\partial \mathbf{q}} = \begin{bmatrix} 2(v_y q_z - v_z q_y) \\ 2(v_y q_y + v_z q_z) \\ -4 v_x q_y + 2 v_y q_x - 2 v_z q_w \\ -4 v_x q_z + 2 v_y q_w + 2 v_z q_x \end{bmatrix}^T$$

Full Jacobian structure:

$$\mathbf{H}_v = \begin{bmatrix} \mathbf{0}_{1\times 3} & \frac{\partial z_v}{\partial \mathbf{q}} & \frac{\partial z_v}{\partial \mathbf{v}} & \mathbf{0}_{1\times 9} \end{bmatrix}$$

5 Filter Implementation

5.1 Prediction Step

Propagate state using high-order dynamics:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \quad \text{(Physics model)}$$

with quaternion propagation:

$$\dot{\mathbf{q}} = \frac{1}{2} \mathbf{q} \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix}$$

5.2 Update Steps

Handle measurements asynchronously:

Position Update: Apply $\mathbf{H}_p$ with Kalman gain

Velocity Update: Apply $\mathbf{H}_v$ using current state estimate

6 Implementation Notes

- **Quaternion Normalization:** Renormalize $\mathbf{q}$ after each update

$$\mathbf{q} \leftarrow \mathbf{q}/\|\mathbf{q}\|$$

- **Adaptive Noise:** Scale R_v inversely with $\|\mathbf{v}\|$

$$R_v \leftarrow R_0(1 + \|\mathbf{v}\|^{-1})$$

7 Conclusion

This sensor-model constraint approach:

- Maintains high-order state dynamics integrity
- Enforces velocity-heading alignment through measurement residuals
- Handles asynchronous sensors naturally
- Requires only 19D matrix operations (sparse Jacobians)