Derivation of Quaternion Update Jacobian

Spencer Maughan

Abstract

This is a derivation of jacobian of quaternion update given 3d angular velocities. Essentially, the purpose of this derivation is to determine $\frac{\partial q^+}{\partial \omega}$. Where q^+ := updated quaternion and ω := the angular velocities along each of the 3 cartesian axes.

Update Model:

$$\begin{aligned}
\omega &:= \text{angular velocity} \\
\delta t &:= \text{time step} \\
q^- &:= \text{previous quaternion} \\
q^+ &= \zeta * q^- \\
\zeta &= \begin{bmatrix} \beta\omega \\ cos(\frac{\alpha}{2}) \end{bmatrix} \\
\beta &= \delta t \frac{sin(\frac{\alpha}{2})}{\alpha} \\
\alpha &= \delta t \|\omega\|
\end{aligned}$$

Partials for intermediate variables:

$$\begin{aligned}
\frac{\partial \zeta}{\partial \omega} &= \begin{bmatrix} \frac{\partial \beta}{\partial \omega} \\ -\frac{1}{2}\frac{\partial \alpha}{\partial \omega} sin(\frac{\alpha}{2}) \end{bmatrix} \\
\frac{\partial \beta}{\partial \omega} &= \delta t \frac{\partial \alpha}{\partial \omega} \frac{\alpha cos(\frac{\alpha}{2}) - 2 sin(\frac{\alpha}{2})}{2\alpha^2} \\
\frac{\partial \alpha}{\partial \omega} &= \frac{\delta t}{\|\omega\|}\omega
\end{aligned}$$

Final Derivation:

$$
\begin{aligned}
\frac{\partial q^+}{\partial \omega} &= \frac{\partial \zeta}{\partial \omega} q^- \\
&= \begin{bmatrix}
\frac{\partial \beta}{\partial \omega_x} & 0 & 0 \\
0 & \frac{\partial \beta}{\partial \omega_y} & 0 \\
0 & 0 & \frac{\partial \beta}{\partial \omega_z} \\
-\frac{1}{2}\frac{\partial \alpha}{\partial \omega_x} sin(\frac{\alpha}{2}) & -\frac{1}{2}\frac{\partial \alpha}{\partial \omega_y} sin(\frac{\alpha}{2}) & -\frac{1}{2}\frac{\partial \alpha}{\partial \omega_z} sin(\frac{\alpha}{2})
\end{bmatrix} q^- \\
&= \begin{bmatrix}
\delta t \frac{\partial \alpha}{\partial \omega_x} \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} & 0 & 0 \\
0 & \delta t \frac{\partial \alpha}{\partial \omega_y} \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} & 0 \\
0 & 0 & \delta t \frac{\partial \alpha}{\partial \omega_z} \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} \\
-\frac{1}{2}\frac{\partial \alpha}{\partial \omega_x} sin(\frac{\alpha}{2}) & -\frac{1}{2}\frac{\partial \alpha}{\partial \omega_y} sin(\frac{\alpha}{2}) & -\frac{1}{2}\frac{\partial \alpha}{\partial \omega_z} sin(\frac{\alpha}{2})
\end{bmatrix} q^- \\
&= \begin{bmatrix}
\delta t \frac{\delta t}{\|\omega\|} \omega_x \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} & 0 & 0 \\
0 & \delta t \frac{\delta t}{\|\omega\|} \omega_y \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} & 0 \\
0 & 0 & \delta t \frac{\delta t}{\|\omega\|} \omega_z \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} \\
-\frac{1}{2}\frac{\delta t}{\|\omega\|} \omega_x sin(\frac{\alpha}{2}) & -\frac{1}{2}\frac{\delta t}{\|\omega\|} \omega_y sin(\frac{\alpha}{2}) & -\frac{1}{2}\frac{\delta t}{\|\omega\|} \omega_z sin(\frac{\alpha}{2})
\end{bmatrix} q^- \\
&= \frac{\delta t}{\|\omega\|} \begin{bmatrix}
\delta t \omega_x \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} & 0 & 0 \\
0 & \delta t \omega_y \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} & 0 \\
0 & 0 & \delta t \omega_z \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} \\
-\frac{1}{2}\omega_x sin(\frac{\alpha}{2}) & -\frac{1}{2}\omega_y sin(\frac{\alpha}{2}) & -\frac{1}{2}\omega_z sin(\frac{\alpha}{2})
\end{bmatrix} q^- \\
&= \frac{\delta t}{\|\omega\|} \begin{bmatrix}
\delta t \frac{\alpha cos(\frac{\alpha}{2}) - 2sin(\frac{\alpha}{2})}{2\alpha^2} \begin{bmatrix} \omega_x & 0 & 0 \\ 0 & \omega_y & 0 \\ 0 & 0 & \omega_z \end{bmatrix} \\
-\frac{1}{2} sin(\frac{\alpha}{2}) \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}
\end{bmatrix} q^-
\end{aligned}
$$