Ray Tracing & Ray Casting

Realistic Graphics Inpsired by Nature

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Bangladesh University of Engineering and Technology (BUET)
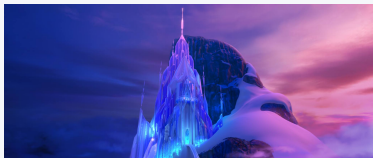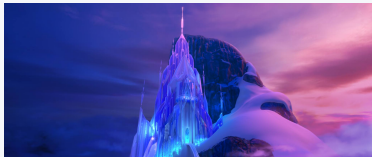
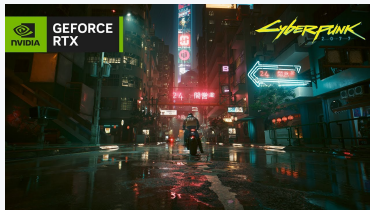# Index

Motivation

Why Learn This?

Elsa's Castle in Frozen

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Elsa's Castle in Frozen

Cyberpunk 2077 with RTX

- **Realistic graphics** of your favourite animated movies are the result of ground-breaking work in Ray Tracing by studios like Disney, Pixar, and DreamWorks. Do you know these films take years to render? 30 hours per frame!
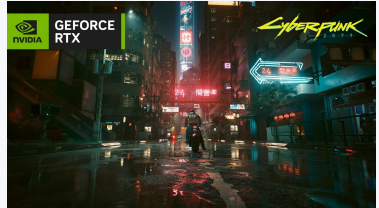
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Cyberpunk 2077 with RTX

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- Lately, **RTX** is all the rage in gaming. New titles boast ray-tracing effects in real-time, not 30 hours per frame!

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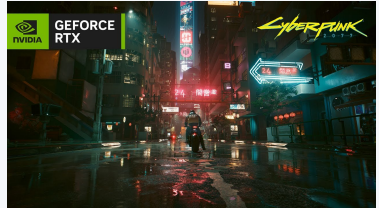


Elsa's Castle in Frozen

Cyberpunk 2077 with RTX

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- Lately, **RTX** is all the rage in gaming. New titles boast ray-tracing effects in real-time, not 30 hours per frame!

- It's fun! You will know when you create your first ray-traced image!

The Story of Light

Light Source
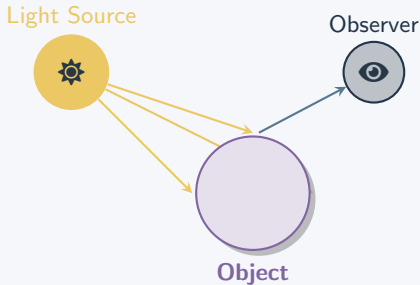
Observer

Object

How Do We See?

Light Source

Observer
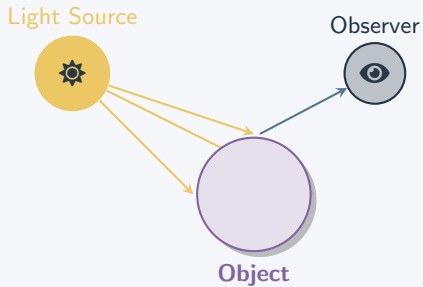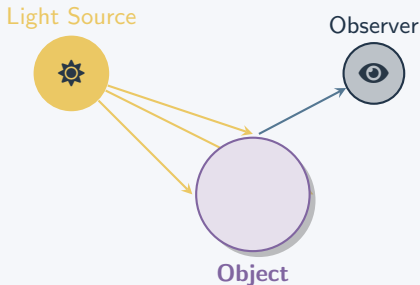
Object

Natural Process

1. Light travels from source
2. Light hits objects
3. Light bounces to our eyes
4. Our brain interprets the signal

How Do We See?

Physical Process

1. Photon is emitted from source
2. Photon hits objects
3. Part of the photon is reflected or absorbed
4. The reflected photons reach our eyes
5. The rods and cones in our retina detect the photons
6. Our brain interprets the signal
7. **Colour**: The wavelength of the photons
8. **Brightness**: The number of photons

How Do We See?
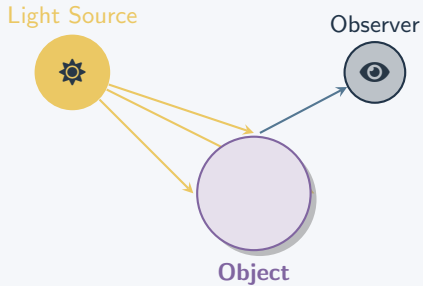
Question: How do we simulate this?
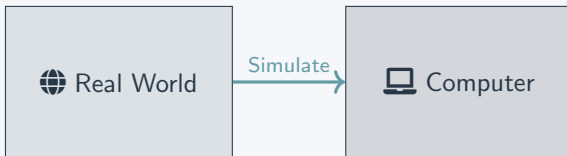
The Computer Graphics Challenge

Infinite Complexity

Finite Pixels

🌐 Real World → _Simulate_ → 🖥 Computer

Challenges:

- Infinite light rays/photons
- Complex physics
- High computational cost

Ray Casting: Foundation

1. Reverse Engineering

Instead of following light rays from light sources —

**Let's trace backwards!
Shoot rays from the eye**, find where it hits and find out how much light reaches there.

This is the opposite of what happens in reality. **Why does this work?**
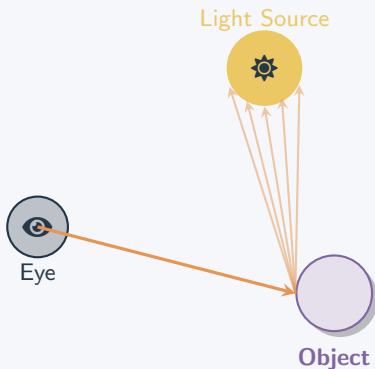
Light Source

Eye

Object

1. Reverse Engineering

Instead of following light rays from light sources —

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- Most light never reaches our eyes

Light Source

Useless ray

Eye

Object

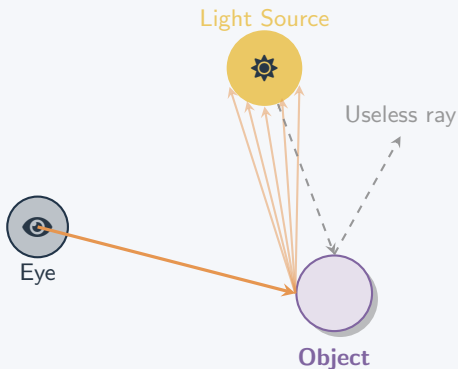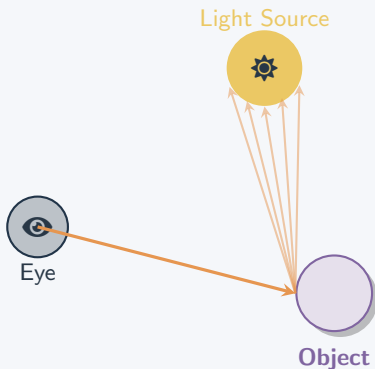# The Key Insight

1. Reverse Engineering

Instead of following light rays from light sources —

Let's trace backwards! Shoot rays from the eye, find where it hits and find out how much light reaches there.

This is the opposite of what happens in reality. **Why does this work?**

- Most light never reaches our eyes
- Only trace rays that matter
- Much more efficient!

Light Source

Eye

Object

From Infinite Rays to Finite Pixels

2. Cutting Costs

Instead of tracing infinite rays —

Trace one ray per pixel.

This comes with little tradeoff, because:

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- An image is just a grid of pixels

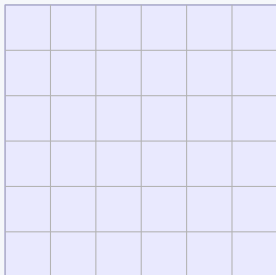## 2. Cutting Costs

Instead of tracing infinite rays —

Trace one ray per pixel.

This comes with little tradeoff, because:

- An image is just a grid of pixels
- Each pixel can only be of one color
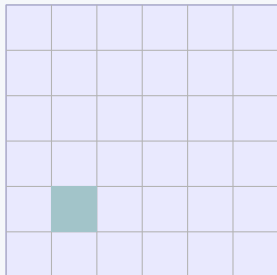
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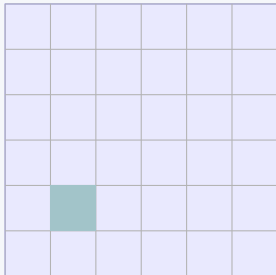- In the end, we just need to know the *color of each pixel*

2. Cutting Costs

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Trace one ray per pixel.

This comes with little tradeoff, because:

- An image is just a grid of pixels
- Each pixel can only be of one color
- In the end, we just need to know the *color of each pixel*
- Hence, one ray from the mid-point of each pixel should be a good approximation*

* We will discuss more advanced techniques later that improve quality
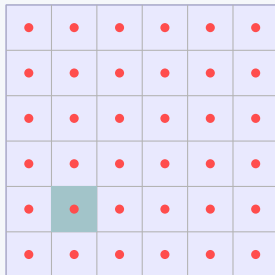
The Full Picture

Light Source

Eye

Object

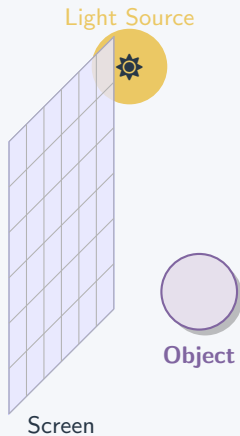Light Source

Place screen in front of eye

Eye

Object

Screen

Place screen in front of eye

For each pixel

Light Source

Eye

Screen

Object

Place screen in front of eye

↓

For each pixel

↓

Shoot ray from eye

Light Source

Eye

Object

Screen

Place screen in front of eye

↓

For each pixel
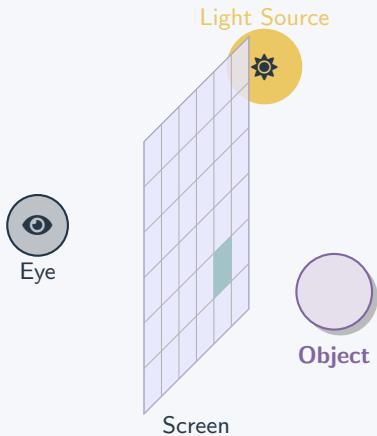
↓

Shoot ray from eye

↓

Find closest intersection

Light Source

Eye

Object

Screen

The Full Picture

Place screen in front of eye

For each pixel

Shoot ray from eye
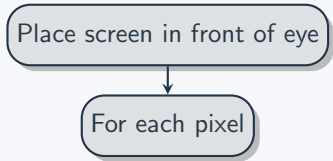
Find closest intersection

Calculate color*

Light Source

Eye

Object

Screen

The Full Picture

- Place screen in front of eye
- For each pixel
- Shoot ray from eye
- Find closest intersection
- Calculate color*
- Set pixel color

Light Source

Eye

Screen

Object

```
Place screen in front of eye
            ↓
      For each pixel
            ↓
    Shoot ray from eye
            ↓
  Find closest intersection
            ↓
    Calculate color*
            ↓
     Set pixel color
```

Light Source

Eye

Object

Screen

Mathematics of Rays

What is a Ray?

Ray Representation

A ray is defined by:

$$\mathbf{P}(t) = \mathbf{R_o} + t \cdot \mathbf{R_d} \quad (1)$$

where:

- $\mathbf{R_o}$ = Origin point
- $\mathbf{R_d}$ = Direction vector
- t = Parameter ($t \geq 0$)
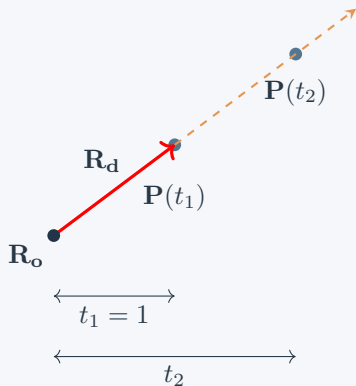
Check out here on desmos.

Finding Intersections

Ray
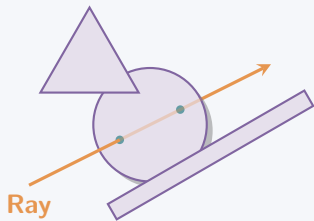
Key Objects:

- **Planes**
- **Spheres**
- **Triangles**
- **General Quadrics**

Challenge: Find the **closest** intersection efficiently!

3D Plane Representation

Plane Definition

A plane is defined by:

- Point $\mathbf{P_0} = (x_0, y_0, z_0)$ on plane
- Normal vector $\mathbf{n} = (A, B, C)$

Implicit equation:

$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P_0}) = 0$

$\boxed{\mathbf{n} \cdot \mathbf{P} + D = 0}$ where $D = -\mathbf{n} \cdot \mathbf{P_0}$

$\boxed{Ax + By + Cz + D = 0}$

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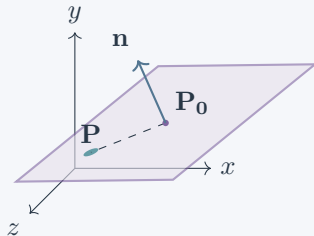$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P_0}) = 0$

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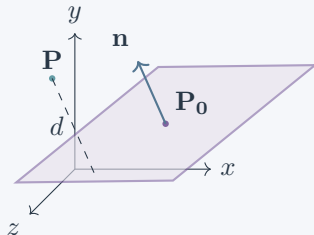
Point-Plane Distance

If $\mathbf{n}$ is normalized: $d = \mathbf{n} \cdot \mathbf{P} + D = \mathbf{n} \cdot (\mathbf{P} - \mathbf{P_0})$

Signed distance: $d > 0$ (front), $d < 0$ (back), $d = 0$ (on plane)

Ray-Plane Intersection

Intersection Method

Step 1: Substitute ray into equation

$$\mathbf{n} \cdot (\mathbf{R_o} + t\mathbf{R_d}) + D = 0$$

$$\mathbf{n} \cdot \mathbf{R_o} + t(\mathbf{n} \cdot \mathbf{R_d}) + D = 0$$

Step 2: Solve for parameter t

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R_o}}{\mathbf{n} \cdot \mathbf{R_d}}$$

Ray-Plane Intersection
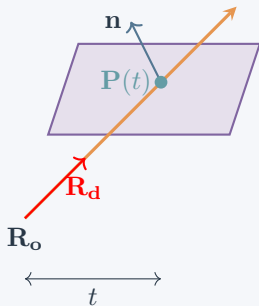
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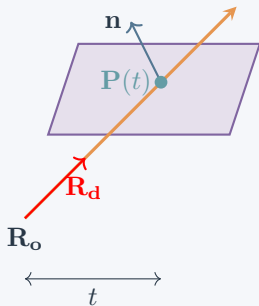
Cases

- If $\mathbf{n} \cdot \mathbf{R_d} = 0$: Ray parallel to plane (0 or infinite)
- If $\mathbf{n} \cdot \mathbf{R_d} < 0$: Ray hits front face
- If $\mathbf{n} \cdot \mathbf{R_d} > 0$: Ray hits back face

Validation Rules

After computing t, verify:

1. **Behind check:** $t > t_{\mathsf{min}}$
2. **Closest check:** $t < t_{\mathsf{current}}$
3. **Valid range:** $t \geq 0$

Where:

- t_{min}: Minimum ray distance (not behind eye/screen)
- t_{current}: Distance to closest intersection so far
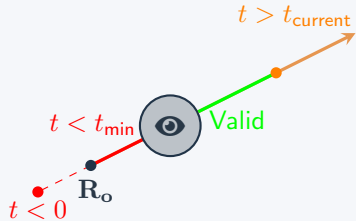
Ray-Triangle Intersection Overview

Two Main Approaches

Method 1: Two-Step Process

1. Ray-plane intersection
2. Inside/outside triangle test

Method 2: Direct Barycentric

1. Set up 3×3 linear system
2. Solve for t, β, γ simultaneously
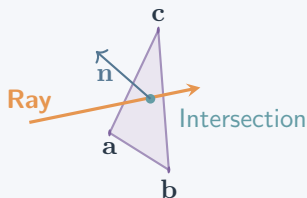
What Are Barycentric Coordinates?

Barycentric Definition

Any point P in the triangle's plane:

$$\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

where: $\alpha + \beta + \gamma = 1$

$\mathbf{c}\ (\gamma = 0.2)$

A_β A_α

P

A_γ

$\mathbf{a}\ (\alpha = 0.3)$ $\mathbf{b}\ (\beta = 0.5)$

Check out the Desmos demo.

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Physical Interpretation:

- α, β, γ are *weights*
- P is the *center of mass*
- Also called *barycenter*

Check out the Desmos demo.

$\mathbf{c}\ (\gamma = 0.2)$

A_β A_α

P

A_γ

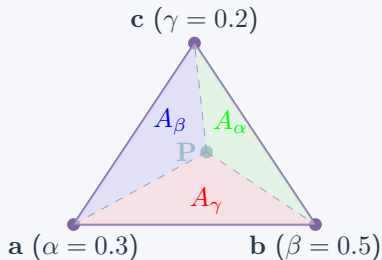$\mathbf{a}\ (\alpha = 0.3)$ $\mathbf{b}\ (\beta = 0.5)$

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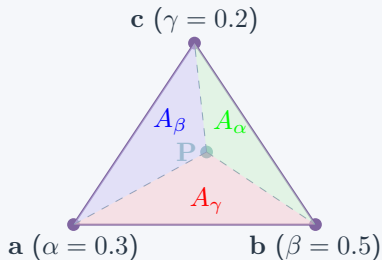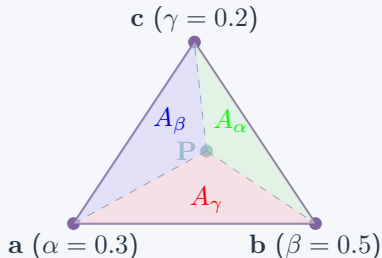
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A_β A_α

P

A_γ

$\mathbf{a}$ ($\alpha = 0.3$) $\mathbf{b}$ ($\beta = 0.5$)

Area Relationship

$\alpha = \frac{A_\alpha}{A_{total}}$, $\beta = \frac{A_\beta}{A_{total}}$,
$\gamma = \frac{A_\gamma}{A_{total}}$

Triangle Interior Test

Point P is **inside** triangle if:

$$\alpha, \beta, \gamma \geq 0$$

Since $\alpha + \beta + \gamma = 1$, we can rewrite as:

$$\beta \geq 0$$
$$\gamma \geq 0$$
$$\alpha \geq 0 \text{ or } \beta + \gamma \leq 1$$

Outside
$\beta < 0$

$\alpha < 0, \beta + \gamma > 1$
Outside

Inside
$0 \geq \beta, \gamma \geq 1$

$\gamma < 0$
Outside

Triangle Interior Test
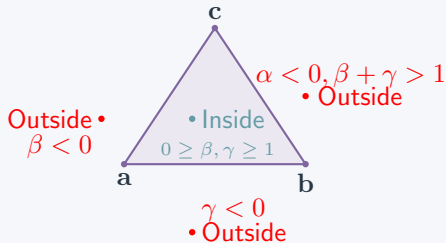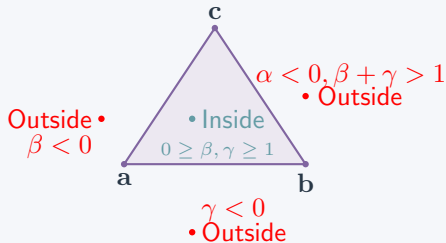
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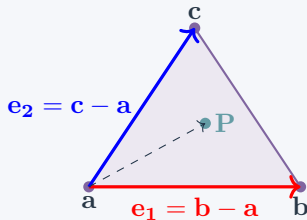$$\alpha \geq 0 \text{ or } \beta + \gamma \leq 1$$

c

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• Outside

Outside •
$\beta < 0$

• Inside
$0 \geq \beta, \gamma \geq 1$

a b

$\gamma < 0$
• Outside

Insight

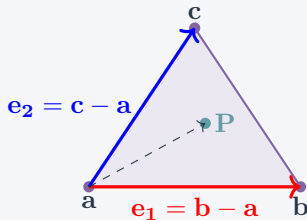Barycentric coordinates doesn't just tell us if a point is inside a triangle, but also it's position with respect to other vertices.

Key Idea

- The sides $e_1 = b - a$ and $e_2 = c - a$ are linearly independent vectors on the triangle's plane.

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- Therefore, any vector in the triangle's plane (e.g. $\mathbf{P} - \mathbf{a}$) can be expressed as a linear combination of these vectors.

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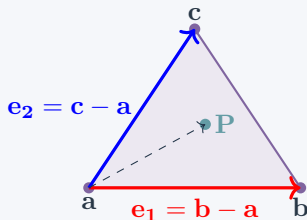
- We can express $\mathbf{P}$ as:

$$\mathbf{P} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$
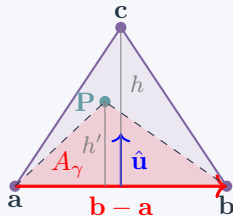$$= \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

Where $\alpha = 1 - \beta - \gamma$.
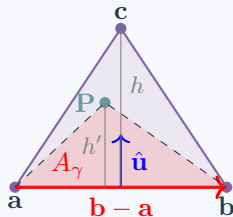
Area Interpretation

- Let $\hat{\mathbf{u}}$ be an unit vector in the direction of the altitude towards C.

Barycentric Coordinates: Derivation

Area Interpretation

- Let $\hat{\mathbf{u}}$ be an unit vector in the direction of the altitude towards C.
- The height of the triangle is $h = \hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})$ (projection).

Barycentric Coordinates: Derivation
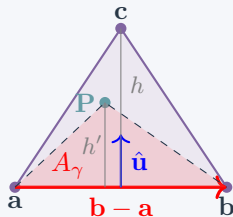
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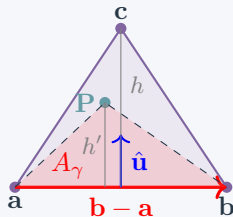- The height of the shaded triangle is $h' = \hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})$.

- Hence,

$$A_\gamma = \frac{1}{2} \cdot h' \cdot |\mathbf{b} - \mathbf{a}|$$

$$= \frac{1}{2} \cdot (\hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}|$$

$$= \frac{1}{2} \cdot \gamma \, (\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}|$$

$$= \gamma \frac{1}{2} \cdot h \cdot |\mathbf{b} - \mathbf{a}| = \gamma A_{total}$$
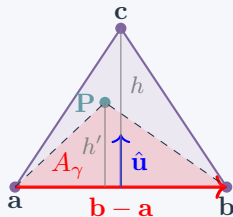
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$$= \frac{1}{2} \cdot (\hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}|$$

$$= \frac{1}{2} \cdot \gamma \left(\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a}) \right) \cdot |\mathbf{b} - \mathbf{a}|$$

$$= \gamma \frac{1}{2} \cdot h \cdot |\mathbf{b} - \mathbf{a}| = \gamma A_{total}$$

Since,

$$\mathbf{P} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}),$$

$$\mathbf{P} - \mathbf{a} = \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

$$\hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a}) = \gamma(\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a}))$$
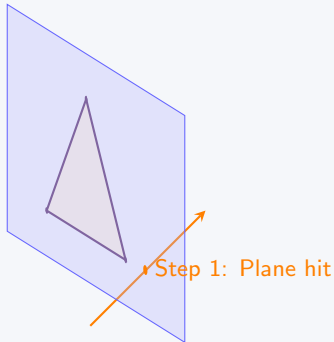
Since $\hat{\mathbf{u}}$ is perpendicular to $\mathbf{b} - \mathbf{a}$.

Algorithm Steps

Step 1: Ray-Plane Intersection

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R_o}}{\mathbf{n} \cdot \mathbf{R_d}}$$

Step 1: Plane hit

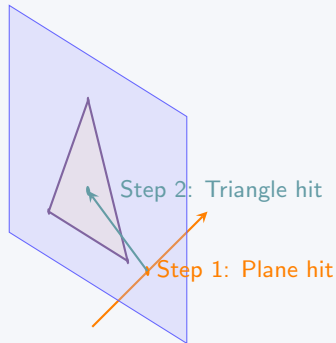# Method 1: Two-Step Ray-Triangle Intersection

Algorithm Steps

Step 1: Ray-Plane Intersection

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R_o}}{\mathbf{n} \cdot \mathbf{R_d}}$$

Step 2: Inside/Outside Test

$$\mathbf{P} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

Solve for β, γ and check bounds.

Step 2: Triangle hit

Step 1: Plane hit

Method 2: Direct Barycentric Intersection
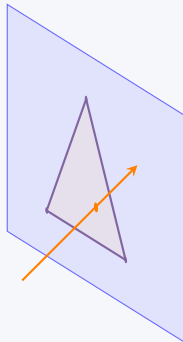
Direct Approach

Set ray equation equal to barycentric form:

$$\mathbf{R_o} + t\mathbf{R_d} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

Rearrange to linear system:

$$\begin{bmatrix} -\mathbf{R_d} & (\mathbf{b} - \mathbf{a}) & (\mathbf{c} - \mathbf{a}) \end{bmatrix} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \mathbf{R_o} - \mathbf{a}$$

Solve using Cramer's rule or
LU decomposition.

Cramer's Rule Solution

Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_{A} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

Cramer's Rule Solution

Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_{A} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

Cramer's Rule

$$t = \frac{1}{|A|} \begin{vmatrix} (R_o - a)_x & (b - a)_x & (c - a)_x \\ (R_o - a)_y & (b - a)_y & (c - a)_y \\ (R_o - a)_z & (b - a)_z & (c - a)_z \end{vmatrix}$$

$$\beta = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix}$$

$$\gamma = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (b - a)_x & (R_o - a)_x \\ -R_{dy} & (b - a)_y & (R_o - a)_y \\ -R_{dz} & (b - a)_z & (R_o - a)_z \end{vmatrix}$$

Cramer's Rule Solution

Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_{A} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

Cramer's Rule

$$t = \frac{1}{|A|} \begin{vmatrix} (R_o - a)_x & (b - a)_x & (c - a)_x \\ (R_o - a)_y & (b - a)_y & (c - a)_y \\ (R_o - a)_z & (b - a)_z & (c - a)_z \end{vmatrix}$$

$$\beta = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix}$$

$$\gamma = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (b - a)_x & (R_o - a)_x \\ -R_{dy} & (b - a)_y & (R_o - a)_y \\ -R_{dz} & (b - a)_z & (R_o - a)_z \end{vmatrix}$$

Checks

- $t_{\min} < t < t_{\text{current}}$
  (valid intersection)

- $\beta, \gamma \geq 0$ and
  $\beta + \gamma \leq 1$
  (inside triangle)

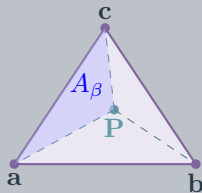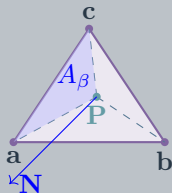## Cramer's Rule Intuition

Determinant Interpretation

Remember, $\beta = \frac{A_\beta}{A_{total}}$?

Determinants measure areas of projected triangles.

$$\begin{vmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{vmatrix} \propto A_{total}$$

$$\begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix} \propto A_\beta$$

Why?

Determinant Interpretation

The areas can be found by cross products of vectors. Consider —

$$\mathbf{N} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x - a_x & b_y - a_y & b_z - a_z \\ c_x - a_x & c_y - a_y & c_z - a_z \end{vmatrix}$$

$$= 2A_{total}\hat{\mathbf{n}}$$

$\mathbf{N}$ is the normal vector of the triangle scaled by twice the area.

Determinant Interpretation

We project the triangle onto the plane perpendicular to the ray direction $\mathbf{R_d}$.
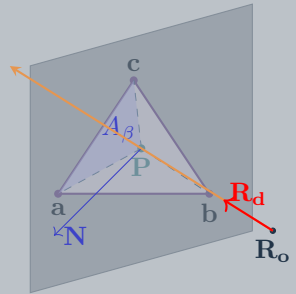
Cramer's Rule Intuition

Determinant Interpretation

$\mathbf{N} \cdot (-\mathbf{R}_d)$ is twices the projected area of the triangle onto the plane perpendicular to the ray direction.

$$\mathbf{N} \cdot (-\mathbf{R}_d) = \begin{vmatrix} -R_x & -R_y & -R_z \\ b_x - a_x & b_y - a_y & b_z - a_z \\ c_x - a_x & c_y - a_y & c_z - a_z \end{vmatrix}$$

$$= \begin{vmatrix} -R_x & b_x - a_x & c_x - a_x \\ -R_y & b_y - a_y & c_y - a_y \\ -R_z & b_z - a_z & c_z - a_z \end{vmatrix}$$

$$\propto A_{total}$$
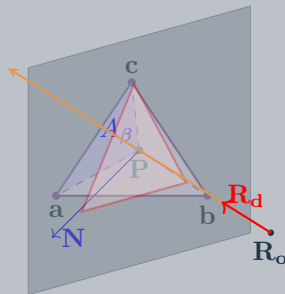
Determinant Interpretation

Now what about the A_γ triangle?
Same idea but a bit different.
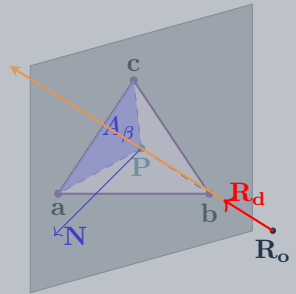
Determinant Interpretation

- Instead of projecting the triangle between $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{P}$, we project the triangle between $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{R}_o$.

- From the perspective of the ray, points $\mathbf{P}$ and $\mathbf{R}_o$ are the same point, as both fall on the ray.
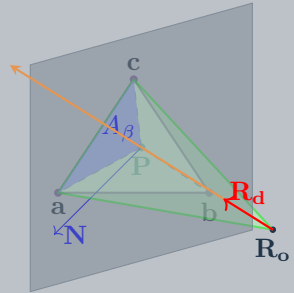
Cramer's Rule Intuition

Determinant Interpretation

(Twice) the area of the triangle between $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{R}_o$ projected onto the plane perpendicular to the ray direction $\mathbf{R}_d$ is given by:

$$(\mathbf{R}_o - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) \cdot (-\mathbf{R}_d)$$

$$= \begin{vmatrix} -R_x & -R_y & -R_z \\ b_x - a_x & b_y - a_y & b_z - a_z \\ R_{ox} - a_x & R_{oy} - a_y & R_{oz} - a_z \end{vmatrix}$$

$$\propto A_\gamma$$
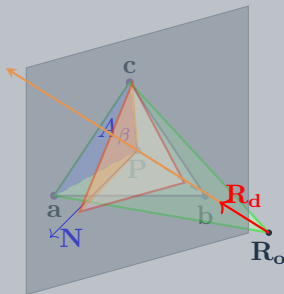
Bonus of Using Barycentric Coordinates

Advantages

- Efficient to compute
- Get Barycentric coordinates for free
- Enables interpolation of vertex attributes
  Used in —
  - Textures
  - Normals
  - Colors

$$\alpha R + \beta G + \gamma B$$
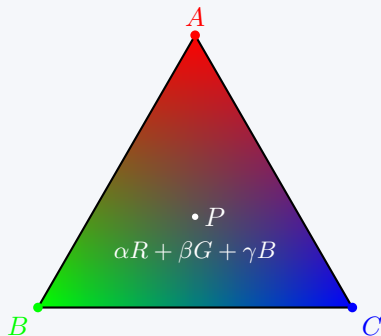
Ray-Sphere Intersection Overview

Two Main Approaches

Method 1: Algebra

1. Setup quadratic equation
2. Solve for t

Method 2: Geometry

1. Use geomety to find intersection step by step
2. Reject early if hit is not possible
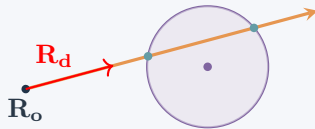
Sphere Representation

Implicit Sphere Equation

Sphere centered at origin:

$$\mathbf{P} \cdot \mathbf{P} - r^2 = 0$$
$$x^2 + y^2 + z^2 - r^2 = 0$$

General sphere at center $\mathbf{C}$:

$$(\mathbf{P} - \mathbf{C}) \cdot (\mathbf{P} - \mathbf{C}) - r^2 = 0$$

Note: Translation to origin simplifies calculation!
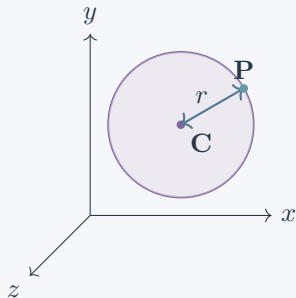
Algebraic Solution

Step 1: Substitute ray equation $\mathbf{P}(t) = \mathbf{R_o} + t\mathbf{R_d}$ into sphere

$$(\mathbf{R_o} + t\mathbf{R_d}) \cdot (\mathbf{R_o} + t\mathbf{R_d}) - r^2 = 0$$
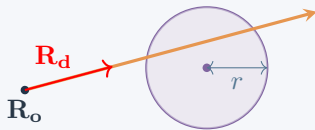
Algebraic Solution

Step 2: Expand and rearrange

$$\mathbf{R_d} \cdot \mathbf{R_d} t^2 + 2\mathbf{R_d} \cdot \mathbf{R_o} t$$
$$+ \mathbf{R_o} \cdot \mathbf{R_o} - r^2 = 0$$
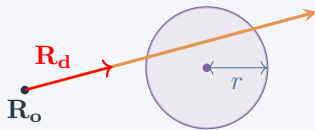
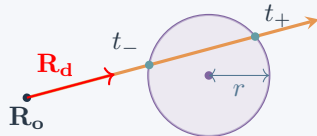# Ray-Sphere Intersection: Algebraic Method

Algebraic Solution

Step 3: Quadratic formula
$(ax^2 + bx + c = 0)$

$a = \mathbf{R_d} \cdot \mathbf{R_d} = 1$ (normalized)
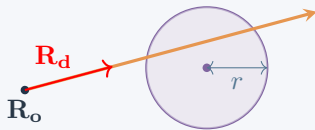
$b = 2\mathbf{R_d} \cdot \mathbf{R_o}$

$c = \mathbf{R_o} \cdot \mathbf{R_o} - r^2$

Ray-Sphere Intersection: Algebraic Method

Algebraic Solution

Step 3: Quadratic formula
$(ax^2 + bx + c = 0)$

$a = \mathbf{R_d} \cdot \mathbf{R_d} = 1$ (normalized)

$b = 2\mathbf{R_d} \cdot \mathbf{R_o}$

$c = \mathbf{R_o} \cdot \mathbf{R_o} - r^2$

Discriminant Analysis

$$\Delta = b^2 - 4ac = (2\mathbf{R_d} \cdot \mathbf{R_o})^2 - 4(\mathbf{R_o} \cdot \mathbf{R_o} - r^2)$$

$$t_\pm = \frac{-b \pm \sqrt{\Delta}}{2a} = -\mathbf{R_d} \cdot \mathbf{R_o} \pm \frac{\sqrt{\Delta}}{2}$$

Algebraic Method: Three Cases

The discriminant Δ determines the number of intersections:

$\Delta > 0$

2 intersections
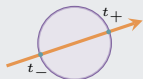
Choose closest positive (usually t_-)

$\Delta = 0$

1 intersection
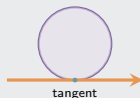
tangent

Ray tangent to sphere

$\Delta < 0$

No intersection

Ray doesn't hit sphere

Additional Check

Remember to check $t_{\min}$ to find closest valid intersection.
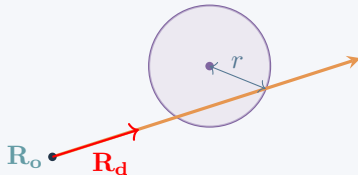
Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 1: Check ray origin position

Inside: $\mathbf{R_o} \cdot \mathbf{R_o} < r^2$

Outside: $\mathbf{R_o} \cdot \mathbf{R_o} > r^2$

On surface: $\mathbf{R_o} \cdot \mathbf{R_o} = r^2$

Geometric Approach

Step 2: Find parameter t_p for the point on the ray closest to the sphere center
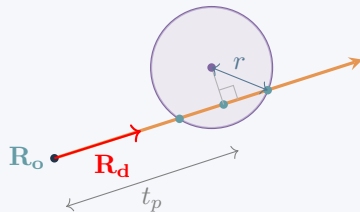
$$t_P = -\mathbf{R_o} \cdot \mathbf{R_d}$$
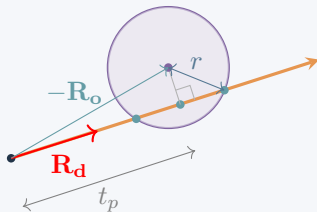
Geometric Approach

Step 3: Early rejection test

If outside & $t_P < 0 \Rightarrow$ no hit

Geometric Approach

Step 4: Find squared distance to sphere center

$$d^2 = \mathbf{R_o} \cdot \mathbf{R_o} - t_P^2$$
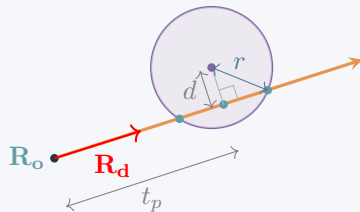
Geometric Approach
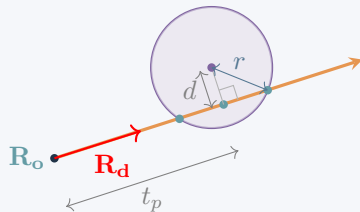
Step 5: Second rejection test

If $d^2 > r^2 \Rightarrow$ no hit

Geometric Approach

Step 6: Find intersection distance

$$t'^2 = r^2 - d^2$$
$$t' = \sqrt{r^2 - d^2}$$
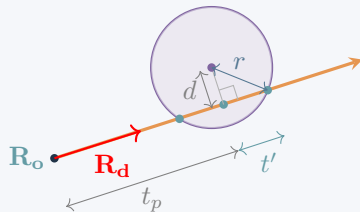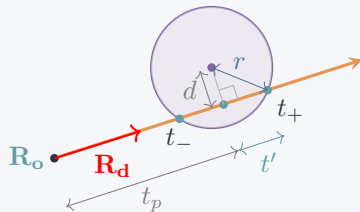
Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 7: Choose correct intersection parameter

$$\textbf{Outside: } t_- = t_P - t'$$
$$\textbf{Inside: } t_+ = t_P + t'$$

$t_{min} < t_\pm < t_{current} \Rightarrow$ hit
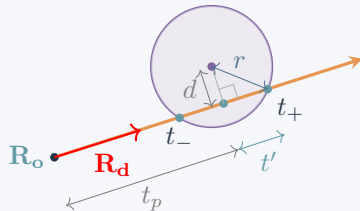
Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 7: Choose correct intersection parameter

$$\textbf{Outside: } t_- = t_P - t'$$
$$\textbf{Inside: } t_+ = t_P + t'$$

$t_{min} < t_{\pm} < t_{current} \Rightarrow$ hit

Benefits of Method

- **Early rejection:** Avoid extra work for rays missing sphere
- **Optimized:** Efficient for rays outside pointing away
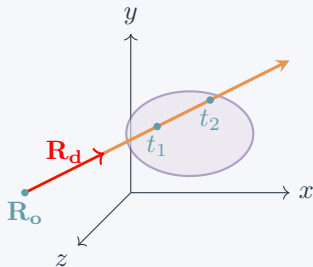
General Quadric Surfaces

Quadric Surface Definition

General equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz$$
$$+Fxz + Gx + Hy + Iz + J = 0$$

Common Quadric Surfaces:

- **Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- **Cone:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$
- **Cylinder:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- **Hyperboloid & Paraboloid**

Reference: Quadric Surfaces in Paul's Online Notes

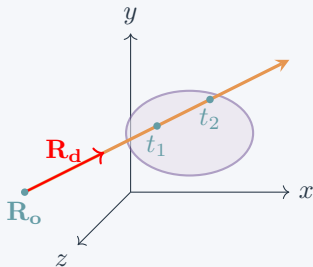# Ray-Quadric Surface Intersection

Intersection Method

Step 1: Substitute ray equation into quadric

$$\mathbf{P}(t) = \mathbf{R_o} + t \cdot \mathbf{R_d}$$
$$P_x = R_{0x} + t \cdot R_{dx}$$
$$P_y = R_{0y} + t \cdot R_{dy}$$
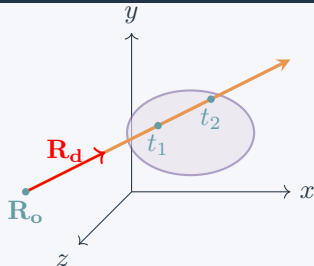$$P_z = R_{0z} + t \cdot R_{dz}$$

Ray-Quadric Surface Intersection

Intersection Method

Step 2: Results in quadratic equation

$$ax^2 + bx + c = 0$$

Solution Cases
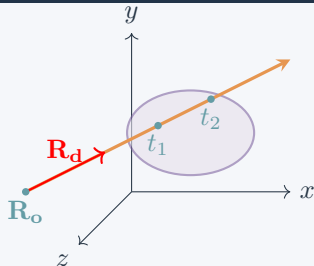
Check the discriminant $b^2 - 4ac$:

- **Discriminant** > 0: Two real solutions (ray intersects surface twice)
- **Discriminant** $= 0$: One solution (ray tangent to surface)
- **Discriminant** < 0: No real solutions (ray misses surface)
- **Accept:** Accept smaller t such that $t_{\mathsf{min}} < t < t_{\mathsf{current}}$

Ray-Quadric Surface Intersection

Intersection Method

Step 3: Solve using quadratic formula

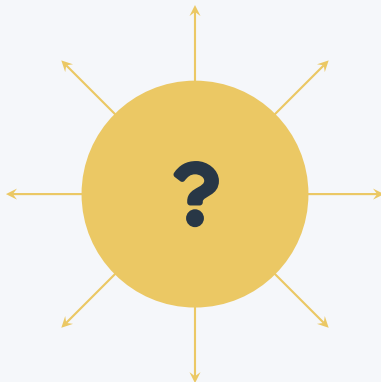$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solution Cases

Check the discriminant $b^2 - 4ac$:

- **Discriminant > 0:** Two real solutions (ray intersects surface twice)
- **Discriminant $= 0$:** One solution (ray tangent to surface)
- **Discriminant < 0:** No real solutions (ray misses surface)
- **Accept:** Accept smaller t such that $t_{\mathsf{min}} < t < t_{\mathsf{current}}$

Questions?

References & Further Reading

📋 Peter Shirley and Steve Marschner et al. *Fundamentals of Computer Graphics (4th Edition)*. CRC Press, 2016.
Available as PDF

📋 Matt Pharr, Wenzel Jakob, and Greg Humphreys. *Physically Based Rendering: From Theory to Implementation (4th Edition)*. Morgan Kaufmann, 2023.
Availabe online

📋 Peter Shirley. *Ray Tracing in One Weekend*. Self-published, 2016–2020.
Project Website

📋 MIT OpenCourseWare: 6.837 Computer Graphics.
ocw.mit.edu/6-837

📋 Scratchapixel: Learn Computer Graphics Programming.
scratchapixel.com