Fractals

Infinite Details from Simple Rules

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Definition

What are Fractals?

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A **fractal** is a geometric shape containing detailed structure at arbitrarily small scales, usually having a **fractal dimension** strictly exceeding the **topological dimension**.

Key Characteristics:

- **Scale invariance**: Details at every level of magnification
- **Fractal dimension**: More than the *conventional* dimension
- **Infinite complexity**: Generated by simple, recursive rules
  - Models the infinite complexities of nature

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- **Infinite complexity**: Generated by simple, recursive rules
    - Models the infinite complexities of nature

"Clouds are not spheres, mountains are not cones, coastlines are not circles..."

— **Benoit Mandelbrot**

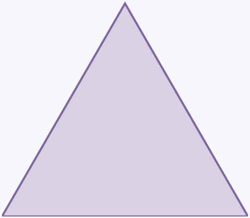
Examples of Fractals

The Sierpinski Triangle

Let's construct the Sierpinski Triangle:

Construction Process

1. Start with an equilateral triangle



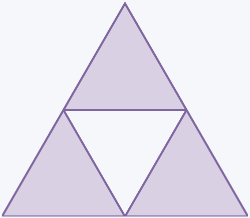
Iteration 0

The Sierpinski Triangle

Let's construct the Sierpinski Triangle:

Construction Process

1. Start with an equilateral triangle
2. Remove the central triangle



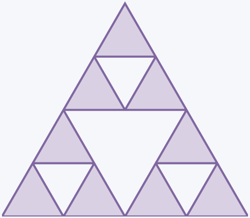
Iteration 1

The Sierpinski Triangle

Let's construct the Sierpinski Triangle:

Construction Process

1. Start with an equilateral triangle
2. Remove the central triangle
3. Repeat for each triangle



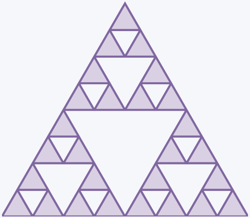
Iteration 2

The Sierpinski Triangle

Let's construct the Sierpinski Triangle:

Construction Process

1. Start with an equilateral triangle
2. Remove the central triangle
3. Repeat for each triangle
4. After infinite iterations, we get the Sierpinski Triangle



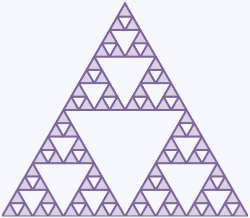
Iteration 3

The Sierpinski Triangle

Let's construct the Sierpinski Triangle:

Construction Process

1. Start with an equilateral triangle
2. Remove the central triangle
3. Repeat for each triangle
4. After infinite iterations, we get the Sierpinski Triangle



Iteration 4

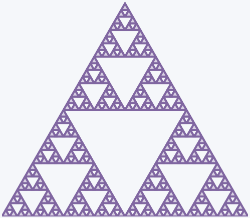
The Sierpinski Triangle

Let's construct the Sierpinski Triangle:

Construction Process

1. Start with an equilateral triangle
2. Remove the central triangle
3. Repeat for each triangle
4. After infinite iterations, we get the Sierpinski Triangle

We get a complex shape with *infinite* details.



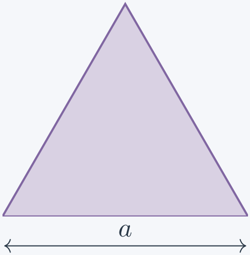
Iteration 5

The Sierpinski Triangle

Now let's find the area of triangle(s):

Area Calculation

$$A_0 = \frac{1}{2} \cdot \sin(60^\circ) \cdot a^2 = \frac{\sqrt{3}}{4} a^2$$



Iteration 0

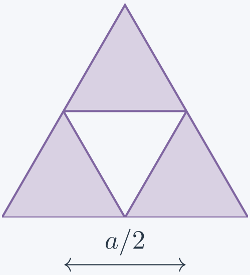
The Sierpinski Triangle

Now let's find the area of triangle(s):

Area Calculation

$$\begin{aligned} A_0 &= \frac{1}{2} \cdot \sin(60^\circ) \cdot a^2 = \frac{\sqrt{3}}{4} a^2 \\ A_1 &= A_0 - \frac{1}{2} \cdot \sin(60^\circ) \cdot \left(\frac{a}{2}\right)^2 \\ &= A_0 - \frac{\sqrt{3}}{16} a^2 = A_0 - \frac{1}{4} A_0 \\ &= \frac{3}{4} A_0 \end{aligned}$$

We cut out one fourth of the area at each step.
So, naturally area is reduced by $\frac{3}{4}$ at each step.



Iteration 1

The Sierpinski Triangle

Now let's find the area of triangle(s):

Area Calculation

$$A_0 = \frac{1}{2} \cdot \sin(60^\circ) \cdot a^2 = \frac{\sqrt{3}}{4} a^2$$
$$A_1 = \left(1 - \frac{1}{4}\right) A_0 = \frac{3}{4} A_0$$
$$A_2 = \left(1 - \frac{1}{4}\right) A_1 = \left(\frac{3}{4}\right)^2 A_0$$

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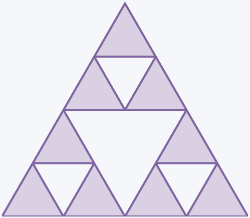


Iteration 2

The Sierpinski Triangle

Now let's find the area of triangle(s):

Area Calculation

$$A_0 = \frac{1}{2} \cdot \sin(60^\circ) \cdot a^2 = \frac{\sqrt{3}}{4} a^2$$

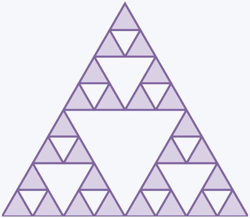
$$A_1 = \left(1 - \frac{1}{4}\right) A_0 = \frac{3}{4} A_0$$

$$A_2 = \left(1 - \frac{1}{4}\right) A_1 = \left(\frac{3}{4}\right)^2 A_0$$

$$\ldots$$

$$A_n = \left(1 - \frac{1}{4}\right)^n A_0 = \left(\frac{3}{4}\right)^n A_0$$

$$\lim_{n \to \infty} A_n = 0$$



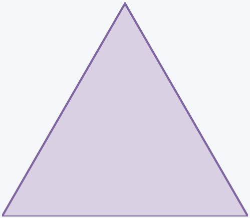
Iteration 3

The area of the Sierpinski Triangle is zero even though it is a 2D shape.

Now let's find the perimeter:

Perimeter Calculation

$$P_0 = 3a$$



Iteration 0

The Sierpinski Triangle

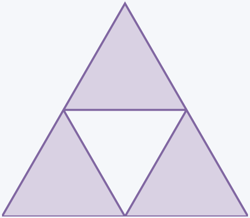
Now let's find the perimeter:

Perimeter Calculation

$$P_0 = 3a$$

$$P_1 = P_0 + 3 \cdot \frac{a}{2} = \frac{9}{2}a = \frac{3}{2}P_0$$

We add half of the previous perimeter at each step. This is because we add the same number of triangles as the last step, but each triangle is half the size of the previous one.
As a result, the perimeter grows by a factor of $\frac{3}{2}$ at each step.



Iteration 1

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Perimeter Calculation

$$P_0 = 3a$$
$$P_1 = P_0 + 3 \cdot \frac{a}{2} = \frac{9}{2}a = \frac{3}{2}P_0$$
$$P_2 = P_1 + 3 \cdot 3 \cdot \frac{a}{4} = P_1 + \frac{1}{2}P_1 = \frac{3}{2}P_1$$

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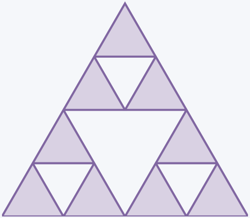


Iteration 2

The Sierpinski Triangle

Now let's find the perimeter:

Perimeter Calculation

$$P_0 = 3a$$

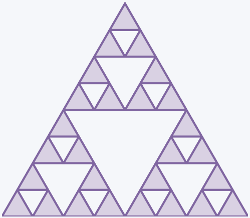
$$P_1 = P_0 + 3 \cdot \frac{a}{2} = \frac{9}{2}a = \frac{3}{2}P_0$$

$$P_2 = P_1 + 3 \cdot 3 \cdot \frac{a}{4} = P_1 + \frac{1}{2}P_1 = \frac{3}{2}P_1$$

$$\dots$$

$$P_n = \frac{3}{2}P_{n-1} = \left(\frac{3}{2}\right)^n P_0$$

$$\lim_{n\to\infty} P_n = \infty$$



Iteration 3

The Sierpinski Triangle has infinite perimeter but zero area.

The Koch Curve

Let's construct the Koch Curve:

Construction Process

1. Start with a line segment of length a

Iteration 0

The Koch Curve

Let's construct the Koch Curve:

Construction Process

1. Start with a line segment of length a
2. Divide it into three equal pieces

Iteration 1

The Koch Curve

Let's construct the Koch Curve:

Construction Process

1. Start with a line segment of length a
2. Divide it into three equal pieces
3. Remove the middle piece and replace it with two sides of an equilateral triangle (side length $a/3$)

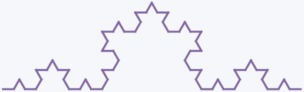
Iteration 2

The Koch Curve

Let's construct the Koch Curve:

Construction Process

1. Start with a line segment of length a
2. Divide it into three equal pieces
3. Remove the middle piece and replace it with two sides of an equilateral triangle (side length $a/3$)
4. Repeat on every straight segment



Iteration 3

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Construction Process

1. Start with a line segment of length a
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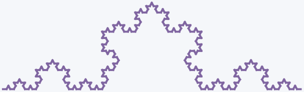
4. Repeat on every straight segment
5. After infinite iterations, we get the Koch Curve



Iteration 4

The Koch Curve

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Construction Process

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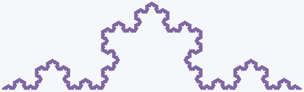


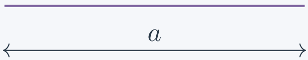
Iteration 5

The Koch Curve

Now let's look at how the length grows:

Length Calculation

$$L_0 = a$$



Iteration 0

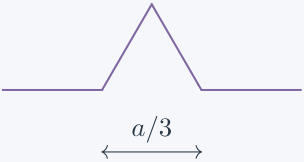
The Koch Curve

Now let's look at how the length grows:

Length Calculation

$$L_0 = a$$
$$L_1 = 4 \cdot \frac{a}{3} = \frac{4}{3} L_0$$

Each step multiplies the number of segments by 4, each $\frac{1}{3}$ as long, so the total length scales by $\frac{4}{3}$ every iteration.



Iteration 1

The Koch Curve

Now let's look at how the length grows:

Length Calculation

$$L_0 = a$$
$$L_1 = 4 \cdot \frac{a}{3} = \frac{4}{3} L_0$$
$$L_2 = 16 \cdot \frac{a}{9} = 4 \frac{L_1}{3} = \frac{4}{3} L_1$$

Each step multiplies the number of segments by 4, each $\frac{1}{3}$ as long, so the total length scales by $\frac{4}{3}$ every iteration.

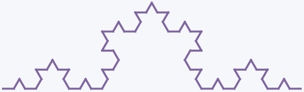
Iteration 2

The Koch Curve

Now let's look at how the length grows:

Length Calculation

$$
\begin{aligned}
L_0 &= a \\
L_1 &= 4 \cdot \frac{a}{3} = \frac{4}{3} L_0 \\
L_2 &= 16 \cdot \frac{a}{9} = 4 \frac{L_1}{3} = \frac{4}{3} L_1 \\
&\dots \\
L_n &= \left(\tfrac{4}{3}\right)^n L_0 \\
\lim_{n\to\infty} L_n &= \infty
\end{aligned}
$$



Iteration 3

The Koch curve has infinite length, despite being bounded in a small area.

Fractal Dimension

Fractal Strangeness

Fractals are not like regular geometric shapes.

Interestingly we saw:

- We saw a 2D fractal (Sierpinski) with **zero area**.
- We saw a 1D fractal (Koch) with **infinite length**.
- No matter how many times we zoom in, we always find more detail.

Regular shapes on the other hand:

- A 2D shape has a **finite area**.
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This suggests that fractals have a dimension in between the traditional dimensions.

Dividing Shapes

Normal Shapes

If we divide a normal shape into smaller pieces, it scales with the dimension.

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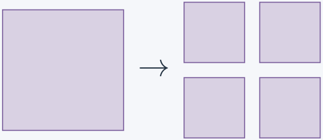
- A **line** divides into 2 pieces, each piece is still a line of scaled by $\frac{1}{2}$.

Dividing Shapes

Normal Shapes

If we divide a normal shape into smaller pieces, it scales with the dimension.

- A **square** divides into $4 = 2^2$ pieces, each piece is still a square scaled by $\frac{1}{2}$.

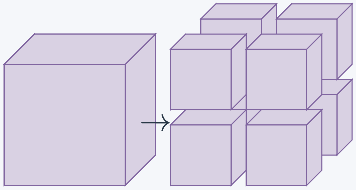


Dividing Shapes

Normal Shapes

If we divide a normal shape into smaller pieces, it scales with the dimension.

- A **cube** divides into $8 = 2^3$ pieces, each piece is still a cube scaled by $\frac{1}{2}$.



Dividing Shapes

From these observations, we can define **dimension** as:

Self-Similarity Dimension

Let,

$$
\begin{aligned}
N &= \text{number of pieces} \\
r &= \text{scaling factor} \\
D &= \text{dimension}
\end{aligned}
$$

Then,

$$
\begin{aligned}
N &= \left(\frac{1}{r}\right)^{D} \\
\log N &= D \cdot \log\left(\frac{1}{r}\right) \\
D &= \frac{\log N}{\log\left(\frac{1}{r}\right)}
\end{aligned}
$$

Dividing Shapes

Fractals

If we divide a fractal into smaller pieces, it doesn't scale with the dimension.

Dividing Shapes

Fractals

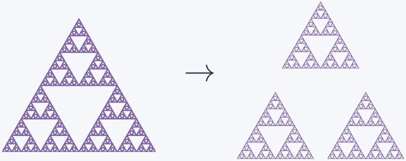
If we divide a fractal into smaller pieces, it doesn't scale with the dimension.

- A **Sierpinski triangle** divides into 3 pieces, each piece is a Sierpinski triangle scaled by $\frac{1}{2}$.

$$N = 3$$

$$r = \frac{1}{2}$$

$$D = \frac{\log 3}{\log 2} \approx 1.585$$



Dividing Shapes

Fractals

If we divide a fractal into smaller pieces, it doesn't scale with the dimension.

- A **Koch curve** divides into 4 pieces, each piece is a Koch curve scaled by $\frac{1}{3}$.

$$N = 4$$
$$r = \frac{1}{3}$$
$$D = \frac{\log 4}{\log 3} \approx 1.262$$

Fractal Dimension Theory

The **Self-Similarity Dimension** works for self-similar fractals. A more general definition is the **Box-Counting Dimension**.

Box-Counting Dimension

The fractal dimension D is defined as:

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

where $N(\varepsilon)$ is the number of boxes of size ε needed to cover the fractal.

Please check this 3blue1brown video for an excellent explanation of fractal dimension.

Nature's Fractal Dimensions

Fractal dimension measures the **complexity** of a shape. We usually assume that when zoomed in things become smooth. Natural objects like fractals, exhibit complexity even at small scales.

Natural Fractals

- Coastline of Britain: $D \approx 1.25$
- Clouds: $D \approx 2.35$
- Lightning: $D \approx 1.7$
- Lung bronchi: $D \approx 2.97$

Coastline of Britain

Questions?

References & Further Reading

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