# Ray Tracing & Ray Casting

Realistic Graphics Inpsired by Nature

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# Motivation

# Why Learn This?

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Elsa's Castle in Frozen

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Cyberpunk 2077 with RTX

- **Realistic graphics** of your favourite animated movies are the result of ground-breaking work in Ray Tracing by studios like Disney, Pixar, and DreamWorks. Do you know these films take years to render? 30 hours per frame!

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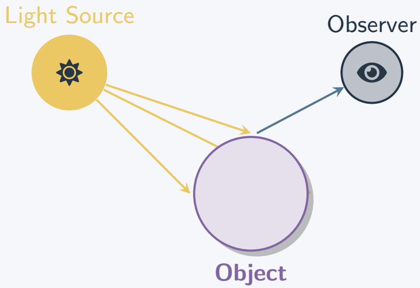
- Lately, **RTX** is all the rage in gaming. New titles boast ray-tracing effects in real-time, not 30 hours per frame!
- It's fun! You will know when you create your first ray-traced image!

# The Story of Light

# How Do We See?



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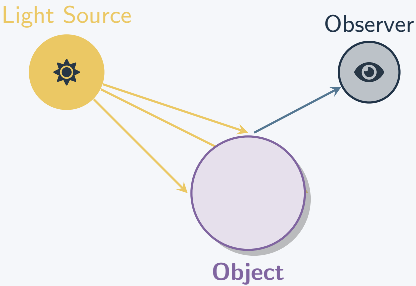
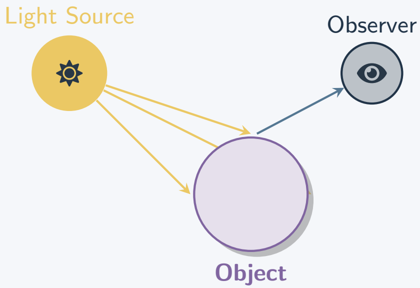



## Natural Process

1. Light travels from source
2. Light hits objects
3. Light bounces to our eyes
4. Our brain interprets the signal

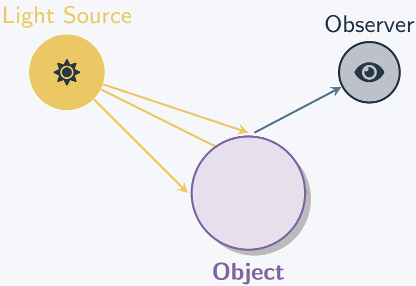
# How Do We See?




## Physical Process

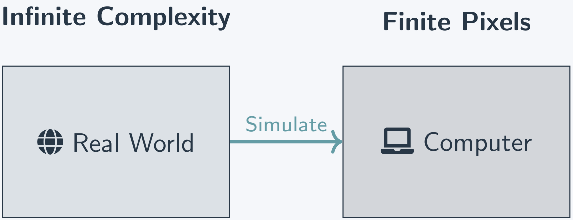
1. Photon is emitted from source
2. Photon hits objects
3. Part of the photon is reflected or absorbed
4. The reflected photons reach our eyes
5. The rods and cones in our retina detect the photons
6. Our brain interprets the signal
7. **Colour**: The wavelength of the photons
8. **Brightness**: The number of photons

# How Do We See?




Question: How do we simulate this?

# The Computer Graphics Challenge




**Challenges:**

- Infinite light rays/photons
- Complex physics
- High computational cost

# Ray Casting: Foundation

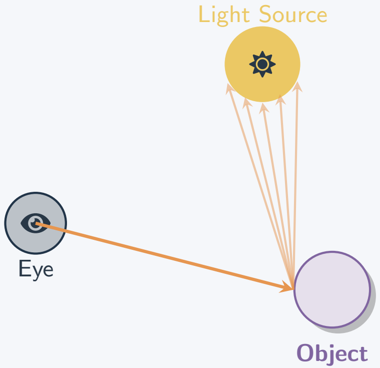
# The Key Insight

## 1. Reverse Engineering

Instead of following light rays from light sources —

**Let's trace backwards!**
**Shoot rays from the eye**, find where it hits and find out how much light reaches there.

This is the opposite of what happens in reality. **Why does this work?**



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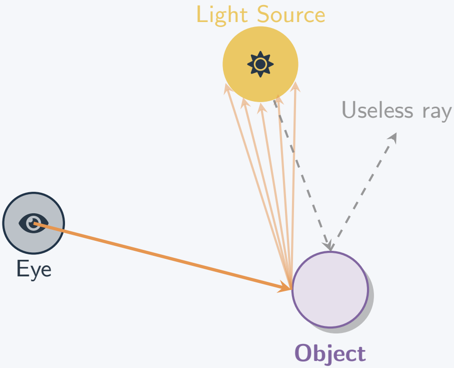


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- Most light never reaches our eyes

# The Key Insight

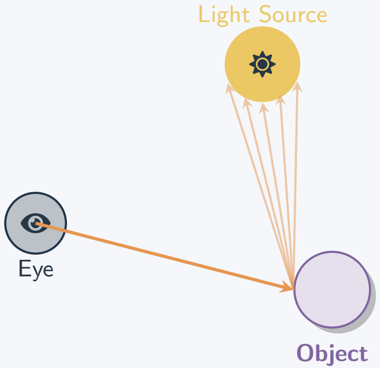
**1. Reverse Engineering**

Instead of following light rays from light sources —

**Let's trace backwards!**
**Shoot rays from the eye**, find where it hits and find out how much light reaches there.

This is the opposite of what happens in reality. **Why does this work?**

- Most light never reaches our eyes
- Only trace rays that matter
- Much more efficient!



**2. Cutting Costs**

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**Trace one ray per pixel.**

This comes with little tradeoff, because:

# From Infinite Rays to Finite Pixels

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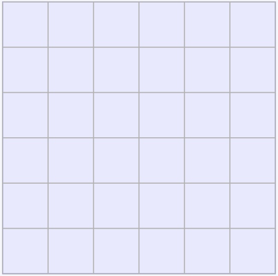

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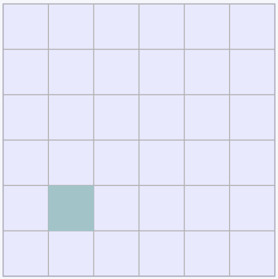

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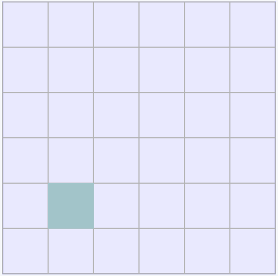
- In the end, we just need to know the *color of each pixel*



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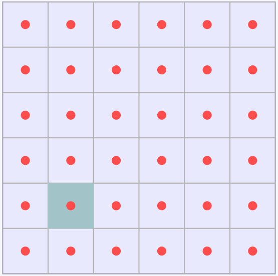
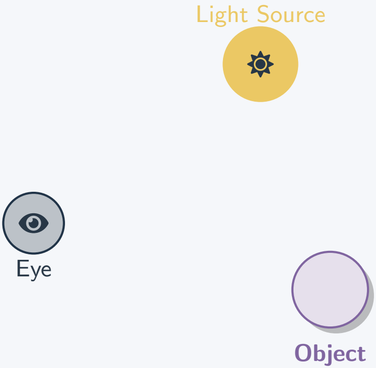


This comes with little tradeoff, because:

- An image is just a grid of pixels
- Each pixel can only be of one color
- In the end, we just need to know the *color of each pixel*
- Hence, one ray from the mid-point of each pixel should be a good approximation*

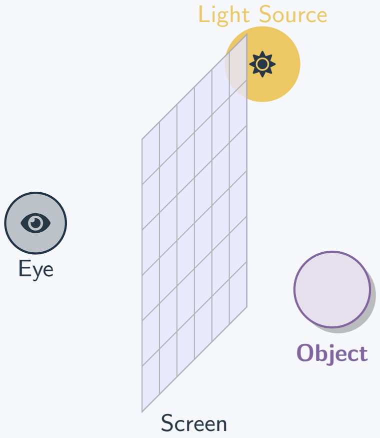
* We will discuss more advanced techniques later that improve quality

# The Full Picture



# The Full Picture

Place screen in front of eye

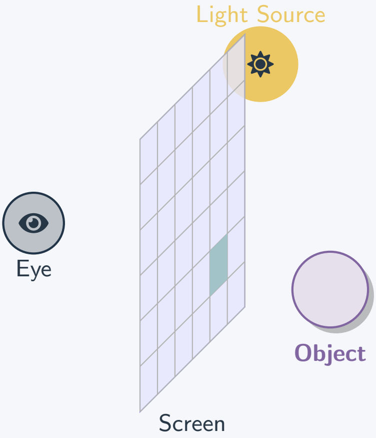


# The Full Picture

Place screen in front of eye

↓

For each pixel



# The Full Picture

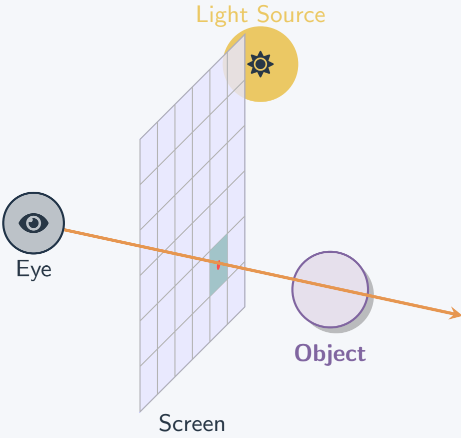
Place screen in front of eye

↓

For each pixel

↓

Shoot ray from eye



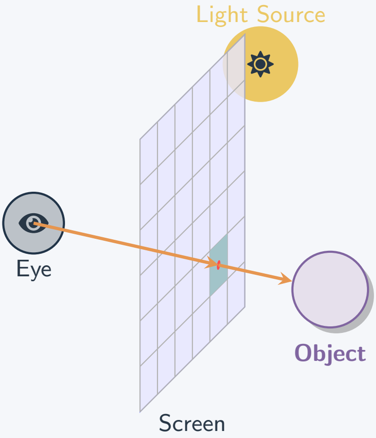
# The Full Picture

Place screen in front of eye

For each pixel

Shoot ray from eye

Find closest intersection



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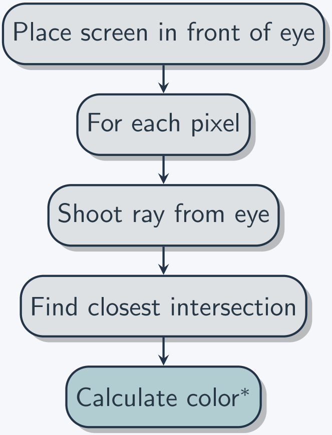
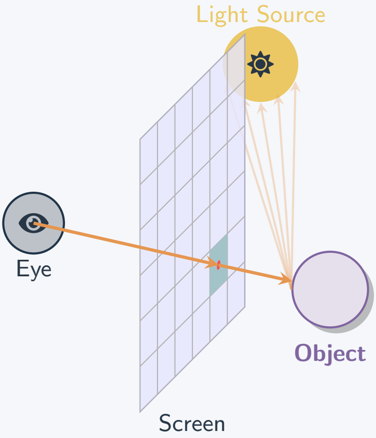
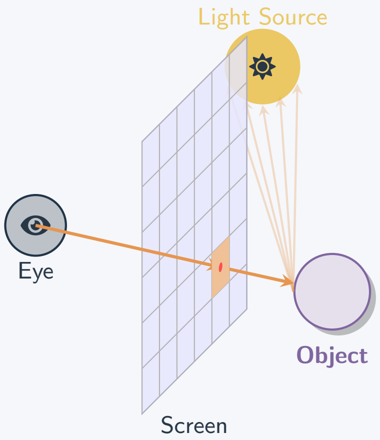
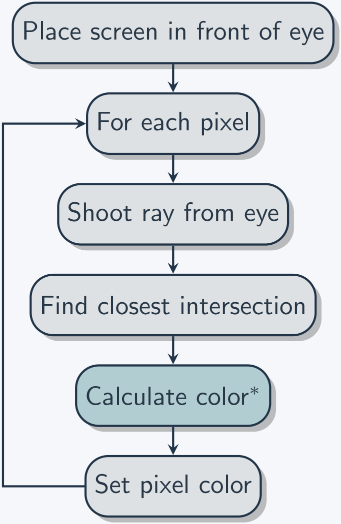


# The Full Picture

- Place screen in front of eye
- For each pixel
- Shoot ray from eye
- Find closest intersection
- Calculate color*
- Set pixel color





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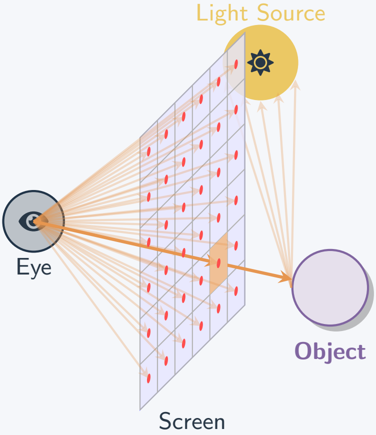


Light Source
Eye
Object
Screen

# Mathematics of Rays

# What is a Ray?

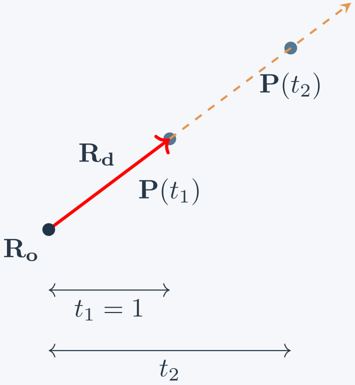
**Ray Representation**

A ray is defined by:

$$\mathbf{P}(t) = \mathbf{R_o} + t \cdot \mathbf{R_d} \quad (1)$$

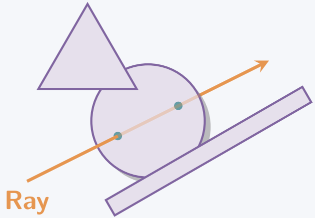
where:

- $\mathbf{R_o}$ = Origin point
- $\mathbf{R_d}$ = Direction vector
- $t$ = Parameter ($t \geq 0$)




Check out here on desmos.

# Finding Intersections




**Key Objects:**

- **Planes**
- **Spheres**
- **Triangles**
- **General Quadrics**

Challenge: Find the **closest** intersection efficiently!

# 3D Plane Representation

## Plane Definition

A plane is defined by:

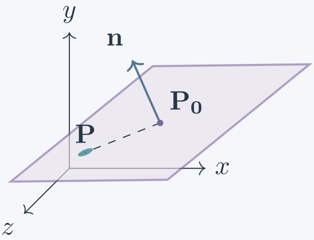
- Point $\mathbf{P_0} = (x_0, y_0, z_0)$ on plane
- Normal vector $\mathbf{n} = (A, B, C)$

**Implicit equation:**

$$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P_0}) = 0$$

$\boxed{\mathbf{n} \cdot \mathbf{P} + D = 0}$ where $D = -\mathbf{n} \cdot \mathbf{P_0}$

$$\boxed{Ax + By + Cz + D = 0}$$



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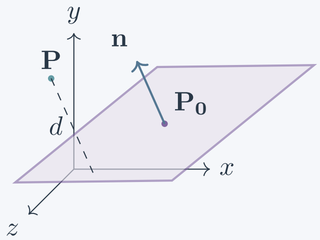



## Point-Plane Distance

If $\mathbf{n}$ is normalized: $d = \mathbf{n} \cdot \mathbf{P} + D = \mathbf{n} \cdot (\mathbf{P} - \mathbf{P_0})$

**Signed distance:** $d > 0$ (front), $d < 0$ (back), $d = 0$ (on plane)

# Ray-Plane Intersection

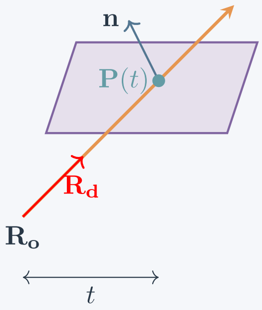
## Intersection Method

**Step 1:** Substitute ray into equation

$$\mathbf{n} \cdot (\mathbf{R_o} + t\mathbf{R_d}) + D = 0$$
$$\mathbf{n} \cdot \mathbf{R_o} + t(\mathbf{n} \cdot \mathbf{R_d}) + D = 0$$

**Step 2:** Solve for parameter $t$

$$\boxed{t = -\frac{D + \mathbf{n} \cdot \mathbf{R_o}}{\mathbf{n} \cdot \mathbf{R_d}}}$$



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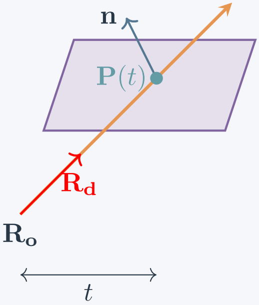



## Cases

- If $\mathbf{n} \cdot \mathbf{R_d} = 0$: Ray parallel to plane (0 or infinite)
- If $\mathbf{n} \cdot \mathbf{R_d} < 0$: Ray hits front face
- If $\mathbf{n} \cdot \mathbf{R_d} > 0$: Ray hits back face

# Additional Checks

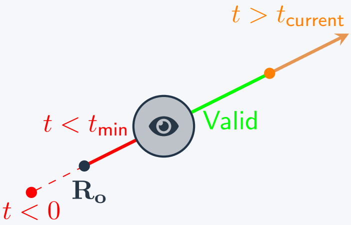
## Validation Rules

After computing $t$, verify:

1. **Behind check:** $t > t_{\text{min}}$
2. **Closest check:** $t < t_{\text{current}}$
3. **Valid range:** $t \geq 0$

Where:

- $t_{\text{min}}$: Minimum ray distance (not behind eye/screen)
- $t_{\text{current}}$: Distance to closest intersection so far



# Ray-Triangle Intersection Overview

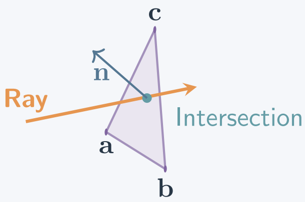
## Two Main Approaches

**Method 1: Two-Step Process**

1. Ray-plane intersection
2. Inside/outside triangle test

**Method 2: Direct Barycentric**

1. Set up 3×3 linear system
2. Solve for t, $\beta$, $\gamma$ simultaneously



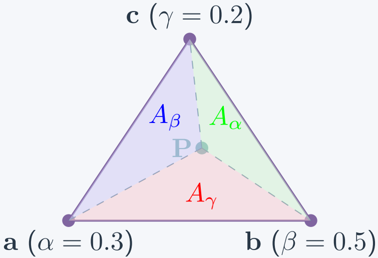
# What Are Barycentric Coordinates?

### Barycentric Definition

Any point P in the triangle's plane:

$$\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

where: $\alpha + \beta + \gamma = 1$

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**Physical Interpretation:**

- $\alpha$, $\beta$, $\gamma$ are *weights*
- P is the *center of mass*
- Also called *barycenter*

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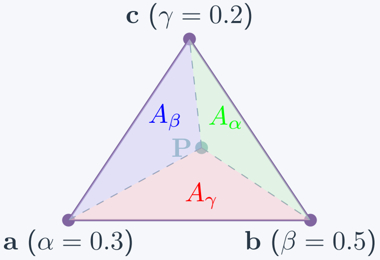

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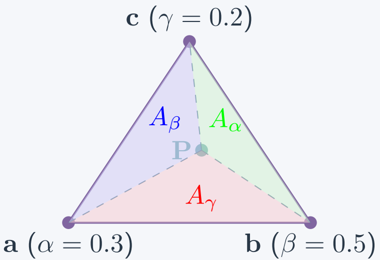



## Area Relationship

$\alpha = \frac{A_\alpha}{A_{total}}$, $\beta = \frac{A_\beta}{A_{total}}$,
$\gamma = \frac{A_\gamma}{A_{total}}$

# Barycentric Coordinates: Inside vs Outside

### Triangle Interior Test

Point P is **inside** triangle if:

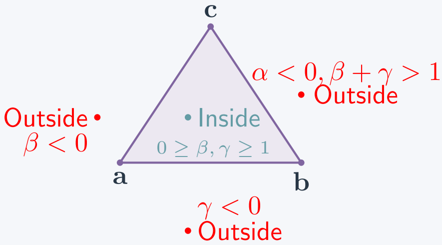
$$\alpha, \beta, \gamma \geq 0$$

Since $\alpha + \beta + \gamma = 1$, we can rewrite as:

$$\beta \geq 0$$

$$\gamma \geq 0$$

$$\alpha \geq 0 \text{ or } \beta + \gamma \leq 1$$



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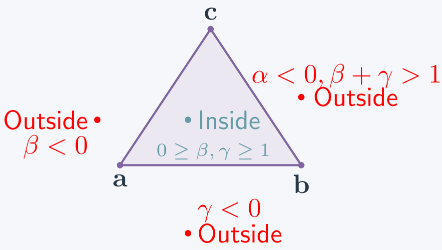


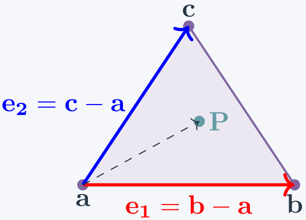

### Insight

Barycentric coordinates doesn't just tell us if a point is inside a triangle, but also it's position with respect to other vertices.

# Barycentric Coordinates: Derivation

## Key Idea

- The sides $\mathbf{e_1} = \mathbf{b} - \mathbf{a}$ and $\mathbf{e_2} = \mathbf{c} - \mathbf{a}$ are linearly independent vectors on the triangle's plane.



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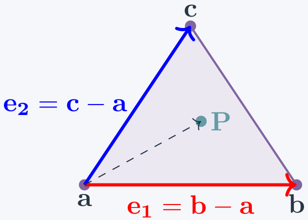
- Therefore, any vector in the triangle's plane (e.g. $\mathbf{P} - \mathbf{a}$) can be expressed as a linear combination of these vectors.



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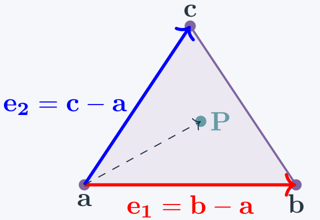
- Therefore, any vector in the triangle's plane (e.g. $\mathbf{P} - \mathbf{a}$) can be expressed as a linear combination of these vectors.
- We can express $\mathbf{P}$ as:

$$\begin{aligned}\mathbf{P} &= \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}) \\ &= \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}\end{aligned}$$

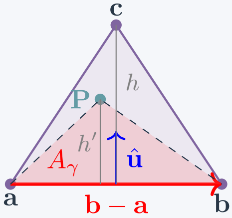
Where $\alpha = 1 - \beta - \gamma$.



# Barycentric Coordinates: Derivation

## Area Interpretation

- Let $\hat{\mathbf{u}}$ be an unit vector in the direction of the altitude towards $C$.



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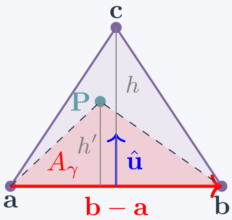
- Let $\hat{\mathbf{u}}$ be an unit vector in the direction of the altitude towards $C$.
- The height of the triangle is $h = \hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})$ (projection).



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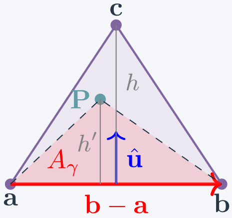
- The height of the triangle is $h = \hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})$ (projection).
- The height of the shaded triangle is $h' = \hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})$.

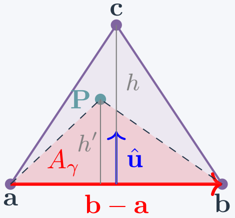


# Barycentric Coordinates: Derivation

## Area Interpretation

- Let $\hat{\mathbf{u}}$ be an unit vector in the direction of the altitude towards $C$.
- The height of the triangle is $h = \hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})$ (projection).
- The height of the shaded triangle is $h' = \hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})$.
- Hence,

$$\begin{aligned} A_\gamma &= \frac{1}{2} \cdot h' \cdot |\mathbf{b} - \mathbf{a}| \\ &= \frac{1}{2} \cdot (\hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}| \\ &= \frac{1}{2} \cdot \gamma \, (\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}| \\ &= \gamma \frac{1}{2} \cdot h \cdot |\mathbf{b} - \mathbf{a}| = \gamma A_{total} \end{aligned}$$



# Barycentric Coordinates: Derivation

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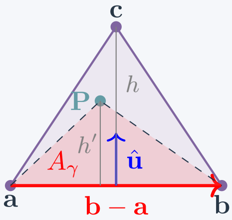



Since,

$$\begin{aligned} \mathbf{P} &= \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}), \\ \mathbf{P} - \mathbf{a} &= \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}) \\ \hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a}) &= \gamma(\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})) \end{aligned}$$

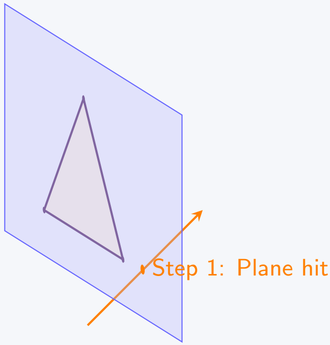
Since $\hat{\mathbf{u}}$ is perpendicular to $\mathbf{b} - \mathbf{a}$.

# Method 1: Two-Step Ray-Triangle Intersection

## Algorithm Steps

**Step 1:** Ray-Plane Intersection

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R_o}}{\mathbf{n} \cdot \mathbf{R_d}}$$



# Method 1: Two-Step Ray-Triangle Intersection

## Algorithm Steps

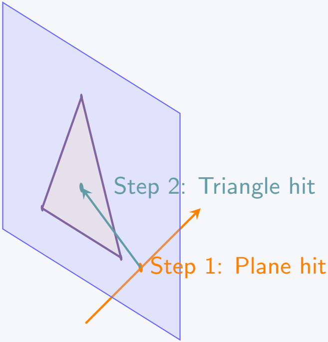
**Step 1:** Ray-Plane Intersection

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R_o}}{\mathbf{n} \cdot \mathbf{R_d}}$$

**Step 2:** Inside/Outside Test

$$\mathbf{P} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

Solve for $\beta$, $\gamma$ and check bounds.



# Method 2: Direct Barycentric Intersection

## Direct Approach

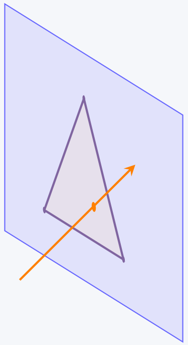
Set ray equation equal to barycentric form:

$$\mathbf{R_o} + t\mathbf{R_d} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

Rearrange to linear system:

$$\begin{bmatrix} -\mathbf{R_d} & (\mathbf{b} - \mathbf{a}) & (\mathbf{c} - \mathbf{a}) \end{bmatrix} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \mathbf{R_o} - \mathbf{a}$$

Solve using Cramer's rule or LU decomposition.



# Cramer's Rule Solution

## Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_{A} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

# Cramer's Rule Solution

## Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_{A} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

## Cramer's Rule

$$t = \frac{1}{|A|} \begin{vmatrix} (R_o - a)_x & (b - a)_x & (c - a)_x \\ (R_o - a)_y & (b - a)_y & (c - a)_y \\ (R_o - a)_z & (b - a)_z & (c - a)_z \end{vmatrix}$$

$$\beta = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix}$$

$$\gamma = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (b - a)_x & (R_o - a)_x \\ -R_{dy} & (b - a)_y & (R_o - a)_y \\ -R_{dz} & (b - a)_z & (R_o - a)_z \end{vmatrix}$$

# Cramer's Rule Solution

## Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_{A} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

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$$\beta = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix}$$

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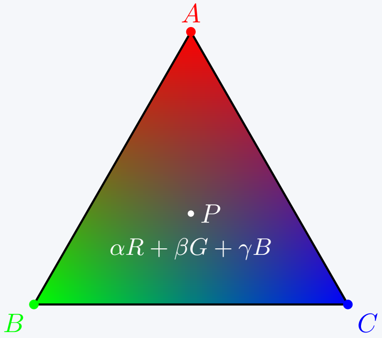
## Checks

- $t_{\text{min}} < t < t_{\text{current}}$ (valid intersection)
- $\beta, \gamma \geq 0$ and $\beta + \gamma \leq 1$ (inside triangle)

# Bonus of Using Barycentric Coordinates

**Advantages**

- Efficient to compute
- Get Barycentric coordinates for free
- Enables interpolation of vertex attributes
  Used in —
  - Textures
  - Normals
  - Colors



# Ray-Sphere Intersection Overview

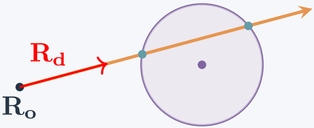
## Two Main Approaches

**Method 1: Algebra**

1. Setup quadratic equation
2. Solve for t

**Method 2: Geometry**

1. Use geomety to find intersection step by step
2. Reject early if hit is not possible



# Sphere Representation

## Implicit Sphere Equation

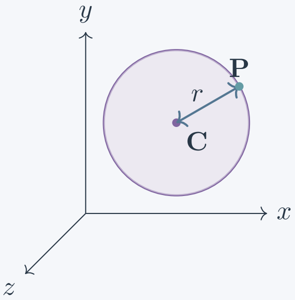
**Sphere centered at origin:**

$$\mathbf{P} \cdot \mathbf{P} - r^2 = 0$$
$$x^2 + y^2 + z^2 - r^2 = 0$$

**General sphere at center $\mathbf{C}$:**

$$(\mathbf{P} - \mathbf{C}) \cdot (\mathbf{P} - \mathbf{C}) - r^2 = 0$$

**Note:** Translation to origin simplifies calculation!

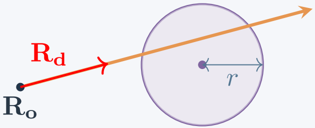


# Ray-Sphere Intersection: Algebraic Method

## Algebraic Solution

**Step 1:** Substitute ray equation $\mathbf{P}(t) = \mathbf{R_o} + t\mathbf{R_d}$ into sphere

$$(\mathbf{R_o} + t\mathbf{R_d}) \cdot (\mathbf{R_o} + t\mathbf{R_d}) - r^2 = 0$$

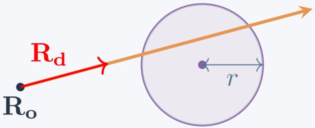


# Ray-Sphere Intersection: Algebraic Method

## Algebraic Solution

**Step 2:** Expand and rearrange

$$\mathbf{R_d} \cdot \mathbf{R_d} t^2 + 2\mathbf{R_d} \cdot \mathbf{R_o} t$$
$$+\mathbf{R_o} \cdot \mathbf{R_o} - r^2 = 0$$



# Ray-Sphere Intersection: Algebraic Method

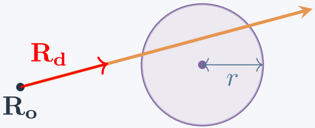
## Algebraic Solution

**Step 3:** Quadratic formula $(ax^2 + bx + c = 0)$

$$a = \mathbf{R_d} \cdot \mathbf{R_d} = 1 \text{ (normalized)}$$

$$b = 2\mathbf{R_d} \cdot \mathbf{R_o}$$

$$c = \mathbf{R_o} \cdot \mathbf{R_o} - r^2$$



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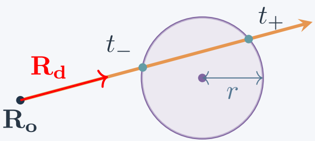



## Discriminant Analysis

$$\Delta = b^2 - 4ac = (2\mathbf{R_d} \cdot \mathbf{R_o})^2 - 4(\mathbf{R_o} \cdot \mathbf{R_o} - r^2)$$

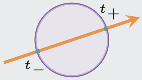
$$t_\pm = \frac{-b \pm \sqrt{\Delta}}{2a} = -\mathbf{R_d} \cdot \mathbf{R_o} \pm \frac{\sqrt{\Delta}}{2}$$

# Algebraic Method: Three Cases

The discriminant $\Delta$ determines the number of intersections:

### $\Delta > 0$

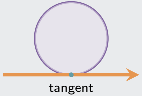
**2 intersections**




Choose closest positive (usually $t_-$)

### $\Delta = 0$

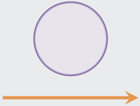
**1 intersection**




Ray tangent to sphere

### $\Delta < 0$

**No intersection**



Ray doesn't hit sphere

### Additional Check

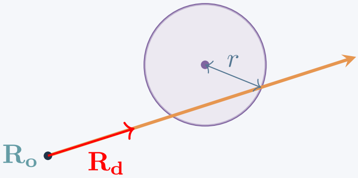
Remember to check $t_{\text{min}}$ to find closest valid intersection.

# Ray-Sphere Intersection: Geometric Method

## Geometric Approach

**Step 1:** Check ray origin position

$$\begin{aligned} \text{Inside: } & \mathbf{R_o} \cdot \mathbf{R_o} < r^2 \\ \text{Outside: } & \mathbf{R_o} \cdot \mathbf{R_o} > r^2 \\ \text{On surface: } & \mathbf{R_o} \cdot \mathbf{R_o} = r^2 \end{aligned}$$

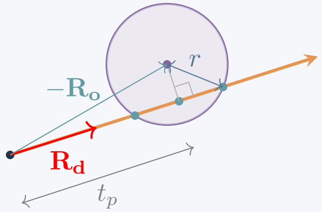


# Ray-Sphere Intersection: Geometric Method

## Geometric Approach

**Step 2:** Find parameter $t_p$ for the point on the ray closest to the sphere center

$$t_P = -\mathbf{R_o} \cdot \mathbf{R_d}$$

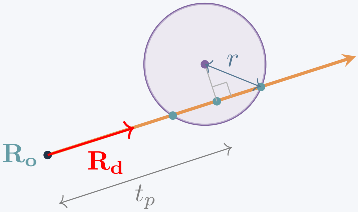


# Ray-Sphere Intersection: Geometric Method

### Geometric Approach

**Step 3:** Early rejection test

If outside & $t_P < 0 \Rightarrow$ no hit

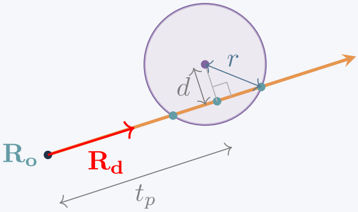


# Ray-Sphere Intersection: Geometric Method

## Geometric Approach

**Step 4:** Find squared distance to sphere center

$$d^2 = \mathbf{R_o} \cdot \mathbf{R_o} - t_P^2$$

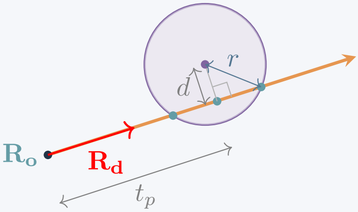


# Ray-Sphere Intersection: Geometric Method

### Geometric Approach

**Step 5:** Second rejection test

$$\text{If } d^2 > r^2 \Rightarrow \text{no hit}$$



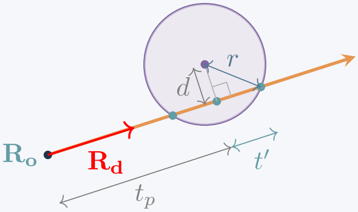
# Ray-Sphere Intersection: Geometric Method

## Geometric Approach

**Step 6:** Find intersection distance

$$t'^2 = r^2 - d^2$$

$$t' = \sqrt{r^2 - d^2}$$



# Ray-Sphere Intersection: Geometric Method

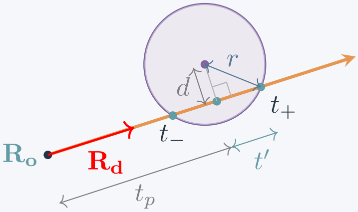
## Geometric Approach

**Step 7:** Choose correct intersection parameter

$$\textbf{Outside: } t_- = t_P - t'$$

$$\textbf{Inside: } t_+ = t_P + t'$$

$t_{min} < t_\pm < t_{current} \Rightarrow$ hit



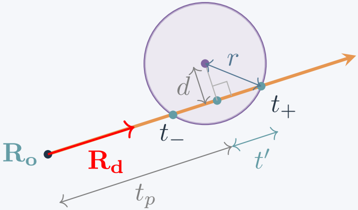
# Ray-Sphere Intersection: Geometric Method

## Geometric Approach

**Step 7:** Choose correct intersection parameter

$$\textbf{Outside: } t_- = t_P - t'$$
$$\textbf{Inside: } t_+ = t_P + t'$$

$t_{min} < t_{\pm} < t_{current} \Rightarrow$ hit




## Benefits of Method

- **Early rejection:** Avoid extra work for rays missing sphere
- **Optimized:** Efficient for rays outside pointing away

# General Quadric Surfaces

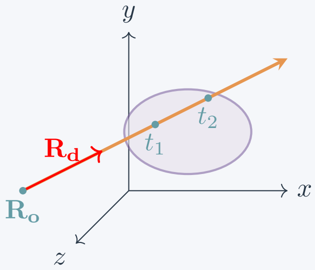
## Quadric Surface Definition

General equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz$$
$$+Fxz + Gx + Hy + Iz + J = 0$$

**Common Quadric Surfaces:**

- **Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- **Cone:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$
- **Cylinder:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- **Hyperboloid & Paraboloid**




Reference: Quadric Surfaces in Paul's Online Notes

# Ray-Quadric Surface Intersection

## Intersection Method

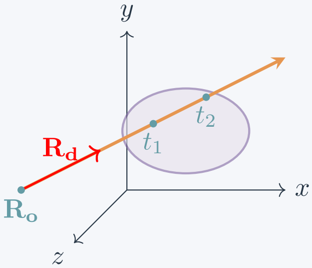
**Step 1:** Substitute ray equation into quadric

$$\mathbf{P}(t) = \mathbf{R_o} + t \cdot \mathbf{R_d}$$

$$P_x = R_{0x} + t \cdot R_{dx}$$

$$P_y = R_{0y} + t \cdot R_{dy}$$

$$P_z = R_{0z} + t \cdot R_{dz}$$

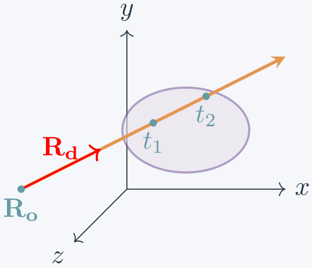


# Ray-Quadric Surface Intersection

## Intersection Method

**Step 2:** Results in quadratic equation

$$ax^2 + bx + c = 0$$




## Solution Cases

Check the discriminant $b^2 - 4ac$:

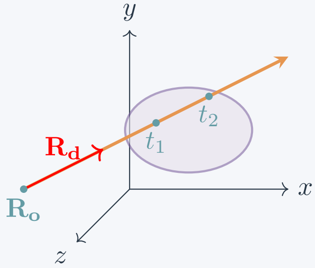
- **Discriminant $> 0$:** Two real solutions (ray intersects surface twice)
- **Discriminant $= 0$:** One solution (ray tangent to surface)
- **Discriminant $< 0$:** No real solutions (ray misses surface)
- **Accept:** Accept smaller $t$ such that $t_{\text{min}} < t < t_{\text{current}}$

# Ray-Quadric Surface Intersection

## Intersection Method

**Step 3:** Solve using quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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# Questions?

# References & Further Reading


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- Scratchapixel: Learn Computer Graphics Programming.
  scratchapixel.com