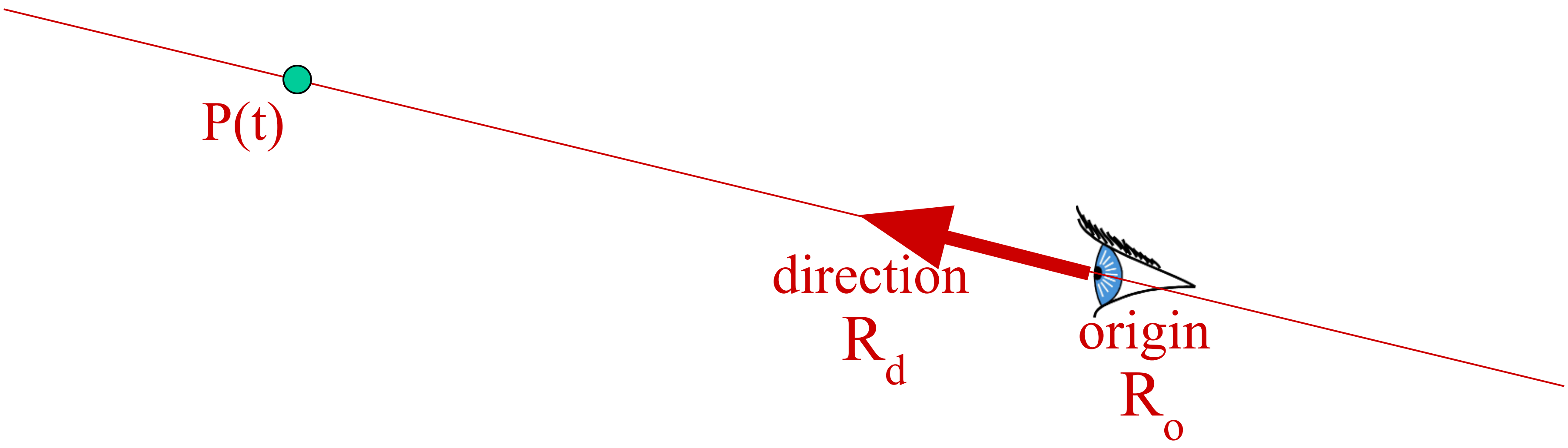
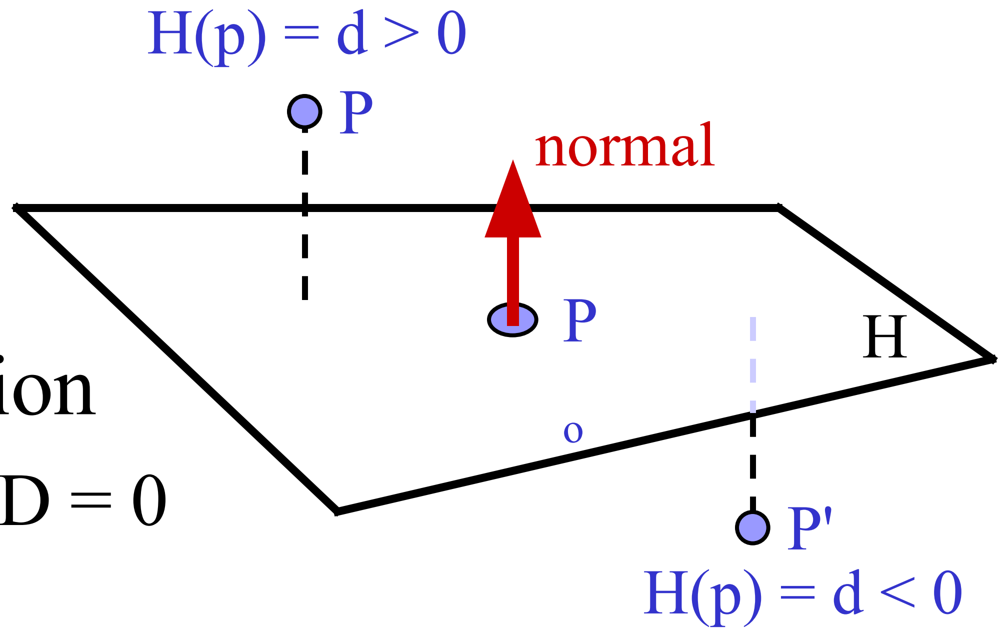
Recall: Ray Representation

- Parametric line
- $P(t) = R_o + t * R_d$
- Explicit representation



3D Plane Representation?

- Plane defined by
  - $P_o = (x,y,z)$
  - $n = (A,B,C)$
- Implicit plane equation
  - $H(P) = Ax+By+Cz+D = 0$
    $= n \cdot P + D = 0$
- Point-Plane distance?
  - If n is normalized,
    distance to plane, $d = H(P)$
  - d is the *signed distance*!

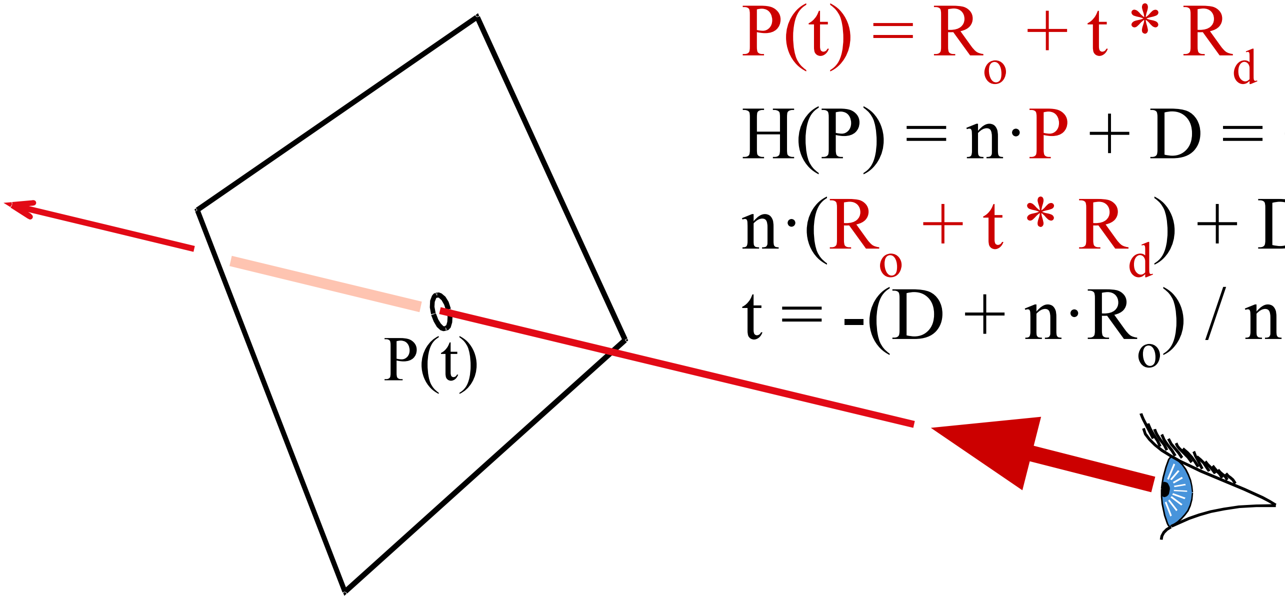


Explicit vs. Implicit?

- Ray equation is explicit $P(t) = R_o + t * R_d$
  - Parametric
  - Generates points
  - Hard to verify that a point is on the ray
- Plane equation is implicit $H(P) = n \cdot P + D = 0$
  - Solution of an equation
  - Does not generate points
  - Verifies that a point is on the plane

Ray-Plane Intersection

- Intersection means both are satisfied
- So, insert explicit equation of ray into implicit equation of plane & solve for t



$$P(t) = R_o + t * R_d$$

$$H(P) = n \cdot P + D = 0$$

$$n \cdot (R_o + t * R_d) + D = 0$$

$$t = -(D + n \cdot R_o) / n \cdot R_d$$

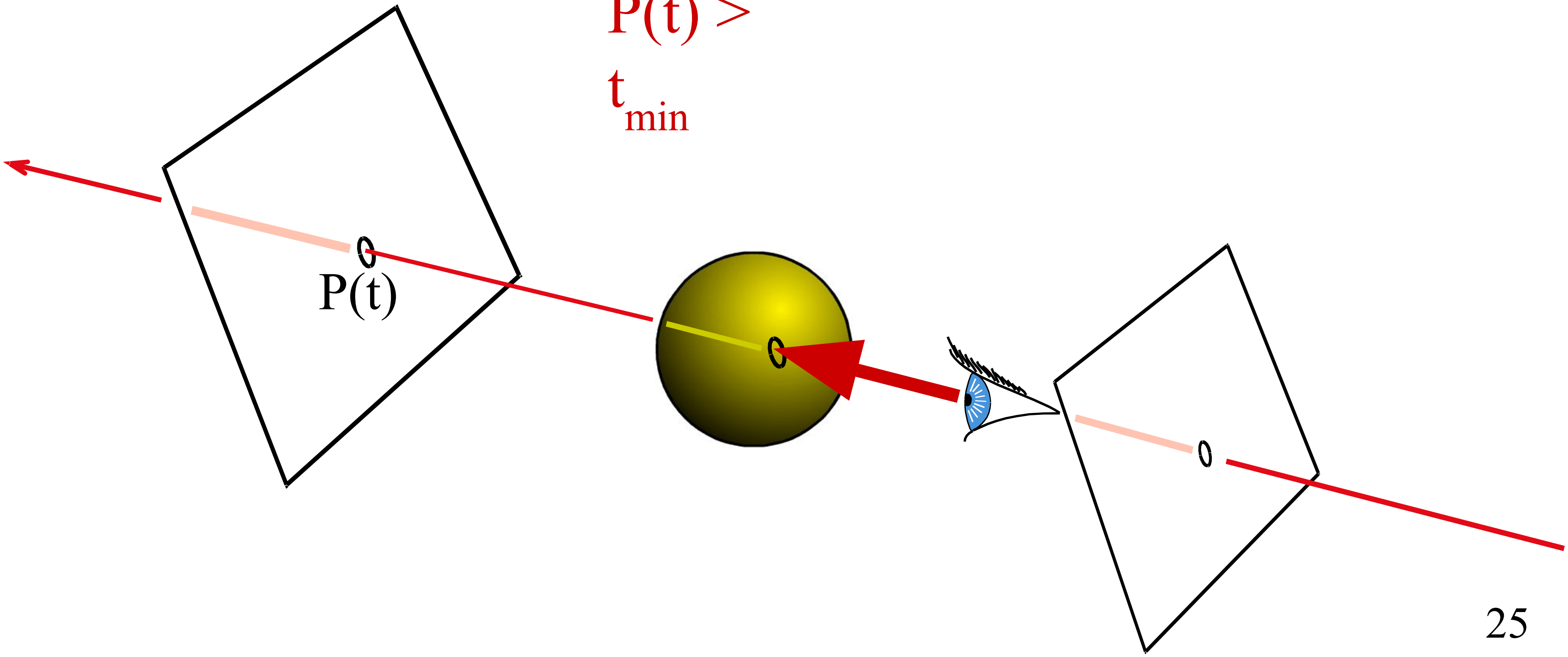
Additional Checks

- Verify that intersection is closer than previous

$P(t) < t_{current}$

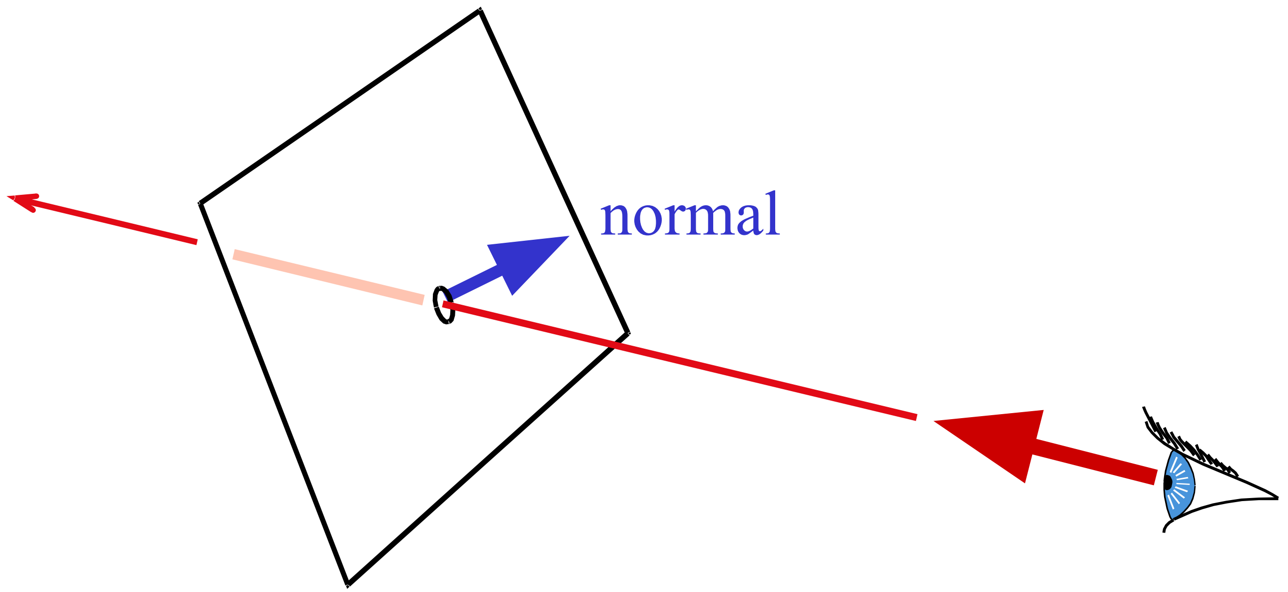
- Verify that it is not out of range (behind eye)

$P(t) > t_{min}$



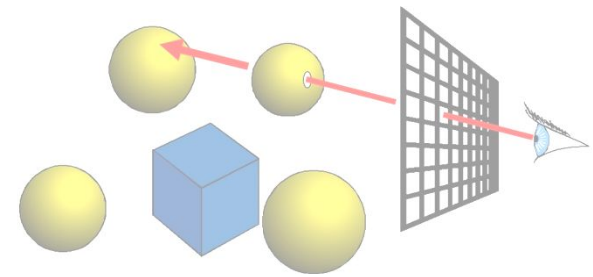
Normal

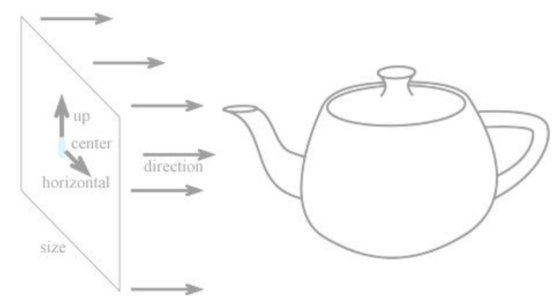
- For shading
  - diffuse: dot product between light and normal
- Normal is constant

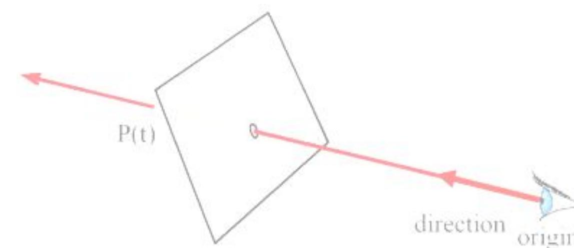


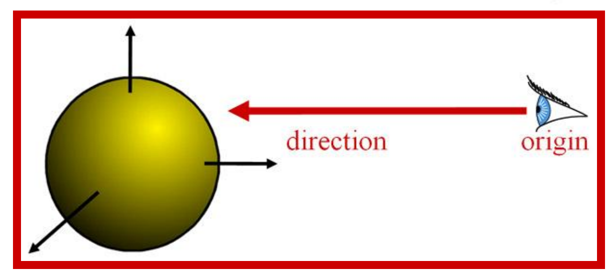
Topics

- Ray Casting Basics
- Camera and Ray Generation
- Ray Object Intersection
  - Plane
  - Sphere
  - Triangle
  - General Quadric Surface
- Recursive Ray Tracing
  - Mirror Reflection
  - Refraction



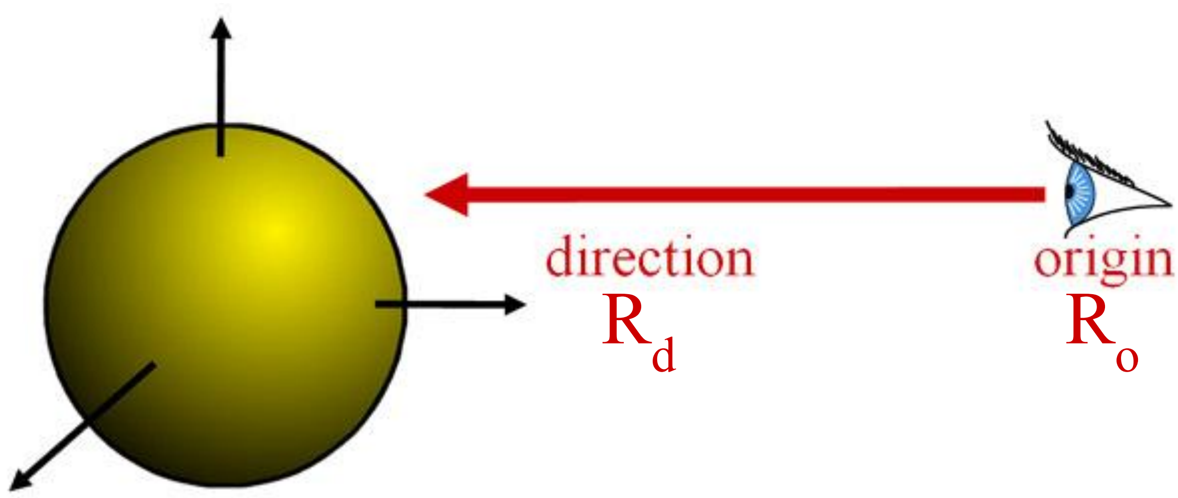






Ray-Sphere Intersection

- Sphere Representation : Implicit sphere equation
  - Assume centered at origin (easy to translate)
  - $H(P) = P \cdot P - r^2 = 0$



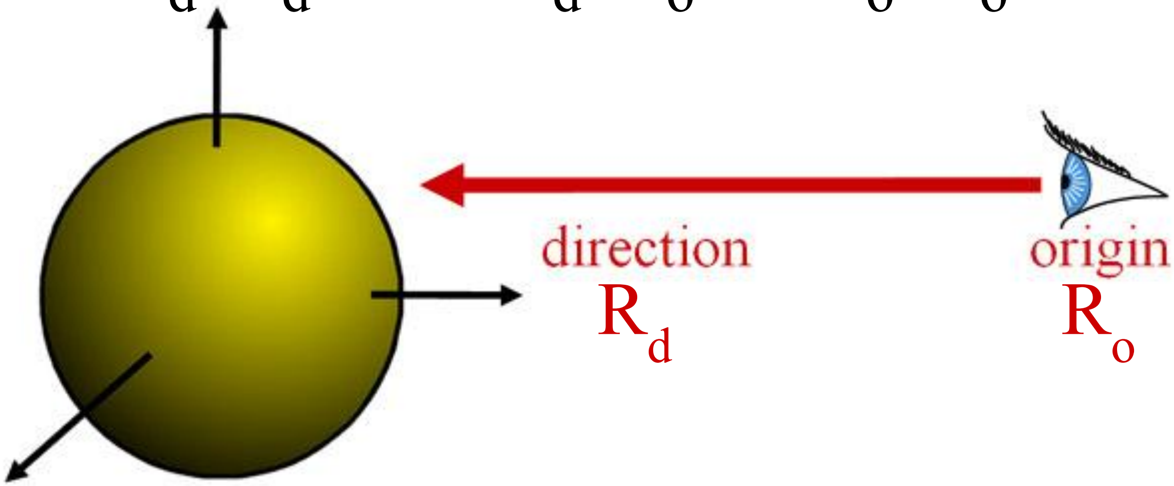
Ray-Sphere Intersection

- Insert explicit equation of ray into implicit equation of sphere & solve for t

$$P(t) = R_o + t^*R_d \qquad H(P) = P \cdot P - r^2 = 0$$

$$(R_o + tR_d) \cdot (R_o + tR_d) - r^2 = 0$$

$$R_d \cdot R_d t^2 + 2R_d \cdot R_o t + R_o \cdot R_o - r^2 = 0$$

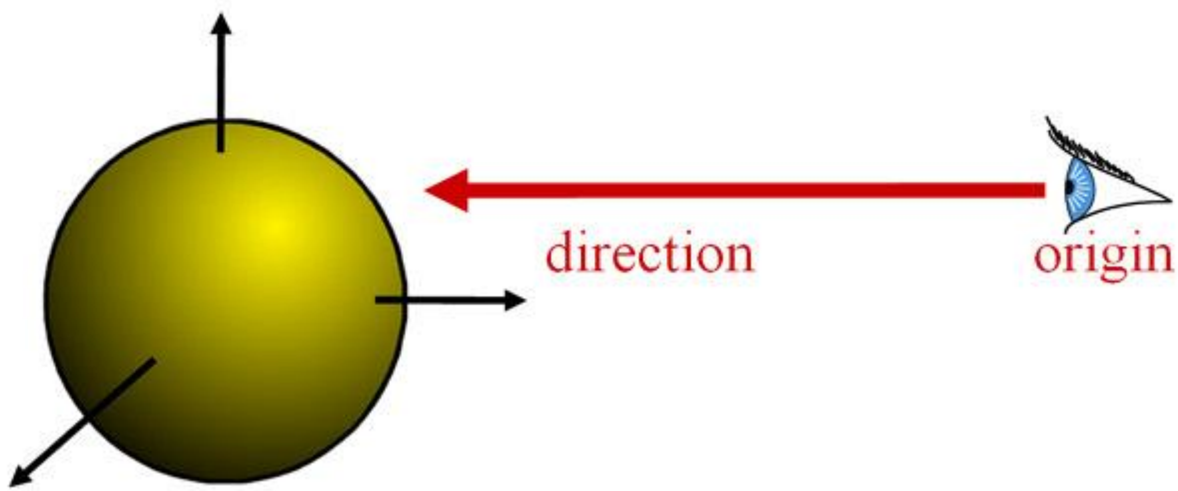


Ray-Sphere Intersection

- Quadratic: $at^2 + bt + c = 0$
  - $a = 1$ (remember, $\|R_d\| = 1$)
  - $b = 2R_d \cdot R_o$
  - $c = R_o \cdot R_o - r^2$
- with discriminant $$d = \sqrt{b^2 - 4ac}$$
- and solutions $$t_{\pm} = \frac{-b \pm d}{2a}$$

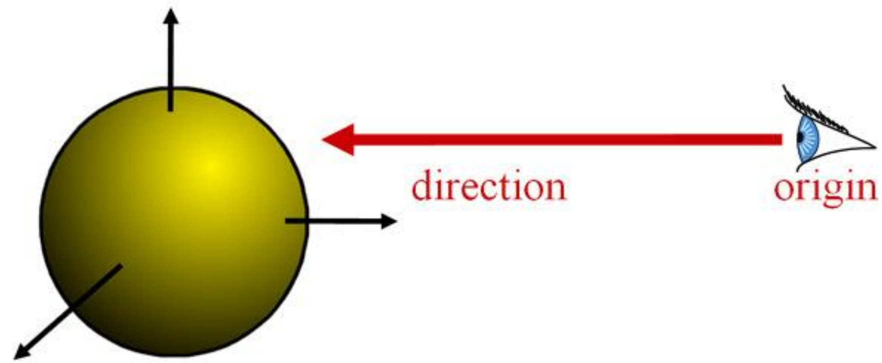
Ray-Sphere Intersection

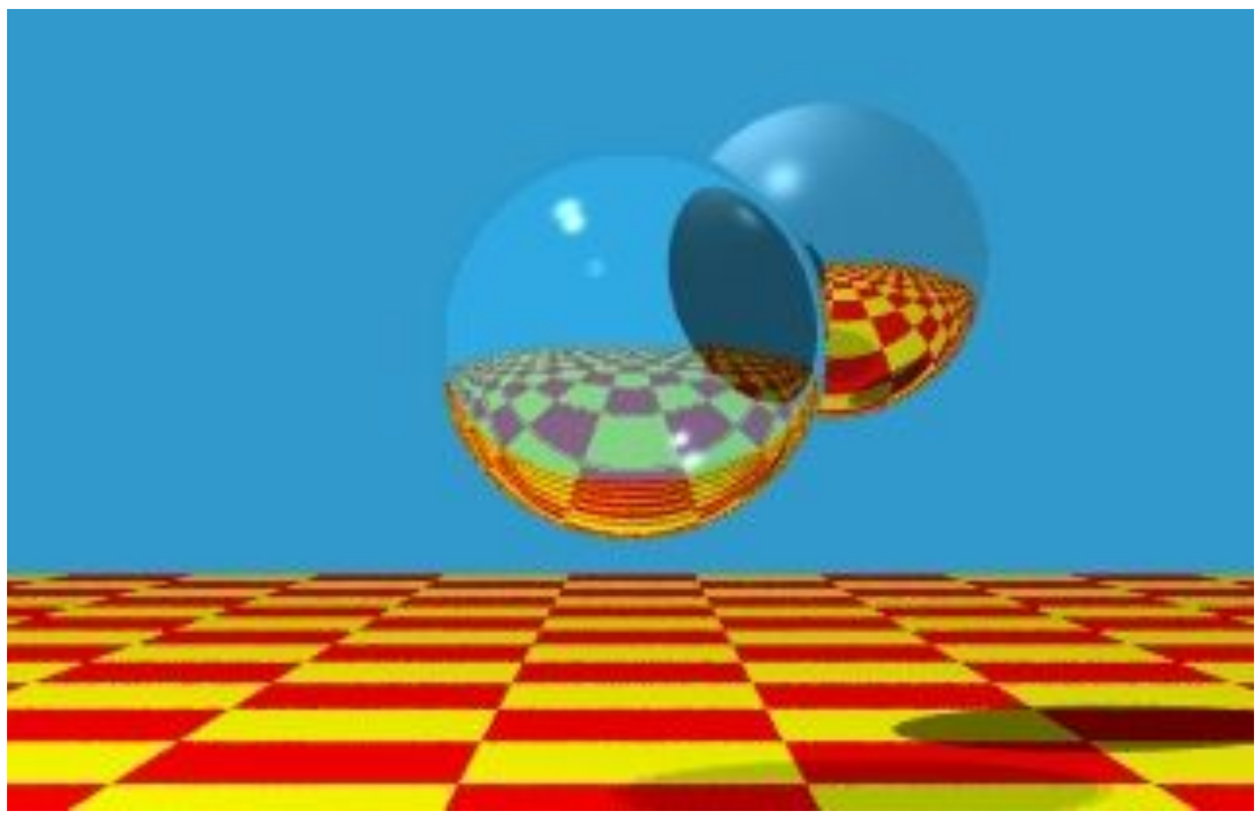
- 3 cases, depending on the sign of $b^2 - 4ac$
- What do these cases correspond to?
- Which root (t+ or t-) should you choose?
  - Closest positive! (usually t-)



Ray-Sphere Intersection

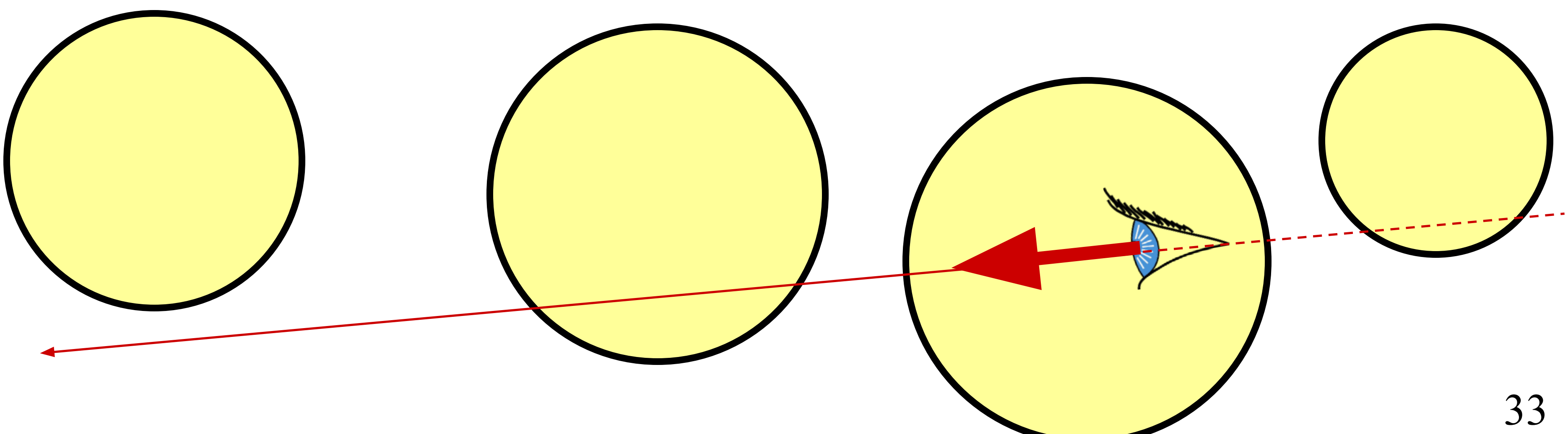
- It's so easy that all ray-tracing images have spheres!





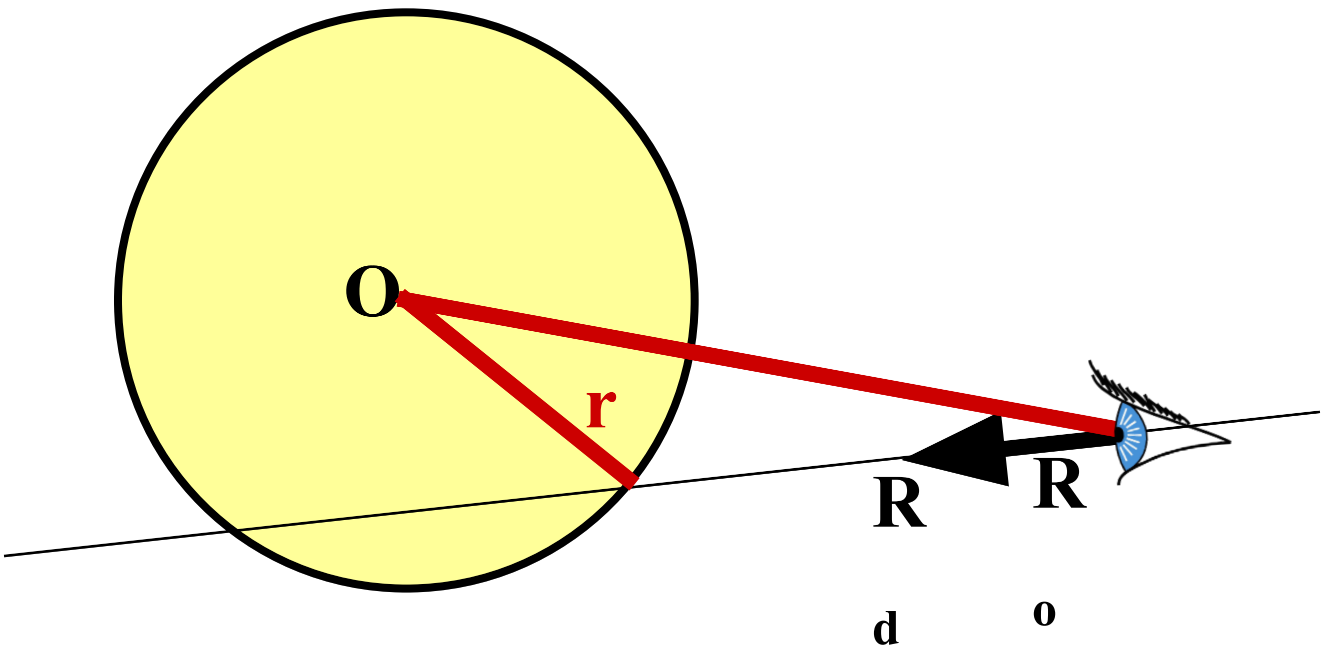
Geometric Ray-Sphere Intersection

- Shortcut / easy reject
- What geometric information is important?
  – Ray origin inside/outside sphere?
  – Closest point to ray from sphere origin?
  – Ray direction: pointing away from sphere?



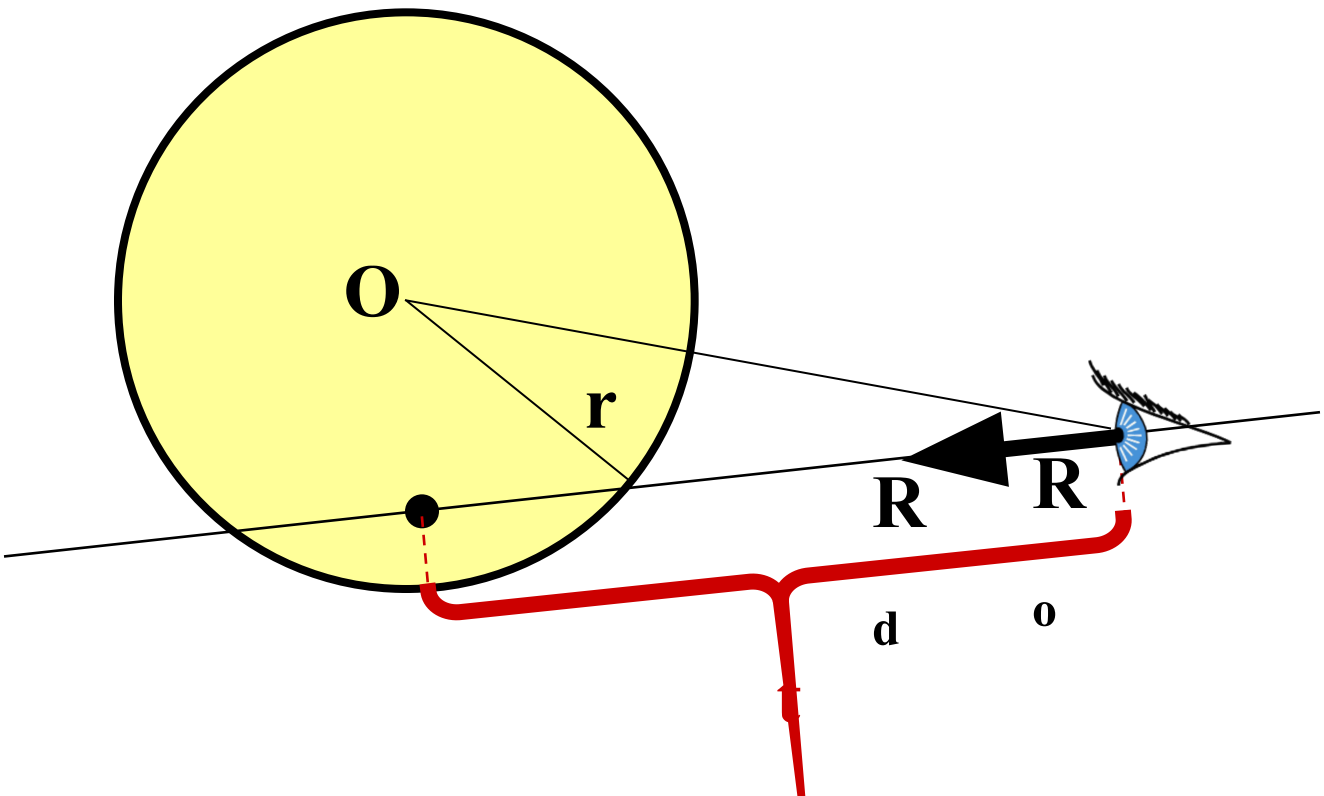
Geometric Ray-Sphere Intersection

- Is ray origin **inside/outside/on** sphere?
  - ($R_o \cdot R_o < r^2$ / $R_o \cdot R_o > r^2$ / $R_o \cdot R_o = r^2$)
  - If origin on sphere, be careful about degeneracies…



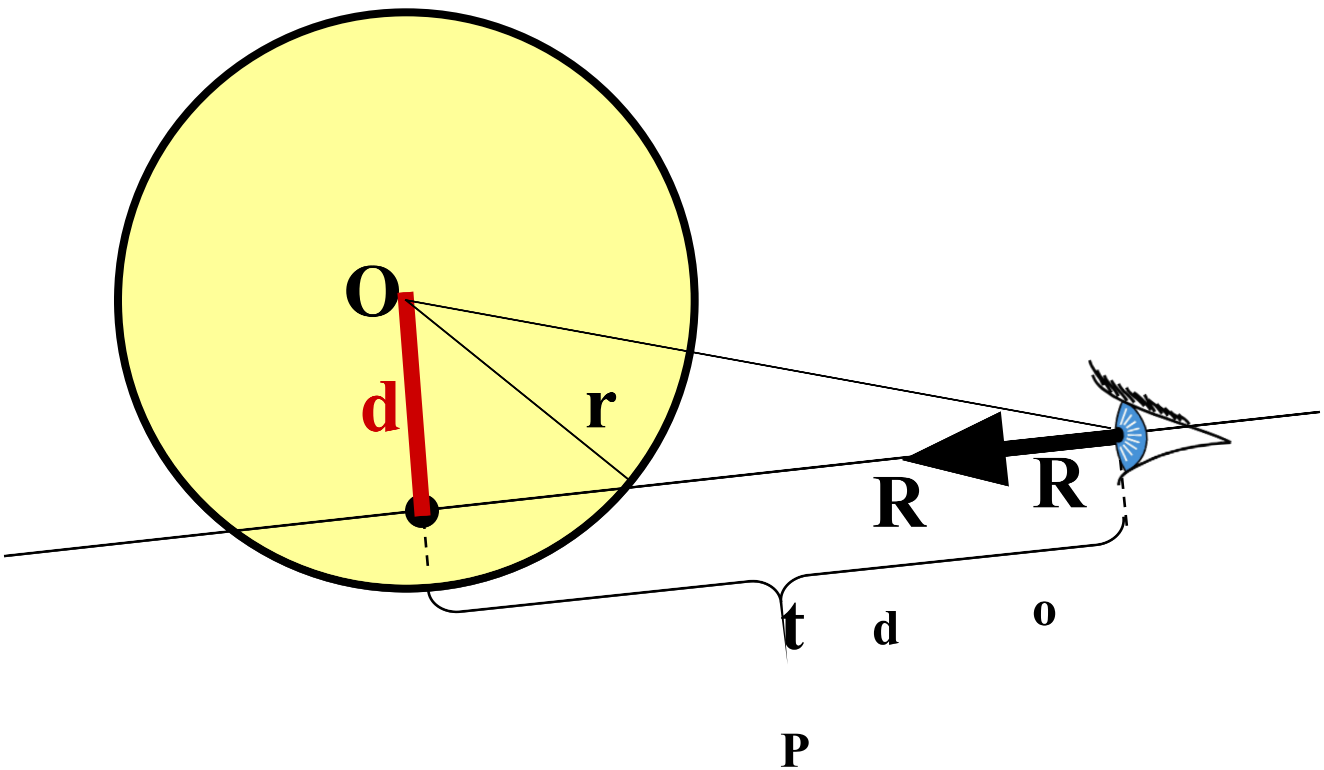
Geometric Ray-Sphere Intersection

- Is ray origin **inside/outside/on** sphere?
- Find closest point to sphere center, $\mathbf{t_P = - R_o \cdot R_d}$
  - If origin outside & $t_P < 0$ $\rightarrow$ **no hit**



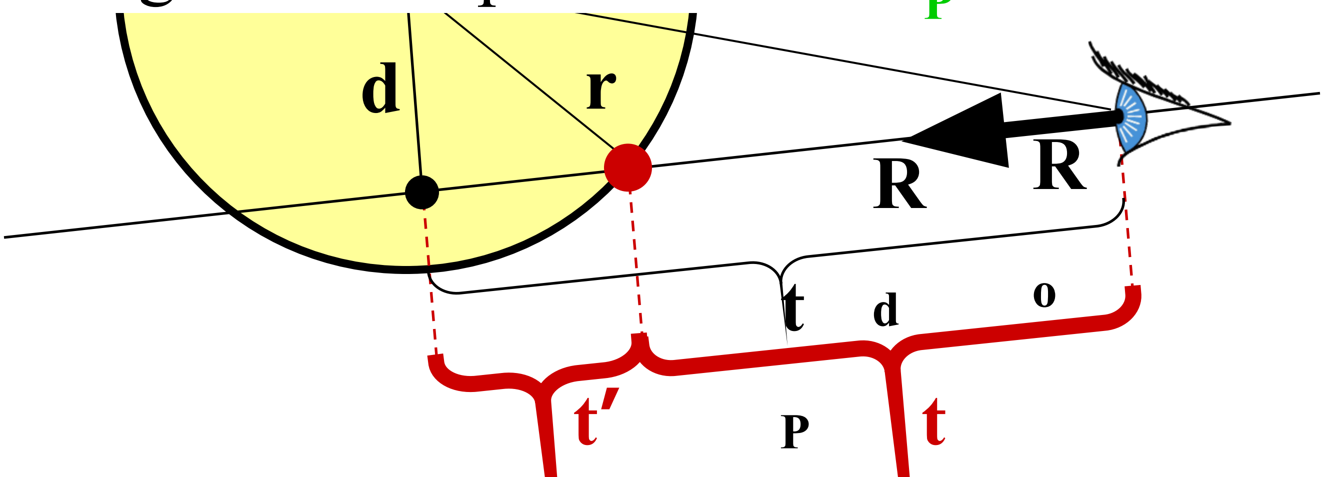
Geometric Ray-Sphere Intersection

- Is ray origin **inside/outside/on** sphere?
- Find closest point to sphere center, $\mathbf{t_P = - R_o \cdot R_d}$.
- Find squared distance, $\mathbf{d^2 = R_o \cdot R_o - t_P{}^2}$
  - If $d^2 > r^2$ $\rightarrow$ **no hit**



Geometric Ray-Sphere Intersection

- Is ray origin **inside/outside/on** sphere?
- Find closest point to sphere center, $\mathbf{t_P = - R_o \cdot R_d}$.
- Find squared distance: $\mathbf{d^2 = R_o \cdot R_o - t_P{}^2}$
- Find distance (t') from closest point (t_P) to correct intersection: $\mathbf{t'^2 = r^2 - d^2}$
  - If origin outside sphere $\rightarrow \mathbf{t = t_P - t'}$
  - If origin inside sphere $\rightarrow \mathbf{t = t_P + t'}$

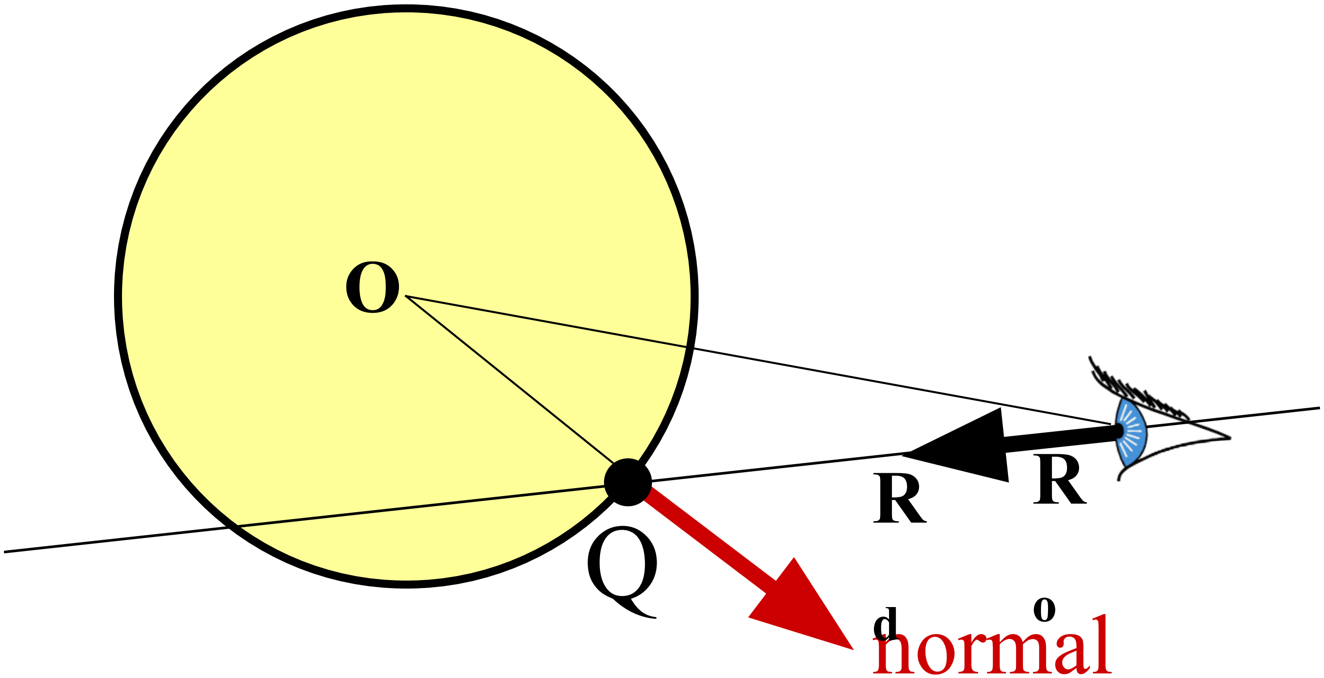


Geometric vs. Algebraic

- Algebraic is simple & generic
- Geometric is more efficient
  - Timely tests
  - In particular for rays outside and pointing away

Sphere Normal

- Simply Q/||Q||
  - Q = P(t), intersection point
  - (for spheres centered at origin)



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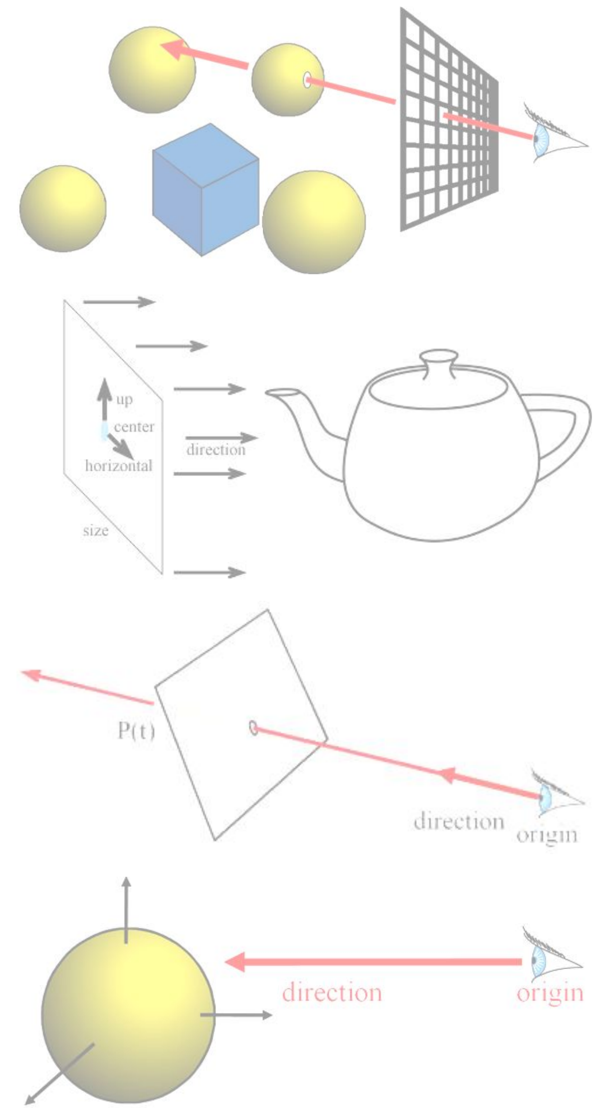
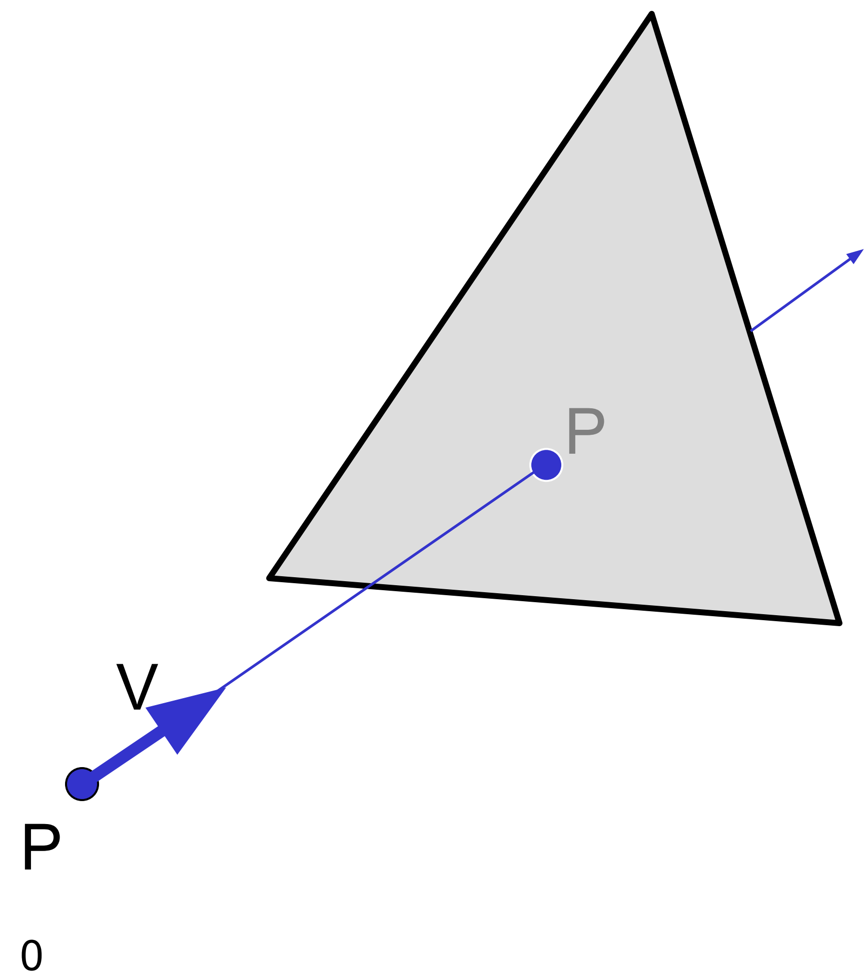


Ray-Triangle Intersection

- First, intersect ray with plane
- Then, check if point is inside triangle



Ray-Triangle Intersection

- Check if point is inside triangle parametrically

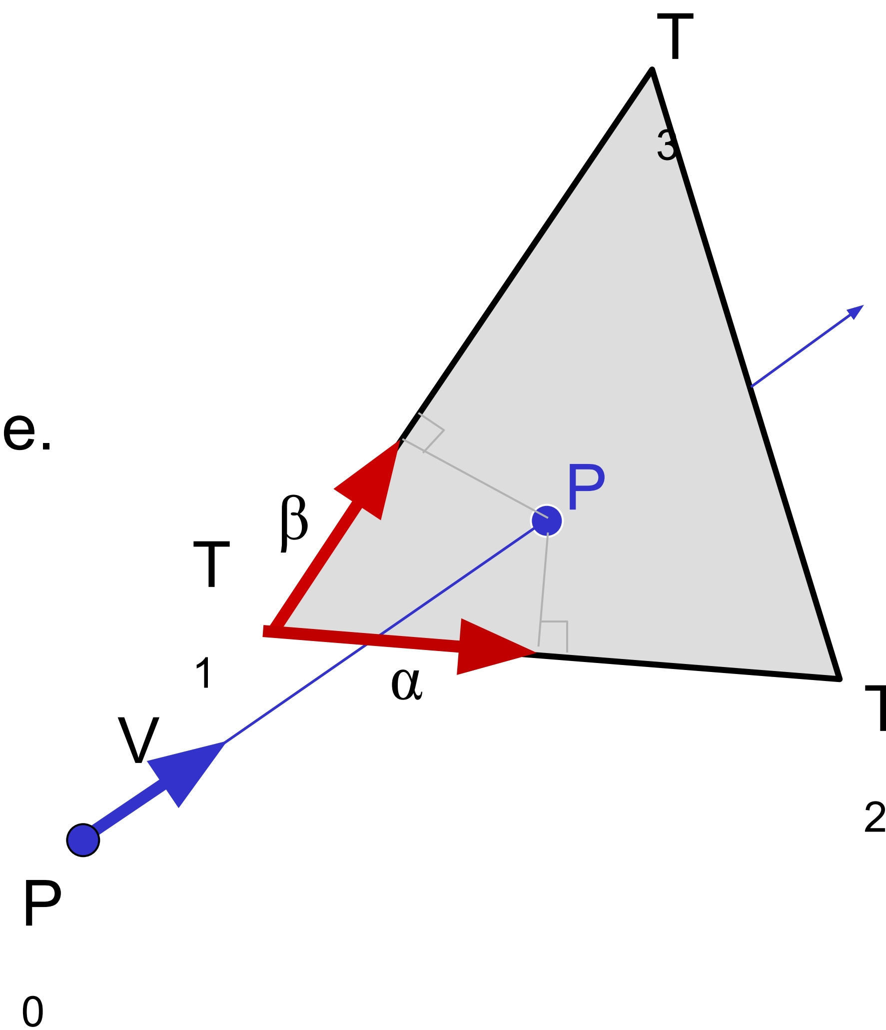
Compute α, β:

$$P = \alpha\,(T_2 - T_1) + \beta\,(T_3 - T_1)$$

Check if point inside triangle.

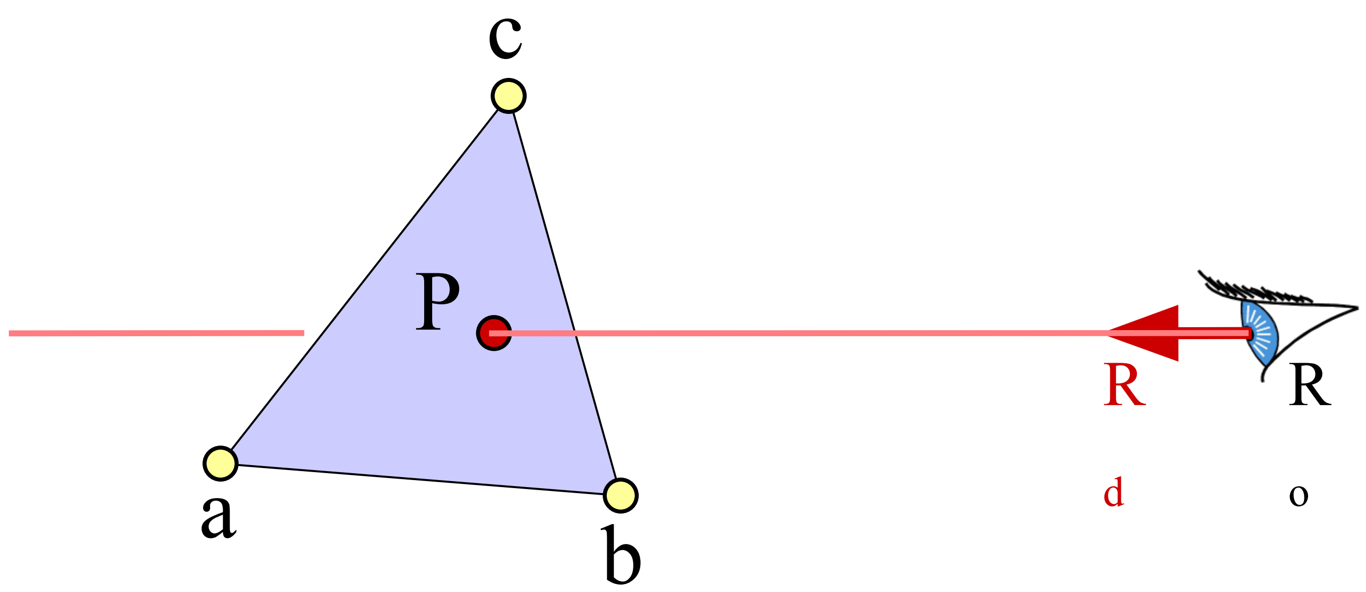
$$0 \leq \alpha \leq 1 \text{ and } 0 \leq \beta \leq 1$$

$$\alpha + \beta \leq 1$$



Ray-Triangle Intersection

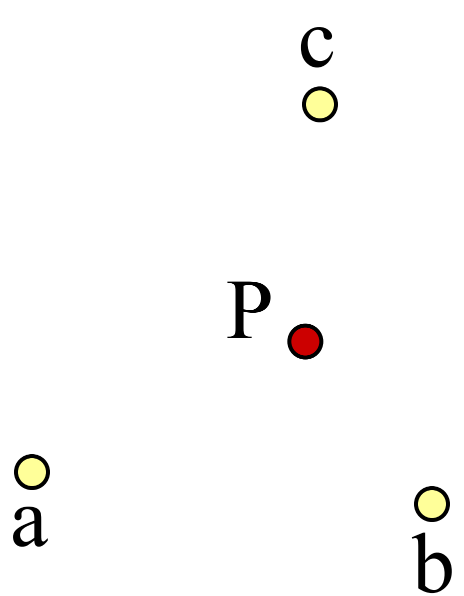
- Use general ray-polygon
- Or try to be smarter
  - Use barycentric coordinates (XM)



Barycentric Definition of a Plane

[Möbius, 1827]

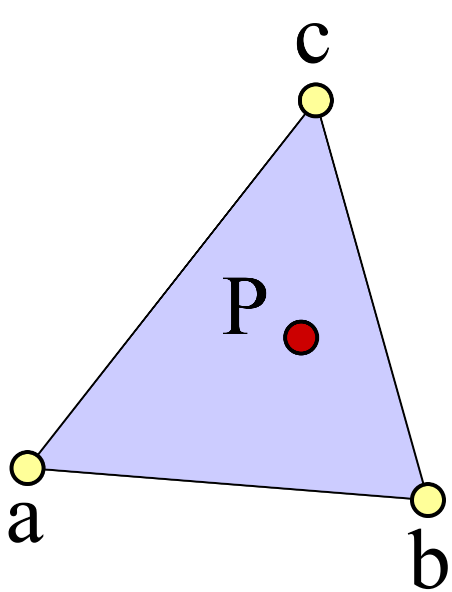
- $P(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$
  with $\alpha + \beta + \gamma = 1$
- Is it explicit or implicit?



P is the *barycenter*:
the single point upon which the plane would balance if weights of size α, β, & γ are placed on points a, b, & c.

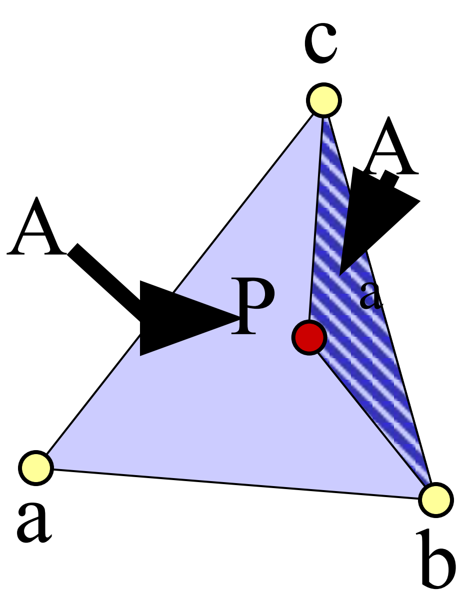
Barycentric Definition of a Triangle

- $P(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$
  with $\alpha + \beta + \gamma = 1$
- AND $0 < \alpha < 1$ & $0 < \beta < 1$ & $0 < \gamma < 1$



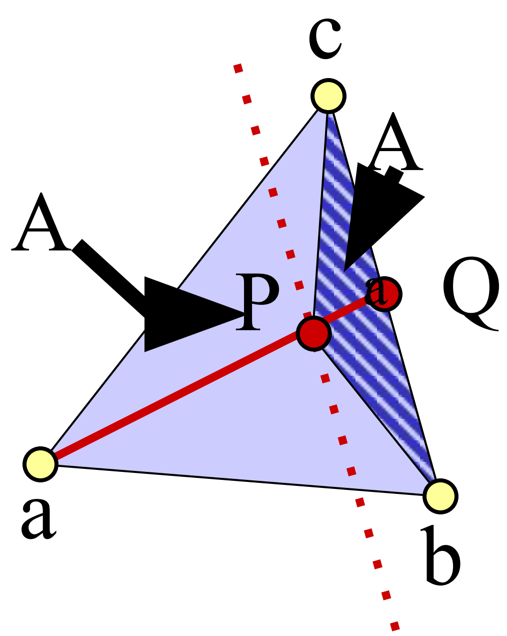
How Do We Compute α, β, γ ?

- Ratio of opposite sub-triangle area to total area
  - $\alpha = A_a/A \qquad \beta = A_b/A \qquad \gamma = A_c/A$
- Use signed areas for points outside the triangle



Intuition Behind Area Formula

- P is barycenter of a and Q
- A_a is the interpolation coefficient on $\overline{aQ}$
- All points on lines parallel to $\overline{bc}$ have the same α (All such triangles have same height/area)



Simplify

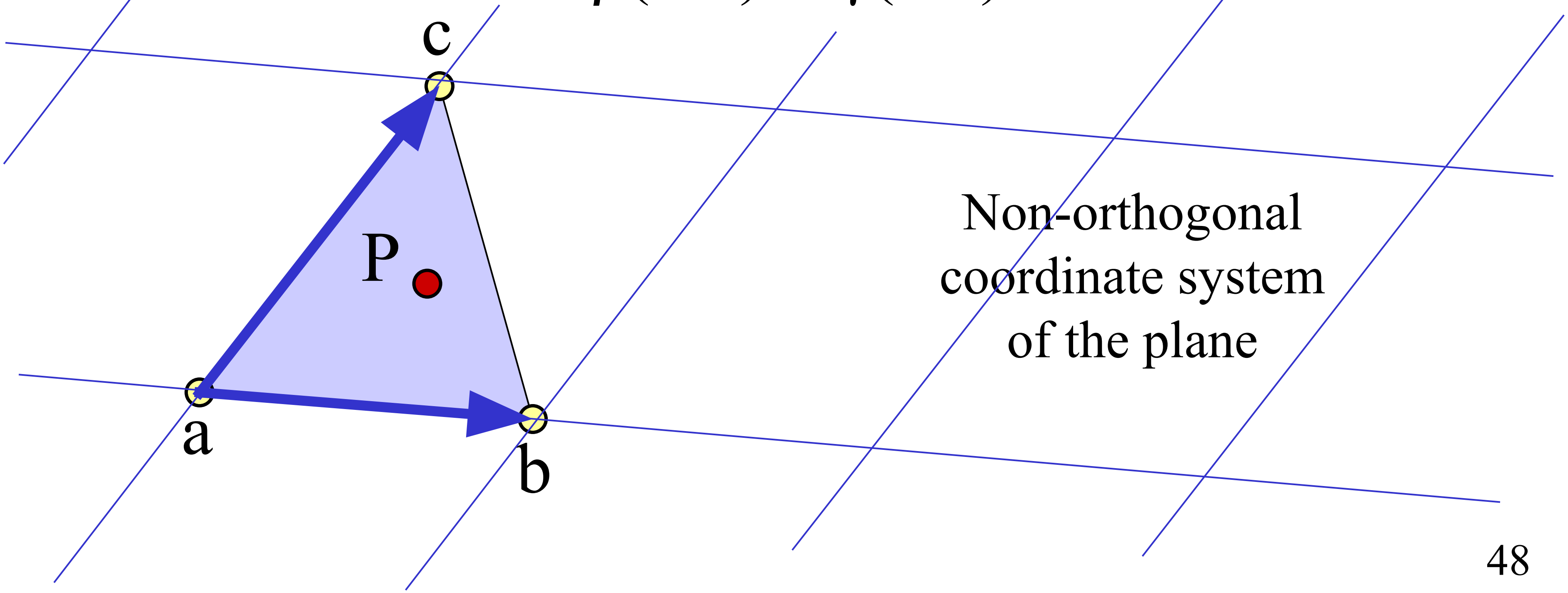
- Since $\alpha + \beta + \gamma = 1$, we can write $\alpha = 1 - \beta - \gamma$

$$P(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$$

rewrite

$$P(\beta, \gamma) = (1-\beta-\gamma)a + \beta b + \gamma c$$

$$= a + \beta(b-a) + \gamma(c-a)$$



Non-orthogonal coordinate system of the plane

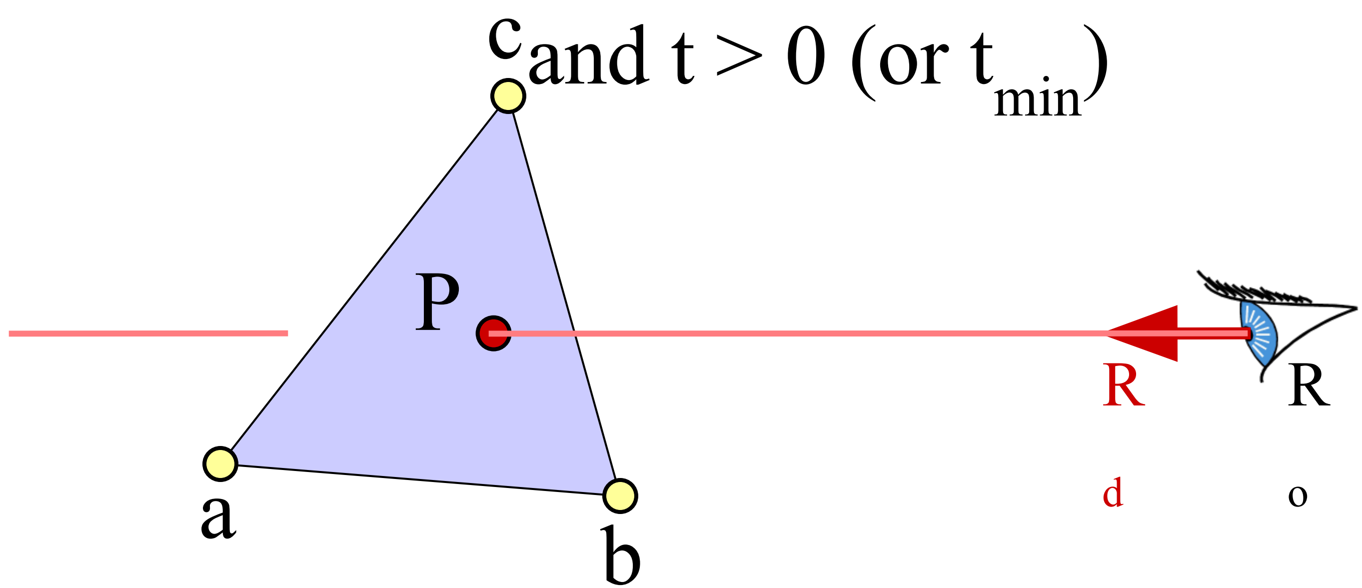
Intersection with Barycentric Triangle

- Set ray equation equal to barycentric equation

$$P(t) = P(\beta, \gamma)$$

$$R_o + t * R_d = a + \beta(b-a) + \gamma(c-a)$$

- Intersection if $\beta + \gamma < 1$ & $\beta > 0$ & $\gamma > 0$ and $t > 0$ (or t_{min})



Intersection with Barycentric Triangle

- $R_o + t * R_d = a + \beta(b\text{-}a) + \gamma(c\text{-}a)$

$$\left.\begin{aligned}
R_{ox} + tR_{dx} &= a_x + \beta(b_x\text{-}a_x) + \gamma(c_x\text{-}a_x)\\
R_{oy} + tR_{dy} &= a_y + \beta(b_y\text{-}a_y) + \gamma(c_y\text{-}a_y)\\
R_{oz} + tR_{dz} &= a_z + \beta(b_z\text{-}a_z) + \gamma(c_z\text{-}a_z)
\end{aligned}\right\}\ \begin{array}{l}\text{3 equations,}\\ \text{3 unknowns}\end{array}$$

- Regroup & write in matrix form:

$$\begin{bmatrix} a_x - b_x & a_x - c_x & R_{d_x} \\ a_y - b_y & a_y - c_y & R_{d_y} \\ a_z - b_z & a_z - c_z & R_{d_z} \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} a_x - R_{o_x} \\ a_y - R_{o_y} \\ a_z - R_{o_z} \end{bmatrix}$$

Cramer's Rule

- Used to solve for one variable at a time in system of equations

$$\beta = \frac{\begin{vmatrix} a_x - R_{ox} & a_x - c_x & R_{dx} \\ a_y - R_{oy} & a_y - c_y & R_{dy} \\ a_z - R_{oz} & a_z - c_z & R_{dz} \end{vmatrix}}{|A|} \qquad \gamma = \frac{\begin{vmatrix} a_x - b_x & a_x - R_{ox} & R_{dx} \\ a_y - b_y & a_y - R_{oy} & R_{dy} \\ a_z - b_z & a_z - R_{oz} & R_{dz} \end{vmatrix}}{|A|}$$

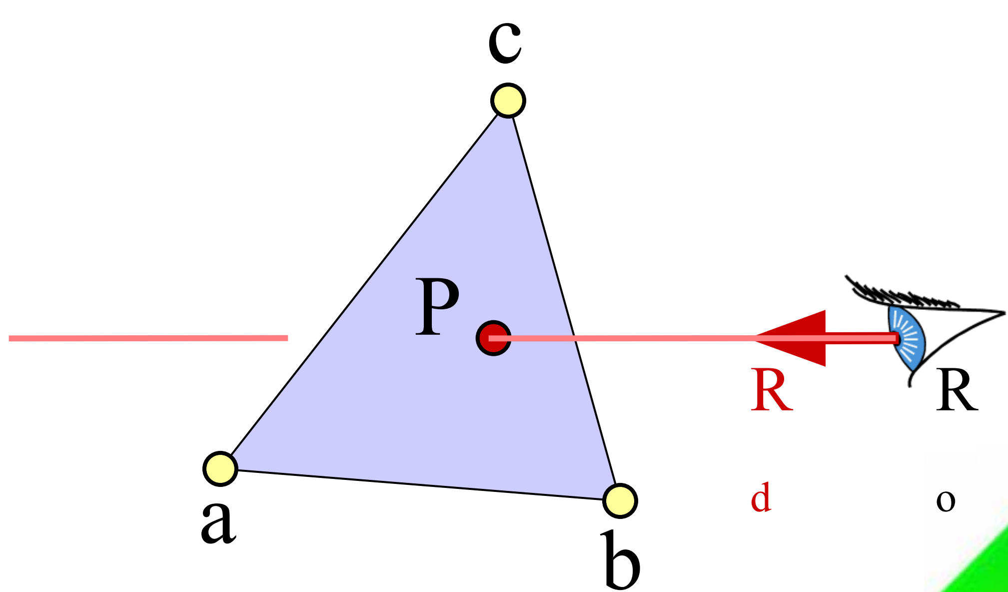
$$t = \frac{\begin{vmatrix} a_x - b_x & a_x - c_x & a_x - R_{ox} \\ a_y - b_y & a_y - c_y & a_y - R_{oy} \\ a_z - b_z & a_z - c_z & a_z - R_{oz} \end{vmatrix}}{|A|}$$

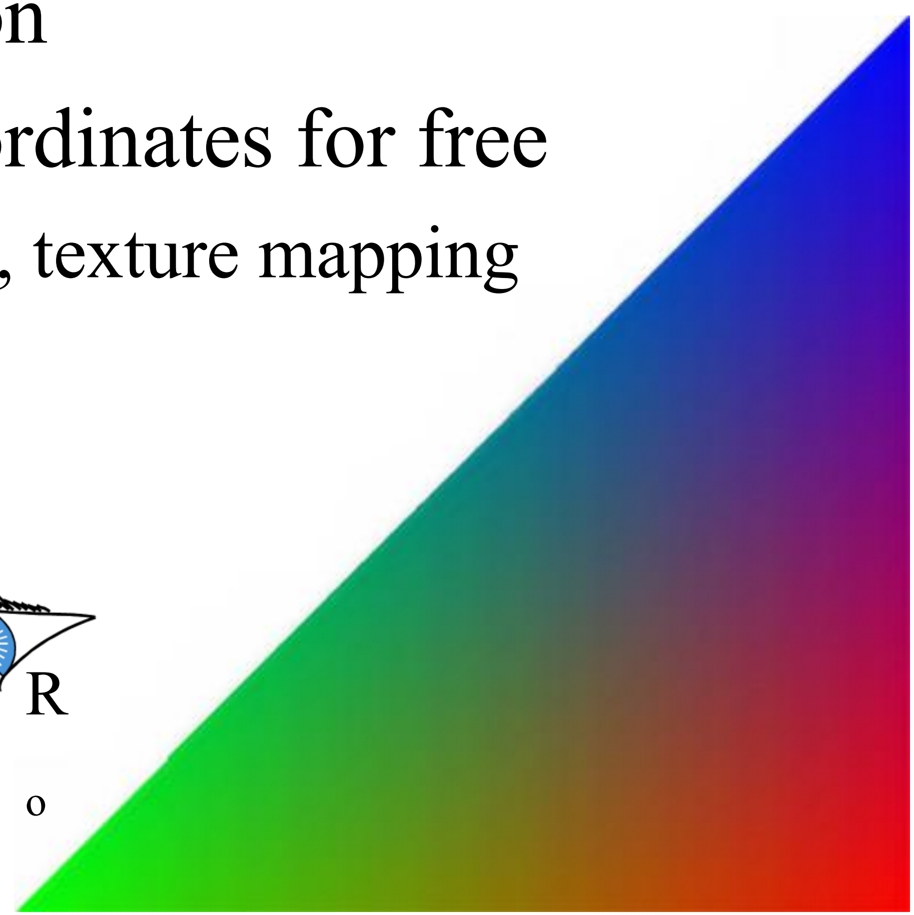
| | denotes the determinant

Can be copied mechanically into code

Advantages of Barycentric Intersection

- Efficient
- Stores no plane equation
- Get the barycentric coordinates for free
  - Useful for interpolation, texture mapping





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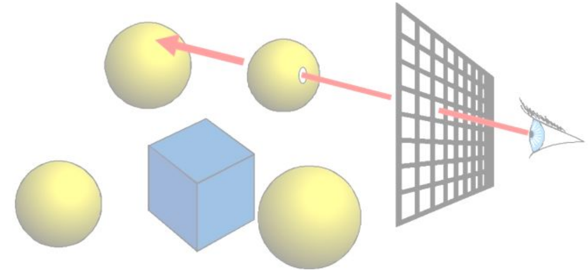

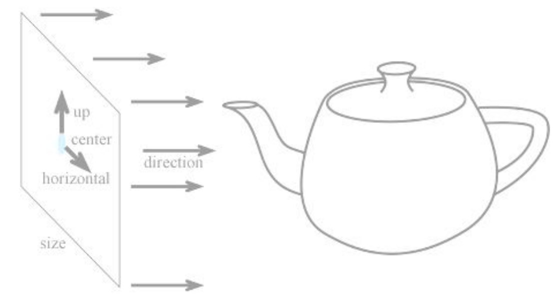

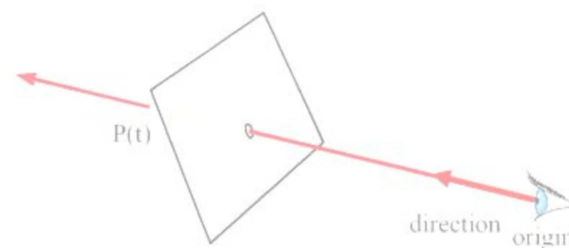

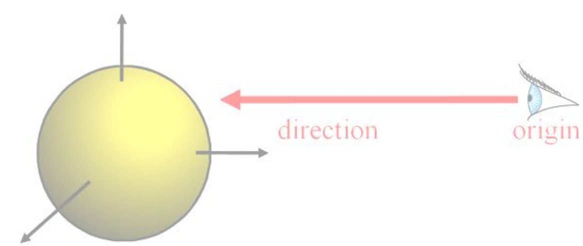

General Quadric Surfaces

- Some Common Quadric Surfaces
  - Ellipsoid $(x^2/a^2 + y^2/b^2 + z^2/c^2 + 1 = 0)$
  - Cone $(x^2/a^2 - y^2/b^2 + z^2/c^2 = 0)$
  - Cylinder
  - Hyperboloid
  - Paraboloid – Eliptic, Hyperbolic etc.

* Check out the following links for the figures & equations:

https://tutorial.math.lamar.edu/Classes/CalcIII/QuadricSurfaces.aspx
https://mrl.cs.nyu.edu/~dzorin/rend05/lecture2.pdf (page12-14)

Ray - Quadric Surface Intersection

- General Quadric Surface Equation

$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0 \ldots \ldots (1)$

- Ray Equation : $P(t) = \mathbf{R_o} + t * \mathbf{R_d}$
- So, $P_x = R_{0x} + t*R_{dx}$, Similar for P_y, P_z
- Put P_x, P_y, P_z as x, y, z in eq.(1) and solve for t
- Accept the smaller non –ve real value of t

- General Quadric Surface Normal
    - Use partial derivatives!