

Ray Tracing & Ray Casting

Realistic Graphics Inspired by Nature

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Motivation

The Story of Light

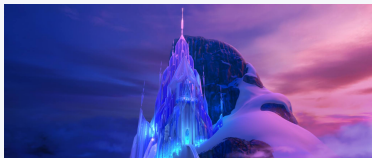
Ray Casting: Foundation

Mathematics of Rays

Motivation

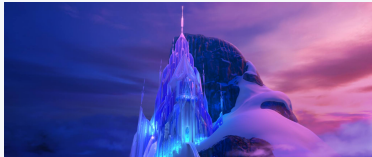
Why Learn This?

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Elsa's Castle in Frozen

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Elsa's Castle in Frozen



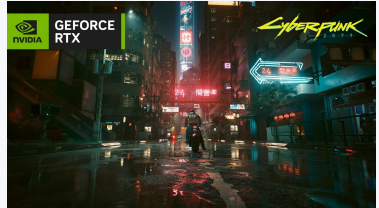
Cyberpunk 2077 with RTX

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Elsa's Castle in Frozen



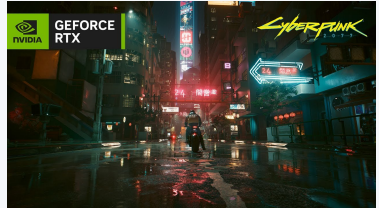
Cyberpunk 2077 with RTX

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Why Learn This?



Elsa's Castle in Frozen

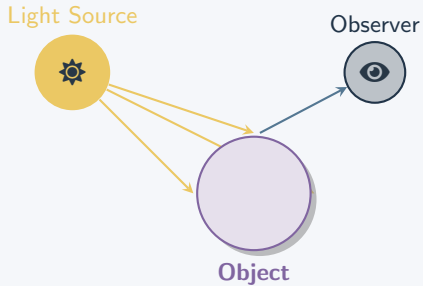


Cyberpunk 2077 with RTX

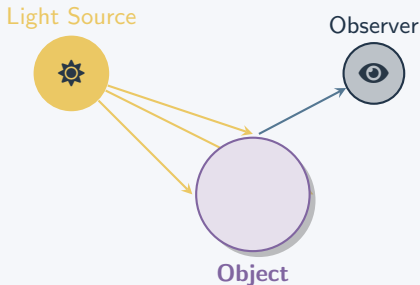
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- Lately, **RTX** is all the rage in gaming. New titles boast ray-tracing effects in real-time, not 30 hours per frame!
- It's fun! You will know when you create your first ray-traced image!

The Story of Light

How Do We See?



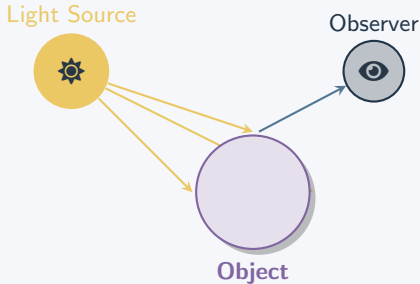
How Do We See?



Natural Process

1. Light travels from source
2. Light hits objects
3. Light bounces to our eyes
4. Our brain interprets the signal

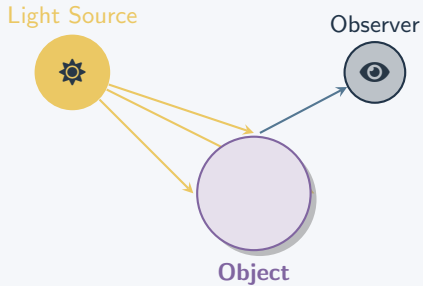
How Do We See?



Physical Process

1. Photon is emitted from source
2. Photon hits objects
3. Part of the photon is reflected or absorbed
4. The reflected photons reach our eyes
5. The rods and cones in our retina detect the photons
6. Our brain interprets the signal
7. **Colour:** The wavelength of the photons
8. **Brightness:** The number of photons

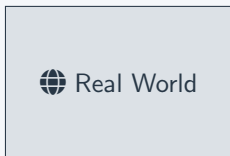
How Do We See?



Question: How do we simulate this?

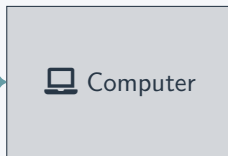
The Computer Graphics Challenge

Infinite Complexity



Simulate

Finite Pixels



Challenges:

- Infinite light rays/photons
- Complex physics
- High computational cost

Ray Casting: Foundation

The Key Insight

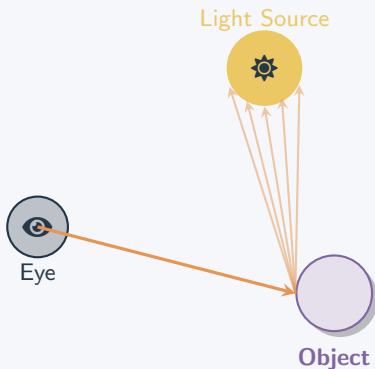
1. Reverse Engineering

Instead of following light rays from light sources —

Let's trace backwards!

Shoot rays from the eye,
find where it hits and find out
how much light reaches there.

This is the opposite of what happens in reality. **Why does this work?**



The Key Insight

1. Reverse Engineering

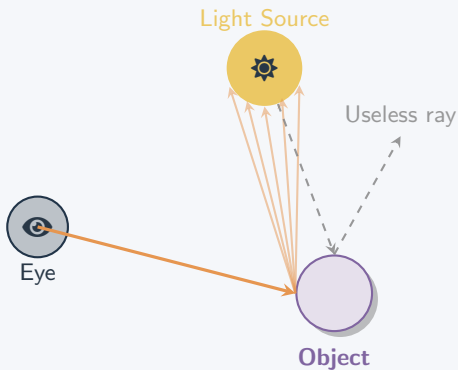
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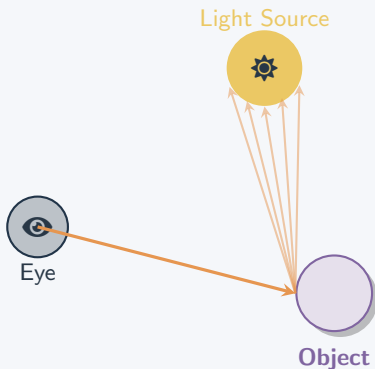
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Shoot rays from the eye,
find where it hits and find out
how much light reaches there.

This is the opposite of what happens in reality. **Why does this work?**

- Most light never reaches our eyes
- Only trace rays that matter
- Much more efficient!



From Infinite Rays to Finite Pixels

2. Cutting Costs

Instead of tracing infinite rays —

Trace one ray per pixel.

This comes with little tradeoff, because:

From Infinite Rays to Finite Pixels

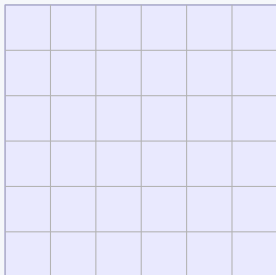
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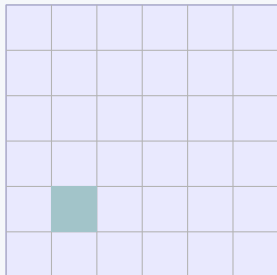
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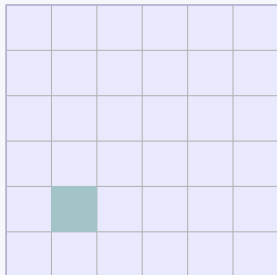
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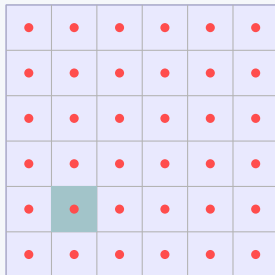
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- An image is just a grid of pixels
- Each pixel can only be of one color
- In the end, we just need to know the *color of each pixel*
- Hence, one ray from the mid-point of each pixel should be a good approximation*

* We will discuss more advanced techniques later that improve quality



The Full Picture

Light Source



Eye



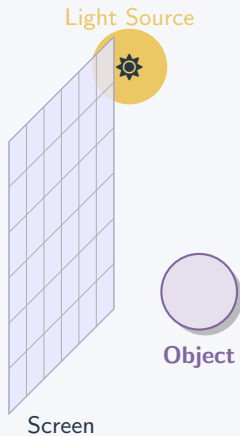
Object

The Full Picture

Place screen in front of eye



Eye

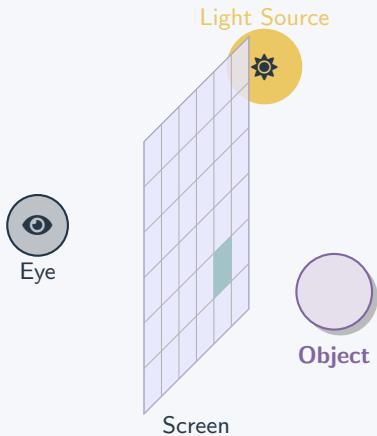
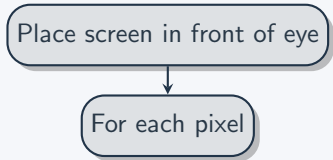


Light Source

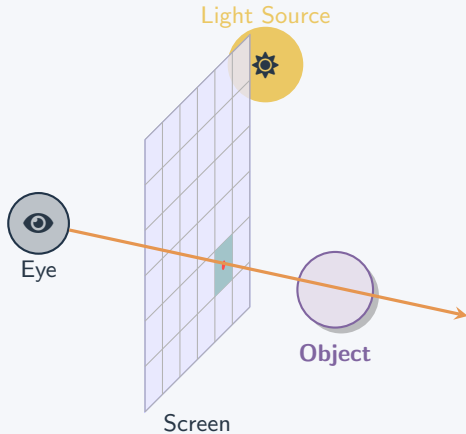
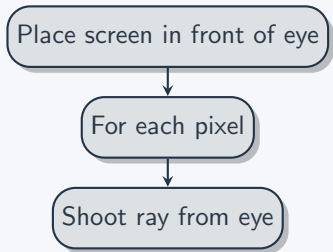
Screen

Object

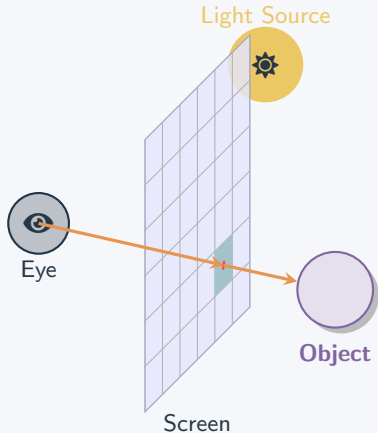
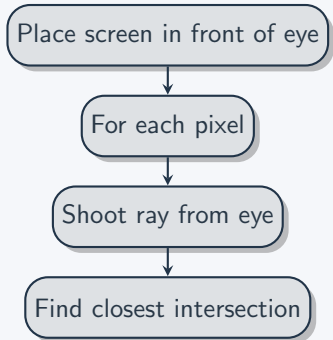
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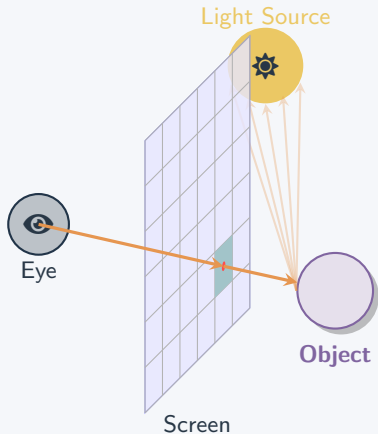
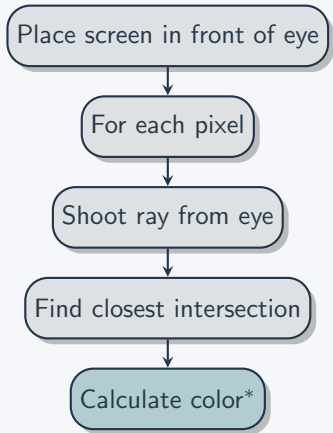
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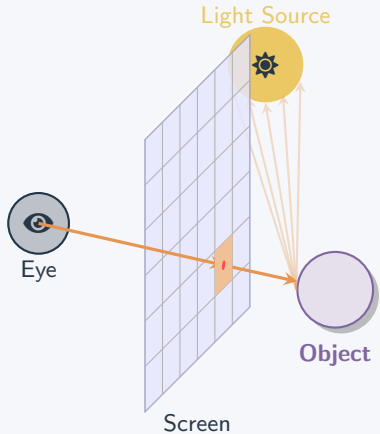
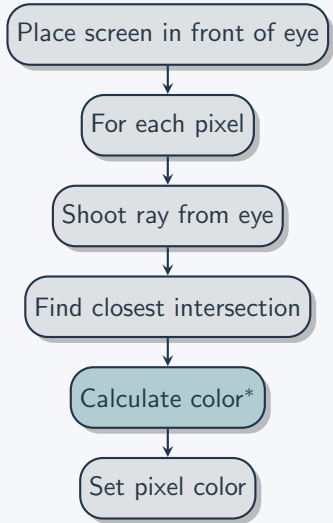
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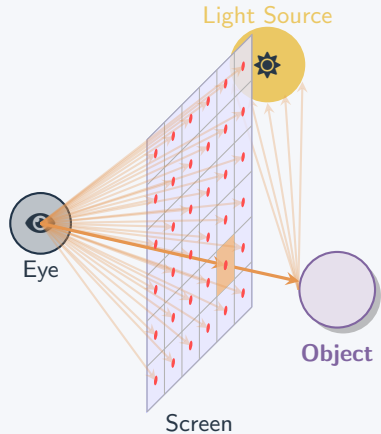
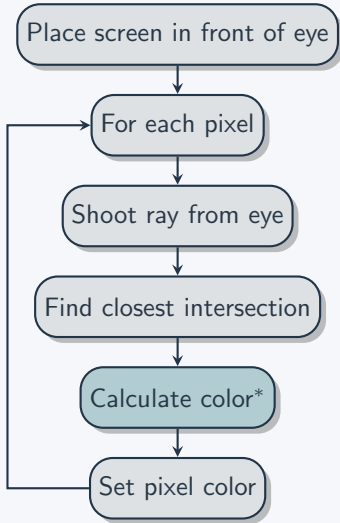
The Full Picture



The Full Picture



The Full Picture



Mathematics of Rays

What is a Ray?

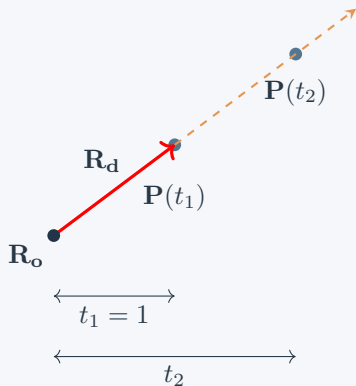
Ray Representation

A ray is defined by:

$$\mathbf{P}(t) = \mathbf{R}_o + t \cdot \mathbf{R}_d \quad (1)$$

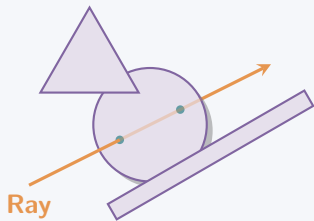
where:

- \mathbf{R}_o = Origin point
- \mathbf{R}_d = Direction vector
- t = Parameter ($t \geq 0$)



Check out here on desmos.

Finding Intersections



Key Objects:

- Planes
- Spheres
- Triangles
- General Quadrics

Challenge: Find the **closest** intersection efficiently!

3D Plane Representation

Plane Definition

A plane is defined by:

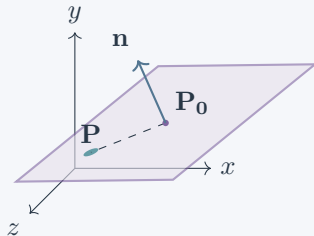
- Point $\mathbf{P}_0 = (x_0, y_0, z_0)$ on plane
- Normal vector $\mathbf{n} = (A, B, C)$

Implicit equation:

$$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_0) = 0$$

$$\boxed{\mathbf{n} \cdot \mathbf{P} + D = 0} \quad \text{where } D = -\mathbf{n} \cdot \mathbf{P}_0$$

$$\boxed{Ax + By + Cz + D = 0}$$



3D Plane Representation

Plane Definition

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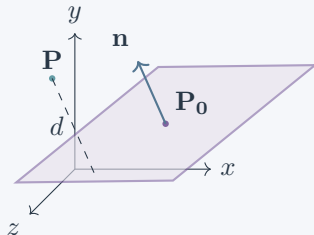
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Point-Plane Distance

If \mathbf{n} is normalized: $d = \mathbf{n} \cdot \mathbf{P} + D = \mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_0)$

Signed distance: $d > 0$ (front), $d < 0$ (back), $d = 0$ (on plane)

Ray-Plane Intersection

Intersection Method

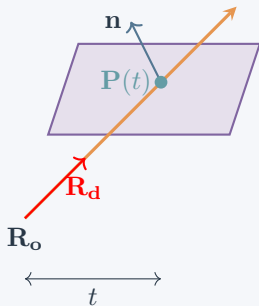
Step 1: Substitute ray into equation

$$\mathbf{n} \cdot (\mathbf{R}_o + t\mathbf{R}_d) + D = 0$$

$$\mathbf{n} \cdot \mathbf{R}_o + t(\mathbf{n} \cdot \mathbf{R}_d) + D = 0$$

Step 2: Solve for parameter t

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R}_o}{\mathbf{n} \cdot \mathbf{R}_d}$$



Ray-Plane Intersection

Intersection Method

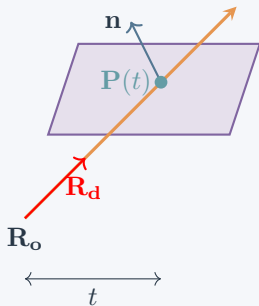
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Cases

- If $\mathbf{n} \cdot \mathbf{R}_d = 0$: Ray parallel to plane (0 or infinite)
- If $\mathbf{n} \cdot \mathbf{R}_d < 0$: Ray hits front face
- If $\mathbf{n} \cdot \mathbf{R}_d > 0$: Ray hits back face

Additional Checks

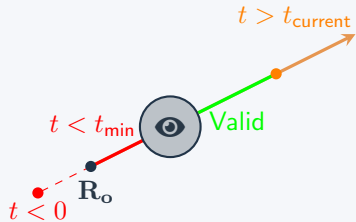
Validation Rules

After computing t , verify:

1. **Behind check:** $t > t_{\min}$
2. **Closest check:** $t < t_{\text{current}}$
3. **Valid range:** $t \geq 0$

Where:

- t_{\min} : Minimum ray distance (not behind eye/screen)
- t_{current} : Distance to closest intersection so far



Ray-Triangle Intersection Overview

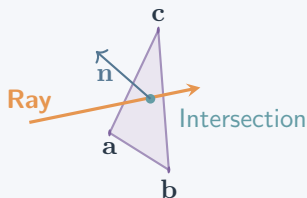
Two Main Approaches

Method 1: Two-Step Process

1. Ray-plane intersection
2. Inside/outside triangle test

Method 2: Direct Barycentric

1. Set up 3×3 linear system
2. Solve for t, β, γ simultaneously



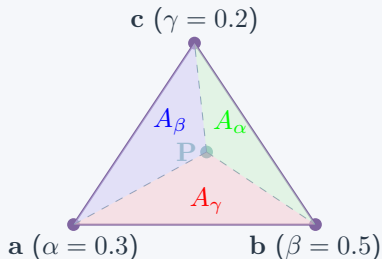
What Are Barycentric Coordinates?

Barycentric Definition

Any point P in the triangle's plane:

$$\mathbf{P}(\alpha, \beta, \gamma) = \alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c}$$

where: $\alpha + \beta + \gamma = 1$



Check out the Desmos demo.

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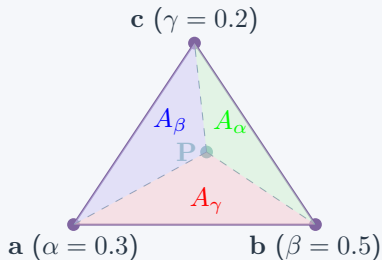
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Physical Interpretation:

- α, β, γ are *weights*
- P is the *center of mass*
- Also called *barycenter*

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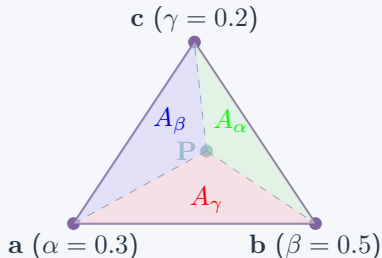
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Area Relationship

$$\alpha = \frac{A_\alpha}{A_{total}}, \quad \beta = \frac{A_\beta}{A_{total}},$$
$$\gamma = \frac{A_\gamma}{A_{total}}$$

Barycentric Coordinates: Inside vs Outside

Triangle Interior Test

Point P is **inside** triangle if:

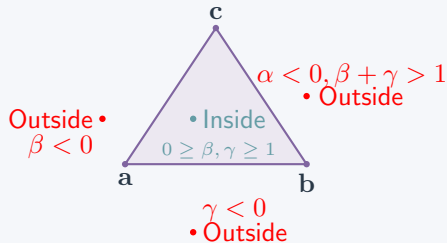
$$\alpha, \beta, \gamma \geq 0$$

Since $\alpha + \beta + \gamma = 1$, we can rewrite as:

$$\beta \geq 0$$

$$\gamma \geq 0$$

$$\alpha \geq 0 \text{ or } \beta + \gamma \leq 1$$



Barycentric Coordinates: Inside vs Outside

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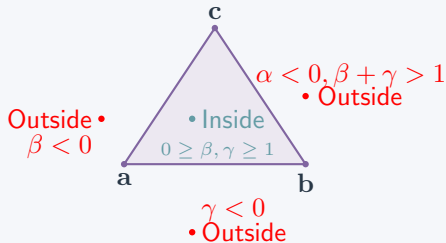
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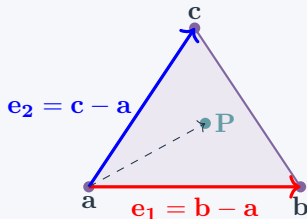
Insight

Barycentric coordinates doesn't just tell us if a point is inside a triangle, but also it's position with respect to other vertices.

Barycentric Coordinates: Derivation

Key Idea

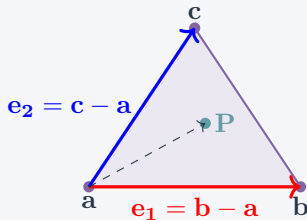
- The sides $e_1 = b - a$ and $e_2 = c - a$ are linearly independent vectors on the triangle's plane.



Barycentric Coordinates: Derivation

Key Idea

- The sides $\mathbf{e}_1 = \mathbf{b} - \mathbf{a}$ and $\mathbf{e}_2 = \mathbf{c} - \mathbf{a}$ are linearly independent vectors on the triangle's plane.
- Therefore, any vector in the triangle's plane (e.g. $\mathbf{P} - \mathbf{a}$) can be expressed as a linear combination of these vectors.



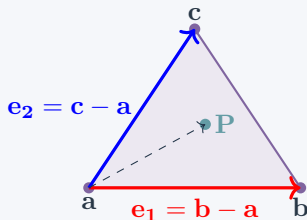
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- Therefore, any vector in the triangle's plane (e.g. $\mathbf{P} - \mathbf{a}$) can be expressed as a linear combination of these vectors.
- We can express \mathbf{P} as:

$$\begin{aligned}\mathbf{P} &= \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}) \\ &= \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}\end{aligned}$$

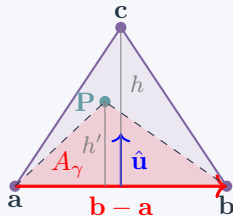
Where $\alpha = 1 - \beta - \gamma$.



Barycentric Coordinates: Derivation

Area Interpretation

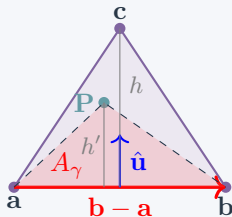
- Let \hat{u} be a unit vector in the direction of the altitude towards C .



Barycentric Coordinates: Derivation

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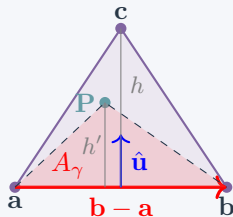
- Let $\hat{\mathbf{u}}$ be a unit vector in the direction of the altitude towards C .
- The height of the triangle is $h = \hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})$ (projection).



Barycentric Coordinates: Derivation

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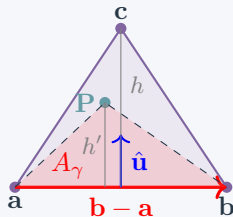


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- The height of the shaded triangle is $h' = \hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})$.
- Hence,

$$\begin{aligned} A_\gamma &= \frac{1}{2} \cdot h' \cdot |\mathbf{b} - \mathbf{a}| \\ &= \frac{1}{2} \cdot (\hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}| \\ &= \frac{1}{2} \cdot \gamma (\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}| \\ &= \gamma \frac{1}{2} \cdot h \cdot |\mathbf{b} - \mathbf{a}| = \gamma A_{total} \end{aligned}$$

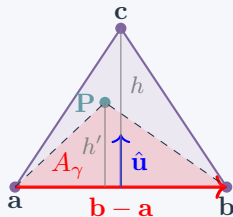


Barycentric Coordinates: Derivation

Area Interpretation

- Let $\hat{\mathbf{u}}$ be a unit vector in the direction of the altitude towards C .
- The height of the triangle is $h = \hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})$ (projection).
- The height of the shaded triangle is $h' = \hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})$.
- Hence,

$$\begin{aligned} A_\gamma &= \frac{1}{2} \cdot h' \cdot |\mathbf{b} - \mathbf{a}| \\ &= \frac{1}{2} \cdot (\hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}| \\ &= \frac{1}{2} \cdot \gamma (\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a})) \cdot |\mathbf{b} - \mathbf{a}| \\ &= \gamma \frac{1}{2} \cdot h \cdot |\mathbf{b} - \mathbf{a}| = \gamma A_{total} \end{aligned}$$



Since,

$$\mathbf{P} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}),$$

$$\mathbf{P} - \mathbf{a} = \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

$$\hat{\mathbf{u}} \cdot (\mathbf{P} - \mathbf{a}) = \gamma(\hat{\mathbf{u}} \cdot (\mathbf{c} - \mathbf{a}))$$

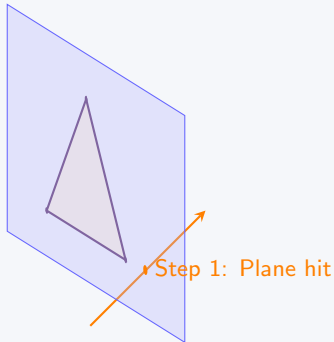
Since $\hat{\mathbf{u}}$ is perpendicular to $\mathbf{b} - \mathbf{a}$.

Method 1: Two-Step Ray-Triangle Intersection

Algorithm Steps

Step 1: Ray-Plane Intersection

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R}_o}{\mathbf{n} \cdot \mathbf{R}_d}$$



Method 1: Two-Step Ray-Triangle Intersection

Algorithm Steps

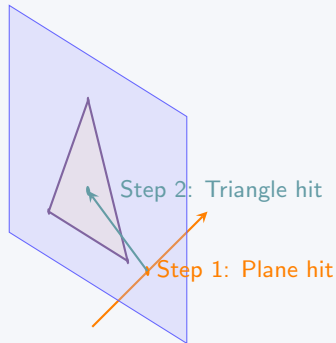
Step 1: Ray-Plane Intersection

$$t = -\frac{D + \mathbf{n} \cdot \mathbf{R}_o}{\mathbf{n} \cdot \mathbf{R}_d}$$

Step 2: Inside/Outside Test

$$\mathbf{P} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

Solve for β , γ and check bounds.



Method 2: Direct Barycentric Intersection

Direct Approach

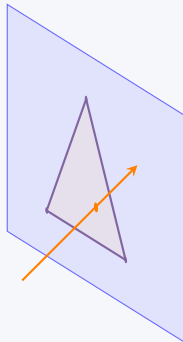
Set ray equation equal to barycentric form:

$$\mathbf{R}_o + t\mathbf{R}_d = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

Rearrange to linear system:

$$\begin{bmatrix} -\mathbf{R}_d & (\mathbf{b} - \mathbf{a}) & (\mathbf{c} - \mathbf{a}) \end{bmatrix} \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \mathbf{R}_o - \mathbf{a}$$

Solve using Cramer's rule or
LU decomposition.



Cramer's Rule Solution

Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_A \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

Cramer's Rule Solution

Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_A \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

Cramer's Rule

$$t = \frac{1}{|A|} \begin{vmatrix} (R_o - a)_x & (b - a)_x & (c - a)_x \\ (R_o - a)_y & (b - a)_y & (c - a)_y \\ (R_o - a)_z & (b - a)_z & (c - a)_z \end{vmatrix}$$

$$\beta = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix}$$

$$\gamma = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (b - a)_x & (R_o - a)_x \\ -R_{dy} & (b - a)_y & (R_o - a)_y \\ -R_{dz} & (b - a)_z & (R_o - a)_z \end{vmatrix}$$

Cramer's Rule Solution

Matrix Form

$$\underbrace{\begin{bmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{bmatrix}}_A \begin{bmatrix} t \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_{ox} - a_x \\ R_{oy} - a_y \\ R_{oz} - a_z \end{bmatrix}$$

Cramer's Rule

$$t = \frac{1}{|A|} \begin{vmatrix} (R_o - a)_x & (b - a)_x & (c - a)_x \\ (R_o - a)_y & (b - a)_y & (c - a)_y \\ (R_o - a)_z & (b - a)_z & (c - a)_z \end{vmatrix}$$

$$\beta = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix}$$

$$\gamma = \frac{1}{|A|} \begin{vmatrix} -R_{dx} & (b - a)_x & (R_o - a)_x \\ -R_{dy} & (b - a)_y & (R_o - a)_y \\ -R_{dz} & (b - a)_z & (R_o - a)_z \end{vmatrix}$$

Checks

- $t_{\min} < t < t_{\text{current}}$
(valid intersection)
- $\beta, \gamma \geq 0$ and
 $\beta + \gamma \leq 1$
(inside triangle)

Cramer's Rule Intuition

Determinant Interpretation

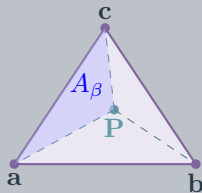
Remember, $\beta = \frac{A_\beta}{A_{total}}$?

Determinants measure areas of projected triangles.

$$\begin{vmatrix} -R_{dx} & (b_x - a_x) & (c_x - a_x) \\ -R_{dy} & (b_y - a_y) & (c_y - a_y) \\ -R_{dz} & (b_z - a_z) & (c_z - a_z) \end{vmatrix} \propto A_{total}$$

$$\begin{vmatrix} -R_{dx} & (R_o - a)_x & (c - a)_x \\ -R_{dy} & (R_o - a)_y & (c - a)_y \\ -R_{dz} & (R_o - a)_z & (c - a)_z \end{vmatrix} \propto A_\beta$$

Why?

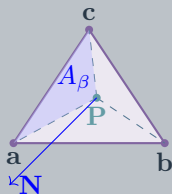


Cramer's Rule Intuition

Determinant Interpretation

The areas can be found by cross products of vectors. Consider —

$$\begin{aligned}\mathbf{N} &= (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x - a_x & b_y - a_y & b_z - a_z \\ c_x - a_x & c_y - a_y & c_z - a_z \end{vmatrix} \\ &= 2A_{total}\hat{\mathbf{n}}\end{aligned}$$

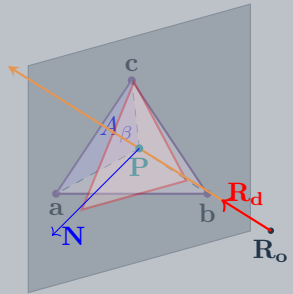


Cramer's Rule Intuition

Determinant Interpretation

- \mathbf{N} is the normal vector of the triangle scaled by twice the area.
- $\mathbf{N} \cdot \mathbf{R}_d$ is the projected area of the triangle onto the plane perpendicular to the ray direction.

$$\mathbf{N} \cdot \mathbf{R}_d = \begin{vmatrix} -R_x & -R_y & -R_z \\ b_x - a_x & b_y - a_y & b_z - a_z \\ c_x - a_x & c_y - a_y & c_z - a_z \end{vmatrix}$$
$$\propto A_{total}$$



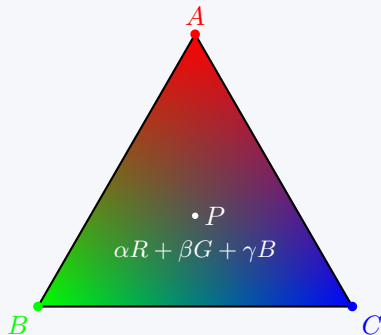
Bonus of Using Barycentric Coordinates

Advantages

- Efficient to compute
- Get Barycentric coordinates for free
- Enables interpolation of vertex attributes

Used in —

- Textures
- Normals
- Colors



Ray-Sphere Intersection Overview

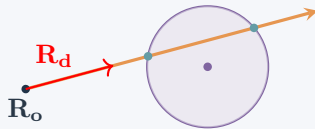
Two Main Approaches

Method 1: Algebra

1. Setup quadratic equation
2. Solve for t

Method 2: Geometry

1. Use geometry to find intersection step by step
2. Reject early if hit is not possible



Sphere Representation

Implicit Sphere Equation

Sphere centered at origin:

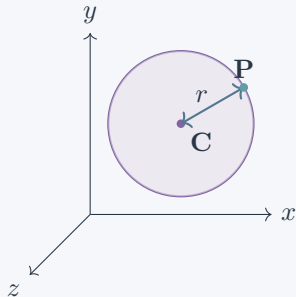
$$\mathbf{P} \cdot \mathbf{P} - r^2 = 0$$

$$x^2 + y^2 + z^2 - r^2 = 0$$

General sphere at center C:

$$(\mathbf{P} - \mathbf{C}) \cdot (\mathbf{P} - \mathbf{C}) - r^2 = 0$$

Note: Translation to origin simplifies calculation!

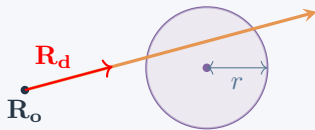


Ray-Sphere Intersection: Algebraic Method

Algebraic Solution

Step 1: Substitute ray equation
 $\mathbf{P}(t) = \mathbf{R}_o + t\mathbf{R}_d$ into sphere

$$(\mathbf{R}_o + t\mathbf{R}_d) \cdot (\mathbf{R}_o + t\mathbf{R}_d) - r^2 = 0$$

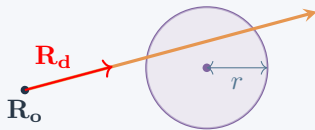


Ray-Sphere Intersection: Algebraic Method

Algebraic Solution

Step 2: Expand and rearrange

$$\begin{aligned} \mathbf{R}_d \cdot \mathbf{R}_d t^2 + 2\mathbf{R}_d \cdot \mathbf{R}_o t \\ + \mathbf{R}_o \cdot \mathbf{R}_o - r^2 = 0 \end{aligned}$$



Ray-Sphere Intersection: Algebraic Method

Algebraic Solution

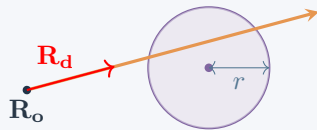
Step 3: Quadratic formula

$$(ax^2 + bx + c = 0)$$

$$a = \mathbf{R}_d \cdot \mathbf{R}_d = 1 \text{ (normalized)}$$

$$b = 2\mathbf{R}_d \cdot \mathbf{R}_o$$

$$c = \mathbf{R}_o \cdot \mathbf{R}_o - r^2$$



Ray-Sphere Intersection: Algebraic Method

Algebraic Solution

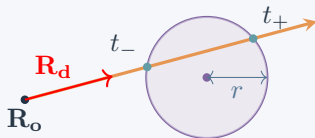
Step 3: Quadratic formula

$$(ax^2 + bx + c = 0)$$

$$a = \mathbf{R}_d \cdot \mathbf{R}_d = 1 \text{ (normalized)}$$

$$b = 2\mathbf{R}_d \cdot \mathbf{R}_o$$

$$c = \mathbf{R}_o \cdot \mathbf{R}_o - r^2$$



Discriminant Analysis

$$\Delta = b^2 - 4ac = (2\mathbf{R}_d \cdot \mathbf{R}_o)^2 - 4(\mathbf{R}_o \cdot \mathbf{R}_o - r^2)$$

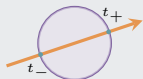
$$t_{\pm} = \frac{-b \pm \sqrt{\Delta}}{2a} = -\mathbf{R}_d \cdot \mathbf{R}_o \pm \frac{\sqrt{\Delta}}{2}$$

Algebraic Method: Three Cases

The discriminant Δ determines the number of intersections:

$$\Delta > 0$$

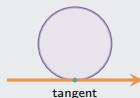
2 intersections



Choose closest
positive
(usually t_-)

$$\Delta = 0$$

1 intersection



Ray tangent to
sphere

$$\Delta < 0$$

No intersection



Ray doesn't hit
sphere

Additional Check

Remember to check t_{\min} to find closest valid intersection.

Ray-Sphere Intersection: Geometric Method

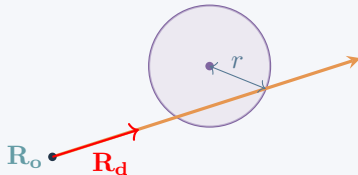
Geometric Approach

Step 1: Check ray origin position

Inside: $\mathbf{R}_o \cdot \mathbf{R}_o < r^2$

Outside: $\mathbf{R}_o \cdot \mathbf{R}_o > r^2$

On surface: $\mathbf{R}_o \cdot \mathbf{R}_o = r^2$

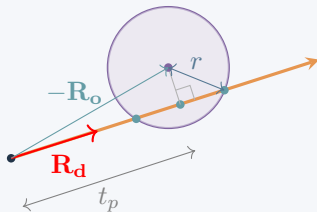


Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 2: Find parameter t_p for the point on the ray closest to the sphere center

$$t_P = -\mathbf{R}_o \cdot \mathbf{R}_d$$

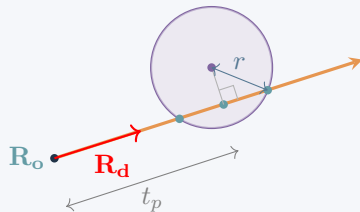


Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 3: Early rejection test

If outside & $t_P < 0 \Rightarrow$ no hit

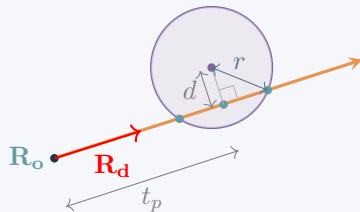


Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 4: Find squared distance to sphere center

$$d^2 = \mathbf{R}_o \cdot \mathbf{R}_o - t_p^2$$

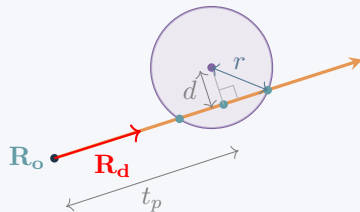


Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 5: Second rejection test

If $d^2 > r^2 \Rightarrow$ no hit



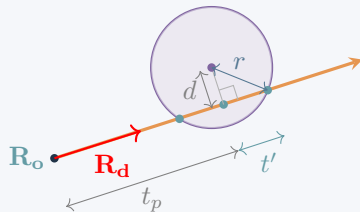
Ray-Sphere Intersection: Geometric Method

Geometric Approach

Step 6: Find intersection distance

$$t'^2 = r^2 - d^2$$

$$t' = \sqrt{r^2 - d^2}$$



Ray-Sphere Intersection: Geometric Method

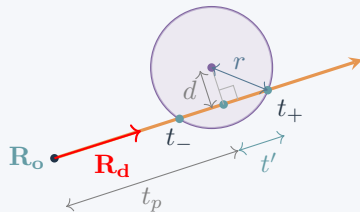
Geometric Approach

Step 7: Choose correct intersection parameter

Outside: $t_- = t_P - t'$

Inside: $t_+ = t_P + t'$

$$t_{min} < t_{\pm} < t_{current} \Rightarrow \text{hit}$$



Ray-Sphere Intersection: Geometric Method

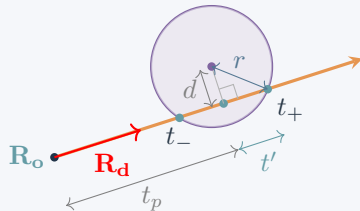
Geometric Approach

Step 7: Choose correct intersection parameter

Outside: $t_- = t_P - t'$

Inside: $t_+ = t_P + t'$

$t_{min} < t_{\pm} < t_{current} \Rightarrow \text{hit}$



Benefits of Method

- **Early rejection:** Avoid extra work for rays missing sphere
- **Optimized:** Efficient for rays outside pointing away

General Quadric Surfaces

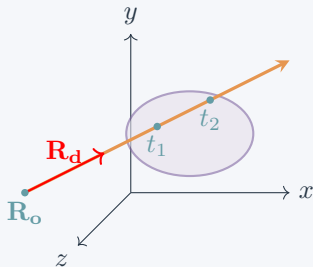
Quadric Surface Definition

General equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Common Quadric Surfaces:

- **Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- **Cone:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$
- **Cylinder:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- **Hyperboloid & Paraboloid**



Reference: Quadric Surfaces in Paul's Online Notes

Ray-Quadric Surface Intersection

Intersection Method

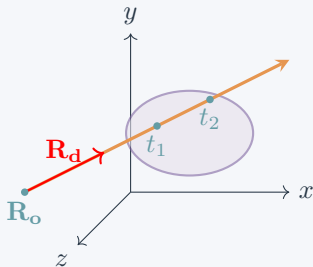
Step 1: Substitute ray equation into quadric

$$\mathbf{P}(t) = \mathbf{R}_o + t \cdot \mathbf{R}_d$$

$$P_x = R_{0x} + t \cdot R_{dx}$$

$$P_y = R_{0y} + t \cdot R_{dy}$$

$$P_z = R_{0z} + t \cdot R_{dz}$$

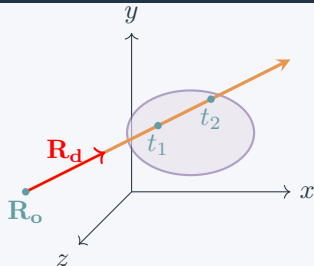


Ray-Quadric Surface Intersection

Intersection Method

Step 2: Results in quadratic equation

$$ax^2 + bx + c = 0$$



Solution Cases

Check the discriminant $b^2 - 4ac$:

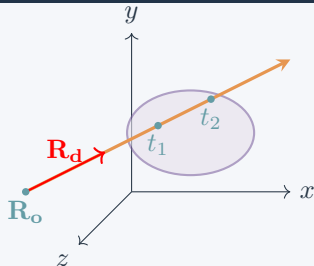
- **Discriminant** > 0 : Two real solutions (ray intersects surface twice)
- **Discriminant** $= 0$: One solution (ray tangent to surface)
- **Discriminant** < 0 : No real solutions (ray misses surface)
- **Accept**: Accept smaller t such that $t_{\min} < t < t_{\text{current}}$

Ray-Quadric Surface Intersection

Intersection Method

Step 3: Solve using quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

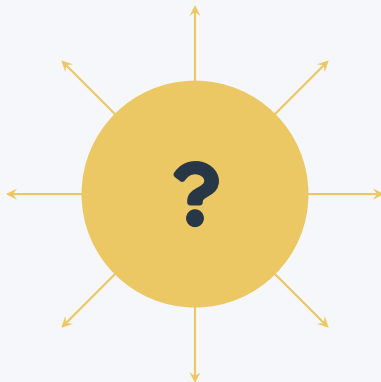


Solution Cases






Check the discriminant $b^2 - 4ac$:

- **Discriminant** > 0 : Two real solutions (ray intersects surface twice)
- **Discriminant** $= 0$: One solution (ray tangent to surface)
- **Discriminant** < 0 : No real solutions (ray misses surface)
- **Accept**: Accept smaller t such that $t_{\min} < t < t_{\text{current}}$

Questions?



References & Further Reading

-  Peter Shirley and Steve Marschner et al. *Fundamentals of Computer Graphics (4th Edition)*. CRC Press, 2016.
Available as PDF
-  Matt Pharr, Wenzel Jakob, and Greg Humphreys. *Physically Based Rendering: From Theory to Implementation (4th Edition)*. Morgan Kaufmann, 2023.
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-  Peter Shirley. *Ray Tracing in One Weekend*. Self-published, 2016–2020.
Project Website
-  MIT OpenCourseWare: 6.837 Computer Graphics.
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-  Scratchapixel: Learn Computer Graphics Programming.
scratchapixel.com