UNIT

4

DYNAMIC PROGRAMMING

GENERAL METHOD

Dynamic programming is a technique that breaks the problems into sub-problems, and saves the result for future purposes so that we do not need to compute the result again. The subproblems are optimized to optimize the overall solution is known as optimal substructure property. The main use of dynamic programming is to solve optimization problems. Here, optimization problems mean that when we are trying to find out the minimum or the maximum solution of a problem. The dynamic programming guarantees to find the optimal solution of a problem if the solution exists.

The definition of dynamic programming says that it is a technique for solving a complex problem by first breaking into a collection of simpler subproblems, solving each subproblem just once, and then storing their solutions to avoid repetitive computations.

Let's understand this approach through an example.

Consider an example of the Fibonacci series. The following series is the Fibonacci series:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ,…

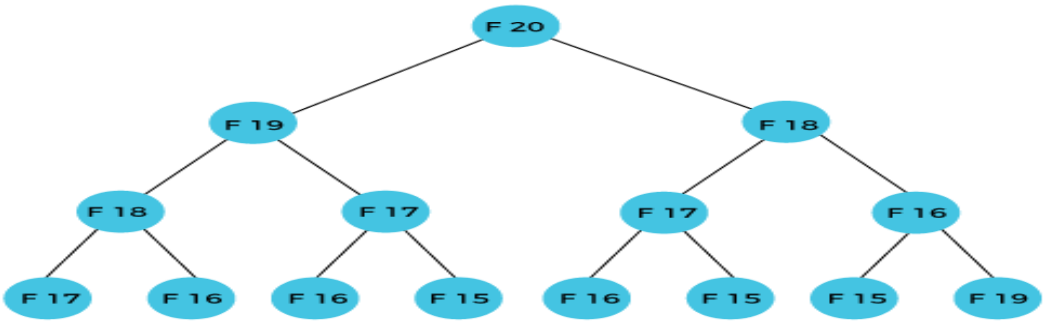
The numbers in the above series are not randomly calculated. Mathematically, we could write each of the terms using the below formula:

F(n) = F(n-1) + F(n-2),

With the base values F(0) = 0, and F(1) = 1. To calculate the other numbers, we follow the above relationship. For example, F(2) is the sum **f(0)** and **f(1),** which is equal to 1.

How can we calculate F(20)?

The F(20) term will be calculated using the nth formula of the Fibonacci series. The below figure shows that how F(20) is calculated.



As we can observe in the above figure that F(20) is calculated as the sum of F(19) and F(18). In the dynamic programming approach, we try to divide the problem into the similar subproblems. We are following this approach in the above case where F(20) into the similar subproblems, i.e., F(19) and F(18). If we recap the definition of dynamic programming that it says the similar subproblem should not be computed more than once. Still, in the above case, the subproblem is calculated twice. In the above example, F(18) is calculated two times; similarly, F(17) is also calculated twice. However, this technique is quite useful as it solves the similar subproblems, but we need to be cautious while storing the results because we are not particular about storing the result that we have computed once, then it can lead to a wastage of resources.

In the above example, if we calculate the F(18) in the right subtree, then it leads to the tremendous usage of resources and decreases the overall performance.

The solution to the above problem is to save the computed results in an array. First, we calculate F(16) and F(17) and save their values in an array. The F(18) is calculated by summing the values of F(17) and F(16), which are already saved in an array. The computed value of F(18) is saved in an array. The value of F(19) is calculated using the sum of F(18), and F(17), and their values are already saved in an array. The computed value of F(19) is stored in an array. The value of F(20) can be calculated by adding the values of F(19) and F(18), and the values of both F(19) and F(18) are stored in an array. The final computed value of F(20) is stored in an array.

How does the dynamic programming approach work?

The following are the steps that the dynamic programming follows:

- It breaks down the complex problem into simpler subproblems.
- It finds the optimal solution to these sub-problems.
- It stores the results of subproblems (memoization). The process of storing the results of subproblems is known as memorization.
- It reuses them so that same sub-problem is calculated more than once.
- Finally, calculate the result of the complex problem.

The above five steps are the basic steps for dynamic programming. The dynamic programming is applicable that are having properties such as:

Those problems that are having overlapping subproblems and optimal substructures. Here, optimal substructure means that the solution of optimization problems can be obtained by simply combining the optimal solution of all the subproblems.

In the case of dynamic programming, the space complexity would be increased as we are storing the intermediate results, but the time complexity would be decreased.

Approaches of dynamic programming

There are two approaches to dynamic programming:

- Top-down approach
- Bottom-up approach

Top-down approach

The top-down approach follows the memorization technique, while bottom-up approach follows the tabulation method. Here memorization is equal to the sum of recursion and caching. Recursion means calling the function itself, while caching means storing the intermediate results.

Advantages

- It is very easy to understand and implement.
- It solves the subproblems only when it is required.
- It is easy to debug.

Disadvantages

It uses the recursion technique that occupies more memory in the call stack. Sometimes when the recursion is too deep, the stack overflow condition will occur.

It occupies more memory that degrades the overall performance.

Let's understand dynamic programming through an example.

```
1. int fib(int n)
```

```
{
  if(n<0)
   error;
 if(n==0)
  return 0;
 if(n==1)
return 1;
sum = fib(n-1) + fib(n-2);
}
```

In the above code, we have used the recursive approach to find out the Fibonacci series. When the value of 'n' increases, the function calls will also increase, and computations will also increase. In this case, the time complexity increases exponentially, and it becomes 2^n.

One solution to this problem is to use the dynamic programming approach. Rather than generating the recursive tree again and again, we can reuse the previously calculated value. If we use the dynamic programming approach, then the time complexity would be O(n).

When we apply the dynamic programming approach in the implementation of the Fibonacci series, then the code would look like:

```
static int count = 0;
int fib(int n)
{
if(memo[n]!= NULL)
return memo[n];
count++;
  if(n<0)
   error;
 if(n==0)
  return 0;
 if(n==1)
return 1;
sum = fib(n-1) + fib(n-2);
memo[n] = sum;
}
```

In the above code, we have used the memorization technique in which we store the results in an array to reuse the values. This is also known as a top-down approach in which we move from the top and break the problem into sub-problems.

Bottom-Up approach

The bottom-up approach is also one of the techniques which can be used to implement the dynamic programming. It uses the tabulation technique to implement the dynamic programming approach. It solves the same kind of problems but it removes the recursion. If we remove the recursion, there is no stack overflow issue and no overhead of the recursive functions. In this tabulation technique, we solve the problems and store the results in a matrix.

There are two ways of applying dynamic programming:

- **Top-Down**
- **Bottom-Up**

The bottom-up is the approach used to avoid the recursion, thus saving the memory space. The bottom-up is an algorithm that starts from the beginning, whereas the recursive algorithm starts from the end and works backward. In the bottom-up approach, we start from the base case to find the answer for the end. As we know, the base cases in the Fibonacci series are 0 and 1. Since the bottom approach starts from the base cases, so we will start from 0 and 1.

Key points

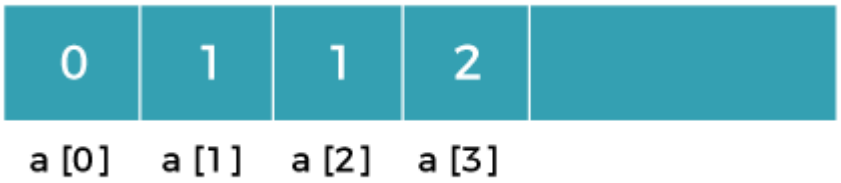
- We solve all the smaller sub-problems that will be needed to solve the larger sub-problems then move to the larger problems using smaller sub-problems.
- We use for loop to iterate over the sub-problems.
- The bottom-up approach is also known as the tabulation or table filling method.

Let's understand through an example.

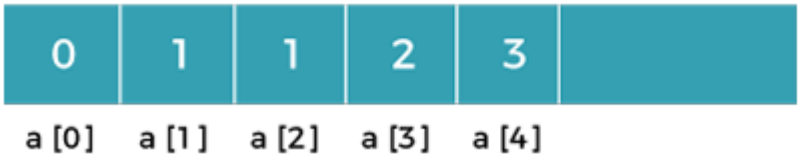
Suppose we have an array that has 0 and 1 values at a[0] and a[1] positions, respectively shown as below:

Since the bottom-up approach starts from the lower values, so the values at a[0] and a[1] are added to find the value of a[2] shown as below:

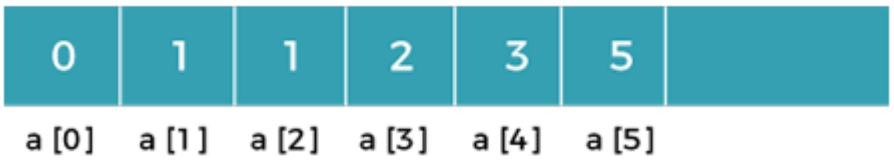
The value of a[3] will be calculated by adding a[1] and a[2], and it becomes 2 shown as below:



The value of a[4] will be calculated by adding a[2] and a[3], and it becomes 3 shown as below:



The value of a[5] will be calculated by adding the values of a[4] and a[3], and it becomes 5 shown as below:



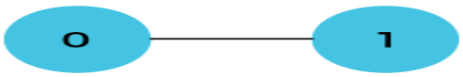
The code for implementing the Fibonacci series using the bottom-up approach is given below:

```
1.  int fib(int n)
2.  {
3.     int A[];
4.     A[0] = 0, A[1] = 1;
5.     for( i=2; i<=n; i++)
6.     {
7.        A[i] = A[i-1] + A[i-2]
8.     }
9.     return A[n];
10. }
```

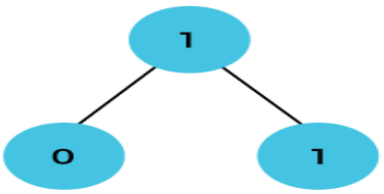
In the above code, base cases are 0 and 1 and then we have used for loop to find other values of Fibonacci series.

Let's understand through the diagrammatic representation.

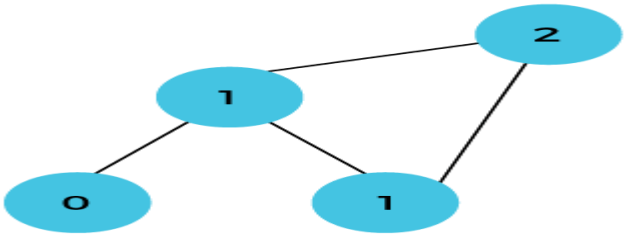
Initially, the first two values, i.e., 0 and 1 can be represented as:



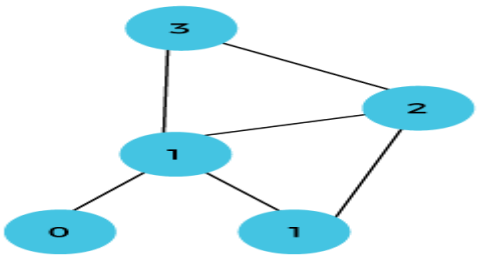
When i=2 then the values 0 and 1 are added shown as below:



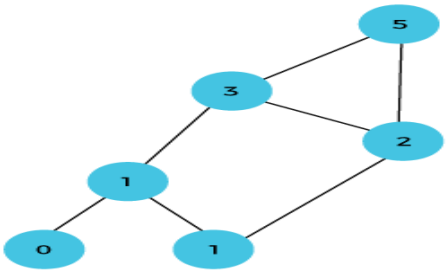
When i=3 then the values 1and 1 are added shown as below:



When i=4 then the values 2 and 1 are added shown as below:



When i=5, then the values 3 and 2 are added shown as below:



Matrix Chain Multiplication

It is a Method under Dynamic Programming in which previous output is taken as input for next.

Here, Chain means one matrix's column is equal to the second matrix's row [always].

In general:

If A = $[a_{ij}]$ is a p x q matrix
B = $[b_{ij}]$ is a q x r matrix
C = $[c_{ij}]$ is a p x r matrix

Then

$$AB = C \text{ if } c_{ij} = \sum_{k=1}^{q} a_{ik}\ b_{kj}$$

Given following matrices $\{A_1,A_2,A_3,...A_n\}$ and we have to perform the matrix multiplication, which can be accomplished by a series of matrix multiplications

$A_1 \times A_2 \times, A_3 \times ..... \times A_n$

Matrix Multiplication operation is **associative** in nature rather commutative. By this, we mean that we have to follow the above matrix order for multiplication but we are free to **parenthesize** the above multiplication depending upon our need.

In general, for $1 \leq i \leq p$ and $1 \leq j \leq r$

$$C[i, j] = \sum_{k=1}^{q} A[i, k]B[k, j]$$

It can be observed that the total entries in matrix 'C' is 'pr' as the matrix is of dimension p x r Also each entry takes O (q) times to compute, thus the total time to compute all possible entries for the matrix 'C' which is a multiplication of 'A' and 'B' is proportional to the product of the dimension p q r.

It is also noticed that we can save the number of operations by reordering the parenthesis.

Example1: Let us have 3 matrices, A_1,A_2,A_3 of order (10 x 100), (100 x 5) and (5 x 50) respectively.

Three Matrices can be multiplied in two ways:

1. **$A_1,(A_2,A_3)$:** First multiplying(A_2 and A_3) then multiplying and resultant withA_1.
2. **$(A_1,A_2),A_3$:** First multiplying(A_1 and A_2) then multiplying and resultant withA_3.

No of Scalar multiplication in Case 1 will be:

1. (100 x 5 x 50) + (10 x 100 x 50) = 25000 + 50000 = 75000

No of Scalar multiplication in Case 2 will be:

1. (100 x 10 x 5) + (10 x 5 x 50) = 5000 + 2500 = 7500

To find the best possible way to calculate the product, we could simply parenthesis the expression in every possible fashion and count each time how many scalar multiplication are required.

Matrix Chain Multiplication Problem can be stated as "find the optimal parenthesization of a chain of matrices to be multiplied such that the number of scalar multiplication is minimized".

Number of ways for parenthesizing the matrices:

There are very large numbers of ways of parenthesizing these matrices. If there are n items, there are (n-1) ways in which the outer most pair of parenthesis can place.

(A_1) (A_2,A_3,A_4,.....A_n)
Or (A_1,A_2) (A_3,A_4 .....A_n)
Or (A_1,A_2,A_3) (A_4 .....A_n)
.....

Or(A_1,A_2,A_3.....A_{n-1}) (A_n)

It can be observed that after splitting the kth matrices, we are left with two parenthesized sequence of matrices: one consist 'k' matrices and another consist 'n-k' matrices.

Now there are 'L' ways of parenthesizing the left sublist and 'R' ways of parenthesizing the right sublist then the Total will be L.R:

$$p(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} p(k)p(n-k) & \text{if } n \geq 2 \end{cases}$$

Also p (n) = c (n-1) where c (n) is the nth **Catalon number**

$$c(n) = \frac{1}{n+1}\binom{2n}{n}$$

On applying Stirling's formula we have

$$c(n) = \Omega\left(\frac{4^n}{n^{1.5}}\right)$$

Which shows that 4^n grows faster, as it is an exponential function, then $n^{1.5}$.

Development of Dynamic Programming Algorithm

1. Characterize the structure of an optimal solution.
2. Define the value of an optimal solution recursively.
3. Compute the value of an optimal solution in a bottom-up fashion.
4. Construct the optimal solution from the computed information.

Dynamic Programming Approach

Let $A_{i,j}$ be the result of multiplying matrices i through j. It can be seen that the dimension of $A_{i,j}$ is p_{i-1} x p_j matrix.

Dynamic Programming solution involves breaking up the problems into subproblems whose solution can be combined to solve the global problem.

At the greatest level of parenthesization, we multiply two matrices

$A_{1.....n}=A_{1....k}$ x $A_{k+1....n})$

Thus we are left with two questions:

- How to split the sequence of matrices?
- How to parenthesize the subsequence $A_{1.....k}$ and$A_{k+1.....n}$?

One possible answer to the first question for finding the best value of 'k' is to check all possible choices of 'k' and consider the best among them. But that it can be observed that checking all possibilities will lead to an exponential number of total possibilities. It can also be noticed that there exists only O (n^2) different sequence of matrices, in this way do not reach the exponential growth.

Step1: Structure of an optimal parenthesization: Our first step in the dynamic paradigm is to find the optimal substructure and then use it to construct an optimal solution to the problem from an optimal solution to subproblems.

Let $A_{i....j}$ where i≤ j denotes the matrix that results from evaluating the product

A_i A_{i+1}....A_j.

If i < j then any parenthesization of the product A_i A_{i+1} .....A_j must split that the product between A_k and A_{k+1} for some integer k in the range i ≤ k ≤ j. That is for some value of k, we first compute the matrices $A_{i.....k}$ & $A_{k+1....j}$ and then multiply them together to produce the final product $A_{i....j}$. The cost of computing $A_{i.....k}$ plus the cost of computing $A_{k+1....j}$ plus the cost of multiplying them together is the cost of parenthesization.

Step 2: A Recursive Solution: Let m [i, j] be the minimum number of scalar multiplication needed to compute the matrix$A_{i....j}$.

If i=j the chain consist of just one matrix $A_{i....i}=A_i$ so no scalar multiplication are necessary to compute the product. Thus m [i, j] = 0 for i= 1, 2, 3....n.

If i<j we assume that to optimally parenthesize the product we split it between A_k and A_{k+1} where i≤ k ≤j. Then m [i,j] equals the minimum cost for computing the subproducts $A_{i....k}$ and $A_{k+1....j}$+ cost of multiplying them together. We know A_i has dimension p_{i-1} x p_i, so computing the product $A_{i....k}$ and $A_{k+1....j}$takes p_{i-1} p_k p_j scalar multiplication, we obtain

m [i,j] = m [i, k] + m [k + 1, j] + p_{i-1} p_k p_j

There are only (j-1) possible values for 'k' namely k = i, i+1.....j-1. Since the optimal parenthesization must use one of these values for 'k' we need only check them all to find the best.

So the minimum cost of parenthesizing the product A_i A_{i+1}.....A_j becomes

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j}\{m[i,k] + m[k+1,j] + p_{i-1}\, p_k p_j\} & \text{if } i < j \end{cases}$$

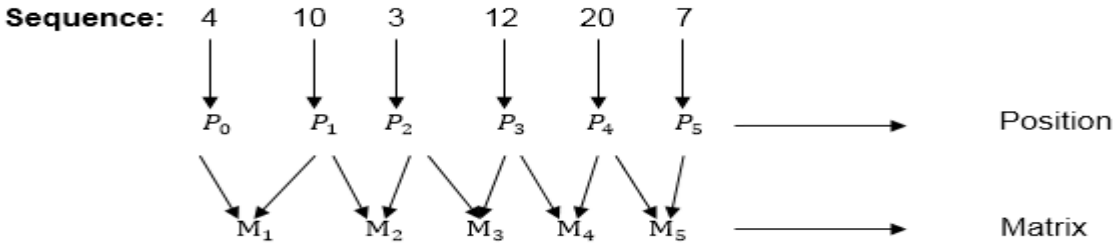
To construct an optimal solution, let us define s [i,j] to be the value of 'k' at which we can split the product $A_i A_{i+1} \ldots A_j$ To obtain an optimal parenthesization i.e. s [i, j] = k such that

$m[i,j] = m[i,k] + m[k+1,j] + p_{i-1}\ p_k\ p_j$

Example: We are given the sequence {4, 10, 3, 12, 20, and 7}. The matrices have size 4 x 10, 10 x 3, 3 x 12, 12 x 20, 20 x 7. We need to compute M [i,j], $0 \le i, j \le 5$. We know M [i, i] = 0 for all i.

1	2	3	4	5	
0					1
	0				2
		0			3
			0		4
				0	5

Let us proceed with working away from the diagonal. We compute the optimal solution for the product of 2 matrices.



Here P_0 to P_5 are Position and M_1 to M_5 are matrix of size (p_i to p_{i-1})

On the basis of sequence, we make a formula

For M_i ⟶ p [i] as column

p [i-1] as row

In Dynamic Programming, initialization of every method done by '0'.So we initialize it by '0'.It will sort out diagonally.

We have to sort out all the combination but the minimum output combination is taken into consideration.

Calculation of Product of 2 matrices:

1. m (1,2) = m_1 x m_2
   = 4 x 10 x 10 x 3
   = 4 x 10 x 3 = 120

2. m (2, 3) = m_2 x m_3
   = 10 x 3 x 3 x 12
   = 10 x 3 x 12 = 360

3. m (3, 4) = m_3 x m_4

= 3 x 12 x 12 x 20

= 3 x 12 x 20 = 720

4. m (4,5) = m_4 x m_5

= 12 x 20 x 20 x 7

= 12 x 20 x 7 = 1680

1	2	3	4	5	
0	120				1
	0	360			2
		0	720		3
			0	1680	4
				0	5

- We initialize the diagonal element with equal i,j value with '0'.
- After that second diagonal is sorted out and we get all the values corresponded to it

Now the third diagonal will be solved out in the same way.

Now product of 3 matrices:

M [1, 3] = M_1 M_2 M_3

1. There are two cases by which we can solve this multiplication: (M_1 x M_2) + M_3, M_1+ (M_2x M_3)
2. After solving both cases we choose the case in which minimum output is there.

$$M[1,3] = \min \begin{cases} M[1,2] + M[3,3] + p_0 p_2 p_3 = 120 + 0 + 4.3.12 = 264 \\ M[1,1] + M[2,3] + p_0 p_1 p_3 = 0 + 360 + 4.10.12 = 840 \end{cases}$$

M [1, 3] =264

As Comparing both output **264** is minimum in both cases so we insert **264** in table and (M_1 x M_2) + M_3 this combination is chosen for the output making.

M [2, 4] = M_2 M_3 M_4

1. There are two cases by which we can solve this multiplication: (M_2x M_3)+M_4, M_2+(M_3 x M_4)
2. After solving both cases we choose the case in which minimum output is there.

$$M[2,4] = \min \begin{cases} M[2,3] + M[4,4] + p_1 p_3 p_4 = 360 + 0 + 10.12.20 = 2760 \\ M[2,2] + M[3,4] + p_1 p_2 p_4 = 0 + 720 + 10.3.20 = 1320 \end{cases}$$

M [2, 4] = 1320

As Comparing both output **1320** is minimum in both cases so we insert **1320** in table and M_2+(M_3 x M_4) this combination is chosen for the output making.

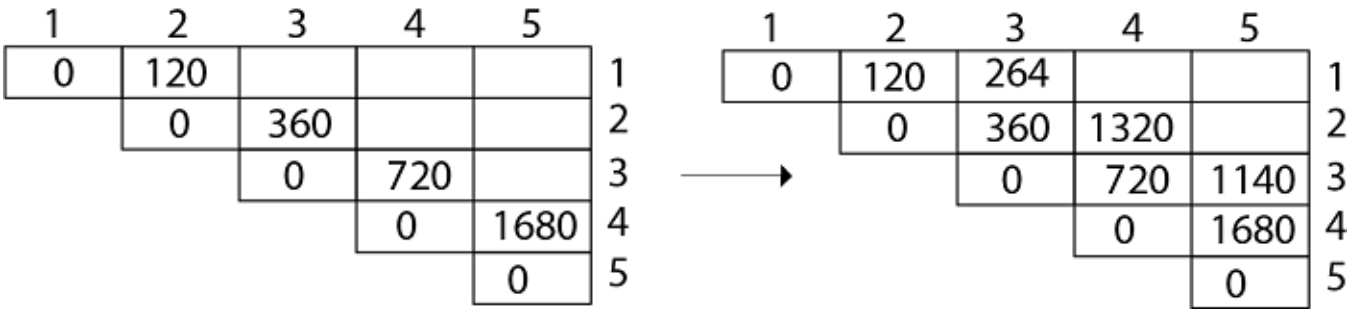
M [3, 5] = M_3 M_4 M_5

1. There are two cases by which we can solve this multiplication: (M_3 x M_4) + M_5, M_3+ (M_4xM_5)
2. After solving both cases we choose the case in which minimum output is there.

$$M[3,5] = \min \begin{Bmatrix} M[3,4] + M[5,5] + p_2p_4p_5 = 720 + 0 + 3.20.7 = 1140 \\ M[3,3] + M[4,5] + p_2p_3p_5 = 0 + 1680 + 3.12.7 = 1932 \end{Bmatrix}$$

M [3, 5] = 1140

As Comparing both output **1140** is minimum in both cases so we insert **1140** in table and (M_3 x M_4) + M_5this combination is chosen for the output making.



Now Product of 4 matrices:

M [1, 4] = M_1 M_2 M_3 M_4

There are three cases by which we can solve this multiplication:

1. (M_1 x M_2 x M_3) M_4
2. M_1 x(M_2 x M_3 x M_4)
3. (M_1 xM_2) x (M_3 x M_4)

After solving these cases we choose the case in which minimum output is there

$$M[1,4] = \min \begin{Bmatrix} M[1,3] + M[4,4] + p_0p_3p_4 = 264 + 0 + 4.12.20 = 1224 \\ M[1,2] + M[3,4] + p_0p_2p_4 = 120 + 720 + 4.3.20 = 1080 \\ M[1,1] + M[2,4] + p_0p_1p_4 = 0 + 1320 + 4.10.20 = 2120 \end{Bmatrix}$$

M [1, 4] =1080

As comparing the output of different cases then '**1080**' is minimum output, so we insert 1080 in the table and (M_1 xM_2) x (M_3 x M_4) combination is taken out in output making,

M [2, 5] = M_2 M_3 M_4 M_5

There are three cases by which we can solve this multiplication:

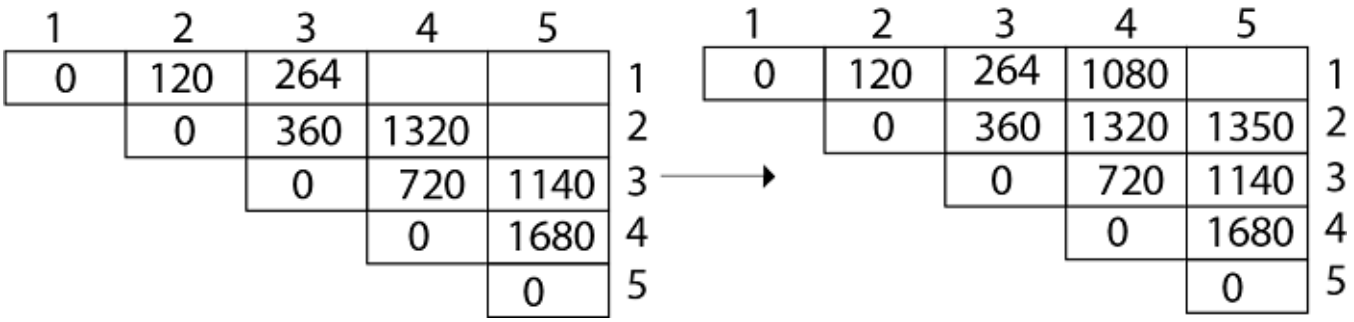
1. (M_2 x M_3 x M_4)x M_5
2. M_2 x(M_3 x M_4 x M_5)
3. (M_2 x M_3)x (M_4 x M_5)

After solving these cases we choose the case in which minimum output is there

$$M[2,5] = \min \begin{Bmatrix} M[2,4] + M[5,5] + p_1p_4p_5 = 1320 + 0 + 10.20.7 = 2720 \\ M[2,3] + M[4,5] + p_1p_3p_5 = 360 + 1680 + 10.12.7 = 2880 \\ M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 1140 + 10.3.7 = 1350 \end{Bmatrix}$$

M [2, 5] = 1350

As comparing the output of different cases then '**1350**' is minimum output, so we insert 1350 in the table and M_2 x(M_3 x M_4 xM_5)combination is taken out in output making.



Now Product of 5 matrices:

M [1, 5] = M_1 M_2 M_3 M_4 M_5

There are five cases by which we can solve this multiplication:

1. (M_1 x M_2 xM_3 x M_4)x M_5
2. M_1 x(M_2 xM_3 x M_4 xM_5)
3. (M_1 x M_2 xM_3)x M_4 xM_5
4. M_1 x M_2x(M_3 x M_4 xM_5)

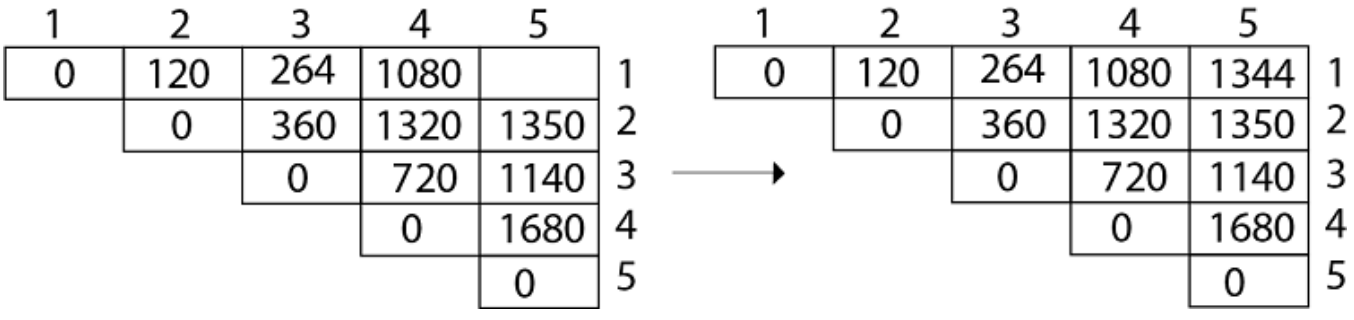
After solving these cases we choose the case in which minimum output is there

$$M[1,5] = \min\begin{cases} M[1,4] + M[5,5] + p_0p_4p_5 = 1080 + 0 + 4.20.7 = 1544 \\ M[1,3] + M[4,5] + p_0p_3p_5 = 264 + 1680 + 4.12.7 = 2016 \\ M[1,2] + M[3,5] + p_0p_2p_5 = 120 + 1140 + 4.3.7 = 1344 \\ M[1,1] + M[2,5] + p_0p_1p_5 = 0 + 1350 + 4.10.7 = 1630 \end{cases}$$

M [1, 5] = 1344

As comparing the output of different cases then '**1344**' is minimum output, so we insert 1344 in the table and M_1 x M_2 x(M_3 x M_4 x M_5)combination is taken out in output making.

Final Output is:



Step 3: Computing Optimal Costs: let us assume that matrix A_i has dimension p_{i-1}x p_i for i=1, 2, 3....n. The input is a sequence (p_0,p_1,.....p_n) where length [p] = n+1. The procedure uses an auxiliary table m [1....n, 1.....n] for storing m [i, j] costs an auxiliary table s [1.....n, 1.....n] that record which index of k achieved the optimal costs in computing m [i, j].

The algorithm first computes m [i, j] ← 0 for i=1, 2, 3.....n, the minimum costs for the chain of length 1.

MATRIX-CHAIN-ORDER (p)

```
1. n   length[p]-1
2. for i ← 1 to n
3. do m [i, i] ← 0
4. for l ← 2 to n     // l is the chain length
5. do for i ← 1 to n-l + 1
6. do j ← i+ l -1
7. m[i,j] ←  ∞
8. for k  ← i to j-1
9. do q  ← m [i, k] + m [k + 1, j] + p_{i-1} p_k p_j
10. If q < m [i,j]
11. then m [i,j] ← q
12. s [i,j] ← k
13. return m and s.
```

There are three nested loops. Each loop executes a maximum n times.

1. l, length, O (n) iterations.
2. i, start, O (n) iterations.
3. k, split point, O (n) iterations

0/1 Knapsack problem

Here knapsack is like a container or a bag. Suppose we have given some items which have some weights or profits. We have to put some items in the knapsack in such a way total value produces a maximum profit.

For example, the weight of the container is 20 kg. We have to select the items in such a way that the sum of the weight of items should be either smaller than or equal to the weight of the container, and the profit should be maximum.

There are two types of knapsack problems:

- 0/1 knapsack problem
- Fractional knapsack problem

We will discuss both the problems one by one. First, we will learn about the 0/1 knapsack problem.

What is the 0/1 knapsack problem?

The 0/1 knapsack problem means that the items are either completely or no items are filled in a knapsack. For example, we have two items having weights 2kg and 3kg, respectively. If we pick the 2kg item then we cannot pick 1kg item from the 2kg item (item is not divisible); we have to pick the 2kg item completely. This is a 0/1 knapsack problem in which either we pick the item completely or we will pick that item. The 0/1 knapsack problem is solved by the dynamic programming.

What is the fractional knapsack problem?

The fractional knapsack problem means that we can divide the item. For example, we have an item of 3 kg then we can pick the item of 2 kg and leave the item of 1 kg. The fractional knapsack problem is solved by the Greedy approach.

Example of 0/1 knapsack problem.

Consider the problem having weights and profits are:

Weights: {3, 4, 6, 5}

Profits: {2, 3, 1, 4}

The weight of the knapsack is 8 kg

The number of items is 4

The above problem can be solved by using the following method:

x_i = {1, 0, 0, 1}

= {0, 0, 0, 1}

= {0, 1, 0, 1}

The above are the possible combinations. 1 denotes that the item is completely picked and 0 means that no item is picked. Since there are 4 items so possible combinations will be:

2^4 = 16; So. There are 16 possible combinations that can be made by using the above problem. Once all the combinations are made, we have to select the combination that provides the maximum profit.

Another approach to solve the problem is dynamic programming approach. In dynamic programming approach, the complicated problem is divided into sub-problems, then we find the solution of a sub-problem and the solution of the sub-problem will be used to find the solution of a complex problem.

How this problem can be solved by using the Dynamic programming approach?

First,

we create a matrix shown as below:

	0	1	2	3	4	5	6
0							
1							
2							
3							
4							

In the above matrix, columns represent the weight, i.e., 8. The rows represent the profits and weights of items. Here we have not taken the weight 8 directly, problem is divided into sub-problems, i.e., 0, 1, 2, 3, 4, 5, 6, 7, 8. The solution of the sub-problems would be saved in the cells and answer to the problem would be stored in the final cell. First, we write the weights in the ascending order and profits according to their weights shown as below:

w_i = {3, 4, 5, 6}

p_i = {2, 3, 4, 1}

The first row and the first column would be 0 as there is no item for w=0

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0						
2	0						
3	0						
4	0						

When i=1, W=1

w_1 = 3; Since we have only one item in the set having weight 3, but the capacity of the knapsack is 1. We cannot fill the item of 3kg in the knapsack of capacity 1 kg so add 0 at M[1][1] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0					
2	0						
3	0						
4	0						

When i = 1, W = 2

w_1 = 3; Since we have only one item in the set having weight 3, but the capacity of the knapsack is 2. We cannot fill the item of 3kg in the knapsack of capacity 2 kg so add 0 at M[1][2] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0				
2	0						
3	0						
4	0						

When i=1, W=3

w_1 = 3; Since we have only one item in the set having weight equal to 3, and weight of the knapsack is also 3; therefore, we can fill the knapsack with an item of weight equal to 3. We put profit corresponding to the weight 3, i.e., 2 at M[1][3] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2			
2	0						
3	0						
4	0						

When i=1, W = 4

w_1 = 3; Since we have only one item in the set having weight equal to 3, and weight of the knapsack is 4; therefore, we can fill the knapsack with an item of weight equal to 3. We put profit corresponding to the weight 3, i.e., 2 at M[1][4] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2		

2	0						
3	0						
4	0						

When i=1, W = 5

w_1 = 3; Since we have only one item in the set having weight equal to 3, and weight of the knapsack is 5; therefore, we can fill the knapsack with an item of weight equal to 3. We put profit corresponding to the weight 3, i.e., 2 at M[1][5] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	
2	0						
3	0						
4	0						

When i =1, W=6

w_1 = 3; Since we have only one item in the set having weight equal to 3, and weight of the knapsack is 6; therefore, we can fill the knapsack with an item of weight equal to 3. We put profit corresponding to the weight 3, i.e., 2 at M[1][6] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0						
3	0						
4	0						

When i=1, W = 7

w_1 = 3; Since we have only one item in the set having weight equal to 3, and weight of the knapsack is 7; therefore, we can fill the knapsack with an item of weight equal to 3. We put profit corresponding to the weight 3, i.e., 2 at M[1][7] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0						
3	0						
4	0						

When i =1, W =8

w_1 = 3; Since we have only one item in the set having weight equal to 3, and weight of the knapsack is 8; therefore, we can fill the knapsack with an item of weight equal to 3. We put profit corresponding to the weight 3, i.e., 2 at M[1][8] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0						
3	0						
4	0						

Now the value of 'i' gets incremented, and becomes 2.

When i =2, W = 1

The weight corresponding to the value 2 is 4, i.e., w_2 = 4. Since we have only one item in the set having weight equal to 4, and the weight of the knapsack is 1. We cannot put the item of weight 4 in a knapsack, so we add 0 at M[2][1] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0					
3	0						
4	0						

When i =2, W = 2

The weight corresponding to the value 2 is 4, i.e., w_2 = 4. Since we have only one item in the set having weight equal to 4, and the weight of the knapsack is 2. We cannot put the item of weight 4 in a knapsack, so we add 0 at M[2][2] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0				
3	0						
4	0						

When i =2, W = 3

The weight corresponding to the value 2 is 4, i.e., w_2 = 4. Since we have two items in the set having weights 3 and 4, and the weight of the knapsack is 3. We can put the item of weight 3 in a knapsack, so we add 2 at M[2][3] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0

1	0	0	0	2	2	2	2
2	0	0	0	2			
3	0						
4	0						

When i =2, W = 4

The weight corresponding to the value 2 is 4, i.e., $w_2 = 4$. Since we have two items in the set having weights 3 and 4, and the weight of the knapsack is 4. We can put item of weight 4 in a knapsack as the profit corresponding to weight 4 is more than the item having weight 3, so we add 3 at M[2][4] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3		
3	0						
4	0						

When i = 2, W = 5

The weight corresponding to the value 2 is 4, i.e., $w_2 = 4$. Since we have two items in the set having weights 3 and 4, and the weight of the knapsack is 5. We can put item of weight 4 in a knapsack and the profit corresponding to weight is 3, so we add 3 at M[2][5] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	
3	0						
4	0						

When i = 2, W = 6

The weight corresponding to the value 2 is 4, i.e., $w_2 = 4$. Since we have two items in the set having weights 3 and 4, and the weight of the knapsack is 6. We can put item of weight 4 in a knapsack and the profit corresponding to weight is 3, so we add 3 at M[2][6] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0						

4	0						

When i = 2, W = 7

The weight corresponding to the value 2 is 4, i.e., $w_2 = 4$. Since we have two items in the set having weights 3 and 4, and the weight of the knapsack is 7. We can put item of weight 4 and 3 in a knapsack and the profits corresponding to weights are 2 and 3; therefore, the total profit is 5, so we add 5 at M[2][7] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	0	3	3	3
3	0						
4	0						

When i = 2, W = 8

The weight corresponding to the value 2 is 4, i.e., $w_2 = 4$. Since we have two items in the set having weights 3 and 4, and the weight of the knapsack is 7. We can put item of weight 4 and 3 in a knapsack and the profits corresponding to weights are 2 and 3; therefore, the total profit is 5, so we add 5 at M[2][7] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0						
4	0						

Now the value of 'i' gets incremented, and becomes 3.

When i = 3, W = 1

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set having weights 3, 4, and 5, and the weight of the knapsack is 1. We cannot put neither of the items in a knapsack, so we add 0 at M[3][1] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0					
4	0						

When i = 3, W = 2

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set having weight 3, 4, and 5, and the weight of the knapsack is 1. We cannot put neither of the items in a knapsack, so we add 0 at M[3][2] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0				
4	0						

When i = 3, W = 3

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set of weight 3, 4, and 5 respectively and weight of the knapsack is 3. The item with a weight 3 can be put in the knapsack and the profit corresponding to the item is 2, so we add 2 at M[3][3] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	2			
4	0						

When i = 3, W = 4

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set of weight 3, 4, and 5 respectively, and weight of the knapsack is 4. We can keep the item of either weight 3 or 4; the profit (3) corresponding to the weight 4 is more than the profit corresponding to the weight 3 so we add 3 at M[3][4] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3		
4	0						

When i = 3, W = 5

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set of weight 3, 4, and 5 respectively, and weight of the knapsack is 5. We can keep the item of either weight 3, 4 or 5; the profit (3) corresponding to the weight 4 is more than the profits corresponding to the weight 3 and 5 so we add 3 at M[3][5] shown as below:

	0	1	2	3	4	5	6

0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	
4	0						

When i =3, W = 6

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set of weight 3, 4, and 5 respectively, and weight of the knapsack is 6. We can keep the item of either weight 3, 4 or 5; the profit (3) corresponding to the weight 4 is more than the profits corresponding to the weight 3 and 5 so we add 3 at M[3][6] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0						

When i =3, W = 7

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set of weight 3, 4, and 5 respectively, and weight of the knapsack is 7. In this case, we can keep both the items of weight 3 and 4, the sum of the profit would be equal to (2 + 3), i.e., 5, so we add 5 at M[3][7] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0						

When i = 3, W = 8

The weight corresponding to the value 3 is 5, i.e., $w_3 = 5$. Since we have three items in the set of weight 3, 4, and 5 respectively, and the weight of the knapsack is 8. In this case, we can keep both the items of weight 3 and 4, the sum of the profit would be equal to (2 + 3), i.e., 5, so we add 5 at M[3][8] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3

3	0	0	0	1	3	3	3
4	0						

Now the value of 'i' gets incremented and becomes 4.

When i = 4, W = 1

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 1. The weight of all the items is more than the weight of the knapsack, so we cannot add any item in the knapsack; Therefore, we add 0 at M[4][1] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0	0					

When i = 4, W = 2

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 2. The weight of all the items is more than the weight of the knapsack, so we cannot add any item in the knapsack; Therefore, we add 0 at M[4][2] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0	0	0				

When i = 4, W = 3

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 3. The item with a weight 3 can be put in the knapsack and the profit corresponding to the weight 4 is 2, so we will add 2 at M[4][3] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0	0	0	2			

When i = 4, W = 4

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 4. The item with a weight 4 can be put in the knapsack and the profit corresponding to the weight 4 is 3, so we will add 3 at M[4][4] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0	0	0	2	3		

When i = 4, W = 5

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 5. The item with a weight 4 can be put in the knapsack and the profit corresponding to the weight 4 is 3, so we will add 3 at M[4][5] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0	0	0	2	3	3	

When i = 4, W = 6

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 6. In this case, we can put the items in the knapsack either of weight 3, 4, 5 or 6 but the profit, i.e., 4 corresponding to the weight 6 is highest among all the items; therefore, we add 4 at M[4][6] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	1	3	3	3
4	0	0	0	2	3	3	4

When i = 4, W = 7

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 7. Here, if we add two items of weights 3 and 4 then it will produce the maximum profit, i.e., (2 + 3) equals to 5, so we add 5 at M[4][7] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	2	3	3	3
4	0	0	0	2	3	3	4

When i = 4, W = 8

The weight corresponding to the value 4 is 6, i.e., $w_4 = 6$. Since we have four items in the set of weights 3, 4, 5, and 6 respectively, and the weight of the knapsack is 8. Here, if we add two items of weights 3 and 4 then it will produce the maximum profit, i.e., (2 + 3) equals to 5, so we add 5 at M[4][8] shown as below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	2	3	3	3
4	0	0	0	2	3	3	4

As we can observe in the above table that 5 is the maximum profit among all the entries. The pointer points to the last row and the last column having 5 value. Now we will compare 5 value with the previous row; if the previous row, i.e., i = 3 contains the same value 5 then the pointer will shift upwards. Since the previous row contains the value 5 so the pointer will be shifted upwards as shown in the below table:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	2	3	3	3
4	0	0	0	2	3	3	4

Again, we will compare the value 5 from the above row, i.e., i = 2. Since the above row contains the value 5 so the pointer will again be shifted upwards as shown in the below table:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	2	3	3	3

4	0	0	0	2	3	3	4

Again, we will compare the value 5 from the above row, i.e., i = 1. Since the above row does not contain the same value so we will consider the row i=1, and the weight corresponding to the row is 4. Therefore, we have selected the weight 4 and we have rejected the weights 5 and 6 shown below:

x = { 1, 0, 0}

The profit corresponding to the weight is 3. Therefore, the remaining profit is (5 - 3) equals to 2. Now we will compare this value 2 with the row i = 2. Since the row (i = 1) contains the value 2; therefore, the pointer shifted upwards shown below:

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	2	2	2	2
2	0	0	0	2	3	3	3
3	0	0	0	2	3	3	3
4	0	0	0	2	3	3	4

Again we compare the value 2 with a above row, i.e., i = 1. Since the row i =0 does not contain the value 2, so row i = 1 will be selected and the weight corresponding to the i = 1 is 3 shown below:

X = {1, 1, 0, 0}

The profit corresponding to the weight is 2. Therefore, the remaining profit is 0. We compare 0 value with the above row. Since the above row contains a 0 value but the profit corresponding to this row is 0. In this problem, two weights are selected, i.e., 3 and 4 to maximize the profit.

OPTIMAL BINARY SEARCH TREE

Let us assume that the given set of identifiers is {a1, . . . , an} with a1 < a2 < . . . . < an. Let p (i) be the probability with which we search for ai. Let q (i) be the probability that the identifier x being searched for is such that ai < x < ai+1, 0 $\leq$ i $\leq$ n (assume a0 = - ~ and an+1 = +oc). We have to arrange the identifiers in a binary search tree in a way that minimizes the expected total access time.

In a binary search tree, the number of comparisons needed to access an element at depth 'd' is d + 1, so if 'ai' is placed at depth 'di', then we want to minimize: *n*

~ *Pi* (1 + *di*) . *i* ~1

Let P (i) be the probability with which we shall be searching for 'ai'. Let Q (i) be the probability of an un-successful search. Every internal node represents a point where a successful search may terminate. Every external node represents a point where an unsuccessful search may terminate.

The expected cost contribution for the internal node for 'ai' is:

P (*i*) * *level* (*ai*) .

Unsuccessful search terminate with I = 0 (i.e at an external node). Hence the cost contribution for this node is:

Q (i) * level ((Ei) - 1)

The expected cost of binary search tree is: *n*

~ *P*(*i*) * *level* (*ai*) + ~ *Q* (*i*) * *level* ((*Ei*) - 1)

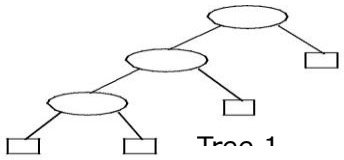
Given a fixed set of identifiers, we wish to create a binary search tree organization. We may expect different binary search trees for the same identifier set to have different performance characteristics.

The computation of each of these c(i, j)'s requires us to find the minimum of m quantities. Hence, each such c(i, j) can be computed in time O(m). The total time for all c(i, j)'s with j – i = m is therefore O($nm – m^2$).

The total time to evaluate all the c(i, j)'s and r(i, j)'s is therefore:

$$\sim (nm - m^2) = O(n^3)\ 1 < m < n$$

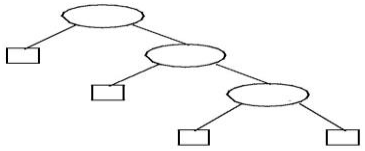
Example 1: The possible binary search trees for the identifier set (a1, a2, a3) = (do, if, stop) are as follows. Given the equal probabilities p (i) = Q (i) = 1/7 for all i, we have: st o p



if

do

Tree 2



do

if

st o p

Tree 3

Cost (tree # 1) = ~ (1 *x* 1 + 1 *x* 2 + 1 *x* 3~ + ~ 7 7 7)

Cost (tree # 3) = ~~ 1 7 *x* 1 + 17 *x* 2 + 1 7*x* 3~ ~ +) (~~ 1 7 *x* 1 + 17 *x* 2 + 17 *x* 3 + 1 7*x* 3~ ~)

= 1 +7 2 + 3 + 1 + 2 +7 3 + 3 ~ 6 +7(19 *x* ~1 15 +7 1

Cost (tree # 4) = ~~~ 1 7 *x* 1 + 17 *x* 2 ~ 1 7*x* 3~ ~ ~) ~~ 7 7 *x* 2 + 17 *x* 3 + 1 7*x* 3~ ~)

= 1 + 7 2 + 3 I- 1 + 2 + 3 + 3 ~6 +7 9 ~15

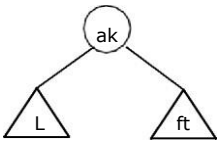
= 7 + 7 ~ 7 ~7

Cost (tree # 2) = ~~ 7 (1 *x* 1 + 1 1 *x* 2~ 7 *x* 2 + 7) + ~~ 7 (1 *x* 2 + 1 *x* 2 + 1 *x* 2 + 1 *x* 2~ 7)

= 1 + 2 + 2 2 + 2 + 2 + 2 5 + 8 13
7 + 7 ~ 7 ~ 7

Huffman coding tree solved by a greedy algorithm has a limitation of having the data only at the leaves and it must not preserve the property that all nodes to the left of the root have keys, which are less etc. Construction of an optimal binary search tree is

harder, because the data is not constrained to appear only at the leaves, and also because the tree must satisfy the binary search tree property and it must preserve the property that all nodes to the left of the root have keys, which are less.
A dynamic programming solution to the problem of obtaining an optimal binary search tree can be viewed by constructing a tree as a result of sequence of decisions by holding the principle of optimality. A possible approach to this is to make a decision as which of the ai's be arraigned to the root node at 'T'. If we choose 'ak' then is clear that the internal nodes for a1, a2, ... ak-1 as well as the external nodes for the classes Eo, E1, ... Ek-1 will lie in the left sub tree, L, of the root. The remaining nodes will be in the right subtree, ft. The structure of an optimal binary search tree is:



$$\text{Cost (L)} = \sum_{i \sim 1}^{K} P(i) * level(ai) + \sum_{i \sim 0}^{K} Q(i) * (level(E_i) - 1)$$

$$\text{Cost (ft)} = \sum_{i \sim K}^{n} P(i) * level(ai) + \sum_{i \sim K}^{n} Q(i) * (level(Ei) - 1)$$

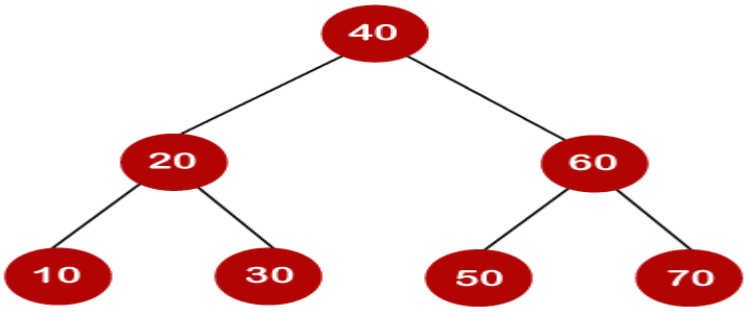
Optimal Binary Search Tree

As we know that in binary search tree, the nodes in the left subtree have lesser value than the root node and the nodes in the right subtree have greater value than the root node.

We know the key values of each node in the tree, and we also know the frequencies of each node in terms of searching means how much time is required to search a node. The frequency and key-value determine the overall cost of searching a node. The cost of searching is a very important factor in various applications. The overall cost of searching a node should be less. The time required to search a node in BST is more than the balanced binary search tree as a balanced binary search tree contains a lesser number of levels than the BST. There is one way that can reduce the cost of a binary search tree is known as an **optimal binary search tree**.

Let's understand through an example.

If the keys are 10, 20, 30, 40, 50, 60, 70



In the above tree, all the nodes on the left subtree are smaller than the value of the root node, and all the nodes on the right subtree are larger than the value of the root node. The maximum time required to search a node is equal to the minimum height of the tree, equal to logn.

Now we will see how many binary search trees can be made from the given number of keys.

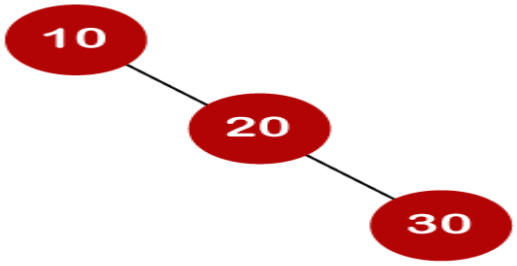
For example: 10, 20, 30 are the keys, and the following are the binary search trees that can be made out from these keys.

The Formula for calculating the number of trees:

$$\frac{{}^{2n}C_n}{n+1}$$

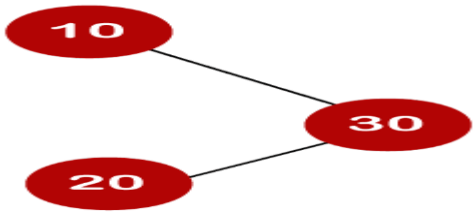
When we use the above formula, then it is found that total 5 number of trees can be created.

The cost required for searching an element depends on the comparisons to be made to search an element. Now, we will calculate the average cost of time of the above binary search trees.



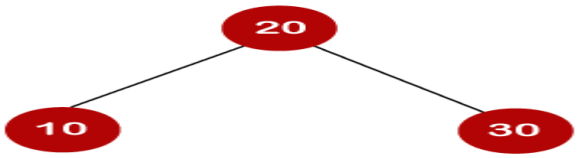
In the above tree, total number of 3 comparisons can be made. The average number of comparisons can be made as:

$$average\ number\ of\ comparisons = \frac{1+2+3}{3} = 2$$



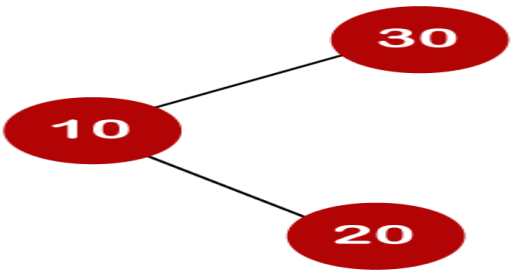
In the above tree, the average number of comparisons that can be made as:

$$average\ number\ of\ comparisons = \frac{1+2+3}{3} = 2$$



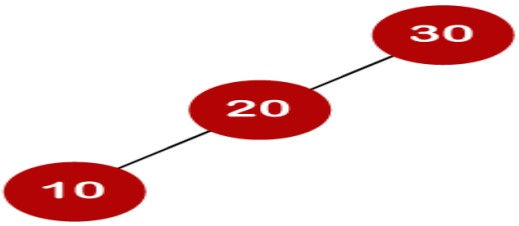
In the above tree, the average number of comparisons that can be made as:

$$average\ number\ of\ comparisons = \frac{1+2+2}{3} = 5/3$$



In the above tree, the total number of comparisons can be made as 3. Therefore, the average number of comparisons that can be made as:

$$average\ number\ of\ comparisons = \frac{1+2+3}{3} = 2$$



In the above tree, the total number of comparisons can be made as 3. Therefore, the average number of comparisons that can be made as:

$$average\ number\ of\ comparisons = \frac{1+2+3}{3} = 2$$

In the third case, the number of comparisons is less because the height of the tree is less, so it's a balanced binary search tree.

Till now, we read about the height-balanced binary search tree. To find the optimal binary search tree, we will determine the frequency of searching a key.

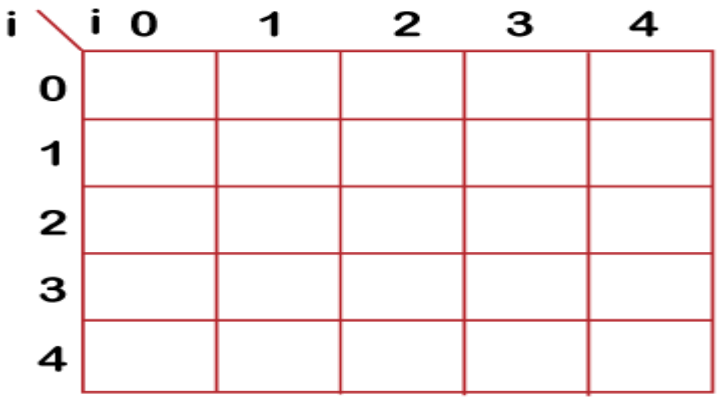
Let's assume that frequencies associated with the keys 10, 20, 30 are 3, 2, 5.

The above trees have different frequencies. The tree with the lowest frequency would be considered the optimal binary search tree. The tree with the frequency 17 is the lowest, so it would be considered as the optimal binary search tree.

Dynamic Approach

Consider the below table, which contains the keys and frequencies.

	1	2	3	4
Keys →	10	20	30	40
Frequency →	4	2	6	3



First, we will calculate the values where j-i is equal to zero.

When i=0, j=0, then j-i = 0

When i = 1, j=1, then j-i = 0

When i = 2, j=2, then j-i = 0

When i = 3, j=3, then j-i = 0

When i = 4, j=4, then j-i = 0

Therefore, c[0, 0] = 0, c[1 , 1] = 0, c[2,2] = 0, c[3,3] = 0, c[4,4] = 0

Now we will calculate the values where j-i equal to 1.

When j=1, i=0 then j-i = 1

When j=2, i=1 then j-i = 1

When j=3, i=2 then j-i = 1

When j=4, i=3 then j-i = 1

Now to calculate the cost, we will consider only the jth value.

The cost of c[0,1] is 4 (The key is 10, and the cost corresponding to key 10 is 4).

The cost of c[1,2] is 2 (The key is 20, and the cost corresponding to key 20 is 2).

The cost of c[2,3] is 6 (The key is 30, and the cost corresponding to key 30 is 6)

The cost of c[3,4] is 3 (The key is 40, and the cost corresponding to key 40 is 3)

	0	1	2	3	4
0	0	4			
1		0	2		
2			0	6	
3				0	3
4					0

Now we will calculate the values where j-i = 2

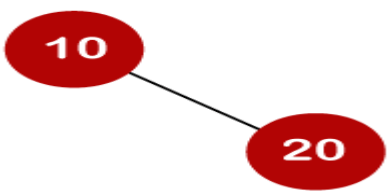
When j=2, i=0 then j-i = 2

When j=3, i=1 then j-i = 2

When j=4, i=2 then j-i = 2

In this case, we will consider two keys.

- When i=0 and j=2, then keys 10 and 20. There are two possible trees that can be made out from these two keys shown below:



In the first binary tree, cost would be: 4*1 + 2*2 = 8

In the second binary tree, cost would be: 4*2 + 2*1 = 10

The minimum cost is 8; therefore, c[0,2] = 8

	0	1	2	3	4
0	0	4	8		
1		0	2		
2			0	6	
3				0	3
4					0

- When i=1 and j=3, then keys 20 and 30. There are two possible trees that can be made out from these two keys shown below:

In the first binary tree, cost would be: 1*2 + 2*6 = 14

In the second binary tree, cost would be: 1*6 + 2*2 = 10

The minimum cost is 10; therefore, c[1,3] = 10

- When i=2 and j=4, we will consider the keys at 3 and 4, i.e., 30 and 40. There are two possible trees that can be made out from these two keys shown as below:

In the first binary tree, cost would be: 1*6 + 2*3 = 12

In the second binary tree, cost would be: 1*3 + 2*6 = 15

The minimum cost is 12, therefore, c[2,4] = 12

i \ 1	0	1	2	3	4
0	0	4	8^{1}		
1		0	2	10^{3}	
2			0	6	12^{3}
3				0	3
4					0

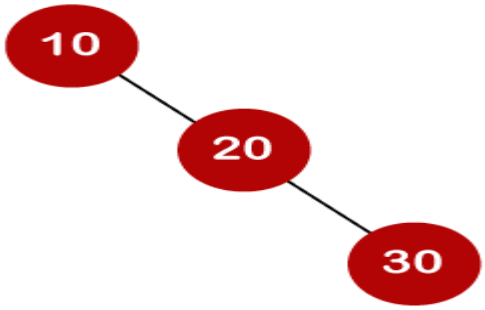
Now we will calculate the values when j-i = 3

When j=3, i=0 then j-i = 3

When j=4, i=1 then j-i = 3

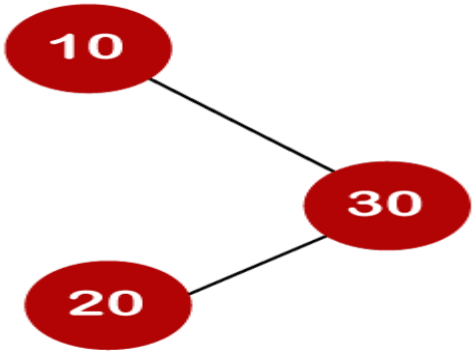
- When i=0, j=3 then we will consider three keys, i.e., 10, 20, and 30.

The following are the trees that can be made if 10 is considered as a root node.



In the above tree, 10 is the root node, 20 is the right child of node 10, and 30 is the right child of node 20.

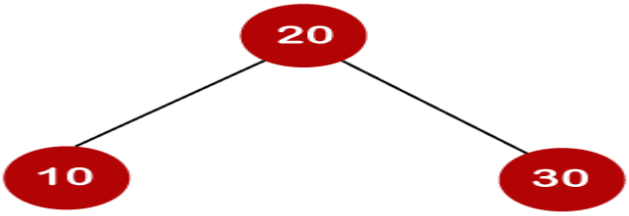
Cost would be: 1*4 + 2*2 + 3*6 = 26



In the above tree, 10 is the root node, 30 is the right child of node 10, and 20 is the left child of node 20.

Cost would be: 1*4 + 2*6 + 3*2 = 22

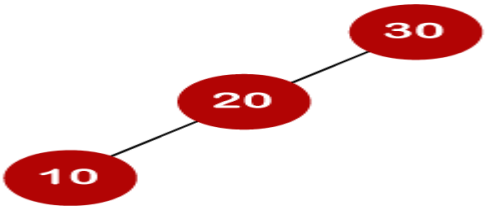
The following tree can be created if 20 is considered as the root node.



In the above tree, 20 is the root node, 30 is the right child of node 20, and 10 is the left child of node 20.

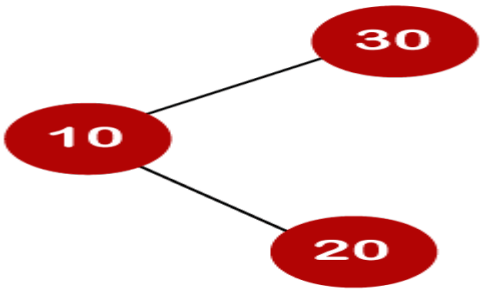
Cost would be: 1*2 + 4*2 + 6*2 = 22

The following are the trees that can be created if 30 is considered as the root node.



In the above tree, 30 is the root node, 20 is the left child of node 30, and 10 is the left child of node 20.

Cost would be: 1*6 + 2*2 + 3*4 = 22



In the above tree, 30 is the root node, 10 is the left child of node 30 and 20 is the right child of node 10.

Cost would be: 1*6 + 2*4 + 3*2 = 20

Therefore, the minimum cost is 20 which is the 3rd root. So, c[0,3] is equal to 20.

- When i=1 and j=4 then we will consider the keys 20, 30, 40

c[1,4] = min{ c[1,1] + c[2,4], c[1,2] + c[3,4], c[1,3] + c[4,4] } + 11

= min{0+12, 2+3, 10+0}+ 11

= min{12, 5, 10} + 11

The minimum value is 5; therefore, c[1,4] = 5+11 = 16

i \ j	0	1	2	3	4
0	0	4	8^1	20^3	
1		0	2	10^3	16^3
2			0	6	12^3
3				0	3
4					0

- **Now we will calculate the values when j-i = 4**

When j=4 and i=0 then j-i = 4

In this case, we will consider four keys, i.e., 10, 20, 30 and 40. The frequencies of 10, 20, 30 and 40 are 4, 2, 6 and 3 respectively.

w[0, 4] = 4 + 2 + 6 + 3 = 15

If we consider 10 as the root node then

C[0, 4] = min {c[0,0] + c[1,4]}+ w[0,4]

= min {0 + 16} + 15= 31

If we consider 20 as the root node then

C[0,4] = min{c[0,1] + c[2,4]} + w[0,4]

= min{4 + 12} + 15

= 16 + 15 = 31

If we consider 30 as the root node then,

C[0,4] = min{c[0,2] + c[3,4]} +w[0,4]

= min {8 + 3} + 15

= 26

If we consider 40 as the root node then,

C[0,4] = min{c[0,3] + c[4,4]} + w[0,4]

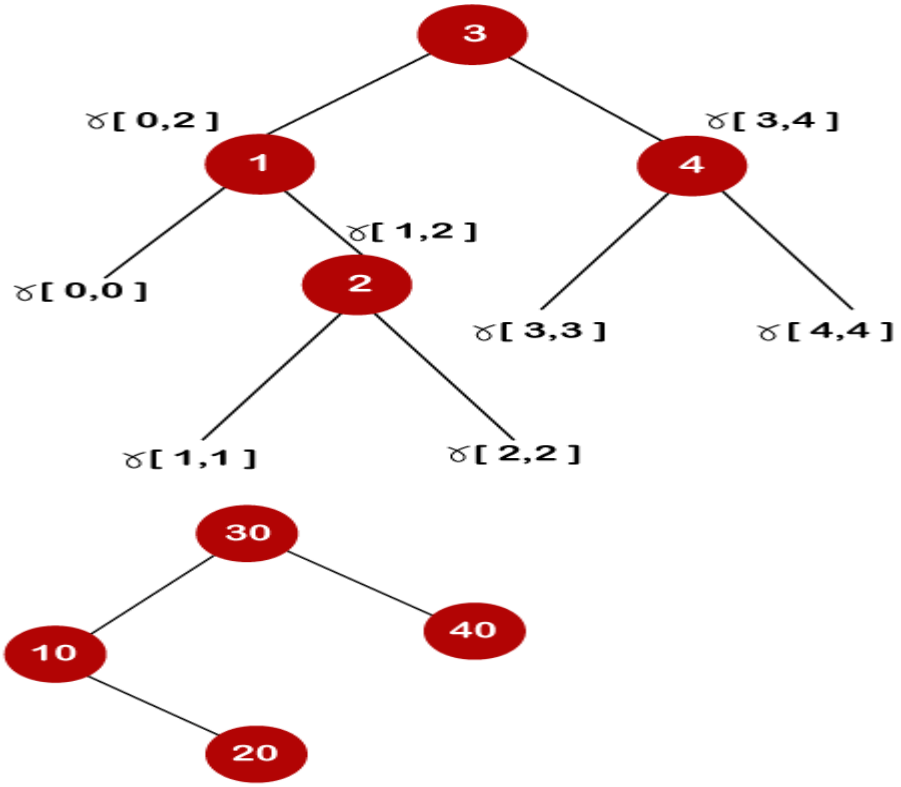
= min{20 + 0} + 15

= 35

In the above cases, we have observed that 26 is the minimum cost; therefore, c[0,4] is equal to 26.

i \ j	0	1	2	3	4
0	0	4	8^1	20^3	26^3
1		0	2	10^3	16^3
2			0	6	12^3
3				0	3
4					0

The optimal binary tree can be created as:



General formula for calculating the minimum cost is:

C[i,j] = min{c[i, k-1] + c[k,j]} + w(i,j)

The algorithm for optimal binary search tree is specified below :

```
Algorithm OBST(p, q, n)

// e[1…n+1, 0…n ] : Optimal sub tree

// w[1…n+1,  0…n] : Sum of probability

// root[1…n, 1…n] : Used to construct OBST



for i ← 1 to n + 1 do

   e[i, i – 1] ← qi – 1

   w[i, i – 1] ← qi – 1

end



for m ← 1 to n do

   for i ← 1 to n – m + 1 do

      j ← i + m – 1

      e[i, j] ← ∞

      w[i, j] ← w[i, j – 1] + pj + qj

      for r ← i to j do

         t ← e[i, r – 1] + e[r + 1, j] + w[i, j]

         if t < e[i, j] then

            e[i, j] ← t

            root[i, j] ← r

         end

      end

   end

end

return (e, root)
```

Complexity Analysis of Optimal Binary Search Tree

It is very simple to derive the complexity of this approach from the above algorithm. It uses three nested loops. Statements in the innermost loop run in Q(1) time. The running time of the algorithm is computed as

Thus, the OBST algorithm runs in cubic time.

MULTISTAGE GRAPH

A multistage graph **G = (V, E)** is a directed graph where vertices are partitioned into **k** (where ***k* > 1**) number of disjoint subsets ***S* = *{s1,s2,…,sk}*** such that edge *(u, v)* is in E, then *u* $\in$ s_i and *v* $\in$ s_{1+1} for some subsets in the partition and $|s_1| = |s_k| = 1$.

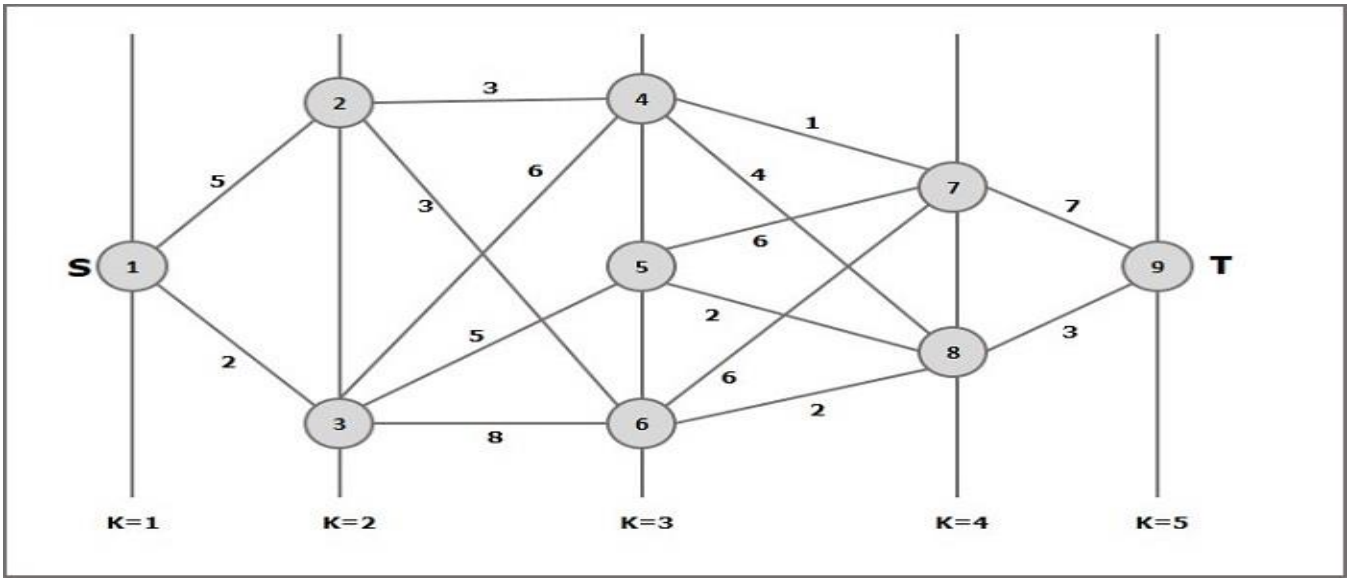
The vertex ***s* $\in$ s_1** is called the **source** and the vertex ***t* $\in$ s_k** is called **sink**.

G is usually assumed to be a weighted graph. In this graph, cost of an edge *(i, j)* is represented by *c(i, j)*. Hence, the cost of path from source ***s*** to sink ***t*** is the sum of costs of each edges in this path.

The multistage graph problem is finding the path with minimum cost from source ***s*** to sink ***t***.

Example

Consider the following example to understand the concept of multistage graph.



According to the formula, we have to calculate the cost **(i, j)** using the following steps

Step 1: Cost (K-2, j)

In this step, three nodes (node 4, 5. 6) are selected as **j**. Hence, we have three options to choose the minimum cost at this step.

Cost(3, 4) = min {c(4, 7) + Cost(7, 9),c(4, 8) + Cost(8, 9)} = 7

Cost(3, 5) = min {c(5, 7) + Cost(7, 9),c(5, 8) + Cost(8, 9)} = 5

Cost(3, 6) = min {c(6, 7) + Cost(7, 9),c(6, 8) + Cost(8, 9)} = 5

Step 2: Cost (K-3, j)

Two nodes are selected as j because at stage k - 3 = 2 there are two nodes, 2 and 3. So, the value i = 2 and j = 2 and 3.

Cost(2, 2) = min {c(2, 4) + Cost(4, 8) + Cost(8, 9),c(2, 6) +

Cost(6, 8) + Cost(8, 9)} = 8

Cost(2, 3) = {c(3, 4) + Cost(4, 8) + Cost(8, 9), c(3, 5) + Cost(5, 8)+ Cost(8, 9), c(3, 6) + Cost(6, 8) + Cost(8, 9)} = 10

Step 3: Cost (K-4, j)

Cost (1, 1) = {c(1, 2) + Cost(2, 6) + Cost(6, 8) + Cost(8, 9), c(1, 3) + Cost(3, 5) + Cost(5, 8) + Cost(8, 9))} = 12

c(1, 3) + Cost(3, 6) + Cost(6, 8 + Cost(8, 9))} = 13

Hence, the path having the minimum cost is **1→ 3→ 5→ 8→ 9**.

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The vertex ***s* Є *s1*** is called the **source** and the vertex ***t* Є *sk*** is called **sink**.

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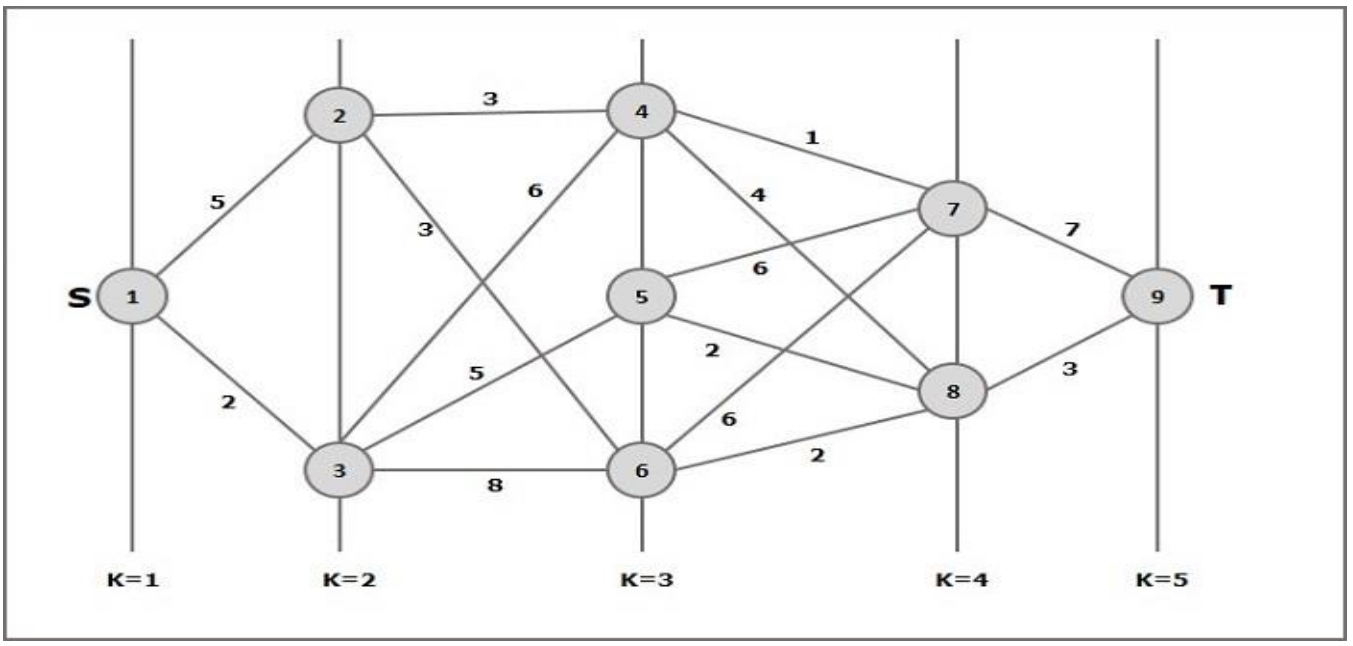

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Reliability Design