harinris_Homework3.rmd

Harin Rishabh

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Exhibit 4.2

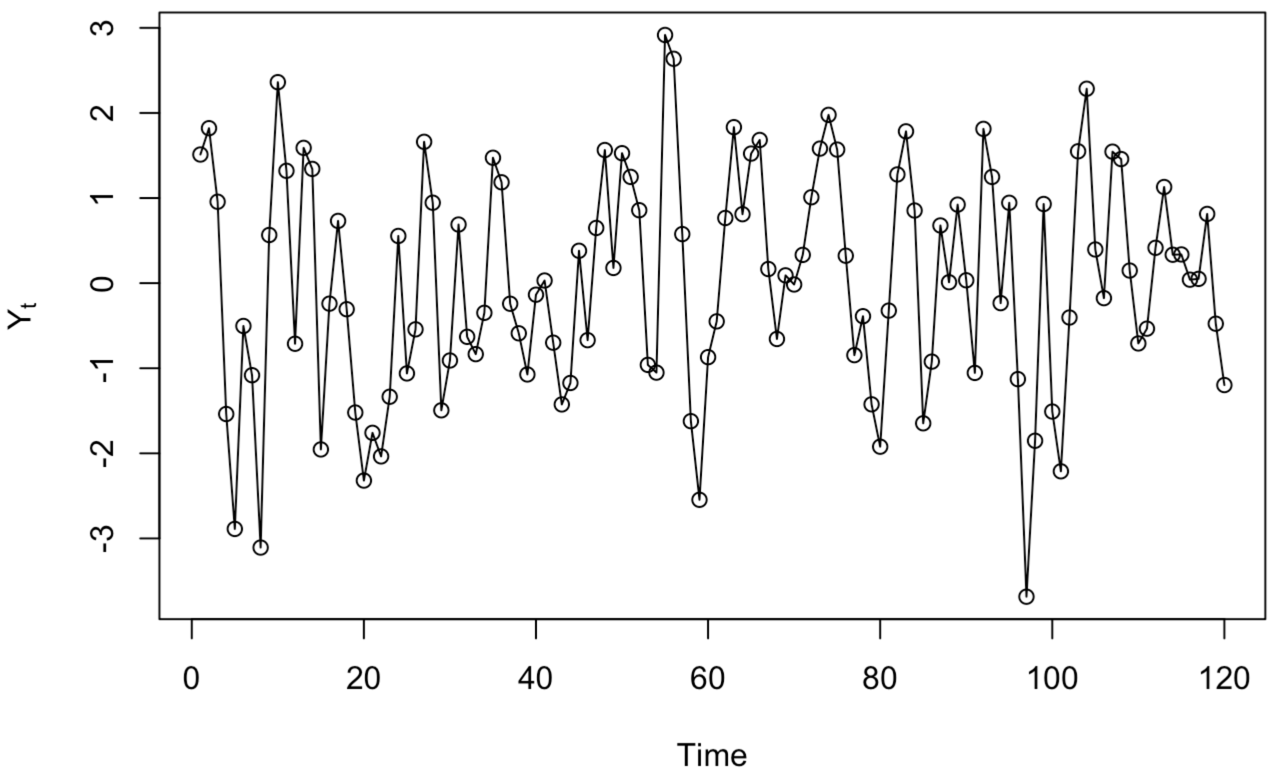
```
library(TSA)
```

```
##
## Attaching package: 'TSA'
```

```
## The following objects are masked from 'package:stats':
##
##     acf, arima
```

```
## The following object is masked from 'package:utils':
##
##     tar
```

```
data(ma1.2.s); plot(ma1.2.s,ylab=expression(Y[t]),type='o')
```



```
set.seed(12345)
y=arima.sim(model=list(ma=-c(-0.9)),n=100)
```

```
list1=list(a=c(1,2,3),b=4,c=ts(c(5,6,7,8), start=c(2006,2),frequency=4))
list1
```

```
## $a
## [1] 1 2 3
##
## $b
## [1] 4
##
## $c
##      Qtr1 Qtr2 Qtr3 Qtr4
## 2006         5    6    7
## 2007    8
```

```
list1$c
```

```
##      Qtr1 Qtr2 Qtr3 Qtr4
## 2006         5    6    7
## 2007    8
```

```
str(list1)
```

```
## List of 3
##  $ a: num [1:3] 1 2 3
##  $ b: num 4
##  $ c: Time-Series [1:4] from 2006 to 2007: 5 6 7 8
```

Exhibit 5.1

```
data(oil.price)
plot(oil.price, ylab='Price per Barrel',type='l')
```

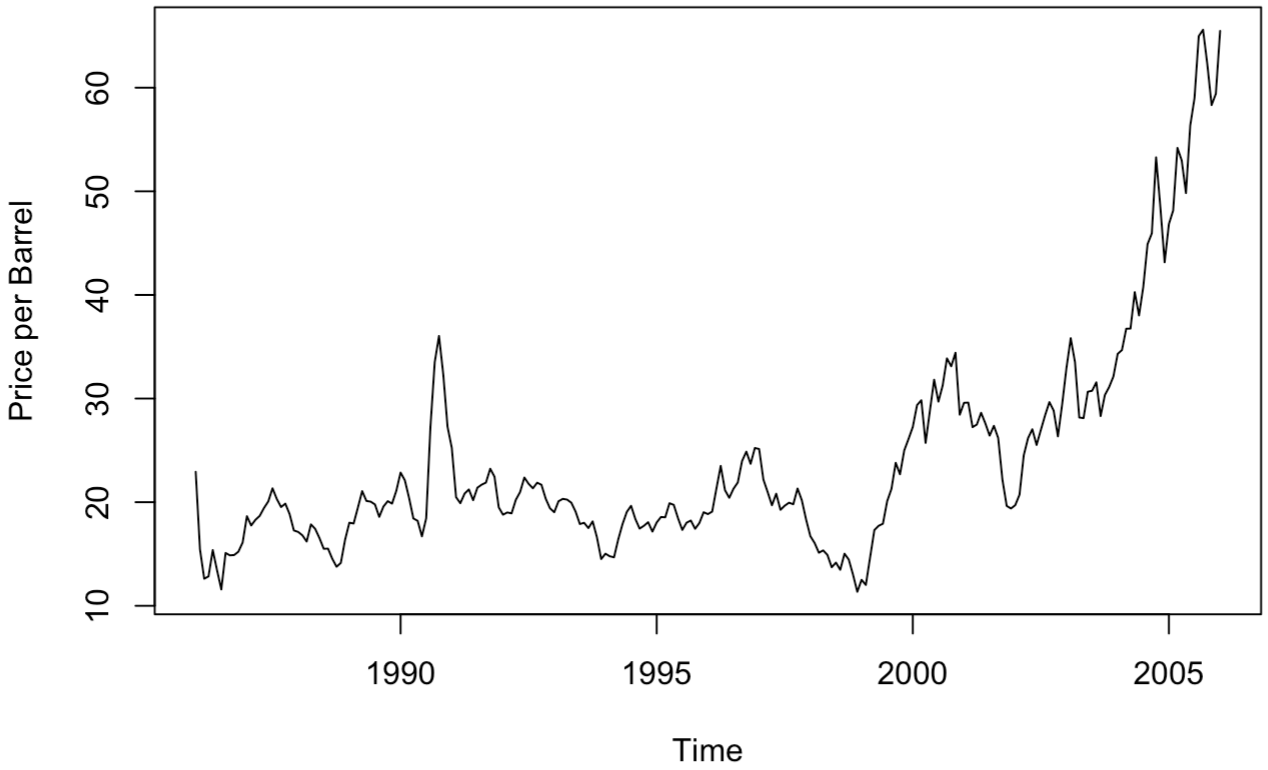
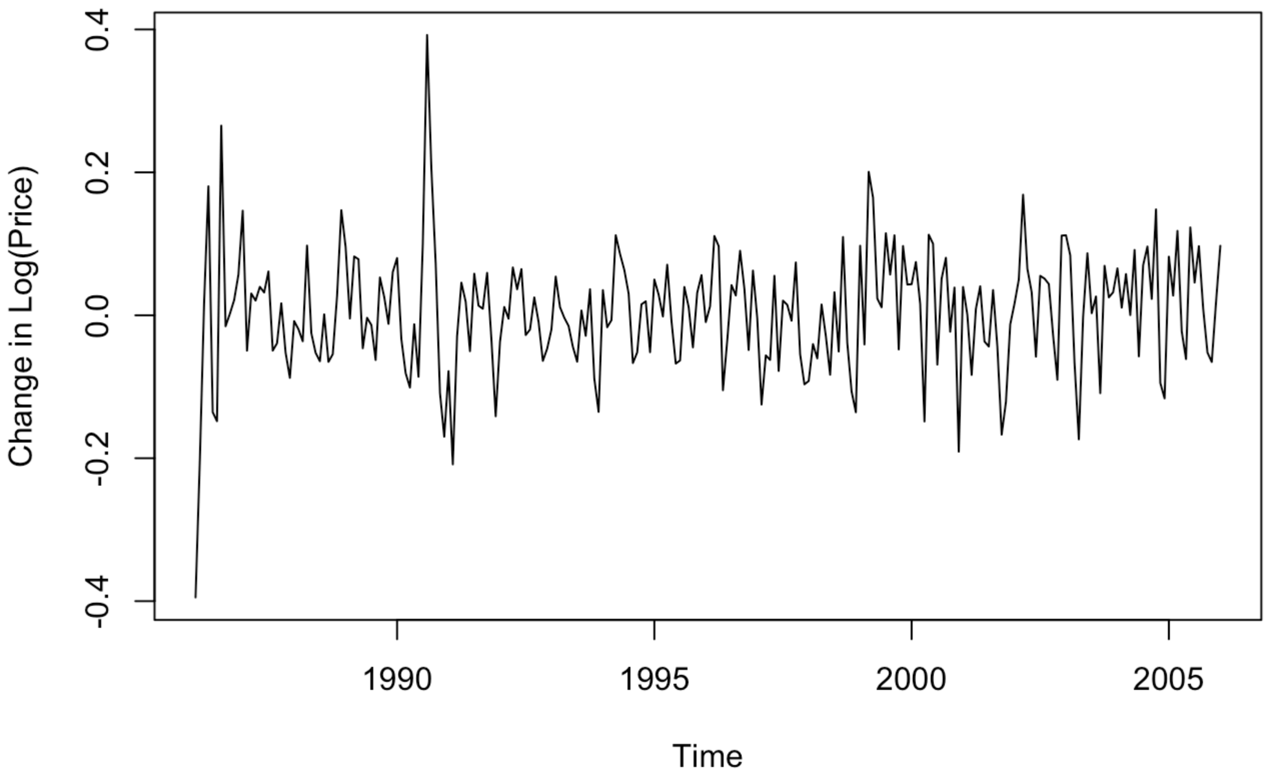


Exhibit 5.4

```
plot(diff(log(oil.price)),ylab='Change in Log(Price)',type='l')
```




```
diff(log(oil.price), differences=2)
```

```
##               Jan           Feb           Mar           Apr           May
## 1986                              0.1917182402  0.2211940017  0.1624275175
## 1987  0.0895761041 -0.1958665791  0.0799761741 -0.0099631710  0.0193269933
## 1988  0.0794024419 -0.0113128461 -0.0169152182  0.1339199774 -0.1224969372
## 1989 -0.0516322442 -0.0998705693  0.0868048371 -0.0038935120 -0.1245978122
## 1990  0.0195502360 -0.1134742814 -0.0476267468 -0.0200794026  0.0885066353
## 1991  0.0917216884 -0.1304648922  0.1798558940  0.0744035920 -0.0266535031
## 1992  0.1042318316  0.0487299952 -0.0163859360  0.0716926804 -0.0305441358
## 1993  0.0270292776  0.0739772134 -0.0428220536 -0.0148342730 -0.0114748211
## 1994  0.1703769936 -0.0519834251  0.0099843959  0.1188029732 -0.0262722873
## 1995  0.1016824311 -0.0210545598 -0.0305726760  0.0724059868 -0.0788618693
## 1996 -0.0657044008  0.0221554913  0.0982981709 -0.0140640061 -0.2017737028
## 1997 -0.0665301256 -0.1208999585  0.0687732326 -0.0063760722  0.1177696366
## 1998  0.0047149163  0.0516595551 -0.0200394386  0.0754104410 -0.0441804485
## 1999  0.2330421941 -0.1380992689  0.2415350755 -0.0359480411 -0.1413887700
## 2000  0.0004256225  0.0310682070 -0.0586772888 -0.1644569061  0.2613400812
## 2001  0.2304810473 -0.0389641858 -0.0841013365  0.0925614817  0.0314969960
## 2002  0.0291794856  0.0336046443  0.1188241447 -0.1036985387 -0.0327774115
## 2003  0.0003927139 -0.0281635795 -0.1507358999 -0.1066446903  0.1714541229
## 2004  0.0333858215 -0.0549203887  0.0469766328 -0.0574307820  0.0911964033
## 2005  0.1983177535 -0.0544715217  0.0905916518 -0.1407571571 -0.0387153658
## 2006  0.0787647155
##               Jun           Jul           Aug           Sep           Oct
## 1986 -0.3160796194 -0.0126345848  0.4136268101 -0.2807642550  0.0173644358
## 1987 -0.0079860025  0.0294637193 -0.1108269533  0.0103082396  0.0559175444
## 1988 -0.0274974556 -0.0118948283  0.0656263787 -0.0665155335  0.0108148833
## 1989  0.0426506911 -0.0100726361 -0.0490277653  0.1155190914 -0.0272329155
## 1990 -0.0734546971  0.1856685257  0.2925329139 -0.1875975987 -0.1318054353
## 1991 -0.0692488638  0.1084313331 -0.0447430869 -0.0042818161  0.0502360486
## 1992  0.0281951481 -0.0922329276  0.0076851983  0.0449393611 -0.0341725327
## 1993 -0.0291388554 -0.0208582346  0.0716080019 -0.0354115399  0.0651959580
## 1994 -0.0223917928 -0.0328645606 -0.0972996615  0.0148907531  0.0672777648
## 1995 -0.0595099658  0.0043801962  0.1028228164 -0.0280340125 -0.0564620657
## 1996  0.0692900199  0.0777900542 -0.0144126959  0.0625368475 -0.0530551922
## 1997 -0.1331790843  0.0984393652 -0.0059126797 -0.0221902343  0.0815108672
## 1998 -0.0540941612  0.1154499379 -0.0829344947  0.1602452766 -0.1482452004
## 1999 -0.0121861136  0.1035789451 -0.0577545581  0.0548700668 -0.1596791981
## 2000 -0.0126922870 -0.1690148954  0.1201402563  0.0292932656 -0.1034750630
## 2001 -0.0772721633 -0.0070551272  0.0790206976 -0.0790143362 -0.1233312749
## 2002 -0.0901762834  0.1131174714 -0.0039508453 -0.0075494671 -0.0717983905
## 2003  0.0889656855 -0.0839023677  0.0233861875 -0.1353095118  0.1782440782
## 2004 -0.1489629870  0.1275738659  0.0261666015 -0.0733476271  0.1253264265
## 2005  0.1842622997 -0.0771793307  0.0509258947 -0.0873678772 -0.0614477954
## 2006
##               Nov           Dec
## 1986  0.0192336870  0.0355807053
## 1987 -0.0689506585 -0.0353472770
## 1988  0.0809265071  0.1205380702
## 1989 -0.0377127396  0.0725775383
## 1990 -0.1814193720 -0.0612079178
## 1991 -0.0931232327 -0.1076128863
## 1992 -0.0546179850  0.0169997925
## 1993 -0.1251353169 -0.0465012196
## 1994  0.0042048629 -0.0712311700
## 1995  0.0764989176  0.0245774449
```

```
## 1996 -0.0858504392  0.1111479342
## 1997 -0.1284218286 -0.0421902035
## 1998 -0.0677748722 -0.0292947543
## 1999  0.1447128080 -0.0538920366
## 2000  0.0617919574 -0.2296436251
## 2001  0.0458476664  0.1083608870
## 2002 -0.0622590131  0.2018601735
## 2003 -0.0441895378  0.0071985252
## 2004 -0.2428408053 -0.0216466560
## 2005 -0.0132699970  0.0838915520
## 2006
```

```
diff(log(oil.price),diff=2)
```

```
##               Jan           Feb           Mar           Apr           May
## 1986                              0.1917182402  0.2211940017  0.1624275175
## 1987  0.0895761041 -0.1958665791  0.0799761741 -0.0099631710  0.0193269933
## 1988  0.0794024419 -0.0113128461 -0.0169152182  0.1339199774 -0.1224969372
## 1989 -0.0516322442 -0.0998705693  0.0868048371 -0.0038935120 -0.1245978122
## 1990  0.0195502360 -0.1134742814 -0.0476267468 -0.0200794026  0.0885066353
## 1991  0.0917216884 -0.1304648922  0.1798558940  0.0744035920 -0.0266535031
## 1992  0.1042318316  0.0487299952 -0.0163859360  0.0716926804 -0.0305441358
## 1993  0.0270292776  0.0739772134 -0.0428220536 -0.0148342730 -0.0114748211
## 1994  0.1703769936 -0.0519834251  0.0099843959  0.1188029732 -0.0262722873
## 1995  0.1016824311 -0.0210545598 -0.0305726760  0.0724059868 -0.0788618693
## 1996 -0.0657044008  0.0221554913  0.0982981709 -0.0140640061 -0.2017737028
## 1997 -0.0665301256 -0.1208999585  0.0687732326 -0.0063760722  0.1177696366
## 1998  0.0047149163  0.0516595551 -0.0200394386  0.0754104410 -0.0441804485
## 1999  0.2330421941 -0.1380992689  0.2415350755 -0.0359480411 -0.1413887700
## 2000  0.0004256225  0.0310682070 -0.0586772888 -0.1644569061  0.2613400812
## 2001  0.2304810473 -0.0389641858 -0.0841013365  0.0925614817  0.0314969960
## 2002  0.0291794856  0.0336046443  0.1188241447 -0.1036985387 -0.0327774115
## 2003  0.0003927139 -0.0281635795 -0.1507358999 -0.1066446903  0.1714541229
## 2004  0.0333858215 -0.0549203887  0.0469766328 -0.0574307820  0.0911964033
## 2005  0.1983177535 -0.0544715217  0.0905916518 -0.1407571571 -0.0387153658
## 2006  0.0787647155
##               Jun           Jul           Aug           Sep           Oct
## 1986 -0.3160796194 -0.0126345848  0.4136268101 -0.2807642550  0.0173644358
## 1987 -0.0079860025  0.0294637193 -0.1108269533  0.0103082396  0.0559175444
## 1988 -0.0274974556 -0.0118948283  0.0656263787 -0.0665155335  0.0108148833
## 1989  0.0426506911 -0.0100726361 -0.0490277653  0.1155190914 -0.0272329155
## 1990 -0.0734546971  0.1856685257  0.2925329139 -0.1875975987 -0.1318054353
## 1991 -0.0692488638  0.1084313331 -0.0447430869 -0.0042818161  0.0502360486
## 1992  0.0281951481 -0.0922329276  0.0076851983  0.0449393611 -0.0341725327
## 1993 -0.0291388554 -0.0208582346  0.0716080019 -0.0354115399  0.0651959580
## 1994 -0.0223917928 -0.0328645606 -0.0972996615  0.0148907531  0.0672777648
## 1995 -0.0595099658  0.0043801962  0.1028228164 -0.0280340125 -0.0564620657
## 1996  0.0692900199  0.0777900542 -0.0144126959  0.0625368475 -0.0530551922
## 1997 -0.1331790843  0.0984393652 -0.0059126797 -0.0221902343  0.0815108672
## 1998 -0.0540941612  0.1154499379 -0.0829344947  0.1602452766 -0.1482452004
## 1999 -0.0121861136  0.1035789451 -0.0577545581  0.0548700668 -0.1596791981
## 2000 -0.0126922870 -0.1690148954  0.1201402563  0.0292932656 -0.1034750630
## 2001 -0.0772721633 -0.0070551272  0.0790206976 -0.0790143362 -0.1233312749
## 2002 -0.0901762834  0.1131174714 -0.0039508453 -0.0075494671 -0.0717983905
## 2003  0.0889656855 -0.0839023677  0.0233861875 -0.1353095118  0.1782440782
## 2004 -0.1489629870  0.1275738659  0.0261666015 -0.0733476271  0.1253264265
## 2005  0.1842622997 -0.0771793307  0.0509258947 -0.0873678772 -0.0614477954
## 2006
##               Nov           Dec
## 1986  0.0192336870  0.0355807053
## 1987 -0.0689506585 -0.0353472770
## 1988  0.0809265071  0.1205380702
## 1989 -0.0377127396  0.0725775383
## 1990 -0.1814193720 -0.0612079178
## 1991 -0.0931232327 -0.1076128863
## 1992 -0.0546179850  0.0169997925
## 1993 -0.1251353169 -0.0465012196
## 1994  0.0042048629 -0.0712311700
## 1995  0.0764989176  0.0245774449
```

```
## 1996 -0.0858504392  0.1111479342
## 1997 -0.1284218286 -0.0421902035
## 1998 -0.0677748722 -0.0292947543
## 1999  0.1447128080 -0.0538920366
## 2000  0.0617919574 -0.2296436251
## 2001  0.0458476664  0.1083608870
## 2002 -0.0622590131  0.2018601735
## 2003 -0.0441895378  0.0071985252
## 2004 -0.2428408053 -0.0216466560
## 2005 -0.0132699970  0.0838915520
## 2006
```

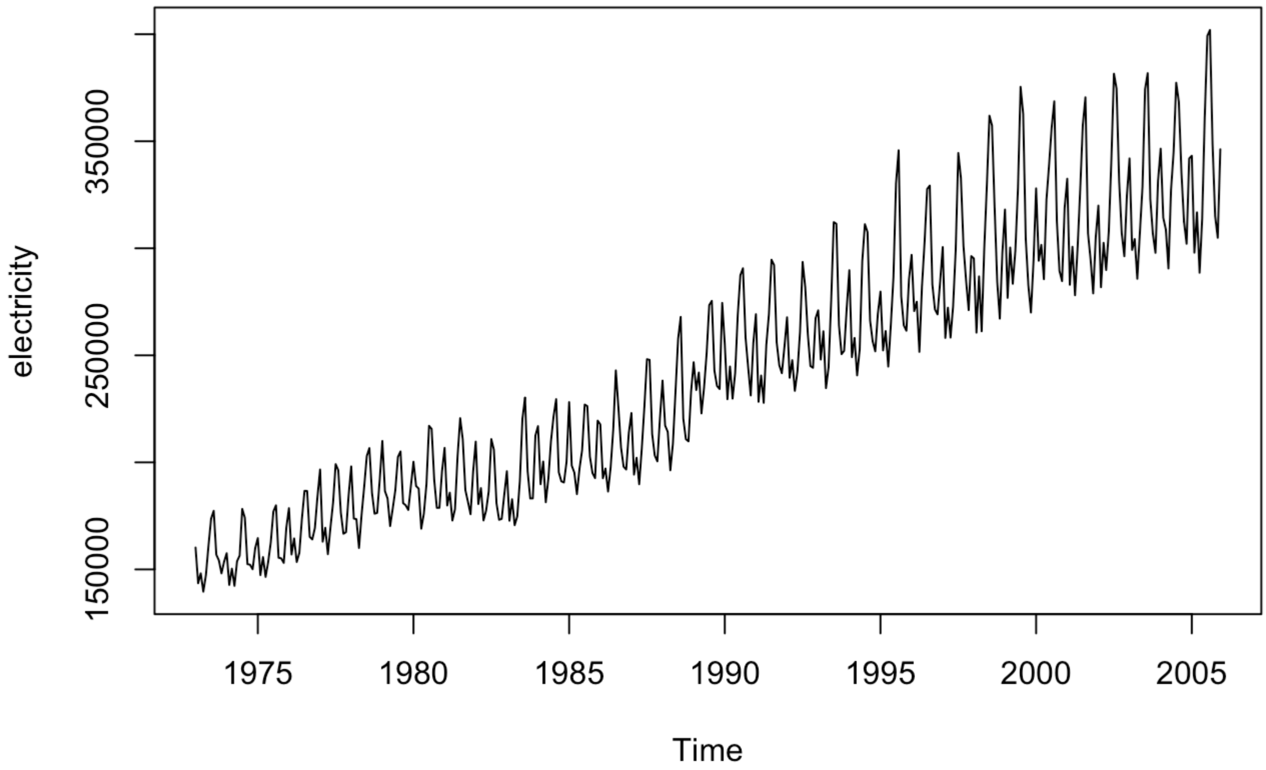
```
data("tempdub")
diff(tempdub, lag=12)
```

```
##        Jan   Feb   Mar   Apr   May   Jun   Jul   Aug   Sep   Oct   Nov   Dec
## 1965  -8.6  -6.6  -6.4  -2.1  -1.6  -2.0  -1.6   0.5  -0.9   2.4  -2.2  11.1
## 1966  -5.7   2.5  13.2  -0.7  -8.1   1.5   1.6  -0.2   0.5  -0.7  -0.2  -6.5
## 1967  11.1  -6.9  -2.4   3.6   0.8   0.2  -4.1  -2.5   0.1  -1.1  -4.0   1.4
## 1968  -2.4   5.9   5.2   1.7   1.3  -0.5   1.1   4.6  -0.2   1.6   2.6  -5.3
## 1969  -5.1   3.5 -10.8  -3.4   3.3  -5.5   1.4   1.4   1.3  -3.1  -1.6  -0.3
## 1970  -5.6  -5.1   2.0   2.1   3.0   5.9   0.1  -1.1   0.6   5.1   2.5   3.4
## 1971   2.8   1.0  -1.8  -1.0  -5.8   5.1  -4.2  -3.5   2.4   4.4   0.9   3.9
## 1972   2.2  -2.8   1.2  -4.0   6.5  -6.8   2.2   4.5  -2.8 -11.1  -4.9 -10.4
## 1973   9.1   8.5  11.5   1.5  -6.8   2.5   2.1   0.7   0.4   9.6   5.3   3.1
## 1974  -4.9  -5.2  -8.1   4.0  -0.7  -5.1   1.7  -5.2  -4.8  -4.8  -1.2   5.1
## 1975   2.8  -0.9  -9.6  -7.9   7.0   4.7  -2.0   4.0  -0.4   1.7   3.8   0.7
```

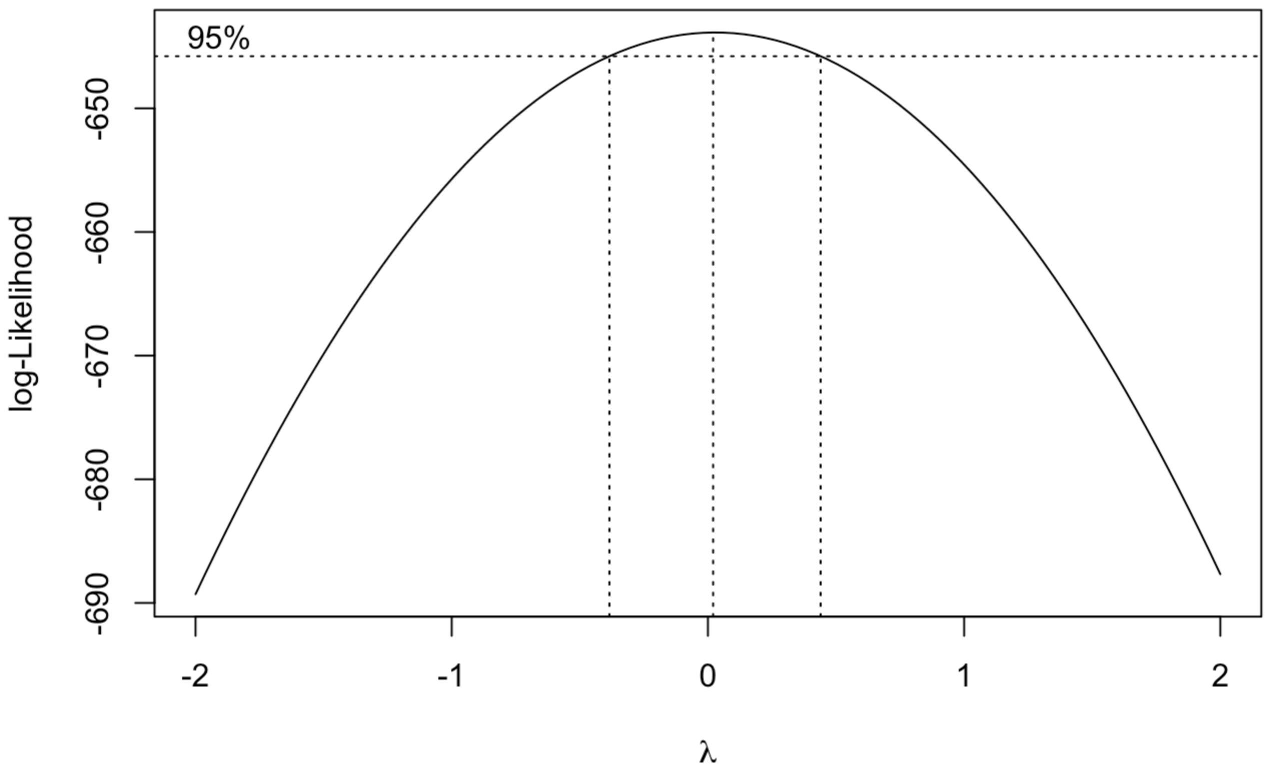
Exhibit 5.11

```
library(MASS)
```

```
data(electricity)
plot(electricity)
```




```
boxcox(lm(electricity~1))
```



Exercise 3

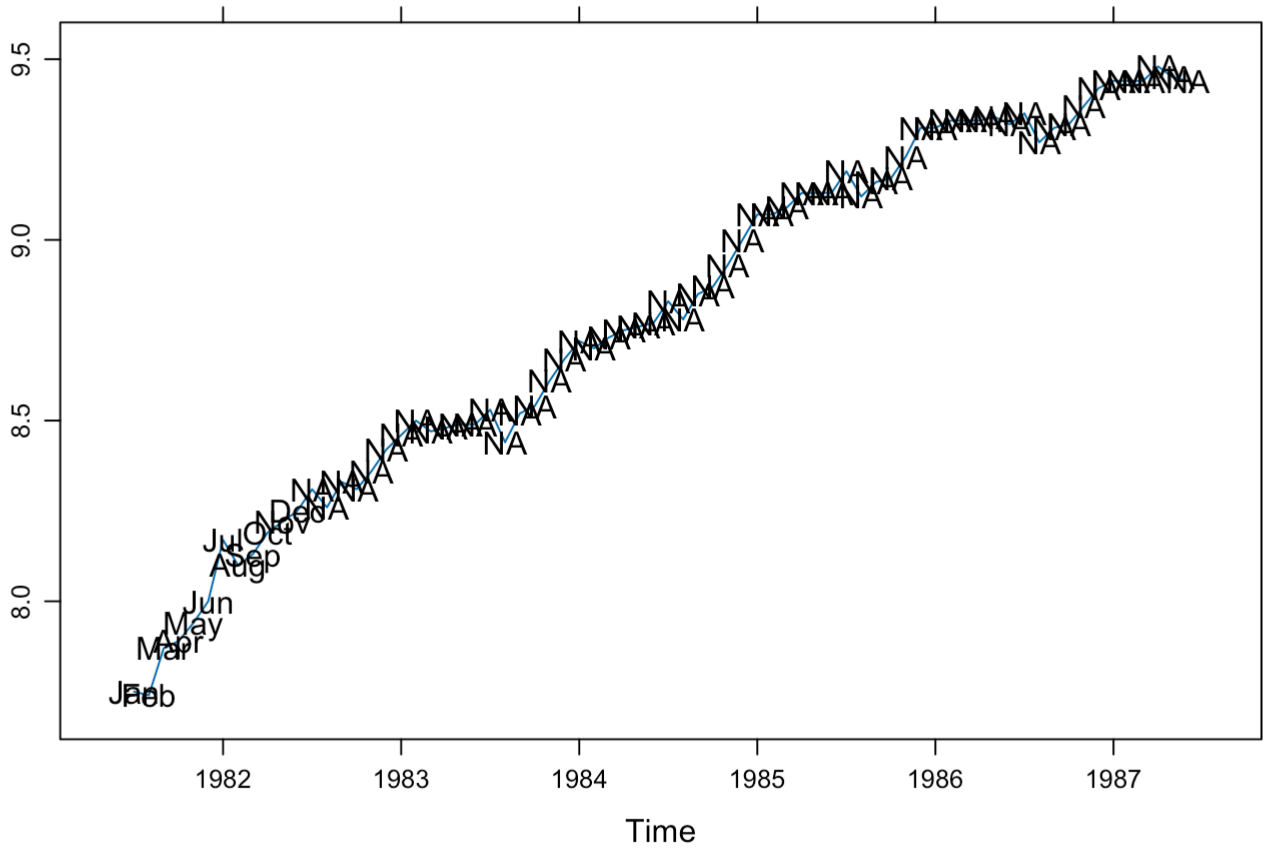
3.5 a)

Answer - this figure shows a clear, smooth, and cyclical seasonal trend. Values are genereally higher for the summer months and there seems to be an exponential increase long-term.

```
library(TSA)
library(lattice)
data("wages")

# Get the month names from the 'wages' time series
months <- month.abb[1:length(wages)]

# Plot the 'wages' time series with month labels
xyplot(wages, panel = function(x, y, ...) {
  panel.xyplot(x, y, ...)
  panel.text(x, y, labels = months)
})
```



3.5 b)

Answer - The monthly percentage difference series looks rather stationary.

```
wages_fit1 <- lm(wages ~ time(wages))
summary(wages_fit1)
```

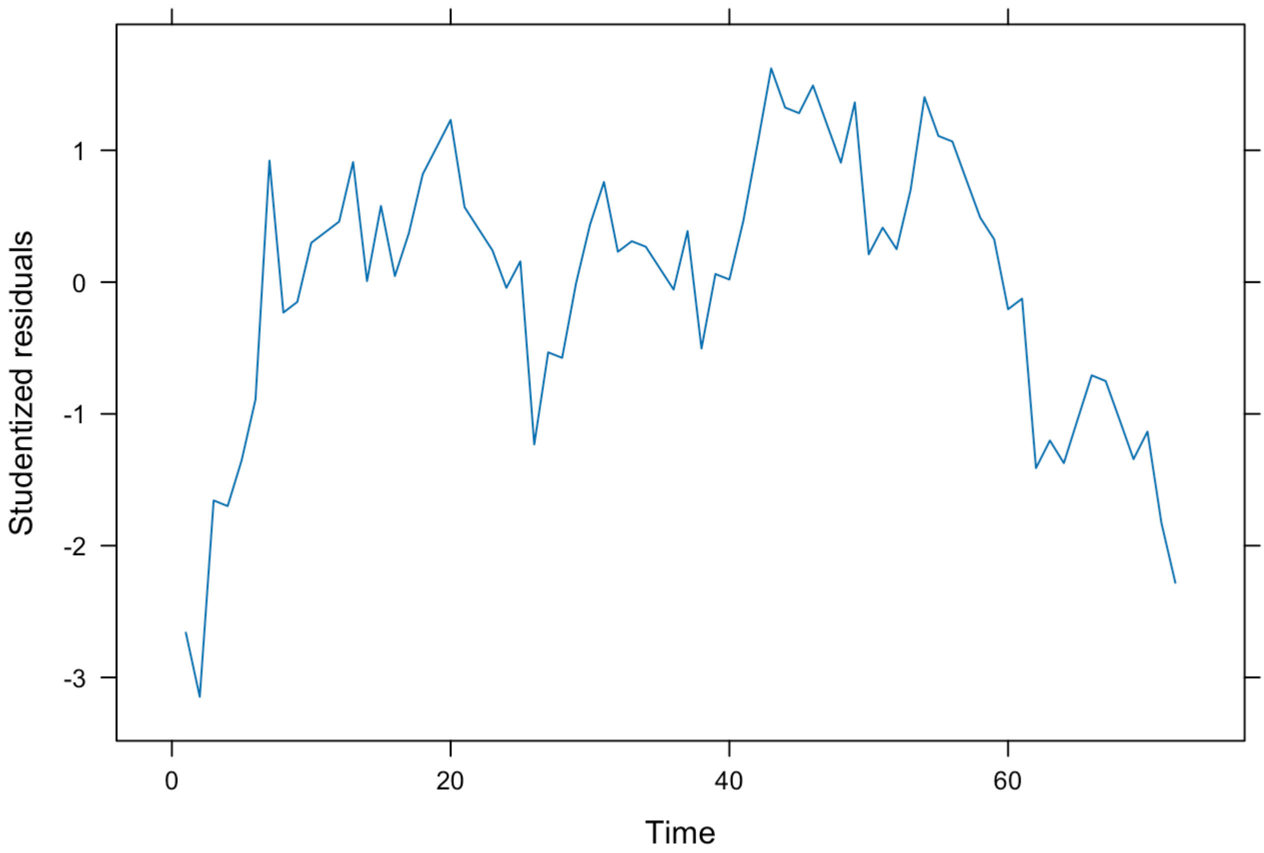
```
##
## Call:
## lm(formula = wages ~ time(wages))
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.23828 -0.04981  0.01942  0.05845  0.13136
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -5.490e+02  1.115e+01  -49.24   <2e-16 ***
## time(wages)  2.811e-01  5.618e-03   50.03   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.08257 on 70 degrees of freedom
## Multiple R-squared:  0.9728, Adjusted R-squared:  0.9724
## F-statistic:  2503 on 1 and 70 DF,  p-value: < 2.2e-16
```

```
wages_rst <- rstudent(wages_fit1)
```

3.5 c)

Answer - We still seem to have autocorrelation related to the time and not white noise.

```
xyplot(wages_rst ~ time(wages_rst), type = "l",
       xlab = "Time", ylab = "Studentized residuals")
```



3.5 d)

Answer - Fit a linear regression model with time and quadratic time term

```
wages_fit2 <- lm(wages ~ time(wages) + I(time(wages)^2))
summary(wages_fit2)
```

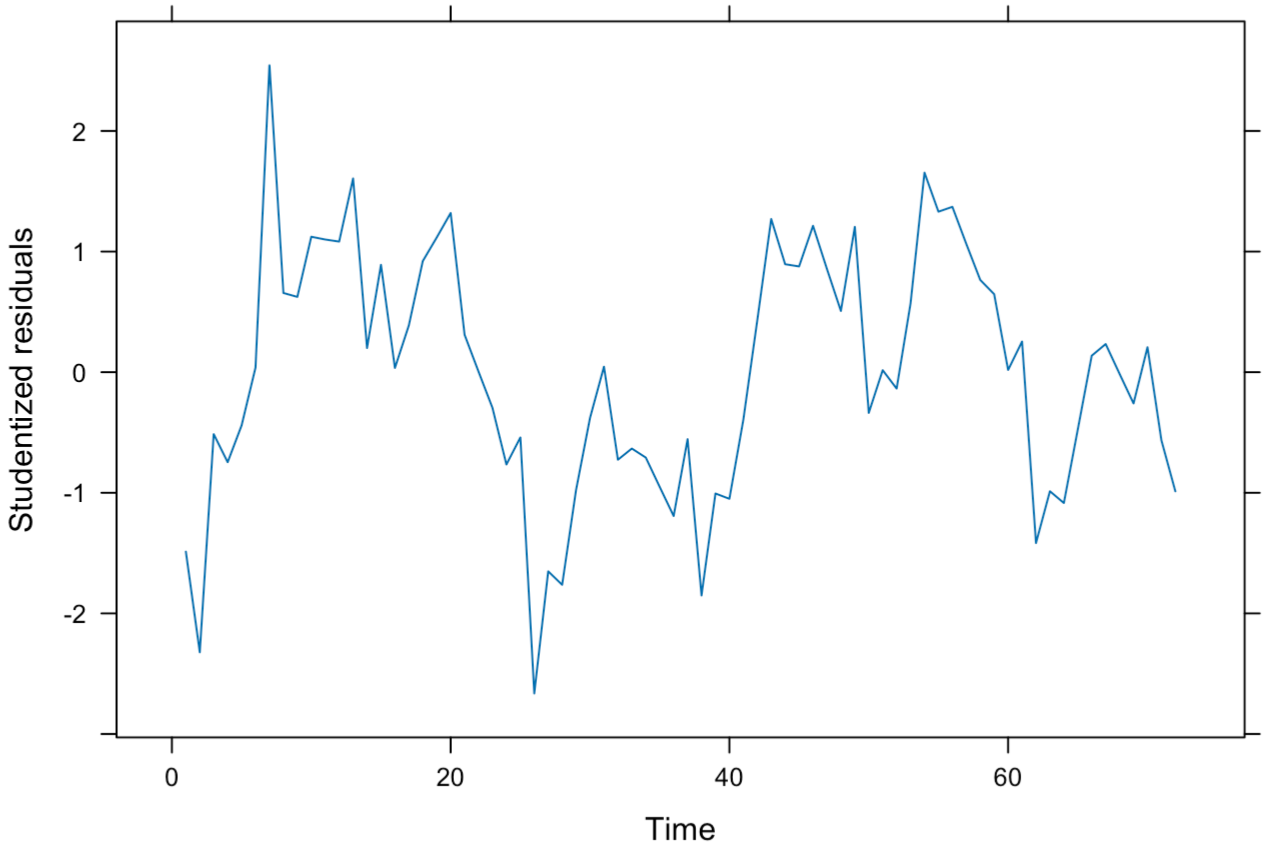
```
##
## Call:
## lm(formula = wages ~ time(wages) + I(time(wages)^2))
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.148318 -0.041440  0.001563  0.050089  0.139839
##
## Coefficients:
##                    Estimate Std. Error t value Pr(>|t|)
## (Intercept)      -8.495e+04  1.019e+04  -8.336 4.87e-12 ***
## time(wages)       8.534e+01  1.027e+01   8.309 5.44e-12 ***
## I(time(wages)^2) -2.143e-02  2.588e-03  -8.282 6.10e-12 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.05889 on 69 degrees of freedom
## Multiple R-squared:  0.9864, Adjusted R-squared:  0.986
## F-statistic:  2494 on 2 and 69 DF,  p-value: < 2.2e-16
```

```
wages_rst2 <- rstudent(wages_fit2)
```

3.5 e)

Answer - This looks more like random noise but there is still clear autocorrelation between the fitted residuals that we have yet to capture in our model.

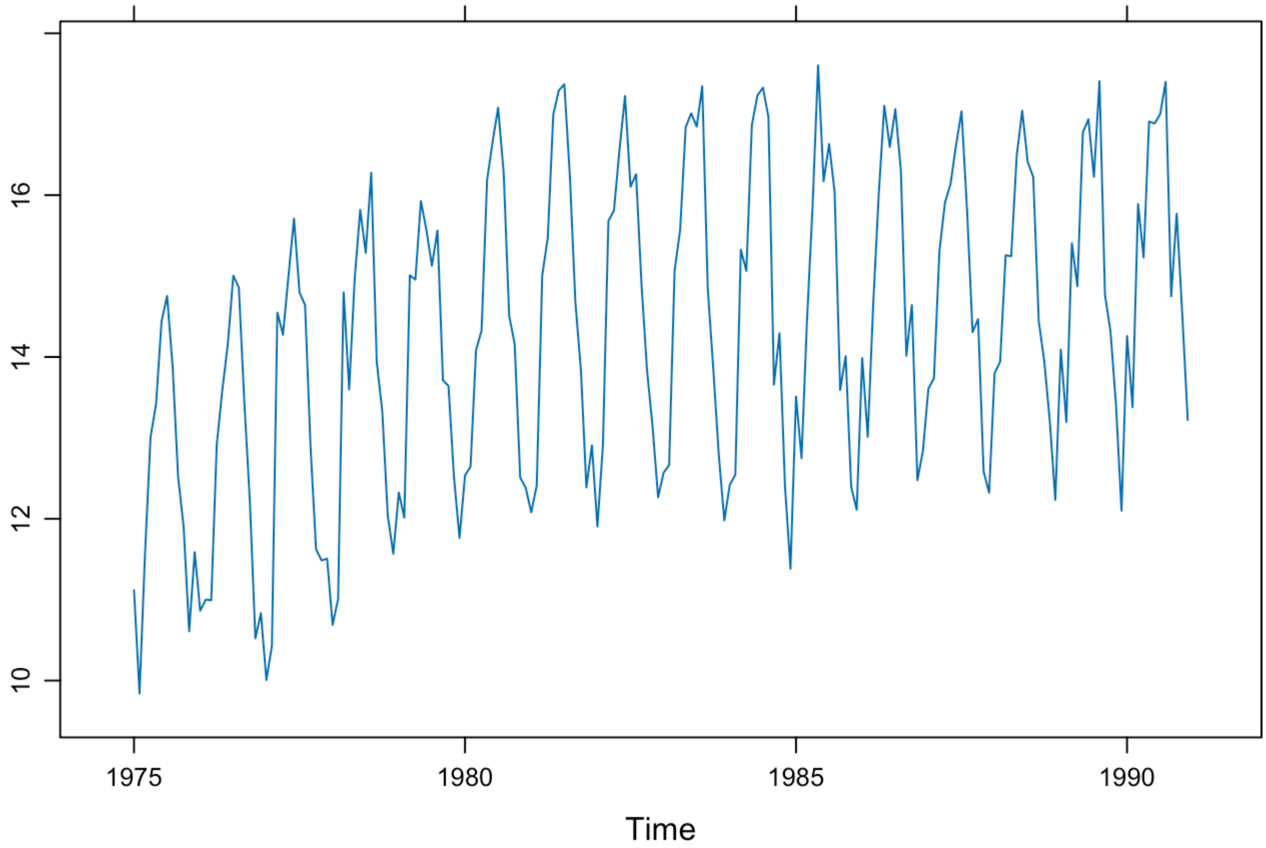
```
xyplot(wages_rst2 ~ time(wages_rst), type = "l",
       xlab = "Time", ylab = "Studentized residuals")
```



3.6 a)

Answer - Clear seasonal trends. There is an initial positive trend from 1975 to around 1981 that then levels out

```
data(beersales)
xyplot(beersales)
```

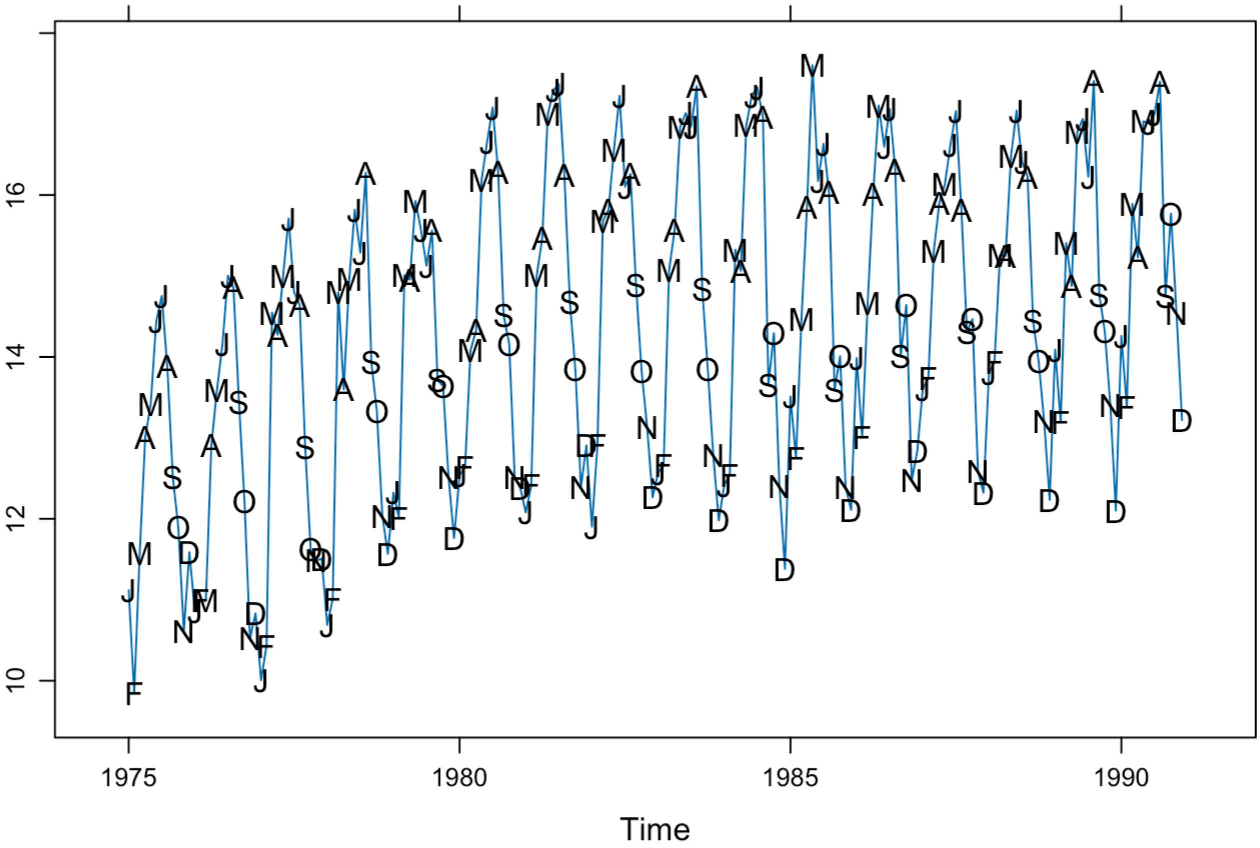


3.6 b)

Answer - It is now evident that the peaks are in the warm months and the slump in the winter and fall months. December is a particular low point, while May, June, and July seem to be the high points.

```
months <- c("J", "F", "M", "A", "M", "J", "J", "A", "S", "O", "N", "D")

xyplot(beersales,
       panel = function(x, y, ...) {
         panel.xyplot(x, y, ...)
         panel.text(x, y, labels = months)
       })
```



3.6 c)

Answer - All comparisons are made against january. The model helpfully explains approximately 0.71 of the variance and is statistically significant. Most of the factors are significant (mostly the winter months as expected).

```
library(pander)
beer_fit1 <- lm(beersales ~ season(beersales))
pander(summary(beer_fit1))
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	12.49	0.2639	47.31	1.786e-103
season(beersales)February	-0.1426	0.3732	-0.382	0.7029
season(beersales)March	2.082	0.3732	5.579	8.771e-08
season(beersales)April	2.398	0.3732	6.424	1.151e-09
season(beersales)May	3.599	0.3732	9.643	5.322e-18
season(beersales)June	3.85	0.3732	10.31	6.813e-20
season(beersales)July	3.769	0.3732	10.1	2.812e-19
season(beersales)August	3.609	0.3732	9.669	4.494e-18
season(beersales)September	1.573	0.3732	4.214	3.964e-05
season(beersales)October	1.254	0.3732	3.361	0.0009484
season(beersales)November	-0.04797	0.3732	-0.1285	0.8979

	Estimate	Std. Error	t value	Pr(>|t|)
season(beersales)December	-0.4231	0.3732	-1.134	0.2585

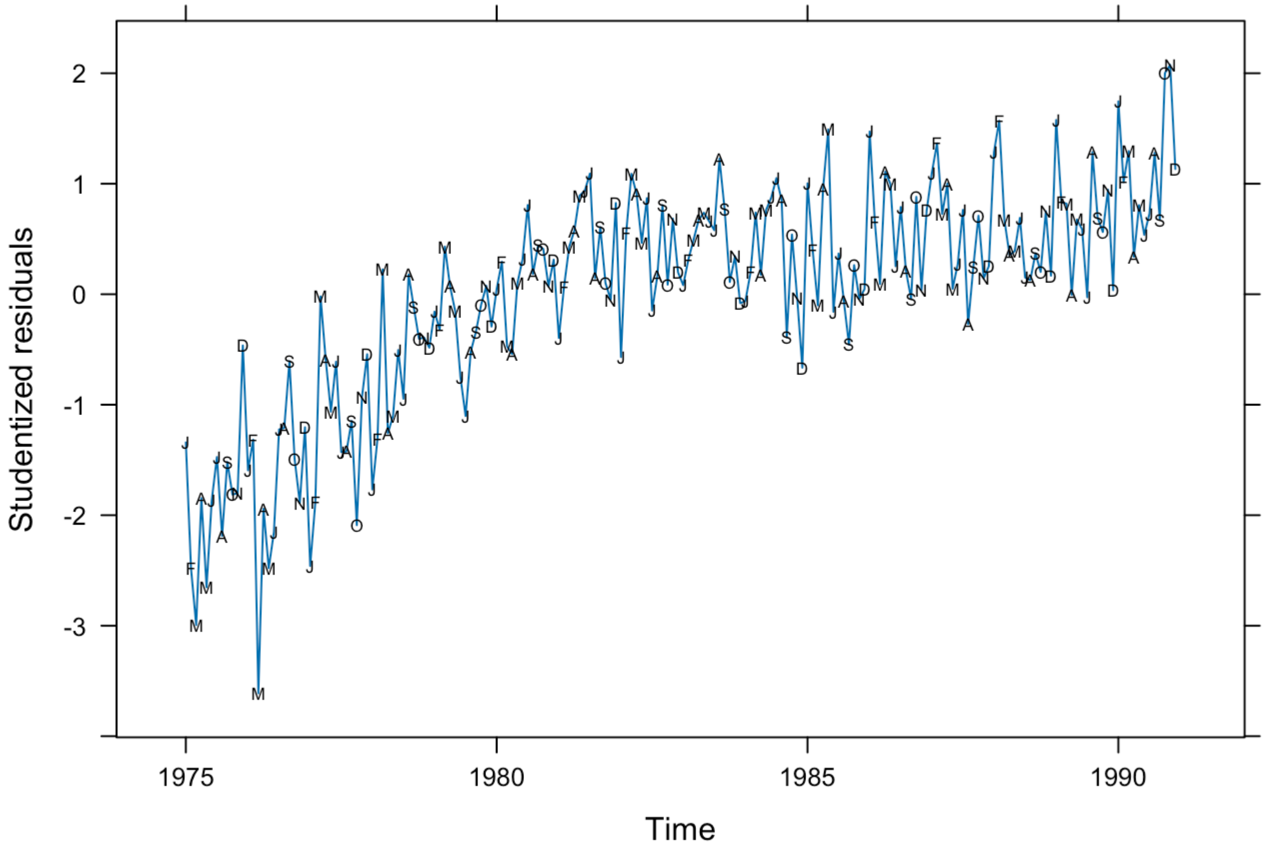
Fitting linear model: beersales ~ season(beersales)

Observations	Residual Std. Error	R^2	Adjusted R^2
192	1.056	0.7103	0.6926

3.6 d)

Answer - We don't have a good fit to our data; in particular, we're not capturing the long-term trend.

```
xyplot(rstudent(beer_fit1) ~ time(beersales), type = "l",
       xlab = "Time", ylab = "Studentized residuals",
       panel = function(x, y, ...) {
         panel.xyplot(x, y, ...)
         panel.xyplot(x, y, pch = as.vector(season(beersales)), col = 1)
       })
```



3.6 e)

Answer - This model fits the data better, explaining roughly 0.91 of the variance.

```
beer_fit2 <- lm(beersales ~ season(beersales) + time(beersales) +
                  I(time(beersales) ^ 2))
pander(summary(beer_fit2))
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-71498	8791	-8.133	6.932e-14
season(beersales)February	-0.1579	0.209	-0.7554	0.451
season(beersales)March	2.052	0.209	9.818	1.864e-18
season(beersales)April	2.353	0.209	11.26	1.533e-22
season(beersales)May	3.539	0.209	16.93	6.063e-39
season(beersales)June	3.776	0.209	18.06	4.117e-42
season(beersales)July	3.681	0.209	17.61	7.706e-41
season(beersales)August	3.507	0.2091	16.78	1.698e-38
season(beersales)September	1.458	0.2091	6.972	5.89e-11
season(beersales)October	1.126	0.2091	5.385	2.268e-07
season(beersales)November	-0.1894	0.2091	-0.9059	0.3662
season(beersales)December	-0.5773	0.2092	-2.76	0.00638
time(beersales)	71.96	8.867	8.115	7.703e-14
I(time(beersales)^2)	-0.0181	0.002236	-8.096	8.633e-14

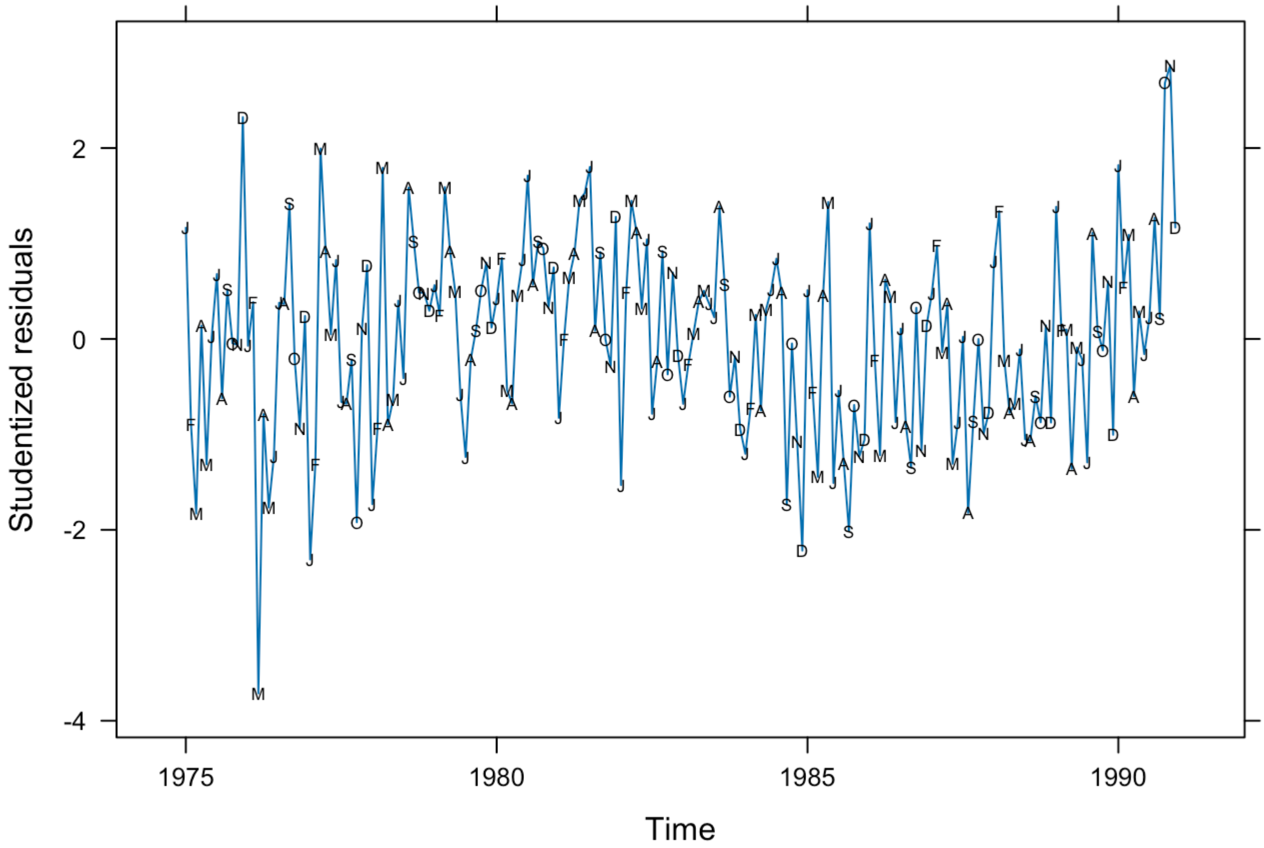
Fitting linear model: beersales ~ season(beersales) + time(beersales) + I(time(beersales)^2)

Observations	Residual Std. Error	R^2	Adjusted R^2
192	0.5911	0.9102	0.9036

3.6 f)

Answer - Many of the values are still not being predicted successfully but at least we’re able to model the long term trend better.

```
xyplot(rstudent(beer_fit2) ~ time(beersales), type = "l",
       xlab = "Time", yla = "Studentized residuals",
       panel = function(x, y, ...) {
         panel.xyplot(x, y, ...)
         panel.xyplot(x, y, pch = as.vector(season(beersales)), col = 1)
       })
```



3.12 a)

Answer - First, we just collect the residuals.

```
data(beersales)
beer_quad_seasonal <- lm(beersales ~ time(beersales) + I(time(beersales)^2) +
                           season(beersales))
beer_resid <- rstudent(beer_quad_seasonal)
```

3.12 b)

Answer - Next, we perform a Runs test.

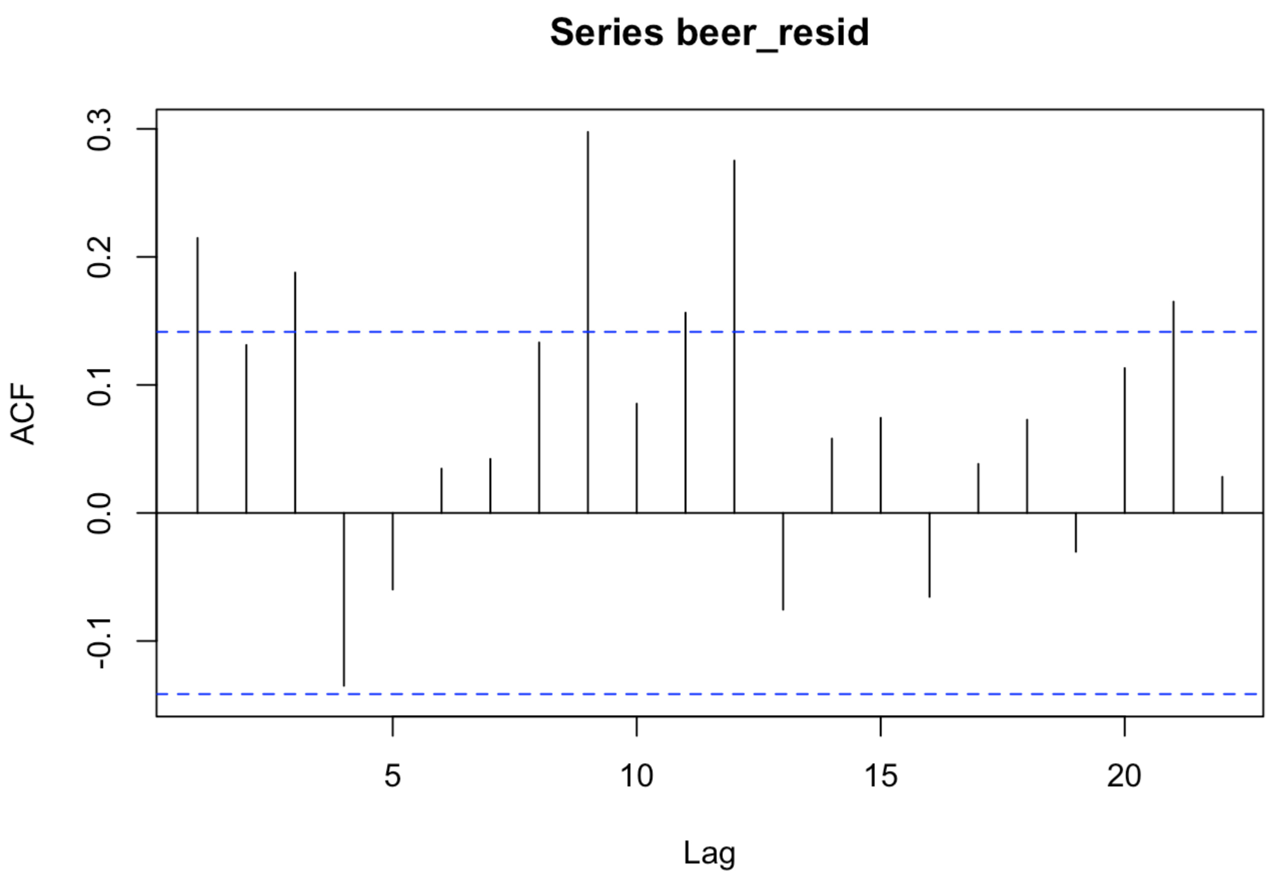
```
runs(beer_resid)
```

```
## $pvalue
## [1] 0.0127
##
## $observed.runs
## [1] 79
##
## $expected.runs
## [1] 96.625
##
## $n1
## [1] 90
##
## $n2
## [1] 102
##
## $k
## [1] 0
```

3.12 c)

Answer - Correlations are significant for several of the lags, leading us to question independence.

```
acf(beer_resid)
```

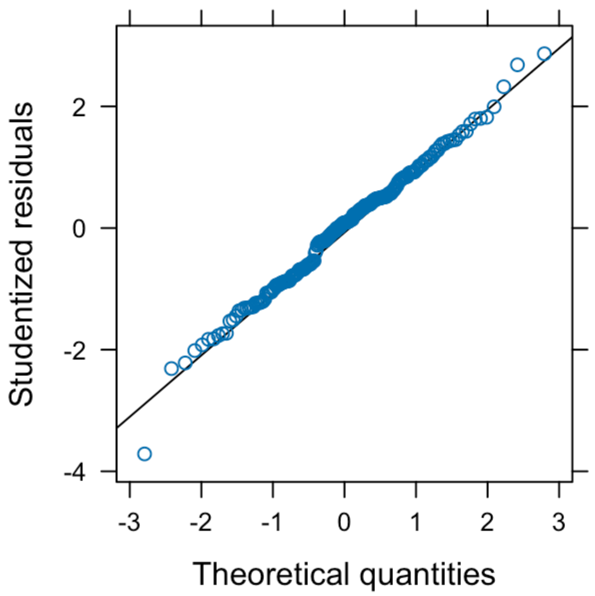


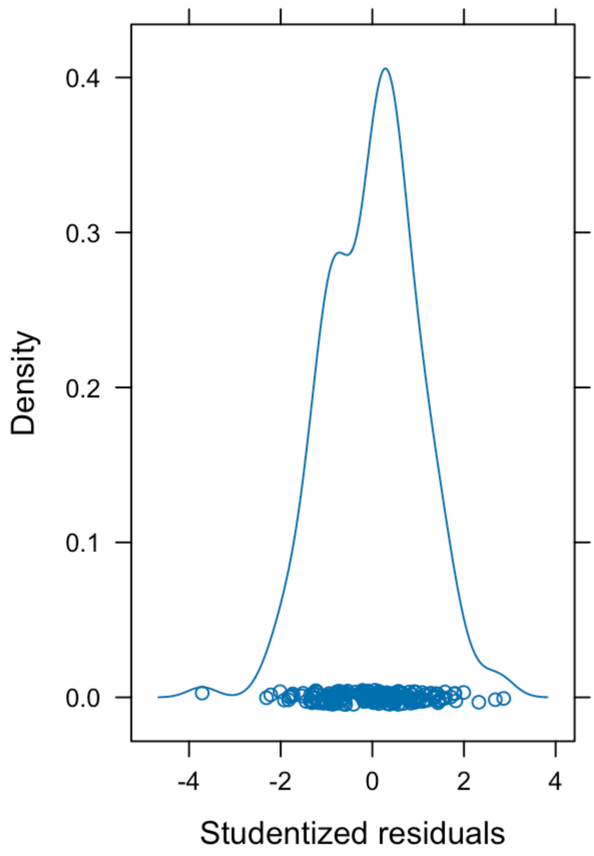
3.12 d)

Answer - Normality plots for the beersales series after a linear, quadratic and seasonal fit.

```
library(gridExtra)
figa <-
  qqmath(beer_resid, xlab = "Theoretical quantities",
       asp = 1,
       ylab = "Studentized residuals",
       panel = function(x, ...) {
         panel.qqmathline(x, ...)
         panel.qqmath(x, ...)
       })

figb <- densityplot(beer_resid, xlab = "Studentized residuals")
gridExtra::grid.arrange(figa, figb, ncol = 2)
```





Homework3

Harin Rishabh - 50540384

March 14, 2024

4.1

We have the process

$$Y_t = 5 + e_t - \frac{1}{2}e_{t-i} + \frac{1}{4}e_{t-2}$$

and begin by working out its variance

$$\begin{aligned}
\text{Var}(Y_t) &= \text{Var}(5 + e_t - \frac{1}{2}e_{t-i} + \frac{1}{4}e_{t-2}) \\
&= \text{Var}(e_t) + \frac{1}{4}\text{Var}(e_t) + \frac{1}{16}\text{Var}(e_t) \\
&= \frac{21}{16}\sigma_e^2
\end{aligned}$$

and then the autocovariance at lag 1

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-1}) &= \text{Cov}(5 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}, 5 + e_{t-1} - \frac{1}{2}e_{t-2} + \frac{1}{4}e_{t-3}) \\
&= \text{Cov}(-\frac{1}{2}e_{t-1}, e_{t-1}) + \text{Cov}(\frac{1}{4}e_{t-2}, -\frac{1}{2}e_{t-2}) \\
&= -\frac{1}{2}\text{Var}(e_{t-1}) - \frac{1}{8}\text{Var}(e_{t-2}) \\
&= -\frac{5}{8}\sigma_e^2
\end{aligned}$$

lag 2

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-2}) &= \text{Cov}(5 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}, 5 + e_{t-2} - \frac{1}{2}e_{t-3} + \frac{1}{4}e_{t-4}) \\
&= \frac{1}{4}\text{Var}(e_{t-2}) \\
&= \frac{1}{4}\sigma_e^2
\end{aligned}$$

and lag 3

$$\text{Cov}(Y_t, Y_{t-2}) = \text{Cov}(5 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}, 5 + e_{t-2} - \frac{1}{2}e_{t-3} + \frac{1}{4}e_{t-4}) = 0$$

which results in the autocorrelation

$$\rho_k = \begin{cases} 1 & k=0 \\ \frac{-\frac{5}{8}\sigma_e^2}{\frac{21}{16}\sigma_e^2} = -\frac{10}{21} & k=1 \\ \frac{\frac{1}{4}\sigma_e^2}{\frac{21}{16}\sigma_e^2} = \frac{4}{21} & k=2 \\ 0 & k=3 \end{cases}$$

4.6

a

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(Y_t - Y_{t-1}, Yt-k-Y_{t-k-1}) \\ &= \mathrm{Cov}(Y_t, Y_{t-k}) - \mathrm{Cov}(Y_{t-1}, Y_{t-k}) - \mathrm{Cov}(Y_t, Y_{t-k-1}) + \mathrm{Cov}(Y_{t-1}, Y_{t-k-1}) \\ &= \frac{\sigma_e^2}{1-\phi^2}(\phi^2 - \phi^{k-1} - \phi^{k+1} + \phi^k) \\ &= \frac{\sigma_e^2}{1-\phi^2}\phi^{k-1}(2\phi - \phi 2 - 1) \\ &= -\frac{\sigma_e^2}{1-\phi^2}(1-\phi)^2\phi^{k-1} \\ &= -\sigma_e^2 \frac{(1-\phi)^2}{(1-\phi)(1+\phi)} \\ &= -\sigma_e^2 \frac{1-\phi}{1+\phi}\phi^{k-1} \end{aligned}$$

as required.

b

$$\begin{aligned} \mathrm{Var}(W_t) &= \mathrm{Var}(Y_t - Y_{t-1}) \\ &= \mathrm{Var}(\phi_1 Y_{t-1} + e_t - Y_{t-1}) \\ &= \mathrm{Var}(Y_{t-1}(\phi - 1) + \sigma_e^2) \\ &= (\phi-1)^2 \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(e_t) \\ &= \frac{\sigma_e^2}{1-\phi^2}(\phi^2 - 2\phi + 1) + \sigma_e^2 \\ &= \frac{\sigma_e^2(\phi^2 - 2\phi + 1 + 1 - \phi^2)}{1-\phi^2} \\ &= \frac{2\sigma_e^2(1-\phi)}{1-\phi^2} \\ &= \frac{2\sigma_e^2}{1+\phi} \end{aligned}$$

4.7

a

Only correlation at lag 1.

b

Only autocorrelation at lag 1 and 2. Shape of process depends on values of coefficients.

c

Exponentially decaying correlation from lag 0.

d

Different patterns in ACF that depends on whether roots are complex or real.

e

Exponentially decaying correlations from lag 1.

4.11

a

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{E}[(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-k}] - \mathrm{E}(Y_t)\mathrm{E}(Y_{t-k}) \\
&= \mathrm{E}(0.8Y_{t-1}Y_{t-k} + Y_{t-k}e_t + 0.7e_{t-1}Y_{t-k} + 0.6e_{t-2}Y_{t-k}) - 0 \\
&= 0.8\mathrm{E}(Y_{t-1}Y_{t-k}) + \mathrm{E}(Y_{t-k}e_t) + 0.7\mathrm{E}(e_{t-1}Y_{t-k}) + 0.7\mathrm{E}(e_{t-2}Y_{t-k}) \\
&= 0.8\mathrm{E}(Y_{t-1}Y_{t-k}) \\
&= 0.8\mathrm{Cov}(Y_t, Y_{t-k+1}) \\
&= 0.8\gamma_{k-1}
\end{aligned}$$

b

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-2}) &= \mathrm{E}[0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-2}] \\
&= \mathrm{E}[(0.8Y_{t-1} + 0.6e_{t-2})Y_{t-2}] \\
&= 0.8\mathrm{Cov}(Y_{t-1}, Y_{t-2}) + 0.6\mathrm{E}(e_{t-2}Y_{t-2}) \\
&= 0.8\gamma_1 + 0.6\mathrm{E}(e_tY_t) \\
&= 0.8\gamma_1 + 0.6\mathrm{E}[e_t(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})] \\
&= 0.8\gamma_1 + 0.6\sigma_e^2 \iff \\
\rho_2 &= 0.8\rho_1 + 0.6\sigma_e^2/\gamma_0
\end{aligned}$$

4.13

$$\mathrm{Var}(Y_{n+1}+Y_n+Y_{n-1}+\cdots+Y_1) = \left((n+1)+2n\rho_1\right)\gamma_0 = \left(1+n(1+2\rho_1)\right)\gamma_0$$
$$\mathrm{Var}(Y_{n+1}-Y_n+Y_{n-1}-\cdots+Y_1) = \left((n+1)-2n\rho_1\right)\gamma_0 = \left(1+n(1-2\rho_1)\right)\gamma_0$$

$[\begin{cases}(1+n(1+2\rho-1))\geq 0\\(1+n(1-2\rho-1))\geq 0\end{cases} \quad \begin{cases}1+n+2\rho_1 n\geq 0\\1+n-2\rho_1 n\geq 0\end{cases} \quad \begin{cases}\rho_1\geq \frac{-(n+1)}{2n}\\ \rho_1\leq \frac{n+1}{2n}\end{cases} \quad \begin{cases}\rho_1\geq -\frac{1}{2}(1+\frac{1}{n})\\ \rho_1\leq \frac{1}{2}(1+\frac{1}{n})\end{cases}]$ where $\rho_1 \geq |1/2|$ for all n.

4.16

a

$$\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \implies \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j} &= 3\left(-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t-1+j}\right) + e_t \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j} &= -\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t-1+j} + \frac{1}{3}e_t \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j} &= -\sum_{j=2}^{\infty}\left(\frac{1}{3}\right)^j e_{t-1+j} \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j} &= -\sum_{j+1=2}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j}
\end{aligned}$$

b

$$\mathrm{E}(Y_t) = \mathrm{E}\left(\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j}\right) = 0$$

since all terms are uncorrelated white noise.

$$\mathrm{Cov}(Y_t, Y_{t-1}) = \mathrm{Cov}\left(-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j}, \sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j-1}\right) =$$
$$\mathrm{Cov}\left(-\frac{1}{3}e_{t+1} - \left(\frac{1}{3}\right)^2 e_{t+2} - \cdots - \left(\frac{1}{3}\right)^n e_{t+n}, -\frac{1}{3}e_t - \left(\frac{1}{3}\right)^2 e_{t+1} - \cdots - \left(\frac{1}{3}\right)^n e_{t+n-1}\right) =$$
$$\mathrm{Cov}\left(-\frac{1}{3}e_{t+1}, -\frac{1}{3^2}e_{t+1}\right) + \mathrm{Cov}\left(-\frac{1}{3}e_{t+2}, -\frac{1}{3^3}e_{t+2}\right) + \cdots + \mathrm{Cov}\left(-\frac{1}{3}e_{t+n}, -\frac{1}{3^{n+1}}e_{t+n}\right) =$$
$$\frac{1}{26}\sigma_e^2\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}\right)$$

which is free of t.

c

It is unsatisfactory because Y_t depends on future observations.

4.25

a

$$\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
&= \phi(\phi Y_{t-2} + e_{t-1}) + e_t = \phi^2 Y_{t-2} + \phi e_{t-1} + e_t \\
&\vdots \\
&= \phi^t Y_{t-t} + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1 + e_t
\end{aligned}$$

b

$$E(Y_t) = E(\phi^t Y_0 + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1 + e_t) = \phi^t \mu_0$$

c

$$\begin{aligned}
\mathrm{Var}(Y_t) &= \mathrm{Var}(\phi^t Y_0 + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1) \\
&= \phi^{2t}\sigma_0^2 + \sigma_e^2 \sum_{k=0}^{t-1} (\phi^2)^k \\
&= \sigma_e^2 \frac{1-\phi^{2n}}{1-\phi^2} + \phi^{2t}\sigma_0^2 \quad \text{if } \phi \neq 1 \text{ else} \\
&= \mathrm{Var}(Y_0) + \sigma_e^2 t = \sigma_0^2 + \sigma_e^2 t
\end{aligned}$$

d

If $\mu_0 = 0$ then $E(Y_t) = 0$ but for $\mathrm{Var}(Y_t)$ to be free of t, ϕ cannot be 1.

e

$$\mathrm{Var}(Y_t) = \phi^2 \mathrm{Var}(Y_{t-1}) + \sigma_e^2 \implies \phi^2 \mathrm{Var}(Y_t) + \sigma_e^2$$

and

$$\mathrm{Var}(Y_{t-1}) = \mathrm{Var}(Y_t)(1-\phi^2) = \sigma_e^2 \implies \mathrm{Var}(Y_t) = \frac{\sigma_e^2}{1-\phi}$$

and then we must have $|\phi| < 1$.