Homework3

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4.1

We have the process

$$Y_t = 5 + e_t - \frac{1}{2}e_{t-i} + \frac{1}{4}e_{t-2}$$

and begin by working out its variance

$$\begin{aligned}
\mathrm{Var}(Y_t) &= \mathrm{Var}(5 + e_t - \frac{1}{2}e_{t-i} + \frac{1}{4}e_{t-2}) \\
&= \mathrm{Var}(e_t) + \frac{1}{4}\mathrm{Var}(e_t) + \frac{1}{16}\mathrm{Var}(e_t) \\
&= \frac{21}{16}\sigma_e^2
\end{aligned}$$

and then the autocovariance at lag 1

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-1}) &= \mathrm{Cov}(5 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}, 5 + e_{t-1} - \frac{1}{2}e_{t-2} + \frac{1}{4}e_{t-3}) \\
&= \mathrm{Cov}(-\frac{1}{2}e_{t-1}, e_{t-1}) + \mathrm{Cov}(\frac{1}{4}e_{t-2}, -\frac{1}{2}e_{t-2}) \\
&= -\frac{1}{2}\mathrm{Var}(e_{t-1}) - \frac{1}{8}\mathrm{Var}(e_{t-2}) \\
&= -\frac{5}{8}\sigma_e^2
\end{aligned}$$

lag 2

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-2}) &= \mathrm{Cov}(5 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}, 5 + e_{t-2} - \frac{1}{2}e_{t-3} + \frac{1}{4}e_{t-4}) \\
&= \frac{1}{4}\mathrm{Var}(e_{t-2}) \\
&= \frac{1}{4}\sigma_e^2
\end{aligned}$$

and lag 3

$$\mathrm{Cov}(Y_t, Y_{t-2}) = \mathrm{Cov}(5 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}, 5 + e_{t-2} - \frac{1}{2}e_{t-3} + \frac{1}{4}e_{t-4}) = 0$$

which results in the autocorrelation

$$\rho_k = \begin{cases} 1 & k=0 \\ \frac{-\frac{5}{8}\sigma_e^2}{\frac{21}{16}\sigma_e^2} = -\frac{10}{21} & k=1 \\ \frac{\frac{1}{4}\sigma_e^2}{\frac{21}{16}\sigma_e^2} = \frac{4}{21} & k=2 \\ 0 & k=3 \end{cases}$$

4.6

a

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(Y_t - Y_{t-1}, Yt - k - Y_{t-k-1}) \\
&= \mathrm{Cov}(Y_t, Y_{t-k}) - \mathrm{Cov}(Y_{t-1}, Y_{t-k}) - \mathrm{Cov}(Y_t, Y_{t-k-1}) + \mathrm{Cov}(Y_{t-1}, Y_{t-k-1}) \\
&= \frac{\sigma_e^2}{1-\phi^2}(\phi^2 - \phi^{k-1} - \phi^{k+1} + \phi^k) \\
&= \frac{\sigma_e^2}{1-\phi^2}\phi^{k-1}(2\phi - \phi 2 - 1) \\
&= -\frac{\sigma_e^2}{1-\phi^2}(1-\phi)^2\phi^{k-1} \\
&= -\sigma_e^2 \frac{(1-\phi)^2}{(1-\phi)(1+\phi)} \\
&= -\sigma_e^2 \frac{1-\phi}{1+\phi}\phi^{k-1}
\end{aligned}$$

as required.

b

$$\begin{aligned}
\mathrm{Var}(W_t) &= \mathrm{Var}(Y_t - Y_{t-1}) \\
&= \mathrm{Var}(\phi_1 Y_{t-1} + e_t - Y_{t-1}) \\
&= \mathrm{Var}(Y_{t-1}(\phi - 1) + \sigma_e^2) \\
&= (\phi-1)^2\mathrm{Var}(Y_{t-1}) + \mathrm{Var}(e_t) \\
&= \frac{\sigma_e^2}{1-\phi^2}(\phi^2 - 2\phi + 1) + \sigma_e^2 \\
&= \frac{\sigma_e^2(\phi^2 - 2\phi + 1 + 1 - \phi^2)}{1-\phi^2} \\
&= \frac{2\sigma_e^2(1-\phi)}{1-\phi^2} \\
&= \frac{2\sigma_e^2}{1+\phi}
\end{aligned}$$

4.7

a

Only correlation at lag 1.

b

Only autocorrelation at lag 1 and 2. Shape of process depends on values of coefficients.

c

Exponentially decaying correlation from lag 0.

d

Different patterns in ACF that depends on whether roots are complex or real.

e

Exponentially decaying correlations from lag 1.

4.11

a

$$
\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{E}[(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-k}] - \mathrm{E}(Y_t)\mathrm{E}(Y_{t-k}) \\
&= \mathrm{E}(0.8Y_{t-1}Y_{t-k} + Y_{t-k}e_t + 0.7e_{t-1}Y_{t-k} + 0.6e_{t-2}Y_{t-k}) - 0 \\
&= 0.8\mathrm{E}(Y_{t-1}Y_{t-k}) + \mathrm{E}(Y_{t-k}e_t) + 0.7\mathrm{E}(e_{t-1}Y_{t-k}) + 0.7\mathrm{E}(e_{t-2}Y_{t-k}) \\
&= 0.8\mathrm{E}(Y_{t-1}Y_{t-k}) \\
&= 0.8\mathrm{Cov}(Y_t, Y_{t-k+1}) \\
&= 0.8\gamma_{k-1}
\end{aligned}
$$

b

$$
\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-2}) &= \mathrm{E}[0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-2}] \\
&= \mathrm{E}[(0.8Y_{t-1} + 0.6e_{t-2})Y_{t-2}] \\
&= 0.8\mathrm{Cov}(Y_{t-1}, Y_{t-2}) + 0.6\mathrm{E}(e_{t-2}Y_{t-2}) \\
&= 0.8\gamma_1 + 0.6\mathrm{E}(e_tY_t) \\
&= 0.8\gamma_1 + 0.6\mathrm{E}[e_t(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})] \\
&= 0.8\gamma_1 + 0.6\sigma_e^2 \iff \\
\rho_2 &= 0.8\rho_1 + 0.6\sigma_e^2/\gamma_0
\end{aligned}
$$

4.13

$$\mathrm{Var}(Y_{n+1}+Y_n+Y_{n-1}+\cdots+Y_1) = ((n+1)+2n\rho_1)\,\gamma_0 = (1+n(1+2\rho_1))\,\gamma_0$$
$$\mathrm{Var}(Y_{n+1}-Y_n+Y_{n-1}-\cdots+Y_1) = ((n+1)-2n\rho_1)\,\gamma_0 = (1+n(1-2\rho_1))\,\gamma_0$$

$[\begin{cases}(1+n(1+2\rho-1))\geq 0\\(1+n(1-2\rho-1))\geq 0\end{cases} \quad \begin{cases}1+n+2\rho_1 n\geq 0\\1+n-2\rho_1 n\geq 0\end{cases} \quad \begin{cases}\rho_1\geq \frac{-(n+1)}{2n}\\ \rho_1\leq \frac{n+1}{2n}\end{cases} \quad \begin{cases}\rho_1\geq -\frac{1}{2}(1+\frac{1}{n})\\ \rho_1\leq \frac{1}{2}(1+\frac{1}{n})\end{cases}$] where $\rho_1 \geq |1/2|$ for all n.

4.16

a

$$\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \implies \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j} &= 3\left(-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t-1+j}\right) + e_t \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j} &= -\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t-1+j} + \frac{1}{3}e_t \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j} &= -\sum_{j=2}^{\infty}\left(\frac{1}{3}\right)^j e_{t-1+j} \\
-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j} &= -\sum_{j+1=2}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t+j}
\end{aligned}$$

b

$$\mathrm{E}(Y_t) = \mathrm{E}(\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j}) = 0$$

since all terms are uncorrelated white noise.

$$\mathrm{Cov}(Y_t, Y_{t-1}) = \mathrm{Cov}\left(-\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j}, \sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j-1}\right) =$$
$$\mathrm{Cov}\left(-\frac{1}{3}e_{t+1} - \left(\frac{1}{3}\right)^2 e_{t+2} - \cdots - \left(\frac{1}{3}\right)^n e_{t+n}, -\frac{1}{3}e_t - \left(\frac{1}{3}\right)^2 e_{t+1} - \cdots - \left(\frac{1}{3}\right)^n e_{t+n-1}\right) =$$
$$\mathrm{Cov}\left(-\frac{1}{3}e_{t+1}, -\frac{1}{3^2}e_{t+1}\right) + \mathrm{Cov}\left(-\frac{1}{3}e_{t+2}, -\frac{1}{3^3}e_{t+2}\right) + \cdots + \mathrm{Cov}\left(-\frac{1}{3}e_{t+n}, -\frac{1}{3^{n+1}}e_{t+n}\right) =$$
$$\frac{1}{26}\sigma_e^2\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}\right)$$

which is free of t.

c

It is unsatisfactory because Y_t depends on future observations.

4.25

a

$$\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
&= \phi(\phi Y_{t-2} + e_{t-1}) + e_t = \phi^2 Y_{t-2} + \phi e_{t-1} + e_t \\
&\vdots \\
&= \phi^t Y_{t-t} + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1 + e_t
\end{aligned}$$

b

$$E(Y_t) = E(\phi^t Y_0 + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1 + e_t) = \phi^t \mu_0$$

c

$$\begin{aligned}
\mathrm{Var}(Y_t) &= \mathrm{Var}(\phi^t Y_0 + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1) \\
&= \phi^{2t}\sigma_0^2 + \sigma_e^2 \sum_{k=0}^{t-1} (\phi^2)^k \\
&= \sigma_e^2 \frac{1-\phi^{2n}}{1-\phi^2} + \phi^{2t}\sigma_0^2 \quad \text{if } \phi \neq 1 \text{ else} \\
&= \mathrm{Var}(Y_0) + \sigma_e^2 t = \sigma_0^2 + \sigma_e^2 t
\end{aligned}$$

d

If $\mu_0 = 0$ then $E(Y_t) = 0$ but for $\mathrm{Var}(Y_t)$ to be free of t, ϕ cannot be 1.

e

$$\mathrm{Var}(Y_t) = \phi^2 \mathrm{Var}(Y_{t-1}) + \sigma_e^2 \implies \phi^2 \mathrm{Var}(Y_t) + \sigma_e^2$$

and

$$\mathrm{Var}(Y_{t-1}) = \mathrm{Var}(Y_t)(1-\phi^2) = \sigma_e^2 \implies \mathrm{Var}(Y_t) = \frac{\sigma_e^2}{1-\phi}$$

and then we must have $|\phi| < 1$.