Functional Analysis Assignment 2

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Questions

Problem 1: Describe the span of $M = \{(1,1,1),(0,0,2)\}$ in $\mathbb{R}^3$.

Solution 1: Span of $M = \{(1,1,1),(0,0,2)\}$:
$S = \alpha \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta \begin{bmatrix}0\\0\\2\end{bmatrix} = \begin{bmatrix}\alpha\\\alpha\\\alpha+2\beta\end{bmatrix}$ where $\alpha, \beta \in \mathbb{R}$. This span can be represented by equation $x = y$ as z can vary, by changing β.

Problem 2: Which of the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$?
[Here, $x = (\xi_1, \xi_2, \xi_3)$.]

1. All x with $\xi_1 = \xi_2$ and $\xi_3 = 0$.
2. All x with $\xi_1 = \xi_2 + 1$.
3. All x with positive ξ_1, ξ_2, ξ_3.
4. All x with $\xi_1 - \xi_2 + \xi_3 = k = const$.

Solution 2:

1. All x with $\xi_1 = \xi_2 = k$ and $\xi_3 = 0$. Here $x = (k,k,0)$ where $k \in \mathbb{R}$. Consider $\alpha x_1 + \beta x_2$ which is $\alpha(k_1,k_1,0) + \beta(k_2,k_2,0) = (\alpha k_1 + \beta k_2, \alpha k_1 + \beta k_2, 0)$. Let $\alpha k_1 + \beta k_2 = k_3$ therefore $\alpha x_1 + \beta x_2 = (k_3,k_3,0) \in \mathbb{R}^3$ and is in the subspace as $\xi_1 = \xi_2$. Hence, it **constitutes a subspace.**
2. All x with $\xi_1 = \xi_2 + 1$. Here $x = (k+1,k,0)$ where $k \in \mathbb{R}$ taking $\xi_2 = k$. Consider $\alpha x_1 + \beta x_2$ which is $\alpha(k_1+1,k_1,0) + \beta(k_2+1,k_2,0) = (\alpha k_1 + \beta k_2 + (\alpha+\beta), \alpha k_1 + \beta k_2, 0)$. Let $\alpha k_1 + \beta k_2 = k_3$ therefore $\alpha x_1 + \beta x_2 = (k_3 + (\alpha+\beta), k_3, 0) \in \mathbb{R}^3$ but **it does not constitute a subspace** because $\xi_1 \neq \xi_2 + 1$.
3. All x with positive ξ_1, ξ_2, ξ_3. Here $x = (k_{11}, k_{12}, k_{13})$ where $k_{ij} \in \mathbb{R}$. Consider $\alpha x_1 + \beta x_2$ which is $\alpha(k_{11},k_{12},k_{13}) + \beta(k_{21},k_{22},k_{23}) = (\alpha k_{11} + \beta k_{21}, \alpha k_{12} + \beta k_{22}, \alpha k_{13} + \beta k_{23})$. Here all elements in $\alpha x_1 + \beta x_2$ can be positive as well as negative depending on sign of $\alpha k_{1i} + \beta k_{2i}$. Therefore, **it does not constitute** a subspace as it should have been all positive rather than posing any sign.

4. All x with $\xi_1 - \xi_2 + \xi_3 = k = const.$ Here $x = (k_{11}, k_{12}, k_{13})$ where $k_{ij} \in \mathbb{R}$. Consider $\alpha x_1 + \beta x_2$ which is $\alpha(k_{11}, k_{12}, k_{13}) + \beta(k_{21}, k_{22}, k_{23}) = (\alpha k_{11} + \beta k_{21}, \alpha k_{12} + \beta k_{22}, \alpha k_{13} + \beta k_{23})$. Let's calculate $\xi_1 - \xi_2 + \xi_3$ and see it if a constant k or not,

$$\begin{aligned} \alpha(k_{11}, k_{12}, k_{13}) + \beta(k_{21}, k_{22}, k_{23}) &= (\alpha k_{11} + \beta k_{21}, \alpha k_{12} + \beta k_{22}, \alpha k_{13} + \beta k_{23}) \\ \xi_1 - \xi_2 + \xi_3 &= (\alpha k_{11} + \beta k_{21}) - (\alpha k_{12} + \beta k_{22}) + (\alpha k_{13} + \beta k_{23}) \\ &= \alpha(k_{11} - k_{12} + k_{13}) + \beta(k_{21} - k_{22} + k_{23}) \\ &= \alpha k + \beta k \\ &= (\alpha + \beta)k \neq k \end{aligned}$$

This is not equal to k for any arbitrary α and β, therefore **it does not constitute** a subspace.

Problem 3: Show that we may replace $(N2)$ by

$$\|x\| = 0 \implies x = 0$$

without altering the concept of a norm. Show that non-negativity of a norm also follows from $(N3)$ and $(N4)$.

Solution 3:

(a) Given any norm $||\cdot||$, as it follows the property that $||\alpha x|| = \alpha||x||$ (i.e. N3), let's take $\alpha = 0$, therefore $||0 \cdot x|| = 0||x||$ or $||0|| = 0$, if $x = 0$. Hence, the back implication that $x = 0 \implies ||x|| = 0$ is trivial, and is redundant in $(N2)$, hence $(N2)$ **can be replaced by** $\|x\| = 0 \implies x = 0$.

(b) Given any norm $||\cdot||$, as it follows the property that $||\alpha x|| = \alpha||x||$ (i.e. N3), let's take $\alpha = 0$, therefore $||0 \cdot x|| = 0||x||$ or $||0|| = 0$, if $x = 0$ ... (*) . And $(N4)$ gives that $||x + y|| \leq ||x|| + ||y||$, let $y = -x$, then $||x + (-x)|| \leq ||x|| + || - x||$ i.e. $||0|| \leq ||x|| + ||x||$ or $0 \leq 2||x|| \implies 0 \leq ||x||$, using (*). Therefore $||x|| \leq 0$ or $||x||$ **is non-negative.**

Problem 4: There are several norms of practical importance on the vector space of ordered n-tuples of numbers (cf. 2.2-2), notably those defined by

$$\begin{aligned} \|x\|_1 &= |\xi_1| + |\xi_2| + ... + |\xi_n| \\ \|x\|_p &= (|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p)^{\frac{1}{p}} \;...\; (1 < p < +\infty) \\ \|x\|_\infty &= max\{|\xi_1|, ..., |\xi_n|\}. \end{aligned}$$

In each case, verify that $(N1)$ to $(N4)$ are satisfied.

Solution 4:

1. $\|x\|_1 = |\xi_1| + |\xi_2| + ... + |\xi_n|$.
   (a) **(N1)** As $\|x\|_1 = |\xi_1| + |\xi_2| + ... + |\xi_n|$ is sum of positive "absolute value" terms, therefore $\|x\|_1 \geq 0$. Hence (N1) **is satisfied**.
   (b) **(N2)** From one hand, let $\|x\|_1 = 0$ which gives $|\xi_1| + |\xi_2| + ... + |\xi_n| = 0$, for this to be 0 and as sum is of all "absolute terms" therefore each $|\xi_i| = 0 \; \forall \; i \in 1, ..., n$, or $\xi_i = 0$ which means $x = 0$ vector i.e. $\|x\|_1 = 0 \implies x = 0$. And from other way round, if $x = 0$ then $\|x\|_1 = |\xi_1| + |\xi_2| + ... + |\xi_n| = 0 + 0 + ... + 0 = 0$ hence $x = 0 \implies \|x\|_1 = 0$. **Combining,** $\|x\|_1 = 0 \iff x = 0$. Hence (N2) **is satisfied**.

(c) **(N3)** $\|\alpha x\|_1 = |\alpha\xi_1| + |\alpha\xi_2| + ... + |\alpha\xi_n| = \alpha(|\xi_1| + |\xi_2| + ... + |\xi_n|) = \alpha\|x\|_1$, as $\alpha x = (\alpha\xi_1, \alpha\xi_2, ..., \alpha\xi_n)$. Hence (N3) **is satisfied.**

(d) **(N4)**. Let $x = (x_i)$ and $y = (y_i)$ for $i \in \{1, ..., n\}$

$$||x + y||_1 = |x_1 + y_1| + |x_2 + y_2| + ...|x_n + y_n|$$

As $|a + b| \leq |a| + |b|$ (proved in previous assignment) or using triangle inequality.

$$\begin{aligned} ||x + y||_1 &\leq |x_1| + |y_1| + ... + |x_n| + |y_n| \\ &= (|x_1| + ... + |x_n|) + (|y_1| + ... + |y_n|) = ||x||_1 + ||y||_1 \end{aligned}$$

Hence,

$$||x + y||_1 \leq ||x||_1 + ||y||_1$$

Therefore, (N4) **is satisfied**.

2. $\|x\|_p = (|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p)^{\frac{1}{p}}$ ... $(1 < p < +\infty)$.

(a) **(N1)** As $\|x\|_p = (|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p)^{\frac{1}{p}}$ is sum of positive "absolute value" terms and $p \neq \infty$ to make the norm 1, therefore $\|x\|_1 \geq 0$. Hence (N1) **is satisfied**.

(b) **(N2)** From one hand, let $\|x\|_1 = 0$ which gives $(|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p)^{\frac{1}{p}} = 0$ or $|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p = 0$, for this to be 0 and as sum is of all "absolute terms" therefore each $|\xi_i| = 0 \; \forall \; i \in 1, ..., n$, or $\xi_i = 0$ which means $x = 0$ vector i.e. $\|x\|_p = 0 \implies x = 0$. And from other way round, if $x = 0$ then $\|x\|_p = (|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p)^{\frac{1}{p}} = (0 + 0 + ... + 0)^{\frac{1}{p}} = 0$ hence $x = 0 \implies \|x\|_p = 0$. **Combining,** $\|x\|_p = 0 \iff x = 0$. Hence (N2) **is satisfied**.

(c) **(N3)** $\|\alpha x\|_1 = (|\alpha\xi_1|^p + |\alpha\xi_2|^p + ... + |\xi_n|^p)^{\frac{1}{p}} = (\alpha^p|\xi_1|^p + \alpha^p|\xi_2|^p + ... + \alpha^p|\xi_n|^p)^{\frac{1}{p}} = \alpha(|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p)^{\frac{1}{p}} = \alpha\|x\|_p$, as $\alpha x = (\alpha\xi_1, \alpha\xi_2, ..., \alpha\xi_n)$. Hence (N3) **is satisfied.**

(d) **(N4)**. Let $x = (x_i)$ and $y = (y_i)$ for $i \in \{1, ..., n\}$

$$||x + y||_p = (|x_1 + y_1|^p + |x_2 + y_2|^p + ... + |x_n + y_n|^p)^{\frac{1}{p}}$$

Using **Minkowski Inequality**, taking $x_i = y_i = 0$ for $i > n$.

$$\begin{aligned} ||x + y||_p &= (|x_1 + y_1|^p + |x_2 + y_2|^p + ... + |x_n + y_n|^p)^{\frac{1}{p}} \\ &\leq (|x_1|^p + ... + |x_n|^p)^{\frac{1}{p}} + (|y_1|^p + ... + |y_n|^p)^{\frac{1}{p}} = ||x||_p + ||y||_p \end{aligned}$$

Hence,

$$||x + y||_p \leq ||x||_p + ||y||_p$$

Therefore, (N4) **is satisfied**.

3. $\|x\|_\infty = max\{|\xi_1|, ..., |\xi_n|\}$.

   (a) **(N1)** As $\|x\|_1 = max\{|\xi_1|, ..., |\xi_n|\}$ is the maximum values among positive "absolute value" terms, therefore $\|x\|_\infty \geq 0$. Hence (N1) **is satisfied**.

   (b) **(N2)** From one hand, let $\|x\|_1 = 0$ which gives $max\{|\xi_1|, ..., |\xi_n|\} = 0$, for this to be 0, the maximum value must be 0, and it comprises of all "absolute terms" therefore each $|\xi_i| = 0 \; \forall \; i \in 1, ..., n$, or $\xi_i = 0$ which means $x = 0$ vector i.e. $\|x\|_\infty = 0 \implies x = 0$. And from other way round, if $x = 0$ then $\|x\|_\infty = max\{|\xi_1|, ..., |\xi_n|\} = 0$ hence $x = 0 \implies \|x\|_\infty = 0$. **Combining,** $\|x\|_\infty = 0 \iff x = 0$. Hence (N2) **is satisfied**.

   (c) **(N3)** $\|\alpha x\|_\infty = max\{|\alpha\xi_1|, ..., |\alpha\xi_n|\} = \alpha max\{|\xi_1|, ..., |\xi_n|\}$, here we can take out α in common, as it's a constant and wouldn't affect the equation so, $= \alpha \|x\|_1$, as $\alpha x = (\alpha\xi_1, \alpha\xi_2, ..., \alpha\xi_n)$. Hence (N3) **is satisfied.**

   (d) **(N4)**. Let $x = (x_i)$ and $y = (y_i)$ for $i \in \{1, ..., n\}$

   $$||x + y||_\infty = max\{|x_1 + y_1|, ..., |x_n + y_n|\}$$

   As $|a+b| \leq |a| + |b|$ (proved in previous assignment) or using triangle inequality. Hence, individual $|x_i + y_i| \leq |x_i| + |y_i| \; \forall \; i \in \{1, ..., n\}$. Hence max of $|x_i + y_i|$ will be less than equal to $|x_i| + |y_i|$ for a particular pair (i), let's assume $\alpha = max|x_i|$ and $\beta = max|y_i|$, now $\alpha + \beta$ must surely be greater than (before pair constraint)

   $$\begin{aligned} ||x + y||_\infty &\leq \max_i |x_i| + |y_i| \text{ for some i} \\ &\leq \max_i |x_i| + \max_i |y_i| = ||x||_\infty + ||y||_\infty \end{aligned}$$

   Hence,

   $$||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$$

   Therefore, (N4) **is satisfied**.

Problem 5: If two norms $\|.\|$ and $\|.\|_0$ on a vector space X are equivalent, show that

$$\|x_n - x\| \to 0 \iff \|x_n - x\|_0 \to 0$$

Solution 5: As $\|.\|$ and $\|.\|_0$ are equivalent and according to the definition of equivalence,

$$\alpha \|x\| \leq \|x\|_0 \leq \beta \|x\| \tag{1}$$

For any arbitrary x in vector space X and some constants $\alpha, \beta \in \mathbb{R}$. From one side, assuming,

$$\|x_n - x\| \to 0$$

now,

$$\frac{1}{\beta} \|x_n - x\|_0 \leq \|x_n - x\|$$

as left side tends to 0, therefore the lesser quantity which is positive (as $\beta > 0$ and $\beta \in \mathbb{R}$ must also tend to 0. Hence,

$$\frac{1}{\beta}\|x_n - x\|_0 \to 0$$
$$\|x_n - x\|_0 \to 0$$

Now, from other side, assuming RHS is correct,

$$\|x_n - x\|_0 \to 0$$

now,

$$\alpha\|x_n - x\| \leq \|x_n - x\|_0$$

as left side tends to 0, therefore the lesser quantity which is positive (as $\alpha > 0$ and $\alpha \in \mathbb{R}$ must also tend to 0. Hence,

$$\alpha\|x_n - x\|_0 \to 0$$
$$\|x_n - x\| \to 0$$

Therefore,

$$\|x_n - x\| \to 0 \iff \|x_n - x\|_0 \to 0$$

Hence proved.

Problem 6: If $\|.\|$ and $\|.\|_0$ are equivalent norms on X, show that the Cauchy sequence in $(X, \|.\|)$ and $(X, \|.\|_0)$ are the same.

Solution 6: As the given norms $||\cdot||$ and $||\cdot||_0$, therefore,

$$\alpha||x||_0 \leq ||x|| \leq \beta||x||_0$$

where α and β are positive real numbers. Considering an arbitrary Cauchy Sequence $x = (x_i)$ where $i \in \{1, ..., n\}$ without loss of generality. As it is a Cauchy sequence, therefore using the definition of Cauchy Sequence, there exists $\epsilon > 0$, and $n, m > N_1$ such that,

$$||x_n - x_m|| < \alpha\epsilon \tag{2}$$

Also,

$$||x_n - x_m||_0 \leq \frac{1}{\alpha}||x_n - x_m|| <$$

for $n, m > N$. Therefore, this is also same Cauchy Sequence i.e. $(X, ||\cdot||_0)$.

Also, from the other side, considering an arbitrary Cauchy Sequence in $(X, ||x||_0)$ $x = (x_i)$ where $i \in \{1, ..., n\}$ without loss of generality. As it is a Cauchy sequence, therefore using the definition of Cauchy Sequence, there exists $\epsilon > 0$, and $n, m > N$ such that,

$$||x_n - x_m||_0 < \frac{1}{\beta}\epsilon \tag{3}$$

Also,

$$||x_n - x_m||_0 \leq \beta||x_n - x_m|| <$$

for $n, m > N_2$. Therefore, this is also same Cauchy Sequence i.e. $(X, ||\cdot||)$.

Hence, both the Cauchy sequences are same.

Problem 7: Show that the operators $T_1, ..., T_4$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by

$$\begin{aligned} T_1 &: (\xi_1, \xi_2) \mapsto (\xi_1, 0) \\ T_2 &: (\xi_1, \xi_2) \mapsto (0, \xi_2) \\ T_3 &: (\xi_1, \xi_2) \mapsto (\xi_2, \xi_1) \\ T_4 &: (\xi_1, \xi_2) \mapsto (\gamma\xi_1, \gamma\xi_2) \end{aligned}$$

respectively, are linear, and interpret these operators geometrically.

Solution 7:

1. $T_1 : (\xi_1, \xi_2) \mapsto (\xi_1, 0)$,
   (a) **Property 1.** Let $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2)$, then $T_1(x_1 + x_2) = T_1(a_1 + a_2, b_1 + b_2) = (a_1 + a_2, 0) = (a_1, 0) + (a_2, 0) = T_1(x_1) + T_1(x_2)$. Hence it satisfies.
   (b) **Property 2.** Let $x = (a, b)$, then $\alpha x = (\alpha a, \alpha b)$. Now, $T_1(\alpha x) = T_1(\alpha a, \alpha b) = (\alpha a, 0) = \alpha(a, 0) = \alpha T_1(x)$. Hence it satisfies.

   Hence, T_1 is a linear operator. **Geometrically** this linear operator maps a vector to it's component on x-axis, or the shadow along x-axis if light is shown in the direction towards $y = 0$.

2. $T_2 : (\xi_1, \xi_2) \mapsto (0, \xi_2)$,
   (a) **Property 1.** Let $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2)$, then $T_2(x_1 + x_2) = T_2(a_1 + a_2, b_1 + b_2) = (0, b_1 + b_2) = (0, b_1) + (0, b_2) = T_2(x_1) + T_2(x_2)$. Hence it satisfies.
   (b) **Property 2.** Let $x = (a, b)$, then $\alpha x = (\alpha a, \alpha b)$. Now, $T_2(\alpha x) = T_2(\alpha a, \alpha b) = (0, \alpha b) = \alpha(0, b) = \alpha T_2(x)$. Hence it satisfies.

   Hence, T_2 is a linear operator. **Geometrically** this linear operator maps a vector to it's component on y-axis, or the shadow along y-axis if light is shown in the direction towards $x = 0$.

3. $T_3 : (\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$,

   (a) **Property 1.** Let $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2)$, then $T_3(x_1+x_2) = T_1(a_1+a_2, b_1+b_2) = (b_1 + b_2, a_1 + a_2) = (b_1, a_1) + (b_2, a_2) = T_3(x_1) + T_3(x_2)$. Hence it satisfies.

   (b) **Property 2.** Let $x = (a, b)$, then $\alpha x = (\alpha a, \alpha b)$. Now, $T_3(\alpha x) = T_3(\alpha a, \alpha b) = (\alpha b, \alpha a) = \alpha(b, a) = \alpha T_3(x)$. Hence it satisfies.

   Hence, T_3 is a linear operator. **Geometrically** this linear operator maps a vector to it's reflection on $x = y$.

4. $T_4 : (\xi_1, \xi_2) \mapsto (\gamma\xi_1, \gamma\xi_2)$,

   (a) **Property 1.** Let $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2)$, then $T_4(x_1+x_2) = T_1(a_1+a_2, b_1+b_2) = (\gamma(a_1 + a_2), \gamma(b_1 + b_2)) = (\gamma a_1, \gamma b_1) + (\gamma a_2, \gamma b_2) = T_4(x_1) + T_4(x_2)$. Hence it satisfies.

   (b) **Property 2.** Let $x = (a, b)$, then $\alpha x = (\alpha a, \alpha b)$. Now, $T_4(\alpha x) = T_4(\alpha a, \alpha b) = (\gamma\alpha a, \gamma\alpha b) = \alpha(\gamma a, \gamma b) = \alpha T_4(x)$. Hence it satisfies.

   Hence, T_4 is a linear operator. **Geometrically** this linear operator increases or decreases the length of the vector along same direction according to γ being greater than 1 or less than 0.

Problem 8:

(a) What are the domain, range and null space of T_1, T_2, T_3 in (Problem 7 of this assignment).

(b) What is the null space of T_4 in (Problem 7 of this assignment).

Solution 8:

(a)

1. $T_1 : (\xi_1, \xi_2) \mapsto (\xi_1, 0)$, here the **domain** for (ξ_1, ξ_2) is whole $\mathbb{R}^2$, **range** for $(\xi_1, 0)$ is that $\xi_1 \in \mathbb{R}$, i.e. $R = \{(\xi, 0) : \xi \in \mathbb{R}\}$. And for **null space**, $(\xi_1, 0)$ or range should be null vector i.e. $(\xi_1, 0) = (0, 0)$ or $\xi_1 = 0$ therefore, $Null\ Space = \{(0, \xi) : \xi \in \mathbb{R}\}$.

2. $T_2 : (\xi_1, \xi_2) \mapsto (0, \xi_2)$, here the **domain** for (ξ_1, ξ_2) is whole $\mathbb{R}^2$, **range** for $(0, \xi_2)$ is that $\xi_2 \in \mathbb{R}$, i.e. $R = \{(0, \xi) : \xi \in \mathbb{R}\}$. And for **null space**, $(0, \xi_2)$ or range should be null vector i.e. $(0, \xi_2) = (0, 0)$ or $\xi_2 = 0$ therefore, $Null\ Space = \{(\xi, 0) : \xi \in \mathbb{R}\}$.

3. $T_3 : (\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$, here the **domain** for (ξ_1, ξ_2) is whole $\mathbb{R}^2$, **range** for (ξ_1, ξ_2) i.e. whole $\mathbb{R}^2$. And for **null space**, (ξ_2, ξ_1) or range should be null vector i.e. $(\xi_2, \xi_1) = (0, 0)$ or $\xi_1 = 0, \xi_2 = 0$ therefore, $Null\ Space = \{(0, 0)\}$.

(b) For null space of $T_4 : (\xi_1, \xi_2) \mapsto (\gamma\xi_1, \gamma\xi_2)$, the vector should map to 0 vector. Therefore, $(\gamma\xi_1, \gamma\xi_2) = (0, 0)$.

1. Assuming, $\gamma = 0$, then all the vectors in $\mathbb{R}^2$ maps to 0 vector, hence then the null space will be complete $\mathbb{R}^2$.

2. Otherwise, if $\gamma \neq 0$, then

$$\begin{aligned}(\gamma\xi_1, \gamma\xi_2) &= (0,0)\\ \gamma\xi_1 = 0 &\implies \xi_1 = 0\\ \gamma\xi_2 = 0 &\implies \xi_2 = 0\end{aligned}$$

Hence, $(\xi_1, \xi_2) = (0,0)$ or **null space** only consists of 0 vector or $(0,0)$.

Problem 9: (Commutativity) Let X be any vector space and $S : X \to X$ and $T : X \to X$ any operators. S and T are said to **commute** if $ST = TS$, that is, $(ST)x = (TS)x$ for all $x \in X$. Do T_1 and T_3 commute?

Solution 9: Given $T_1 : (\xi_1, \xi_2) \mapsto (\xi_1, 0)$ and $T_3 : (\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$, so

$$\begin{aligned}(T_1T_3)x &= T_1(T_3x)\\ &= T_1(T_3(\xi_1, \xi_2))\\ &= T_1(\xi_2, \xi_1) = (\xi_2, 0) \;...(1)\end{aligned}$$

Now,

$$\begin{aligned}(T_3T_1)x &= T_3(T_1x)\\ &= T_3(T_1(\xi_1, \xi_2))\\ &= T_3(\xi_1, 0) = (0, \xi_1) \;...(2)\end{aligned}$$

Here (1) & (2) are not equal $(\xi_2, 0) \neq (0, \xi_1)$ i.e. $(T_3T_1)x \neq (T_1T_3)x$ therefore T_1 and T_3 **do not commute.**