

Functional Analysis

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Chapter 1

Metric Spaces

1.0.1 Notes

- "Soft Analysis" instead of "Hard Analysis"
- Harmonic Analysis - Study of Fourier Series
- Hilbert Spaces and Banach Spaces
- Limit Function ? (Major topic for analysis of sequences)
- Spaces of function? It is a linear space equipped with a norm, akin to a vector.
- Complete Space - (Limit actually exists)
- Spaces of functions can be infinite dimensional

1.0.2 Metric Spaces

- In Real Analysis, functions are defined on the Real Line \mathbb{R} .
- Limit and other functions use the distance function

$$d(x, y) = |x - y|$$

- For more general spaces, Real Line \mathbb{R} is replaced with abstract set \mathbb{X} .

Definition 1.0.1 A *Metric Space* is defined as an ordered pair (\mathbb{X}, d) . Where d (distance function) follows following properties:

1. d is a **real-valued, finite** (can also be infinite, if two functions are infinitely placed) and **non-negative** function
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$

4. *Triangle Inequality:* $d(x, y) \leq d(x, z) + d(z, y)$ or in general,

$$d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{n-1}, x_n)$$

Definition 1.0.2 Suppose $Y \subset X$, the **restriction of the Metric in X to Y** is $\tilde{d} = d|_{Y \times Y}$. Where \tilde{d} is the metric induced on Y by d and the new metric space is (Y, \tilde{d})

1.0.3 Examples of Metric Spaces

- Some examples: Real Line \mathbb{R} , Euclidean Plane $\mathbb{R} \times \mathbb{R}$, Sequence Spaces, Unitary Spaces (Complex Numbers)
- Distance function for \mathbb{R}^n , $d = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$
- For \mathbb{C} (complex numbers), it is called n -dimensional unitary Space \mathbb{C}^n consisting of ordered n -tuple complex numbers, having form

$$v = (x_1 + y_1 i, x_2 + y_2 i, \dots, x_n + y_n i) \text{ and } d(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}|$$

Sequences

- Sequence Space l^∞ , consisting of sequences which are bounded.

Definition 1.0.3 Let $\zeta_j \in l^\infty$ which is a **Bounded Sequence** then $\forall j, |\zeta_j| \leq c_x$ where c_x is a real number.

- Example : $\{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$, here $c_x = 1$ i.e. $|\zeta_j| \leq 1$.

Definition 1.0.4 **Distance Function** for sequences is defined as

$$d(x, y) = \sup_{j \in \mathbb{N}} |\zeta_j - \eta_j|$$

where $x, y \in l^\infty$ with $\zeta_j \in x$ and $\eta_j \in y$. Also, \sup (supremum) is the least upper bound.

Claim. Distance function of a sequence can be defined as \sup (supremum).

Proof 1 Let $x = \{\zeta_j\}$, $y = \{\eta_j\}$ and $z = \{\gamma_j\}$. Then using triangular inequality,

$$|\zeta_j - \eta_j| \leq |\zeta_j - \gamma_j| + |\gamma_j - \eta_j|$$

Taking supremum both sides,

$$\begin{aligned} &= \sup_{j \in \mathbb{N}} |\zeta_j - \eta_j| \leq \sup_{j \in \mathbb{N}} |\zeta_j - \gamma_j| + \sup_{j \in \mathbb{N}} |\gamma_j - \eta_j| \\ &= f(x, y) \leq f(x, z) + f(z, y) \end{aligned}$$

Hence proved.

Chapter 2

Examples of Metric Spaces

2.0.1 Some more Examples of Metric Spaces

Definition 2.0.1 *Function Space* $\mathbb{C}[a, b]$. where \mathbb{C} suggests continuous. It contains the abstract set X of functions which are based on a parameter t i.e. $f(t)$ and distance function (d) is defined as,

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|$$

where J is $[a, b]$.

Proof 2 It satisfies all the first 3 properties of distance functions. So, let's prove **Triangular Inequality**:

$$\begin{aligned} |x(t) - y(t)| &\leq |x(t) - z(t)| + |z(t) - y(t)| \\ &\leq \sup |x(t) - z(t)| + \sup |z(t) - y(t)| \end{aligned}$$

as RHS is independent of t , take sup on both sides

$$d(x, y) = \sup |x(t) - y(t)| \leq d(x, z) + d(z, y)$$

Hence Proved.

- **Note:** Sometimes sup may exist but not max.

Definition 2.0.2 *Discrete Metric Space*. Consider any set X , the discrete metric d ,

$$\begin{aligned} d(x, x) &= 0 \\ d(x, y) &= 1, \text{ for } (x \neq y) \end{aligned}$$

Proof 3 Let a, b, c be three distinct points. It already follows first 3 properties of distance.

$$d(a, c) = 1, \quad d(a, b) = 1 \quad \text{and} \quad d(b, c) = 1 \quad (2.1)$$

Therefore,

$$d(a, c) = 1 \leq (d(a, b) = 1) + (d(b, c) = 1) = 2 \quad (2.2)$$

Hence Proved.

- If we use the notion of circle i.e. all points which have equal distance from some single point. Then that circle varies according to the Space in consideration.
- **Comparison test** is to bound the series sum, and show that a particular series is convergent.

2.0.2 Some Problems

- **Problem 1:** Consider a distance function between two series $\tilde{x}, \tilde{y} \in l^\infty$:

$$d(\tilde{x}, \tilde{y}) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$$

Prove that it is a valid metric distance.

Proof 4 This function is bounded by $\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$

Let $f(x) = \frac{x}{1+x}$, as $f'(t) = \frac{1}{1+t^2} > 0$ f is monotonically increasing

$$\begin{aligned} t_1 &< t_2 \\ f(t_1) &< f(t_2) \\ \text{as, } |a+b| &\leq |a| + |b| \\ f(|a+b|) &\leq f(|a| + |b|) \\ \frac{|a+b|}{1+|a+b|} &\leq \frac{|a|+|b|}{1+|a|+|b|} \\ &\leq \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \\ &\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \end{aligned}$$

Let $a = x_j - y_j$ and $b = y_j - z_j$, where $z = (z_j)$

$$\begin{aligned} a+b &= x_j - z_j \\ \frac{|x_j - z_j|}{1+|x_j - z_j|} &\leq \frac{|x_j - y_j|}{1+|x_j - y_j|} + \frac{|y_j - z_j|}{1+|y_j - z_j|} \end{aligned}$$

Multiplying by $\frac{1}{2^j}$ on both sides and taking summation,

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - z_j|}{1+|x_j - z_j|} &\leq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1+|x_j - y_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|y_j - z_j|}{1+|y_j - z_j|} \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

Hence Proved.

- **(Prob-7 ex-1.1, Kreyzig):** If A is the sub-sequence l^∞ consisting of all sequences of 0's and 1's. What is the induced metric on A ?

Solution: It is the discrete metric distance.

Definition 2.0.3 Hamming Distance: The number of places two ordered tuples (let's of length 3) of 0's and 1's differs.

Proof 5 Hint: We can prove (if this is a valid distance metric) using enumeration for every possible bit possible i.e. 0 or 1.

2.0.3 Space of Bounded Functions

- Space of Bounded Functions is denoted by $B(A)$.
- $d(x, y) = \sup_{t \in A} |x(t) - y(t)|$

Definition 2.0.4 The ℓ^p space where $p \geq 1$ is a fixed real number, is called **Hilbert Sequence Space**. ℓ^p is a sequence space

$$\tilde{x} = (x_1, x_2, \dots, x_n, \dots) \quad (2.3)$$

$|x_1|^p + |x_2|^p + |x_3|^p + \dots$ converges,

$$\sum_{j=1}^{\infty} |x_j|^p < \infty$$

$$d(x, y) = \left(\sum_{j=1}^{\infty} |x_j - y_j|^p \right)^{\frac{1}{p}}$$

Definition 2.0.5 *Hilbert Space, Hölder's Inequality:*

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} |y_m|^q \right)^{\frac{1}{q}} \quad (2.4)$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Chapter 3

Inequalities

3.0.1 Some Inequalities

Definition 3.0.1 *Young's Inequality*

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad (3.1)$$

Proof 6 Let $f : [0, \infty) \mapsto \mathbb{R}$

$$\begin{aligned} f(\alpha) &= \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta \text{ for a fixed } \beta \\ f'(\alpha) &= \alpha^{p-1} - \beta = 0 \\ \alpha &= \beta^{\frac{1}{p-1}} \end{aligned}$$

now $\frac{1}{p} + \frac{1}{q} = 1$, therefore $\frac{q}{p} = \frac{1}{p-1}$

$$\begin{aligned} \alpha &= \beta^{\frac{q}{p}} \\ f''(\alpha) &= (p-1)\alpha^{p-2} > 0 \end{aligned}$$

$f(\alpha)$ has a min at $\alpha = \beta^{\frac{q}{p}}$.

$$\begin{aligned} 0 = f(\beta^{\frac{q}{p}}) &\leq f(\alpha) = \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta \\ \alpha\beta &\leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \end{aligned}$$

Hence Proved.

Definition 3.0.2 *Hölder's inequality*

Proof 7 Let $(x_n) \in \ell^p$ and $(y_n) \in \ell^q$. Then trivial case:

$$\sum_{k=1}^{\infty} |x_k|^p = 0 \text{ or } \sum_{k=1}^{\infty} |y_k|^q = 0 \quad (3.2)$$

Assume both these sums are not equal to 0, and set in **Young's Inequality**

$$\alpha = \frac{|x_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}} \text{ and } \beta = \frac{|y_k|}{(\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} \quad (3.3)$$

$$\alpha\beta = \frac{|x_k y_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} \leq \frac{1}{p} \cdot \frac{|x_k|^p}{\sum_{k=1}^{\infty} |x_k|^p} + \frac{1}{q} \cdot \frac{|y_k|^q}{\sum_{k=1}^{\infty} |y_k|^q}$$

Sum both sides from 1 to ∞

$$\begin{aligned} \frac{\sum_{k=1}^{\infty} |x_k y_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} &\leq \frac{1}{p} \cdot \frac{\sum_{k=1}^{\infty} |x_k|^p}{\sum_{k=1}^{\infty} |x_k|^p} + \frac{1}{q} \cdot \frac{\sum_{k=1}^{\infty} |y_k|^q}{\sum_{k=1}^{\infty} |y_k|^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} |x_k y_k| \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}$$

Hence Proved.

Definition 3.0.3 Minkowski Inequality for sequences $p > 1$, (x_n) and (y_n) are both sequence in ℓ^p .

$$(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} \quad (3.4)$$

Proof 8 Let $q = \frac{p}{p-1}$ and $\sum_{k=1}^{\infty} |x_k + y_k|^p \neq 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k + y_k|^p &= \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |x_k + y_k| \\ &\leq \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |x_k| + \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |y_k| \text{ using triangle ineq.} \end{aligned}$$

From the **Hölder's inequality** $\sum_{m=1}^{\infty} |x_m y_m| \leq (\sum_{m=1}^{\infty} |x_m|^p)^{\frac{1}{p}} (\sum_{m=1}^{\infty} |y_m|^q)^{\frac{1}{q}}$, taking $|x_k + y_k|^{p-1}$ as y_m and $|x_k|$ as x_m ,

$$\leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} (|x_k + y_k|^{p-1})^q)^{\frac{1}{q}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} (|x_k + y_k|^{p-1})^q)^{\frac{1}{q}}$$

As $\frac{1}{p} + \frac{1}{q} = 1$, therefore $(p-1)q = p$,

$$\begin{aligned}
 &\leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{q}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{q}} \\
 &\leq \left(\left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} \right) \left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{q}} \\
 \left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{1-\frac{1}{q}} &\leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}}
 \end{aligned}$$

Therefore,

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} \quad (3.5)$$

Hence Proved.

3.0.2 Sequence Metric Space

Definition 3.0.4 l^p where $p \geq 1$ is a metric space where,

$$d(x, y) = \left(\sum_{j=1}^{\infty} |x_j - y_j|^p \right)^{\frac{1}{p}} \quad (3.6)$$

provided $\sum_{j=1}^{\infty} |x_j|^p < \infty$.

Proof 9 To prove triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$ for metric distance, i.e. to prove:

$$\left(\sum_{j=1}^{\infty} |x_j - y_j|^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{\infty} |x_j - z_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} |z_j - y_j|^p \right)^{\frac{1}{p}} \quad (3.7)$$

Hint: Substitute $x_k = x_j - z_j$ and $y_k = z_j - y_j$ in **Minkowski Inequality**.

- **(Ex - 1.2, Kreyzig):** Consider a distance function between two series $\tilde{x}, \tilde{y} \in l^{\infty}$:

$$d(\tilde{x}, \tilde{y}) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$$

Prove that by replacing $\frac{1}{2^j}$ by $\mu_j > 0$, such that $\sum \mu_j$ converges, it is a metric space.

Proof 10 We need to show that these infinite terms converge. As $\sum \mu_j$ converges and the term multiplied is < 1 therefore, the series converge.

Chapter 4

Convergence, Cauchy Sequence and Completeness

4.0.1 Notes

- As n increases the distance between the terms (at very large n) comes closer and closer.

4.0.2 Some Definitions

Definition 4.0.1 Limit of a sequence. A sequence in (X, d) is said to converge or a convergent sequence if \exists an $x \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. x is called the limit of the sequence. If x doesn't exist in X (the metric space) then limit does not exist.

Definition 4.0.2 Bounded Set. A non-empty set $M \subset X$ is said to be bounded if its diameter is finite. Where diameter (δ) is -

$$\delta(M) = \sup_{x, y \in M} d(x, y) \quad (4.1)$$

Definition 4.0.3 Bounded Sequence. A convergent sequence in X is bounded if:

1. Its limit is unique.
2. If $x_n \rightarrow x$ (at ∞) and $y_n \rightarrow y$ then $d(x_n, y_n) = d(x, y)$

Definition 4.0.4 Cauchy Sequence. A sequence is a Cauchy Sequence in a metric space (X, d) if $\forall \epsilon \exists$ an N such that $d(x_m, x_n) < \epsilon \forall m, n > N$. **Note:** ϵ may not belong to the sequence and can be any arbitrarily small number.

Definition 4.0.5 Complete Metric Space. A metric space is complete if every Cauchy sequence converge.

Example 4.1: $X = \mathbb{R} - \{a\}$ is an incomplete metric space because we can find any cauchy sequence which converges at a but $a \notin X$.

Example 4.2: Real Numbers after removal of irrational numbers i.e. a set \mathbb{Q} of rational numbers.

Example 4.3: (a, b) is not complete metric space because there can exist a cauchy sequence whose limit is a or b which are not included.

Note: $d(x, y) = |x - y|$ is said to be **usual metric**.

Example 4.4: $(0, 1]$ with usual metric, and let the cauchy sequence be $x_n = \frac{1}{n}$ which converges to 0 and 0 does not exist in the metric space. So convergence is a property of metric space "more".

Lemma 1 *Every convergent sequence is a cauchy sequence.*

Proof 11 *If $x_n \rightarrow x \forall \epsilon > 0$ and $x_n > N$ the sequence converges.*

$$\begin{aligned} d(x_n, x) &< \frac{\epsilon}{2} \text{ where } n > N \\ d(x_m, x_n) &\leq d(x_m, x) + d(x_n, x) \\ d(x_m, x_n) &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ where } n, m > N \end{aligned}$$

Problem 4.1: Prove that \mathbb{R}^n and \mathbb{C}^n are complete metric spaces.

Proof 12 *Hint: Use the fact that \mathbb{R} is complete.*

Problem 4.2: Show that the set \mathbb{X} of all the integers with metric d defined as $d(x, y) = |x - y|$ is a complete metric space.

Proof 13 *To proof completeness,*

1. **Find all cauchy sequence.** Note: $1, 2, 3, \dots$ is not a cauchy sequence. A constant sequence is a cauchy sequence or a sequence which eventually becomes a constant sequence is a cauchy sequence (these are the only cauchy sequence).

Note: Discrete Metric is a complete metric space, as the only cauchy sequence is the sequence of constants which is convergent.