Set Theory $\bigcap$ Functional Analysis

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November 17, 2020

Overview

- Introduction
- Theorems Overview
- Proof of Equivalences

Introduction

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Axiom of Choice is a fundamental axiom in set theory which is very powerful and often used in proving a lot of important results in Mathematics. Yet it is the most controversial axiom in Mathematics, the reason being the paradoxes which result in its application.

In our discussion we are going to present some equivalence of Axiom of Choice and its use in proving fundamental results in Functional Analysis such as **Existence of Hamel basis** and **Hahn-Banach Theorem**.

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It is known that Existence of Hamel basis $\implies$ Axiom of Choice, but the proof is quite involved and so we do not include it.[2] It is also know that Hahn-Banach theorem is strictly weaker than Axiom of Choice.[5]

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Now we discuss the statements of each of them with appropriate definitions one after the other.

Axiom of Choice

Two formulations of **Axiom of Choice** are given.

- The Cartesian product of a non-empty family of non-empty sets is non-empty.
- For every non-empty set X, there exists a choice function f defined on X.

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These are the terms that need definitions:

1. Cartesian Product
2. Family of sets
3. Choice Function

Axiom of Choice – Definitions

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Family of Sets

Suppose I is a set, called as the **index set**, and with each $i \in I$ we associate a set A_i. Then, $\{A_i : i \in I\}$ is defined as the family of sets. This can also be denoted by $\{A_i\}_{i \in I}$

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Example

Consider a collection $\{\{1,2\},\{3,4\}\}$ and a function f defined as $f(\{1,2\}) = 2$ and $f(\{3,4\}) = 3$. Then f is a choice function.

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1. Partially-ordered set
2. Chain
3. Upper bound and maximal element

Partially-Ordered Set

A **partially ordered set** is a set together with a partial order on it $(X, \preccurlyeq)$. Where partial order in X is defined as a relation $\preccurlyeq$ in X such that, for all $x, y, z \in X$ it follows

1. $x \preccurlyeq x$
2. $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$
3. if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$

Zorn's Lemma – Partially-Ordered Set

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Note

It is not necessary for all $x, y \in X$ to have a partial order defined between them.

Zorn's Lemma – Chain

Chain

A set together was a total order on it is a a **chain** or **totally ordered set**. Where a relation $\preccurlyeq$ is **totally ordered** if for every $x, y \in X$ either $x \preccurlyeq y$ or $y \preccurlyeq x$.

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Zorn's Lemma – Upper Bound and Maximal Element

Upper Bound

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Maximum Element

Let X be a partially ordered set, then an element $a \in X$ is **maximum (or largest)** if $x \preccurlyeq a \ \forall x \in X$.

Well-Ordering Principle

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Well-ordering

A poset P is called **well-ordered** if it is a chain, and every non-empty subset $S \subseteq P$ has a minimum.

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Hamel Basis

For a vector space X, a set $B \subseteq X$ is called a **basis** (or **Hamel basis**) if B is a linearly independent set and $\text{span}(B) = X$.

Hahn-Banach Theorem

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Let X be a real vector space and p a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$f(x) \leq p(x) \ \forall x \in Z$$

Then f has a linear extension of $\tilde{f}$ from Z to X satisfying

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that is, $\tilde{f}$ is a linear functional on X, satisfying above inequality on X and $\tilde{f}(x) = f(x)$ for every $x \in Z$.

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We know about linear functional from class.

- What is sublinear functional?

Hahn-Banach Theorem – Sublinear Functional

Sublinear Functional

Let X be a linear space. A **Sublinear Functional** is a function $p : X \to \mathbb{R}$ that follows following properties

1. **Subadditive.** $p(x+y) \leq p(x) + p(y)\ \forall x, y \in X$.
2. **Nonnegatively Homogeneous.** $p(\lambda x) = \lambda p(x)\ \forall \lambda \geq 0$ where $\lambda \in \mathbb{R}, x \in X$

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Now we will discuss the results giving rough sketch of the proofs.

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Notice the that Zorn's Lemma is used in proving 4 of 6 results.

Proof sketch

Let X be a non-empty set.

1. Construct a poset $(P, \preccurlyeq)$ as follows

$$P = \{(Y, f) : \ Y \subseteq X \text{ and f is choice function on } Y\},$$

and $(Y, f) \preccurlyeq (Y', f')$ whenever $Y \subseteq Y'$ and $f = f'|_Y$.

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2. For any chain $C \subseteq P$, $(\tilde{Y}, \tilde{f})$ is an upper bound, where $\tilde{Y} = \bigcup_{(Y,f)\in C} Y$ and $\tilde{f}(S) = f(S)$ for any S such that f is defined on S. By Zorn's Lemma there is some maximal element in P, say (Y^*, f^*).

Zorn's Lemma $\implies$ Axiom of Choice

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3. Notice that $Y^* = X$, otherwise we can extend the f^* to $Y^* \cup \{x\}$, $x \in X \setminus Y^*$ as $f^*(S) = x$ for any S containing x which is a contradiction. Thus f^* is the choice function on X.

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2. For any chain $C \subseteq P$, $(\tilde{Y}, \leq_{\tilde{Y}})$ is an upper bound, where $\tilde{Y} = \bigcup_{(Y, \leq_Y) \in C} Y$ and extending $\leq_{\tilde{Y}}$ to $\tilde{Y}$. By Zorn's Lemma there is some maximal element in P, say $(Y^*, \leq_{Y^*})$.

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3. Notice that $Y^* = X$, otherwise we can extend the $\leq_{Y^*}$ to $Y^* \cup \{x\}$, $x \in X \setminus Y^*$ by defining x to be greater than every element in Y^* which is a contradiction. Thus $Y^* = X$ and $\leq_{Y^*}$ is the required well-ordering on X.

Well-ordering principle $\implies$ Axiom of choice

Proof.

Suppose X is a non-empty set, and $\leq$ is a well-ordering of X. Then $f(S) = \min S$, defines a choice function on X which is guaranteed to exist for any set S by Well-ordering principle. $\square$

Proof of Axiom of Choice $\implies$ Zorn's Lemma

Proof:

1. Let's assume there exist a non-empty partially ordered set P such that every chain in P has an upper bound, but does not contain a maximal element.
2. Considering axiom of choice is true, there must exist a choice function f on P, and let $x_0 := f(P)$. Also, let the set of *strict* upper bounds on a chain C in P be

$$Upp(C) := \{u \notin C : \forall x \in C, x \prec u\}$$

Lemma

For any chain C, the set $Upp(C)$ is non-empty.

Proof – Continue

A sub-chain C' is is an initial segment of a chain C such that $x \in C, y \in C'$ and $x \prec y$ implies $c \in C'$.

Intuition: For all $y \in C \setminus C'$, and for all $x \in C'$, $y \prec x$.

Now, let's define a function g, such that for any chain C,

$$g(C) := f(Upp(C))$$

Also, let's define an **attempt** as a well ordered set $A \subset P$ satisfying following:

1. $\min A = x_0$
2. For every proper initial segment $C \subset A$, $\min A \setminus C = g(C)$

Proof – Continue

Lemma

If A and A' are two attempts, then either $A \subseteq A'$ or $A' \subseteq A$.

As, for any two attempts A, A' either $A \subseteq A'$ or $A' \subseteq A$, therefore $A \cup A'$ is either A or A' which is an attempt. Let $\mathcal{A}$ be the set of all attempts then $A := \bigcup_{\tilde{A} \in \mathcal{A}} \tilde{A}$. Then A is also an attempt.

However, $A \cup \{g(A)\}$ is also an attempt and must have belonged in the previous set of attempts $\mathcal{A}$, and also $A \subseteq A \cup \{g(A)\}$ therefore $A \cup \{g(A)\} := \bigcup_{A \in \mathcal{A}} A$ but this is not the case, therefore a contradiction. And, **hence there must exist a maximal element of** P.

Conclusion – Part 1

Axiom of Choice $\iff$ **Zorn's Lemma** $\iff$ **Well-ordering principle**

Functional Analysis results

- Zorn's Lemma $\implies$ Existence of Hamel Basis.
- Zorn's Lemma $\implies$ Hahn-Banach Theorem

Proof sketch

Let $X \neq \{0\}$ be a vector space.

1. Construct a poset $(P, \preccurlyeq)$ where P is the set of subsets of X which are linearly independent and for every $B, B' \in P$, $B \preccurlyeq B'$ whenever $B \subseteq B'$.

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2. For any chain $C \subseteq P$, $\tilde{B} = \bigcup_{B \in C} B$ is an upper bound. By Zorn's Lemma there is some maximal element in P, say B^*.
3. Notice that $\text{span}(B^*) = X$ we can extend B^* by including $x \in X \setminus \text{span}(B^*)$ in it. Notice that the extended set is an element in P which is greater than B^* under the subset relation. This is a contradiction. Hence $\text{span}(B^*) = X$ and B^* is a linearly independent set, thus B^* is a basis for X.

- Zorn's Lemma $\implies$ Hahn-Banach Theorem

Proof:
Let's proof this in 2 parts,

Part 1: Let's define M as the partial order set of pairs (Z, f_Z) where

1. Z is a subspace of X containing Y.
2. $f_Z : Z \to \mathbb{R}$ is a linear functional extending f, satisfying

$$f_Z(z) \leq p(z) \; \forall \; z \in Z$$

with partial ordering defined as $(Z_1, f_{Z_1}) \preccurlyeq (Z_2, f_{Z_2})$ if $Z_1 \subset Z_2$ and $(f_{Z_2})\big|_{Z_1} = f_{Z_1}$. Since, $(Y, f) \in M$, M is a non-empty set. Let's choose any arbitrary chain $C = \{(Z_\alpha, f_{Z_\alpha})\}_{\alpha \in \Lambda}$ in M, with Λ being some indexing set.

Continue

Let $W = \bigcup_{\alpha \in \Lambda} Z_\alpha$ and construct a functional $f_W : W \implies \mathbb{R}$ defined as follow: If $w \in W$, then $w \in Z_\alpha$ for some $\alpha \in \Lambda$ and set $f_W(w) = f_{Z_\alpha}(w)$ for that particular α.

Lemma

(W, f_W) is an upperbound of C in M.

Proof:

- Notice that, W clearly contains Y, and we show that W is a subspace of X and f_W is a linear functional on W. Choose any $w_1, w_2 \in W$, then $w_1 \in Z_{\alpha_1}, w_2 \in Z_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in \lambda$. If $Z_{\alpha_1} \subset Z_{\alpha_2}$, say, then for any scalars $\beta, \gamma \in \mathbb{R}$ we have

$$w_1, w_2 \in Z_{\alpha_2} \implies \beta w_1 + \gamma w_2 \in Z_{\alpha_2} \subset W$$

Continue

Also, with $f_W(u) = f_{Z_{\alpha_1}}(u)$ and $f_W(v) = f_{Z_{\alpha_2}}(v)$,

$$
\begin{aligned}
f_w(\beta u + \gamma v) &= f_{Z_{\alpha_2}}(\beta u + \gamma v) \\
&= \beta f_{Z_{\alpha_2}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ linearity} \\
&= \beta f_{Z_{\alpha_1}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ because in same chain} \\
&= \beta f_{Z_W}(u) + \gamma f_{Z_W}(v)
\end{aligned}
$$

The case $Z_{\alpha_2} \subset Z_{\alpha_1}$ follows from a symmetric argument.

- Choose any $w \in W$,then $w \in Z_\alpha$ for some $\alpha \in \Lambda$ and

$$f_W(w) = f_{Z_\alpha}(w) \leq p(w) \text{ since } (w, Z_\alpha) \in M$$

Continue

Hence, (W, f_W) is an element of M and an upper bound of C. **By Zorn's lemma**, M has a maximal element $(Z, f_Z) \in M$, and f_Z is (by definition) a linear extension of f satisfying $f_Z(z) \leq p(z)$ for all $z \in Z$.

Part 2:
The proof is complete if we can show that $Z = X$. Suppose not, then there exists an $\theta \in X \setminus Z$; note $\theta \neq 0$ since Z is a subspace of X. Consider the subspace $Z_\theta = \text{span}\{Z, \{\theta\}\}$. Any $x \in Z_\theta$ has a unique representation $x = z + \alpha\theta$, $z \in Z$, $\alpha \in \mathbb{R}$. Indeed, if

$$x = z_1 + \alpha_1\theta = z_2 + \alpha_2\theta, z_1, z_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{R}$$

then $z_1 - z_2 = (\alpha_2 - \alpha_1)\theta \in Z$ since Z is a subspace of X. Since $\theta \notin Z$, we must have $\alpha_2 - \alpha_1 = 0$ and $z_1 - z_2 = \theta$. Next, we construct a functional $f_{Z_\theta} : Z_\theta \to \mathbb{R}$ defined by

$$f_{Z_\theta}(x) = f_{Z_\theta}(z + \alpha\theta) = f_Z(z) + \alpha\delta, \ \ldots \ (1)$$

where δ is any real number. It can be shown that f_{Z_θ} is linear and f_{Z_θ} is a proper linear extension of f_Z ; indeed, we have, for $\alpha = 0, f_{Z_\theta}(x) = f_{Z_\theta}(z) = f_Z(x)$. Also, it can be shown that

$$f_{Z_\theta}(x) \leq p(x) \ \forall x \in Z_\theta \ ... \ (2)$$

then $(Z_\theta, f_{Z_\theta}) \in M$ satisfying $(Z, f_Z) \leq (Z_\theta, f_{Z_\theta})$, thus contradicting the maximality of (Z, f_Z).

Therefore, by using **Zorn's Lemma** (as using in 1st part) we proved Hahn-Banach Theorem.

Conclusion

Thank You!

Proof for (2):

From (1), observe that (2) is trivial if $\alpha = 0$, so suppose $\alpha \neq 0$. We do have a single degree of freedom, which is the parameter δ in (1), thus the problem reduces to showing the existence of a suitable $\delta \in \mathbb{R}$ such that (2) holds. Consider any $x = z + \alpha\theta \in Z_\theta, z \in Z, \alpha \in \mathbb{R}$. Assuming $\alpha > 0$,(2) is equivalent to

$$
\begin{aligned}
f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = \alpha p(z/\alpha + \theta) \\
f_Z(z/\alpha) + \delta &\leq p(z/\alpha + \theta) \\
\delta &\leq p(z/\alpha + \theta) - f_Z(z/\alpha)
\end{aligned}
$$

Since the above must holds for all $z \in Z, \alpha \in \mathbb{R}$, we need to choose δ such that

$$\delta \leq \inf_{z_1 \in Z}(p(z_1 + \theta) - f_Z(z_1)) = m_1 \ \ldots \ (3)$$

Assuming $\alpha < 0$, (2) is equivalent to

$$\begin{aligned} f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = -\alpha p(-z/\alpha - \theta) \\ -f_Z(z/\alpha) - \delta &\leq p(-z/\alpha - \theta) \\ \delta &\geq -p(-z/\alpha - \theta) - f_Z(z/\alpha) \end{aligned}$$

Since the above must holds for all $z \in Z, \alpha \in \mathbb{R}$, we need to choose δ such that

$$\delta \geq \sup_{z_2 \in Z}(-p(z_2 + \theta) - f_Z(z_2)) = m_0 \ \ldots \ (4)$$

We are left with showing condition (3), (4) are compatible, i.e

$$-p(-z_2 - \theta) - f_Z(z_2) \leq p(z_1 + \theta) - f_Z(z_1) \ \forall z_1, z_2 \in Z$$

The inequality above is trivial if $z_1 = z_2$, so suppose not. We have that

$$
\begin{aligned}
Let, PP &= p(z_1+\theta) - f_Z(z_1) + p(-z_2-\theta) + f_Z(z_2) \\
&= p(z_1+\theta) + p(-z_2-\theta) + f_Z(z_2-z_1) \\
&\geq f_Z(z_2-z_1) + p(z_1+\theta-z_2-\theta) \\
&= f_Z(z_2-z_1) + p(z_1-z_2) \\
&= -f_Z(z_1-z_2) + p(z_1-z_2) \geq 0 \\
PP &\geq 0
\end{aligned}
$$

where linearity of f_Z and subadditivity of p are used. Hence, the required condition on δ is $m_0 \leq \delta \leq m_1$

References

A.A. Fraenkel, Y. Bar–Hillel and A. Levy. Foundations of Set Theory. 2nd ed., North–Holland, 1973.

Existence of Basis implies AC

Axiom of Choice equivalents

Hahn-Banach Theorem

Wikipedia, Hahn-Banach Theorem