Functional Analysis

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Contents

Chapter 1

Metric Spaces

1.0.1 Notes

- "Soft Analysis" instead of "Hard Analysis"
- Harmonic Analysis - Study of Fourier Series
- Hilbert Spaces and Banach Spaces
- Limit Function ? (Major topic for analysis of sequences)
- Spaces of function? It is a linear space equipped with a norm, akin to a vector.
- Complete Space - (Limit actually exists)
- Spaces of functions can be infinite dimensional

1.0.2 Metric Spaces

- In Real Analysis, functions are defined on the Real Line $\mathbb{R}$.
- Limit and other functions use the distance function

$$d(x, y) = |x - y|$$

- For more general spaces, Real Line $\mathbb{R}$ is replaced with abstract set $\mathbb{X}$.

Definition 1.0.1 *A **Metric Space** is defined as an ordered pair* $(\mathbb{X}, d)$. *Where* d *(distance function) follows following properties:*

1. d *is a **real-valued, finite** (can also be infinite, if two functions are infinitely placed) and **non-negative** function*
2. $d(x, y) = 0$ *iff* $x = y$
3. $d(x, y) = d(y, x)$

4. *Triangle Inequality:* $d(x, y) \leqslant d(x, z) + d(z, y)$ *or in general,*

$$d(x_0, x_n) \leqslant d(x_0, x_1) + d(x_1, x_2) + ... + d(x_{n-1}, x_n)$$

Definition 1.0.2 *Suppose $\mathbb{Y} \subset \mathbb{X}$, the **restriction of the Metric in $\mathbb{X}$ to $\mathbb{Y}$** is $\tilde{d} = d_{|YxY}$. Where $\tilde{d}$ is the metric induced on $\mathbb{Y}$ by d and the new metric space is $(Y, \tilde{d})$*

1.0.3 Examples of Metric Spaces

- Some examples: Real Line $\mathbb{R}$, Euclidean Plane $\mathbb{R} \times \mathbb{R}$, Sequence Spaces, Unitary Spaces (Complex Numbers)
- Distance function for $\mathbb{R}^n$, $d = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
- For $\mathbb{C}$ (complex numbers), it is called n-dimensional unitary Space $\mathbb{C}^n$ consisting of ordered n-tuple complex numbers, having form

$$\mathbf{v} = (x_1 + y_1 i, x_2 + y_2 i, ..., x_n + y_n i) \text{ and } d(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}|$$

Sequences

- Sequence Space l^∞, consisting of sequences which are bounded.

Definition 1.0.3 *Let $\zeta_j \in l^\infty$ which is a **Bounded Sequence** then $\forall j$, $|\zeta_j| \leqslant c_x$ where c_x is a real number.*

- Example : $\{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...\}$, here $c_x = 1$ i.e. $|\zeta_j| \leqslant 1$.

Definition 1.0.4 ***Distance Function** for sequences is defined as*

$$d(x, y) = \sup_{j \in \mathbb{N}} |\zeta_j - \eta_j|$$

where $x, y \in l^\infty$ with $\zeta_j \in x$ and $\eta_j \in y$. Also, sup *(supremum) is the least upper bound.*

Claim. Distance function of a sequence can be defined as sup (supremum).

Proof 1 *Let $x = \{\zeta_j\}$, $y = \{\eta_j\}$ and $z = \{\gamma_j\}$. Then using triangular inequality,*

$$|\zeta_j - \eta_j| \leqslant |\zeta_j - \gamma_j| + |\gamma_j - \eta_j|$$

Taking supremum both sides,

$$\begin{aligned} &= \sup_{j \in \mathbb{N}} |\zeta_j - \eta_j| \leqslant \sup_{j \in \mathbb{N}} |\zeta_j - \gamma_j| + \sup_{j \in \mathbb{N}} |\gamma_j - \eta_j| \\ &= f(x, y) \ \leqslant \ f(x, z) + f(z, y) \end{aligned}$$

Hence proved.

Chapter 2

Examples of Metric Spaces

2.0.1 Some more Examples of Metric Spaces

Definition 2.0.1 ***Function Space*** $\mathbb{C}[a, b]$. *where* C *suggests continuous. It contains the abstract set* X *of functions which are based on a parameter* t *i.e.* $f(t)$ *and distance function (d) is defined as,*

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|$$

where J *is* $[a, b]$.

Proof 2 *It satisfies all the first 3 properties of distance functions. So, let's prove* ***Triangular Inequality:***

$$\begin{aligned} |x(t) - y(t)| &\leqslant |x(t) - z(t)| + |z(t) - y(t)| \\ &\leqslant \sup |x(t) - z(t)| + \sup |z(t) - y(t)| \\ &\text{as RHS is independent of } t\text{, take sup on both sides} \\ d(x, y) = \sup |x(t) - y(t)| &\leqslant d(x, z) + d(z, y) \end{aligned}$$

Hence Proved.

- **Note:** Sometimes sup may exist but not max.

Definition 2.0.2 ***Discrete Metric Space****. Consider any set* X*, the discrete metric d,*

$$\begin{aligned} d(x, x) &= 0 \\ d(x, y) &= 1, \textit{ for } (x \neq y) \end{aligned}$$

Proof 3 *Let* a, b, c *be three distinct points. It already follows first* 3 *properties of distance.*

$$d(a, c) = 1,\ d(a, b) = 1 \text{ and } d(b, c) = 1 \tag{2.1}$$

Therefore,

$$d(a, c) = 1 \leqslant (d(a, b) = 1)\ + (d(b, c) = 1) = 2 \tag{2.2}$$

Hence Proved.

- If we use the notion of circle i.e. all points which have equal distance from some single point. Then that circle varies according to the Space in consideration.
- **Comparison test** is to bound the series sum, and show that a particular series is convergent.

2.0.2 Some Problems

- **Problem 1:** Consider a distance function between two series $\tilde{x}, \tilde{y} \in l^{\infty}$:

  $$d(\tilde{x}, \tilde{y}) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$$

  Prove that it is a valid metric distance.

  Proof 4 *This function is bounded by* $\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$
  Let $f(x) = \frac{x}{1+x}$, *as* $f'(t) = \frac{1}{1+t^2} > 0$ f *is monotonically increasing*

  $$\begin{aligned}
  t_1 &< t_2 \\
  f(t_1) &< f(t_2) \\
  \textit{as, } |a+b| &\leqslant |a| + |b| \\
  f(|a+b|) &\leqslant f(|a| + |b|) \\
  \frac{|a+b|}{1+|a+b|} &\leqslant \frac{|a|+|b|}{1+|a|+|b|} \\
  &\leqslant \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \\
  &\leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}
  \end{aligned}$$

  Let $a = x_j - y_j$ *and* $b = y_j - z_j$, where $z = (z_j)$

  $$\begin{aligned}
  a + b &= x_j - z_j \\
  \frac{|x_j - z_j|}{1 + |x_j - z_j|} &\leqslant \frac{|x_j - y_j|}{1 + |x_j - y_j|} + \frac{|y_j - z_j|}{1 + |y_j - z_j|}
  \end{aligned}$$

  Multiplying by $\frac{1}{2^j}$ *on both sides and taking summation,*

  $$\begin{aligned}
  \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - z_j|}{1 + |x_j - z_j|} &\leqslant \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|y_j - z_j|}{1 + |y_j - z_j|} \\
  d(x, z) &\leqslant d(x, y) + d(y, z)
  \end{aligned}$$

  Hence Proved.

- **(Prob-7 ex-1.1, Kreyzig):** If A is the sub-sequence l^{∞} consisting of all sequences of $0'$s and $1'$s. What is the induced metric on A?
  Solution: It is the discrete metric distance.

Definition 2.0.3 ***Hamming Distance:*** *The number of places two ordered tuples (let's of length* 3*) of* $0'\text{s and }1'\text{s}$ *differs.*

Proof 5 ***Hint:*** *We can prove (if this is a valid distance metric) using enumeration for every possible bit possible i.e.* 0 or 1.

2.0.3 Space of Bounded Functions

- Space of Bounded Functions is denoted by $B(A)$.
- $d(x, y) = \sup_{t \in A} |x(t) - y(t)|$

Definition 2.0.4 *The l^p space where $p \geqslant 1$ is a fixed real number, is called* ***Hilbert Sequence Space****. l^p is a sequence space*

$$\tilde{x} = (x_1, x_2, ..., x_n, ...) \tag{2.3}$$

$$|x_1|^p + |x_2|^p + |x_3|^p + ... \textit{ converges,}$$

$$\sum_{j=1}^{\infty} |x_j|^p < \infty$$

$$d(x, y) = (\sum_{j=1}^{\infty} |x_j - y_j|^p)^{\frac{1}{p}}$$

Definition 2.0.5 ***Hilbert Space, Hölder's Inequality:***

$$\sum_{j=1}^{\infty} |x_j y_j| \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{m=1}^{\infty} |y_m|^q)^{\frac{1}{q}} \tag{2.4}$$

where $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Chapter 3

Inequalities

3.0.1 Some Inequalities

Definition 3.0.1 ***Young's Inequality***

$$\alpha\beta \leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q} \tag{3.1}$$

Proof 6 *Let* $f : [0, \infty) \mapsto \mathbb{R}$

$$\begin{aligned} f(\alpha) &= \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta \textit{ for a fixed } \beta \\ f'(\alpha) &= \alpha^{p-1} - \beta = 0 \\ \alpha &= \beta^{\frac{1}{p-1}} \end{aligned}$$

now $\frac{1}{p} + \frac{1}{q} = 1$*, therefore* $\frac{q}{p} = \frac{1}{p-1}$

$$\begin{aligned} \alpha &= \beta^{\frac{q}{p}} \\ f''(\alpha) &= (p-1)\alpha^{p-2} > 0 \end{aligned}$$

$f(\alpha)$ *has a min at* $\alpha = \beta^{\frac{q}{p}}$.

$$\begin{aligned} 0 = f(\beta^{\frac{q}{p}}) \leqslant f(\alpha) &= \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta \\ \alpha\beta &\leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q} \end{aligned}$$

Hence Proved.

Definition 3.0.2 ***Hölder's inequality***

Proof 7 *Let* $(x_n) \in l^p$ *and* $(y_n) \in l^q$. *Then trivial case:*

$$\sum_{k=1}^{\infty} |x_k|^p = 0 \text{ or } \sum_{k=1}^{\infty} |y_k|^q = 0 \tag{3.2}$$

Assume both these sums are not equal to 0, *and set in* ***Young's Inequality***

$$\alpha = \frac{|x_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}} \text{ and } \beta = \frac{|y_k|}{(\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} \tag{3.3}$$

$$\alpha\beta = \frac{|x_k y_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} \leqslant \frac{1}{p} \cdot \frac{|x_k|^p}{\sum_{k=1}^{\infty} |x_k|^p} + \frac{1}{q} \cdot \frac{|y_k|^q}{\sum_{k=1}^{\infty} |y_k|^q}$$

Sum both sides from 1 *to* ∞

$$\begin{aligned} \frac{\sum_{k=1}^{\infty} |x_k y_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} &\leqslant \frac{1}{p} \cdot \frac{\sum_{k=1}^{\infty} |x_k|^p}{\sum_{k=1}^{\infty} |x_k|^p} + \frac{1}{q} \cdot \frac{\sum_{k=1}^{\infty} |y_k|^q}{\sum_{k=1}^{\infty} |y_k|^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} |x_k y_k| \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}$$

Hence Proved.

Definition 3.0.3 ***Minkowski Inequality*** *for sequences* $p > 1$, (x_n) *and* (y_n) *are both sequence in* l^p.

$$(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{p}} \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} \tag{3.4}$$

Proof 8 *Let* $q = \frac{p}{p-1}$ *and* $\sum_{k=1}^{\infty} |x_k + y_k|^p \neq 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k + y_k|^p &= \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |x_k + y_k| \\ &\leqslant \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |x_k| + \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |y_k| \text{ using triangle ineq.} \end{aligned}$$

From the ***Hölder's inequality*** $\sum_{m=1}^{\infty} |x_m y_m| \leqslant (\sum_{m=1}^{\infty} |x_m|^p)^{\frac{1}{p}} (\sum_{m=1}^{\infty} |y_m|^q)^{\frac{1}{q}}$, *taking* $|x_k + y_k|^{p-1}$ *as* y_m *and* $|x_k|$ *as* x_m,

$$\leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} (|x_k + y_k|^{p-1})^q)^{\frac{1}{q}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} (|x_k + y_k|^{p-1})^q)^{\frac{1}{q}}$$

As $\frac{1}{p}+\frac{1}{q}=1$, *therefore* $(p-1)q=p$,

$$\leqslant (\sum_{k=1}^{\infty}|x_k|^p)^{\frac{1}{p}}(\sum_{k=1}^{\infty}|x_k+y_k|^p)^{\frac{1}{q}} + (\sum_{k=1}^{\infty}|y_k|^p)^{\frac{1}{p}}(\sum_{k=1}^{\infty}|x_k+y_k|^p)^{\frac{1}{q}}$$

$$\leqslant ((\sum_{k=1}^{\infty}|x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty}|y_k|^p)^{\frac{1}{p}})(\sum_{k=1}^{\infty}|x_k+y_k|^p)^{\frac{1}{q}}$$

$$(\sum_{k=1}^{\infty}|x_k+y_k|^p)^{1-\frac{1}{q}} \leqslant (\sum_{k=1}^{\infty}|x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty}|y_k|^p)^{\frac{1}{p}}$$

Therefore,

$$(\sum_{k=1}^{\infty}|x_k+y_k|^p)^{\frac{1}{p}} \leqslant (\sum_{k=1}^{\infty}|x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty}|y_k|^p)^{\frac{1}{p}} \tag{3.5}$$

Hence Proved.

3.0.2 Sequence Metric Space

Definition 3.0.4 l^p *where* $p \geqslant 1$ *is a metric space where,*

$$d(x,y) = (\sum_{j=1}^{\infty}|x_j-y_j|^p)^{\frac{1}{p}} \tag{3.6}$$

provided $\sum_{j=1}^{\infty}|x_j|^p < \infty$.

Proof 9 *To prove triangle inequality* $d(x,y) \leqslant d(x,z)+d(z,y)$ *for metric distance, i.e. to prove:*

$$(\sum_{j=1}^{\infty}|x_j-y_j|^p)^{\frac{1}{p}} \leqslant (\sum_{j=1}^{\infty}|x_j-z_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{\infty}|z_j-y_j|^p)^{\frac{1}{p}} \tag{3.7}$$

Hint: *Substitute* $x_k = x_j - z_j$ *and* $y_k = z_j - y_j$ *in **Minkowski Inequality**.*

- **(Ex - 1.2, Kreyzig):** Consider a distance function between two series $\tilde{x},\tilde{y} \in l^{\infty}$:

  $$d(\tilde{x},\tilde{y}) = \sum_{j=1}^{\infty}\frac{1}{2^j}\frac{|x_j-y_j|}{1+|x_j-y_j|}$$

  Prove that by replacing $\frac{1}{2^j}$ by $\mu_j > 0$, such that $\sum \mu_j$ converges, it is a metric space.

Proof 10 ***We need to show that these infinite terms converge.*** *As* $\sum \mu_j$ *converges and the term multiplied is* < 1 *therefore, the series converge.*

Chapter 4

Convergence, Cauchy Sequence and Completeness

4.0.1 Notes

- As n increases the distance between the terms (at very large n) comes closer and closer.

4.0.2 Some Definitions

Definition 4.0.1 ***Limit of a sequence.*** *A sequence in $(\mathbb{X}, d)$ is said to converge or a convergent sequence if $\exists$ an $x \in \mathbb{X}$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. x is called the limit of the sequence. If x doesn't exist in $\mathbb{X}$ (the metric space) then limit does not exist.*

Definition 4.0.2 ***Bounded Set.*** *A non-empty set $M \subset \mathbb{X}$ is said to be bounded if it's diameter is finite. Where diameter (δ) is -*

$$\delta(M) = \sup_{x,y\in M} d(x, y) \tag{4.1}$$

Definition 4.0.3 ***Bounded Sequence.*** *A convergent sequence in $\mathbb{X}$ is bounded if:*

1. *Its limit is unique.*
2. *If $x_n \to x$ (at ∞) and $y_n \to y$ then $d(x_n, y_n) = d(x, y)$*

Definition 4.0.4 ***Cauchy Sequence.*** *A sequence is a Cauchy Sequence in a metric space $(\mathbb{X}, d)$ if $\forall \epsilon$ $\exists$ an N such that $d(x_m, x_n) < \epsilon$ $\forall$ $m, n > N$.* ***Note:*** *ϵ may not belong to the sequence and can be any arbitrarily small number.*

Definition 4.0.5 ***Complete Metric Space.*** *A metric space is complete if every Cauchy sequence converge.*

Example 4.1: $\mathbb{X} = \mathbb{R} - \{a\}$ is an incomplete metric space because we can find any cauchy sequence which converges at a but $a \notin \mathbb{X}$.

Example 4.2: Real Numbers after removal of irrational numbers i.e. a set $\mathbb{Q}$ of rational numbers.

Example 4.3: (a, b) is not complete metric space because there can exist a cauchy sequence whose limit is a or b which are not included.

Note: $d(x, y) = |x - y|$ is said to be **usual metric**.

Example 4.4: $(0, 1]$ with usual metric, and let the cauchy sequence be $x_n = \frac{1}{n}$ which converges to 0 and 0 does not exist in the metric space. So convergence is a property of metric space "more".

Lemma 1 *Every convergent sequence is a cauchy sequence.*

Proof 11 *If* $x_n \to x \;\forall\; \epsilon > 0$ *and* $x_n > N$ *the sequence converges.*

$$\begin{aligned} d(x_n, x) &< \frac{\epsilon}{2} \text{ where } n > N \\ d(x_m, x_n) &\leqslant d(x_m, x) + d(x_n, x) \\ d(x_m, x_n) &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ where } n, m > N \end{aligned}$$

Problem 4.1: Prove that $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete metric spaces.

Proof 12 ***Hint:*** *Use the fact that* $\mathbb{R}$ *is complete.*

Problem 4.2: Show that the set $\mathbb{X}$ of all the integers with metric d defined as $d(x, y) = |x - y|$ is a complete metric space.

Proof 13 *To proof completeness,*

1. ***Find all cauchy sequence.*** *Note:* $1, 2, 3, \ldots$ *is not a cauchy sequence. A constant sequence is a cauchy sequence or a sequence which eventually becomes a constant sequence is a cauchy sequence (these are the only cauchy sequence).*

Note: Discrete Metric is a complete metric space, as the only cauchy sequence is the sequence of constants which is convergent.

Chapter 5

Normed Linear Spaces

5.0.1 Vector Space

Definition 5.0.1 ***(Vector Space)***. *A vector space (or linear space) over a field* K *is a non-empty set* X *of elements* $x, y, \ldots$ *(called vectors) together with two algebraic operations. There operations are called vector addition and multiplication of vectors by scalars, that is, by elements of* K.

Vector addition associates with every ordered pair (x, y) of vectors a vector $x + y$, called the sum of x and y, in such a way that the following properties hold. Vector addition is commutative and associative, that is for all vectors we have

$$x + y = y + x$$
$$x + (y + z) = (x + y) + z;$$

furthermore, there exists a vector 0, called the zero vector, and for every x there exists a vector $-x$, such that for all vectors we have

$$x + 0 = x$$
$$x + (-x) = 0$$

Multiplication by scalars associates with every vector x and scalar α a vector αx (also written as $x\alpha$), alled the product of α and x, in such a way that for all vectors x, y and scalars α, β we have

$$\alpha(\beta x) = (\alpha\beta)x$$
$$1x = x$$

and the distributive laws

$$\alpha(x + y) = \alpha x + \alpha y$$
$$(\alpha + \beta)x = \alpha x + \beta x$$

Definition 5.0.2 ***Field.*** *A field is a set* F*, containing at least two elements, on which two operations* $+$ *and* $\cdot$ *(called addition and multiplication, respectively) are defined so that for each pair of elements* x, y *in* F *there are unique elements* $x + y$ *and* $x \cdot y$ *(often written as* xy*) in* F *for which the following conditions hold for all elements* x, y, z *in* F*:*

1. $x + y = y + x$ *(commutativity of addition)*
2. $(x + y) + z = x + (y + z)$ *(associativity of addition)*
3. *There is an element* $0 in F$*, called zero, such that* $x + 0 = x$*. (existence of an additive identity)*
4. *For each* x*, there is an element* $-x \in F$ *such that* $x + (-x) = 0$*. (existence if additive inverses)*
5. $xy = yx$ *(commutativity of multiplication)*
6. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(associaitivity of multiplication)*
7. $(x + y) \cdot z = x \cdot z + y \cdot z$ *and* $x \cdot (y + z) = x \cdot y + x \cdot z$ *(distributivity)*
8. *There is an element* $1 \in F$*, such that* $1 \neq 0$ *and* $x\dot{1} = x$ *(existence of a multiplicative identity)*
9. *Id* $x \neq 0$*, then there is an element* $x^{-1} \in F$ *such that* $x \cdot x^{-1} = 1$ *(existence of multiplicative inverses)*

From the definition we see that vector addition is a mapping $X \times X \to X$, whereas multiplication by scalars is a mapping $K \times X \to X$.

K is called the **scalar field** (or coefficient field) of the vector space X, and X is called a **real vector space** if $K = \mathbb{R}$ (the field of real numbers), and a **complex vector space** if $K = \mathbb{C}$ (the field of complex numbers).

The use of 0 for the scalar 0 as well as for the zero vector should cause no confusion, in general. If desirable for clarity, we can denote the zero vector by θ.

Definition 5.0.3 ***Eigen Value of Transformation****. Let* $T(x) = \alpha x$ *where* T *is transformation and* x *is vector then* α *is eigen value of transformation.*