Quiz 1: ICT Answer Key

Question 1a: Consider a single-use of an additive Gaussain noise channel, with

$$Y = X + W$$

where $X \in \{-a, +a\}$, for some $a > 0$, and $W \sim \mathcal{N}(0, \frac{N_0}{2})$ is Gaussian Noise. Write down the ML decoder for the channel, and simplify it to obtain an equivalent characterization of the ML decoder. (3 marks)

Solution 1a: For ML Decoder, we need to find

$$\hat{x} = \arg\max_{x \in \mathbb{C}} Pr(y|x)$$

where received vector is $y \in Y$ and $x \in X$. Now,

$$Pr(y| + a) = Pr_W(y - a) = \frac{1}{\sqrt{\pi N_0}} exp(-\frac{(y-a)^2}{N_0})$$
$$Pr(y| - a) = Pr_W(y + a) = \frac{1}{\sqrt{\pi N_0}} exp(-\frac{(y+a)^2}{N_0})$$

therefore,

1. $\hat{x} = +a$ if $Pr(y| + a) > Pr(y| - a)$
2. $\hat{x} = -a$ if $Pr(y| - a) > Pr(y| + a)$

Hence, further simplifying these expressions can be written as

$$\hat{x} = \begin{cases} +a & \text{, if } (y-a)^2 < (y+a)^2 \\ -a & \text{, if } (y-a)^2 > (y+a)^2 \end{cases}$$

Therefore,

$$\hat{x} = \begin{cases} +a & \text{, if } y > 0 \\ -a & \text{, if } y < 0 \end{cases}$$

Question 1b: Let $C = \{000, 011, 101, 110\}$, i.e., the 3-length simple parity-check code. Suppose that the codewords, $\underline{c}$ of C, are passed through an erasure channel. Consider the following decoder $\mathcal{D} : \mathbb{F}_2^3 \mapsto C \cup \{e\}$ that operates as follows:

$$\mathcal{D}(\underline{y}) = \begin{cases} \underline{c} & \text{, if } \underline{y} \text{ agrees with exactly one } \underline{c} \in C \text{ on the non-`?' entries,} \\ \text{`e'} & \text{, otherwise.} \end{cases}$$

If $\epsilon = 0.1$ were the erasure probability of the channel, compute the probability that $\mathcal{D}$ produces an 'e', given that all codewords equally likely. (3 marks)

Solution 1b: Here, we need to calculate probability of error or when the output of the decoder is 'e' given any codeword. Due to the symmetric nature of all the elements of $\mathbb{F}_2^3$ we can say that,

$$\text{Probability of Error } = Pr_e = 4 \times \frac{Pr_e(\underline{c}, \underline{c} \in C)}{4} = Pr_e(\underline{c})$$

Note, this fact can be observed too (if it is not apparent by symmetry argument). Here it is evident that if the number of '?' is greater than or equal to 2 after coming through erasure channel, then the output of Decoder $\mathcal{D}$ will be 'e' as then it won't be possible for exactly one $\underline{c} \in C$ on the non-'?' entries.

Hence, we can take any $c \in \mathbb{F}_2^3$ let's say $\{c_1 c_2 c_3\}$ which can be equal to 000 or 011 or 101 or 110. So, via symmetric argument let's take $Pr_e(000)$ without loss of generality, therefore

$$\text{Probability of Error } = Pr_e = Pr_e(000)$$

now,

$$Pr_e(\underline{c} = x = 000) = Pr(\underline{y} = 0`?'`?'|x) + Pr(\underline{y} = `?'0`?'|x) + Pr(\underline{y} = `?'`?'0|x) + Pr(\underline{y} = `?'`?'`?'|x)$$

which is

$$Pr_e(\underline{c} = x = 000) = (1-\epsilon)\epsilon^2 + (1-\epsilon)\epsilon^2 + (1-\epsilon)\epsilon^2 + \epsilon^3$$

as $Pr(`?'|0) = \epsilon$ and $Pr(0|0) = 1 - \epsilon$, and after substituting $\epsilon = 0.1$

$$Pr_e(\underline{c} = x = 000) = 3 \times (1 - 0.1) \times (0.1)^2 + (0.1)^3 = 0.028$$

Therefore answer i.e. the probability of obtaining 'e' is Probability of Error $= Pr_e = Pr_e(000) = 0.028$ or

$$\text{Probability of Error or obtaining `e' } = 0.028$$

Question 1c: Let C be an (n, M, d) code over $\mathbb{F}$ and we want to use the code as follows:

1. If the number of errors is τ or less, then the errors will be recovered correctly,
2. If $\tau <$ no. of errors $< \tau + \sigma$, then they will be detected.

What is the condition relating τ, σ, and d? (3 marks)

Solution 1c:

1. It is known as a fact that the maximum number of errors, so it can be recovered correctly is $\lfloor \frac{d-1}{2} \rfloor$, and it is given that if the number of errors is less than or equal to τ then the error can be recovered correctly, therefore
$$\tau \leq \lfloor \frac{d-1}{2} \rfloor$$
2. It is also a known fact that if the number of errors is less than d, then they can be detected, so we can say that
$$\tau + \sigma \leq d$$