Mathematics and Indian Classical Music

Team Members
- Siddharth Jain - 20171169
- Rishabh Singhal - 20171213
- Divyesh Jain - 20171171

"*Music is a secret exercise in arithmetic of the soul, unaware of its act of counting.*"
-- Gottfried Leibniz, philosopher and mathematician.

Let's start with the most elementary question.

What is music?
According to simple google search,
The word derives from Greek μουσική (*mousike*; "art of the Muses") and more formally " *vocal or instrumental sounds (or both) combined in such a way as to produce beauty of form, harmony, and expression of emotion.* " is music.

General definitions of music include common elements such as pitch (which governs melody and harmony), rhythm (and its associated concepts tempo, meter, and articulation), dynamics (loudness and softness), and the sonic qualities of timbre and texture (which are sometimes termed the "color" of a musical sound).

And apparently, Music is an extremely subjective experience. Some of the sound waves that reach the human ear are perceived to be pleasant while others are unpleasant and merely termed as noise. Thus, music is the art of combining sounds with a view to the beauty of expression of emotion. Musically good melodies are thus harmonious in character.

Historians also document early Indian and Chinese theorists who sought to show the correlation between the mathematical laws of harmonics and rhythms and its relationship towards human well-being.

Music, both vocal and instrumental, is primarily made up of beats. Musical beats are pulses in which time is marked. This is then played or sung as a series of notes in accordance to a pattern. Notes can be combined in an endless variety of groupings, but the specific number of notes that exist are finite. Similarly, in mathematics, the result always remains finite despite the various ways in which you can add, multiply, subtract, and divide numbers. It is these patterns and combinations that make music and mathematics very similar, for the art of calculation primarily lies in the understanding of the pattern.

Reiterating this fact is **Shankar Mahadevan**, famous Indian singer, music director and composer. *"I never really connected the dots between music and maths*," says Shankar Mahadevan, who holds an engineering degree in Computer Science from Mumbai University, India. "*But maybe learning to play the harmonium and veena before I was five, helped develop my mathematical skills without my realizing it.*'

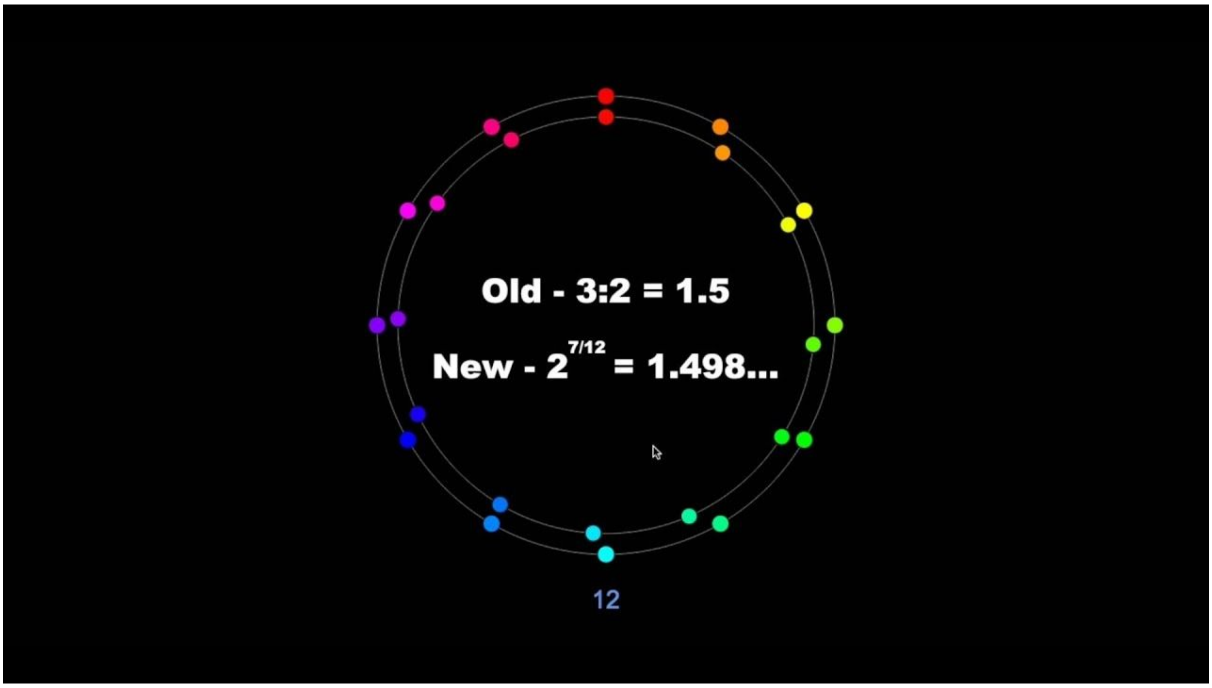
Mathematics is the basis of sound wave propagation, and a pleasant sound consists of harmony arising out of musical scales in terms of numerical ratios, particularly those of small integers. Mathematics is music for the mind while music is mathematics for the soul.

A suitable permutation and combination of some basic notes gives rise to melodious music which enthrals and transports one to a new enjoyable experience. While theoretically infinite possibilities can be thought of, only 279 had been in vogue. In Carnatic music, seven notes or *swaras* as they are called, form the foundation of the various permutations and combinations. Theoretically, one can mathematically think of $7! = 5040$ possibilities. But only 72 of them, called Janaka ragas, have been analyzed and found to have practical usage from the melody point of view. A raga thus has a set of rules that specify what notes of the octave must be used under the given rule and how to move from one note to the other. A Melakartha raga must necessarily have Sa and Pa and one of the Mas, one each of the Ris and Ga's, and one each of the Dhas and Nis, and further Ri must precede Ga and Dha must precede Ni, and thus we have $2 \times 6 \times 6 = 72$ possibilities of the Melakartha ragas.

It was Venkata Makhi who first thought of classifying these 72 *ragas* or combinations of *swaras*, and later Govindacharya adopted a slightly different combination. The basic *swaras* in Carnatic music and those that correspond to the basic notes in Western music are as follows:

Western system:	C	D	E	F	G	A	B
Carnatic music:	Sa	Ri	Ga	Ma	Pa	Dha	Ni

Why Are There Only Twelve Notes in Music



https://www.youtube.com/watch?v=IT9CPoe5LnM

“Rhythm is, after all, ‘the subdivision of a beat’. It's about ratios and proportions, the relationship between a part and a whole - all material from math classes.”

-- Frances Rauscher

A psychologist at the University of Wisconsin at Oshkosh

In the equal tempered system of Western music, the successive notes have a ratio of twelfth root of 2 as shown below:

Note	Ratio
C	1
Minor second C	1.059463
Major second C	1.122462
Minor third	1.189207
Major third	1.259921
Perfect fourth	1.334840
Augmented fourth	1.414214
Perfect fifth	1.498307
Minor sixth	1.587401
Major sixth	1.681793
Minor seventh	1.781797
Major seventh	1.887749

In the Carnatic music system also, just as in the Western music system, some of the *swaras* have small variants, and according to some scholars, 22 such notes are possible with the ratios as given below to the fundamental Sa.

Swara	Ratio	Frequency I Hz
Sa	1	240
Ri1	32/31	248
Ri2	16/15	256
Ri3	10/9	266.6
Ri4	9/8	270
Ga1	32/27	284.4
Ga2	6/5	288
Ga3	5/4	300
Ga4	81/64	303.7
Ma1	4/3	320
Ma2	27/20	324
Ma3	45/32	337.5
Ma4	64/45	341.3
Pa	3/2	360
Dha1	128/81	379
Dha2	8/5	384
Dha3	5/3	400
Dha4	27/16	405
Ni1	16/9	426.6
Ni2	9/5	432
Ni3	115/8	450
Ni4	31/16	465

And hence together,

Swara	Note	Frequency Ratio	Frequency 1	Frequency 2
S	C	1	220	261.6
R	C#	16/15	234.70	279.1
R	D	9/8	247.5	294.3
G	D#	6/5	264	314
G	E	5/4	275	327
M	F	4/3	293.3	348.8
M	F#	45/32	312.9	372.1
P	G	3/2	330	392.5
D	G#	8/5	352	418.6
D	A	27/16	371.3	441.5
N	A#	9/5	396	470
N	B	15/8	412.5	490.1

The 72 *Melakartha ragas*, according to Venkata Makhi, consisting of all the 7 *swaras* in the ascending as well as descending mode in proper order, are called *Janaka ragas* or parent *ragas*, and other *ragas* which arose out of them with absence of one or more of the 7 *swaras* were called *Janya ragas* (offspring *ragas*). The *Janya* ragas are derived from the 72 fundamental set by the permutation and combination of various ascending and descending notes, and mathematically about 3000 such *janya* ragas are possible. The 72 Melakartha raga system was responsible for the transformation of the raga system of carnatic music. Many new ragas came into existence and were popularised by great musical savants like Saint Thyagaraja, Muthuswami Dikshitar, etc. and many different kinds of musical compositions were developed with different structural arrangements. The musical forms included *Varnam, Kriti, Padam, Javali, Tillana, Swarangal, Swarajati,* etc.

The genius of the musician consists in how he can move to intermediate frequencies between those specified by the seven basic notes, and create microtones melodious and pleasing to the ear, called *gamakas*. (When one talks of the intermediate frequencies one is reminded of the Raman Effect discovered by the renowned Indian Nobel Laureate in Physics, Sir C.V.

Raman. While studying the spectra of fluids, he found that there were frequencies intermediate between those predicted according to Bohr's postulates, and he was able to explain them as being due to intermediate energy levels arising out of further degrees of freedom of the fluid molecules giving rise to vibrational, rotational etc. spectra).

(The list of the Melakarta ragas and the number of the Melakartha to which the raga belongs is decided according to the *katapayadi* system of numeration as prevalent in ancient India, which is as follows:

The following verse found in Śaṅkaravarman's Sadratnamāla explains the mechanism of the system.

नञ्ज्ञावचश्च शून्यानि संख्या: कटपयादय:|
मिश्रे तूपान्त्यहल् संख्या न च चिन्त्यो हलस्वर:||

nanyāvacaśca śūnyāni saṃkhyāḥ kaṭapayādayaḥ
miśre tūpāntyahal saṃkhyā na ca cintyo halasvaraḥ

Translation: na (न), nya (ञ) and a (अ)-s i.e. vowels represent zero. The (nine) integers are represented by consonant group beginning with ka, ṭa, pa, ya. In a conjunct consonant, the last of the consonants alone will count. A consonant without vowel is to be ignored.

Consonants have numerals assigned as per the above table. For example, ba (ब) is always three 3 whereas 5 can be represented by either nga (ङ) or ṇa (ण) or ma (म) or śha (श).

All stand-alone vowels like a (अ) and ṛ (ऋ) are assigned to zero 0.
In case of a conjunct, consonants attached to a non-vowel will not be valueless. For example, kya (क्या) is formed by k (क्) + ya (य) + a (अ). The only consonant standing with a vowel is ya (य). So the corresponding numeral for kya (क्या) will be 1.

There is no way of representing Decimal separator in the system.
Indians used the Hindu-Arabic numeral system for numbering, traditionally written in increasing place values from left to right. This is as per the rule

aṅkānām vāmato gati (अङ्कानाम् वामतो गति) which means numbers go from left to right.

The moment the name of a raga is given, the above system is used to find the Melakartha of that raga. Sometimes to fix the correct number, the name of the raga is slightly changed, as for instance, Sankarabharanam is called Dheera Sankarabharanam, and Kalyani is called Mechakalyani and so on.

Let us consider some examples. Take MayamalavaGowla. Here, Ma stands for 5 and ya stands for 1. So, the number we get is 51, and as per the reversing rule, the number of the Melkartha is 15.

The question arises which variant of Ri, Ga, Ma, Dha, Ni figures in what parent raga.

For ragas whose Melakartha number is 36 or less, M1 is chosen and for ragas whose Melakartha number is 37 or more, M2 is chosen.

MANODHARMASANGEETHA

- **The Pallavi,** of a Ragam-Tanam-Pallavi is highly mathematical, requiring the musician to sing the same line in different degrees of speed, gati etc. It is also usually mounted with a lot of mathematical structures in the kalpanswaram.
- **Kalpanaswaram** (also called manodharma swara or just swaras), is raga improvisation within a specific tala in which the musician improvises in the Indian music solfege (sa, ri, ga, ma, pa, da ni) after completing a composition.
- **Prastara,** the permutations of the infinite and finite varieties of Swara or Tala is called Prastara. Prastara of a tala anga means splitting up the anga into its possible component angas and presenting them with all possible varieties. If a chaturasa laghu (4 aksharakala) is taken as an example its given 8 possible prastaras in the case are:

  3+1, 2+2, 2+1+1, 1+3, 1+2+1, 1+1+2,1+1+1+1

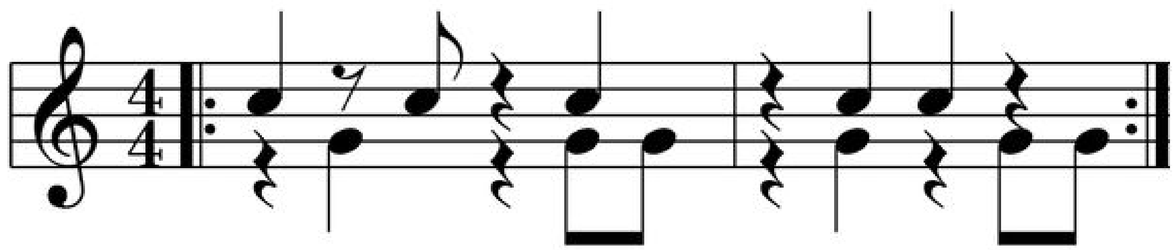
Rhythmic Loops

What can mathematics say about one of the most common features of contemporary music – rhythmic loops?

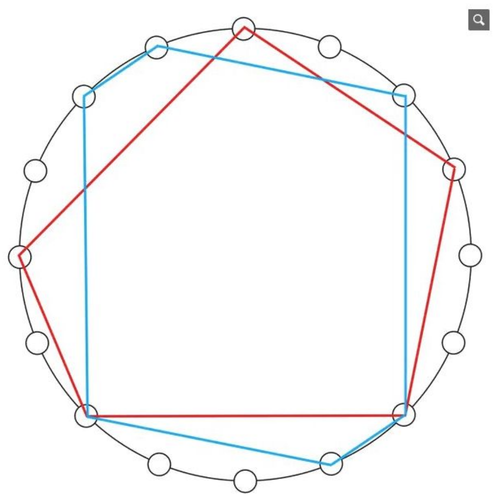
A natural geometrical characterisation of a periodic structure, such as a rhythmic loop, is as a circular arrangement of points. You can travel clockwise around a circle but inevitably you come back to where you started.

A common feature of rhythmic loops is that they are multilevel. For example, in Latin percussion, different instruments play different interlocking patterns that may or may not coincide. Such rhythms can be depicted by multiple polygons on the same circle.

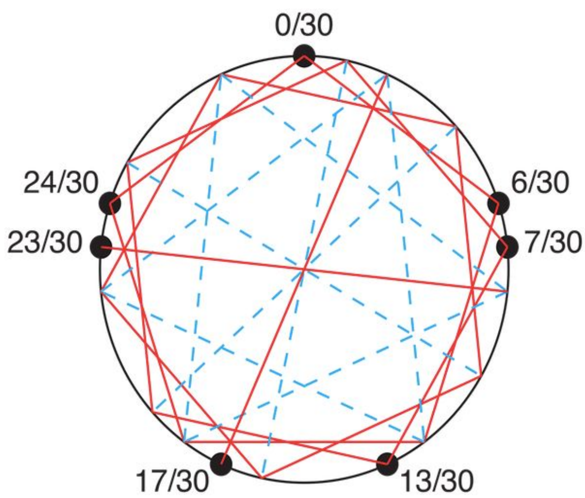
A simple geometrical representation is to draw lines that make each of these independent levels into an independent polygon. In this way, a multilevel rhythm becomes a collection of inscribed polygons.



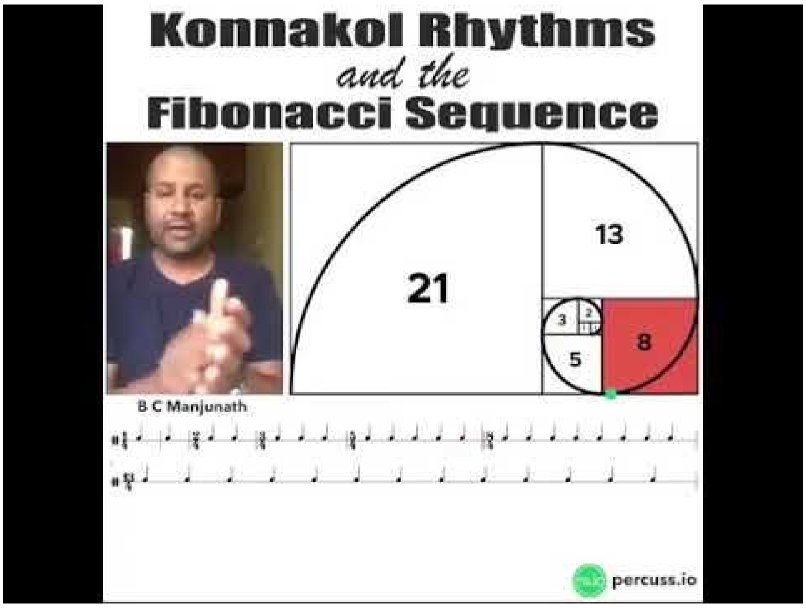
Clave (top) + Conga (bottom) rhythm in score notation. Andrew Milne, Author provided



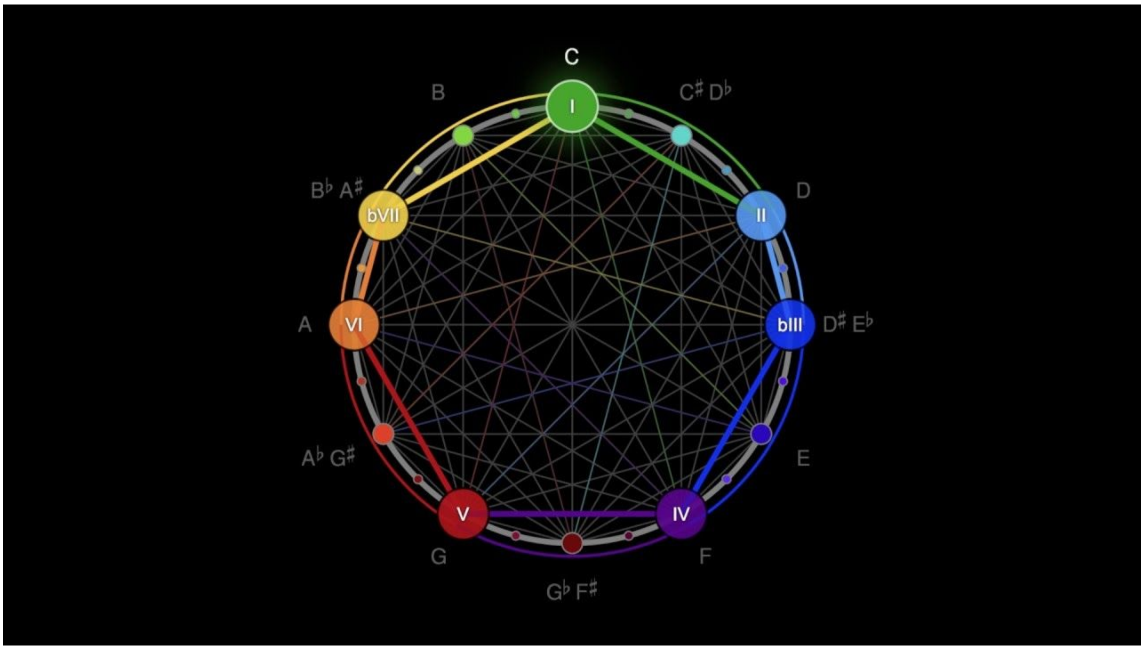
Clave (red) + Conga (blue) rhythm as polygons. Andrew Milne, Author provided



An irregular elemental polygon: 2 digons + 3 pentagons - 3 digons - 2 triangles. Only the labelled vertices are sounded.
Andrew Milne, Author provided



https://youtu.be/Zc7WMbVW-3s
When Mathematical beauties meet rhythm and beats…



https://www.youtube.com/watch?v=ZWzwb4BumIk
What does a rectangle sound like? A square, a circle, a pentagon? This short video introduces the geometry of music.

Mathematical Modeling

Set theory

Musical set theory uses the language of mathematical set theory in an elementary way to organize musical objects and describe their relationships. To analyze the structure of a piece of (typically atonal) music using musical set theory, one usually starts with a set of tones, which could form motives or chords. By applying simple operations such as transposition and inversion, one can discover deep structures in the music. Operations such as transposition and inversion are called isometries because they preserve the intervals between tones in a set.

Abstract algebra

Expanding on the methods of musical set theory, some theorists have used abstract algebra to analyze music. For example, the pitch classes in an equally tempered octave form an abelian group with 12 elements. It is possible to describe just intonation in terms of a free abelian group.

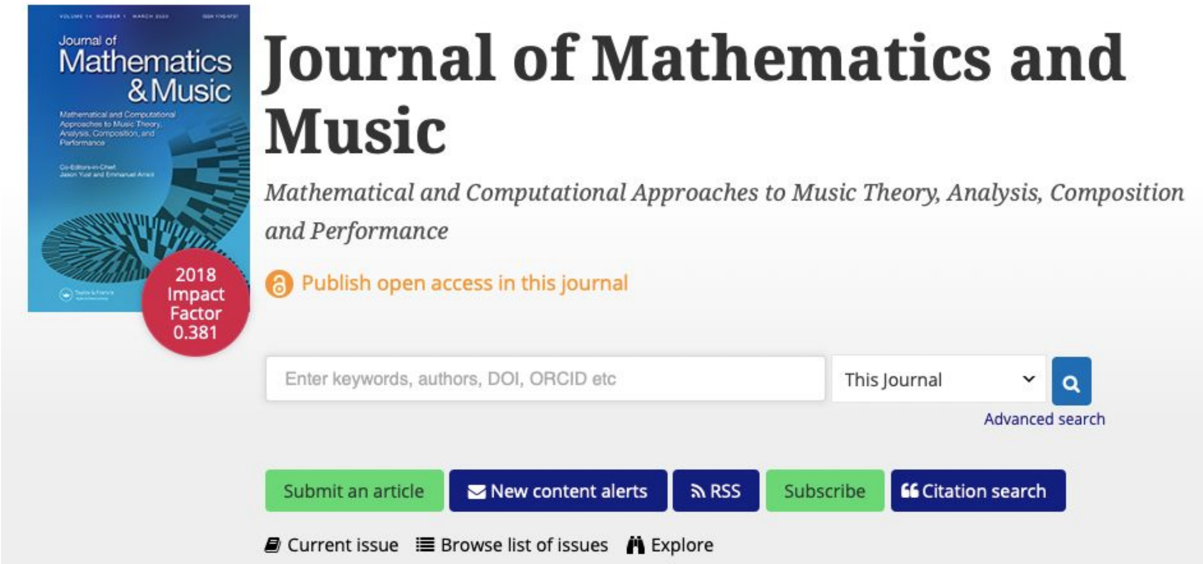
Transformational theory is a branch of music theory developed by David Lewin. The theory allows for great generality because it emphasizes transformations between musical objects, rather than the musical objects themselves.

Theorists have also proposed musical applications of more sophisticated algebraic concepts. The theory of regular temperaments has been extensively developed with a wide range of sophisticated mathematics, for example by associating each regular temperament with a rational point on a Grassmannian.

Category theory

The mathematician and musicologist Guerino Mazzola has used category theory (topos theory) for a basis of music theory, which includes using topology as a basis for a theory of rhythm and motives, and differential geometry as a basis for a theory of music performance[disambiguation needed], tempo, and intonation.

Journal of Mathematics and Music



This journal aims to advance the use of mathematical modelling and computation in music theory. The Journal focuses on mathematical approaches to musical structures and processes, including mathematical investigations into music-theoretic or compositional issues as well as mathematically motivated analyses of musical works or performances. In consideration of the deep unsolved ontological and epistemological questions concerning knowledge about music, the Journal is open to a broad array of methodologies and topics, particularly those outside of established research fields such as acoustics, sound engineering, auditory perception, linguistics etc.

One of the articles in this journal is "**An algorithmic approach to South Indian classical music**"

Where they developed a theoretical framework for representation and automated generation of South Indian classical music. The foundational part of the latter is based on symbolic dynamics and is implemented by translating the lexicographic rules for a rāga to constraints on a Markov chain whose state space is a layered graph. They analyzed the statistical properties of this Markov chain from the point of view of information theory. They also develop several tools in music signal processing, such as, (a) a procedure for automated generation of *gamakas* or ornamental notes, unique to South Indian classical music, (b) rhythm synchronization, and (c) an algorithm for perceptual scale shifts. The Online Supplement has computer synthesized music from Śankarābharaṇa rāga, rhythm synchronized music for a Mōhana rāga composition, original and synthesized gamakas over madhyama for Begaḍa and Nīlāmbari rāgas, and perceptually scale shifted rāgas Hindoḷa, Madhyamāvati, Śuddha sāveri, and Udaya ravi chandrikā rāgas from the base Mōhana rāga.

"*Composing music is like problem solving-you have to put text and rhythm together*"

-- Anita Cooper, faculty member at Peabody Prep

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