Advanced Optimization: Assignment

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Question 1. *Consider Half Space:* $\{x|a^T x \geq b\}$

1. *Is half space convex?*

2. *What is the difference between half space and half plane.*

Question 2. *Find the* ***interior*** *points of the following sets as a subsets of* $\mathbb{R}$.

1. $\{x| -1 \leq x \leq 1\}$

2. $\{1/n | n \in \mathbb{R}\}$

Question 3. *Find the* ***closure*** *of the following sets as a subsets of* $\mathbb{R}$.

1. $\{x|1 < x < 2\}$

2. $\{1, 2, 3\}$

Question 4. *Find the* ***boundary*** *of the following sets as a subsets of* $\mathbb{R}$.

1. $\{x| -1 < x < 1\}$

2. $\mathbb{Q}$

Question 5. *Consider* ***Log-sum-exp:*** $f(x) = \log(e^{x_1} + \cdots + e^{x_n})$ *which is convex on* $\mathbb{R}^n$. *Calculate* ***hessian*** *of* $f(x)$ *and verify it is equal to*

$$\nabla^2 f(x) = \frac{1}{(1^T z)^2}((1^T z)diag(z) - zz^T)$$

where $z = (e^{x_1}, \cdots, e^{x_n})$. *Also show that*

$$v^T \nabla^2 f(x) v \leq 0$$

Question 6. *Consider the problem of minimizing*

$$f_0(x) = (1/2)x^T P x + q^T z + r$$

where $P \in S^n_+$. *Find the* ***necessary and sufficient*** *condition for* x *to be minimizer of* f_0.

Question 7. *Consider the following problem*

$$\begin{aligned} &\textit{minimize } x^T x \\ &\textit{subject to } Ax = b \end{aligned}$$

Write out the ***Slater Condition***.

Question 8. *Prove that the loss function for the recommender system is non-convex.*

Question 9. *Consider* ***Perceptron Learning Algorithm:***

$$w = w + \eta(d(x) - y(x))x$$

where

- $d(x)$: *desired output in response to input* x
- $y(x)$: *actual output in response to* x

Is this gradient descent? What is the gradient?

Question 10. *Prove that* $||w_0||_0 \leq s$ *i.e.* $w \in B_0(s)$ *is a* ***non-convex*** *set.*

Question 11. *Prove that* $S = \{x|rank(x) \leq r\}$ *is a* ***non-convex*** *set.*

Question 12. *Prove the following:*

1. *Joint Convexity* $\implies$ *Marginal*
2. *Marginal* $\not\Longrightarrow$ *Joint Convexity*

Question 13. *Prove that partial derivatives must vanish at a bistable point given that the function is differentiable.*

Question 14. *Show that for the Gaussian Mixture modeling problem, AM-LVM reduces to the popular Llyod's algorithm for k-means clustering.*

Question 15. *Recall the law-rank matrix completion problem in recommendation systems from before Show that the objective in this optimization problem is not jointly convex in* U *and* V*. Then show that the objective is nevertheless, marginally convex in both the variables.*

Question 16. *Show that a function that is jointly convex is necessarily marginally convex as well. Similarly show that a (jointly) strongly convex and smooth function is marginally so as well.*

Question 17. *Marginal strong convexity does not imply convexity. Show this by giving an example of a function* $f : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}$ *that is marginally strongly convex in both its variables, but non-convex.*

Question 18. *Show that* $(\mathbf{x}^*, \mathbf{y}^*) \in argmin_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} f(\mathbf{x}, \mathbf{y})$ *must be a bistable point for any function even if* f *is non-convex.*

Question 19. *Let* $f : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}$ *be a differentiable, jointly convex function. Show that any bistable point of* f *is a global minimum for* f*.*

Hint: first show that directional derivatives vanish at bistable points.

Question 20. *The alternating minimization procedure may oscillate if the optimization problem is not well-behaved. Suppose for an especially nasty problem, the gAM procedure enters into the following loop*

$$(x^t, y^t) \to (x^{t+1}, y) \to (x^{t+1}, y^{t+1}) \to (x^t, y^{t+1}) \to (x^t, y^t)$$

Show that all four points in the loop are bistable and share the same function value. Can you draw a hypothetical set of marginally optimal coordinate curves which ,ay cause this to happen?