Functional Analysis - Project

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Topic

The equivalence of following:

1. Axiom of Choice
2. Zorn's Lemma
3. Existence of Hamel basis
4. Hahn-Banach Theorem
5. Well-Ordering Principle

Resources

1. P. R. Halmos - Naive Set Theory (Section 15, 16, 17)
2. Kreyszig - FA (Section 4.1, 4.2)

1 Jayadev - Notes

2 Rishabh - Notes

Axiom of extension. Two sets are equal if and only if they have the same elements.

Q1. What is an extension?

This is a not trivial property of sets, which can not be applied to humans $x \in y$ (if y is ancestor of x) per-se.

Some properties of set inclusion: Anti-symmetric, Reflexive and Transitive.

Aussonderungsaxiom

Axiom of specification. Given something specified (some property) and a set there exists one of the subset of the set following that property. (in layman terms)

Note: Think about the weak points of the box analogy of sets. It's true that something in a box is not equal to that thing without box, similarly $x \neq \{x\}$.

A sentence. Atomic parts: $x \in A$ or $A = B$. (other are logical operators).

2.1 Nothing contains everything

Consider an arbitrary set A which is assumed to contain everything (or is the universe). Then consider another set B, such that

$$y \in B \iff (y \in A \cap y \notin y). \tag{1}$$

Now consider if $B \in A$ or not. Hints: ***Russell's Paradox.***

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The **empty set** $\emptyset$ comes into picture, once we know there exists a set and by using axiom of specification.

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To **prove** that something is true about the empty set, prove that it cannot be false.

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Axiom of pairing. For any two sets there exists a set that they both belong to.

EXERCISE: Are all the sets obtained in this way distinct from one another? (We can use axiom of extension I guess).

// Read page 11. ending paragraph once more.

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Axiom of unions. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

Using axiom of specification => exact Union can be found out.

$A \cup A = A$ (idempotence).

Relations. If xRy then $(x, y) \in R$. The projection of R onto the first and second coordinates is known as domain (dom R) and range (ran R) respectively.

Principle of duality. Replace

1. $A \mapsto A'$
2. $\cap \mapsto \cup$
3. $\subset \mapsto \supset$

Symmetric difference or Boolean Sum. $A + B = (A - B) \cup (B - A)$ – Boolean sum because the elements appearing twice are removed in $A + B$.

Axiom of Powers. A set (or collection) exists for every set, which have all the subsets of the given set.

Orderings as a set. $\{a, b, c\} = \{\{a\}, \{a, b\}, \{a, b, c\}\}$. A nice way to see an ordering, as unique ordering can be reconstructed from RHS.

It can be seen that ordered pair $(a, b) = \{\{a\}, \{a, b\}\} \in \mathbb{P}(\mathbb{P}(A \cup B))$, hence axiom of specification can be used to produce the set of all such (a, b), where $a \in A$ and $b \in B$. And such a set is defined as $A \times B$

Functions.

1. Uniqueness condition: $(x, y) \in f$ and $(x, z) \in f \implies y = z$.
   xfy or $f(x) = y$ or $(x, y) \in f$.
2. y is value, and x is the argument (typical C++)..
3. Set of all functions is a subset of Power Set $P(X \times Y) == Y^X$.
4. The set of ordered pairs is called the *graph of the functions.*
5. *image* $f = Y$ *then* f *maps* X *onto* Y.
6. *inclusion map:* $f(x) = x$
7. $g = f|X$, *something like function projected on* X.
8. $f((x, y)) = f(x, y)$
9. *index, indexed set, family, index set, term of the family* x_i.

Numbers.

1. **Axiom of Infinity.** there exists a set containing 0 and containing the successor of each of its elements.

Piano's Axiom.

5^{th} property: If n and w are in ω, and if $n^+ = m^+$ then $n = m$.

Proposition 1. No natural number is a subset of any of it's elements.
Proof 1: Consider the set S with the given properties, now as $0 = \emptyset$, therefore no elements in 0 exists hence 0 is not a subset of any of it's elements or $0 \in S$.
Induction Step: Consider $n \in S$, now as $n = n$ therefore $n \subset n$ hence n can not be an element of n i.e. $n \notin n$ as otherwise $n \notin S$. Considering, $n^+ = n \cup \{n\}$, as $n \notin n$ therefore $n^+ \not\subset n$ — (1).
Now let $n^+ \subset x$ therefore $n \subset x$, which gives $x \notin n$, therefore n^+ is not a subset of any $x \in n$. Hence, n^+ is not a subset of any element of n^+ showing that $n^+ \in S$.
Now by mathematical induction, $S = \omega$ (the set of natural numbers).

Proposition 2. Every element of a natural number is a subset of it i.e. a natural number is a transitive set.
Proof 2. Similar proof using mathematical induction.

Proof of 5^{th} property becomes natural after using 1, 2 propositions proved above.

Recursion theorem. If a is an element,.... define $u(0) = a, u(n^+) = f(u(n))$ .. exists. This is applied as a definition for induction.

Arithmetic.
Two natural numbers are comparable iff $m \in n$ or $n \in m$ or $m = n$.

Order.
If X is a partially ordered set, and if $a \in X$, the set $s(a) = \{x \in X : x < a\}$ is the initial segment while $\bar{s}(a) = \{x \in X : x \leq a\}$ is weak initial segment.