# Functional Analysis Assignment 3

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**Note:** For $l^2$ space, sequences of numbers belonging to $\mathbb{C}$ (complex numbers) is considered **WLOG**.

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## Questions

**Problem 1:** Show that the subset $M = \{y = (\eta_j) | \Sigma \eta_j = 1\}$ of complex space $\mathbb{C}^n$ (cf. 3.1-4) is complete and convex. Find the vector of minimum norm in $M$.

**Solution 1:**

1. **Completeness**
   As $M$ is a subset of $\mathbb{C}^n$ then let's assume the induced inner product $< x, y >= \sum_{i=1}^{n} x_i \overline{y_i}$ where $x = (x_i)$ and $y = (y_i)$ for $i \in \{1, ..., n\}$. Now, metric is defined as,

$$\begin{aligned} d(x, y) = ||x - y|| &= \sqrt{< x - y, x - y >} \\ &= \sqrt{\sum_{i=1}^{n} (x_i - y_i)\overline{(x_i - y_i)}} \\ &= \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \\ &= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \end{aligned}$$

   where $x = (x_i)$ and $y = (y_i)$, now consider any Cauchy sequence $(x_m)$ in $M$, where $(x_m) = (\xi_1^{(m)}, ..., \xi_n^{(m)}$ (by definition of Cauchy sequence) for every $\epsilon > 0$ there is a $N$ such that

$$d(x_m, x_r) = \sqrt{\sum_{i=1}^{n} (\xi_i^{(m)} - \xi_i^{(r)})^2} < \epsilon \qquad (m, r > N) \tag{1}$$

   Now let's square both side,

$$\begin{aligned} \sum_{i=1}^{n} (\xi_i^{(m)} - \xi_i^{(r)})^2 < \epsilon^2 \\ (\xi_i^{(m)} - \xi_i^{(r)})^2 < \epsilon^2 \quad \forall m \in [n] \\ |\xi_i^{(m)} - \xi_i^{(r)}|^2 < \epsilon \end{aligned}$$

Therefore, from this the sequence $(\xi_j^{(1)}, \xi_k^{(2)}, ...)$ is a Cauchy sequence of numbers in $\mathbb{C}$ and it converges that is $\xi_j^{(m)} \to \xi_j$. Considering all these limits, let $x = (\xi_j)$ where $\xi_j$ are the individual limits, using (1),

$$d(x_m, x_r) = \sqrt{\sum_{i=1}^{n} (\xi_i^{(m)} - \xi_i^{(r)})^2} < \epsilon \qquad (m, r > N) d(x_m, x_r) \leq \epsilon$$

Let's see if this limit i.e. $(x)$ exists in $M$ or not, as the Cauchy sequence was constructed from the elements of $M$, it is true that $x_i = (\xi_j^i)$ and $\sum \xi_j^i = 1$. Now consider the limit as $i \to \infty$, i.e. $\lim_{i\to\infty} \sum \xi_j^i = \lim_{i\to\infty} 1$ which gives,

$$\sum \xi_j = 1 \tag{2}$$

where $x = (\xi_j)$, therefore it is in $M$. Hence as the limit is in the set itself (for all Cauchy sequences), therefore subspace $M$ is complete.

2. **Convexity**
For $M$ to be convex, it is needed to prove that if $y_1, y_2 \in M$ then $\alpha y_1 + (1-\alpha) y_2 \in M \ \forall \alpha \in [0, 1]$. Let, $y_1, y_2 \in M$ where $y_1 = (\eta_{j1}), y_2 = (\eta_{j2})$ and $\Sigma \eta_{j1} = 1$ and $\Sigma \eta_{j2} = 1$. Then, $\alpha y_1 + (1-\alpha) y_2 = (\alpha * \eta_{j1} + (1-\alpha) * \eta_{j2})$, now consider, $\Sigma(\alpha \eta_{j1} + (1-\alpha)\eta_{j2})$.

$$\begin{aligned}
\Sigma(\alpha\eta_{j1} + (1-\alpha)\eta_{j2}) &= \Sigma(\alpha\eta_{j1}) + \Sigma((1-\alpha)\eta_{j2}) \\
&= \alpha\Sigma\eta_{j1} + (1-\alpha)\Sigma\eta_{j2} \\
&= \alpha \times 1 + (1-\alpha) \times 1 \\
&= \alpha + 1 - \alpha = 1 + (\alpha - \alpha) \\
&= 1 + 0 = 1
\end{aligned}$$

Therefore, $\Sigma(\alpha\eta_{j1} + (1-\alpha)\eta_{j2}) = 1$, hence $\alpha y_1 + (1-\alpha) y_2 \in M$. Hence, the given subset $M$ is **convex.**

3. **Vector of Minimum Norm**
For the Vector of Minimum Norm (let's say of norm $\delta$), $||y|| \geq \delta \ \forall y \in M$, now for complex space $\mathbb{C}^n$ considering $l^p$ norm where $1 \leq p < +\infty$ (**for the case of unitary space (cf. 3.1-4 consider p = 2** given in this question).

$$||y|| = (\sum_{i=1}^{n} |\eta_i|^p)^{\frac{1}{p}}$$

Using Hölder's Inequality, i.e. $\sum_{k=1}^{\infty} |x_k y_k| \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}$. In this let, $x_k = 1 \ \forall k \in \{1, ..., n\}$ and $x_k = 0$ for $k > n$. And, let $y_k = \eta_k \ \forall k \in \{1, ..., n\}$, and $y_k = 0$ for $k > n$, and set $p = q$.

$$\begin{aligned}\sum_{k=1}^{n}|\eta_k| &\leq (\sum_{k=1}^{n}1^p)^{\frac{1}{p}}(\sum_{k=1}^{n}|\eta_k|^q)^{\frac{1}{q}}\\ &\leq (n)^{\frac{1}{p}}(\sum_{k=1}^{n}|\eta_k|^p)^{\frac{1}{p}}\\ &\leq n^{\frac{1}{p}}||y||\end{aligned}$$

Now, as $|\sum_{k=1}^{n}\eta_k| \leq \sum_{k=1}^{n}|\eta_k|$ hence,

$$|\sum_{k=1}^{n}\eta_k| \leq \sum_{k=1}^{n}|\eta_k| \leq n^{\frac{1}{p}}||y||$$

$$1 \leq n^{\frac{1}{p}}||y|| \text{ as} \sum_{k=1}^{n}\eta_k = 1$$

Therefore,

$$||y|| \geq \frac{1}{n^{\frac{1}{p}}}$$

and, it is true that for the vector $y = (\eta_k)$ where $\eta_k = \frac{1}{n}$ $\forall k \in \{1, ..., n\}$, $||y|| = \frac{1}{n^{\frac{1}{p}}}$. Hence the vector with the minimum norm is,

$$y = (\eta_k), \text{ where } \eta_k = \frac{1}{n} \ \forall k \in \{1, ..., n\}$$

as $\sum \eta_k = 1$, $y \in M$.

**Problem 2:** Show that the vector space $X$ of all real-valued continuous functions on $[-1, 1]$ is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on $[-1, 1]$.

**Solution 2:**
For a vector space $X$ to be called as a direct sum of two vector sub-spaces $Y$ and $Z$, i.e. $X = Y \oplus Z$ if for every $x \in X$ there exists a unique representation $x = y + z$ where $y \in Y, z \in Z$.

So, let's consider an arbitrary function $f(x) \in X$ and let $f_e(x) = \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$,

**Claim 1:** $f_o(x) = \frac{f(x)-f(-x)}{2}$ belongs to the set of all odd continuous functions on $[-1, 1]$ for, $f(x) \in X$.

**Proof 1:** Let's calculate $f_o(-x)$,

$$\begin{aligned}
f_o(x) &= \frac{f(x) - f(-x)}{2} \\
f_o(-x) &= \frac{f(-x) - f(-(-x))}{2} \\
&= \frac{f(-x) - f(x)}{2} \\
&= -\frac{-f(-x) + f(x)}{2} = -\frac{f(x) - f(-x)}{2} \\
&= -f_o(x)
\end{aligned}$$

Hence, $f_o(-x) = -f_o(x)$, therefore $f_o$ is an odd function defined for $[-1, 1]$, as $f(x)$ is itself defined on $[-1, 1]$ and continuous (as it is a linear sum of $f(x)$)

**Claim 2:** $f_e(x) = \frac{f(x)+f(-x)}{2}$ belongs to the set of all even continuous functions on $[-1, 1]$ for, $f(x) \in X$.
**Proof 2:** Let's calculate $f_e(-x)$,

$$\begin{aligned}
f_e(x) &= \frac{f(x) + f(-x)}{2} \\
f_e(-x) &= \frac{f(-x) + f(-(-x))}{2} \\
&= \frac{f(-x) + f(x)}{2} \\
&\frac{f(x) + f(-x)}{2} \\
&= f_e(x)
\end{aligned}$$

Hence, $f_e(-x) = f_e(x)$, therefore $f_e$ is an even function defined for $[-1, 1]$, as $f(x)$ is itself defined on $[-1, 1]$ and continuous (as it is a linear sum of $f(x)$)

Let's calculate $f_o(x) + f_e(x)$,

$$\begin{aligned}
f_o(x) + f_e(x) &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\
&= \frac{f(x) + f(-x) + f(x) - f(-x)}{2} \\
&= \frac{2f(x)}{2} = f(x)
\end{aligned}$$

Hence, $f_o(x) + f_e(x) = f(x)$ therefore it can be a direct sum given it is unique.

**Claim 3:** $f(x) = f_o(x) + f_e(x)$ is a unique representation.
**Proof 3:** Note : $f'$ is just another function here (for the notation part) rather than the derivative of $f$ for the purpose of the given proof.
Let's assume there exists another two function $f'_o(x) + f'_e(x)$ such that $f(x) = f'_o(x) + f'_e(x)$, now for this to happen,

$$f(x) = f'_o(x) + f'_e(x) = f_o(x) + f_e(x)$$

replacing $x$ by $-x$,

$$\begin{aligned} f_o'(-x) + f_e'(-x) &= f_o(-x) + f_e(-x) \\ -f_o'(x) + f_e'(x) &= -f_o(x) + f_e(x) \\ f_o(x) - f_o'(x) &= f_e(x) - f_e'(x) \end{aligned}$$

Now, $f_o(x) - f_o'(x)$ belongs to subspace of even functions while $f_e(x) - f_e'(x)$ belongs to subspace of even functions, hence let, $z(x) = f_e(x) - f_e'(x) = f_o(x) - f_o'(x)$. Here z(x) should follow both properties of even and odd functions. Therefore,

$$z(-x) = z(x) \text{ and } z(-x) = -z(x)$$

Hence, $z(x) = -z(x)$ or $2z(x) = 0 \implies z(x) = 0$, therefore, $f_e(x) - f_e'(x) = f_o(x) - f_o'(x) = 0$ or $f_e(x) = f_e'(x)$ and $f_o(x) = f_o'(x)$. Therefore by contradiction $f(x) = f_o(x) + f_e(x)$ is **a unique representation**.

And hence, it's **a direct sum** (by definition).

**Problem 3:** Let $X = \mathbb{R}^2$. Find $M^\perp$ if $M$ is (a) $\{x\}$, where $x = (\xi_1, \xi_2) \neq 0$, (b) a linearly independent set $\{x_1, x_2\} \subset X$.

**Solution 3:**

(a) $M = \{x\}$ where $x = (\xi_1, \xi_2) \neq 0$
For any $y \in M^\perp$, $< y, x >= 0$ let $y = (y_1, y_2)$, then

$$\begin{aligned} < y, x > &= 0 \\ y_1\xi_1 + y_2\xi_2 &= 0 \end{aligned}$$

Let $y_2 = \alpha$, then

$$\begin{aligned} y_1\xi_1 + y_2\xi_2 &= 0 \\ y_1\xi_1 + \alpha\xi_2 &= 0 \\ y_1\xi_1 &= -\alpha\xi_2 \\ y_1 &= -\frac{\alpha\xi_2}{\xi_1} \end{aligned}$$

There a general $y = (-\frac{\alpha\xi_2}{\xi_1}, \alpha)$. It is defined as $x = (\xi_1, \xi_2) \neq 0$ therefore, $\xi_1 \neq 0$.
So, $M^\perp = \{(-\frac{\alpha\xi_2}{\xi_1}, \alpha) | \alpha \in \mathbb{R}\}$.

(b) $M$ is a linearly independent set $\{x_1, x_2\} \subset X = \mathbb{R}^2$
Let an arbitrary $y \in M^\perp$ where $y = (y_1, y_2)$ then $< y, x_1 >=< y, x_2 >= 0$ where $< x_1, x_2 >=$

0 as they are linearly independent, then we need to solve these equations,

$$< y, x_1 > = y_1x_{11} + y_2x_{12} = 0 - (1)$$
$$< y, x_2 > = y_1x_{21} + y_2x_{22} = 0 - (2)$$
$$< x_1, x_2 > = x_{11}x_{21} + x_{12}x_{22} = 0$$

where, $x_1 = (x_{11}, x_{12})$ and $x_2 = (x_{21}, x_{22})$, from (1),

$$y_1x_{11} + y_2x_{12} = 0$$
$$y_1 = -\frac{y_2x_{12}}{x_{11}}$$

Similarly from (2),

$$y_1x_{21} + y_2x_{22} = 0$$
$$y_1 = -\frac{y_2x_{22}}{x_{21}}$$

now equating both,

$$y_1 = -\frac{y_2x_{12}}{x_{11}} = -\frac{y_2x_{22}}{x_{21}}$$

or,

$$x_{12}x_{21} - x_{22}x_{11} = 0 \tag{3}$$

but from this equation and $x_{11}x_{21} + x_{12}x_{22} = 0$ it seems that $x_{11}^2 + x_{12}^2 = 0$ or $y = (0, 0)$. Let's consider $x_1 = 0$ then $x_1, x_2$ are not independent (as stated in the question), therefore first possibility is completely ruled out. Hence, $M^\perp = \{(0, 0)\}$, where the inner product of $(0, 0)$ and both $x_1, x_2$ is 0.

**Problem 4:** Show that $Y = \{x | x = (\xi_j) \in l^2, \xi_{2n} = 0, n \in \mathbb{N}\}$ is a closed subspace of $l^2$ and find $Y^\perp$. What is $Y^\perp$ if $Y = span\ \{e_1, ..., e_n\} \subset l^2$, where $e_j = (\delta_{jk})$?

**Solution 4:**

1. $Y = \{x | x = (\xi_j) \in l^2, \xi_{2n} = 0, n \in \mathbb{N}\}$ is a closed subspace of $l^2$.

   (a) $Y$ is a vector subspace
   Let $y_1, y_2 \in Y$ and $\lambda \in \mathbb{C}$, then $y_1 + y_2 = \alpha_j = (\xi_{1j} + \xi_{2j})$, now $\xi_{1j}, \xi_{2j} \in \mathbb{C}$ therefore $\xi_{1j} + \xi_{2j} \in \mathbb{C}$. And as $\alpha_{2n} = \xi_{1(2n)} + \xi_{2(2n)} = 0 + 0 = 0\ \forall n \in \mathbb{N}$, therefore $y_1 + y_2 \in Y$ whenever $_1, y_2 \in Y$. — (*)
   Now, consider $y \in Y$, then $y = (\lambda\xi_j)$ where $\lambda\xi_j \in \mathbb{C}$ for which $\xi_{2n} = \lambda \times 0 = 0\ \forall n \in \mathbb{N}$. Hence, $y \in Y \implies \lambda y \in Y$. — (**)
   Combining both (*) and (**), $Y$ is a vector subspace.

(b) $Y$ is closed subspace

As it is true that $l^2$ is a complete metric space, then using **1.4 - 7 Theorem (Complete subspace)** given subspace $(Y)$ is a closed subspace if and only if it is complete. So, let's prove that $Y$ is a complete subspace,

As $M$ is a subset of $l^2$ then let's assume the induced inner product $< x, y >= \sum_{i=1}^{\infty} x_i \overline{y_i}$ where $x = (x_i)$ and $y = (y_i)$ for $i \in \mathbb{N}$. Now, metric is defined as,

$$\begin{aligned} d(x,y) = ||x-y|| &= \sqrt{< x-y, x-y >} \\ &= \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)\overline{(x_i - y_i)}} \\ &= \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2} \\ &= \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} \end{aligned}$$

where $x = (x_i)$ and $y = (y_i)$, now consider any Cauchy sequence $(x_m)$ in $M$, where $(x_m) = (\xi_1^{(m)}, ..., \xi_n^{(m)}$ (by definition of Cauchy sequence) for every $\epsilon > 0$ there is a $N$ such that

$$d(x_m, x_r) = \sqrt{\sum_{i=1}^{\infty} (\xi_i^{(m)} - \xi_i^{(r)})^2} < \epsilon \qquad (m, r > N) \tag{4}$$

Now let's square both side,

$$\begin{aligned} \sum_{i=1}^{\infty} (\xi_i^{(m)} - \xi_i^{(r)})^2 &< \epsilon^2 \\ (\xi_i^{(m)} - \xi_i^{(r)})^2 &< \epsilon^2 \quad \forall m \in \mathbb{N} \\ |\xi_i^{(m)} - \xi_i^{(r)}|^2 &< \epsilon \end{aligned}$$

Therefore, from this the sequence $(\xi_j^{(1)}, \xi_k^{(2)}, ...)$ is a Cauchy sequence of numbers in $\mathbb{F} = \mathbb{C}$ and it converges that is $\xi_j^{(m)} \to \xi_j$. Considering all these limits, let $x = (\xi_j)$ where $\xi_j$ are the individual limits, using (1),

$$d(x_m, x_r) = \sqrt{\sum_{i=1}^{\infty} (\xi_i^{(m)} - \xi_i^{(r)})^2} < \epsilon \qquad (m, r > N) d(x_m, x_r) \leq \epsilon$$

Let's see if this limit i.e. $(x)$ exists in $M$ or not, as the Cauchy sequence was constructed from the elements of $M$, it is true that $x_i = (\xi_j^i)$ and $\xi_{2n}^i = 0, n \in \mathbb{N}$. Now consider the limit as $i \to \infty$, i.e. $\lim_{i\to\infty} \xi_{2n}^i = \lim_{i\to\infty} 0$ which gives,

$$\xi_{2n} = 0 \tag{5}$$

where $x = (\xi_j)$, therefore it is in $Y$ as for all the even indices, the limit vector $x$ have 0 values, and for all the odd they belong to the the field $\mathbb{F}$ which is assumed (WLOG) to be $\mathbb{C}$ (complex numbers) as said in note. Hence, as the limit of all possible Cauchy sequence in $Y$, exists in subspace $Y$ itself, $Y$ is a complete subspace.

(c) $Y^{\perp}$
$Y^{\perp} = \{y | y \perp x \ \forall x \in Y\}$, now consider any arbitrary $y \in Y^{\perp}$, then $< y, x >= 0 \ \forall x \in Y$ so, let $x = (\alpha_j)$ then,

$$< y, x > = \sum_{j=1}^{\infty} \alpha_j \xi_j = 0$$

Now, it's given that $\xi_{2n} \ \forall n \in \mathbb{N}$, then only the odd (indexed term i.e. $1, 3, ...$) will be left (as other are 0).

$$< y, x > = \sum_{j=1}^{\infty} \alpha_{2j-1} \xi_{2j-1} = 0$$

where $2j - 1$ where $j \in \mathbb{N}$ are odd indexed terms, now consider the fact that all $\xi_{2j-1} \in \mathbb{R}$ and are independent from each other (as they can be any element in $\mathbb{R}$ irrespective of others). Hence, they act like linearly independent terms making all $\alpha_{2j-1} = 0$ individually. Hence,

$$Y^{\perp} = \{x | x = (\xi_j) \in l^2, \xi_{2n-1} = 0, n \in \mathbb{N}\}$$

2. $Y^{\perp}$ if $Y = span \ \{e_1, ..., e_n\} \subset l^2$, where $e_j = (\delta_{jk})$
Consider the $Y = span \ \{e_1, ..., e_n\} \subset l^2$, for any $y \in Y$, $y = \sum_{i=1}^{n} x_i e_i$ where $x_i \in \mathbb{C}$. Now, consider $Y^{\perp} = \{y | y \perp x \ \forall x \in Y\}$, and arbitrary $y = (_i) \in Y^{\perp}$ then for that $< y, x >= 0 \ \forall x \in Y$ i.e.

$$< y, x >= \sum_{i=1}^{n} y_i \alpha_i + 0 = 0$$

where $\alpha_i \in \mathbb{C} \ \forall i \in [n]$, given that $x = \sum_{i=1}^{n} \alpha_i e_i$ and as $e_j = (\delta_{jk})$ they are all linearly independent vectors (unit). Now, going by similar argument given in previous part making all $y_i = 0 \forall i \in [n]$. Hence,

$$Y^{\perp} = \{x | x = (\xi_i) \in l^2, \xi_i = 0 \ \forall i \in [n]\}$$

where $[n] = \{1, ..., n\}$ ($n$ being a natural number).

**Problem 5:** Let $A$ and $B \supset A$ be nonempty subsets of an inner product space $X$. Show that

(a) $A \subset A^{\perp\perp}$

(b) $B^{\perp} \subset A^{\perp}$

(c) $A^{\perp\perp\perp} = A^{\perp}$

**Solution 5:**

(a) $A \subset A^{\perp\perp}$
Let take some arbitrary $x \in A$, now, to define $A^{\perp\perp}$, $A^{\perp}$ needs to defined first, so

$$A^{\perp} = \{y \in X | y \perp z \ \forall z \in A\}$$

or we can say that for all $y \in A^{\perp}$, the inner product between $x$ and $y$ is 0 i.e. $< x, y >= 0$. Now consider the definition of $A^{\perp\perp}$,

$$A^{\perp\perp} = \{y \in X | y \perp z \ \forall z \in A^{\perp}\}$$

Let's take an arbitrary $y \in A^{\perp}$, then it is sure that $< x, y >= 0$ as $x \in A$. So, we can say that for all $y \in A^{\perp}$, $< x, y >= 0$ for all $x \in A$. But according to the definition of $A^{\perp\perp}$, $x \in A^{\perp\perp}$, as it is the set of all $\alpha$ where $< \alpha, y >= 0$ for all $y \in A^{\perp}$. Hence, $x \in A^{\perp\perp}$ so (according to set theory), $A \subset A^{\perp\perp}$ as for any arbitrary $x \in A \implies x \in A^{\perp\perp}$.

(b) $B^{\perp} \subset A^{\perp}$
Given that $B \supset A$ that is $x \in A \implies x \in B$. Now, consider the definitions of $B^{\perp}$ and $A^{\perp}$,

$$A^{\perp} = \{y \in X | y \perp z \ \forall z \in A\}$$
$$B^{\perp} = \{y \in X | y \perp z \ \forall z \in B\}$$

Let's consider an arbitrary element $\alpha$ of $B^{\perp}$, then it must follow that $\alpha \perp z \ \forall z \in B$, but as $B \supset A$, then $\alpha$ must be perpendicular to all the elements in $A$ or we can say that $\alpha \in A^{\perp}$ according to the definition of $A^{\perp}$.
Hence, $\alpha \in B^{\perp} \implies \alpha \in A^{\perp}$ i.e. $B^{\perp} \subset A^{\perp}$ (according to the definitions from set theory).

(c) $A^{\perp\perp\perp} = A^{\perp}$
Let's define $A^{\perp}, A^{\perp\perp}, A^{\perp\perp\perp}$

$$A^{\perp} = \{y \in X | y \perp z \ \forall z \in A\}$$
$$A^{\perp\perp} = \{y \in X | y \perp z \ \forall z \in A^{\perp}\}$$
$$A^{\perp\perp\perp} = \{y \in X | y \perp z \ \forall z \in A^{\perp\perp}\}$$

(1) To show $A^{\perp\perp\perp} \subset A^{\perp}$
Let's take an arbitrary $\alpha \in A^{\perp\perp\perp}$, therefore according to how $A^{\perp\perp\perp}$ is defined, $\alpha \perp z \ \forall z \in A^{\perp\perp}$ or $< \alpha, z >= 0$. Now, for this $z$ as it is in $A^{\perp\perp}$, it is perpendicular to all $k \in A^{\perp}$ for all possible $k \in X$.
But, we can say that $\alpha$ also belongs to $A^{\perp}$, as for any $\beta \in A^{\perp}$ $z \perp \beta$. Therefore, $\alpha \in A^{\perp\perp\perp} \implies \alpha \in A^{\perp}$ or $A^{\perp\perp\perp} \subset A^{\perp}$.

(2) To show $A^{\perp} \subset A^{\perp\perp\perp}$
Let's consider $\alpha \in A^{\perp}$, then for any arbitrary $x \in A^{\perp\perp}$, $\alpha \perp x$. Now, consider the set $A^{\perp\perp\perp}$, it consists of all elements whose inner product with all elements in $A^{\perp\perp}$ is 0 (or they are perpendicular). But, in the first line it is seen that $\alpha \perp x \ \forall x \in A^{\perp\perp}$, and $A^{\perp\perp\perp}$ consists of such elements (like $\alpha$). Hence, $\alpha \in A^{\perp\perp\perp}$, or $\alpha \in A^{\perp} \subset \alpha \in A^{\perp\perp\perp}$ or $A^{\perp} \subset A^{\perp\perp\perp}$.

From (1) and (2), as $A^{\perp\perp\perp} \subset A^{\perp}$ and $A^{\perp} \subset A^{\perp\perp\perp}$. Hence, $A^{\perp} = A^{\perp\perp\perp}$.

**Problem 6:** Show that the annihilator $M^{\perp}$ of a set $M \neq \emptyset$ in an inner product space $X$ is a closed subspace of $X$.

**Solution 6:**
The definition of $M^{\perp}$,

$$A^{\perp} = \{y \in X | y \perp z \ \forall z \in M\} \tag{6}$$

1. $M^{\perp}$ is a vector subspace
   Let $x, y \in M^{\perp}$, then $< x, z >=< y, z >= 0 \ \forall z \in M$, now consider $\alpha x + \beta y$

$$\begin{aligned} < \alpha x + \beta y, z > &=< \alpha x, z > + < \beta y, z > \\ &= \alpha < x, z > + \beta < y, z > \\ &= \alpha \cdot 0 + \beta \cdot 0 \\ &= 0 + 0 = 0 \end{aligned}$$

   Hence, $< \alpha x + \beta y, z >= 0 \ \forall z \in M$, hence $\alpha x + \beta y \in M^{\perp}$ therefore $M^{\perp}$ is a vector subspace.

2. $M^{\perp}$ is closed
   For this, consider a functional (associated with each $z \in M$), let's say it as $f_z : X \mapsto \mathbb{F}, f_z(x) =< z, x >$, where $\mathbb{F}$ is the field using which $X$ is defined (or made up of). Now this is a continuous linear (because of the axiomatic properties) functional.
   From closed set point of view too, $|f_z(x)| = | < z, x > | \leq ||z||||x||$ (using Cauchy-Schwarz inequality). Now consider the set $g_z = \{x | f_z(x) = 0, x \in X\}$ it is a closed subspace of $X$ as $f_z \ \forall z \in M$ are continuous linear functionals.
   Another fact to notice is that the intersections of closed subspace is closed itself.

   **Claim:** $M^{\perp} = \bigcap_{z \in M} g_z$
   **Proof:** As $M^{\perp}$ is the set of all $x$ such that $< x, z >= 0 \ \forall z \in M$, therefore $x$ must belong to all $g_z$ (as they are set of all $x \in X$ such that $< x, z >= 0$). Or hence, $x \in \cap_{z \in M} g_z$. Therefore, $M^{\perp} \subset \bigcap_{z \in M} g_z$ — (1)

   Let's assume $x \in X$ and also $x \in \bigcap_{z \in M} g_z$, then by definition of $g_z$, $< x, z >= 0 \ \forall z \in M$, which is exactly the criteria to be in $M^{\perp}$. Hence, $x \in M^{\perp}$ or $M^{\perp} \supset \bigcap_{z \in M} g_z$ — (2)

   Combining $(1), (2)$ gives $M^{\perp} = \bigcap_{z \in M} g_z$.

   Hence, $M^{\perp}$ is an intersection of all $g_z$ where $g_z$ are closed subspaces for all $z \in M$. Therefore, $M^{\perp}$ is also a closed subspace.

Combininig Part 1 and 2, the annihilator $M^{\perp}$ is a closed subspace of $X$ when $M \neq \emptyset$.