Functional Analysis Assignment 1

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Questions

Problem 1: Show that the real line is a metric space.

Solution 1: Claim: For real line, the metric space can be defined as $(\mathbb{R}, d)$. Where $d(x, y) = |x - y|$. Let's see if the following distance metric follows all the properties :-

1. as $d(x, y) = |x - y|$, it is an absolute function therefore the value should be **non-negative**, also as domain is $\mathbb{R}$ it is **real valued** and it is **finite** as distance between two "well defined, non-infinite" numbers x and y should be finite (or the limit must exist).
2. Let $d(x, y) = 0$, therefore, $|x - y| = 0$ this is true only when $x = y$.
3. as $|x - y| = |(-1) * (y - x)| = |-1||y - x| = |y - x|$ hence $d(x, y) = d(y, x)$
4. **Triangle Inequality**, Let

$$
\begin{aligned}
|x - y| &= |x - z + y - z| \\
|x - y|^2 &= |x - z + y - z|^2 \text{ by taking square on both sides} \\
|x - y|^2 &= |x - z|^2 + |y - z|^2 + 2(x - z)(y - z) \\
|x - y|^2 &\leq |x - z|^2 + |y - z|^2 + 2|x - z||y - z| \text{ (as } x \leq |x|) \\
|x - y|^2 &\leq (|x - z| + |y - z|)^2 \\
|x - y| &\leq |x - z| + |y - z| \\
d(x, y) &\leq d(x, z) + d(z, y)
\end{aligned}
$$

Hence Proved.

Problem 2: Does $d(x, y) = (x - y)^2$ define a metric on the set of all the real numbers?

Solution 2: To be defined as a distance metric it should follow these properties :-

1. as $d(x, y) = (x - y)^2$, it is a square function therefore the value should be **non-negative**, also as domain is $\mathbb{R}$ it is **real valued** and it is **finite** as distance between two "well defined, non-infinite" numbers x and y should be finite (or the limit must exist).
2. Let $d(x, y) = 0$, therefore, $(x - y)^2 = 0$ this is true only when $x = y$.
3. as $(x - y)^2 = ((-1) * (y - x))^2 = (-1)^2(y - x)^2 = (y - x)^2$ hence $d(x, y) = d(y, x)$

4. **Triangle Inequality**,

$$\begin{aligned}
(x-y)^2 &= (x-z+y-z)^2 \\
(x-y)^2 &= (x-z)^2 + (y-z)^2 + 2(x-z)(y-z) \\
d(x,y) &= d(x,z) + d(z,y) + 2(x-z)(y-z)
\end{aligned}$$

   Now $d(x,y) \leq d(x,z) + d(z,y)$ only when $2(x-z)(y-z) \leq 0$ but let's say $z > y$ & $z > x$, then in that case $2(x-z)(y-z) > 0$ and hence the given distance function can not be defined as a metric on the set of all the real numbers.

Problem 3: Show that $d(x,y) = \sqrt{|x-y|}$ defines a metric on the set of all real numbers.

Solution 3: To be defined as a distance metric it should follow these properties :-

1. as $d(x,y) = \sqrt{|x-y|}$, it has an absolute function therefore the value should be **non-negative**, also as domain is $\mathbb{R}$ it is **real valued** and it is **finite** as distance between two "well defined, non-infinite" numbers x and y should be finite (or the limit must exist).
2. Let $d(x,y) = 0$, therefore, $\sqrt{|x-y|} = 0$ or $|x-y| = 0|$ this is true only when $x = y$.
3. as $\sqrt{|x-y|} = \sqrt{|(-1)(y-x)|} = \sqrt{|y-x|}$ hence $d(x,y) = d(y,x)$
4. **Triangle Inequality**, As proved in **(Problem 1)**

$$\begin{aligned}
|x-y| &\leq |x-z| + |y-z| \\
(\sqrt{|x-y|})^2 &\leq (\sqrt{|x-z|})^2 + (\sqrt{|y-z|})^2 \\
(\sqrt{|x-y|})^2 &\leq (\sqrt{|x-z|})^2 + (\sqrt{|y-z|})^2 + 2 * \sqrt{|x-z|}\sqrt{|y-z|} \text{ as it is a non-negative quantity} \\
(\sqrt{|x-y|})^2 &\leq (\sqrt{|x-z|} + \sqrt{|y-z|})^2
\end{aligned}$$

   Taking square-root both sides,

$$\begin{aligned}
\sqrt{|x-y|} &\leq \sqrt{|x-z|} + \sqrt{|y-z|} \\
d(x,y) &\leq d(x,z) + d(z,y)
\end{aligned}$$

Hence Proved.

Problem 4: Find all metrics on a set X consisting of two points. Consisting of one point.

Solution 4:

1. **Two Points.** For this, as $d(x,y) = 0$ only when $x = y$, therefore $d(p_i, p_i) = 0$ and $d(p_1, p_2) = d(p_2, p_1) = C$ where C is a real value, finite and positive (i.e. greater than 0).

   It follows triangle inequality too, as $d(p_1, p_x) + d(p_2, p_x) = C + 0$ or $0 + C$ which is greater than equal to $d(p_1, p_2) = C$.

2. One Point. $d(p,p) = 0$, as there is only one point and distance metric d is 0 for two same elements (therefore $d(p,p) = 0$). It is **non-negative, real valued and finite.** As $0 \leq 0$ therefore $d(x,x) \leq d(x,x) + d(x,x)$ **(Triangle inequality)**. And **reflexive property is trivial.**

1.1-7 Function space $C[a,b]$. As a set X we take the set of all real-valued functions $x, y, ...$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = [a,b]$. Choosing the metric defined by

$$d(x,y) = \max_{t \in J} |x(t) - y(t)|, \tag{1}$$

where max denoted the maximum, we obtain a metric space which is denoted by $C[a,b]$. (The letter C suggests "continuous.") This is a *function space* because every point of $C[a,b]$ is function.

Problem 5: Show that another metric $\tilde{d}$ on the set X in 1.1-7 is define by

$$\tilde{d}(x,y) = \int_a^b |x(t) - y(t)| dt$$

Solution 5: To be defined as a distance metric it should follow these properties :-

1. as $\tilde{d}(x,y) = \int_a^b |x(t) - y(t)| dt$, as only positive quantities are added therefore the value should be **non-negative**, it is **real valued** as the functions are real valued and it is **finite** as distance between two "well defined, non-infinite" real-valued functions x and y should be finite (or the integral is).

2. Let $d(x,y) = 0$, therefore, $\int_a^b |x(t) - y(t)| dt$ this means for every real value in interval $[a,b]$, $|x(t) - y(t)| = 0$ which in turn means that $x(t) = y(t) \ \forall \ t \ \in [a,b]$, or both functions are same i.e. $x = y$.

3. as $d(x,y) = \int_a^b |x(t) - y(t)| dt = \int_a^b |(-1)||y(t) - x(t)| dt = \int_a^b |y(t) - x(t)| dt = d(y,x)$ hence $d(x,y) = d(y,x)$

4. **Triangle Inequality** As proved in **(Problem 1)** where x_1, y_1, z_1 are real-valued.

$$|x_1 - y_1| \leq |x_1 - z_1| + |y_1 - z_1|$$
$$|x(t) - y(t)| \leq |x(t) - z(t)| + |y(t) - z(t)|$$

As $x(t), y(t), z(t)$ where $t \in [a,b]$ are real-valued and can replace x_1, y_1, z_1. Taking integral (continuous sum) from a to b on both side.

$$\int_a^b |x(t) - y(t)| dt \leq \int_a^b |x(t) - z(t)| dt + \int_a^b |y(t) - z(t)| dt \tag{2}$$
$$d(x,y) \leq d(x,z) + d(z,y) \tag{3}$$

Hence Proved.

Problem 6 (Axioms of a metric): $(M1)$ to $(M4)$ could be replaced by other axioms (without changing the definition). For instance, show that $(M3)$ and $(M4)$ could be obtained from $(M2)$ and

$$d(x, y) \leq d(z, x) + d(z, y)$$

Solution 6: Given that $d(x, y) \leq d(z, x) + d(z, y)$ - (1) is correct and $d(x, y) = 0$ iff $x = y$ $(M2)$.

To prove: $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$

Proof:

1. Let $z = y$ in equation (1)

$$\begin{aligned} &d(x, y) \leq d(y, x) + d(y, y) \\ &d(x, y) \leq d(y, x) + 0 \text{ as } d(y, y) = 0 \text{ (M2)} \\ &d(x, y) \leq d(y, x) - \text{equation 2} \end{aligned}$$

   Also in –equation (1) we can switch the positions of x and y, and take $z = x$

$$\begin{aligned} &d(y, x) \leq d(x, y) + d(x, x) \\ &d(y, x) \leq d(x, y) + 0 \text{ as } d(y, y) = 0 \text{ (M2)} \\ &d(y, x) \leq d(x, y) - \text{equation 3} \end{aligned}$$

   Combining equation 2 & 3 gives $d(x, y) = d(y, x)$ (M3).

2. now using result from (point 1) i.e. $d(x, y) = d(y, x)$, $d(z, x) = d(x, z)$ and this can be replaced in equation 1.

$$\begin{aligned} &d(x, y) \leq d(z, x) + d(z, y) \\ &d(x, y) \leq d(x, z) + d(z, y) \text{ which is (M4)} \end{aligned}$$

Problem 7: Show that non-negativity of a metric follows from $(M2)$ to $(M4)$.

Solution 7: Given (M2), (M3) and (M4). Let's substitute $y = x$ in (M4),

$$\begin{aligned} d(x, x) &\leq d(x, z) + d(z, x) \\ 0 &\leq d(x, z) + d(x, z) \text{ using (M2)} \\ 0 &\leq d(x, z) + d(x, z) \text{ using (M3)} \\ 0 &\leq 2 * d(x, z) \\ 0 &\leq d(x, z) \end{aligned}$$

Hence distance metric d is non-negative.

Problem 8: Using (6), show that the geometric mean of two positive numbers does not exceed the arithmetic mean.

Solution 8: As referenced from book, (6) refers to **Young's Inequality** i.e. $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.
Putting $p = q = 2$ in the equation,

$$\alpha\beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2}$$
$$2\alpha\beta \leq \alpha^2 + \beta^2$$

adding $2\alpha\beta$ on both sides,

$$4\alpha\beta \leq \alpha^2 + \beta^2 + 2\alpha\beta$$
$$4\alpha\beta \leq (\alpha + \beta)^2$$

Taking square-root both sides (as α and β are positive it is possible),

$$2\sqrt{\alpha\beta} \leq |\alpha + \beta|$$
$$\sqrt{\alpha\beta} \leq \frac{|\alpha + \beta|}{2}$$
$$\sqrt{\alpha\beta} \leq \frac{\alpha + \beta}{2} \text{ as } |x| = x \text{ if } x > 0$$
$$\text{Geometric Mean} \ \leq \ \text{Arithmetic Mean}$$

Problem 9: Show that the Cauchy-Schwarz inequality (11) implies

$$(|\xi_1| + ... + |\xi_n|)^2 \leq n(|\xi_1|^2 + ... + |\xi_n|^2) \tag{4}$$

Solution 9: Cauchy-Schwarz inequality :-

$$\sum_{i=1}^{\infty} |\xi_i \eta_i| \leq \sqrt{\sum_{i=1}^{\infty} |\xi_i|^2} \sqrt{\sum_{i=1}^{\infty} |\eta_i|^2} \tag{5}$$

Substituting $\xi_{n+1} = \xi_{n+2} = ... = 0$, $\eta_1 = ... = \eta_n = 1$ and $\eta_{n+1} = \eta_{n+2} = ... = 0$

$$\sum_{i=1}^{n} |\xi_i| \leq \sqrt{\sum_{i=1}^{n} |\xi_i|^2}\sqrt{\sum_{i=1}^{n} 1}$$
$$\sum_{i=1}^{n} |\xi_i| \leq \sqrt{n\sum_{i=1}^{n} |\xi_i|^2}$$

Taking square on both sides,

$$(\sum_{i=1}^{n} |\xi_i|)^2 \leq n\sum_{i=1}^{n} |\xi_i|^2$$
$$(|\xi_1| + ... + |\xi_n|)^2 \leq n(|\xi_1|^2 + ... + |\xi_n|^2)$$

Hence Proved.

Problem 10 (Space l^p): Find a sequence which converges to 0, but is not in any space l^p, where $1 \leq p < +\infty$.

Solution 10: Consider the sequence: $x_i = \frac{1}{\ln(i+1)}$. As $n \to \infty$ this sequence converges to 0. But it's sum $\sum_{i=1}^{\infty} |x_i|^p$ diverges hence it does not belong to any space l^p, where $1 \leq p < +\infty$.

Proof.

$$|\ln(k+1)| < |k+1|$$
$$0 < \frac{1}{|k+1|} < \frac{1}{|\ln(k+1)|}$$
$$\frac{1}{|k+1|} < \frac{1}{|\ln(k+1)|} < |\frac{1}{\ln(k+1)}|^p$$
$$\sum_{i=1}^{\infty} \frac{1}{k+1} < \sum_{i=1}^{\infty} |\frac{1}{\ln(k+1)}|^p$$

As the left side is divergent therefore the series sum limit does not exists (or is divergent). □

Hence Proved.

Problem 11: Find a sequence x which is in l^p with $p > 1$ but $x \notin l^1$.

Solution 11: Let's consider a sequence: "$1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$". But to qualify to belong to a space l^p, where $1 < p$ $1 + \frac{1}{2} + \frac{1}{3} + ...$ should diverge.

Proof. Now, any number is less than the perfect power of two which comes first after that number for example, $3 < 4$ and $11 < 16$.

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + ... \tag{6}$$
$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} > 1 + 1 + 1 + ... \text{ log(n) times} \tag{7}$$
$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} > \log(n) \tag{8}$$

Taking $n \to \infty$ both sides,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} > \lim_{n\to\infty} \log(n) \tag{9}$$

This clearly shows, the series diverges and hence cannot be a part of l^1. But there exists a proof that this sequence exists in l^p where $p > 1$ for example the sum of series for $p = 2$ is $\frac{\pi^2}{6}$. □

Hence Proved.

Problem 12: Let X be the space of all ordered n-tuples $x = (\xi_1, ..., \xi_n)$ of real numbers and

$$d(x, y) = \max_i |\xi_i - \eta_i|$$

where $y = (\eta_i)$. Show that (X, d) is complete.

Solution 12: Consider a general cauchy sequence (x_k) where $x_k = (\xi_1^k, \xi_2^k, ..., \xi_n^k)$. Then as the sequence is cauchy, for all ϵ $\exists$ an N such that $d(x_m, x_n) < \epsilon$ and $m, n > N$. From this:

$$\begin{aligned} d(x_m, x_n) = \max_i |\xi_i^m - \xi_i^n| &< \epsilon \\ |\xi_i^m - \xi_i^n| &< \epsilon \ \forall\ i, n, m \text{ with } n, m > N \end{aligned}$$

Which states that $\xi_i^1, \xi_i^2, ..., \xi_i^p, ...$ is a cauchy sequence for all i in $\mathbb{R}$ and using as $\mathbb{R}$ is complete it converges, let's say to ξ_i as $p \to \infty$. Therefore,

$$|\xi_i^n - \xi_i| < \epsilon \ \forall\ i, n, m \text{ with } n, m > N$$

and let's define $x = (\xi_1, ..., \xi_n)$, where ξ_i are the individual limits. Then,

$$\begin{aligned} \max_i |\xi_i^n - \xi_i| &< \epsilon \ \forall\ i, n, m \text{ with } n, m > N \\ d(x_i, x) &< \epsilon \end{aligned}$$

Hence the given cauchy sequence converges to x. As an arbitary cauchy sequence was taken, it shows that any cauchy sequence in X converges which proves that (X, d) is complete.

Problem 13: Let X be the set of all positive integers and $d(m,n) = |m^{-1} - n^{-1}|$. Show that (X, d) is not complete.

Solution 13: Consider a sequence $1, 2, 3, ...$ surely it is a cauchy sequence as $n, m \to \infty$ $d(x_m, x_n)$ becomes lesser and lesser, i.e. we can use any ϵ and it's associated N exists. But for any positive integer k, $\lim_{x\to\infty} d(x,k) = k^{-1}$ hence the given sequence does not converge in X, that in turn proves that (X, d) is **not complete.**

Problem 14: (Space $C[a,b]$) Show that the subspace $Y \subset C[a,b]$ consisting of all $x \in C[a,b]$ such that $x(a) = x(b)$ is complete.

Solution 14: As a subspace M of a complete metric space X is itself complete if and only if the set M is closed in X. So, in this problem given $Y \subset C[a,b]$ consisting of all $x \in C[a,b]$ such that $x(a) = x(b)$ if we are able to prove it is closed then the given Y is complete.

Proof. Let's consider a sequence of functions belonging to Y such that these functions, $f_n \to f$ in $C[a,b]$. Therefore, if limit exists, then for any ϵ there exists N such that $n > N$.

$$d(f_n, f) = \max_{t\in[a,b]} |f_n(t) - f(t)| < \epsilon$$
$$|f_n(t) - f(t)| < \epsilon$$

for all t. Therefore $f_n(t)$ converges to $f(t)$ for all $t \in [a,b]$. Also limit function f is continuous on $[a,b]$ because f_n are continuous functions. Now according to triangle inequality,

$$\begin{aligned}
|f(a) - f(b)| &\le |f(a) - f_n(a)| + |f_n(a) - f_n(b)| + |f_n(b) - f(b)| \\
|f(a) - f(b)| &\le |f(a) - f_n(a)| + |f_n(b) - f(b)| \\
&\le 2 \max_{t\in[a,b]} |f_n(t) - f(t)| \\
&= 2d(f_n, f)
\end{aligned}$$

which approaches to 0 as $n \to \infty$, hence $|f(a) - f(b)| \le 0$ or $|f(a) - f(b)| = 0$ hence the $f \in Y$ or it is closed. Hence, it is complete subspace. □

1.2-1 Sequence Space s. This space consists of the set of all (bounded or unbounded) sequences of complex numbers and the metric d denoted by

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} \tag{10}$$

where $x = (\xi_j)$ and $y = (\eta_j)$

Problem 15: (Space s) Show that in the space s (cf. 1.2-1) we have $x_n \to x$ if and only if $\xi_j^{(n)} \to \xi_j$ for all $j = 1, 2, ...$, where $x_n = (\xi_j^{(n)})$ and $x = (\xi_j)$

Solution 15: Let's consider $x_n \to x$ to be true where $x_n = (\xi_1, \xi_2, ...)$, then $\lim_{n\to\infty} d(x_n, x) = 0$ or for any ϵ there exists an N such that for all $n > N$,

$$\frac{1}{2^i}\frac{|\xi_i^n - \xi_i|}{1+|\xi_i^n - \xi_i|} \leq d(x_n, x) < \frac{1}{2^i}\frac{\epsilon}{1+\epsilon}$$
$$\frac{1}{2^i}\frac{|\xi_i^n - \xi_i|}{1+|\xi_i^n - \xi_i|} < \frac{1}{2^i}\frac{\epsilon}{1+\epsilon}$$
$$\frac{|\xi_i^n - \xi_i|}{1+|\xi_i^n - \xi_i|} < \frac{\epsilon}{1+\epsilon}$$
$$|\xi_i^n - \xi_i|(1+\epsilon) < \epsilon(1+|\xi_i^n - \xi_i|)$$
$$|\xi_i^n - \xi_i| < \epsilon$$

Therefore $\xi_i^n \to \xi_i$ as $n \to \infty$. And conversely, if $\xi_i^n \to \xi_i$ for all i then, according to the definition of convergent sequence, $x_n = (\xi_j^{(n)})$ must approach $x = (\xi_j)$ using second property i.e. if $x_n \to x$ and $y_m \to y$ then $d(x_n, y_m) \to d(x, y)$.

Problem 16: Using problem 11 (problem 15 in this assignment), show that the sequence space s in 1.2-1 is complete.

Solution 16: Consider any cauchy sequence (x_n) such that $x_n = \xi_i^{(n)}$, as it is cauchy for any ϵ there exists N such that $n, m > N$,

$$\frac{1}{2^i}\frac{|\xi_i^n - \xi_i|}{1+|\xi_i^n - \xi_i|} \leq d(x_n, x) < \frac{1}{2^i}\frac{\epsilon}{1+\epsilon} \tag{11}$$

which is equivalent to $|\xi_i^n - \xi_i^m| < \epsilon$ (as shown in solution to problem 15), which shows that for all i, $(\xi_i^1, \xi_i^2, ...$ is a cauchy sequence but as the elements of this sequence are in $\mathbb{R}$ therefore it must converge. Now using (problem 15 solution) we can state that, the original cauchy sequence (x_n) also converges i.e. $(x_n) \to (x)$ therefore as an arbitrary cauchy sequence was considered it proves that all cauchy sequence converges. Hence, **the sequence space is complete.**

Problem 17: Let X be the metric space of all real sequences $x = (\xi_i)$ each of which has only finitely many non-zero terms, and $d(x, y) = \sum |\xi_j - \eta_j|$, where $y = (\eta_j)$. Note that this is a finite sum but the number of terms depends on x and y. Show that (x_n) with $x_n = (\xi_j^{(n)})$,

$$\xi_j^{(n)} = j^{-2} \text{ for } j = 1, ..., n \text{ and } \xi_j^{(n)} = 0 \text{ for } j > n \tag{12}$$

is Cauchy but does not converge.

Solution 17:
First: To prove that it is a cauchy sequence. Consider the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$, this series converges and have a finite sum that is $\frac{\pi^2}{6}$. Therefore as the series converges, we can say that for every ϵ there exists N such that for $n > N$,

$$\sum_{i=1}^{\infty} \frac{1}{i^2} < \epsilon \tag{13}$$

now consider the distance function for two real sequences x_n, x_m where $n > m > N$

$$d(x_n, x_m) = \sum_{i=m+1}^{n} \frac{1}{i^2} \leq \sum_{i=m+1}^{\infty} \frac{1}{i^2} \leq \sum_{i=N+1}^{\infty} \frac{1}{i^2} < \epsilon$$

therefore, such sequence is a cauchy sequence.
Second: To prove that it does not converge, consider the distance between x_m and x, where in $m < N$ and $x_i = 0$ for $i > n$. Therefore,

$$d(x_m, x) = |xi_1 - 1| + |xi_2 - \frac{1}{4}| + ... + |xi_m - \frac{1}{m^2}| + \frac{1}{(1+N)^2} + ... + \frac{1}{n^2} \tag{14}$$

This will not approach to 0 when $n \to \infty$, even if all the modulo terms becomes 0. Hence, the given sequcence is cauchy but not converge.