Functional Analysis

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Contents

Chapter 1

Metric Spaces

1.0.1 Notes

- "Soft Analysis" instead of "Hard Analysis"
- Harmonic Analysis - Study of Fourier Series
- Hilbert Spaces and Banach Spaces
- Limit Function ? (Major topic for analysis of sequences)
- Spaces of function? It is a linear space equipped with a norm, akin to a vector.
- Complete Space - (Limit actually exists)
- Spaces of functions can be infinite dimensional

1.0.2 Metric Spaces

- In Real Analysis, functions are defined on the Real Line $\mathbb{R}$.
- Limit and other functions use the distance function

$$d(x, y) = |x - y|$$

- For more general spaces, Real Line $\mathbb{R}$ is replaced with abstract set $\mathbb{X}$.

Definition 1.0.1 *A* ***Metric Space*** *is defined as an ordered pair* $(\mathbb{X}, d)$*. Where* d *(distance function) follows following properties:*

1. d *is a* ***real-valued, finite*** *(can also be infinite, if two functions are infinitely placed) and* ***non-negative*** *function*
2. $d(x, y) = 0$ *iff* $x = y$
3. $d(x, y) = d(y, x)$

4. *Triangle Inequality:* $d(x, y) \leqslant d(x, z) + d(z, y)$ *or in general,*

$$d(x_0, x_n) \leqslant d(x_0, x_1) + d(x_1, x_2) + ... + d(x_{n-1}, x_n)$$

Definition 1.0.2 *Suppose $\mathbb{Y} \subset \mathbb{X}$, the* ***restriction of the Metric in*** $\mathbb{X}$ ***to*** $\mathbb{Y}$ *is* $\tilde{d} = d_{|YxY}$. *Where* $\tilde{d}$ *is the metric induced on* $\mathbb{Y}$ *by* d *and the new metric space is* $(Y, \tilde{d})$

1.0.3 Examples of Metric Spaces

- Some examples: Real Line $\mathbb{R}$, Euclidean Plane $\mathbb{R} \times \mathbb{R}$, Sequence Spaces, Unitary Spaces (Complex Numbers)
- Distance function for $\mathbb{R}^n$, $d = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
- For $\mathbb{C}$ (complex numbers), it is called n-dimensional unitary Space $\mathbb{C}^n$ consisting of ordered n-tuple complex numbers, having form

$$\mathbf{v} = (x_1 + y_1 i, x_2 + y_2 i, ..., x_n + y_n i) \text{ and } d(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}|$$

Sequences

- Sequence Space l^∞, consisting of sequences which are bounded.

Definition 1.0.3 *Let* $\zeta_j \in l^\infty$ *which is a* ***Bounded Sequence*** *then* $\forall j$, $\left|\zeta_j\right| \leqslant c_x$ *where* c_x *is a real number.*

- Example : $\{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...\}$, here $c_x = 1$ i.e. $\left|\zeta_j\right| \leqslant 1$.

Definition 1.0.4 ***Distance Function*** *for sequences is defined as*

$$d(x, y) = \sup_{j \in \mathbb{N}} \left|\zeta_j - \eta_j\right|$$

where $x, y \in l^\infty$ *with* $\zeta_j \in x$ *and* $\eta_j \in y$. *Also,* $\sup$ *(supremum) is the least upper bound.*

Claim. Distance function of a sequence can be defined as sup (supremum).

Proof 1 *Let* $x = \{\zeta_j\}$, $y = \{\eta_j\}$ *and* $z = \{\gamma_j\}$. *Then using triangular inequality,*

$$\left|\zeta_j - \eta_j\right| \leqslant \left|\zeta_j - \gamma_j\right| + \left|\gamma_j - \eta_j\right|$$

Taking supremum both sides,

$$\begin{aligned} &= \sup_{j \in \mathbb{N}} \left|\zeta_j - \eta_j\right| \leqslant \sup_{j \in \mathbb{N}} \left|\zeta_j - \gamma_j\right| + \sup_{j \in \mathbb{N}} \left|\gamma_j - \eta_j\right| \\ &= f(x, y) \ \leqslant \ f(x, z) + f(z, y) \end{aligned}$$

Hence proved.

Chapter 2

Examples of Metric Spaces

2.0.1 Some more Examples of Metric Spaces

Definition 2.0.1 ***Function Space*** $\mathbb{C}[a, b]$. *where* C *suggests continuous. It contains the abstract set* X *of functions which are based on a parameter* t *i.e.* $f(t)$ *and distance function (d) is defined as,*

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|$$

where J *is* $[a, b]$.

Proof 2 *It satisfies all the first 3 properties of distance functions. So, let's prove* ***Triangular Inequality:***

$$\begin{aligned} |x(t) - y(t)| &\leqslant |x(t) - z(t)| + |z(t) - y(t)| \\ &\leqslant \sup |x(t) - z(t)| + \sup |z(t) - y(t)| \\ &\textit{as RHS is independent of } t\textit{, take sup on both sides} \\ d(x, y) = \sup |x(t) - y(t)| &\leqslant d(x, z) + d(z, y) \end{aligned}$$

Hence Proved.

- **Note:** Sometimes sup may exist but not max.

Definition 2.0.2 ***Discrete Metric Space***. *Consider any set* X*, the discrete metric d,*

$$d(x, x) = 0$$
$$d(x, y) = 1, \textit{ for } (x \neq y)$$

Proof 3 *Let* a, b, c *be three distinct points. It already follows first 3 properties of distance.*

$$d(a, c) = 1, \ d(a, b) = 1 \text{ and } d(b, c) = 1 \tag{2.1}$$

Therefore,

$$d(a, c) = 1 \leqslant (d(a, b) = 1) \ + (d(b, c) = 1) = 2 \tag{2.2}$$

Hence Proved.

- If we use the notion of circle i.e. all points which have equal distance from some single point. Then that circle varies according to the Space in consideration.
- **Comparison test** is to bound the series sum, and show that a particular series is convergent.

2.0.2 Some Problems

- **Problem 1:** Consider a distance function between two series $\tilde{x}, \tilde{y} \in l^{\infty}$:

$$d(\tilde{x}, \tilde{y}) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$$

Prove that it is a valid metric distance.

Proof 4 *This function is bounded by* $\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$
Let $f(x) = \frac{x}{1+x}$, *as* $f'(t) = \frac{1}{1+t^2} > 0$ f *is monotonically increasing*

$$\begin{aligned}
t_1 &< t_2 \\
f(t_1) &< f(t_2) \\
\textit{as, } |a+b| &\leqslant |a| + |b| \\
f(|a+b|) &\leqslant f(|a|+|b|) \\
\frac{|a+b|}{1+|a+b|} &\leqslant \frac{|a|+|b|}{1+|a|+|b|} \\
&\leqslant \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \\
&\leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}
\end{aligned}$$

Let $a = x_j - y_j$ *and* $b = y_j - z_j$, where $z = (z_j)$

$$\begin{aligned}
a + b &= x_j - z_j \\
\frac{|x_j - z_j|}{1 + |x_j - z_j|} &\leqslant \frac{|x_j - y_j|}{1 + |x_j - y_j|} + \frac{|y_j - z_j|}{1 + |y_j - z_j|}
\end{aligned}$$

Multiplying by $\frac{1}{2^j}$ *on both sides and taking summation,*

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - z_j|}{1 + |x_j - z_j|} &\leqslant \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|y_j - z_j|}{1 + |y_j - z_j|} \\
d(x, z) &\leqslant d(x, y) + d(y, z)
\end{aligned}$$

Hence Proved.

- **(Prob-7 ex-1.1, Kreyzig):** If A is the sub-sequence l^{∞} consisting of all sequences of $0'$s and $1'$s. What is the induced metric on A?
  Solution: It is the discrete metric distance.

Definition 2.0.3 ***Hamming Distance:*** *The number of places two ordered tuples (let's of length* 3*) of* $0'\mathrm{s}$ and $1'\mathrm{s}$ *differs.*

Proof 5 ***Hint:*** *We can prove (if this is a valid distance metric) using enumeration for every possible bit possible i.e.* 0 or 1.

2.0.3 Space of Bounded Functions

- Space of Bounded Functions is denoted by $B(A)$.
- $d(x,y) = \sup_{t\in A} |x(t) - y(t)|$

Definition 2.0.4 *The l^p space where $p \geqslant 1$ is a fixed real number, is called* ***Hilbert Sequence Space****. l^p is a sequence space*

$$\tilde{x} = (x_1, x_2, ..., x_n, ...) \tag{2.3}$$

$$|x_1|^p + |x_2|^p + |x_3|^p + ... \textit{ converges,}$$

$$\sum_{j=1}^{\infty} |x_j|^p < \infty$$

$$d(x,y) = (\sum_{j=1}^{\infty} |x_j - y_j|^p)^{\frac{1}{p}}$$

Definition 2.0.5 ***Hilbert Space, Hölder's Inequality:***

$$\sum_{j=1}^{\infty} |x_j y_j| \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{m=1}^{\infty} |y_m|^q)^{\frac{1}{q}} \tag{2.4}$$

where $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Chapter 3

Inequalities

3.0.1 Some Inequalities

Definition 3.0.1 ***Young's Inequality***

$$\alpha\beta \leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q} \tag{3.1}$$

Proof 6 *Let* $f : [0, \infty) \mapsto \mathbb{R}$

$$\begin{aligned} f(\alpha) &= \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta \textit{ for a fixed } \beta \\ f'(\alpha) &= \alpha^{p-1} - \beta = 0 \\ \alpha &= \beta^{\frac{1}{p-1}} \end{aligned}$$

now $\frac{1}{p} + \frac{1}{q} = 1$*, therefore* $\frac{q}{p} = \frac{1}{p-1}$

$$\begin{aligned} \alpha &= \beta^{\frac{q}{p}} \\ f''(\alpha) &= (p-1)\alpha^{p-2} > 0 \end{aligned}$$

$f(\alpha)$ *has a min at* $\alpha = \beta^{\frac{q}{p}}$.

$$\begin{aligned} 0 = f(\beta^{\frac{q}{p}}) \leqslant f(\alpha) &= \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta \\ \alpha\beta &\leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q} \end{aligned}$$

Hence Proved.

Definition 3.0.2 ***Hölder's inequality***

Proof 7 *Let* $(x_n) \in l^p$ *and* $(y_n) \in l^q$*. Then trivial case:*

$$\sum_{k=1}^{\infty} |x_k|^p = 0 \text{ or } \sum_{k=1}^{\infty} |y_k|^q = 0 \tag{3.2}$$

Assume both these sums are not equal to 0*, and set in* ***Young's Inequality***

$$\alpha = \frac{|x_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}} \text{ and } \beta = \frac{|y_k|}{(\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} \tag{3.3}$$

$$\alpha\beta = \frac{|x_k y_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} \leqslant \frac{1}{p} \cdot \frac{|x_k|^p}{\sum_{k=1}^{\infty} |x_k|^p} + \frac{1}{q} \cdot \frac{|y_k|^q}{\sum_{k=1}^{\infty} |y_k|^q}$$

Sum both sides from 1 *to* ∞

$$\begin{aligned} \frac{\sum_{k=1}^{\infty} |x_k y_k|}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}} &\leqslant \frac{1}{p} \cdot \frac{\sum_{k=1}^{\infty} |x_k|^p}{\sum_{k=1}^{\infty} |x_k|^p} + \frac{1}{q} \cdot \frac{\sum_{k=1}^{\infty} |y_k|^q}{\sum_{k=1}^{\infty} |y_k|^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} |x_k y_k| \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}$$

Hence Proved.

Definition 3.0.3 ***Minkowski Inequality*** *for sequences* $p > 1$, (x_n) *and* (y_n) *are both sequence in* l^p*.*

$$(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{p}} \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} \tag{3.4}$$

Proof 8 *Let* $q = \frac{p}{p-1}$ *and* $\sum_{k=1}^{\infty} |x_k + y_k|^p \neq 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k + y_k|^p &= \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |x_k + y_k| \\ &\leqslant \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |x_k| + \sum_{k=1}^{\infty} |x_k + y_k|^{p-1} |y_k| \text{ using triangle ineq.} \end{aligned}$$

From the ***Hölder's inequality*** $\sum_{m=1}^{\infty} |x_m y_m| \leqslant (\sum_{m=1}^{\infty} |x_m|^p)^{\frac{1}{p}} (\sum_{m=1}^{\infty} |y_m|^q)^{\frac{1}{q}}$*, taking* $|x_k + y_k|^{p-1}$ *as* y_m *and* $|x_k|$ *as* x_m*,*

$$\leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} (|x_k + y_k|^{p-1})^q)^{\frac{1}{q}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} (|x_k + y_k|^{p-1})^q)^{\frac{1}{q}}$$

As $\frac{1}{p} + \frac{1}{q} = 1$, *therefore* $(p-1)q = p$,

$$\leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{q}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{q}}$$

$$\leqslant ((\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}})(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{q}}$$

$$(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{1-\frac{1}{q}} \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}}$$

Therefore,

$$(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{p}} \leqslant (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} \tag{3.5}$$

Hence Proved.

3.0.2 Sequence Metric Space

Definition 3.0.4 l^p *where* $p \geqslant 1$ *is a metric space where,*

$$d(x, y) = (\sum_{j=1}^{\infty} |x_j - y_j|^p)^{\frac{1}{p}} \tag{3.6}$$

provided $\sum_{j=1}^{\infty} |x_j|^p < \infty$.

Proof 9 *To prove triangle inequality* $d(x, y) \leqslant d(x, z) + d(z, y)$ *for metric distance, i.e. to prove:*

$$(\sum_{j=1}^{\infty} |x_j - y_j|^p)^{\frac{1}{p}} \leqslant (\sum_{j=1}^{\infty} |x_j - z_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{\infty} |z_j - y_j|^p)^{\frac{1}{p}} \tag{3.7}$$

Hint: *Substitute* $x_k = x_j - z_j$ *and* $y_k = z_j - y_j$ *in* ***Minkowski Inequality***.

- **(Ex - 1.2, Kreyzig):** Consider a distance function between two series $\tilde{x}, \tilde{y} \in l^{\infty}$:

$$d(\tilde{x}, \tilde{y}) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$$

Prove that by replacing $\frac{1}{2^j}$ by $\mu_j > 0$, such that $\sum \mu_j$ converges, it is a metric space.

Proof 10 ***We need to show that these infinite terms converge***. *As* $\sum \mu_j$ *converges and the term multiplied is* < 1 *therefore, the series converge.*

Chapter 4

Convergence, Cauchy Sequence and Completeness

4.0.1 Notes

- As n increases the distance between the terms (at very large n) comes closer and closer.

4.0.2 Some Definitions

Definition 4.0.1 ***Limit of a sequence.*** *A sequence in* $(\mathbb{X}, d)$ *is said to converge or a convergent sequence if* $\exists$ *an* $x \in \mathbb{X}$ *such that* $\lim_{n\to\infty} d(x_n, x) = 0$. x *is called the limit of the sequence. If* x *doesn't exist in* $\mathbb{X}$ *(the metric space) then limit does not exist.*

Definition 4.0.2 ***Bounded Set.*** *A non-empty set* $M \subset \mathbb{X}$ *is said to be bounded if it's diameter is finite. Where diameter* (δ) *is -*

$$\delta(M) = \sup_{x,y \in M} d(x, y) \tag{4.1}$$

Definition 4.0.3 ***Bounded Sequence.*** *A convergent sequence in* $\mathbb{X}$ *is bounded if:*

1. *Its limit is unique.*

2. *If* $x_n \to x$ *(at* ∞*) and* $y_n \to y$ *then* $d(x_n, y_n) = d(x, y)$

Definition 4.0.4 ***Cauchy Sequence.*** *A sequence is a Cauchy Sequence in a metric space* $(\mathbb{X}, d)$ *if* $\forall \epsilon\ \exists$ *an* N *such that* $d(x_m, x_n) < \epsilon\ \forall\ m, n > N$. ***Note:*** ϵ *may not belong to the sequence and can be any arbitrarily small number.*

Definition 4.0.5 ***Complete Metric Space.*** *A metric space is complete if every Cauchy sequence converge.*

Example 4.1: $\mathbb{X} = \mathbb{R} - \{a\}$ is an incomplete metric space because we can find any cauchy sequence which converges at a but $a \notin \mathbb{X}$.

Example 4.2: Real Numbers after removal of irrational numbers i.e. a set $\mathbb{Q}$ of rational numbers.

Example 4.3: (a, b) is not complete metric space because there can exist a cauchy sequence whose limit is a or b which are not included.

Note: $d(x, y) = |x - y|$ is said to be **usual metric**.

Example 4.4: $(0, 1]$ with usual metric, and let the cauchy sequence be $x_n = \frac{1}{n}$ which converges to 0 and 0 does not exist in the metric space. So convergence is a property of metric space "more".

Lemma 1 *Every convergent sequence is a cauchy sequence.*

Proof 11 *If* $x_n \to x \; \forall \; \epsilon > 0$ *and* $x_n > N$ *the sequence converges.*

$$
\begin{aligned}
d(x_n, x) &< \frac{\epsilon}{2} \textit{ where } n > N \\
d(x_m, x_n) &\leqslant d(x_m, x) + d(x_n, x) \\
d(x_m, x_n) &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \textit{ where } n, m > N
\end{aligned}
$$

Problem 4.1: Prove that $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete metric spaces.

Proof 12 ***Hint:*** *Use the fact that* $\mathbb{R}$ *is complete.*

Problem 4.2: Show that the set $\mathbb{X}$ of all the integers with metric d defined as $d(x, y) = |x - y|$ is a complete metric space.

Proof 13 *To proof completeness,*

1. ***Find all cauchy sequence.*** *Note:* $1, 2, 3, \ldots$ *is not a cauchy sequence. A constant sequence is a cauchy sequence or a sequence which eventually becomes a constant sequence is a cauchy sequence (these are the only cauchy sequence).*

Note: Discrete Metric is a complete metric space, as the only cauchy sequence is the sequence of constants which is convergent.

Chapter 5

Normed Linear Spaces

5.1 Vector Space

Definition 5.1.1 ***(Vector Space)****. A vector space (or linear space) over a field* K *is a non-empty set* X *of elements* $x, y, ...$ *(called vectors) together with two algebraic operations. There operations are called vector addition and multiplication of vectors by scalars, that is, by elements of* K.

Vector addition associates with every ordered pair (x, y) of vectors a vector $x + y$, called the sum of x and y, in such a way that the following properties hold. Vector addition is commutative and associative, that is for all vectors we have

$$\begin{aligned} x + y &= y + x \\ x + (y + z) &= (x + y) + z; \end{aligned}$$

furthermore, there exists a vector 0, called the zero vector, and for every x there exists a vector $-x$, such that for all vectors we have

$$\begin{aligned} x + 0 &= x \\ x + (-x) &= 0 \end{aligned}$$

Multiplication by scalars associates with every vector x and scalar α a vector αx (also written as $x\alpha$), alled the product of α and x, in such a way that for all vectors x, y and scalars α, β we have

$$\begin{aligned} \alpha(\beta x) &= (\alpha\beta)x \\ 1x &= x \end{aligned}$$

and the distributive laws

$$\begin{aligned} \alpha(x + y) &= \alpha x + \alpha y \\ (\alpha + \beta)x &= \alpha x + \beta x \end{aligned}$$

Definition 5.1.2 ***Field.*** *A field is a set* F*, containing at least two elements, on which two operations* $+$ *and* $\cdot$ *(called addition and multiplication, respectively) are defined so that for each pair of elements* x, y *in* F *there are unique elements* $x + y$ *and* $x \cdot y$ *(often written as* xy*) in* F *for which the following conditions hold for all elements* x, y, z *in* F*:*

1. $x + y = y + x$ *(commutativity of addition)*
2. $(x + y) + z = x + (y + z)$ *(associativity of addition)*
3. *There is an element* $0 in F$*, called zero, such that* $x + 0 = x$*. (existence of an additive identity)*
4. *For each* x*, there is an element* $-x \in F$ *such that* $x + (-x) = 0$*. (existence if additive inverses)*
5. $xy = yx$ *(commutativity of multiplication)*
6. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(associaitivity of multiplication)*
7. $(x + y) \cdot z = x \cdot z + y \cdot z$ *and* $x \cdot (y + z) = x \cdot y + x \cdot z$ *(distributivity)*
8. *There is an element* $1 \in F$*, such that* $1 \neq 0$ *and* $x\dot{1} = x$ *(existence of a multiplicative identity)*
9. *Id* $x \neq 0$*, then there is an element* $x^{-1} \in F$ *such that* $x \cdot x^{-1} = 1$ *(existence of multiplicative inverses)*

From the definition we see that vector addition is a mapping $X \times X \to X$, whereas multiplication by scalars is a mapping $K \times X \to X$.

K is called the **scalar field** (or coefficient field) of the vector space X, and X is called a **real vector space** if $K = \mathbb{R}$ (the field of real numbers), and a **complex vector space** if $K = \mathbb{C}$ (the field of complex numbers).

The use of 0 for the scalar 0 as well as for the zero vector should cause no confusion, in general. If desirable for clarity, we can denote the zero vector by θ.

Definition 5.1.3 ***Eigen Value of Transformation****. Let* $T(x) = \alpha x$ *where* T *is transformation and* x *is vector then* α *is eigen value of transformation.*

5.2 Closed Set and Closure

Definition 5.2.1 ***Closed Set.*** *A set containing all of it's limit points is called closed set.*

Definition 5.2.2 ***Closure of a Set.*** *A set with all of it's limit points* $\bar{M}$ *is called closed set.*

If there is a set, then a sequence can always be constructed such that it's limit is some arbitrary limit point. For e.g. take a $x_i \in \mathtt{Neighbour}(\mathtt{dis}(\frac{1}{2^i}))$ for all $i \in \mathbb{N}$.

Lemma 2 *A subspace* M *of a complete metric space* X *is itself complete iff* M *is closed.*

5.3 Vector Subspaces

Definition 5.3.1 *A* ***subspace of a vector space*** X *is a non-empty subset* Y *i.e.* $Y \subset X$ *if* $\forall\, y_1, y_2 \in Y$, $\alpha y_1 + \beta y_2 \in Y$.

Definition 5.3.2 *A* ***span of*** **M** X *is the set of all linear combination of vectors of* M*, i.e.* $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$.

Definition 5.3.3 *A set of vectors is called linear independent iff* $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0 \implies \alpha_1 = \alpha_2 = ... = \alpha_n = 0$ *where* $\alpha_i s$ *are scalars. So parallel vectors are not independent.*

If the number of dimensions (or $\dim(X)$) is n, then $n+1$ vectors are surely linearly dependent.

Definition 5.3.4 ***Hamel Basis.*** *If* X *is any vector space not necessarily finite dimensional and* B *is a linearly independent subset of* X *which spans* X *then* B *is called Hamel Basis. "Can be infinitely dimensional."*

Definition 5.3.5 *A* ***Normed Space*** X *is a vector space with a norm defined on it.*

Note: Functional is different from Function.

Definition 5.3.6 *A* ***norm*** *is a real valued function* X *whose value at any* $x \in X$ *is denoted by* $\|\bar{x}\|$ *with following properties :-*

N_1 : $\|\bar{x}\| \geqslant 0$

N_2 : $\|\bar{x}\| = 0 \iff \bar{x} = 0$

N_3 : $\|\alpha\bar{x}\| = \alpha \|\bar{x}\|$

N_4 : $\|\bar{x} + \bar{y}\| \geqslant \|\bar{x}\| + \|\bar{y}\|$

where $\bar{x}, \bar{y}$ *are arbitrary vectors.*

Definition 5.3.7 *The norm defines a metric* d *on* X.

$$d(x, y) = \|x - y\| \tag{5.1}$$

which is the metric induced by the norm $(X, \|\cdot\|)$.

Note: $\|x\|$ is a generalisation of length of a vector $\|x\| = |x|$
Note: Normed and Banach spaces are metric spaces.

Some examples:

1. $\mathbb{R}^n$ with $\|x\| = (\sum_{j=1}^{n} |x_i|^2)^{\frac{1}{2}}$

2. l^p with $\|x\| = (\sum_{j=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$

3. l^∞ with $\|x\| = \sup_j |x_j|$

4. $C[a, b]$ with $\|x\| = \max_{tin} |x_t|$

Definition 5.3.8 *A metric* d *induced by a norm in a normed space* X *satisfies*

$$\begin{aligned} d(x + a, y + a) &= d(x, y) \\ d(\alpha x, \alpha y) &= |\alpha| d(x, y) \end{aligned}$$

$\forall x, y, a \in X$ *and every scalar* α*, as,*

$$\begin{aligned} d(x + a, y + a) &= \|(x + a) - (y + a)\| = \|x - y\| = d(x, y) \\ d(\alpha x, \alpha y) &= \|\alpha x - \alpha y\| \\ &= |\alpha| \, \|x - y\| = |\alpha| d(x, y) \end{aligned}$$

Definition 5.3.9 *A subspace* Y *of a normed space* X *is a subspace of* X *considered as a vector space with the norm obtained by restricting the norm on* X *to the subset* Y*. The norm is the* ***induced norm.***

Problem 5.3.1: Describe the span of $M = \{(1, 1, 1), (0, 0, 2)\}$ in $\mathbb{R}^3$.

Solution 5.3.1: Span of $M = \{\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} : \alpha, \beta \in \mathbb{R}\} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha + 2\beta \end{bmatrix} : \alpha, \beta \in \mathbb{R}$. Or the subspace $x = y$.

Problem 5.3.2: Which of the subsets of $\mathbb{R}^3$ consists of a subspace of $\mathbb{R}^3$. All: x with $x_1 = x_2$ & $x_3 = 0$.

Solution 5.3.2: Let $x = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_1 \\ 0 \end{bmatrix}$ where $x, y \in w$. Then $\alpha x + \beta y = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_1 + \beta y_1 \\ 0 \end{bmatrix} \in w$. Therefore, $\alpha x + \beta y \in w$ hence it is a subspace.

Chapter 6

Vector Basis

Note: If a vector under study is ∞ dimensional, then the Hamel Basis is **uncountable.**

Definition 6.0.1 ***Schauder Basis.*** *$e_1, e_2, ...$ is an infinite basis.*

6.1 Incomplete Normed Space

Definition 6.1.1 *An incomplete normed space and its completion $L^2[a, b]$ (square integrable norm). The vector space of all continuous real-valued functions on $[a, b]$ where the norm is defined as*

$$\|x\| = \{\int_a^b (x(t))^2 dt\}^{\frac{1}{2}} \tag{6.1}$$

Distance induced by this norm

$$\|x_n - x_m\|^2 = \int_0^1 (x_n(t) - x_m(t))^2 dt \tag{6.2}$$

Note: Here basically prove that the given normed space is incomplete by finding a cauchy sequence which doesn't have a limit inside the metric space X. And then complete it using $L^p[a, b]$.

6.2 Finite Dimensional Normed Space

Theorem 1 *Let X be a finite dimensional linear space with basis $\{x_1, x_2, ..., x_n\}$. Then there is a constant $m > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, ...$*

$$m \sum_{j=0}^{n} |\alpha_j| \leqslant \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \tag{6.3}$$

Proof 14 *If $\sum_{j=0}^{n} |\alpha_j| = 0$ then $|\alpha_j| = 0$ for all $j = 1, 2, ..., n$. Assume that $\sum_{j=0}^{n} |\alpha_j| \neq 0$. Let's prove the theorem for scalars $\{\alpha_1, \alpha_2, ...\}$ with the condition that $\sum_{j=1}^{n} |\alpha_j| = 1$. Let $A = \{(\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n | \sum_{j=1}^{n} |\alpha_j| = 1\}$.*

Define $f(\alpha_1, \alpha_2, ..., \alpha_n) = \left\|\sum_{j=1}^{n} \alpha_j x_j\right\|$. *For any* $(\alpha_1, \alpha_2, ..., \alpha_n)$ *and* $(\beta_1, \beta_2, ..., \beta_n)$ *consider* $|f(\alpha_1, \alpha_2, ..., \alpha_n) - f(\beta_1, \beta_2, ..., \beta_n)|$ *which is*

$$\begin{aligned}
\left| \left\|\sum_{j=1}^{n} \alpha_j x_j\right\| - \left\|\sum_{j=1}^{n} \beta_j x_j\right\| \right| &\leqslant \left\|\sum_{j=1}^{n} \alpha_j x_j - \sum_{j=1}^{n} \beta_j x_j\right\| = \left\|\sum_{j=1}^{n} (\alpha_j - \beta_j) x_j\right\| \\
&\leqslant \sum_{j=1}^{n} \left\|(\alpha_j - \beta_j) x_j\right\| = \sum_{j=1}^{n} |\alpha_j - \beta_j| \left\|x_j\right\| \\
&\leqslant \max_{1 \leqslant j \leqslant n} \left\|x_j\right\| \sum_{j=1}^{n} |\alpha_j - \beta_j|
\end{aligned}$$

Let $(\mu_1, \mu_2, ..., \mu_n) \in A$ *be such that* $f(\mu_1, \mu_2, ..., \mu_n) = \inf\{f(\alpha_1, \alpha_2, ..., \alpha_n) | (\alpha_1, \alpha_2, ..., \alpha_n) \in A\}$. *Let* $m = f(\mu_1, \mu_2, ..., \mu_n)$. *In case this minimum is* 0.

$$\left\|\sum_{j=1}^{n} \mu_j x_j\right\| = 0 \implies \sum_{j=1}^{n} \mu_j x_j = 0$$

but $x_1, x_2, ...$ *are linearly independent therefore* $\mu_j = 0$ *for all* $j = 1, 2, ..., n$ *i.e. it is a zero vector also summation must have been* 1*, hence a contradiction. Therefore* $0 < m \leqslant f(\alpha_1, \alpha_2, ..., \alpha_n)$.

$$\left\|\sum_{j=1}^{n} \mu_j x_j\right\| \leqslant \left\|\sum_{j=1}^{n} \alpha_j x_j\right\|$$

Let $\alpha_1, \alpha_2, ..., \alpha_n$ *be any collection of scalars, set* $\beta = \sum_{j=1}^{n} |\alpha_j|$. *Let* $\beta > 0$*, then* $(\frac{\alpha_1}{\beta}, \frac{\alpha_2}{\beta}, ..., \frac{\alpha_n}{\beta}) \in A$ *as*

$$\frac{\alpha_i}{\beta} = \frac{\alpha_i}{\sum_{j=1}^{n} |\alpha_j|}$$

taking summation on both sides, gives 1 *hence it is in* A. *Therefore,*

$$f(\alpha_1, \alpha_2, ..., \alpha_n) = \left\|\sum_{j=1}^{n} \alpha_j x_j\right\| \tag{6.4}$$

$$= \left\|\sum_{j=1}^{n} \frac{\alpha_j}{\beta} x_j\right\| \beta \tag{6.5}$$

$$= \beta f(\frac{\alpha_1}{\beta}, \frac{\alpha_2}{\beta}, ..., \frac{\alpha_n}{\beta}) > m\beta \tag{6.6}$$

Therefore,

$$m\beta = m \sum_{j=1}^{n} |\alpha_j| \leqslant \left\|\sum_{j=1}^{n} \alpha_j x_j\right\|$$

Hence Proved.

Note: Unit sphere is $S(0) = \{x \in X : \|x\| = 1\}$.

Definition 6.2.1 *A subset* A *of a vector space is said to be convex if for any* $x, y \in A \implies M = \{z \in X : z = \alpha x + (1-\alpha)y,\ 0 \leqslant \alpha \leqslant 1 \subset A$. M *is a closed segment with boundary points* x, y.

Definition 6.2.2 ***Equivalence of Norm.*** *A norm* $\|\cdot\|$ *on a vector space* X *is said to be equivalent to* $\|\cdot\|_0$ *on* X, *if there are positive numbers* a, b *such that* $\forall\ x \in X$ *we have,*

$$a\|x\|_0 \leqslant \|x\| \leqslant b\|x\|_0$$