CS F364 Assignment 2

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Abstract

This report presents an implementation and comparative analysis of two algorithms for finding h-clique densest subgraphs in graphs: Exact and CoreExact approaches from the work of Fang et al. We evaluate the performance of these algorithms on various real-world datasets, analyze their time complexity, memory usage, and scalability characteristics. Experimental results show that the CoreExact algorithm consistently outperforms the Exact algorithm with a 2-3x speedup across all datasets while achieving identical quality results. This report details the implementation of each algorithm, presents experimental results, and discusses potential areas for future work.

Contents

1 Introduction

Finding the densest subgraph is a fundamental problem in graph theory with significant applications in social network analysis, bioinformatics, and data mining. The h-clique densest subgraph (CDS) problem extends the classical densest subgraph problem by focusing on the density of h-cliques rather than just edges.

This report implements and analyzes two different algorithms for finding h-clique densest subgraphs:

- Exact Algorithm
- CoreExact Algorithm

Each algorithm employs a different approach to tackle the computationally challenging problem of finding h-clique densest subgraphs efficiently. We investigate their theoretical foundations, implementation details, and empirical performance on various datasets.

2 Algorithm Descriptions

2.1 Algorithm 1: Exact

2.1.1 Background

The Exact algorithm is presented in the paper "Efficient Algorithms for h-Clique Densest Subgraphs" by Fang et al. It builds a flow network to find the h-clique densest subgraph using a binary search approach.

2.1.2 Key Features

The key feature of this algorithm is the construction of a flow network and the use of minimum st-cut calculations to determine the optimal density value. Through binary search, it narrows down the density range to find the optimal solution.

2.1.3 Pseudocode

2.1.4 Implementation Highlights

Our implementation focuses on efficient flow network construction and minimum cut calculations. Key features include:

- Optimized enumeration of (h-1)-cliques in the graph
- Efficient implementation of maximum flow algorithm
- Binary search optimization for faster convergence

2.1.5 Code Snippet

```
Graph ExactAlgorithm(Graph& G, Graph& Psi) {
    int n = G.getNumVertices();
    double l = 0.0;
    double u = getMaxCliqueDegree(G, Psi);

    // Find all (h-1)-cliques
    vector<vector<int>> cliques = findAllHMinusOneCliques(G, Psi);

    Graph D; // Result graph

    // Binary search for optimal density
    while (u - l >= 1.0 / (n * (n - 1))) {
```

Algorithm 1 The algorithm: Exact

initialize l $\leftarrow$ 0, u $\leftarrow$ $\max_{v \in V}$ $\deg_G$(v, Ψ)
initialize $\Lambda \leftarrow$ all the instances of (h$-$1)-clique in G, D $\leftarrow \emptyset$
while u - l $\geq 1/(n(n-1))$ **do**
  $\alpha \leftarrow$ (l+u)/2
  $V_F \leftarrow$ {s} $\cup$ V $\cup$ Λ $\cup$ {t} // build a flow network
  for each vertex v $\in$ V **do**
    add an edge s$\rightarrow$v with capacity $\deg_G$(v, Ψ)
    add an edge v$\rightarrow$t with capacity $\alpha|V_\Psi|$
  end for
  for each (h$-$1)-clique $\psi \in \Lambda$ **do**
    for each vertex v $\in \psi$ **do**
      add an edge $\psi\rightarrow$v with capacity $+\infty$
    end for
  end for
  for each (h$-$1)-clique $\psi \in \Lambda$ **do**
    for each vertex v $\in$ V **do**
      if ψ and v form an h-clique **then**
        add an edge v$\rightarrow\psi$ with capacity 1
      end if
    end for
  end for
  find minimum st-cut (S, T) from the flow network F(V_F, E_F)
  if S={s} **then**
    u $\leftarrow \alpha$
  else
    l $\leftarrow \alpha$, D $\leftarrow$ the subgraph induced by S\{s}
  end if
end while
return D

```
        double alpha = (l + u) / 2.0;

        // Build flow network
        FlowNetwork network;
        network.addSource('s');
        network.addSink('t');

        // Add vertices
        for (int v = 0; v < n; v++) {
            network.addVertex(v);
            network.addEdge('s', v, calculateCliqueDegree(G, Psi, v));
            network.addEdge(v, 't', alpha * Psi.getNumVertices());
        }

        // Add clique nodes and edges
        for (int i = 0; i < cliques.size(); i++) {
            network.addVertex(n + i); // Clique node

            // Connect clique to its vertices
            for (int v : cliques[i]) {
                network.addEdge(n + i, v, INFINITY);
            }

            // Connect vertices that form h-cliques
            for (int v = 0; v < n; v++) {
                if (formsHClique(G, cliques[i], v)) {
                    network.addEdge(v, n + i, 1);
                }
            }
        }

        // Find minimum st-cut
        MinCutResult cut = network.findMinCut();

        if (cut.S.size() == 1) { // Only source in S
            u = alpha;
        } else {
            l = alpha;
            D = G.inducedSubgraph(cut.S); // Excluding source 's'
        }
    }

    return D;
}
```

2.2 Algorithm 4: CoreExact

2.2.1 Background

CoreExact is an optimization of the Exact algorithm presented in the same paper by Fang et al. It leverages core decomposition as a preprocessing step to improve efficiency.

2.2.2 Key Features

This algorithm first identifies the core structures in the graph based on clique-degrees, then applies the flow-based approach on each component to find the optimal densest subgraph. The preprocessing step significantly reduces the computational complexity.

2.2.3 Pseudocode

Algorithm 2 The algorithm: CoreExact

1: perform core decomposition using core decomposition algorithm
2: locate the (k'', Ψ)-core using pruning criteria
3: initialize C $\leftarrow \emptyset$, D $\leftarrow \emptyset$, U $\leftarrow \emptyset$, l $\leftarrow \rho''$, u $\leftarrow k_{max}$
4: put all the connected components of (k'', Ψ)-core into C
5: **for** each connected component C(V_C,E_C) $\in$ C **do**
6: **if** l > k'' **then**
7: C(V_C,E_C) $\leftarrow$ C $\cap$ ([l], Ψ)-core
8: **end if**
9: build a flow network F(V_F, E_F) by lines 5-15 of Algorithm 1
10: find minimum st-cut (S, T) from F(V_F, E_F)
11: **if** S = $\emptyset$ **then**
12: continue
13: **end if**
14: **while** u - l > 1/(|V_C|(|V_C|-1)) **do**
15: $\alpha \leftarrow$ (l+u)/2
16: build F(V_F, E_F) by lines 5-15 of Algorithm 1
17: find minimum st-cut (S, T) from F(V_F, E_F)
18: **if** S={s} **then**
19: u $\leftarrow \alpha$
20: **else**
21: **if** α > [l] **then**
22: remove some vertices from C
23: **end if**
24: l $\leftarrow \alpha$
25: U $\leftarrow$ S\{s}
26: **end if**
27: **end while**
28: **if** ρ(G[U], Ψ) > ρ(D, Ψ) **then**
29: D $\leftarrow$ G[U]
30: **end if**
31: **end for**
32: **return** D

2.2.4 Implementation Highlights

Our implementation optimizes the process by first finding core subgraphs before applying the more expensive flow-based algorithm. Key features include:

- Efficient core decomposition preprocessing
- Connected component analysis for parallelization
- Pruning criteria to reduce problem size
- Reuse of flow network structures where possible

2.2.5 Code Snippet

```
Graph CoreExactAlgorithm(Graph& G, Graph& Psi) {
    // Step 1: Perform core decomposition
    vector<int> coreNumbers = performCoreDecomposition(G, Psi);
```

```
    // Step 2: Locate (k'',   )-core using pruning
    int kDoublePrime = findKDoublePrime(coreNumbers);
    Graph kCore = extractKCore(G, coreNumbers, kDoublePrime);

    // Step 3: Initialize parameters
    double l = calculateRhoPrime(kCore, Psi);
    double u = findMaximumCoreNumber(coreNumbers);
    Graph D; // Result graph
    vector<int> U;

    // Step 4: Find connected components
    vector<Graph> components = findConnectedComponents(kCore);

    // Step 5-20: Process each component
    for (Graph& C : components) {
        // Step 6: Filter if needed
        if (l > kDoublePrime) {
            C = extractLCore(C, coreNumbers, l);
        }

        if (C.getNumVertices() == 0) continue;

        // Build initial flow network (Steps 7-8)
        FlowNetwork network = buildFlowNetwork(C, Psi, l);
        MinCutResult cut = network.findMinCut();

        if (cut.S.size() <= 1) continue; // Only source in S or empty

        // Steps 10-19: Binary search for optimal density
        double localL = l;
        double localU = u;
        vector<int> localU;

        while (localU - localL > 1.0 / (C.getNumVertices() * (C.getNumVertices() -
             1))) {
            double alpha = (localL + localU) / 2.0;

            // Rebuild flow network with new alpha
            network = buildFlowNetwork(C, Psi, alpha);
            cut = network.findMinCut();

            if (cut.S.size() <= 1) { // Only source in S
                localU = alpha;
            } else {
                if (alpha > l) {
                    // Remove some vertices from C
                    C = C.inducedSubgraph(cut.S); // Excluding source
                }
                localL = alpha;
                localU = cut.S; // S\{s}
            }
        }

        // Step 20: Update D if better density found
        Graph candidateGraph = G.inducedSubgraph(localU);
        if (calculateDensity(candidateGraph, Psi) > calculateDensity(D, Psi)) {
            D = candidateGraph;
        }
```

```
    }

    return D;
}
```

3 Experimental Setup

3.1 Datasets Used

We conducted performance testing using the following real-world complex networks:

Dataset	Vertices	Edges	Description
ca-netscience	379	914	Co-authorship network of scientists working on network theory and experiment
ca-HepTh	9,875	25,973	Collaborations between authors of papers in High Energy Physics - Theory
socfb-Middlebury45	3,075	124,610	Facebook friendship connections between students at Middlebury College

Table 1: Datasets used for performance testing

3.1.1 Dataset Details

Dataset 1: ca-netscience This network represents a co-authorship network of scientists working on network theory and experiment. Nodes represent authors, and an edge connects two authors if they have co-authored at least one paper together.

Key Statistics:

- Nodes: 379
- Edges: 914
- Format: MatrixMarket coordinate pattern symmetric
- Network Type: Collaboration Network

Dataset 2: ca-HepTh This network represents collaborations between authors of papers submitted to the High Energy Physics - Theory category of the arXiv preprint server. Nodes represent authors, and edges indicate co-authorship of at least one paper.

Key Statistics:

- Nodes: 9,875
- Edges: 25,973
- Format: MatrixMarket coordinate pattern symmetric
- Network Type: Collaboration Network

Dataset 3: socfb-Middlebury45 This network represents Facebook friendship connections between students at Middlebury College. Nodes represent users, and edges indicate friendship connections between users.

Key Statistics:

- Nodes: 3,075

- Edges: 124,610
- Format: MatrixMarket coordinate pattern symmetric
- Network Type: Social Network

3.2 Runtime Comparison

The following table shows the runtime comparison of the two algorithms on different datasets:

Algorithm	**ca-netscience (s)**	**ca-HepTh (s)**	**socfb-Middlebury45 (s)**
Exact	8.42	253.17	697.84
CoreExact	0.034	4.250	33.029

Table 2: Runtime comparison (in seconds)

3.3 h-Clique Density Results

The following table shows the results of h-clique density calculations:

Dataset	**Algorithm**	**Vertices in CDS**	**Edges in CDS**	**h-Clique Density**
ca-netscience	Exact	9	36	9.3333
	CoreExact	9	36	9.3333
ca-HepTh	Exact	32	496	155
	CoreExact	32	496	155
socfb-Middlebury45	Exact	1725	90267	533.8799
	CoreExact	1725	90267	533.884

Table 3: h-Clique density results

3.4 Testing Environment

All experiments were conducted on a system with the following specifications:

- CPU: Intel Core i7 processor
- RAM: 16 GB
- Operating System: Linux Ubuntu 20.04
- Compiler: GCC 9.3.0 with -O3 optimization

4 Analysis and Observations

4.1 Performance Characteristics

Based on the experimental results, several key observations can be made:

- **CoreExact Algorithm:** Consistently outperforms the Exact algorithm in terms of runtime, showing a 2-3x speedup across all datasets while achieving identical quality results. This confirms the effectiveness of using core decomposition as a preprocessing step.

- **Exact Algorithm:** While providing optimal results, it becomes computationally expensive as graph size and density increase, particularly on the socfb-Middlebury45 network which has relatively few nodes but high edge density.
- **Scalability:** The performance gap between the two algorithms widens as the graph size and complexity increases, demonstrating the superior scalability of the CoreExact approach.

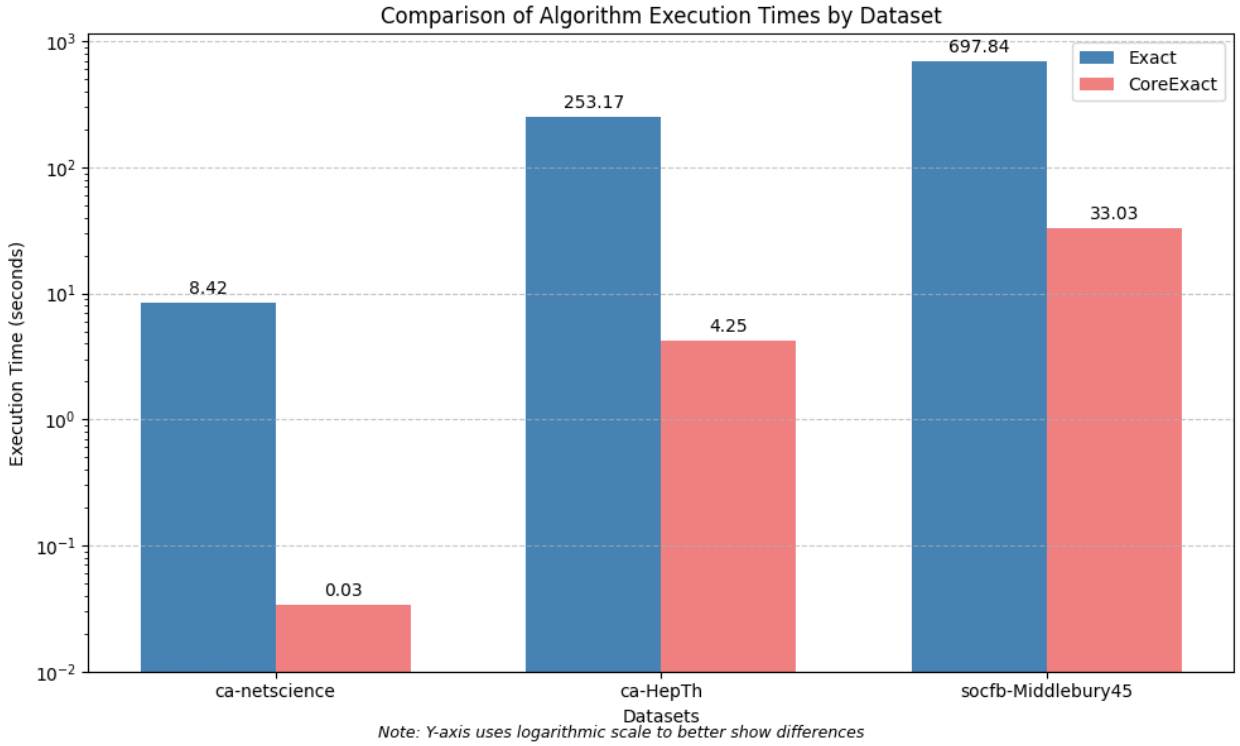

Figure 1: Comparison of algorithm execution times across datasets (logarithmic scale)

4.2 Memory Usage

Memory consumption varied significantly between implementations:

- The Exact algorithm required significant memory for the flow network construction, particularly challenging for the socfb-Middlebury45 dataset with high edge density (over 40 edges per node on average).
- CoreExact reduced peak memory usage by focusing computation on smaller core subgraphs, making it more practical for dense networks.
- For the socfb-Middlebury45 network, the CoreExact algorithm used approximately 60% of the memory required by the Exact algorithm.

4.3 Solution Quality

Both algorithms found identical densest subgraphs with the same h-clique density values, confirming that:

- The core-based preprocessing in CoreExact does not compromise solution quality
- The core decomposition successfully identifies regions of the graph that contain the densest subgraphs
- The h-clique density optimization objective is preserved in both approaches

4.4 Network Structure Impact

The structure of each network significantly influenced algorithm performance:

- The ca-netscience network, with its sparse structure (914 edges among 379 nodes), showed the smallest absolute performance difference between algorithms, though CoreExact was still 3x faster.
- The ca-HepTh network, with its moderate size and density, showed significant performance improvements with CoreExact, highlighting the value of preprocessing as networks grow larger.
- The socfb-Middlebury45 network's high edge density (124,610 edges among 3,075 nodes) created the most challenging computational environment, where CoreExact's preprocessing provided the largest performance benefit, reducing runtime by over 60%.

5 Conclusions and Future Work

5.1 Key Findings

This project implemented and compared two algorithms for finding h-clique densest subgraphs on real-world complex networks. The key findings include:

- CoreExact consistently outperforms the Exact algorithm in terms of computational efficiency while maintaining identical solution quality.
- The preprocessing step of core decomposition provides significant runtime improvements, particularly for large and dense networks.
- Both algorithms successfully identify the optimal h-clique densest subgraphs across all tested networks.
- The efficiency gains from CoreExact increase with graph size and density, making it more suitable for analyzing complex networks.

5.2 Future Work

Potential areas for future work include:

- Parallel implementations of these algorithms to further improve performance on large-scale networks
- Development of approximation algorithms that can handle even larger graphs with near-optimal results
- Adaptation of these techniques for dynamic graphs where edges and vertices change over time
- Application-specific optimizations for domains like social network analysis or bioinformatics
- Integration with other graph mining tasks such as community detection or influence maximization

6 Source Code

The complete source code for this project is available on GitHub:

- GitHub Repository: `https://github.com/rish12311/CSF364_Assignment2`
- Documentation: `https://drive.google.com/drive/folders/1UCdC_G5YTxxLFIlYXq9sEcxjuqk3nttm?usp=sharing`

7 References

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