Q1

06 December 2022 20:02

Question 1:

The file **F16s.mat** gives a linear perturbation model for an F16 aircraft in level flight at 10,000 ft. The outputs of this model are the aircraft's speed and climb rate, both in ft/sec. The two inputs to the model are stochastic processes, $d_1(t)$ and $d_2(t)$ representing wind gust effects on the vertical component of the aircraft velocity, and on the pitch rate of the aircraft, respectively.

Using the Dryden model for vertical wind turbulence, we can model $d_1(t)$ and $d_2(t)$ as zero mean WSS processes with spectra

$$S_{d_1d_1}(\omega) = \frac{K(1+3(a\omega)^2)}{(1+(a\omega)^2)^2}$$

and

$$S_{d_2d_2}(\omega) = \left[\frac{\omega^2}{1+(b\omega)^2}\right] S_{d_1d_1}(\omega)$$

Moreover, these disturbances are assumed to be correlated, which can be modeled by using a single white noise source to drive the shaping filters for both disturbances. (Hence d_2 can be modeled by passing d_1 through an additional shaping filter, given the structure of its spectrum.) For the flight conditions above, the values $a = 3$, $b = 0.1$, are appropriate.

a.) The constant K controls the RMS (root-mean-square) magnitude of the vertical gust component. Choose this constant so that $\sqrt{E[d_1(t)^2]} = 7$.

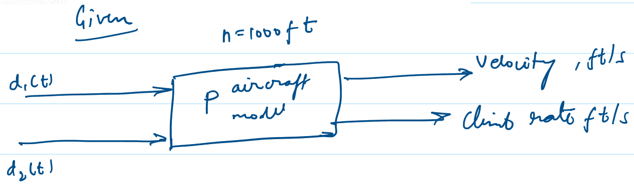
b.) With the value of K found in a.), compute the RMS magnitude of the variations in speed and climb rate that result from the disturbance inputs.

c.) Compute the components F and Q for the stochastic discretization of the complete system. Assume a sample rate of 20Hz.

d.) Show that the discretization computed in c.) is consistent with the calculations in b.). That is, show that the discrete output sequence $\mathbf{y}_k$ has the same steady-state variance as found in b.)

a)

Given



zero mean
WSS

$$S_{d_1d_1}(\omega) = \frac{K(1+3(a\omega)^2)}{(1+(a\omega)^2)^2}$$

where $a = 3$, $\omega^2 = -s^2$

$$\Rightarrow S_{dd}(s) = \frac{K(1+3a^2(-s^2))}{(1-9s^2)^2}$$

$$= \frac{K(1-3a^2s^2)}{(1-9s^2)^2}$$

$$= \frac{K(1-27s^2)}{(1-9s^2)^2}$$

$\because$ WSS,

$$R(\tau) = \mathcal{L}^{-1}\{S_{d_1d_1}(s)\}$$

```
syms s K
f = K*(1 - 27*s^2)/((1 - 9*s^2)^2)
R_tau = ilaplace(f)
```

f =

$$-\frac{K\,(27\,s^2-1)}{(9\,s^2-1)^2}$$

R_tau =

$$\frac{K\,e^{-\frac{t}{3}}}{3} - \frac{K\,e^{t/3}}{3} - \frac{K\,t\,e^{-\frac{t}{3}}}{18} - \frac{K\,t\,e^{t/3}}{18}$$

$$= k\left\{\frac{e^{-\tau/3}}{3} - \frac{\tau e^{-\tau/3}}{18}\right\} \mathbb{1}(\tau)$$

$$+ k\left\{\frac{e^{-\tau/3}}{3} - \frac{\tau e^{-\tau/3}}{18}\right\} \mathbb{1}(-\tau)$$

$$R(\tau) = \begin{cases} k\left(\dfrac{e^{-\tau/3}}{3} - \dfrac{\tau e^{-\tau/3}}{18}\right), & \tau > 0 \\ k\left(\dfrac{e^{\tau/3}}{3} - \dfrac{\tau e^{\tau/3}}{18}\right), & \tau < 0 \end{cases}$$

$$\boxed{\therefore \quad E[d_1(t)^2] = R(0) = k/3} \quad (1)$$

Given,

$$\sqrt{R(0)} = 7, \quad \therefore R(0) = 49$$

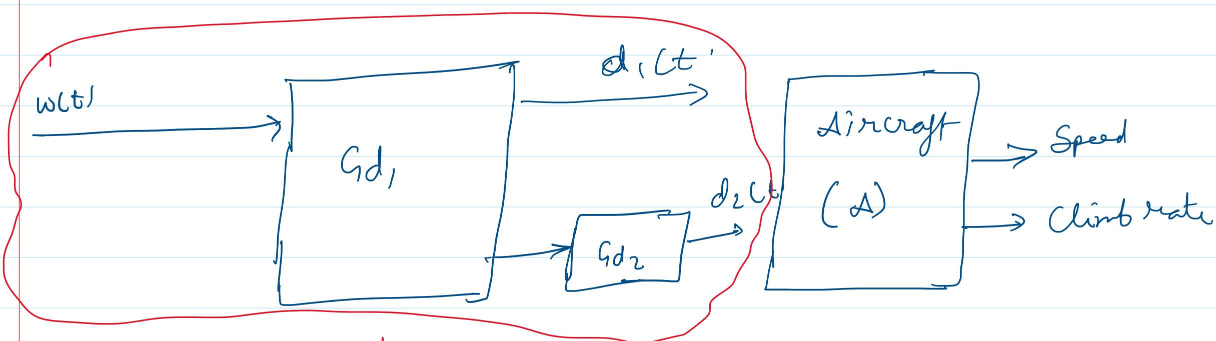
putting above,

$$k/3 = 49$$

$$\boxed{\therefore \quad k = 147}$$

b)

We know that the augmented model for observer will be given by,



Assuming Z_1 & Z_2 to be the state vectors we get,

$$\dot{z}_1 = A_{d_1} Z_1 + B_{d_1} w, \quad Y_{d_1} = C_{d_1} Z$$

$\dot{z}_1 = \dots\quad Bd_1 w \quad , \quad Y_{d_1} = Cd_1 z$

$$\dot{z}_2 = Ad_2 z_2 + Bd_2 w \quad , \quad Y_{d_2} = C_{d_2} z$$

Here, Ad_1, Bd_1, Cd_1 for d_1 output comes from

Gd_1,

$$S_{d_1 d_1} = \frac{3\sqrt{3}s+1}{(3s+1)^2} \times 147 \times \frac{(1-3\sqrt{3}s)}{(1-3s)^2}$$

stable part

taking $Q = 147$ for S_{ww}

($\because$ $w(t)$ is white)

```
bd1 = [0 3*sqrt(3) 1]
ad1 = [9 6 1]

[Ad1, Bd1, Cd1, Dd1] = tf2ss(bd1,ad1)
```

```
bd2 = [0 5.1962 1 0]
ad2 = [0.9 9.6 6.1 1]

[Ad2, Bd2, Cd2, Dd2] = tf2ss(bd2,ad2)
```

Augmented model for disturbance

```
bd1 = 1×3
     0    5.1962    1.0000

ad1 = 1×3
     9     6     1

Ad1 = 2×2
   -0.6667   -0.1111
    1.0000         0

Bd1 = 2×1
     1
     0

Cd1 = 1×2
    0.5774    0.1111
```

and Ad_2, Bd_2, Cd_2 for d_2 output comes from

$Gd_2 \times Gd_1$,

$$\left(\frac{s}{1+bs}\right)\left(\frac{-s}{1-bs}\right)$$

$$S_{d_2 d_2}(s) = \left[\frac{-s^2}{1-b^2 s^2}\right] S_{d_1 d_1}(s) \quad , \quad b = 0.1$$

$$S_{d_2 d_2}(s) = \frac{(3\sqrt{3}s^2 + s)}{(0.9s^3 + 9.6s^2 + 6.1s + 1)} \times 147 \times \frac{(3\sqrt{3}s^2 - s)}{(-0.9s^3 + 9.6s^2 - 6.1s + 1)}$$

Stable part

Q

$\because$ $w(t)$ is white

```
bd2 = [0 5.1962 1 0]
ad2 = [0.9 9.6 6.1 1]

[Ad2, Bd2, Cd2, Dd2] = tf2ss(bd2,ad2)
```

```
bd2 = 1×4
     0    5.1962    1.0000 ...

ad2 = 1×4
    0.9000    9.6000    6.1000 ...

Ad2 = 3×3
  -10.6667   -6.7778   -1.1111
    1.0000         0         0
         0    1.0000         0

Bd2 = 3×1
     1
     0
     0

Cd2 = 1×3
```

```
bd2 = [0 5.1962 1 0]
ad2 = [0.9 9.6 6.1 1]

[Ad2, Bd2, Cd2, Dd2] = tf2ss(bd2,ad2)
```

```
bd2 = 1×4
         0    5.1962    1.0000 ...

ad2 = 1×4
    0.9000    9.6000    6.1000 ...

Ad2 = 3×3
  -10.6667   -6.7778   -1.1111
    1.0000         0         0
         0    1.0000         0

Bd2 = 3×1
     1
     0
     0

Cd2 = 1×3
    5.7736    1.1111         0

Dd2 = 0
```

: w(t) is white

Stacking $\dot z_1$, $\dot z_2$ into z, we get,

$$\begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix} = \overbrace{\left[\begin{array}{c|c} Ad_1 & 0 \\ \hline 0 & Ad_2\end{array}\right]}^{Ad}\begin{bmatrix} z_1 \\ z_2\end{bmatrix} + \overbrace{\begin{bmatrix} Bd_1 \\ Bd_2\end{bmatrix}}^{Bd} w$$

$$\overset{Y}{\begin{bmatrix} d_1 \\ d_2\end{bmatrix}} = \underbrace{\left[\begin{array}{c|c} Cd_1 & 0 \\ \hline 0 & Cd_2\end{array}\right]}_{Cd}\begin{bmatrix} z_1 \\ z_2\end{bmatrix}$$

Augmented model for disturbance

```
Ad = [Ad1 zeros(2,3); zeros(3,2) Ad2]
Bd = [Bd1;Bd2]
Cd = [Cd1 zeros(1,3); zeros(1,2) Cd2]
Dd = [0;0]
```

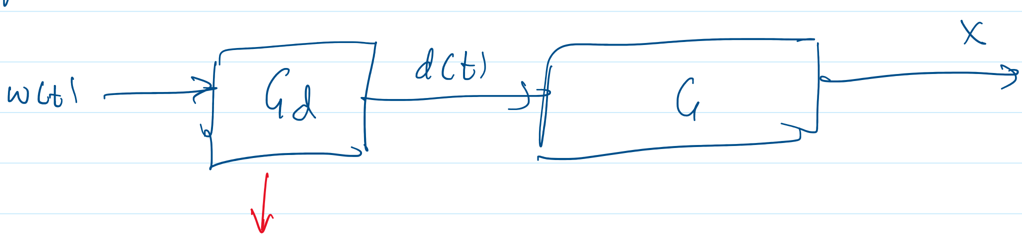
```
Ad = 5×5
   -0.6667   -0.1111         0         0         0
    1.0000         0         0         0         0
         0         0  -10.6667   -6.7778   -1.1111
         0         0    1.0000         0         0
         0         0         0    1.0000         0

Bd = 5×1
     1
     0
     1
     0
     0

Cd = 2×5
    0.5774    0.1111         0         0         0
         0         0    5.7736    1.1111         0

Dd = 2×1
     0
     0
```

So, now basically we can write in simple form as,



corresponds to A_d found above.

$\therefore$ Augmented System will be given by,

$$\frac{d}{dt}\begin{bmatrix} \underline{X} \\ \underline{Z} \end{bmatrix} = \underbrace{\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix}}_{A_a} \begin{bmatrix} \underline{X} \\ \underline{Z} \end{bmatrix} + \overbrace{\begin{bmatrix} BD_d \\ B_d \end{bmatrix}}^{B_a} \underline{w}$$

$$Y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_a} \begin{bmatrix} \underline{X} \\ \underline{Z} \end{bmatrix}$$

Augmented oberver matrix for

```
Aa = [A B*Cd; zeros(5,4) Ad]
Ba = [B*Dd; Bd]
Ca = [C zeros(2,5)]
```

```
Aa = 9×9
   -0.0083   -0.0703   -0.0581    0.0277    0.0344    0.0066   -0.1026   -0.0198         0
    0.0770   -0.0082   -0.0750    0.0345   -0.0293   -0.0056    0.1669    0.0321         0
   -0.0612    0.0714   -1.5824    2.0070    0.1731    0.0333    0.5542    0.1067         0
    0.0150   -0.0282   -1.1581   -0.7484   -0.0129   -0.0025    0.8395    0.1616         0
         0         0         0         0   -0.6667   -0.1111         0         0         0
         0         0         0         0    1.0000         0         0         0         0
         0         0         0         0         0         0  -10.6667   -6.7778   -1.1111
         0         0         0         0         0         0    1.0000         0         0
         0         0         0         0         0         0         0    1.0000         0

Ba = 9×1
     0
     0
     0
     0
     1
     0
     1
     0
     0

Ca = 2×9
    0.0365   -0.0415   -0.0087    0.0175         0         0         0         0         0
    0.0504    0.0410    0.3147   -0.1461         0         0         0         0         0
```

Now, to find the variance by applying Stochastic Disc., and Kalman Filter,

$$Q\_c = B_a \times \overset{147}{Q} \times B_a^T$$

Variance of the process using lyapunov equation

```
Q_c = Ba*147*Ba'
Pss = lyap(Aa, Q_c)

%%RMS values
Var = Ca*Pss*Ca'
rms_speed = sqrt(Var(1,1))
rms_climbrate = sqrt(Var(2,2))
```

```
     0

Q_c = 9×9
     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0
     0     0     0     0   147     0   147     0     0
     0     0     0     0     0     0     0     0     0
     0     0     0     0   147     0   147     0     0
     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0

Pss = 9×9
   30.7618    0.0234   -0.3238    0.2390   -9.2926   76.0061   -0.2802   -0.9012    7.6907
    0.0234   27.1250    0.1807   -0.2737    1.6062  -38.5397    0.4234    0.1183   -3.8658
   -0.3238    0.1807    1.1326    0.0337    9.1116    0.0073    0.0310    0.9081   -0.0901
    0.2390   -0.2737    0.0337    0.9012   -2.1216  -14.5201    0.8174   -0.2939   -1.4226
   -9.2926    1.6062    9.1116   -2.1216  110.2500    0.0000    6.9982   10.3252   -1.0325
   76.0061  -38.5397    0.0073  -14.5201    0.0000  992.2500  -10.3252    1.0325   99.1217
   -0.2802    0.4234    0.0310    0.8174    6.9982  -10.3252    6.9982   -0.0000   -1.0325
   -0.9012    0.1183    0.9081   -0.2939   10.3252    1.0325   -0.0000    1.0325    0.0000
    7.6907   -3.8658   -0.0901   -1.4226   -1.0325   99.1217   -1.0325    0.0000    9.9122

Var = 2×2
    0.0891   -0.0037
   -0.0037    0.2463

rms_speed = 0.2984
rms_climbrate = 0.4963
```

```
0.0891   -0.0037
-0.0037   0.2463
rms_speed = 0.2984
rms_climbrate = 0.4963
```

∴ RMS for variation in speed,

$$= 0.2984$$

RMS for variation in climb rate

$$= 0.4963$$

c)

Given, Sample rate $= 20\,Hz$

$$\therefore \tau = 1/20 \; s = 0.05 \; s$$

using van-loan's algorithm,

$$A = \left[\begin{array}{c|c} -A_a & B_a Q B_a^T \\ \hline 0 & A^T \end{array}\right]$$

$$e^{A\tau} = \left[\begin{array}{c|c} \text{<mess>} & F^{-1}\bar{Q} \\ \hline 0 & F^T \end{array}\right]$$

Variance of the process using lyapunov equation

```
Q_c = Ba*147*Ba';
Pss = lyap(Aa, Q_c);

%%RMS values
Var = Ca*Pss*Ca';
rms_speed = sqrt(Var(1,1));
rms_climbrate = sqrt(Var(2,2));
```

```
%%c part
lam = [-Aa Q_c; zeros(9,9) Aa']
Tau = 0.05
stat_trans = expm(lam*Tau)
F = stat_trans(10:18,10:18)'
Q_bar = F*stat_trans(1:9,10:18)
```

```
Tau = 0.0500
stat_trans = 18×18
    1.0004    0.0035    0.0030   -0.0016   -0.0018   -0.0003    0.0067    0.0020    0.0002 ..
   -0.0038    1.0004    0.0038   -0.0019    0.0015    0.0003   -0.0110   -0.0033   -0.0003
    0.0032   -0.0038    1.0792   -0.1063   -0.0091   -0.0018   -0.0350   -0.0106   -0.0009
   -0.0007    0.0013    0.0613    1.0351    0.0004    0.0001   -0.0570   -0.0171   -0.0014
         0         0         0         0    1.0338    0.0056         0         0         0
         0         0         0         0   -0.0508    0.9999         0         0         0
         0         0         0         0         0         0    1.6925    0.4448    0.0732
         0         0         0         0         0         0   -0.0659    0.9898   -0.0017
         0         0         0         0         0         0    0.0015   -0.0498    1.0000
         0         0         0         0         0         0         0         0         0
    :

F = 9×9
    0.9996   -0.0035   -0.0028    0.0012    0.0017    0.0003   -0.0040   -0.0003    0.0001
    0.0039    0.9996   -0.0037    0.0015   -0.0015   -0.0003    0.0065    0.0004   -0.0002
   -0.0029    0.0034    0.9212    0.0946    0.0082    0.0016    0.0223    0.0014   -0.0007
    0.0008   -0.0015   -0.0546    0.9605   -0.0009   -0.0002    0.0313    0.0019   -0.0010
         0         0         0         0    0.9671   -0.0055         0         0         0
         0         0         0         0    0.0492    0.9999         0         0         0
         0         0         0         0         0         0    0.5807   -0.2631   -0.0429
         0         0         0         0         0         0    0.0386    0.9928   -0.0012
         0         0         0         0         0         0    0.0011    0.0499    1.0000

Q_bar = 9×9
    0.0000   -0.0000   -0.0002   -0.0002   -0.0095   -0.0003   -0.0069   -0.0003   -0.0000
   -0.0000    0.0001    0.0004    0.0004    0.0201    0.0007    0.0146    0.0005    0.0000
   -0.0002    0.0004    0.0025    0.0026    0.1162    0.0039    0.0840    0.0032    0.0001
   -0.0002    0.0004    0.0026    0.0027    0.1214    0.0040    0.0881    0.0033    0.0001
   -0.0095    0.0201    0.1162    0.1214    7.1097    0.1777    5.5950    0.1515    0.0026
   -0.0003    0.0007    0.0039    0.0040    0.1777    0.0060    0.1279    0.0050    0.0001
   -0.0069    0.0146    0.0840    0.0881    5.5950    0.1279    4.4975    0.1098    0.0018
   -0.0003    0.0005    0.0032    0.0033    0.1515    0.0050    0.1098    0.0042    0.0001
   -0.0000    0.0000    0.0001    0.0001    0.0026    0.0001    0.0018    0.0001    0.0000
```

→ F

→ Q

d) To apply dlyap we need to check the magnitude of eigen values first,

$$x_{k+1} = F_k x_k + w_k$$
$$E[w_j w_k^T] = Q_k \delta_{jk}$$

d part

```
abs(eig(F))
```

```
ans = 9×1
    0.9996
    0.9996
    0.9434
    0.9434
    0.9835
    0.9835
    0.6065
    0.9835
    0.9835
```

$\rightarrow |\lambda| < 1$

∴ using "dlyap" in matlab,

```
G = eye(9,9)
Q_d = G*Q_bar*G'
```

```
G = 9×9
     1     0     0     0     0     0     0     0     0
     0     1     0     0     0     0     0     0     0
     0     0     1     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0
     0     0     0     0     1     0     0     0     0
     0     0     0     0     0     1     0     0     0
     0     0     0     0     0     0     1     0     0
     0     0     0     0     0     0     0     1     0
     0     0     0     0     0     0     0     0     1

Q_d = 9×9
    0.0000   -0.0000   -0.0002   -0.0002   -0.0095   -0.0003   -0.0069   -0.0003   -0.0000
   -0.0000    0.0001    0.0004    0.0004    0.0201    0.0007    0.0146    0.0005    0.0000
   -0.0002    0.0004    0.0025    0.0026    0.1162    0.0039    0.0840    0.0032    0.0001
   -0.0002    0.0004    0.0026    0.0027    0.1214    0.0040    0.0881    0.0033    0.0001
   -0.0095    0.0201    0.1162    0.1214    7.1097    0.1777    5.5950    0.1515    0.0026
   -0.0003    0.0007    0.0039    0.0040    0.1777    0.0060    0.1279    0.0050    0.0001
   -0.0069    0.0146    0.0840    0.0881    5.5950    0.1279    4.4975    0.1098    0.0018
   -0.0003    0.0005    0.0032    0.0033    0.1515    0.0050    0.1098    0.0042    0.0001
   -0.0000    0.0000    0.0001    0.0001    0.0026    0.0001    0.0018    0.0001    0.0000
```

```
G = eye(9,9)
Q_d = G*Q_bar*G'
P_disc = dlyap(F,Q_d)
var_discrete = Ca*P_disc*Ca'
rms_speed = sqrt(var_discrete(1,1))
rms_pitchrate = sqrt(var_discrete(2,2))
```

R

```
G = 9×9
     1     0     0     0     0     0     0     0     0
     0     1     0     0     0     0     0     0     0
     0     0     1     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0
     0     0     0     0     1     0     0     0     0
     0     0     0     0     0     1     0     0     0
     0     0     0     0     0     0     1     0     0
     0     0     0     0     0     0     0     1     0
     0     0     0     0     0     0     0     0     1

Q_d = 9×9
    0.0000   -0.0000   -0.0002   -0.0002   -0.0095   -0.0003   -0.0069   -0.0003   -0.0000
   -0.0000    0.0001    0.0004    0.0004    0.0201    0.0007    0.0146    0.0005    0.0000
   -0.0002    0.0004    0.0025    0.0026    0.1162    0.0039    0.0840    0.0032    0.0001
   -0.0002    0.0004    0.0026    0.0027    0.1214    0.0040    0.0881    0.0033    0.0001
   -0.0095    0.0201    0.1162    0.1214    7.1097    0.1777    5.5950    0.1515    0.0026
   -0.0003    0.0007    0.0039    0.0040    0.1777    0.0060    0.1279    0.0050    0.0001
   -0.0069    0.0146    0.0840    0.0881    5.5950    0.1279    4.4975    0.1098    0.0018
   -0.0003    0.0005    0.0032    0.0033    0.1515    0.0050    0.1098    0.0042    0.0001
   -0.0000    0.0000    0.0001    0.0001    0.0026    0.0001    0.0018    0.0001    0.0000

P_disc = 9×9
   30.7618    0.0234   -0.3238    0.2390   -9.2926   76.0061   -0.2802   -0.9012    7.6907
    0.0234   27.1250    0.1807   -0.2737    1.6062  -38.5397    0.4234    0.1183   -3.8658
   -0.3238    0.1807    1.1326    0.0337    9.1116    0.0073    0.0310    0.9081   -0.0901
    0.2390   -0.2737    0.0337    0.9012   -2.1216  -14.5201    0.8174   -0.2939   -1.4226
   -9.2926    1.6062    9.1116   -2.1216  110.2500    0.0000    6.9982   10.3252   -1.0325
   76.0061  -38.5397    0.0073  -14.5201    0.0000  992.2500  -10.3252    1.0325   99.1217
   -0.2802    0.4234    0.0310    0.8174    6.9982  -10.3252    6.9982    0.0000   -1.0325
   -0.9012    0.1183    0.9081   -0.2939   10.3252    1.0325    0.0000    1.0325   -0.0000
    7.6907   -3.8658   -0.0901   -1.4226   -1.0325   99.1217   -1.0325   -0.0000    9.9122

var_discrete = 2×2
    0.0891   -0.0037
   -0.0037    0.2463

rms_speed = 0.2984
rms_pitchrate = 0.4963
```

RMS variation in speed = 0.2984

RMS variation in climb rate = 0.4963

The values are consistent to what we got in 'b' part.