FOURTH SEMESTER [B.TECH]

APPLIED MATHEMATICS-IV [ETMA-202]

Time : 1: 30 hrs. **M.M. : 30**

Note: Attempt Q. No. 1 which is compulsory and any two more questions from remaining. All questions carry equal marks.

Q.1. (a) Solve $\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + y\cos x$ (2.5)

Ans.

$$\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = y\cos x$$

$$(D-2D') = y\cos x$$

put $\quad D = m, D' = 1$

A.E $\quad m - 2 = 0$

$$\Rightarrow m = 2.$$

$$\text{C.F.} = f_1(y+2x).$$

$$\text{P.I.} = \frac{1}{D-2D'} y\cos x$$

$$= \int (c-2x)\cos x\, dx$$

Here $\quad m = 2$, where c is replaced by $y + mx = y + 2x$.

$$= (c-2x)\sin x - (-2)(-\cos x)$$

$$= (c-2x)\sin x - 2\cos x$$

$$= (y+2x-2x)\sin x - 2\cos x$$

$$= y\sin x - 2\cos x$$

$\therefore$ Complete solution is

$$z = f_1(y+2x) + y\sin x - 2\cos x.$$

Q.1. (b) Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = z$ (2.5)

Ans. $\quad (D^2 + DD' + D' - 1)z = 0$

A.E $\quad (D^2 - 1) + D'(D+1) = 0$

$\Rightarrow \quad (D-1)(D+1) + D'(D+1) = 0$

$\Rightarrow \quad (D+1)(D+D'-1) = 0$

$$D = -1$$

for $\quad (D+D'-1), b = 1\ a = -1, c = 1$

C.F $\quad e^{-x}\phi_1(y) + e^{x}\phi_2(y-x)$

Thus, solution is

$$z = e^{-x}\phi_1(y) + e^{x}\phi_2(y-x)$$

Q.1. (c) If the probability that the man aged 60 will live 70 is 0.6, What is the probability that out of 10 men aged 60, 9 men will live upto 70. (2.5)

Ans. Let, probability of success that man will live

upto $70 = 0.6$

ie $p = 0.6$

probability of failure $q = 1 - p = 1 - 0.6 = 0.4$

Let $n = 10$

Let X be the binomial variate

$$\therefore \quad f_X(x) = p[X = x] = {}^{n}C_x\, p^x q^{n-x}$$

$$= {}^{10}C_x (0.6)^x (0.4)^{10-x}$$

$$p[X = 9] = {}^{10}C_9 (0.6)^9 (0.4) = 10 \times 0.4 \times (0.6)^9$$

$$= 0.0403$$

Q.1. (d) Determine the value of k, if the probability function of a random variable X is given by

$$p(x) = \begin{cases} \dfrac{kx}{20}, x = 1, 2, 3, 4 \\ 0, \text{other integers} \end{cases}$$ (2.5)

Ans. Since p_X is the probability distribution function

$$\sum p_X(x) = 1$$

$$\Rightarrow \quad \frac{k}{20} + \frac{2k}{20} + \frac{3k}{20} + \frac{4k}{20} = 1$$

$$10k = 20$$

$$k = 2$$

Q.2. (a) Find the solution of the partial differential equation $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x + 2y)$ (5)

Ans. $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x + 2y)$

Replace D by m and D^1 by 1

A.E $m^3 - 7m - 6 = 0$

$$(m + 1)(m^2 - m - 6) = 0$$

$$\Rightarrow \quad (m + 1)(m^2 - 3m + 2m - 6) = 0$$

$$\Rightarrow \quad m = -1, (m + 2)(m - 3) = 0$$

$$\Rightarrow \quad m = -1, -2, 3$$

$$\text{C.F} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$$

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x + 2y)$$

Replace D by 1 and D' by 2

$$= \frac{1}{1 - 28 - 48} \iiint \sin u \, du\, du\, du$$

$$= \frac{-1}{75} \cos u = \frac{-1}{75} \cos(x + 2y)$$

complete solution is

$$\therefore \quad z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) \frac{-1}{75}\cos(x+2y)$$

Q.2. (b) Use the method of separation of variable to solve the partial differential equation **(5)**

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \text{ given } u = 3e^{-y} - e^{-5y} \text{ when } x = 0$$

Ans. Given equation is $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ ...(A)

Let
$$u = X(x)\,Y(y) = XY$$

$\therefore$
$$\frac{\partial u}{\partial x} = \frac{\partial (XY)}{\partial x} = Y\frac{dX}{dx}$$

and
$$\frac{\partial u}{\partial y} = \frac{\partial (XY)}{\partial y} = X\frac{dY}{dy}$$

$\therefore$
$$A \Rightarrow \quad 4\frac{YdX}{dx} + X\frac{dY}{dy} = 3XY$$

$$4X'Y + XY' = 3XY$$

$\Rightarrow$
$$4X'Y = (3Y - Y')X$$

$\Rightarrow$
$$\frac{4X'}{X} = \frac{3Y - Y'}{Y}$$

$\Rightarrow$
$$\frac{4X'}{X} = 3 - \frac{Y'}{Y} = a \text{ (say)}$$

$\Rightarrow$

As LHS is a function of x and RHS is a function of y only

$\therefore$
$$\frac{4X'}{X} = a \Rightarrow 4\frac{dX}{dx}.\frac{1}{X} = a$$

On integrating,
$$\int\frac{dX}{X} = \int\frac{a}{4}dx$$

$\Rightarrow$
$$\log X = \frac{ax}{4} + \log c_1 \qquad ...(1)$$

$\Rightarrow$
$$X = c_1 e^{ax/4}$$

Now
$$3 - \frac{Y'}{Y} = a \Rightarrow \frac{Y'}{Y} = 3 - a$$

On integrating,
$$\int\frac{dY}{Y} = \int(3-a)dy$$

$\Rightarrow$
$$\log Y = (3-a)y + \log c_2$$

$$y = c_2 e^{(3-a)y}$$

$\Rightarrow$ $$U = XY = c_1 c_2 e^{ax/4} e^{(3-a)y}$$

As $$U(0,y) = 3e^{-y} - e^{-5y}$$

As given from (B) $$U(0,y) = c_1 c_2 e^{(3-a)y}$$

comparing two, we get

$$3e^{-y} - e^{-5y} = c_1 c_2 e^{(3-a)y}$$

$$c_1 c_2 = 3,\ 3 - a = -1 \text{ and } c_1 c_2 = -1,\ 3 - a = -5$$

$\Rightarrow$ $$c_1 c_2 = 3,\ a = 4 \Rightarrow c_1 c_2 = -1,\ a = 8$$

Now, equation (B) becomes

$$U = 3e^{4x/4} e^{(3-4)y} - e^{8x/4} e^{-5y}$$

$$U = 3e^{x} e^{-y} - e^{2x} e^{-5y}.$$

$$U = 3e^{x-y} - e^{2x-5y}$$

Q.3. (a) Find the solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions

(i) $u \to 0$ as $y \to \infty$ for all x (ii) $u = 0$ at $x = 0$ for all y

(iii) $u = 0$ at $x = l$ for all y (iv) $u = lx - x^2$ if $y = 0$ for all $x \in (0,l)$

Ans. Given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ...(1)

The boundary conditions are

$$\left.\begin{aligned} u(0,y) &= 0 \\ u(l,y) &= 0 \end{aligned}\right\} \text{for all } y$$

$$u(x,\infty) = 0\ \forall x$$

$$u(x,0) = lx - x^2 \quad 0 < x < l$$

The three possible solutions are

(i) $$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

(ii) $$u(x,y) = (c_5 \cos px + c_6 \sin px)\, c_7 e^{py} + c_8 e^{-py}$$

(iii) $$u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12})$$

from the condition that $u \to 0$ as $y \to \infty$ for all value of x, solutions (i) and (iii) lead to trivial solutions and hence (ii) is the only suitable one.

i.e $u(x,y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py})$ ...(2)

using boundary conditions $u(0,y) = 0$ in (2) gives

$$0 = A(ce^{py} + De^{-py})$$

$\Rightarrow$ $$A = 0$$

$\therefore$ (2) reduces to

$$u(x,y) = B \sin px\,(Ce^{py} + De^{-py})$$

$$u(x,y) = \sin px\,(C'e^{py} + D'e^{-py})$$

Using the condition $\quad u(l,y) = 0 \qquad$ ...(3)

$$0 = \sin pl\,(C'e^{py} + D'e^{-py})$$

$\Rightarrow$
$$\sin pl = 0 \text{ or } p = \frac{n\pi}{l}, n \text{ being an integer.}$$

Also, using $u(x, \infty) = 0$ in (3), we get $C' = 0$

By (3), we get

$$u(x,y) = \sin\frac{n\pi x}{l}\,.\,D\,e^{-n\pi y/l}, n \text{ is an integer}$$

$\therefore$ General solution is of the form

$$u(x,y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l}\,e^{-ny\pi/l} \qquad ...(4)$$

Using, condition $u(x, 0) = lx - x^2$, we get

$$lx - x^2 = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l}\quad, 0 < x < l$$

which is half range sine series

$\therefore$
$$D_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx\,.$$

$$= \frac{2}{l}\left[\left|lx\left(-\cos\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi} - l\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^l\right.$$

$$-\left(x^2\left(-\cos\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi} - 2x\left(-\sin\frac{n\pi x}{l}\right)\cdot\frac{l^2}{n^2\pi^2}\right.$$

$$\left.\left.+ 2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right\}_0^l\right]$$

$$= \frac{2}{l}\left[\frac{-l^3}{n\pi}\cos n\pi + \frac{l^3}{n\pi}\cos n\pi - \frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4l^2}{n^3\pi^3}\left[-(-1)^n + 1\right]$$

$$D_n = \begin{cases} \dfrac{8l^2}{n^3\pi^3}, n \text{ is odd} \\ 0, n \text{ is even} \end{cases}$$

$$u(x,y) = \frac{8l^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l} \ldots$$

Q.3. (b) An insurance company insured 2000 scooter drivers, 4000 car and 6000 truck drivers.

The probability of an accident involving a scoter driver, a car driver, truck driver is 0.01, 0.03 and 0.15 respectively. One of the insured person with an accident. What is the probability that he is a scoter driver.

Ans. Let E_1, E_2, E_3 denote the events that an insured person at random is car and truck drivers. respectively.

Let H denote the event person met with an accident.

Then $P(E_1) = \frac{2000}{12000}, P(E_2) = \frac{4000}{12000} P(E_3) = \frac{6000}{12000}$

Probability of insured person met with an accident is scooter driver $P(H/E_1) =$

$$P(H/E_2) = 0.03$$

similarly

$$P(H/E_3) = 0.15$$

By Baye's theorem, we have

$$P(E_1/H) = \frac{P(E_1)P(H/E_1)}{P(E_1)P(H)(E_1) + P(E_2)P(H/E_2) + P(E_3)P(H/E_3)}$$

$$= \frac{\frac{1}{6}\times 0.01}{\frac{1}{6}\times 0.01 + \frac{1}{3}\times 0.03 + \frac{1}{2}\times 0.15}$$

$$= \frac{1}{6}\times\frac{6}{52} = \frac{1}{52} = 0.0192$$

Q.4. (a) Find the moment generating function of the distribution $f(x) = \frac{1}{C}e^{-x/c}, 0 \le x < \infty, c > 0$ about origin. Hence find its mean and standard deviation (5)

Ans. $$f(x) = \frac{1}{C}e^{-x/c} \quad 0 \le x < \infty, c > 0$$

M.G.F (about origin) $= E[e^{tx}]$

$$= \int_0^\infty e^{tx} \cdot f_X(x)\,dx$$

$$= \int_0^\infty e^{tx}\frac{1}{c}e^{-x/c}dx$$

$$= \frac{1}{C}\int_0^\infty e^{x(t-1/c)}dx$$

$$= \frac{1}{C}\left[\frac{e^{x(t-1/c)}}{t-1/c}\right]_0^\infty$$

$$M_X(t) = \frac{-1}{c\,\frac{(t\,c-1)}{C}} = \frac{-1}{t\,c-1} = -(tc-1)^{-1}$$

Now $$E[X] = \frac{d}{dt}\text{M.G.F.}$$

$$= (t\,c-1)^{-2}.C$$

Mean $$= E[X] = \frac{d}{dt}Mx(t)\Big]_{t=0}$$

$$= \left[(t\,c-1)^{-2}.c\right]_{t=0} = 0$$

$$E(X^2) = \frac{d^2}{dt^2}MX(t)\Big|_{t=0}$$

$$= -2c^2(t\,c-1)^{-3}\Big|_{t=0} = -2C^2$$

$$\text{var}\,X = E(X^2) - [E(X)]^2 = -2c^2 - c^2$$

$$= -3c^2$$

$$\text{S.D} = \sqrt{\text{var}\,X} = \sqrt{-3c^2} = c\sqrt{-3}$$

Q.4. (*b*) A manufacture of pins knowns that on an average 5% of his product is defative. He sells pins in boxes of 100 and gurantees that not more than 4 pins will be defective what is the probability that the box will fail to meet the guaranteed quality? ($e^{-5} = 0.0067$) (5)

Ans.Let X : no of defective pins

$$X = p(\lambda)$$

Let $p =$ the probability that a pin is defective

$$= 5\% = 0.05$$

also $n = 100$

$\Rightarrow$ $\lambda = np = 100 \times 0.05 = 5$

$$\text{p.}(x = r) = \frac{e^{-\lambda}\lambda^r}{r!}, r = 0,1,2,...$$

The box will meet guarantee if it contains atmost 4 defective pins.

$\therefore$ Required probability $= p(X \le 4)$

$$= p[X=0] + p[X=1] + p[X=2] + p[X=3] + p[X=4]$$

$$= e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda}\frac{\lambda^2}{2!} + e^{-\lambda}\frac{\lambda^3}{3!} + e^{-\lambda}\frac{\lambda^4}{4!}$$

$$= e^{-5}\left[1+5+\frac{25}{2}+\frac{125}{6}+\frac{625}{24}\right]$$

$$= e^{-5}(6 + 12.5 + 20.83 + 26.04)$$

$$= e^{-5} \times 65.37 = 0.0067 \times 67.37$$

$$= 0.44$$

Box fails to meet guarantee

$$1 - 0.44 = 0.5619.$$