

# Research Plan

## Area of Focus

- In their paper on Proximal Policy Optimization, Schulman et. al. [1] propose the clipped surrogate loss function for a fixed parameter  $\epsilon$ :

$$L^{CLIP}(\theta) = \hat{\mathbb{E}}_t \left[ \min \left( r_t(\theta) \hat{A}_t, \text{clip}(r_t(\theta), 1 - \epsilon, 1 + \epsilon) \hat{A}_t \right) \right]$$

where  $r_t(\theta) = \frac{\pi_\theta(a_t|s_t)}{\pi_{\theta_{old}}(a_t|s_t)}$  and  $\hat{A}_t$  is the generalized advantage estimator. For simplicity, let  $r_t(\theta) = r_t$ .  $\hat{A}_t$  can be replaced with a number of other “ $\gamma$ -just” estimators that must satisfy certain conditions [2]. Generalizing  $\hat{A}_t$  to these estimators, which will be denoted  $\hat{G}_t$ , yields:

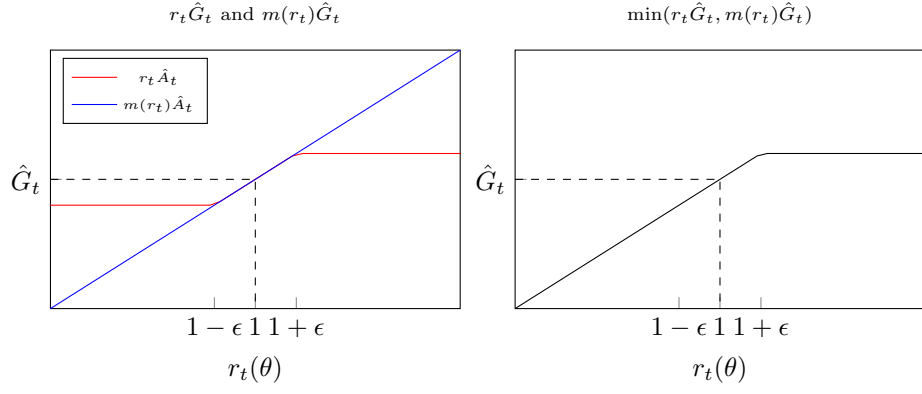
$$L^{CLIP}(\theta) = \hat{\mathbb{E}}_t \left[ \min \left( r_t \hat{G}_t, \text{clip}(r_t, 1 - \epsilon, 1 + \epsilon) \hat{G}_t \right) \right]$$

- The goal is to investigate replacements for the clipper function  $\text{clip}(r_t, 1 - \epsilon, 1 + \epsilon)$ . Let us refer to these replacements as “min-filters,” and let  $m(r_t)$  denote an arbitrary min-filter.
- In this experimental framework, we have the loss function:

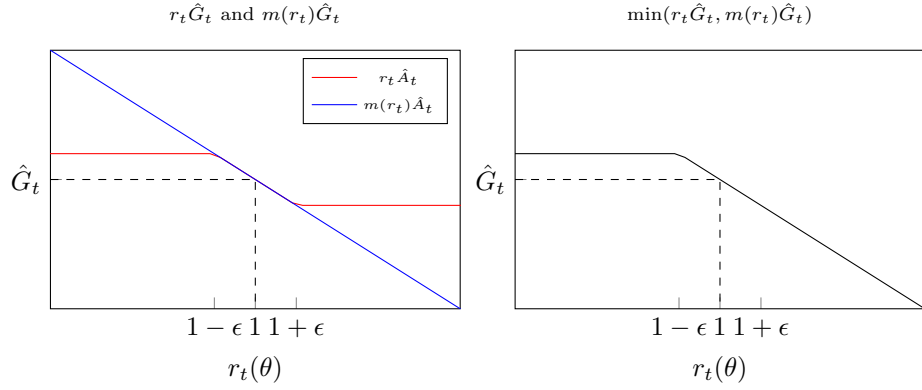
$$L^m(\theta) = \hat{\mathbb{E}}_t \left[ \min \left( r_t(\theta) \hat{G}_t, m(r_t) \hat{G}_t \right) \right]$$

- $L^{CLIP}$  is simply an instance of this where  $m(r_t) = \text{clip}(r_t, 1 - \epsilon, 1 + \epsilon)$ .

- Illustrating minimization under  $L_{CLIP}$  on individual expectation components:



Expectation component,  $\hat{G}_t > 0$



Expectation component,  $\hat{G}_t < 0$

- The paper on Trust Region Policy Optimization by Schulman et. al. [3] proposes a target function whose maximization guarantees monotonic improvement:

$$targ(\theta) = L_{\theta_{old}}(\theta) - CD_{KL}^{max}(\theta, \theta_{old})$$

where  $C$  is a fixed positive constant (see paper for specifics) and it is shown that

$$L_{\theta_{old}}(\theta) = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim p_{\theta_{old}}, a \sim \theta_{old}} \left[ \frac{\pi_{\theta}(a|s)}{\pi_{\theta_{old}}(a|s)} A_{\theta_{old}}(s, a) \right]$$

where  $p_{\theta_{old}}$  is the normalized discounted visitation frequency distribution.

- Assuming that the on-policy distribution matches the normalized dis-

counted visitation frequency distribution, we can write:

$$L_{\theta_{old}}(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{t \in (1, \dots, \infty)} \left[ r_t \hat{A}_t \right]$$

- By definition, any  $\gamma$ -just estimator can replace  $\hat{A}_t$  because doing so only adds a constant to  $targ(\theta)$ . Therefore, we can redefine  $L_{\theta_{old}}(\theta)$  as:

$$L_{g, \theta_{old}}(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{t \in (1, \dots, \infty)} \left[ r_t \hat{G}_t \right]$$

- Plugging into the target function, multiplying by  $1-\gamma$ , and absorbing  $1-\gamma$  into  $C$  leaves us with the gradient-equivalent target function:

$$\begin{aligned} targ_g(\theta) &= \mathbb{E}_t \left[ r_t \hat{G}_t \right] - CD_{KL}^{max}(\theta, \theta_{old}) \\ \nabla_{\theta} targ_g(\theta) &= \nabla_{\theta} targ(\theta) \end{aligned}$$

- Consider the case where  $\forall t \in (1, \dots, \infty)$ ,  $\hat{G}_t > 0$  and  $r_t < 1 + \epsilon$  and let  $\theta \neq \theta_{old}$ . In this case, no penalty is applied and the clipped loss is a strict overestimate without the same gradient:

$$\begin{aligned} L^{CLIP}(\theta) &= \hat{\mathbb{E}}_t \left[ \min \left( r_t \hat{G}_t, \text{clip}(r_t, 1 - \epsilon, 1 + \epsilon) \hat{G}_t \right) \right] \\ &= \hat{\mathbb{E}}_t \left[ r_t \hat{G}_t \right] \\ &\geq \mathbb{E}_t \left[ r_t \hat{G}_t \right] - CD_{KL}^{max}(\theta, \theta_{old}) \\ &= targ_g(\theta) \\ \nabla_{\theta} L^{CLIP}(\theta) &= \nabla_{\theta} \hat{\mathbb{E}}_t \left[ r_t \hat{G}_t \right] \\ &\neq \nabla_{\theta} \mathbb{E}_t \left[ r_t \hat{G}_t \right] - C \nabla_{\theta} D_{KL}^{max}(\theta, \theta_{old}) \\ &= \nabla_{\theta} targ_g(\theta) \end{aligned}$$

- Removing the assumption that  $\hat{G}_t > 0$ , the above still holds only if, for all positive  $\hat{G}_t$ ,  $r_t < 1 + \epsilon$ , and for all negative  $\hat{G}_t$ ,  $r_t > 1 - \epsilon$ .
- If  $r_t$  is independent of the sign of  $\hat{G}_t$ , this is generally a harder condition to meet. Experimentally, I found that, on almost every batch, the number of timesteps  $t$  where  $r_t < 1 + \epsilon$  was greater than the number of timesteps where  $(\hat{A}_t < 0 \text{ and } r_t > 1 - \epsilon)$  or  $(\hat{A}_t > 0 \text{ and } r_t < 1 + \epsilon)$ . This means that, if  $\hat{G}_t$  can be both positive and negative, penalties become more possible, allowing  $L^{CLIP}(\theta)$  to better approximate  $targ_g(\theta)$ , better guaranteeing monotonic improvement.
- This reasoning could explain the preference for advantage estimators over value estimators, because the condition that  $\mathbb{E}_t(\hat{A}_t) = 0$  requires that advantage estimators be negative half the time, while value functions are typically always positive or always negative.
- Research question: In some cases, it is simpler to implement a value estimator than an advantage estimator. Can we design a min-filter that specifically addresses the above concerns to make it more feasible to use a value estimator in Proximal Policy Optimization?

## $L^{CLIP}$ Penalty Differences for Different Estimate Models

- Consider the set of expectation-component parameters  $((r_1, \hat{G}_1), (r_2, \hat{G}_2), \dots (r_T, \hat{G}_T))$ , where all  $r$  and  $\hat{G}$  are uncorrelated.
- Under  $L^{CLIP}$ , a particular timestep  $t$  will be penalized in either of two cases:
  - $\hat{G}_t$  is positive and  $r_t > 1 + \epsilon$ .
  - $\hat{G}_t$  is negative and  $r_t < 1 - \epsilon$ .
- Therefore, by the assumption of independence of  $\hat{G}_t$  and  $r_t$ , we have the expected number of penalized timesteps:  
 $(p(\hat{G}_t > 0)p(r_t > 1 + \epsilon) + p(\hat{G}_t < 0)p(r_t < 1 - \epsilon))T$ .
- Let  $\hat{G}_{1,t}$  and  $\hat{G}_{2,t}$  be two alternate estimators. To understand differences in the expected number of penalized timesteps as we modify the sign of  $\hat{G}$ , define the ratio:

$$\begin{aligned} r_{diff} &= \frac{(p(\hat{G}_{1,t} > 0)p(r_t > 1 + \epsilon) + p(\hat{G}_{1,t} < 0)p(r_t < 1 - \epsilon))T}{(p(\hat{G}_{2,t} > 0)p(r_t > 1 + \epsilon) + p(\hat{G}_{2,t} < 0)p(r_t < 1 - \epsilon))T} \\ &= \frac{p(\hat{G}_{1,t} > 0)p(r_t > 1 + \epsilon) + p(\hat{G}_{1,t} < 0)p(r_t < 1 - \epsilon)}{p(\hat{G}_{2,t} > 0)p(r_t > 1 + \epsilon) + p(\hat{G}_{2,t} < 0)p(r_t < 1 - \epsilon)} \end{aligned}$$

- If  $p(r_t > 1 + \epsilon) = p(r_t < 1 - \epsilon)$ , this ratio degenerates to 1 regardless of the sign distributions of  $\hat{G}_1$  and  $\hat{G}_2$ .
- Consider the example where  $p(\hat{G}_{1,t} > 0) = p(\hat{G}_{1,t} < 0) = 0.5$  and  $p(\hat{G}_{2,t} > 0) = 1, p(\hat{G}_{2,t} < 0) = 0$ . Finding the conditions under which  $r_{diff} > 1$ :

$$\begin{aligned} r_{diff} &> 1 \\ \frac{0.5(p(r_t > 1 + \epsilon) + p(r_t < 1 - \epsilon))}{p(r_t > 1 + \epsilon)} &> 1 \\ \frac{p(r_t > 1 + \epsilon) + p(r_t < 1 - \epsilon)}{p(r_t > 1 + \epsilon)} &> 2 \\ p(r_t > 1 + \epsilon) + p(r_t < 1 - \epsilon) &> 2p(r_t > 1 + \epsilon) \\ p(r_t < 1 - \epsilon) &> p(r_t > 1 + \epsilon) \end{aligned}$$

- Consider a continuous action space and gaussian policies with trainable but state-independent standard deviations.
- In general, on a particular state  $s$ , the standard deviation encoded by  $\theta$  will decrease as the agent becomes more certain of its actions, and the mean will get further from the mean encoded by  $\theta_{old}$ . Visualizing the overlaid gaussians, both of these actions will make it more likely that the above condition is true.
- Therefore, as training progresses in a single iteration, we expect that the  $\hat{G}_1$  estimate will begin to induce penalties on more timesteps than the  $\hat{G}_2$  estimate.

- Following similar logic as above, we have that if  $p(\hat{G}_{2,t} > 0) = 0$ ,  $p(\hat{G}_{2,t} < 0) = 1$ , the condition  $r_{diff} > 1$  requires that  $p(r_t < 1 - \epsilon) < p(r_t > 1 + \epsilon)$ . However, because we have just reasoned that the opposite relation tends to be true as an iteration progresses, it must be the case that  $r_{diff} < 1$  - that is, using such an estimator results in more penalized timesteps.
- Testing this theory empirically on the InvertedPendulum-v2 environment with a standard PPO agent and an advantage estimate  $\hat{G}_t$ , I observed that, in a single iteration, the number of  $(\hat{G}_t, r_t)$  where  $(\hat{G}_t > 0 \text{ and } r_t > 1 + \epsilon)$  or  $(\hat{G}_t < 0 \text{ and } r_t < 1 - \epsilon)$  became consistently greater than the number of  $r_t$  where  $r_t > 1 + \epsilon$ , and consistently less than the number of  $r_t$  where  $r_t < 1 - \epsilon$ . This is in agreement with the above theoretical results.

## Expected Loss Contributions

- Assume a gaussian action space with fixed standard deviations and clipping min-filter.
- It can be shown that the point at which  $r_t = 1 + \epsilon$  is:

$$x^+ = \frac{(\mu^2 - \mu_{old}^2) + 2\sigma^2 \ln(1 + \epsilon)}{2(\mu - \mu_{old})}$$

- Similarly, it can be shown that the point at which  $r_t = 1 - \epsilon$  is:

$$x^- = \frac{(\mu^2 - \mu_{old}^2) + 2\sigma^2 \ln(1 - \epsilon)}{2(\mu - \mu_{old})}$$

- Let  $p(\mu, x)$  be the probability of  $x$  given a gaussian distribution with fixed standard deviation  $\sigma$  and mean  $\mu$ .
- Solving for the expected ratio coefficients for positive estimators:

$$\begin{aligned} E[r_{t,CLIP}^+] &= \int_{-\infty}^{x^+} p(\mu_{old}, x) r_t(x) dx + \int_{x^+}^{\infty} p(\mu_{old}, x) (1 + \epsilon) dx \\ &= \int_{-\infty}^{x^+} p(\mu_{old}, x) \frac{p(\mu, x)}{p(\mu_{old}, x)} dx + (1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\ &= \int_{-\infty}^{x^+} p(\mu, x) dx + (1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \end{aligned}$$

- Finding the expected penalty contribution:

$$\begin{aligned} 1 - E[r_{t,CLIP}^+] &= 1 - \int_{-\infty}^{x^+} p(\mu, x) dx - (1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\ &= \int_{x^+}^{\infty} p(\mu, x) dx - (1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\ &= \int_{x^+}^{\infty} p(\mu, x) - (1 + \epsilon) p(\mu_{old}, x) dx \end{aligned}$$

- Similarly, solving for the expected ratio coefficients for negative estimators:

$$\begin{aligned} E[r_{t,CLIP}^-] &= \int_{-\infty}^{x^-} p(\mu_{old}, x) (1 - \epsilon) dx + \int_{x^-}^{\infty} p(\mu_{old}, x) r_t(x) dx \\ &= (1 - \epsilon) \int_{-\infty}^{x^-} p(\mu_{old}, x) dx + \int_{x^-}^{\infty} p(\mu, x) dx \end{aligned}$$

– Finding the expected penalty contribution:

$$\begin{aligned}
E[r_{t,CLIP}^-] - 1 &= \int_{-\infty}^{x^-} p(\mu_{old}, x)(1 - \epsilon)dx + \int_{x^-}^{\infty} p(\mu, x)dx - 1 \\
&= - \left( 1 - \int_{-\infty}^{x^-} p(\mu_{old}, x)(1 - \epsilon)dx - \int_{x^-}^{\infty} p(\mu, x)dx \right) \\
&= - \left( - \int_{-\infty}^{x^-} p(\mu_{old}, x)(1 - \epsilon)dx + \int_{-\infty}^{x^-} p(\mu, x)dx \right) \\
&= \int_{-\infty}^{x^-} p(\mu_{old}, x)(1 - \epsilon)dx - \int_{-\infty}^{x^-} p(\mu, x)dx \\
&= \int_{-\infty}^{x^-} p(\mu_{old}, x)(1 - \epsilon) - p(\mu, x)dx
\end{aligned}$$

– Alternatively, this can be written as:

$$\begin{aligned}
E[r_{t,CLIP}^-] - 1 &= \int_{-\infty}^{x^-} p(\mu_{old}, x)(1 - \epsilon)dx - \int_{-\infty}^{x^-} p(\mu, x)dx \\
&= (1 - \epsilon) \left( 1 - \left( \int_{x^-}^{x^+} p(\mu_{old}, x)dx + \int_{x^+}^{\infty} p(\mu_{old}, x)dx \right) \right) - \\
&\quad \left( 1 - \left( \int_{x^-}^{x^+} p(\mu, x)dx + \int_{x^+}^{\infty} p(\mu, x)dx \right) \right) \\
&= (1 - \epsilon) \left( 1 - \int_{x^-}^{x^+} p(\mu_{old}, x)dx - \int_{x^+}^{\infty} p(\mu_{old}, x)dx \right) - \\
&\quad \left( 1 - \int_{x^-}^{x^+} p(\mu, x)dx - \int_{x^+}^{\infty} p(\mu, x)dx \right) \\
&= 1 - \epsilon - (1 - \epsilon) \left( \int_{x^-}^{x^+} p(\mu_{old}, x)dx + \int_{x^+}^{\infty} p(\mu_{old}, x)dx \right) - 1 \\
&\quad + \int_{x^-}^{x^+} p(\mu, x)dx + \int_{x^+}^{\infty} p(\mu, x)dx \\
&= -\epsilon - (1 - \epsilon) \left( \int_{x^-}^{x^+} p(\mu_{old}, x)dx + \int_{x^+}^{\infty} p(\mu_{old}, x)dx \right) \\
&\quad + \int_{x^-}^{x^+} p(\mu, x)dx + \int_{x^+}^{\infty} p(\mu, x)dx
\end{aligned}$$

– Finding the differences between these two values

$$\begin{aligned}
(1 - E[r_{t,CLIP}^+]) - (E[r_{t,CLIP}^-] - 1) &= \int_{x^+}^{\infty} p(\mu, x) dx - (1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\
&\quad + \epsilon + (1 - \epsilon) \left( \int_{x^-}^{x^+} p(\mu_{old}, x) dx + \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right) \\
&\quad - \int_{x^-}^{x^+} p(\mu, x) dx - \int_{x^+}^{\infty} p(\mu, x) dx \\
&= -(1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\
&\quad + \epsilon + (1 - \epsilon) \left( \int_{x^-}^{x^+} p(\mu_{old}, x) dx + \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right) \\
&\quad - \int_{x^-}^{x^+} p(\mu, x) dx \\
&= -\epsilon \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\
&\quad + \epsilon + \left( (1 - \epsilon) \int_{x^-}^{x^+} p(\mu_{old}, x) dx - \epsilon \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right) \\
&\quad - \int_{x^-}^{x^+} p(\mu, x) dx \\
&= -2\epsilon \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\
&\quad + \epsilon + (1 - \epsilon) \int_{x^-}^{x^+} p(\mu_{old}, x) dx \\
&\quad - \int_{x^-}^{x^+} p(\mu, x) dx \\
&= \epsilon + (1 - \epsilon) \int_{x^-}^{x^+} p(\mu_{old}, x) dx \\
&\quad - \left( \int_{x^-}^{x^+} p(\mu, x) dx + 2\epsilon \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right)
\end{aligned}$$



– Finding the addition of these two values

$$\begin{aligned}
(1 - E[r_{t,CLIP}^+]) + (E[r_{t,CLIP}^-] - 1) &= \int_{x^+}^{\infty} p(\mu, x) dx - (1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\
&\quad - \epsilon - (1 - \epsilon) \left( \int_{x^-}^{x^+} p(\mu_{old}, x) dx + \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right) \\
&\quad + \int_{x^-}^{x^+} p(\mu, x) dx + \int_{x^+}^{\infty} p(\mu, x) dx \\
&= 2 \int_{x^+}^{\infty} p(\mu, x) dx - (1 + \epsilon) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\
&\quad - \epsilon - (1 - \epsilon) \left( \int_{x^-}^{x^+} p(\mu_{old}, x) dx + \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right) \\
&\quad + \int_{x^-}^{x^+} p(\mu, x) dx \\
&= 2 \left( \int_{x^+}^{\infty} p(\mu, x) dx - \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right) \\
&\quad - \epsilon - (1 - \epsilon) \int_{x^-}^{x^+} p(\mu_{old}, x) dx \\
&\quad + \int_{x^-}^{x^+} p(\mu, x) dx
\end{aligned}$$

– Generalizing the results to two  $\epsilon$ :

$$\begin{aligned}
(1 - E[r_{t,CLIP}^+]) - (E[r_{t,CLIP}^-] - 1) &= \epsilon^- + (1 - \epsilon^-) \int_{x^-}^{x^+} p(\mu_{old}, x) dx \\
&\quad - \left( \int_{x^-}^{x^+} p(\mu, x) dx + (\epsilon^+ + \epsilon^-) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \right) \\
(1 - E[r_{t,CLIP}^+]) + (E[r_{t,CLIP}^-] - 1) &= 2 \int_{x^+}^{\infty} p(\mu, x) dx - (2 + \epsilon^+ - \epsilon^-) \int_{x^+}^{\infty} p(\mu_{old}, x) dx \\
&\quad - \epsilon^- - (1 - \epsilon^-) \int_{x^-}^{x^+} p(\mu_{old}, x) dx + \int_{x^-}^{x^+} p(\mu, x) dx
\end{aligned}$$

## References

- [1] <https://arxiv.org/abs/1707.06347>
- [2] <https://arxiv.org/abs/1506.02438>
- [3] <https://arxiv.org/abs/1502.05477>