Econ/Math C103

Individual Choice Theory I: Choice, Revealed Preference & Ordinal Utility

Haluk Ergin

1 Binary Relations

Binary relations turn out to be important mathematical tools in economic theory. Most importantly, preferences of individuals are typically modelled as binary relations as we will discuss in more detail in the next section. Let X be a set. $X \times X$ denotes the Cartesian product, set of all ordered pairs (x, y) where $x, y \in X$. A **binary relation** B on X is a subset $B \subset X \times X$. We usually write xBy instead of $(x, y) \in B$ and read it as x *stands in relation* B to y. (We will also sometimes use R, P, I, $\succsim$, $\succ$, $\sim$,... to denote binary relations.)

Examples of binary relations:

- $X = \{c(hicken), f(ish), s(teak)\}$,

  $$X \times X = \{(c, c), (c, f), (c, s), (f, c), (f, f), (f, s), (s, c), (s, f), (s, s)\}$$

  and for example $B = \{(c, c), (c, s), (s, f)\}$; or for example if B' is "is a lighter lunch than", $B' = \{(f, c), (c, s), (f, s)\}$.

- X is the set of people in the world, B is "is a brother of", "is taller than", "is at least as tall as," "is just as tall as," "is the aunt of," "is married to," "weighs at least twice as," "is a relative of"...

- $X = [0, 1]$, B is $\geq$, $>$, $=$, ...

- $X = [0, 1]$, B is defined by: $xBy \Leftrightarrow x \geq y - 1/4$ for any $x, y \in X$.

- $X = [0, 1]$, B is defined by: $xBy \Leftrightarrow x \geq y + 1/4$ for any $x, y \in X$.

- $X = [0, 1]$, B is defined by: $xBy \Leftrightarrow |x - y| \leq 1/3$ for any $x, y \in X$.

Some properties of binary relations:

Definition 1 Let B be a binary relation on X. Then

- B is **reflexive** if xBx for any $x \in X$. (every element stands in B-relation with itself)
- B is **complete** if xBy or yBx for any $x, y \in X$. (all elements are B-comparable)
- B is **transitive** if xBy and yBz imply xBz for any $x, y, z \in X$. (a consistency condition on B)
- B is **symmetric** if xBy implies yBx for any $x, y \in X$.
- B is **antisymmetric** if xBy and yBx imply that $x = y$, for any $x, y \in X$.

Note that completeness $\Rightarrow$ reflexivity.

Some important types of binary relations

Definition 2 Let B be a binary relation on X. Then

- B is **weak-order** if it is complete and transitive.
- B is a **linear/strict order** if it is complete, transitive, and antisymmetric.

Note that every linear order is a weak order but not vice versa.

2 Preferences

In economics, when X is an arbitrary set of alternatives/outcomes that could be faced by an individual, we model the individual's preference over X as a binary relation. For example for a senior facing a career choice, the set of alternatives X could be {apply to a PhD program in economics, be an investment banker, apply for an MBA, be a consultant,...}. If we interpret the binary relation R as a preference relation, then we read xRy as "x is weakly preferred to y," or "x is at least as preferable as y".

Definition 3 A preference R over alternatives is **rational** if it is complete and transitive (i.e. if it is a weak order).

Given a preference R, define two new binary relations: strict preference P and indifference I on X, by for any $x, y \in X$:

$$xPy \Leftrightarrow xRy \ \& \text{ not } yRx$$

$$xIy \Leftrightarrow xRy \ \& \ yRx$$

We read xPy as "x is strictly preferred to y" or as "x is strictly better than y". We read xIy as "x is indifferent to y".

3 Utility Functions and Ordinal Representations

An often convenient way of representing a binary relation is by using a real valued function that we will call a **utility function**. Let R be a preference on X. Let $u : X \to \mathbb{R}$ be real valued function (the utility function). We say that u **represents** R if:

$$\forall x, y \in X : \qquad xRy \Leftrightarrow u(x) \geq u(y).$$

Proposition 1 *Let X be an arbitrary set of alternatives and let R be a preference relation on X.If there is a utility function $u : X \to \mathbb{R}$ that represents R, then R is rational.*

Proof: Covered in class. □

The converse is also always true when X is finite. That is:

Proposition 2 *Let X be a finite set of alternatives and let R be a rational preference relation on X. Then there exists a utility function $u : X \to \mathbb{R}$ that represents R.*

Proof: For any $x \in X$, let $L(x) := \{z \in X : xRz\}$ and set $u(x) = |L(x)|$. Note that by transitivity of of R, $xRy \Rightarrow L(x) \supset L(y)$, for any $x, y \in X$. We will show that u defined above represents R.

For the "$\Rightarrow$" direction, let xRy. Then $u(x) \geq u(y)$ follows from $L(x) \supset L(y)$.

For the "$\Leftarrow$" direction, let $u(x) \geq u(y)$. Suppose for a contradiction that xRy does not hold. Then completeness of R implies that yRx, hence $L(x) \subset L(y)$. By reflexivity $y \in L(y)$ and since xRy does not hold $y \notin L(x)$. Hence $L(x) \subsetneq L(y)$, implying that $u(x) < u(y)$, a contradiction to our initial assumption. Therefore xRy must hold. □

When X is infinite for example when $X = [0, 1]$, the above proof no longer works. (Can you see why?) In that case in order to obtain a utility representation, one needs to impose additional technical conditions that we will not get into.

The two results above have the following immediate corollary:

Corollary 1 *Let X be a finite set of alternatives and let R be a preference relation on X. Then R can be represented by a utility function if and only if it is rational.*

Can a preference be represented by more than one utility function? The answer is yes, the utility function contains much more information than the preference. In particular if a utility function u represents a preference relation R, then any monotone increasing transformation of u also represents R, as we make precise in the following Proposition:

Proposition 3 *Let X be an arbitrary set and let R be a preference relation on X represented by the utility function u. If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then the utility function v defined by:*

$$v(x) = f\left(u(x)\right) \qquad \textit{for any } x \in X$$

also represents R.

Proof: Covered in class. $\square$

The cardinality of the utility function has no meaning in this setup. Nor can any notion of strength or intensity of preference be captured in this framework. The utility function represents a preference relation R as long as they are "ordinally equivalent". We will be able to give some meaning to cardinal utility when we discuss choice under uncertainty.

4 Choice functions and correspondences

From now on suppose that the set of potential alternatives X is finite. Denote the set of nonempty subsets of X by $\mathcal{P}(X) = \{A \subset X | A \neq \emptyset\}$. We can perceive every such subset of alternatives $A, B, C \in \mathcal{P}(X)$ as a potential decision-problem or choice set.

For example when X is a finite set of potential meals, $A = \{steak, fish, salad\}$ corresponds to a decision problem where the decision-maker needs to choose a lunch out of the three meals in the menu A. Alternatively when X is a finite set of potential cars, $B = \{$*Toyota-Corolla*, *Ford-Taurus*, *Mitsubishi-Lancer*, *Honda-Civic*$\}$ corresponds to a decision problem where the decision-maker needs to buy one out of these four possible cars.

A choice function/correspondence tells us what the decision maker chooses when she is faced with any such decision problem A. Namely, she chooses the alternatives in $c(A)$ when she is faced with A. If our decision-maker always chooses *exactly one* element out of every choice set A, then we can represent her choices by a choice function:

Definition 4 A **choice function** is function $c : \mathcal{P}(X) \to X$ such that $c(A) \in A$ for any $A \in \mathcal{P}(X)$.

If our decision-maker may sometimes be indifferent among a subset of choices, we may also want to allow for the possibility where she/he chooses more than one element from every choice set A (to be interpreted as she is just as willing to choose any element in that subset), in that case we can represent her/his choices by:

Definition 5 A **choice correspondence** is function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $c(A) \subset A$ for any $A \in \mathcal{P}(X)$.

Note that a choice function is a single-valued choice correspondence.

Examples of choice procedures:

- (The first-best procedure) Let X be a set of potential candidates for a job. For each candidate x let $e(x)$ denote the years of experience that candidate x has in similar jobs. For any set of applicants $A \in \mathcal{P}(X)$, suppose that the employer chooses the most experienced applicants in A.

  More generally let X be an arbitrary finite set of alternatives. Let $u : X \to \mathbb{R}$ be a function over the alternatives and let $c(A)$ be the set of maximizers of u in A. That is $c(A) = \{x \in A \,|\, \forall z \in A : u(x) \geq u(z)\}$. Check that c is a choice correspondence (chooses from A, non-empty valued). Note that if u is one-to-one then c is a choice function.

- (The second-best procedure) Let X be an arbitrary finite set of alternatives. Let $u : X \to \mathbb{R}$ be a one-to-one function over the alternatives. For any set A with at least two elements, let $c(A)$ be the second from top alternative in A w.r.t. u. This is a choice function.

- (Examples of procedures with two criteria) Let X be again a set of potential candidates for a job. For each candidate x let $e(x)$ denote the years of experience that candidate x has in similar jobs, and let $g(x)$ be the undergraduate GPA of the candidate. Assume that no two applicants have the same GPA nor the same experience. Let e^* denote a critical level of experience for the employer. For any set of applicants $A \in \mathcal{P}(X)$:

  - The employer first considers the applicant x_{gpa} with the highest GPA in A. If x_{gpa} has an experience level of at least e^*, then he gives the job to x_{gpa}. If x_{gpa}'s experience is less than e^*, then he gives the job to the most experienced candidate x_e in A.
  - The employer first considers the most experienced applicant x_e in A. If x_e has an experience level at least e^*, then he gives the job to x_e. Otherwise he gives the job to the the applicant x_{gpa} with the highest GPA in A.

  By our simplifying tie-breaking assumption that no two applicants have the same GPA nor the same experience, each of the above procedures lead to well-defined choice function. These are also known as (u, v) procedures (Kalai, Rubinstein, and Spiegler, 2002).

- (Satisfycing, Herbert Simon) Suppose that the set of alternatives is ordered $x_1, \dots, x_n$ where $n = |X|$. Let $s\colon X \to \mathbb{R}$ be a function that denotes the agent's "satisfaction level" from the alternative. Assume that there is a cutoff value $s^* \in \mathbb{R}$ such that the decision-maker finds an alternative x_i satisfactory if and only if $s(x_i) \geq s^*$.

  Faced with a decision problem A, the decision-maker uses the following procedure to make a choice out of A. He considers the alternatives in A in order, starting from the alternative with the lowest index, then the one with the second lowest index, and so on. The first time he encounters an alternative x_i with $s(x_i) \geq s^*$, he stops and chooses x_i. If he runs through all alternatives in A and does not find a satisfactory one, then he just chooses the last alternative he considered.

  A decision-maker who "satisfices" is looking for a satisfactory alternative (that is, an alternative $x_i \in A$ such that $s(x_i) \geq s^*$) instead of an optimal one (that would be an alternative $x_i \in A$ such that $s(x_i) \geq s(x_j)$ for any $x_j \in A$). This is the main contrast between satisfycing and optimizing.

5 Rationalizing choices: Revealed Preference

Given a preference relation R on X, define the choice correspondence c^R induced by R through:

$$c^R(A) = \{x \in A : xRz \text{ for any } z \in A\} \qquad A \in \mathcal{P}(X)$$

Note that the above definition implies that $c^R(A) \subset A$ for any decision problem A. In order to guarantee that c^R is a well-defined choice correspondence, we should also make sure that $c^R(A) \neq \emptyset$ for any such A. This is guaranteed if the preference relation is rational:

Proposition 4 *Let R be a preference relation over a finite set of alternatives X. If R is rational, then c^R is a well-defined choice correspondence. Moreover if R is a linear order, then c^R is a choice function (i.e. it is single-valued).*

Proof: Let R be a rational preference relation over a finite set of alternatives X. To prove that c^R is a well-defined choice correspondence, we need to verify that (i) $c^R(A) \subset A$ and (ii) $c^R(A) \neq \emptyset$ for any decision problem $A \in \mathcal{P}(A)$.

Definition of $c^R(A)$ above immediately implies that $c^R(A) \subset A$, so part (i) is verified. To see (ii), note that by Proposition 2, there exists a utility function $u : X \to \mathbb{R}$ that represents R since X is finite and R is rational. Then we can rewrite $c^R(A)$ as $c^R(A) = \{x \in A : u(x) \geq u(z) \text{ for any } z \in A\}$, so $c^R(A)$ corresponds to the set of maximizers of u in A. Since there always is at least one maximizer of a real-valued function over a finite set we conclude that $c^R(A) \neq \emptyset$.

If there is a decision problem $A \in \mathcal{P}(X)$ such that there are two elements $x, y \in c^R(A)$ with $x \neq y$, then it must be that xRy and yRx, so in this case R can not be antisymmetric. Therefore if R is a linear order (remember that a linear order is an antisymmetric rational preference), then c^R must be single-valued. □

The natural converse to the above proposition is the following "When can we rationalize a choice correspondence by a preference, i.e. what condition(s) should a choice correspondence c satisfy so that there exists a rational preference relation R with $c = c^R$?"

Consider the following properties:

Definition 6 A choice correspondence c satisfies **Sen's condition** α if

$$x \in B \subset A \ \& \ x \in c(A) \Rightarrow x \in c(B).$$

Sen's explanation of condition α: If the world champion in some game is Pakistani, then he is also the champion of Pakistan.

Definition 7 A choice correspondence c satisfies **Sen's condition** β if

$$B \subset A,\ x, y \in c(B) \ \& \ y \in c(A) \Rightarrow x \in c(A).$$

Sen's explanation of condition β: If a champion of Pakistan is also a world champion, then all champions of Pakistan are world champions.

Theorem 1 *("Fundamental Theorem of Revealed Preference") Assume that the set of alternatives X is finite. A choice correspondence c satisfies Sen's condition α and β if and only if there exists a unique rational preference relation R over X such that $c = c^R$.*

Proof: Covered in class. □

6 Optional Additional Reading

Lectures 1–3 of **Ru**, Chapter 1 of **MWG**:

Ru Rubinstein, *Lecture Notes in Microeconomic Theory*, Princeton University Press, 2005. (Available for free download at: http://arielrubinstein.tau.ac.il/)

MWG Mas-Colell, Whinston, and Green, *Microeconomic Theory*, Oxford University Press, 1995.

7 List of Symbols

$\emptyset$	the empty set
$\in$	is an element of
$\notin$	is not an element of
$A \times B$	$\{(x, y) \vert x \in A, y \in B\}$, ordered pairs of elements from sets A and B
$\subset$	is a subset of.
$\supset$	is a superset of, includes.
$\subsetneq$	is a subset of but it is not equal to, is a proper subset of
$\vert A\vert$	number of elements of a set A
$\Rightarrow$	implies, only if
$\Leftarrow$	is implied by, if
$\Leftrightarrow$	implies and is implied by, if and only if
$=$	is equal to
$:=$	is defined as, ($x := 3$ means x is defined to be 3)
$\neq$	is not equal to
$\mathcal{P}(X)$	$\{A \subset X \vert A \neq \emptyset\}$, the set of all non-empty subsets of the set X
$A \setminus B$	$\{x \in A \vert x \notin B\}$, set subtraction
$f : A \to B$	notation for a function from a set A into a set B. A function f associates an element $f(x) \in B$ to each element $x \in A$.
$\forall$	for all
$\exists$	there exists

TEX commands:

$\succsim$	\succsim
$\succ$	\succ
$\sim$	\sim