Econ/Math C103

Non-Cooperative Game Theory I:

Normal Form Games

Haluk Ergin

1. Overview
2. Knowledge
3. Normal form games
4. Dominant strategy equilibrium
5. Iterated elimination of strictly dominated strategies (IESDS)
6. Nash Equilibrium
7. Nash Equilibrium Applications
    1. Cournot quantity competition
    2. Commons problem

Non-cooperative Game Theory

How may a group of self-interested individuals behave if each of them is affected by the others' actions?

We need to specify

- Who are the players?
- What are the actions that each player can take?
- What is each player’s payoff resulting from everybody’s actions?

Prisoners' Dilemma

Player 1 \ Player 2	Cooperate	Defect
Cooperate	3,3	0,5
Defect	5,0	1,1

A Three Player Game

Player 1 chooses the row: T, B; Player 2 chooses the column: L, M, R; & Player 3 chooses the matrix: **M1, M2**. Payoffs:

M1

	L	M	R
T	3,0,2	2,1,1	1,0,3
B	0,0,3	1,1,1	0,3,0

M2

	L	M	R
T	3,0,1	1,1,-1	0,2,3
B	0,1,3	1,1,2	3,0,0

Levels of interactive knowledge

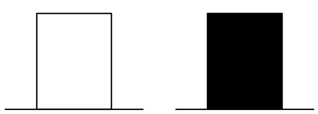
E: an event or statement (Examples: E=Jim is wearing a white hat; E=it rains outside; E=John is rational;…)

Levels of interactive knowledge:

1. Each player knows E
2. Each player knows that each player knows E
3. Each player knows that each player knows that each player knows E

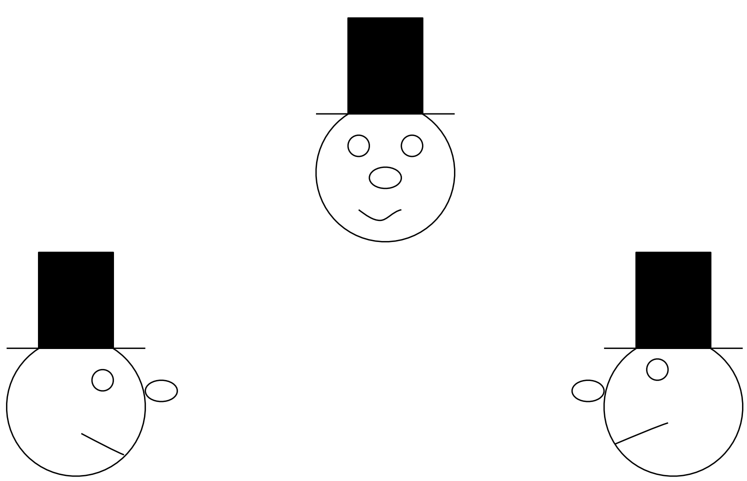
   ….

E is **common knowledge** among players if these hold ad infinitum.



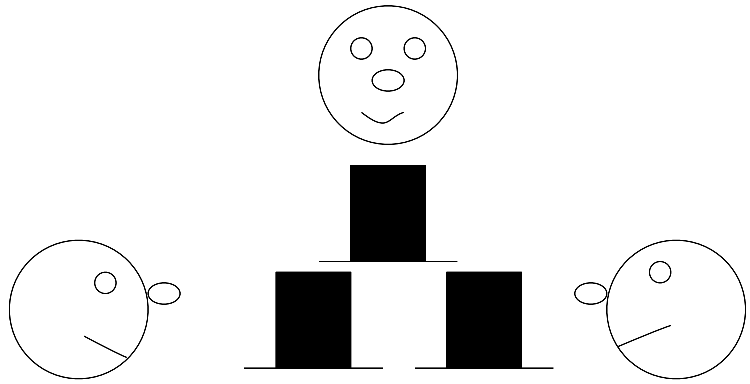
An example

E=There is at least one black hat



Note: 1. Each person knows E, 2. Each person knows that each person knows E. However E is not common knowledge.

Common knowledge



E is common knowledge

Normal form games

A **normal form game** consists of:

- a set of players $N=\{1,2,\ldots,n\}$
- a set of actions/strategies S_i for each i in N
- a vNM utility function:

  $u_i: S_1 \times S_2 \times \ldots \times S_n \rightarrow \mathbb{R}$ for each i in N

S_i is also called the **pure strategies** of player i

Assumptions: -The game is common knowledge among the players.

-Players choose their actions simultaneously.

Frequently used notations

- A strategy of player i typically denoted by s_i in S_i
- Strategy profiles of all players:

$$S=S_1\times S_2\times\ldots\times S_n$$

with a typical member $s=(s_1,s_2,\ldots,s_n)$ in S

- The strategy profile of all players except player i:

$$S_{-i}=S_1\times\ldots\times S_{i-1}\times S_{i+1}\times\ldots\times S_n$$

with a typical member $s_{-i}=(s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$ in S_{-i}

How to play?

Player i is ***rational*** if she maximizes the expected value of u_i, given her knowledge and beliefs about how the others will play.

Prisoners' Dilemma

Player 2 / Player 1	Cooperate	Defect
Cooperate	3,3	0,5
Defect	5,0	1,1

Dominant-strategy equilibrium

- A pure strategy s_i^* **weakly dominates** s_i if

  $$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for any } s_{-i} \text{ in } S_{-i}$$

  and the inequality is strict for some s_{-i}.
- A strategy s_i^* is a **dominant strategy** if s_i^* weakly dominates every other strategy s_i.
- A strategy profile s^* is a **dominant-strategy equilibrium** if s_i^* is a dominant strategy for each player i.

Second-price auction

- An object is auctioned to two bidders 1 and 2.
- The value of the object to bidder i is v_i. Utility of player i from not buying the object is 0, and from buying the object at price p is $v_i - p$.
- Each bidder i bids b_i without seeing the other's bid.
- The highest bidder gets the object and pays the other's bid (=second highest bid).
- If bids are the same, then one of the bidders is chosen with equal probability, he gets the object and pays $p=b_1=b_2$.

Proposition: It is a dominant strategy equilibrium for each player to bid her valuation.

Proof: Covered in class.

A 2x3 Game

Player 1 \ Player 2	Left	Middle	Right
Top	2,0	0,1	1,2
Bottom	0,0	1,2	2,1

Randomization & mixed strategies

A **mixed strategy** σ_i of player i is a probability distribution over S_i.

Let $\sigma=(\sigma_1,\sigma_2,\ldots, \sigma_n)$, when each player j in N plays the mixed strategy σ_j the expected utility of player i is:

$U_i(\sigma)= \sum_{s=(s_1, s_2,\ldots,s_n) \text{ in } S} [\sigma_1(s_1)\sigma_2(s_2)\ldots\sigma_n(s_n)]\, u_i(s)$

where $\sigma_j(s_j)$ is the probability that player j plays the pure strategy s_j. (when S_j is finite)

Example: Prisoners' Dilemma

If $\sigma_1(C)=2/3$, $\sigma_1(D)=1/3$, $\sigma_2(C)=\sigma_2(D)=1/2$, then:

$$U_1(\sigma)= \sigma_1(C)\sigma_2(C)u_1(C,C) + \sigma_1(C)\sigma_2(D)u_1(C,D) + \sigma_1(D)\sigma_2(C)u_1(D,C) + \sigma_1(D)\sigma_2(D)u_1(D,D) =2$$

Pl 1 \ Pl 2	C	D
C	3,3	0,5
D	5,0	1,1

Strict domination

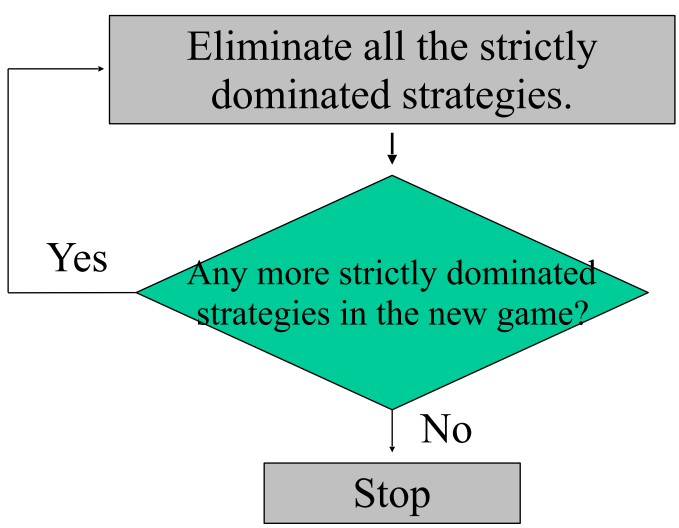
A mixed strategy σ_i^* **strictly dominates** a pure strategy s_i if

$$U_i(\sigma_i^*, s_{-i}) > U_i(s_i, s_{-i}) \quad \text{for any } s_{-i}$$

Player i is *rational* if she maximizes her expected utility, given her knowledge and beliefs about how the others will play.

RATIONALITY implies Never play a strictly dominated strategy.

Iterated Elimination of Strictly Dominated Strategies (IESDS)



Prediction relies on the **common knowledge of rationality**

A 3x3 game

Player 2 / Player 1

	L	m	R
T	3,0	1,1	0,3
M	1,0	0,10	1,0
B	0,3	1,1	3,0

Assume: Players are rational and 2 knows that 1 is rational.

Player 2 / Player 1

	L	M	R
T	3,0	1,1	0,3
M	1,0	0,10	1,0
B	0,3	1,1	3,0

1 is rational:

	L	M	R
T	3,0	1,1	0,3
B	0,3	1,1	3,0

2 knows 1 is rational and 2 is rational:

	L	R
T	3,0	0,3
B	0,3	3,0

Simplified price-competition

Firm 1 \ Firm 2	High	Medium	Low
High	6,6	0,10	0,8
Medium	10,0	5,5	0,8
Low	8,0	8,0	4,4

Summary

- If players are rational (and cautious), then they play the dominant-strategy equilibrium whenever it exists
  - But typically, it does not exist
- If it is common knowledge that players are rational, then they will play a strategy-profile that survives IESDS:
  - Typically, there are too many strategies that survive IESDS
- Next, a stronger assumption: The players are rational and they hold correct beliefs about the other players' strategies.

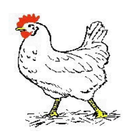
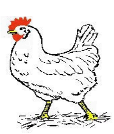
Nash Equilibrium in pure strategies

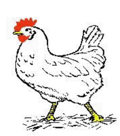
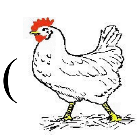
A pure strategy-profile $s^*=(s_1^*,\ldots,s_n^*)$ is a **Nash equilibrium (NE)** if no player has an incentive to deviate when the others play according to s^*, i.e. if for any i and s_i in S_i :

$$u_i(s_i^*,s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

Assumption: Players are rational and have correct conjectures about others' strategies.

Chicken

		
	-1,-1	1,0
	0,1	1/2,1/2

Two pure strategy NE: ( , ) and ( , )

Stag Hunt

		
	2,2	4,0
	0,4	6,6

Two pure strategy NE:

( , ) and ( , )

Matching Pennies

Player 2 Player 1	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

No pure strategy NE.

Nash Equilibrium in mixed strategies

A mixed strategy-profile $\sigma^*=(\sigma_1{}^*,\ldots,\sigma_n{}^*)$ is a **Nash equilibrium** if no player has an incentive to deviate when the others randomize according to σ^*, i.e. if for any i and σ_i :

$$U_i(\sigma_i{}^*, \sigma_{-i}{}^*) \geq U_i(\sigma_i, \sigma_{-i}{}^*)$$

Assumption: Players are rational and have correct conjectures about others' strategies.

Best reply

A mixed strategy σ_i of player i is a ***best reply*** to σ_{-i} if it maximizes $U_i(\cdot, \sigma_{-i})$.

Notation: Let $B_i(\sigma_{-i})$ denote the set of best replies of player i to σ_{-i}.

Important Note: A mixed strategy profile $\sigma^*=(\sigma_1{}^*,\ldots, \sigma_n{}^*)$ is a Nash equilibrium if and only if for every player i, $\sigma_i{}^*$ is a best reply to $\sigma_{-i}{}^*$.

Proposition: σ_i is a best reply to σ_{-i} iff for any pure strategy s_i played with positive probability (i.e. $\sigma_i(s_i)>0$), s_i gives i more expected payoff than any other pure strategy:

$$U_i(s_i,\sigma_{-i}) \geq U_i(s_i', \sigma_{-i}) \text{ for any } s_i' \text{ in } S_i$$

Proof: $U_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \text{ in } S_i} \sigma_i(s_i)\, U_i(s_i,\sigma_{-i})$, therefore σ_i is an optimal response to σ_{-i} if and only if each s_i with $\sigma_i(s_i)>0$ is an optimal response.

In particular when σ_i is a best reply to σ_{-i}, i is *indifferent* between any two strategies that he plays with positive probability.

Nash equilibrium existence

Theorem: (John F. Nash, 1950) Every normal form game with finitely many pure strategies has a mixed strategy Nash equilibrium.

Example: (A game without a mixed strategy Nash equilibrium) Consider two players where each player i chooses an integer s_i. Player i receives 1 if $s_i > s_j$, and 0 otherwise ($j \neq i$).

Stag Hunt

		
	2,2	4,0
	0,4	6,6

Mixed strategy equilibrium in the stag hunt game

Suppose that player 1 believes that player 2 plays Rabbit with probability β.

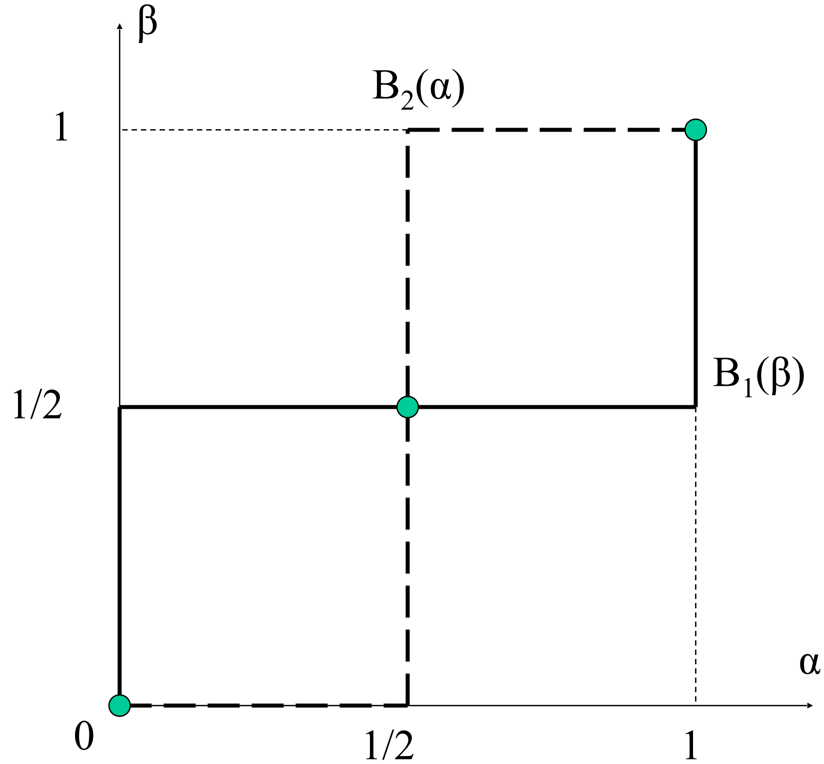
Player 1's payoff from playing Rabbit or Stag:

$U_1(R,\beta)=2\beta +4(1- \beta)$; $U_1(S,\beta)=0\beta +6(1- \beta)$

$$U_1(R, \beta) > U_1(S, \beta) \quad , \beta > 1/2$$

α: the probability with which 1 plays Rabbit.

Best replies in the Stag-Hunt game



A 3x3 game

Player 2

Player 1

	L	m	R
T	3,0	1,1	0,3
M	1,0	0,10	1,0
B	0,3	1,1	3,0

Nash equilibria and IESDS, Fact I

Proposition 1: Let $\sigma^*=(\sigma_1^*,\ldots, \sigma_n^*)$ be a mixed strategy Nash equilibrium. Then any pure strategy profile s that comes about with positive probability under σ^* (i.e. $\sigma_i^*(s_i)>0$ for any i) survives IESDS.

Proof: Covered in class.

Simplified price-competition

Firm 1 \ Firm 2	High	Medium	Low
High	6,6	0,10	0,8
Medium	10,0	5,5	0,8
Low	8,0	8,0	4,4

Nash equilibria and IESDS, Fact II

Proposition 2: If only one pure strategy profile s^* survives IESDS, then s^* is a Nash equilibrium of the game. Moreover, the game has no other (pure or mixed strategy) Nash equilibrium.

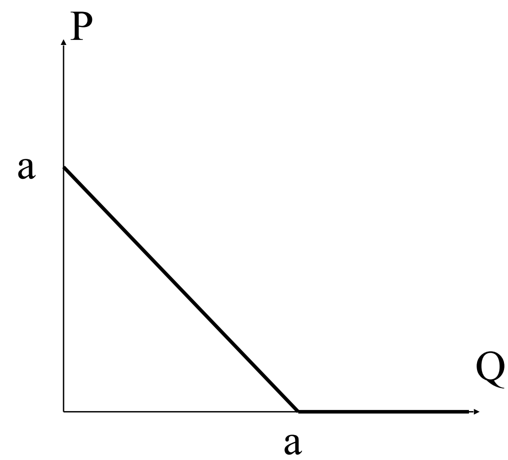
Proof: Covered in class.

Cournot duopoly

- $N = \{1,2\}$ two firms;
- Simultaneously, each firm i produces $q_i \geq 0$ units of a good at marginal cost $c > 0$,
- and sells the good at price

  $P(Q) = \max\{0, a-Q\}$ $a>c$

  where $Q = q_1+q_2$.



Profits of firm i:

$$\pi_i(q_1,q_2) = q_i[P(q_1+q_2)-c] = \begin{cases} q_i[a-q_1-q_2-c] & \text{if } q_1+q_2<a \\ -q_i c & \text{otherwise} \end{cases}$$

Cournot duopoly best replies

$$B_i(q_j)=\begin{cases} 0 & \text{if } q_j \geq a-c \\ (a-q_j-c)/2 & \text{otherwise} \end{cases}$$

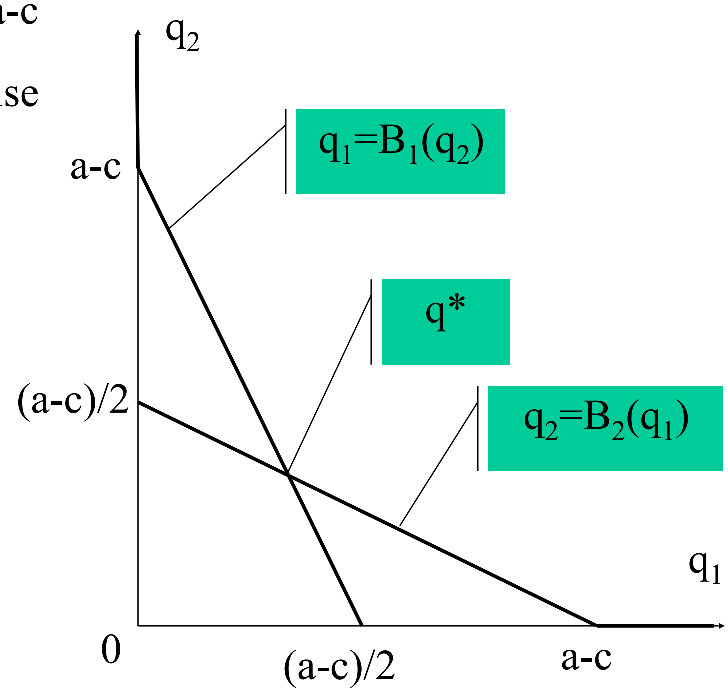
Nash equilibrium:

$$q_1^*=(a-q_2^*-c)/2$$

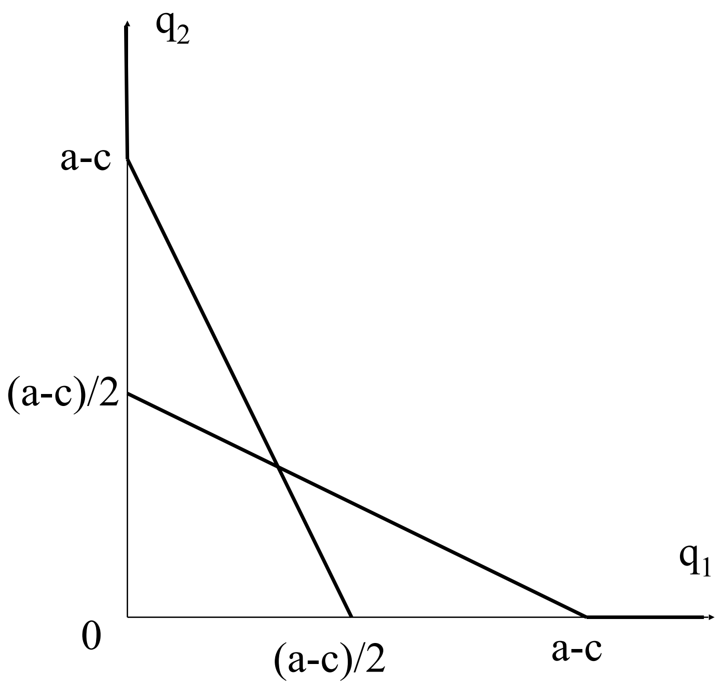
$$q_2^*=(a-q_1^*-c)/2$$

imply:

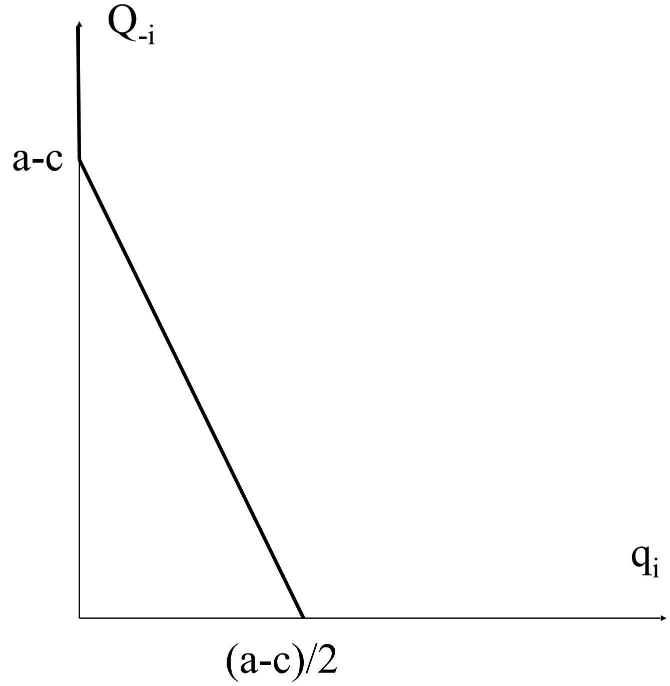
$$q_1^*=q_2^*=(a-c)/3$$



IESDS in the Cournot duopoly



IESDS in the Cournot Oligopoly with n>2



Commons problem

- N = {1,2,…,n} players, each with unlimited money;
- Simultaneously, each player i contributes $x_i \geq 0$ to produce $y = x_1 + \ldots + x_n$ unit of some public good, yielding payoff

$$u_i(x_i,y) = y^{1/2} - x_i.$$

Optional Additional Reading

- Chapters 1-4 of Osborne and Rubinstein.
- Chapter 7-8 of MWG