Econ/Math C103

First and Second Price Auctions

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1 Auctions as Bayesian Games

In these lecture notes, we will formulate auctions as Bayesian games and analyze the equilibrium outcome in two widely used auction mechanisms: the first price and the second price auctions.

We have a set of n bidders $N = \{1, 2, \ldots, n\}$ bidding for a single indivisible object (e.g. land for oil wells, a house, a painting, a book, a bottle of wine, radio frequency spectrum,...). Bidder i's valuation for object i is denoted by X_i or x_i. The bidders' valuations are i.i.d. with density f and c.d.f. F, in a closed interval $[\underline{x}, \bar{x}] \subset \mathbb{R}$. We will assume that the density f is continuous and is strictly positive, i.e. $f(x) > 0$ for any $x \in [\underline{x}, \bar{x}]$. The auctions we will analyze here are *sealed-bid*. That is, the bids are assumed to be made simultaneously once and for all.[1] A strategy of bidder i is a bid function $b_i : [\underline{x}, \bar{x}] \to \mathbb{R}$, where $b_i(x_i)$ specifies how much bidder i bids for the object when his type/valuation is x_i.

Our conclusions in these lecture notes rely on a set of important assumptions, most of which are already made implicitly through the above formulation. These are specifically: independence, private values, symmetry, and risk neutrality.

Independence refers to the assumption that the valuations of the bidders are distributed independently. This assumption rules out correlations in valuations: whether my valuation is high or low does not tell me anything about how the other bidders value the object.

Symmetry refers to the assumption that bidders are symmetric: ex-ante the distribution of bidder valuations are the same. This is a reasonable assumption if for example bidders' identities are anonymous. On the other hand it is questionable when certain bidders, say for a government contract, are known to be significantly larger firms than the others.

[1] Certain dynamic auctions, like the eBay, or the ascending price English auction, turn out to be strategically equivalent to sealed-bid auctions under our assumptions. Hence most of what is said here will extend to such dynamic auctions.

Private values refers to the assumption that each bidder knows his own valuation. Hence information about the others' valuations has no effect on a bidder's valuation. This assumption rules out situations where there might be asymmetric information about factors that have similar effects on bidder preferences. For instance, one important situation that the private values assumption rules out is one where the bidders are interested in the potential resale value of the object, and they have asymmetric signals about the resale price. Therefore, this assumption by itself restricts the applicability of the above model to many interesting financial markets. On the other hand, private values is a reasonable assumption if there is no resale possibility, or if the resale value is common knowledge and the the randomness in valuations are due to idiosyncratic differences across the bidders' needs and tastes.

Risk neutrality says that individuals maximize expected value, ruling out other interesting risk attitudes like risk aversion or risk loving behavior. Hence the vNM utility function is assumed to be linear over valuations/payments.

It is important to understand the nature of the above assumptions in order to judge the applicability of the subsequent results to different situations. There are many other interesting auction models that investigate what happens if part of the above assumptions are dropped. The appendix reviews the necessary calculus and statistics results used in the subsequent equilibrium analysis.

2 The Second-Price Auction

The rules of the second-price auction are that the highest bidder wins the object and pays the second highest bid. If there are multiple highest bidders, then one of them is chosen at random to win the object and pay the second highest bid (which would be the same as the highest bid when there is such a tie). Formally the payoff function is given by:

$$u_i(b_1, \ldots, b_n; x_1, \ldots, x_n) = \begin{cases} \frac{1}{|\{j \in N : b_j = b^1\}|}(x_i - b^2) & \text{if } b_i = b^1 \\ 0 & \text{otherwise.} \end{cases}$$

Above, b^1 denotes the highest bid (the first-order statistic) and b^2 denotes the second highest bid (the second-order statistic) among $b_1, \ldots, b_n$. The term $|\{j \in N : b_j = b^1\}|$ is the number of bidders that make the highest bid, so 1 over that term is the probability with which any highest bidder wins the object and pays his bid. Conditional on winning the object, a bidder's utility is his value minus his payment: $x_i - b^2$. The reservation utility is still zero. The model given in Section 1 along with the payoff function defined above defines the Second Price Auction as a Bayesian game.

There are many BNE of the second price auction.[2] One of these BNE is a dominant strategy equilibrium in which bidders bid their own valuation.

Theorem 1 *In the second-price auction, it is a dominant strategy for a bidder i with valuation $x_i \in [\underline{x}, \bar{x}]$ to bid:*

$$b_i(x_i) = x_i. \tag{1}$$

That is, for each b_i':

$$u_i(x_i, b_{-i}; x_i) \geq u_i(b_i', b_{-i}; x_i) \tag{2}$$

for all b_{-i} and the inequality is strict for some b_{-i}.

Proof: For any b_{-i}, let b_{-i}^1 denote the highest bid of all the bidders other than i. In the second price auction, i's bid doesn't affect his payoff $x_i - b_{-i}^1$ conditional on winning, it just affects whether i wins or not. By bidding $b_i(x_i) = x_i$, i ensures to win if his payoff from winning is positive, i.e. if $x_i - b_{-i}^1 > 0$; and to lose if his payoff from winning is negative, i.e. if $x_i - b_{-i}^1 < 0$. This establishes Equation (2). If $x_i \neq b_i'$, then note that inequality in Equation (2) is satisfied strictly for any b_{-i} such that b_{-i}^1 lies strictly in between x_i and b_i'. $\square$

Because of the plausibility of dominant-strategy equilibria (when they exist), we will assume in subsequent comparative statistics that bidders in a second price auction bid their valuations.

3 The First-Price Auction

The rules of the first-price auction are simple: the highest bidder wins the object and pays his bid. If there are multiple highest bidders, then one of them is chosen at random to win the object and pay his bid. Formally the payoff function is given by:

$$u_i(b_1, \ldots, b_n; x_1, \ldots, x_n) = \begin{cases} \frac{1}{|\{j \in N : b_j = b^1\}|}(x_i - b_i) & \text{if } b_i = b^1 \\ 0 & \text{otherwise.} \end{cases}$$

Again, b^1 denotes the highest bid (or the first-order statistic, see the appendix) among $b_1, \ldots, b_n$. Conditional on winning the object, a bidder's utility is his value minus his payment: $x_i - b_i$. The reservation utility when the object is not received and no payment is made is normalized to be zero. The model given in Section 1 along with the payoff function defined above defines the First Price Auction as a Bayesian game. We next solve for its Bayesian Nash equilibrium.

[2] For instance, for every bidder i^* and $x^* \geq \bar{x}$, the bid profile b where $b_i(x_i) = x^*$ for all $x_i \in [\underline{x}, \bar{x}]$ and $b_j(x_j) = \underline{x}$ for all $j \neq i$ and $x_j \in [\underline{x}, \bar{x}]$ is a BNE. Exercise: Construct other BNE of the second price auction.

3.1 The Bayesian Nash Equilibrium

Our strategy for finding a BNE of this game will be "conjecture and verify". More specifically we will conjecture that there is a symmetric BNE, i.e. one where every bidder uses the same bid function $b_i(x_i) = b(x_i)$, such that $b(\cdot)$ is differentiable with a strictly positive derivative $b'(x) > 0$ for any $x \in [\underline{x}, \bar{x}]$.[3] We will derive $b(\cdot)$ by using these conjectures and then show that the derived $b(\cdot)$ indeed gives us a BNE with the conjectured properties.

The expected payoff of bidder i with valuation x_i when he bids b_i is equal to the probability with which he wins the object when he bids b_i times $(x_i - b_i)$. Since the probability that any other bidder j bids $b_j \equiv b(X_j) = b_i$ is $\mathbb{P}(X_j = b^{-1}(b_i)) = 0$, we can ignore ties in bids when we calculate expected payoffs. Hence the expected payoff of i can be written as:

$$\begin{aligned}\mathbb{P}(\forall j \neq i : b_j \leq b_i)(x_i - b_i) &= \mathbb{P}(\forall j \neq i : b(X_j) \leq b_i)(x_i - b_i) \\ &= \mathbb{P}(\forall j \neq i : X_j \leq b^{-1}(b_i))(x_i - b_i) \\ &= F^{n-1}(b^{-1}(b_i))(x_i - b_i)\end{aligned}$$

A *necessary* condition for b_i to be maximizing i's expected payoff when he has valuation x_i, is that the derivative of the above term w.r.t. b_i is zero. This gives us the first order condition:

$$\begin{aligned}0 &= \frac{d}{db_i}\left[F^{n-1}(b^{-1}(b_i))(x_i - b_i)\right] \\ &= (n-1)F^{n-2}(b^{-1}(b_i))f(b^{-1}(b_i))\frac{1}{b'(b^{-1}(b_i))}(x_i - b_i) - F^{n-1}(b^{-1}(b_i)).\end{aligned}$$

According to our conjecture that the optimal bid for bidder i with valuation x_i is $b(x_i)$, $b_i = b(x_i)$ must satisfy the above first order condition, i.e.:

$$0 = (n-1)F^{n-2}(x_i)f(x_i)\frac{1}{b'(x_i)}(x_i - b(x_i)) - F^{n-1}(x_i).$$

since $b^{-1}(b_i) = b^{-1}(b(x_i)) = x_i$. A manipulation of the above equation gives us:

$$(n-1)F^{n-2}(x_i)f(x_i)x_i + F^{n-1}(x_i) - F^{n-1}(x_i) = b'(x_i)F^{n-1}(x_i) + (n-1)F^{n-2}(x_i)f(x_i)b(x_i),$$

which by the product rule of differentiation, can be rewritten as:

$$\frac{d}{dx_i}\left[F^{n-1}(x_i)x_i\right] - F^{n-1}(x_i) = \frac{d}{dx_i}\left[b(x_i)F^{n-1}(x_i)\right].$$

[3] Note that this implies in particular that $b(\cdot)$ is strictly increasing.

Integrating the above equation from $\underline{x}$ to some $x \in [\underline{x}, \bar{x}]$, over α we obtain:

$$F^{n-1}(x)x - F^{n-1}(\underline{x})\underline{x} - \int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha = b(x)F^{n-1}(x) - b(\underline{x})F^{n-1}(\underline{x}).$$

Note that $F(\underline{x}) = 0$, so we can simplify and rearrange the above equation to obtain the solution for the symmetric bid function of the first price-auction:

$$b(x) = x - \frac{1}{F^{n-1}(x)} \int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha. \tag{3}$$

Note that the above bid function is only defined on $(\underline{x}, \bar{x}]$. We extend it to $[\underline{x}, \bar{x}]$ by setting $b(\underline{x}) := \underline{x} = \lim_{x \searrow \underline{x}} b(x)$. It can be checked that $b : [\underline{x}, \bar{x}] \to \mathbb{R}$ is indeed differentiable with strictly positive derivative. To conclude that it gives a BNE we need to verify that the $b_i = b(x)$ maximizes bidder i's expected payoff when his valuation is x and the others bid according to (3).[4]

First note that since the bid function is continuous and strictly increasing, its image is the closed interval $b([\underline{x}, \bar{x}]) = [b(\underline{x}), b(\bar{x})]$. It can clearly not be optimal for i to bid strictly greater than $b(\bar{x})$ since by reducing his bid to $b(\bar{x})$, he would still win the object with probability one and pay strictly less. Also, any bid strictly less than $b(\underline{x})$ gives bidder i an expected payoff of zero, which is the same expected payoff he gets from bidding $b(\underline{x})$.

To conclude that $b(x)$ is the optimal bid for bidder i with valuation x, let us next show that he does not find it profitable to bid $b(y)$ for any $y \in [\underline{x}, \bar{x}]$ such that $y \neq x$. Bidder i's expected payoff, when the others bid according to (3) and if i bids $b(y)$ is given by:

$$[x - b(y)]\mathbb{P}(\forall j \neq i : b(X_j) \leq b(y)) = [x - b(y)]F^{n-1}(b^{-1}(b(y))) = [x - b(y)]F^{n-1}(y)$$

We can think of this payoff as the payoff of type x from bidding like a type y or payoff

[4] Note that we know from our derivation of $b(\cdot)$ that $b_i = b(x)$ satisfies the necessary first order condition for a maximum, but we do not yet know whether the sufficient (second-order) condition for a maximum is satisfied. If we do not check this, we can not ensure whether the solution of the first order condition gives us a minimum, a maximum, or just a point of inflection.

of type x from "imitating" a type y. This can be rewritten as:

$$\begin{aligned}
[x-b(y)]F^{n-1}(y) &= [y-b(y)]F^{n-1}(y)+(x-y)F^{n-1}(y) \\
&= \int_{\underline{x}}^{y} F^{n-1}(\alpha)d\alpha+(x-y)F^{n-1}(y) \\
&= \int_{\underline{x}}^{y} F^{n-1}(\alpha)d\alpha+\int_{y}^{x} F^{n-1}(y)d\alpha \\
&= \int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha+\int_{y}^{x} [F^{n-1}(y)-F^{n-1}(\alpha)]d\alpha \\
&= [x-b(x)]F^{n-1}(x)+\int_{y}^{x} [F^{n-1}(y)-F^{n-1}(\alpha)]d\alpha
\end{aligned}$$

where the second and last equalities follow from (3). Note that since F is strictly increasing, the final integral is negative if $x<y$ or if $y<x$. This implies that:

$$[x-b(y)]F^{n-1}(y)<[x-b(x)]F^{n-1}(x) \quad \text{if } y \neq x.$$

That is, bidder i with valuation x finds it uniquely optimal to bid $b(x)$, rather than "pretending to have another valuation y" and bidding $b(y)$. This finishes the proof that the symmetric bid function given in (3) is a BNE.

Theorem 2 *Consider the bid function*

$$b(x)=x-\frac{1}{F^{n-1}(x)}\int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha. \tag{4}$$

Then, the first-price auction has an "essentially unique" BNE, where bidders with types $x>\underline{x}$, bid according to the above symmetric pure strategy bid function. The bids of $\underline{x}$ types are not pinned down any further than the support of $b_i(\underline{x})$ lies in $(-\infty,\underline{x}]$.

Proof: We already proved that the above described bidding strategies are BNE. Our equilibrium derivation also implies that any symmetric pure strategy BNE in differentiable bid functions with strictly positive derivative must be of the above form. Corollary 4 in Lebrun (1999), included in the the optional additional readings, implies that any BNE of the first-price auction must be in symmetric pure strategy BNE in differentiable bid functions with strictly positive derivative for types $x>\underline{x}$. Together with our derivation, this implies that there is no other BNE. □

4 Revenue Equivalence

How do seller's expected revenues compare in the two types of auctions? Given specific bids, the first price auction clearly generates more revenue to the seller. On the other hand as we have shown, the first price auction also gives bidders incentives to bid below their valuations (this is called "shading" one's bid) whereas there are no such incentives in the second price auction. Hence the answer to the question is not obvious. We will show next that the two effects cancel out exactly, so that the expected revenues of the seller and the expected payoffs of the bidders are the same in the two types of auctions. Our findings here constitute a special case of a more general result known as Myerson's "Revenue Equivalence Theorem" which we will study towards the end of the semester when we cover optimal auctions.

Let's denote the first and second price bid functions in (3) and (1), by $b^I(\cdot)$ and $b^{II}(\cdot)$ respectively. Similarly let $Revenue^I$ and $Revenue^{II}$ denote the seller's expected revenues in the respective auctions. Note that since the bid functions in each of the two auction equilibria are symmetric and strictly increasing, the winner of the auction is always the bidder with the highest valuation. Hence the expected revenue is equal to the expected payment of the highest bidder. In the first price auction:

$$Revenue^I = \mathbb{E}[b^I(X^1)] = \int_{\underline{x}}^{\bar{x}} b^I(x) f_{X^1}(x)dx$$

Integrating by parts with $u = b^I(x)$ and $dv = f_{X^1}(x)dx$, we can rewrite this as:

$$\begin{aligned} Revenue^I &= b^I(\bar{x})F_{X^1}(\bar{x}) - b^I(\underline{x})F_{X^1}(\underline{x}) - \int_{\underline{x}}^{\bar{x}} \left[\frac{d}{dx}b^I(x)\right] F_{X^1}(x)dx \\ &= \bar{x} - \int_{\underline{x}}^{\bar{x}} F^{n-1}(x)dx - (n-1)\int_{\underline{x}}^{\bar{x}} f(x)\left(\int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha\right)dx \end{aligned}$$

where the second equality follows from $F_{X^1}(\bar{x}) = 1$, $F_{X^1}(\underline{x}) = 0$ and the expression for $b^I(\cdot)$ in (3). Applying a second integration by parts with $u = \int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha$ and $dv = f(x)dx$ to the final integral and simplifying further, we obtain:

$$\begin{aligned} Revenue^I &= \bar{x} - \int_{\underline{x}}^{\bar{x}} F^{n-1}(x)dx - (n-1)\left[\int_{\underline{x}}^{\bar{x}} F^{n-1}(x)dx - \int_{\underline{x}}^{\bar{x}} F^{n}(x)dx\right] \\ &= \bar{x} - \int_{\underline{x}}^{\bar{x}} \left[nF^{n-1}(x) - (n-1)F^n(x)\right]dx \\ &= \bar{x} - \int_{\underline{x}}^{\bar{x}} F_{X^2}(x)dx. \end{aligned}$$

In the second-price auction, the revenue of the seller is equal to the expected value of the second highest bid which is the second highest valuation:

$$\begin{aligned} Revenue^{II} &= \mathbb{E}[X^2] \\ &= \int_{\underline{x}}^{\bar{x}} x f_{X^2}(x)dx \\ &= \bar{x} - \int_{\underline{x}}^{\bar{x}} F_{X^2}(x)dx. \end{aligned}$$

where the last equality follows from an integration by parts with $u = x$ and $dv = f_{X^2}(x)dx$. This proves that $Revenue^I = Revenue^{II}$.

We next show that the expected payoffs of the bidders are also the same in the two auctions. Let $U^I(x)$ and $U^{II}(x)$ denote the first and second-price auction equilibrium payoffs of a bidder with valuation $x \in [\underline{x}, \bar{x}]$. Then:

$$\begin{aligned} U^I(x) &= \mathbb{P}(x \text{ is the highest valuation})(x - b^I(x)) \\ &= F^{n-1}(x)(x - b^I(x)) \\ &= \int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha \end{aligned}$$

Let X^1_{-i} denote the highest valuation among the bidders other than i. Since bidder i does not observe the others' valuations, to him, X^1_{-i} is a random variable with c.d.f. $F_{X^1_{-i}}(\alpha) = F^{n-1}(\alpha)$. Then:

$$U^{II}(x) = \int_{\underline{x}}^{x} [x - \alpha] f_{X^1_{-i}}(\alpha)d\alpha = \int_{\underline{x}}^{x} F^{n-1}(\alpha)d\alpha.$$

where the first equality corresponds to the expected value of the bidder i's payoff: 0 if $X^1_{-i} > x$ and $x - X^1_{-i}$ if $X^1_{-i} \leq x$. The second equality follows from integrating by parts with $u = x - \alpha$ and $dv = f_{X^1_{-i}}(\alpha)d\alpha$ and further simplifications.

5 Optional Additional Reading

Chapters 1 and 2 of **Kr** are directly related to the material in these lecture notes. See Lebrun (1999) if you are interested in the complete argument for the uniqueness part of the first-price auction BNE in Theorem 2.

Kr Krishna, *Auction Theory*, Academic Press, 2002.

- Lebrun (1999) "First Price Auctions in the Asymmetric N Bidder Case" in *International Economic Review*, 40, 125–42.

A Appendix

This appendix reviews some probability and calculus that is needed for the equilibrium analysis in these lecture notes.

A.1 Continuous random variables

Let X denote a random variable that takes values in some bounded closed interval $[\underline{x}, \bar{x}]$ in $\mathbb{R}$. Particular realizations of X are denoted by x. The random structure of a continuous random variable X is summarized by what is called a **density of** X, which is a function $f\colon [\underline{x}, \bar{x}] \to \mathbb{R}_+$ such that $\int_{\underline{x}}^{\bar{x}} f(\alpha)d\alpha = 1$. If X has density f, the probability that X takes a value in $A \subset [\underline{x}, \bar{x}]$ is given by:

$$\mathbb{P}(X \in A) = \int_A f(\alpha)d\alpha.$$

The **cumulative distribution function (c.d.f.)** of X is a function $F : [\underline{x}, \bar{x}] \to [0, 1]$ that gives the probability that X is less than or equal to some $x \in [\underline{x}, \bar{x}]$. The c.d.f. can be derived from the density by integrating:

$$F(x) = \mathbb{P}(X \leq x) = \int_{\underline{x}}^{x} f(\alpha)d\alpha.$$

Note that F is a non-decreasing function with $F(\underline{x}) = 0$ and $F(\bar{x}) = 1$. Moreover if $f(x) > 0$ for any $x \in [\underline{x}, \bar{x}]$ then F is *strictly increasing* on $[\underline{x}, \bar{x}]$.

When the density f is continuous, we can apply the Leibniz's rule to derive the density back from the c.d.f.:

$$f(x) = \frac{d}{dx}F(x) \equiv \frac{d}{dx}\int_{\underline{x}}^{x} f(\alpha)d\alpha.$$

Let $h\colon [\underline{x}, \bar{x}] \to \mathbb{R}$ be a function defined on the set of possible values that the random variable X can take. Then $h(X)$ itself is a random variable and its expectation is defined by:

$$\mathbb{E}[h(X)] = \int_{\underline{x}}^{\bar{x}} h(\alpha)f(\alpha)d\alpha.$$

Note that, unlike discrete random variables, the probability that a continuous random variable X takes any particular value $x \in [\underline{x}, \bar{x}]$ is zero: $\mathbb{P}(X = x) = \int_x^x f(\alpha)d\alpha = 0$. This implies that if X has a density then $F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X < x)$. Finally when we are given n random variables $X_1, \ldots, X_n$ we say that they are **independent and identically distributed (i.i.d.)** if $X_1, \ldots, X_n$ are independent and each of them is distributed according to the same density f (or c.d.f. F).

A.2 Distribution of order statistics

Let $X_1, \dots, X_n$ be n i.i.d. continuous random variables with continuous density f and c.d.f. F. For $k = 1, \dots, n$, the **kth order statistic** is the random variable X^k that takes the kth largest value among $X_1, \dots, X_n$, counting repetitions of the same values. The first order statistic is $X^1 = \max\{X_1, \dots, X_n\}$, and the second order statistic takes the second highest value in $X_1, \dots, X_n$. For example if $n = 4$ and (X_1, X_2, X_3, X_4) take the value $(5, 6, 10, 6)$, then the order statistics (X^1, X^2, X^3, X^4) take the values $(10, 6, 6, 5)$.

We can compute the c.d.f. of the first order statistic as:

$$\begin{aligned} F_{X^1}(x) &= \mathbb{P}(X^1 \leq x) \\ &= \mathbb{P}(\max\{X_1, \dots, X_n\} \leq x) \\ &= \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) \\ &= \mathbb{P}(X_1 \leq x)\mathbb{P}(X_2 \leq x) \dots \mathbb{P}(X_n \leq x) \\ &= F^n(x). \end{aligned}$$

We can then find the density by differentiating the c.d.f.:

$$f_{X^1}(x) = \frac{d}{dx} F_{X^1}(x) = nF^{n-1}(x) f(x).$$

Similarly when $n \geq 2$, the c.d.f. of the second-order statistic can be found as:

$$\begin{aligned} F_{X^2}(x) &= \mathbb{P}(X^2 \leq x) \\ &= \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) + \sum_{i=1}^{n} \mathbb{P}(X_i > x \text{ and } \forall j \neq i : \ X_j \leq x) \\ &= F^n(x) + n[1 - F(x)]F^{n-1}(x) \\ &= nF^{n-1}(x) - (n-1)F^n(x). \end{aligned}$$

Differentiating the c.d.f. gives the density:

$$f_{X^2}(x) = \frac{d}{dx} F_{X^2}(x) = n(n-1)[F^{n-2}(x) - F^{n-1}(x)] f(x).$$

A.3 Derivative of an inverse function

The following is a useful result about differentiating inverse functions. Let (a, b), (c, d) be open intervals in $\mathbb{R}$. Let $g \colon (a, b) \to (c, d)$ be an onto, differentiable function such that $g'(x) > 0$ for any $x \in (a, b)$. Note that g must be strictly increasing since its derivative is strictly positive.

The inverse of g is the function $g^{-1}\colon (c,d) \to (a,b)$ defined by $g^{-1}(y) = x$ if and only if $g(x) = y$. The inverse is well-defined since g is monotonically increasing and onto. Then the derivative of the inverse function is one over the derivative of the original function:

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{g'\left(g^{-1}(y)\right)}.$$