Econ/Math C103

Individual Choice Theory II: Decision-making under Uncertainty

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1 Lotteries

Now suppose that we can construct choice experiments in which we can ask the decision-maker to choose not only among alternatives in X, but also among lotteries that yield alternatives in X. Consider a set of prizes or outcomes X. For expositional simplicity, we will assume here that X is finite. We represent uncertainty by lotteries:

Definition 1 A **lottery** p is a probability distribution over prizes, that is:

1. $p : X \to [0, 1]$, and
2. $\sum_{x \in X} p(x) = 1$.

Let $\triangle(X)$ be the set of lotteries over X.

Above $p(x)$ corresponds to the probability of receiving the prize p when one faces the lottery p. typical lotteries are denoted by $p, q, r, \ldots \in \triangle(X)$. A lottery that yields an outcome $x \in X$ with probability 1 is called a **degenerate lottery** and is denoted by δ_x.

Some examples of lotteries:

- with 1/2 probability receive \$100; with 1/2 probability lose \$30,
- Consider a house that is worth \$200,000 which has probability 0.001 risk of burning down in which case it has (for simplicity) \$0 value,
  - Owning that house without insurance translates to the lottery: with probability 0.999 you have \$200,000; with probability 0.001 you have \$0.
  - Owning that house with full fire insurance at price \$2,000, translates to the degenerate lottery: with probability one you will have \$198,000.

As we have seen in class, lotteries can be viewed in tree forms with probabilities attached to the branches and prizes attached to the endpoints. The Machina-Marschak triangle also turns out to be a useful way of visualizing the set of lotteries.

For any two lotteries p and q and some $\alpha \in [0,1]$, we can construct a new lottery as follows: with probability α we will execute the lottery p and with probability $1-\alpha$ we will execute the lottery q. The process is called **compounding** two lotteries. We will denote the newly obtained lottery by $\alpha p + (1-\alpha)q$. It is formally defined as:

$$[\alpha p + (1-\alpha)q](x) = \alpha p(x) + (1-\alpha)q(x), \qquad x \in X$$

since the probability with which the compound lottery $\alpha p + (1-\alpha)q$ yields a particular prize x is $\alpha p(x) + (1-\alpha)q(x)$.

Let R be a preference relation over $\triangle(X)$. One may think of many plausible preferences among lotteries: (pessimistic) compare only on the basis of the worst possible outcome; (optimistic) compare only on the basis of the best possible outcome; or when $X \subset \mathbb{R}$ maximize the expected value...

2 Expected Utility: The vNM Theorem

Here we will concentrate on a very particular model of decision-making under uncertainty in economics. This is known as the expected utility model.

Definition 2 Given a utility function $u : X \to \mathbb{R}$, the expected utility of a lottery $p \in \triangle(X)$ is defined by:

$$\sum_{x \in X} u(x)p(x).$$

The function u above is called the von Neuman Morgenstern (vNM) utility function or vNM utility index.

Definition 3 A function $U : \triangle(X) \to \mathbb{R}$ is **linear** if

$$U(\alpha p + (1-\alpha)q) = \alpha U(p) + (1-\alpha)U(q)$$

for any $p, q \in \triangle(X)$ and $\alpha \in [0,1]$.

Our first result states that expected utility and linearity are mathematically the same thing.

Proposition 1 *A function $U : \triangle(X) \to \mathbb{R}$ is linear if and only if there exists a function $u : X \to \mathbb{R}$ such that:*

$$U(p) = \sum_{x \in X} p(x)u(x) \qquad p \in \triangle(X).$$

Proof: The "$\Leftarrow$" part, that is the fact that any expected utility function is linear follows from:

$$\begin{aligned} U(\alpha p + (1-\alpha)q) &= \sum_{x \in X} [\alpha p + (1-\alpha)q](x)u(x) \\ &= \sum_{x \in X} \left[\alpha p(x) + (1-\alpha)q(x)\right] u(x) \\ &= \alpha \sum_{x \in X} p(x)u(x) + (1-\alpha) \sum_{x \in X} q(x)u(x) \\ &= \alpha U(p) + (1-\alpha)U(q). \end{aligned}$$

For the "$\Rightarrow$" part, for any linear function U we can define $u(x) = U(\delta_x)$. We have verified in class that the linearity of U guarantees (by induction) that the equality in the Proposition is satisfied. $\square$

We will next analyze the choices induced by the expected utility criterion. Suppose that R represents our decision-maker's choices among lotteries from a revealed preference perspective. The question that vNM asks is the following: "Under what conditions on R is there a utility function u over prizes (equivalently a linear function U: $\triangle(X) \to \mathbb{R}$) such that the individual evaluates lotteries according to their expected utility $U(p) = \sum_{x \in X} p(x)u(x)$?". Consider the following conditions on R:

Independence For any $p, q, r \in \triangle(X)$ and $\alpha \in (0,1)$:

$$pRq \Leftrightarrow \alpha p + (1-\alpha)r \, R \, \alpha q + (1-\alpha)r.$$

Solvability For any $p, q, r \in \triangle(X)$, if $pPqPr$ then there exist $\alpha \in (0,1)$ such that:

$$q \, I \, \alpha p + (1-\alpha)r.$$

Theorem 1 *(von Neumann-Morgenstern) Let X be finite. A rational preference R over $\triangle(X)$ satisfies Independence and Solvability if and only if there is a $u : X \to \mathbb{R}$ such that:*

$$\forall p, q \in \triangle(X): \qquad pRq \Leftrightarrow \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x).$$

Proof: Covered in class. $\square$

3 The cardinal uniqueness of the vNM utility

We had seen in our earlier discussion of utility functions that, many different utility functions lead to the same choices over sure outcomes, as long as they give the same ordinal ranking. In particular, we could not capture cardinal properties such as strength of preference in that context. This is no longer the case when there is uncertainty and we analyze the choices induced by the expected utility criterion. We will illustrate that two vNM utility functions yield the same choices over lotteries if they are affinely related (i.e. linear transformation plus a constant). More importantly we will show that, if the two vNM functions are not cardinally, but possibly ordinally, related, then they lead to different choice behavior. The latter means that we can find a set of lotteries such that the two vNM functions lead to different choices out of this set.

Theorem 2 *Let X be finite. Let u and v be two vNM utility functions on X. Define the expected utility functions U and V by:*

$$U(p) = \sum_{x \in X} u(x)p(x) \quad \text{and} \quad V(p) = \sum_{x \in X} v(x)p(x) \qquad p \in \triangle(X).$$

Then U and V represent the same preference over $\triangle(X)$, i.e.:

$$U(p) \geq U(q) \Leftrightarrow V(p) \geq V(q) \qquad p, q \in \triangle(X),$$

if and only if $v = au + b$ for two real numbers a and b with $a > 0$.

Proof: One part of the proof is relatively easy. Suppose that $v = au + b$ for two real numbers a and b with $a > 0$. Then it is straightforward to verify that $V = aU + b$. But then since $a > 0$:

$$U(p) \geq U(q) \Leftrightarrow V(p) \geq V(q).$$

For the other part of the proof, let x^* maximize u and x_* minimize u, over X. If $u(x^*) = u(x_*)$ then u is constant. Since U and V represent the same preference over lotteries v must also be constant, therefore $a = 1$ and $b = v - u$ do the job.

The other possibility is $u(x^*) > u(x_*)$. Since U and V represent the same preference over lotteries $u(x^*) > u(x_*)$ implies that $v(x^*) > v(x_*)$, so we can find $a > 0$ and b such that $v(x^*) = au(x^*) + b$ and $v(x_*) = au(x_*) + b$. Now take any $x \in X$, since $u(x^*) \geq u(x) \geq u(x_*)$, there is $\alpha \in [0, 1]$ such that $u(x) = \alpha u(x^*) + (1 - \alpha)u(x_*)$, i.e. a decision maker who is using the vNM function u is indifferent between x and the the lottery p that gives x^* with probability α and x_* with probability $1 - \alpha$. Since U and V represent the same preference, a decision maker who is using the vNM function v must also be indifferent between x and the lottery p, i.e. $v(x) = \alpha v(x^*) + (1-\alpha)v(x_*)$. This brings us to:

$$\begin{aligned}
v(x) &= \alpha v(x^*) + (1-\alpha)v(x_*) \\
&= \alpha[au(x^*) + b] + (1-\alpha)[au(x_*) + b] \\
&= a[\alpha u(x^*) + (1-\alpha)u(x_*)] + b \\
&= au(x) + b
\end{aligned}$$

which was what we wanted to show. □

4 Infinite X

We have assumed so far that the set of prizes X is finite. However, in many applications of interest X is an infinite set. For example, when we consider monetary prizes it is natural to assume that $X = \mathbb{R}$. Fortunately, there is a simple way to extend the previous analysis to the infinite case. Let X be an arbitrary (possibly infinite) set of prizes:

Definition 4 A **simple lottery** p over X, is a probability distribution over prizes that has finite support, that is:

1. $p \colon X \to [0, 1]$,
2. $|\{x \in X : p(x) > 0\}| < \infty$, and
3. $\sum_{x \in X} p(x) = 1$.

Simple lotteries are interpreted just like ordinary lotteries, the additional restriction (the "finite support" condition) being that they give positive probability only to finitely many outcomes in X. Let $\triangle(X)$ denote the set of simple lotteries.[1] With a little extra work, all the results we proved in earlier sections can be extended to infinite X when $\triangle(X)$ denotes the set of simple lotteries. This is a relatively straightforward extension that we will take as given.

[1] Note that if X is finite, then the finite support condition (condition 2) in the definition of a simple lottery is satisfied automatically for all lotteries. Therefore, when X is finite the above definition of $\triangle(X)$ is the same as the definition we gave in Section 1.

5 Discussion

Modeling agents as expected utility maximizers was commonly used in economics before von Neumann and Morgenstern. So if vNM did not introduce the expected utility model, what is the value-added of the above theorem? Suppose that, as economic modelers, we had the capacity of opening up the brain of our decision-maker and recover the formula that rules how he makes choices among lotteries. Then, we would indeed do that and recover the true economic model of our individual. Unfortunately, we can not directly observe whether an individual is an expected utility maximizer, let alone recover the particular u. However since in principle we can offer him choice experiments among lotteries, we can observe his choice behavior, namely the revealed preference R.

From a descriptive point of view, the vNM result identifies the set of behavioral implications (rationality, independence, and solvability) of the expected utility model. The necessity part ("expected utility $\Rightarrow$ rationality, independence, and solvability") tells us that these three conditions constitute a behavioral test of the expected utility model with objective probabilities. That is, if the choices of an individual violate one of these three properties, then we can confidently reject the expected utility hypothesis. The sufficiency part ("rationality, independence, and solvability $\Rightarrow$ expected utility") says that there is no other conceivable property 4, that is (i) logically independent from the three above and (ii) that constitutes a behavioral test of expected utility. Hence the three properties jointly constitute the *maximal test* of expected utility. Finally in case our decision maker's behavior is consistent with all the three properties above, then the Theorem can identify u from R up to a positive affine transformation.

6 Optional Additional Reading

Lectures 8–9 of **Ru**, Chapter 6 of **MWG**.