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FIRST PRICE AUCTIONS IN THE ASYMMETRIC N BIDDER CASE

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I consider the first price auction when the bidders' valuations may be differently distributed. I show that every Bayesian equilibrium is an 'essentially' pure equilibrium formed by bid functions whose inverses are solutions of a system of differential equations with boundary conditions. I then prove the existence of an equilibrium. I prove its uniqueness when the valuation distributions have a mass point at the lower extremity of the support. I give sufficient conditions for uniqueness when every valuation distribution is one of two atomless distributions. I establish inequalities between equilibrium strategies when relations of stochastic dominance exist between valuation distributions.

1. INTRODUCTION

I study the first price auction in the general asymmetric framework. An indivisible item is offered for sale to $n \geq 2$ bidders. I denote bidder 1's valuation of the item by v_1, bidder 2's valuation by v_2, etc. I assume that the bidders' valuations $v_1, v_2, \ldots, v_n$ are chosen randomly by nature according to commonly known independent probability measures $F_1, F_2, \ldots, F_n$ (respectively). Only bidder i is informed of v_i. The participation to the auction is voluntary. In Lebrun (1997), I also consider the case of mandatory bidding. If at least one bidder takes part in the auction, the item goes to the highest bidder who has to pay the price equal to his bid.

Most of the literature in this 'independent private value model' has dealt with the symmetric case where the measures $F_1, F_2, \ldots, F_n$ are all the same measure, or with the case where there are only two bidders. However, asymmetry arises naturally in many examples. Consider a first price auction with more than two bidders where bidder j is reputed to be very interested in objects of the same style as the object being sold. The other bidders, however, are reputed to have only little interest in such objects. In this example, the measure F_j has to give more probability to high valuations than F_i does, for $i \neq j$.

Riley and Samuelson (1981) prove the existence of an equilibrium, and give a mathematical expression for the equilibrium strategy, in the symmetric case where the measures $F_1, \ldots, F_n$ are equal to the same absolutely continuous measure.[2] In the asymmetric case, Griesmer et al. (1967) consider a first price auction with two

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[2] Throughout my paper, an absolutely continuous measure means a measure absolutely continuous with respect to the Lebesgue measure.

bidders whose valuations are uniformly distributed over possibly different intervals. Vickrey (1961) analyzes the asymmetric two bidder case where one bidder knows the other bidder's valuation with certainty. Plum (1992) gives necessary and sufficient conditions for the existence of a pure equilibrium in the two bidder case. Maskin and Riley (1996a) examine several asymmetric two bidder examples.

The general model with mandatory bidding where the measures F_i are required to have compact supports, but are otherwise arbitrary, has been studied in Lebrun (1996). In this general case, a Nash equilibrium, even in mixed strategies, does not always exist.[3] However, it has been shown that if the supports of the measures F_i have the same minimum, and if this minimum is not a mass point of any of these measures, then there exists a Nash equilibrium. It is possible to obtain general positive results if the rules of the first price auction game are modified slightly. The existence theorems in Lebrun (1996) are proved in an indirect way by approximating the first price auction game with a sequence of games with a finite number of pure strategies. No characterization of the equilibria is given.

Asymmetric n bidder examples, where all bidders, except one, have the same valuation probability measure, have been numerically examined by Marshall et al. (1994). More numerical analysis can be found in Li and Riley (1997). Maskin and Riley (1996b) consider the existence of an equilibrium in the asymmetric n bidder case by relying on discrete approximations and passing to the limit. They prove the existence of an equilibrium in the cases of valuation measures absolutely continuous everywhere and of measures with finite supports. Maskin and Riley (1994) study properties and descriptions of the equilibria when they exist. In Maskin and Riley (1994 and 1996b), the uniqueness of the equilibrium is stated in the case of absolutely continuous valuation distributions, with possible mass points at the lower extremities of the supports, and density functions whose continuous extensions exist and are strictly positive everywhere. Unfortunately, in the versions at my disposal the proof of even the case with common support is not complete.[4]

In the present paper, I analyze the asymmetric n bidder case where the measures $F_1, F_2, \ldots, F_n$ have their supports equal to the same interval $[\underline{c}, \bar{c}]$, and are either absolutely continuous over the whole interval $[\underline{c}, \bar{c}]$, or are absolutely continuous over the interval $(\underline{c}, \bar{c}]$ with mass points at the lower extremity $\underline{c}$. This last case can be used to model situations where the reserve price set by the auctioneer is larger than the lower extremity of the valuation interval, and is not covered by the existence results in Lebrun (1996) concerning the unaltered first price auction game, nor by the results in Maskin and Riley (1996b). In the atomless case, the continuous extensions of the density functions are not required to exist at $\underline{c}$.

[3] In the frameworks of Lebrun (1996) and this present paper, it can be easily seen that an equilibrium of the auction with mandatory bidding is an equilibrium of the auction with voluntary bidding. However, the reverse is not generally true. For a counter-example, see the Introduction in Lebrun (1996).

[4] For example, when the measures are atomless the authors use, but do not prove, the differentiability of the bid functions at the lower extremity of the support. When a measure has a mass point at this lower extremity, the existence result in Maskin and Riley (1996b) does not apply. The endpoint conditions are not precisely specified, nor fully proved.

The approach I follow is, in a sense, reverse to and more direct than the approaches of Lebrun (1996) and Maskin and Riley (1994 and 1996b). First I give a characterization of the equilibrium strategies as solutions to a system of differential equations with boundary conditions. Then I proceed directly from this characterization in order to obtain the existence and other important properties of the equilibria, such as uniqueness.

I prove the existence of an equilibrium when all valuation distributions have a mass point at $\underline{c}$, and when the distributions are atomless. The difficulty of this proof stems from the singularity of the differential system at $\underline{c}$. I circumvent it by considering the solution of the differential system as a function of the initial condition at the upper extremity $\bar{c}$.

When $\underline{c}$ is a mass point of all distributions, I prove the uniqueness of the equilibrium. When every bidder's valuation distribution is one of two distributions, I give assumptions under which the equilibrium is unique in the atomless case. When the initial distributions are identical, the collusion of several bidders into one cartel results in distributions satisfying these assumptions. These results can be applied to all examples studied by Marshall et al. (1994). They can also be applied to situations where the bidders can be divided into two groups, with the bidders of one group being more interested in the object than the bidders of the other.

I establish inequalities that hold between bidders' equilibrium strategies when relations of stochastic dominance exist between valuation probability distributions. As a consequence of these results, I show that if two bidders' valuation distributions are equal, then their equilibrium strategies are equal and the uniqueness of the equilibrium in the symmetric n bidder case follows from the known uniqueness of the symmetric equilibrium in this case.

In Section 2, I introduce the model and give necessary and sufficient conditions for an n-tuple of strategies to be an equilibrium. In Section 3, I prove the existence of an equilibrium and investigate other properties of the equilibria. Section 4 concludes the paper. Proofs can be found in the Appendix.

2. THE MODEL AND THE CHARACTERIZATION OF THE EQUILIBRIA

The support[5] of the probability measures $F_1, F_2, \ldots, F_n$ are equal to the same interval $[\underline{c}, \bar{c}]$, with $\underline{c} < \bar{c}$. For the sake of convenience, I also denote by $F_1, F_2, \ldots, F_n$ the cumulative distribution functions continuous from the right. I assume that these functions are differentiable over the half-open interval $(\underline{c}, \bar{c}]$[6] and that their derivatives—the density functions $f_1, f_2, \ldots, f_n$—are locally bounded away from zero over this same interval.[7] In the rest of the paper, this set of assumptions will be referred to as 'the assumptions of Section 2'. Notice that under these assumptions some density functions may not admit any extensions continuous everywhere in $[\underline{c}, \bar{c}]$.

[5] The support of a probability measure μ is the largest closed set of μ-measure one.

[6] The derivative at $\bar{c}$ is a left-hand derivative.

[7] This last assumption is satisfied if, for example, the density functions are continuous and strictly positive over $(\underline{c}, \bar{c}]$.

In the case with a reserve price $r > \underline{c}$, the bidders with valuations not larger than r will bid as low as possible or will stay out, and will behave as if their valuations were equal to r. This case will be equivalent to the case where the valuations are distributed over the interval $[r, \bar{c}]$, with the lower extremity r of this interval which is a mass point of the distributions $F_1, F_2, \dots, F_n$.

After having observed his valuation v_i, bidder i can submit a bid $b_i \in \mathbb{R}$ at least as high as $\underline{c}$ or stay out. I thus assume that $\underline{c}$ is the reserve price.[8] I denote the decision of bidder i to stay out by[9] $b_i = OUT$. Bidder i wins the auction if his bid b_i is strictly larger than the bids submitted by the other bidders and his payoff is equal to $(v_i - b_i)$. If bidder i stayed out of the auction or if at least one other bidder has submitted a bid strictly larger than b_i, he is not awarded the item and his payoff is equal to zero. If bidder i and at least one other bidder have submitted the highest bid, then there is a tie which is solved by a fair lottery. I also assume that the bidders are risk neutral.

A strategy of bidder i tells him what bid probability measure he should use as a function of his valuation. I formally define the strategies as 'regular conditional probability distributions' ('stochastic kernels' or 'transition probability distributions' —for the formal definition see Appendix 4 of Lebrun 1997). It enables us to consider the expected values of random variables of interest, such as the bidders' payoffs. For v in $[\underline{c}, \bar{c}]$, I denote by $\beta_i(v, \cdot)$ the bid probability distribution, over $\{OUT\} \cup [\underline{c}, +\infty)$. I say that a strategy β_i is pure if and only if, for all v in $[\underline{c}, \bar{c}]$, $\beta_i(v, \cdot)$ is concentrated at one point that I denote by $\beta_i(v)$. In this case, I identify the strategy β_i with the bid function from $[\underline{c}, \bar{c}]$ to $\{OUT\} \cup [\underline{c}, +\infty)$, whose value at v is equal to $\beta_i(v)$, for all v in $[\underline{c}, \bar{c}]$.

An n-tuple of strategies $(\beta_1, \dots, \beta_n)$ forms a Bayesian equilibrium if and only if the bid probability distribution $\beta_i(v, \cdot)$ gives bidder i the highest possible expected payoff against the other bidders' strategies β_j, $j \neq i$, when his valuation is equal to v, for all $1 \leq i \leq n$ and for all v in $[\underline{c}, \bar{c}]$ (a more formal definition can be found in Lebrun 1997).

Theorem 1 below provides a characterization of all Bayesian equilibria.

THEOREM 1. *Under the assumptions of Section* 2, *an n-tuple of strategies* $(\beta_1, \dots, \beta_n)$ *is a Bayesian equilibrium of the first price auction if and only if the strategies* $\beta_1, \dots, \beta_n$ *are equal to pure strategies over* $(\underline{c}, \bar{c}]$, *and there exists* $\underline{c} < \eta < \bar{c}$, *such that the inverses* $\alpha_1 = \beta_1^{-1}, \dots, \alpha_n = \beta_n^{-1}$ *exist, are strictly increasing, and form a solution over* $(\underline{c}, \eta]$ *of the system of differential equations* (1)—*considered in the domain* $D = \{(b, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1} \mid \underline{c}, b < \alpha_i \leq \bar{c},$ *for all* $1 \leq i \leq n\}$—*and satisfy the boundary conditions* (2):

$$(1) \qquad \frac{d}{db}\alpha_k(b) = \frac{F_k(\alpha_k(b))}{(n-1)f_k(\alpha_k(b))}\left\{\frac{(-1)(n-2)}{\alpha_k(b)-b} + \sum_{\substack{l=1 \\ l \neq k}}^{n} \frac{1}{\alpha_l(b)-b}\right\}, \qquad 1 \leq k \leq n,$$

[8] Although this assumption is convenient, it is not necessary.

[9] For example, OUT can be any real number strictly smaller than $\underline{c}$.

(2) $\alpha_k(\eta)=\bar{c}$, *for all* $1\le k\le n$, *and* $\alpha_j(\underline{c})=\underline{c}$, *for all, but at most one, j between* 1 *and* n,

and the distributions $\beta_1(\underline{c},\cdot),\ldots,\beta_n(\underline{c},\cdot)$ *have their supports included in* $\{OUT,\underline{c}\}$, *and are such that* (3) *holds true*:

(3) *if there exists j such that* $\alpha_j(\underline{c})>\underline{c}$, *then* $F_i(\underline{c})>0$ *and* $\beta_i(\underline{c},\cdot)$ *is concentrated at OUT, for all* $i\neq j$, *and* $\beta_j(v,\cdot)$ *is concentrated at* $\underline{c}$, *for all* v *in* $(\underline{c},\alpha_j(c)]$.

In Theorem 1 above, and in what follows, $\alpha_j(\underline{c})$ denotes the value of the continuous extension of α_j at $\underline{c}$; that is, $\alpha_j(\underline{c})=\lim_{v \underset{>}{\to} \underline{c}}\alpha_j(v)$.

From Theorem 1, Bayesian equilibria can be obtained by taking the inverses of the solutions of the differential system (1). As I show in Lebrun (1997), the boundary conditions are different when bidding is mandatory. In this case, the continuous extensions of the inverses of the bid functions must all be equal to $\underline{c}$ at $\underline{c}$. Here, when bidding is voluntary and when there is a mass point at $\underline{c}$, there can be at most one bid function[10] such that the continuous extension of its inverse takes a value different from $\underline{c}$ at $\underline{c}$. However, when the distributions are atomless, (2) and (3) reduce to $\alpha_1(\underline{c})=\cdots=\alpha_n(\underline{c})=\underline{c}$. In all cases, all bid functions are strictly increasing over $(\max_{1\le k\le n}\alpha_k(\underline{c}),\bar{c}]$ and there exists η, such that $\alpha_k(\eta)=\bar{c}$ and thus $\beta_k(\bar{c})=\eta$ for all $1\le k\le n$, and η is the common value of the bid functions at the upper extremity $\bar{c}$ of the valuation interval.

The proof that an n-tuple of strategies satisfying the conditions given in Theorem 1 is an equilibrium is very short and can be found in Section A of the Appendix, as well as a sketch of the proof of the 'necessity part' of Theorem 1. It can be found in complete detail in Lebrun (1997). I will now explain briefly one of the main difficulties of this proof.

Once it is assumed that the equilibrium strategies are continuous bid functions, it is relatively simple to prove that they must be differentiable, and to obtain a differential system with boundary conditions they form a solution of. Without any assumption on the equilibrium strategies, it is readily obtained that the supports of the bid distributions depend monotonically on the valuations. However, after this property has been established, it is much simpler to rule out mixed and discontinuous strategies in the two bidder case than in the n bidder case, where $n>2$. In fact, in the two bidder case this is done by noticing that no 'jump' can occur in any bidder's equilibrium strategy. If a bidder's strategy displayed such a jump, then his opponent's equilibrium strategy would also display a jump, and the range of bids used by the bidders would be 'cut' by a nonempty open interval. It is easily seen that, if such are the strategies, then it is in the interest of one of the bidders to deviate.

However, a priori, such discontinuity jumps could already occur in the three bidder case. One bidder's equilibrium strategy could present a jump in which both

[10] From Corollary 3(i) in Section 3, I can show that if $F_n(x)\le F_j(x)$ for all i and x, only bidder n can have such a bid function, and only when (from (3)) $F_j(\underline{c})>0$ for all $i\neq n$ and no other bidder has the same valuation distribution as bidder n, or in other words, $F_n\neq F_i$ for all $i\neq n$. From Corollary 3(iv) and Corollary 6 in Lebrun (1997), I can show that if $\underline{c}$ is a mass point of all distributions, such a bid function can exist only when there are more than two different distributions.

other bidders bid with a certain strictly positive probability. These two bidders would prevent each other from bidding lower (see Figure 1, Section A of the Appendix). Let us first consider the possibility of such discontinuities and prove some of their properties—if they exist—before obtaining a contradiction.

3. EXISTENCE, UNIQUENESS, AND OTHER PROPERTIES OF THE EQUILIBRIA

I obtain the existence of a Bayesian equilibrium directly from the characterization given in Theorem 1 (Section 2).[11]

THEOREM 2. *Let the assumptions of Section* 2 *be satisfied. If* $F_1(\underline{c}) = \cdots = F_n(\underline{c}) = 0$, *or if the right-hand derivatives of* $F_1, \ldots, F_n$ *at* $\underline{c}$ *exist and are such that* $d/dv F_1 = f_1, \ldots, d/dv F_n = f_n$ *are bounded*[12] *away from zero over* $[\underline{c}, \bar{c}]$ *and* $F_1(\underline{c}), \ldots, F_n(\underline{c}) > 0$, *there then exists a Bayesian equilibrium of the first price auction.*

The proof of Theorem 2 is long but straightforward and can be found in Lebrun (1997). I outline it in Section B of the Appendix; here, I will give its main idea. Remember that from Theorem 1 (Section 2), the existence of a Bayesian equilibrium reduces to the existence of a parameter η for which there exists a solution $(\alpha_1, \ldots, \alpha_n)$ of (1,2,3). The system (1) considered in D is equivalent to the system (4)—considered in $\mathscr{D} = \{(b, \psi_1, \ldots, \psi_n) \in \mathbb{R}^{n+1} | F_i(\underline{c}),\ F_i(b) < \psi_i \leq 1, \text{ for all } 1 \leq i \leq n\}$—in the unknown functions $\psi_1 = F(\alpha_1), \ldots, \psi_n = F_n(\alpha_n)$:

$$(4) \qquad \frac{d}{db}\psi_k(b) = \frac{\psi_k(b)}{(n-1)}\left\{\frac{(-1)(n-2)}{F_k^{-1}(\psi_k(b)) - b} + \sum_{\substack{l=1\\ l\neq k}}^{n} \frac{1}{F_l^{-1}(\psi_l(b)) - b}\right\}, \qquad 1 \leq k \leq n.$$

Under the assumptions of Section 2, F_k^{-1} is locally Lipschitz over $(F_k(\underline{c}), 1]$ for all $1 \leq k \leq n$, and the system (4) thus satisfies over $\mathscr{D}$ the standard requirements of the theory of ordinary differential equations.

Under the boundary conditions (2), the system (4) presents a singularity at $\underline{c}$. In fact, $F_i^{-1}(\psi_i(\underline{c})) - c = \alpha_i(\underline{c}) - \underline{c} = 0$ for at least $n-1$ values of the index i, and there may exist $1 \leq j \leq n$ such that F_j^{-1} is not locally Lipschitz at $\underline{c}$. As a particular consequence, I cannot apply the classic theorems from the theory of ordinary differential equations to the system (4) with the initial condition at $\underline{c}$. Furthermore, the boundary conditions (2, 3) do not always provide me with a complete initial condition at $\underline{c}$. Rather I will consider the system (1), or its equivalent expression (4), with the initial condition (5) below:

$$(5) \qquad \alpha_k(\eta) = \bar{c}, \quad \text{or, equivalently,} \quad \psi_k(\eta) = 1, \qquad \text{for all } 1 \leq k \leq n.$$

[11] In Corollary 3(v) (Section 3), my existence results are extended somewhat in the symmetric case.

[12] In the case of the closed interval $[\underline{c}, \bar{c}]$, being locally bounded away from zero implies being 'uniformly' bounded away from zero.

For any parameter η, such that $\underline{c} < \eta < \bar{c}$, the system (4) does not present a singularity at this initial condition. I thus apply the theorems from the theory of ordinary differential equations to the problem (1,5) through the system (4). I prove Theorem 2 by proving the existence of a parameter $\underline{c} < \eta < \bar{c}$, for which the solution of the problem (1,5) consists of strictly increasing functions defined over $(\underline{c}, \eta]$, such that the conditions (2, 3) are satisfied. To this end, I first study the system (1) when there is no mass point. When $\underline{c}$ is a mass point of all distributions, I come back to the atomless case by extending all density functions to the left, and by considering a larger common support.

In Corollaries 1 and 2 below, I gather some first consequences of Theorem 2 and its proof. The uniqueness of the equilibrium when all distributions have a mass point at $\underline{c}$ can be easily proved (see Lebrun 1997) from the monotonicity of the solutions of (1,5) with respect to η (Lemma A2-8, Lebrun 1997) and equation (A.1) in Section A of the Appendix.

COROLLARY 1. *Let the assumptions of Section 2 be satisfied. If the right-hand derivatives of $F_1, \ldots, F_n$ at $\underline{c}$ exist and are such that $d/dv F_1 = f_1, \ldots, d/dv F_n = f_n$ are bounded away from zero over $[\underline{c}, \bar{c}]$ and $F_1(\underline{c}), \ldots, F_n(\underline{c}) > 0$, there then exists an 'essentially' unique Bayesian equilibrium $(\beta_1, \ldots, \beta_n)$ of the first price auction. Any equilibrium coincides with $(\beta_1, \ldots, \beta_n)$ over $(\underline{c}, \bar{c}]$ and any n-tuple of strategies which coincides with $(\beta_1, \ldots, \beta_n)$ over $(\underline{c}, \bar{c}]$ and which satisfies* (3) *is an equilibrium.*

The first statement of Corollary 2 below follows immediately from Theorem 2 and Corollary 1, since when there is a reserve price $r \in (\underline{c}, \bar{c})$, I consider that the valuations are distributed over $[r, \bar{c}]$, with the probability weights previously spread over $[\underline{c}, r]$, which are now concentrated at r. The second statement follows from the observation (Lemma A2-4, Lebrun 1997) made while proving Theorem 2 that when there is no mass point at $\underline{c}$, for every solution $(\alpha_1, \ldots, \alpha_n)$ of (1,5) that can be extended over $(\underline{c}, \eta]$[13] we have $\alpha_1(\underline{c}), \ldots, \alpha_n(\underline{c}) > \underline{c}$ or $\alpha_1(\underline{c}) = \cdots = \alpha_n(\underline{c}) = \underline{c}$. The third statement follows again from the property of monotonicity. Still more could be said about, for example, the continuity of the equilibrium with respect to the reserve price.

COROLLARY 2. *Let the assumptions of Section 2 be satisfied. (i). For all $r \in (\underline{c}, \bar{c})$, there exists an essentially unique Bayesian equilibrium of the first price auction with a reserve price equal to r. (ii). If $F_1(\underline{c}) = \cdots = F_n(\underline{c}) = 0$, and if $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction, then $\beta_1, \ldots, \beta_n$ are strictly increasing over $(\underline{c}, \bar{c}]$. (iii). If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium when the reserve price is r, and if $(\beta_1', \ldots, \beta_n')$ is a Bayesian equilibrium when the reserve price is r', with $\underline{c} \leq r < r' < \bar{c}$, then $\beta_i'(v) > \beta(v)$, for all v in $(r', \bar{c}]$.*

[13] In Section B of the Appendix and in Lebrun (1997), I refer to such solutions as type I solutions. This property of type I solutions implies that when there is no mass point at $\underline{c}$, an n-tuple of strategies is an equilibrium of the first price auction with voluntary bidding if and only if it is an equilibrium of the first price auction with mandatory bidding (see Corollary 3 in Lebrun 1997).

I now give some properties the Bayesian equilibria display when there exists a relation of stochastic dominance between valuation distributions. Again, these properties mainly follow from results I have already proved in the course of the proof of Theorem 2 in Section B of the Appendix (and in Lebrun 1997). Their complete proofs can be found in Lebrun (1997).

COROLLARY 3. *Let the assumptions of Section 2 be satisfied. Let* $(\beta_1, \ldots, \beta_n)$ *be a Bayesian equilibrium of the first price auction, and let i and j be two indices such that* $1 \le i, j \le n$. *(i). If* $F_j(v) \le F_i(v)$, *for all* v *in* $[\underline{c}, \bar{c}]$, *then we have* $F_j(\alpha_j(b)) \le F_i(\alpha_i(b))$, *for all b in* $[\underline{c}, \eta]$, *with* $\eta = \beta_1(\bar{c}) = \cdots = \beta_n(\bar{c})$; *or, equivalently,* $\beta_i(v) \le \beta_j(F_j^{-1}(F_i(v)))$, *for all* v *in* $(\underline{c}, \bar{c}]$. *(ii). If* F_i/F_j *is nonincreasing over* $(\underline{c}, \bar{c}]$, *then we have* $\beta_j(v) \le \beta_i(v)$, *for all* v *in* $(\underline{c}, \bar{c}]$. *(iii). If* $d/dv(F_i/F_j)(v) < 0$, *for all* v *in* $(\underline{c}, \bar{c}]$, *then we have* $\beta_j(v) < \beta_i(v)$, *for all* v *in* $(\underline{c}, \bar{c})$. *(iv). If* $F_i(v) = F_j(v)$, *for all* v *in* $[\underline{c}, \bar{c}]$, *then we have* $\beta_j(v) = \beta_i(v)$, *for all* v *in* $(\underline{c}, \bar{c}]$. *(v). If* $F_1 = \cdots = F_n = F$, *then we have* $\beta_1(v) = \cdots = \beta_n(v) = \beta(v) = v - \int_{\underline{c}}^{v} F^{n-1}(w)\,dw / F^{n-1}(v)$, *for all* v *in* $(\underline{c}, \bar{c}]$; *every n-tuple of strategies satisfying these equalities and such that the supports of* $\beta_1(\underline{c}, \cdot), \ldots, \beta_n(\underline{c}, \cdot)$ *are included in* $\{OUT, \underline{c}\}$ *is an equilibrium.*

Statement (i) of Corollary 3 tells us that the same relation of stochastic dominance passes from the valuation probability distributions to the bid probability distributions (for a related result in the case of two bidders, see Proposition 2.2(ii) in Maskin and Riley, 1996a). As can be easily seen, the assumption of (ii) is stronger than the assumption of (i). In addition to the competition from the other bidders, bidder j faces the competition from bidder i, who is likely to have only little interest in the item. Bidder i faces, likely, a more fierce competition from bidder j, and, under the assumption of (ii), bidder i bids higher. Statement (iii) is useful in the proof of Corollary 4. From (iv) two bidders whose valuations are identically distributed follow the same equilibrium strategy. From (v) the equilibrium $(\beta_1, \ldots, \beta_n)$ is essentially unique when the valuations are identically distributed. Corollary 3(v) thus extends this uniqueness in the set of symmetric n-tuples of (pure) strategies, proved by Riley and Samuelson (1981), to the set of all (symmetric and asymmetric) n-tuples of strategies.

In my final corollary, I obtain a uniqueness result in the atomless case for the class of asymmetric n-tuples of distributions $(F_1, \ldots, F_n)$, for which every bidder's probability distribution is one of two distributions. Without loss of generality for such an n-tuple I can assume that there exist $1 \le m \le n$, G_1 and G_2 such that:

$$(6) \qquad F_i = G_1, \qquad \text{for all } 1 \le i \le m, \quad \text{and} \quad F_i = G_2, \qquad \text{for all } m < i \le n.$$

Simple considerations of collusion, from a symmetric setting, lead to n-tuples in this class. Assume that initially the bidders' valuations are identically distributed according to F. Suppose that $k > 1$ bidders collude into one surplus maximizing cartel, with perfect information about its members' valuations, and perfect control over their actions. Since when it wins, the cartel will allocate the item to its member with the highest valuation, it is equivalent to a single bidder whose valuation is the maximum of k independent, random variables distributed according to F. I thus obtain[14] an asymmetric situation where one bidder's valuation is distributed according to $G_2 = F^k$ and the other bidders' valuations are distributed according to $G_1 = F$.

In the previous example, if the distribution function F is differentiable with a strictly positive derivative over $(\underline{c}, \bar{c}]$, the assumption (7) is satisfied:

$$\frac{d}{dv}\frac{G_1}{G_2}(v) < 0, \qquad \text{for all } v \text{ in } (\underline{c}, \bar{c}]. \tag{7}$$

In Corollary 4, I show that the equilibrium is unique under the assumption of the stochastic dominance relation (7) between atomless distributions G_1 and G_2. The examples studied by Marshall et al. (1994) satisfy this requirement; thus the equilibria obtained by these authors were unique equilibria.

If (6) holds true, Corollary 3(iv) implies that any equilibrium is determined by only two bid functions: β_1' and β_2', used by the bidders whose valuations are distributed according to G_1 and G_2, respectively. The system (1) is thus reduced to a system of two equations in the unknown functions $\alpha_1' = \beta_1'^{-1}$ and $\alpha_2' = \beta_2'^{-1}$. If I divide these two equations by each other and simplify, I see that the differences $(\alpha_1'(b) - b)$ and $(\alpha_2'(b) - b)$ appear only in a quotient of two polynomials of degree one. For this reason, it is advantageous in the proof to consider the differential system the functions $\phi_{21}' = \alpha_2' \beta_1'$ and β_1' form a solution of. The complete proof of Corollary 4 can be found in Lebrun (1997). I sketch it here in Section C of the Appendix. Note that Corollary 4 does not require any strengthening at $\underline{c}$ of the regularity conditions of Section 2.

COROLLARY 4. *Let the assumptions of Section 2 be satisfied. Assume that there exist $1 \le m \le n$ and two distributions G_1, G_2 absolutely continuous over $[\underline{c}, \bar{c}]$, such that (6) and (7) hold true. There then exists an essentially unique equilibrium of the first price auction.*

4. CONCLUSION

Without assumption of symmetry, and with an arbitrary number n of bidders, I obtained a characterization of the Bayesian equilibria of the first price auction game. Proceeding directly from this characterization, I proved the existence of a Bayesian equilibrium. I proved inequalities between equilibrium strategies when there exist relations of stochastic dominance between valuation distributions; as a consequence of these inequalities, two bidders have the same equilibrium strategy if their valuations are identically distributed. When the distributions have a mass point at the lower extremity of the support, I prove the uniqueness of the equilibrium. When there are no more than two different valuation distributions and when there exists a relation of stochastic dominance between them, I proved the uniqueness of

[14] Notice that I would still obtain an n-tuple of distributions from the class I consider in this paragraph if several cartels of the same size m formed, or if all bidders colluded into cartels of two different sizes.

the Bayesian equilibrium in the atomless case. The valuation distributions that result from a symmetric situation, after some bidders have colluded, satisfy these assumptions.

APPENDIX

A. *Proof of the 'Sufficiency Part' of Theorem* 1 (*Section* 2). One can immediately see that if $\alpha_1, \alpha_2, \dots, \alpha_n$ satisfy (1), then one has

$$\frac{d}{db} \sum_{\substack{j=1 \\ j \neq i}}^{n} \ln F_j(\alpha_j(b)) = \frac{1}{\alpha_i(b) - b}, \tag{A.1}$$

for all b in $(\underline{c}, \eta]$ and all $1 \leq i \leq n$. I have to prove that $\beta_i(v, \cdot)$ maximizes bidder i's payoff when the other bidders bid according to β_j, $j \neq i$, for all v in $[\underline{c}, \bar{c}]$ and all $1 \leq i \leq n$. It is easily seen that a bid larger than η is never a best response. It can also be checked that if $v = \underline{c}$, bidding in $\{OUT, \underline{c}\}$ is a best response. The probability distribution $\beta_i(\underline{c}, \cdot)$ is thus a best response.

Suppose then that $v > \underline{c}$. Since $b = (v + \underline{c})/2$ gives a strictly positive expected payoff, bidding $b > v$ can never be a best response, and bidding $b = \underline{c}$ is not a best response when $i \neq j$, where j is as in (3). Bidder i's expected payoff if he bids $b \in (\underline{c}, \eta]$ is equal to $(v - b)\prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(\alpha_j(b))$ and is strictly positive. When $i = j$ as in (3), this product is strictly positive and continuous at $b = \underline{c}$ and it again gives bidder i's expected payoff if $b = \underline{c}$. Since it is strictly positive, one can consider its logarithm. From (A.1), the derivative of this logarithm is equal[15] to $(-1)/(v - b) + 1/(\alpha_i(b) - b)$, for $v > \underline{c}$ and $b < v$. Since α_i is strictly increasing over $[\underline{c}, \eta]$ and such that $\alpha_i(\beta_i(v)) = v > \beta_i(v)$ when $\beta_i(v) > \underline{c}$, one can see that this derivative is strictly positive for $\underline{c} < b < \beta_i(v)$. Since $\alpha_i(\beta_i(v)) \geq v$ (it is equal except when $i = j$ as in (3) and $v < \alpha_i(\underline{c})$), for all v in $(\underline{c}, \bar{c}]$, and α_j is strictly increasing, one can see that the derivative above is strictly negative for $v > b > \beta_i(v)$. Consequently, the global maximum of bidder i's expected payoff is obtained at $b = \beta_i(v)$, and the suffciency part of Theorem 1 is proved. □

Next, I give the main steps of the proof of the necessity part of Theorem 1 (Section 2). The complete proof can be found in Lebrun (1997). I use arguments which are now standard in the study of auctions (see Griesmer et al., 1967) and of other games (see Baye, et al., 1992). I also use arguments from the theory of incentive compatible mechanisms (see Myerson 1981).

[15] When $b = \underline{c}$ and $i = j$ as in (3), I use the fact that if a function f is continuous over an interval $[a, b]$, and is differentiable over the interval (a, b), and if the limit of the derivative of f at x for $x \underset{>}{\to} a$ exists, then the function f is differentiable on the right at a, and the right-hand derivative at a is equal to the limit of the derivatives $f'(x)$.

Proof (Outline) of the 'Necessity Part' of Theorem 1 (*Section* 2). Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium. The strategy β_j of bidder i and the valuation probability distribution F_i determine a probability measure $\beta_i * F_i$ over the product $[\underline{c}, \bar{c}] \times (\{OUT\} \cup [\underline{c}, +\infty))$ of the set of possible valuations with the set of possible actions. I denote by $[\beta_i * F_i]_2$ the marginal distribution of $\beta_i * F_i$ over the second component space. This marginal distribution should be interpreted as the 'ex ante' probability distribution of the bid from bidder i prior to the choice by Nature of bidder i's valuation. I denote by b_i the random variable, whose probability measure is $[\beta_i * F_i]_2$, and by $b_i(v_i)$ the random variable, whose probability measure is $\beta_i(v_i, \cdot)$. I denote by $P(i|v)$ the expected payoff of bidder i when his valuation is equal to v; by $P(i|v,b)$ the payoff of bidder i if his valuation is equal to v and if his bid is equal to b; by Prob(i wins$|v$) the probability that bidder i wins if his valuation is equal to v; and by Prob(i wins$|b$) the probability that bidder i wins if his bid is equal to b. Thus, $P(i|v,b) = (v-b)\,\mathrm{Prob}(i \text{ wins}|b)$ when $b \neq OUT$.

I define the two functions b_{il} and b_{iu} as follows:

$$b_{il}(v) = \inf\{b \in [\underline{c}, +\infty) \,|P(i|v) = P(i|v,b)\}, \tag{A.2}$$

$$b_{iu}(v) = \sup\{b \in [\underline{c}, +\infty) \,|P(i|v) = P(i|v,b)\}. \tag{A.3}$$

Notice that since a bidder gets a zero payoff when he does not take part in the auction and when he submits a bid equal to his valuation, the sets in the definitions (A.2) and (A.3) are always nonempty.

I prove in Lemma A1-1 (Lebrun 1997) that every bidder's equilibrium payoff $P(i|v)$ is strictly positive and thus so is also his probability of winning Prob(i wins$|v$) when his valuation v is strictly larger than $\underline{c}$. It is then not difficult to prove that, at a Bayesian equilibrium, $b_{1l}(\underline{c}) = \cdots = b_{nl}(\underline{c}) = b_{1u}(\underline{c}) = \cdots = b_{nu}(\underline{c}) = \underline{c}$ (Lemma A1-3, Lebrun 1997). Moreover, I show in Lemma A1-9 (Lebrun 1997) that the functions b_{il} and b_{iu} take strictly larger values than $\underline{c}$ over $(\underline{c}, \bar{c}]$, for all i except possibly one. If $b_{jl}(v) = \underline{c}$ for some $v > \underline{c}$ and j then $F_i(\underline{c}) > 0$, for all $i \neq j$, and there exists $w' > \underline{c}$ such that $b_{jl}(v) = \underline{c}$, for all v in $[\underline{c}, w']$, and $b_{jl}(v) > \underline{c}$, for all v in $(w', \bar{c}]$. One can easily understand why there cannot be such a w' for more than one bidder. If this was the case, then with strictly positive probability, there would be a tie at $\underline{c}$ and the bidders bidding $\underline{c}$ for valuations larger than $\underline{c}$ would be better off if they bid slightly higher instead.

It is rather straightforward to prove (see Lemma A1-8, Lebrun 1997) the following 'monotonicity' property of the two functions b_{il} and b_{iu}: $b_{iu}(v) \leq b_{il}(v')$, for all v, v' such that $\underline{c} \leq v < v' \leq \bar{c}$. This property implies, in particular, that both functions b_{il} and b_{iu} are nondecreasing (Lemma A1-11, Lebrun 1997). A useful property of the equilibrium strategies $\beta_1, \ldots, \beta_n$, which is not much more difficult to establish in the n bidder case than in the two bidder case, is that the probability distributions $[\beta_1 * F_1]_2, \ldots, [\beta_n * F_n]_2$ have no mass point $b > \underline{c}$ (see Lemma A1-7, Lebrun 1997). As a byproduct, one can see that if $b > \underline{c}$, Prob(i wins$|b$) is equal to Prob($b_j = OUT$ or $b_j \leq b$, for all $j \neq i) = \prod_{j \neq i}[\beta_n * F_n]_2(\{OUT\} \cup [\underline{c}, b])$ and is a continuous function of $b > \underline{c}$, and thus $P(i|v,b) = (v-b)\,\mathrm{Prob}(i \text{ wins}|b)$ is a continuous function of v and $b > \underline{c}$. Consequently, in the definitions of b_{il} and b_{iu}, one can substitute 'min' and 'max' for 'inf' and 'sup', respectively (see Lemma A1-10, Lebrun 1997).

By comparing the bidders' behaviors at $\bar{c}$, I also prove that $b_{1l}(\bar{c}) = \cdots = b_{nl}(\bar{c}) = b_{1u}(\bar{c}) = \cdots = b_{nu}(\bar{c}) < \bar{c}$ (Lemma A1-12, Lebrun 1997). This sequence of equalities implies that every bidder submits the same bid when his valuation is equal to the upper extremity $\bar{c}$ of the valuation interval. I denote this common bid by η.

I show (Lemma A1-4, Lebrun 1997) that the expected payoff $P(i|v)$ of bidder i, conditional on the valuation v, is a continuous function of v in $[\underline{c}, \bar{c}]$, for all $1 \leq i \leq n$. As a consequence (see Lemma A1-13, Lebrun 1997), the functions b_{il} and b_{iu} are continuous from the left and from the right respectively, for all $1 \leq i \leq n$. Moreover, b_{iu} can be obtained from b_{il} by taking the limit from the right of b_{il}. Similarly, b_{il} is equal to the limit from the left of b_{iu}. One can then see that the functions $b_{1l}, \ldots, b_{nl}, b_{1u}, \ldots, b_{nu}$ are strictly increasing when they are larger than $\underline{c}$ (see Lemma A1-14, Lebrun 1997). For example, if b_{il} was equal to a constant larger than $\underline{c}$ on a nondegenerate interval, it would be continuous and thus equal to b_{iu} over this interval, and bidder i would bid the same bid when his valuation belongs to this interval. This bid would then be a mass point of the bid probability distribution, which is impossible at an equilibrium, as seen earlier. Figure 1 shows how these functions may look according to what is known so far.[16]

Next, I give some intuition about how I rule out discontinuities. Imagine that $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium. I want to show that the equilibrium strategies are pure and that the bid functions are continuous. This will be done if I show that the functions $b_{1l}, \ldots, b_{nl}$ are continuous. Because $b_{1l}, \ldots, b_{nl}$ are strictly increasing, if one of them is discontinuous at a valuation v, the discontinuity is of the 'jump' kind.

I first rule out situations like those in Figure 1, where bidder i's b_{il} jumps at v and where other bidders bid within the jump for valuations strictly smaller than v. If bidder $j \neq i$ submits b at w, the cost of any change from b, and in particular the

[16] The graphs of the functions b_{il} may cross each other. In the diagrams I represented simple cases where they do not and also where $F_i(\underline{c}) = 0$ for all i.

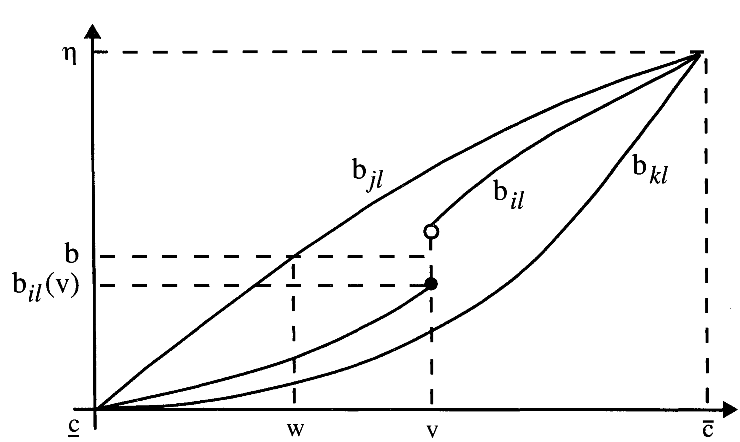

FIGURE 1

change to $b_{il}(v)$, must outweigh its benefit. Note that maximizing the expected payoff $(w-b)\mathrm{Prob}(j \text{ wins}|b)$ (which, under my assumptions, is strictly positive) is equivalent to maximizing its logarithm $\ln(w-b)+\ln \mathrm{Prob}(j \text{ wins}|b)$. Thus the percentage decrease of the probability of winning, that is, the decrease of the term $\ln \mathrm{Prob}(j \text{ wins}|b)$ due to a decrease of his bid to $b_{il}(v)$, must be at least as large as the percentage increase of payoff in case of winning; in other words the increase in the term $\ln(w-b)$, and I find (by using obvious notations):

$$|\Delta \ln \mathrm{Prob}(j \text{ wins})| \geq |\Delta \ln(w-b)|. \tag{A.4}$$

Bidder i's maximal expected payoff is reached at $b_{il}(v)$. Thus, if he increases his bid to b, the percentage decrease in his payoff if he wins is not smaller than the percentage increase of his probability of winning; that is:

$$|\Delta \ln(v-b)| \geq |\Delta \ln \mathrm{Prob}(i \text{ wins})|. \tag{A.5}$$

However, the percentage change of bidder i's probability of winning is larger than the percentage change of bidder j's probability of winning; that is:

$$|\Delta \ln \mathrm{Prob}(i \text{ wins})| \geq |\Delta \ln \mathrm{Prob}(j \text{ wins})|. \tag{A.6}$$

In fact, bidder i has to take into account the increase in the probability of losing the auction to bidder j. On the other hand, the probability that bidder j looses the auction to bidder i does not change when bidder j decreases his bid to $b_{il}(v)$.[17] As a consequence, we see from (A.4), (A.5), and (A.6) that the percentage change in bidder i's payoff in case of winning, when he increases his bid from $b_{il}(v)$ to b, must be at least as large as the percentage change in bidder j's payoff in case of winning when bidder j decreases his bid from b to $b_{il}(v)$; that is, $|\Delta \ln(v-b)| \geq |\Delta \ln(w-b)|$. Since the absolute changes in the payoffs are given by the difference between the two bids, and are thus equal, we can see that the only way this is possible is if $v \leq w$, and an example such as that in Figure 1 is impossible.

Before ruling out the only remaining possible case of discontinuity, the following result is needed. I prove in Lemma A1-18 (Lebrun 1997) that when b_{il} is continuous over a neighborhood of v, the probability $\mathrm{Prob}(i \text{ wins}|b)$ is a differentiable function of b over a neighborhood of $b_{il}(v)$. Consequently, I can simply take the derivative, with respect to b, of the objective function (in its logarithmic form) $\ln(v-b)+\ln \mathrm{Prob}(i \text{ wins}|b)$ and set this derivative equal to zero at the best choice of bidder i. I find the equation:

$$\frac{\frac{d}{db}\mathrm{Prob}(i \text{ wins}|b)}{\mathrm{Prob}(i \text{ wins}|b)} = \frac{1}{(v-b)}, \tag{A.7}$$

which holds true at $b = b_{il}(v)$, and obtain the mathematical expression for the equality of the 'marginal benefit' of a change in the bid with its 'marginal cost'.

The only possible type of equilibria with discontinuities still to be examined is the type shown in Figure 2.

[17] This statement is correct when $b_{il}(v) > \underline{c}$. When $b_{il}(v) = \underline{c}$, it suffices in the proof to consider the limit $b' \underset{>}{\to} b_{il}(v)$ (see Lemma A1-24 and the proof of Lemma A1-22 in Lebrun 1997).

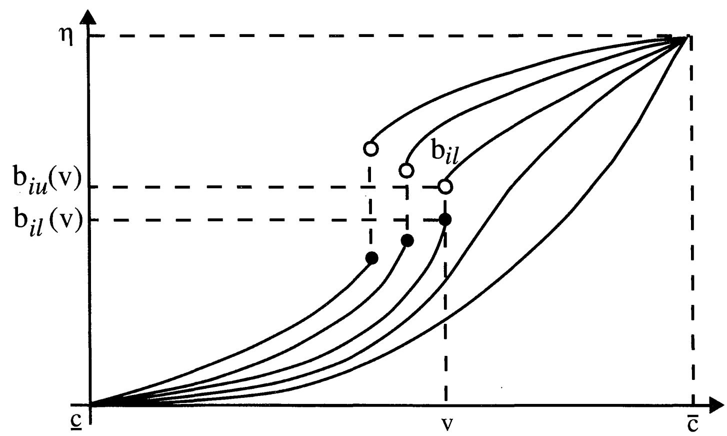

FIGURE 2

In this example, bidder i's b_{il} is discontinuous at v and all bidders k bidding within the discontinuity jump do so for valuations not smaller than v. Moreover, I have assumed in this example that these latter bidders k have their functions b_{kl} continuous over the ranges of valuations where they bid inside the jump. If this was not true, a function b_{kl}, $k \neq i$, would exhibit a jump included in the jump of b_{il}, and I would focus on b_{kl} instead of b_{il}. If necessary by repeating this argument, one can see that my assumption does not imply any loss of generality. Assume that the bidders k bidding continuously within the jump are bidders $i+1, \ldots, n$. The bidders $1, \ldots, i-1$ have a discontinuity jump including b_{jl}'s jump at v.

When bidder k's b_{kl} is strictly increasing within a certain neighborhood of valuations, then bidder k's marginal cost of changing his bid is equal to his marginal benefit (see equation (A.7)). As a consequence, equation (A.7) holds for all b in $(b_{il}(v), b_{iu}(v))$ and for all bidders k, with $k \geq i+1$. Taking the limit for b tending towards the lower extremity $b_{il}(v)$, one can see that the same equation also holds at $b_{il}(v)$, if the derivative is interpreted as a right-hand derivative. Similarly, the equation holds at $b_{iu}(v)$ when the derivative is the left-hand derivative (see footnote 15 for a property similar to the one used here).

For bids b from bidder k inside the jump of b_{il}, the probability that bidders $1, \ldots, i$ bid lower is constant since it is equal to the probability that all these bidders do not bid larger than $b_{il}(v)$. I thus write $\ln \text{Prob}(k \text{ wins} \mid b)$ as follows: $\ln \text{Prob}(k \text{ wins} \mid b) = C + \Sigma_{j=i+1}^{n} \ln \text{Prob}(b_j \leq b)$, where C is a constant, for all $k \geq i + \text{i}$. Summing these equalities and dividing by $(n-i-1)$, I find the equality $1/(n-i-1)\Sigma_{k=i+1}^{n} \ln \text{Prob}(k \text{ wins} \mid b) - K = \Sigma_{\substack{j=i+1 \\ j \neq k}}^{n} \ln \text{Prob}(b_j \leq b)$, where K is also a constant. Reasoning as in the beginning of this paragraph, see that up to an additive constant, the R.H.S. of the equality above is nothing but $\ln \text{Prob}(i \text{ wins} \mid b)$. I thus obtain the equality $\ln \text{Prob}(i \text{ wins} \mid b) = 1/(n-i-1)\Sigma_{k=i+1}^{n} \ln \text{Prob}(k \text{ wins} \mid b) + L$, where L is a constant, for all b in $[b_{il}(v), b_{iu}(v)]$. Taking the derivative of the last equality and using equation (A.7), which holds for bidders $i+1, \ldots, n$, one can see that $d/db \ln \operatorname{Prob}(i \text{ wins} \mid b)$ exists and find:

$$\text{(A.8)} \qquad \frac{d}{db} \ln \operatorname{Prob}(i \text{ wins} \mid b) = \frac{1}{(n-i-1)} \sum_{k=i+1}^{n} \frac{1}{\alpha_k(b) - b}$$

for all b in $[b_{il}(v), b_{iu}(v)]$, where the derivative at $b_{il}(v)$ is a right-hand derivative, the derivative at $b_{iu}(v)$ is a left-hand derivative, and where α_k is the inverse of b_{kl}, or, $\alpha_k = b_{kl}^{-1}$.

Bidder i with valuation v reaches his maximum expected payoff when he bids $b_{il}(v)$. Consequently, the marginal percentage increase of probability $d/db \ln \operatorname{Prob}(i \text{ wins} \mid b_{il}(v))$ when he increases his bid, must be offset by the corresponding marginal percentage decrease $1/(v - b_{il}(v))$ of his payoff if he wins. I thus obtain $d/db \ln \operatorname{Prob}(i \text{ wins} \mid b_{il}(v)) \leq 1/(v - b_{il}(v))$. Using equation (A.8) and rearranging, I find:

$$\text{(A.9)} \qquad \sum_{k=i+1}^{n} \frac{v - b_{il}(v)}{\alpha_k(b_{il}(v)) - b_{il}(v)} \leq (n-i-1).$$

Similarly, because the maximum expected payoff of bidder i with valuation v is also reached at $b_{iu}(v)$, I find:

$$\text{(A.10)} \qquad (n-i-1) \leq \sum_{k=i+1}^{n} \frac{v - b_{iu}(v)}{\alpha_k(b_{iu}(v)) - b_{iu}(v)}.$$

However, as it can be easily checked (see the proof of Lemma A1-25, Lebrun 1997), the functions $(v-b)/(\alpha_k(b)-b)$ are strictly decreasing functions of b over the domain $v > b$ and $\alpha_k(b) \geq v$. Since $b_{il}(v) < b_{iu}(v)$ and $\alpha_k(b_{il}(v)) \geq v$, I have $\sum_{k=i+1}^{n}(v - b_{il}(v))/(\alpha_k(b_{il}(v)) - b_{il}(v)) > \sum_{k=i+1}^{n}(v - b_{iu}(v))/(\alpha_k(b_{iu}(v)) - b_{iu}(v))$, which contradicts (A.9) and (A.10). I have ruled out the only possible type of discontinuity (see Figure 2) in the equilibrium strategies, and consequently, the equilibrium strategies have to be continuous bid functions.

Once it is known that the equilibrium strategies are continuous, the differentiability over $(\underline{c}, \eta]$ of the inverses $\alpha_1 = \beta_1^{-1}, \ldots, \alpha_n = \beta_n^{-1}$ follows from the already mentioned Lemma A1-18 (Lebrun 1997). The system (1) (Section 2) is simply obtained by solving for $d/db\, \alpha_k(b)$, $1 \leq k \leq n$, the equations (A.7), with $1 \leq i \leq n$, where $\prod_{k \neq i} F_k(\alpha_k(b))$ has been substituted for $\operatorname{Prob}(i \text{ wins} \mid b)$ (see Lemmas A1-16 and A1-25, Lebrun 1997).

B. *Outline of the Proof of Theorem* 2 (*Section* 3). First assume that $F_1(\underline{c}) = \cdots = F_n(\underline{c}) = 0$. In Lebrun (1997), I prove (Lemma A2-2) that for every $\underline{c} < \eta < \bar{c}$, the solution of (1,5) in the domain D consists of strictly increasing functions. I then look at the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of (1,5) over $(\underline{c}, \eta]$: that is, according to the terminology from Birkhoff and Rota (1978, p. 162) the solution of (1,5) that cannot be defined over a larger subinterval of $(\underline{c}, \eta]$ and still be a solution of (1,5) in the domain D. Following Pontryagin (1962, p. 21), I refer to the definition interval $(\underline{\gamma}, \eta] \subseteq (\underline{c}, \eta]$ of the maximal solution as the maximal interval of existence, or simply as the maximal interval. I prove that only two cases are possible. In the first case, the maximal interval is equal to the whole $(\underline{c}, \eta]$; in other words, $\underline{\gamma} = \underline{c}$. In this case,

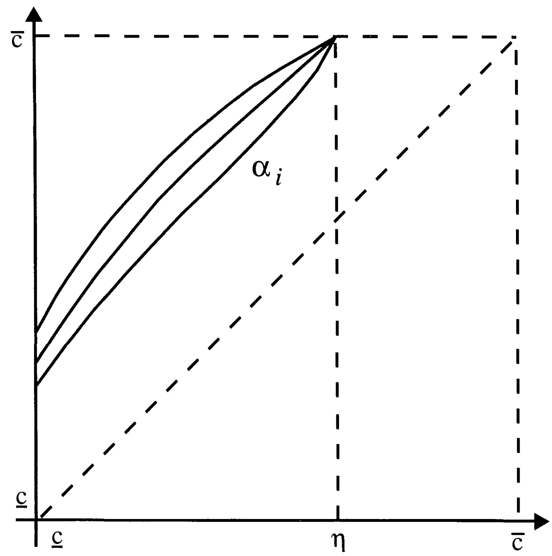

FIGURE 3

I have (Lemma A2-4, Lebrun 1997) either $\alpha_1(\underline{c}),\ldots,\alpha_n(\underline{c}) > \underline{c}$ or $\alpha_1(\underline{c}) = \cdots = \alpha_n(\underline{c}) = \underline{c}$ (see Figure 3). I refer to this solution as a type I solution.

In the second case, the maximal interval is a subinterval $(\underline{\gamma}, \eta]$ strictly smaller than $(\underline{c}, \eta]$; or, $\underline{\gamma} > \underline{c}$. In this case, I show (Lemma A2-7, Lebrun 1997) that all functions α_i, except possibly one, are such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$ (see Figure 4). I refer to this solution as a type II solution.[18]

An important property of the system (1) is that the solution $(\alpha_1,\ldots,\alpha_n)$ of the problem $(1,5)$ depends monotonically on η (Lemma A2-8, Lebrun 1997); that is, if $\eta' > \eta$ and if $(\alpha_1,\ldots,\alpha_n)$ is the solution corresponding to η and $(\alpha'_1,\ldots,\alpha'_n)$, the solution corresponding to η', then $\alpha'_i \leq \alpha_i$, over the common definition domain of $(\alpha'_1,\ldots,\alpha'_n)$ and $(\alpha_1,\ldots,\alpha_n)$. Furthermore, I prove (Lemma A2-13, Lebrun 1997) that when η tends towards $\bar{c}$ the corresponding solution is of type II and $\underline{\gamma}$ tends towards $\bar{c}$, and that a solution corresponding to η close to $\underline{c}$ is of type I. By using continuity arguments (see Lemma A2-12 and the proof of Theorem 3 in Lebrun 1997), I then show that there exists η, such that the solution of $(1,5)$ is such that $\alpha_1(\underline{c}) = \cdots = \alpha_n(\underline{c}) = \underline{c}$. This solution is thus also a solution of the boundary value problem $(1,2,3)$; and, by Theorem 1 (Section 2), it corresponds to a Bayesian equilibrium, and Theorem 2 is proved when $F_1(\underline{c}) = \cdots = F_n(\underline{c}) = 0$.

Assume now that the right-hand derivatives of $F_1,\ldots,F_n$ at $\underline{c}$ exist and are such that $d/dv\,F_1 = f_1,\ldots,d/dv\,F_n = f_n$ are bounded away from zero over $[\underline{c},\bar{c}]$, and $\underline{c}$ is a mass point of all distributions $F_1,\ldots,F_n$. The existence can now be proved simply by extending the density functions (for example, in a piecewise linear way) to an

[18] A type II solution whose definition interval is $(\underline{\gamma},\bar{c}]$ can be interpreted as a Bayesian equilibrium of the first price auction with a reserve price equal to $\underline{\gamma}$.

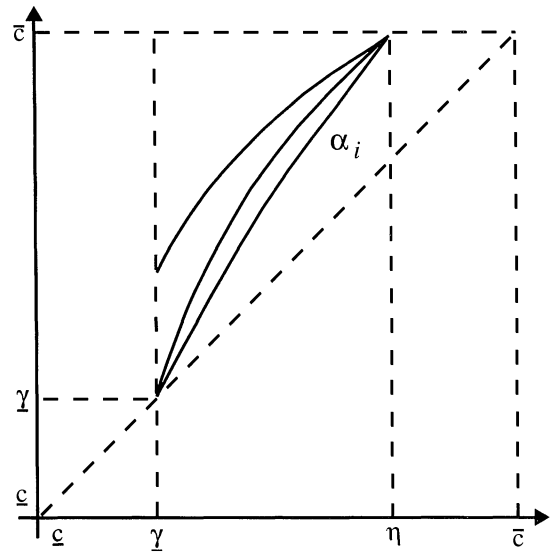

FIGURE 4

interval $[\underline{c}_0, \bar{c}]$, with $\underline{c}_0 < \underline{c}$, in a such a way that they define new atomless probability distributions. From the continuity (Lemma A2-13, Lebrun 1997) of the lower extremity $\underline{\gamma}$ of the maximal interval with respect to η, one can see that there exists η such that the corresponding $\underline{\gamma}$ is equal to $\underline{c}$. The solution of (1,5) with this η is a type II solution, and therefore the initial condition (2) is immediate. I also prove (Lemma A2-7, Lebrun 1997) that the condition (3) is satisfied. This value of η thus determines an equilibrium and the proof of Theorem 2 is complete.[19]

C. *Proof* (*Outline*) *of Corollary* 4 (*Section* 3). As explained in Section 3, the proof in Lebrun (1997) of Corollary 4 proceeds by considering the system of differential equations the functions ϕ'_{21} and β'_1 satisfy. If there is an equilibrium that differs over $(\underline{c}, \bar{c}]$, the corresponding functions $\tilde{\phi}'_{21}$ and $\tilde{\beta}'_1$ will form another solution of this system. Let η and $\tilde{\eta}$ be the values of the parameter in (5) that correspond to the two equilibria. Without loss of generality,[20] one can assume that $\tilde{\eta} < \eta$. By examination of the differential system, and from the property of monotonicity (Lemma A2-8 in Lebrun 1997), the assumption of stochastic dominance, Corollary 3(iii), and a variant (Lemma A5-1 in Lebrun 1997) of Lemma 2 in Milgrom and Weber (1982), I obtain that if $\tilde{\phi}'_{21} > \phi'_{21}$ at one point v, then $\tilde{\phi}'_{21} > \phi'_{21}$ over $(\underline{c}, v]$. The differential equations and the assumption of stochastic dominance imply that the $d/d \ln G_1 \ln G_2(\tilde{\phi}'_{21})(\bar{c}) = d/d \ln G_1 \ln G_2(\phi'_{21}(\bar{c}) = 1$ and that the derivatives

[19] Bounds on such a η are given in Lemma A2-13 in Lebrun (1997).

[20] As it is the case of the system (1), the system in the functions ϕ'_{21} and β'_1 is equivalent to a system satisfying the standard requirements of the theory of ordinary differential equations, which ensure the existence and uniqueness of the solution when the initial condition is in the domain.

$d^2/d(\ln G_1)^2 \ln G_2(\tilde{\phi}'_{21})(\bar{c})$ and $d^2/d(\ln G_1)^2 \ln G_2(\phi'_{21})(\bar{c})$ exist and are such that $d^2/d(\ln G_1)^2 \ln G_2(\tilde{\phi}'_{21})(\bar{c}) > d^2/d(\ln G_1)^2 \ln G_2(\phi'_{21})(\bar{c})$. Thus, $\tilde{\phi}'_{21} > \phi'_{21}$ over an nonempty open interval with $\bar{c}$ as its upper extremity. Consequently, $\tilde{\phi}'_{21} > \phi'_{21}$ over $(\underline{c}, \bar{c})$.

From Lemma A2-6 in Lebrun (1997), the equality

$$d/dv\{(v - \beta'_2(v))G_1^m(\phi'_{12}(v))G_2^{(n-m-1)}(v)\} = G_1^m(\phi'_{12}(v))G_2^{(n-m-1)}(v),$$

where $\phi'_{12} = \phi'^{-1}_{21}$, holds true for both equilibria over $(\underline{c}, \bar{c}]$. By integrating this equality over this interval, I find $\int_{\underline{c}}^{\bar{c}} G_1^m(\tilde{\phi}'_{12}(v))G_2^{(n-m-1)}(v)\,dv = \bar{c} - \tilde{\eta}$ and $\int_{\underline{c}}^{\bar{c}} G_1^m(\phi'_{12}(v))G_2^{(n-m-1)}(v)\,dv = \bar{c} - \eta$. From the previous paragraph, $\tilde{\phi}'_{21} > \phi'_{21}$ and thus $\tilde{\phi}'_{12} < \phi'_{12}$, and the first integral should be strictly smaller than the second one. However, this conclusion contradicts my initial assumption $\tilde{\eta} < \eta$ and Corollary 4 is proved.

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