HW3: Computational Complexity

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Q. Given the following definitions:

- 1) A subgraph *G'* of a graph *G* consists of a subset of vertices and edges of *G*. A subgraph *G'* is complete if there is an edge between any pair of vertices of *G'*.
- 2) An independent set of vertices of a graph *G* is defined as a set *S* of vertices of *G* such that there is no edge between any two vertices of *S*.

Show that finding a complete subgraph of *G* with *k* vertices is NP-complete by reducing from the *q*-Indepedent Set problem (i.e., finding an independent set with *q* vertices).

Note: you should first present the problem in its decision version and then prove the NP-completeness.

Ans.

First show that the Clique problem is NP. Given a Graph G(V,E), If we provide the set of vertices $V' \subseteq V$ as a certificate of a clique in G, then we can check if V' induces a clique in polynomial time. We can check whether each pair of vertices $(x, y) \epsilon V'$ also belongs in E(G). This can be done in $o(|V'|^2)$ time.

To show Clique problem is NP-Complete, we reduce it from Independent set problem, i.e., $Indp \leq_P Clique$. We assume that Finding independent set in a graph is NP-hard.

Take an instance of Independent Set Problem, Graph G(V,E) and q. Decision problem is - does there exists a Independent Set of size q in G? We see that G has an Independent Set of Size q iff G^c has a clique of size q. Construct the Graph G^c as follows. Let $y \epsilon V$. For each vertex $x \epsilon V$ and if $(x, y) \notin E(G)$, then draw edge $(x, y) \epsilon E(G^c)$. If $(x, y) \epsilon E(G)$, then we do not draw any edge in $E(G^c)$. We look at $|V|^2$ such pairs (x, y). Hence we can construct G^c in polynomial time.

Instance of the Clique problem - $G^c(V, E')$ and q. Does there exists a clique of size q?

We have reduced the Independent Set Problem to Clique Problem in Polynomial time. If we can find answer to clique problem in polynomial time, then essentially, we would have found answer to Independent Set Problem in polynomial time, but Independent Set Problem is NP-hard. Hence Clique Problem must also NP-hard. We have shown that Clique problem is NP and NP hard, hence it is NP-Complete.