Continuous Random Variable

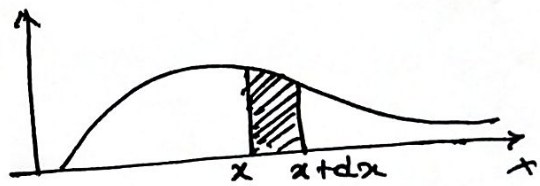
Probability density function : Concept and definition :

Let X be a continuous random variable.
Then X assumes values within the interval (a, b)
i.e., $a \leq X \leq b$.
As $P(X = x) = 0$ if X is a continuous random variable.

$\therefore \quad P(a \leq X \leq b) = P(a < X < b)$.

Let us consider a small interval $(x, x+dx)$ of length dx

Let $f(x)$ be the graph in the fig.



Then $f(x)\,dx$ represents the area under the curve $y = f(x)$, x-axis and the ordinates at the points x and $x + dx$.

Now if $P(x \leq X \leq x+dx) = f_X(x)\,dx$.

Then the fn. $f_X(x)$ is said to be the probability density fn. of the random variable X. and satisfies the following two conditions.

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

[Note:- if $a \leq X \leq b$ is the range of the r.v X then the condition (ii) can be written as $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx = 0 + \int_{a}^{b} f(x)dx + 0 = 1$

i.e $\int_{a}^{b} f(x)\,dx = 1$.

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(i) $E(X) = \int_a^b x f_X(x)\, dx$.

(ii) $E(X^2) = \int_a^b x^2 f_X(x)\, dx$

(iii) $E(g(X)) = \int_a^b g(x) f_X(x)\, dx$.

(iv) $\mu_r' =$ rth moment about origin : $\int_a^b x^r f_X(x)\, dx = E(X^r)$

(v) $\mu_r =$ rth moment about $x = A$: $\int_a^b (x-A)^r f_X(x)\, dx = E(X-A)^r$

(vi) $\mu_r =$ rth moment about mean $= \int_a^b (x-\bar{x})^r f_X(x)\, dx$

$= E(X-\bar{x})^r$, where $\bar{x} = E(X)$.

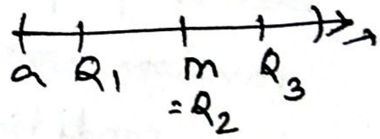
(vii) <u>Median</u> :

Let M be the median, then

$$\underbrace{\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}}$$

↳ solving this eqns. we can obtain M.

(viii) <u>Quartiles</u> (Q_1 & Q_3)



Let Q_1 be the 1st quartile

Q_3 be the 3rd quartile

Then $\int_a^{Q_1} f(x)\, dx = \frac{1}{4}$ and $\int_a^{Q_3} f(x)\,dx = \frac{3}{4}$.

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(ix) **Moment generating fn.**

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

(x) **Characteristic fn.**

$$\phi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx.$$

(a) $$|\phi_X(t)| = \left|\int_{-\infty}^{\infty} e^{itx} f(x)\,dx\right| \le \int_{-\infty}^{\infty} |e^{itx}|\,|f(x)|\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\,dx \quad \text{as } |e^{itx}| = |\cos tx + i\sin tx| = 1.$$

$$= 1$$

Hence, $|\phi_X(t)| \le 1$.

(xi) **Continuous dist. fn. (cdf).**

$$F_X(x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f(x)\,dx.$$

(a) $0 \le F(x) \le 1$.

Hence, $f_X(x) = \frac{d}{dx} F_X(x)$.

Example:- ~~Let~~ X be a cont. r.v. with cdf given by $F_X(x) = \begin{cases} 1-e^{-x}, & \text{if } x \geq 0 \\ 0, & \text{otherwis} \end{cases}$

Find the corresponding pdf.

Soln. $f_X(x) = \frac{d}{dx} F_X(x) = e^{x}$ for $x \geq 0$ & 0 otherwise

$$\therefore f_X(x) = \begin{cases} e^{x}, & x \geq 0 \\ 0, & \text{otherwise}. \end{cases}$$

Let X be a continuous r.v. with the pdf $f_X(x)$, then the cumulative distribution fn. of X, denoted by $F_X(x)$, is defined by

$$F_X(x) = P(-\infty < X \le x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$-\infty < x < \infty.$$

Properties of cdf.

1. $0 \le F_X(x) \le 1$, $-\infty < x < \infty$
2. $F_X'(x) = \frac{d}{dx} F_X(x) = f(x) \geq 0$.
3. $F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} \int_{-\infty}^{x} f(x)\,dx = 0$.

   or $F_X(\infty) = \lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} \int_{-\infty}^{x} f_X(x)\,dx = 1$.
4. $F(x)$ is a non-decreasing cont fn. of x (on right)
5. The discontinuity of $F_X(x)$ are at most countable.
6. $P(a \le X \le b) = \int_a^b f_X(x)\,dx = \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx$

   $= P(X \le b) - P(X \le a) = F_X(b) - F_X(a)$.

Similarly,

$$P(a < X < b) \equiv P(a \le X < b) \equiv P(a < X \le b) \equiv P(a \le X \le b).$$

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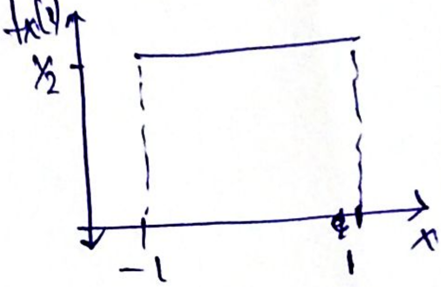
Functions of random variable (Continuous r.v.)

Example:- Let X be a continuous r.v. that is uniformly distributed in the interval $[-1, +1]$. Let $Y = \alpha X + \beta$, with $\alpha > 0$, be the derived r.v. Find the pdf of Y.



Soln. The pdf of X is given by

$$f_X(x) = \begin{cases} 1/2, & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

The cdf of X is given by

$$F_X(x) = \begin{cases} 0, & x \le -1 \\ \frac{(x+1)}{2}, & -1 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx \quad -1 \le x \le 1$$

$$= \int_{-\infty}^{-1} + \int_{-1}^{x}$$

$$= 0 + \int_{-1}^{x} \frac{1}{2}\,dx$$

$$= \left[\frac{x}{2}\right]_{-1}^{x} = \frac{x+1}{2}$$

$-1 \le x \le 1$

Let us now consider the derived r.v. $Y = \alpha X + \beta$

[# If $Y = g(X)$ is an increasing or decreasing fn. of X then

$$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|]$$

Using the above formula we can write

$$f_Y(y) = \begin{cases} 1/2\alpha, & \beta - \alpha \le y \le \beta + \alpha \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ Y is also a uniformly distributed r.v.

The cdf of Y is given by

$$F_Y(y) = \begin{cases} 0, & y \le \beta - \alpha \\ (y + \alpha - \beta)/2\alpha, & \beta - \alpha \le y \le \beta + \alpha \\ 1, & y \ge \beta + \alpha. \end{cases}$$

Q. The diameter, say X, of an electric cable, is assumed to be a continuous random variable with p.d.f. :

$f(x) = 6x(1-x)$, $0 \le x \le 1$.

(i) Check that the above is a p.d.f.

(ii) Obtain an expression for the c.d.f. of X,

(iii) Compute $P\left(X \le \frac{1}{2} \Big/ \frac{1}{3} \le X \le \frac{2}{3}\right)$, and

(iv) Determine the number k such that $P(X < k) = P(X > k)$.

(i) $\int_0^1 6x(1-x)dx = \int_0^1 (6x - 6x^2)dx = \left[3x^2 - 2x^3\right]_0^1$

$= 3 - 2 = 1$. Hence proved

(ii) $dF(x) = f(x)dx \Rightarrow F(x) = \int_{-\infty}^{x} f(x)dx$

$\Rightarrow F(x) = \int_0^x f(x)dx = \int_0^x [6x - 6x^2]dx$, $0 \le x < 1$

$F(x) = 3x^2 - 2x^3$, $0 < x \le 1$.

$$\therefore F(x) \begin{cases} = 0 ; & x \le 0 \\ = x^2(3-2x) ; & 0 < x \le 1. \\ = 1 , & \text{if } x > 1. \end{cases}$$

$P(A/B) = \frac{P(AB)}{P(B)}$

(iii) $P\left(X \le \frac{1}{2} \,\Big|\, \frac{1}{3} \le X \le \frac{2}{3}\right)$

$$= \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)} = \frac{\int_{1/3}^{1/2} 6x(1-x)dx}{\int_{1/3}^{2/3} 6x(1-x)dx}$$

$$= \frac{11/54}{13/27} = \frac{11}{26}.$$

(iv) $P(X < k) = P(X > k)$

$\Rightarrow \int_0^k 6x(1-x)dx = \int_k^1 6x(1-x)dx$

$\Rightarrow k = \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2}$.

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Q. Let X be a continuous r.v with p.d.f.

$$f(x) = \begin{cases} ax & , 0 \le x \le 1 \\ a & , 1 \le x \le 2 \\ -ax+3a & , 2 \le x \le 3 \\ 0 & , \text{elsewhere.} \end{cases}$$

(i) Determine the constant a, (ii) compute $P(X \le 1.5)$.

Soln. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1.$$

$$\Rightarrow 0 + a\int_{0}^{1} x\,dx + a\int_{1}^{2} dx + \int_{2}^{3} (-ax+3a)\,dx + 0 = 1$$

$$\Rightarrow a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[-\frac{ax^2}{2} + 3ax\right]_2^3 = 1$$

$$\Rightarrow \frac{a}{2} + 2a - a + \left[-\frac{9a}{2} + 9a + \frac{\cancel{4}^{2}a}{\cancel{2}} - 6a\right] = 1$$

$$\Rightarrow \frac{3a}{2} + \left[-\frac{9a}{2} + 5a\right] = 1$$

$$\Rightarrow \frac{3a}{2} + \frac{a}{2} = 1 \Rightarrow 2a = 1 \Rightarrow \boxed{a = \frac{1}{2}}$$

(ii) $$P(X \le 1.5) = \int_{-\infty}^{1.5} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{1.5} f(x)dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{1.5} a\,dx + 0$$

$$= \frac{a}{2} + [ax]_1^{1.5} = \frac{a}{2} + \frac{3a}{2} - a$$

$$= 2a - a = a. = 1/2$$

$$\therefore \boxed{P(X \le 1.5) = 1/2}$$

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Some Special functions

Let n be an integer then $\Gamma(n) = (n-1)!$

$$\Gamma(1) = 0! = 1.$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad Re(x) > 0.$$

$\hookrightarrow$ under this condition the integral converges absolutely.

i) $\Gamma(x+1) = x\Gamma(x)$

ii) $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) \cdots = n(n-1)\cdots 2.1.\Gamma(1)$

$= n(n-1)\cdots 2.1. = n!$

iii) $\Gamma(1/2) = \sqrt{\pi}$

(a) $\Gamma(5/2) = \Gamma(1+3/2) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\Gamma(1+\frac{1}{2}) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)$

$= \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}.$

$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ for $Re(x) > 0$, $Re(y) > 0$.

$= 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}$ "

$= \int_0^\infty \frac{t^{x-1}\,dt}{(1+t)^{x+y}}$, "

Relation between $\Gamma(x)$, $\Gamma(y)$ and $\beta(x,y)$.

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

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