IMA-102

Relations :

1. Reflexive : $xRx \quad \forall x \in X$
2. Symmetric : if xRy implies yRx
3. Anti Symmetric : If xRy and yRx implies $x=y$
4. Transitive : if xRy and yRz imply xRz.

- If R is Reflexive, Symmetric and transitive, then it is called Equivalence Relation.
- If R is Reflexive, Anti-symm. and transitive, then it is called Partial Order Relation.

Equivalence class :

Let X be a set and E is an equivalence relation on X. let $a \in X$ &
$E(a) = \{ x \in X \mid xRa \}$. Then $E(a)$ is an equivalence class of a, determined by E.

If $C \subset X$ is an equivalence class of X determined by E then $C = E(a)$ for same $a \in X$.

Quotient set $= X/E = \{ c \mid c = E(a) \text{ for same } a \in X \}$

$\downarrow$

collection of sets such that each set is an eq. class.

Functions :

- well defined func

$f : X \longrightarrow Y$

$x_1, x_2 \in X$

$x_1 = x_2$

$\Downarrow$

$f(x_1) = f(x_2)$

check whether function is well defined or not.

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Binary Operations :

X is a non-empty set.

A map $* : X \times X \longrightarrow X$

i.e. $*(a,b) \longrightarrow a*b$.

Eg : $+ : Z \times Z \longrightarrow Z$

$- : N \times N \not\longrightarrow N$

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Algebraic Structures :

A non empty set X, together with a binary operation $*$ i.e. $(X, *)$ is called an algebraic structure.

- → Group
- → Field
- → Semi Ring
- → Ring
- → Vector Space.

Group := A non empty set G with binary operation $*$ is called a group if the following conditions hold :

(1) Associative property

$$a*(b*c) = (a*b)*c \quad \forall\, a,b,c \in G$$

(2) Existence of identity

i.e. $\exists\, e \in G$; s.t

$$a*e = e*a = a \;;\; \forall\, a \in G$$

unique

e is called identity of G.

(3) Existence of inverse

$\forall\, a \in G \;\exists$ a unique element $b \in G$

s.t. $a*b = b*a = e$

identity of G

further if,

$$a*b = b*a$$

Then $(G,*)$ is called Abelian Group.

Also,

(4) $\boxed{\text{Check Closure's Law.}}$

Group Definition

- Set of elements G
- Operation : $+, \times, *$
- closed under operation

$x,y \in G \Rightarrow x*y \in G$

- Inverses : x^{-1} exists for all x

$x * x^{-1} = e$

- Identity : $y*e = e*y = y$
- Associativity : $(a*b)*c = a*(b*c)$.

G may not be commutative. Possible for $x*y \neq y*x$

If G is commutative, called:
- Commutative / Abelian group.

If G is not commutative, called
- Non-commutative group.

- $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

$(\mathbb{R}^2, +)$

$(x_1,y_1) + (x_2,y_2) = (x_1+x_2,\, y_1+y_2)$

Proving associativity :

$$(x_1,y_1) + \big((x_2,y_2)+(x_3,y_3)\big) = \big((x_1,y_1)+(x_2,y_2)\big) + (x_3,y_3)$$

$$\text{LHS} = (x_1,y_1) + (x_2+x_3,\, y_2+y_3)$$

$$= \big((x_1+(x_2+x_3)),\, (y_1+(y_2+y_3))\big)$$

$$= \big((x_1+x_2)+x_3,\, (y_1+y_2)+y_3\big) = (x_1+x_2,\, y_1+y_2) + (x_3,y_3)$$

$$= \big((x_1,y_1)+(x_2,y_2)\big) + (x_3,y_3) = \text{RHS}.$$

- This is an additive identity as $(0,0) \in \mathbb{R}^2$ when added gives the same.
- Existence of additive inverse :-

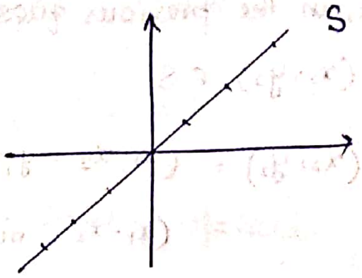
$$\forall \ (x_1, y_1) \in \mathbb{R}^2, \ \exists \ (-x_1, -y_1) \in \mathbb{R}^2.$$

Q. S = Collection of all points of $\mathbb{R}^2$ that lie on a straight line with slope = m (real) passing through the origin.

i.e. $S = \{(x,y) \in \mathbb{R}^2 \mid y = mx$; m is a real and x is an arbitrary real$\}$.

Is $(S,+)$ group?

<u>Soln</u> : Let $(x_1, y_1), (x_2, y_2) \in S$



<u>Closure's law</u> :

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2, m(x_1 + x_2)) \in S.$$

$(0,0) \in S$ as $(0,0) \equiv (0, m.0) \longrightarrow$ Existence of identity.

If $(x_1, y_1) \in S$, then, $(-x_1, -y_1) \equiv (-x_1, m(-x_1)) \in S \longrightarrow$ Existence of inverse.

Commutative property can be proved as shown in the previous example.

Associative " can also be " " " " " " " .

<u>Subgroup</u> :-

G is a group.

A non-empty subset H of a group G is called a ~~subset~~ subgroup of G if H is itself a group with respect to binary operations of G.

<u>Example</u> :

In the previous example :

S is a subgroup of $\mathbb{R}^2$ as $S \subseteq \mathbb{R}^2$ and S is itself a group with respect to the operation '+'.

<u>THEOREM</u> :

A non-empty subset H of a group G is a subgroup of G iff —

(i) $\forall \ x, y \in H \Rightarrow x - y \in H$

(ii) $xy^{-1} \in H$ (if operation is multiplication).

Proving this proves three properties —
1. Closure property
2. Existence of identity
3. Existence of inverse.

Proof : ① $x, y \in H \Rightarrow x, -y \in H$

$\Rightarrow x-(-y) \in H$

$\Rightarrow x+y \in H$.

② $x \in H$

$\Rightarrow x-x \in H$

$\Rightarrow 0 \in H$

③ $0, x \in H$

$\Rightarrow 0-x \in H$

$\Rightarrow -x \in H$.

Way to proof :

As for example, in the previous question :

(x_1, y_1) , $(x_2, y_2) \in S$

$\therefore (x_1, y_1) - (x_2, y_2) = (x_1-x_2, y_1-y_2)$

$= (x_1-x_2, m(x_1-x_2)) \in S$.

Ring :-

A non-empty set R together with two binary operations [say, addition '+' and multiplication '•'] is called a 'Ring', if the following conditions hold —

(i) $(R, +)$ is commutative group, or, Abelian group.

(ii) $\forall a, b, c \in R$

→ $a \cdot b \in R$

→ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(iii) → $a \cdot (b+c) = a \cdot b + a \cdot c$

→ $(a+b) \cdot c = a \cdot c + b \cdot c$

Ring :
- Set of elements
- Two operations : + and ×
- Commutative group under '+'
- Multiplication is associative
  $a \times (b \times c) = (a \times b) \times c$
- Distributive property: $a \times (b+c) = a \times b + a \times c$

- If further, $a \cdot b = b \cdot a$ $\forall a, b \in R$; then $(R, +, \cdot)$ is called Commutative Ring.
- If further, $1 \cdot a = a$ $\forall a \in R$, $(R, +, \cdot)$ is called commutative Ring with identity, and, 1 is called multiplicative identity of R.

Field :-

A commutative Ring with unity in which every non-zero element has its multiplicative inverse is called a 'Field'.

$(Z_5, +_5, \times_5)$

$Z_5 = \{0, 1, 2, 3, 4\}$

$Z_2 = \{0, 1\}$

Example of Modulo operation : $4 \times 4 \equiv 1 \mod 5$.

Vector Space := V (F) $\rightarrow$ Vector Space 'V' over Field 'F'.

A non-empty V together with binary operation say '+' and scalar multiplication say '.',

[$\because F \times V \longrightarrow V$ and image of (α, v) under '.' denoted as αv, F is a Field].

is called a Vector Space over a field F if the following conditions hold :

I> $(V, +)$ is an Abelian Group.

II> (i) $\alpha.(v+u) = \alpha v + \alpha u$; $\forall\ \alpha, \beta \in F$ & $u, v \in V$.

(ii) $(\alpha+\beta).u = \alpha u + \beta u$

III> $\alpha(\beta v) = (\alpha\beta)v$; $\forall\ \alpha, \beta \in F$ & $v \in V$.

IV> $1.v = v$; $\forall\ v \in V$.

1 is called the 'Multiplicative Identity' of Field.

$\left.\begin{array}{l} \mathbb{R}^2(\mathbb{R}) \\ \mathbb{R}(\mathbb{R}) \\ \mathbb{C}(\mathbb{C}) \\ \mathbb{Q}(\mathbb{Q}) \end{array}\right\}$ Vector Spaces with respect to '+' and '.'.

$\mathbb{R}(\mathbb{C}) \longrightarrow$ Not a vector space (Not satisfying $F \times V \longrightarrow V$).

$V = \{0\} \Rightarrow V(\mathbb{R})$ is a Vector Space.

$\longrightarrow$ Zero space.

$V = \{1\} \Rightarrow V(\mathbb{R})$ is NOT a Vector Space.

$Z_2 = \{0, 1\}$

00000 $\longrightarrow$ $\boxed{\text{Channel}}$ $\longrightarrow$ { 10000, 01000, 00100, 00010, 00001.

Noisy

A $\longrightarrow$ 00 $\longrightarrow$ 000 $\longrightarrow$ 00000

B $\longrightarrow$ 01 $\longrightarrow$ 011 $\longrightarrow$ 01110

C $\longrightarrow$ 10 $\longrightarrow$ 101 $\longrightarrow$ 10101

D $\longrightarrow$ 11 $\longrightarrow$ 110 $\longrightarrow$ 11011

$\mathbb{C}(Z_2)$ $\qquad \mathbb{C} = \{00000, 01110, 10101, 11011\}$

- A code $\mathbb{C}$ with distance $d(\mathbb{C})$ correct 't' errors iff : $d \geq (2t+1)$.

$d(\mathbb{C}) = \min\{d(x,y) \mid x, y \in C, x \neq y\}$.

Some more examples of Vector Space :

① $F_2^3(F_2)$

② Is $M_{n \times n}$ a vector space?

$M_{n \times n}$ = set of non-singular matrices over the real number (Sum of two non-singular matrices need not be a non-singular matrix.)

③ $P_n(x)$: The set of all polynomials in x with coefficient of real no. of degree less than or equal to n.

i.e. $P_n(x) = \{a_0 + a_1x + \dots + a_nx^n \mid a_0, a_1, \dots, a_n \in \mathbb{R}\}$

Eg (Vector Space).

$C[0,1]$ Set of all real-valued continuous functions defined on $[0,1]$.

$f\ [0,1] \longrightarrow \mathbb{R}$

$g\ [0,1] \longrightarrow \mathbb{R}$

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x).$$

Theorem: Let V be a vector space over a field F

Then

(i) $\alpha \cdot \bar{0} = \bar{0}$

(ii) $0 \cdot v = \bar{0}$

(iii) $\alpha \cdot v = 0 \Rightarrow \alpha = 0$, or, $v = \bar{0}$, or, both

(iv) $(-1)\, v = -v$.

Proof: (i) Since $\bar{0} + \bar{0} = \bar{0}$

$\therefore\ \alpha(\bar{0}+\bar{0}) = \alpha \cdot \bar{0}$

$\Rightarrow \alpha \cdot \bar{0} + \alpha \bar{0} = \alpha \cdot \bar{0}$

$\Rightarrow (\alpha \cdot \bar{0} + \alpha \cdot \bar{0}) + (-\alpha \cdot \bar{0}) = \alpha \cdot \bar{0} + (-\alpha \cdot \bar{0})$

$\Rightarrow \alpha \cdot \bar{0} + (\alpha \cdot \bar{0} + (-\alpha \cdot \bar{0})) = \bar{0}$

$\Rightarrow \alpha \cdot \bar{0} + \bar{0} = \bar{0}$

$\Rightarrow \alpha \cdot \bar{0} = \bar{0}$.

(iii) $\alpha \cdot v = 0$

Suppose $\alpha \neq 0$ & $\alpha \in F$

$\exists\ \alpha^{-1} \in F$ st $\alpha^{-1}\alpha = 1$

$\alpha^{-1}(\alpha \cdot v) = \alpha^{-1} \cdot \bar{0}$

$\Rightarrow (\alpha^{-1}\alpha) \cdot v = \bar{0}$

$\Rightarrow 1 \cdot v = \bar{0}$

$\Rightarrow v = \bar{0}$

(iv) $\bar{0} = 0 \cdot v$

$= (1 + (-1))\, v$

$= 1 \cdot v + (-1) \cdot v$

$\bar{0} = v + (-1)\, v$

$\bar{0} + (-v) = [v + (-1)v] + (-v)$

$(-v) = v + [(-v) + (-1)v]$

$= [v + (-v)] + (-1)\, v$

$= \bar{0} + (-1)\, v$

$= (-1)\, v$.

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Sub space :

Def^n : A non empty subset H of a vector space V(F) is called a subspace of V if H is itself a vector space w.r.t operations like vector addition and scalar multiplication that is defined in V.

eg: $\mathbb{C}(\mathbb{R}) \longrightarrow$ vector space.

$\mathbb{R}(\mathbb{R}) \longrightarrow$ Subspace.

$\mathbb{R} \subseteq \mathbb{C}$

$\mathbb{R}^2(\mathbb{R})$

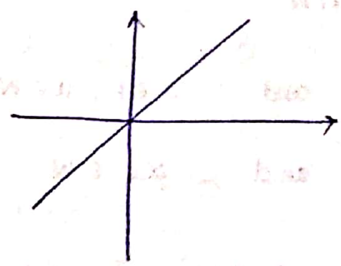
Operations : (i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

(ii) $\alpha(x, y) = (\alpha x, \alpha y)$.

$H = \{ (x,y) \in \mathbb{R}^2 \mid y = mx$, m is a real no. and x is an arbitrary real$\}$

$v = (x,y) \in H$

$$(\alpha+\beta)(x,y) = ((\alpha+\beta)x, (\alpha+\beta)y)$$
$$= (\alpha x + \beta x, \alpha y + \beta y)$$
$$= (\alpha x, \alpha y) + (\beta x, \beta y)$$
$$= \alpha(x,y) + \beta(x,y).$$



→ H is proper subspace of $\mathbb{R}^2(\mathbb{R})$.

→ $H = \{ (x,y,z) \in \mathbb{R}^3 \mid ax+by+cz=0,\ a,b,c \in \mathbb{R}\}$ is a proper subspace of $\mathbb{R}^3(\mathbb{R})$.

- Trivial subspace or Improper subspace.
- Proper subspace.

Theorem :-

A non empty subset H of a vector space V(F) is a subspace of V iff

(I) $u, v \in H \Rightarrow u + v \in H$

(II) $\forall \alpha \in F, \forall u \in H \Rightarrow \alpha u \in H$.

$\Rightarrow$ obvious

$\Leftarrow$ let $u \in H$

then, $u \in V$

$0 \cdot u = \bar{0} \in H$ (by (II) cond^n)

If $u \in H$

then,

$(-1)u = -u \in H$. (II cond^n)

e.g. $C'[0,1]$ is a subspace of $C[0,1] \rightarrow \{$set of all real-valued continuous fns defined on $[0,1]\}$

(proper)

$P_n(x)$ is a subspace of $C[0,1]$.

(proper)

{Set of all real valued continuous fn defined on [0,1] whose first derivative is continuous}

{Set of polynomials of degree less than or equal to n}

- Let H and W be two subspaces of a vector space $V(F)$.

Q ① Is $H \cap W$ a subspace? (yes)

② Is $H \cup W$ a subspace? (Need not be).

① let $u, v \in H \cap W$

$u, v \in H$ & $u, v \in W$

$\Rightarrow u+v \in H$ & $u+v \in W$

(as H and W are given subspaces)

$\Rightarrow u+v \in H \cap W$.

$\alpha \in F$, $u \in H \cap W$

$\Rightarrow \alpha \in F, u \in H$ and $\alpha \in F, u \in W$

$\Rightarrow \alpha u \in H$ and $\alpha u \in W$

$\Rightarrow \alpha u \in H \cap W$

② $H \cup W$ is a subspace of V iff either $H \subseteq W$ or $W \subseteq H$.

$\Leftarrow$ obvious

$\Rightarrow$ Suppose $H \not\subseteq W$ and $W \not\subseteq H$

$\therefore \exists\ u, v \in V$ (s.t) [i.e. $u, v \in H \cup W$]

$u \in H$, $u \notin W$

& $v \in W$, $v \notin H$

$u \in H$ & $v \notin H$ $\Rightarrow u+v \notin H$. — (I)

$v \in W$ & $u \notin W$ $\Rightarrow v+u \notin W$. — (II)

By (I) and (II):

$u+v \notin H \cup W$ [Contradiction].

$\therefore H \cup W$ is not a subspace

Linear Combination:

An expression as:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

is called a linear combination of $v_1, v_2, \ldots, v_n$:

where, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars

and, $v_1, v_2, \ldots, v_n$ are vectors of a vector space $V(F)$.

$\{v_1, v_2, \ldots, v_n\}$: set of vectors of a vector space $V(F)$.

$\rightarrow$ If $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not all zero);
Such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Then the set $\{v_1, v_2, \ldots, v_n\}$ is called <u>linearly dependent</u>.

$\rightarrow$ If $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

$\Rightarrow$ all $\alpha_i = 0$

Then $\{v_1, v_2, \ldots, v_n\}$ is called <u>linearly independent</u>.

→ Any non-empty set which contains Null vector $(\bar{0})$ is linearly dependent always.

Q. Prove : Any n vectors in $\mathbb{R}^m$ are always linearly dependent if $n > m$.

Ans :-

Let $v_1, v_2, v_3, \ldots, v_n$ are n vectors in $\mathbb{R}^m$.

Let $c_1, c_2, c_3, \ldots, c_n$ are scalars such that :

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

$$v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} ; \quad v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} ; \quad \ldots$$

$$\Rightarrow c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots = 0$$

$$\Rightarrow \begin{aligned} a_{11} c_1 + a_{12} c_2 + \cdots + a_{1n} c_n &= 0 \\ a_{21} c_1 + a_{22} c_2 + \cdots + a_{2n} c_n &= 0 \\ &\vdots \\ a_{m1} c_1 + a_{m2} c_2 + \cdots + a_{mn} c_n &= 0. \end{aligned}$$

$\because n > m$, it means no. of unknowns is greater than the no. of equns; so above homogeneous system of equns have non-trivial solution.

Span (Generate) :-

Let $v_1, v_2, \ldots, v_n$ are n vectors of a vector space $V(F)$. These vectors are said to span V if every $v \in V$ can be written as a linear combination of $v_1, v_2, \ldots, v_n$.

$V = \text{span}\{v_1, v_2, \ldots, v_n\}$

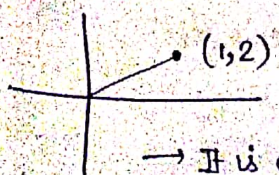
Let $v_1, v_2, \ldots, v_k \in V$

$$\text{Span}\{v_1, v_2, \ldots, v_k\} = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in F\}$$

Span$\{v_1, v_2, \ldots, v_k\}$ is a subspan of V.

Ex ① $V = \mathbb{R}^2$, $F = \mathbb{R}$

Span$\{(1,2)\}$



→ It is a st. line.

② $V = \mathbb{R}^3$, $F = \mathbb{R}$

Span$\{(1,0), (0,1)\}$

→ It is a plane.

$S = \text{span}\{v_1, v_2, \ldots, v_K\}$ is a subspace of V.

Let $u, v \in S$.

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_K v_K$$

$$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_K v_K$$

$$u + v = (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_K + \beta_K) v_K$$

$$= \gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_K v_K$$

$$\gamma_i = \alpha_i + \beta_i$$

Basis :=

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ of a vector space $V(F)$ is said to be a basis of V if

$\{v_1, v_2, \ldots, v_n\}$ is linearly independent, and,

$\{v_1, v_2, \ldots, v_n\}$ spans V.

$\mathbb{R}^2(\mathbb{R})$

$$\underbrace{\{(1,0), (0,1)\} \text{ or, } \{(1,1), (1,0)\} \text{ or, } \{(1,1), (0,1)\}}_{\text{Basis of } \mathbb{R}^2(\mathbb{R}).}$$

The no. of elements in the basis is called dimension of the space.

$$P_2(x) = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in F\}$$

$$S = \{1, x, x^2\}$$

Cardinal no. $= 3$

Dimension of $P_2(x) = 3$

Dimension of $P_n(x) = n+1$.

$$M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\}$$

Dimension $= 3$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Dimension :-

The no. of vectors (elements) in a basis of a vector space V is called the dimension of V(F).

Q. Every field over itself is a vector space. What is its dimension?

→ Let R (R) be the vector space. Its basis is $\{1\}$.

If $\{2\} \in \mathbb{R}$

$c = \alpha \cdot 2$

Let $c = \sqrt{3}$

$\alpha = \frac{\sqrt{3}}{2}$

$C(\mathbb{R}) \longrightarrow$ Basis $= \{1, i\}$

Dimension $= 2$.

$\mathbb{R}(\mathbb{Q}) \longrightarrow$ Infinite dimensional.

Basis : $\{1, \sqrt{2}, \text{------}\}$

S = set of all polynomials in x is infinite dimensional.

Theorem : If $\{v_1, v_2, \text{------}, v_k\}$ is a basis of a vector space V(F). Then for any $v \in V$, $\exists$ a unique sector of scalars $c_1, c_2, \text{---}, c_k \in F$ such that

$v = c_1 v_1 + c_2 v_2 + \text{------} + c_k v_k$.

Suppose, $v = c_1 v_1 + c_2 v_2 + \text{------} + c_k v_k = d_1 v_1 + d_2 v_2 + \text{------} + d_k v_k$

$\Rightarrow (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + \text{------} + (c_k - d_k) v_k = 0$

$\Rightarrow c_i = d_i$ (uniqueness).

Theorem :

If $\{u_1, u_2, \text{------}, u_m\}$ and $\{v_1, v_2, \text{------}, v_n\}$ are bases of a vector space V(F) then $m = n$.

Pf :- Let $S_1 = \{u_1, u_2, \text{------}, u_m\}$

$S_2 = \{v_1, v_2, \text{------}, v_n\}$

Suppose $m > n$ and S_2 is basis.

Each u_i can then be written as a linear combination of v_i vectors.

$\therefore u_1 = a_{11} v_1 + a_{12} v_2 + \text{------} + a_{1n} v_n$

$u_2 = a_{21} v_1 + a_{22} v_2 + \text{------} + a_{2n} v_n$

$\vdots$

$u_m = a_{m1} v_1 + a_{m2} v_2 + \text{------} + a_{mn} v_n$

Take, $c_1 u_1 + c_2 u_2 + \text{------} + c_m u_m = 0$

$\Rightarrow c_1 (a_{11} v_1 + a_{12} v_2 + \text{----} + a_{1n} v_n) + c_2 (a_{21} v_1 + a_{22} v_2 + \text{----} + a_{2n} v_n) + \text{------} + c_m (a_{m1} v_1 + \text{------} + a_{mn} v_n) = 0$

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$(a_{11}c_1 + a_{21}c_2 + \cdots + a_{m1}c_m)v_1 + (a_{12}c_1 + a_{22}c_2 + \cdots + a_{m2}c_m)v_2 + \cdots$

$\cdots + (a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{mn}c_m)v_n = 0$

$\Rightarrow$

$$a_{11}c_1 + a_{21}c_2 + \cdots + a_{m1}c_m = 0$$
$$a_{12}c_1 + a_{22}c_2 + \cdots + a_{m2}c_m = 0$$
$$\vdots$$
$$a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{mn}c_m = 0$$

$\because$ No. of equns is less than the no. of unknowns, so system has non-trivial solutions i.e. some c_i are non-zero $\Rightarrow m \not\leq n$ contradiction since we assumed $n \geq m$.

Theorem : A set of n linearly independent vectors of a vector space V of dimension n constitute a basis of V.

Theorem : A set of vectors $v_1, v_2, \ldots, v_m$ of n dimension vector space are linearly independent if $m \leq n$.

Theorem : Let H be a subspace of a vector space V, then

$$\dim H \leq \dim V.$$

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Q. (1) Find a basis and dimension of the following vector space :

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} 2x - y + 3z = 0 \\ x, y, z \in \mathbb{R} \end{array} \right\}$$

$\boxed{\text{ANS}}$:

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V$

$$\left.\begin{array}{l} 2x - y + 3z = 0 \\ \Rightarrow y = 2x + 3z \end{array}\right\} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x+3z \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3z \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}.$$

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$$

Dimension = 2 .

- Basis of an n-dimensional ~~plane~~ space $(\mathbb{R}^n)$

$$\left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}, \cdots \underset{e_i}{\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \ldots, \underset{e_n}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}} \right\} \longrightarrow \text{Standard Basis.}$$

- Standard Basis of $P_n(x)$ is $\{1, x, x^2, \ldots, x^n\}$.

Q. (2) Only proper subspace of $\mathbb{R}^2$ is straight line passing through the origin.

$\boxed{\text{ANS}}$: [If H is a proper subspace of finite dimensional vector space V ; then :

$$\text{dim. } H < \text{dim. } V.$$]

> For a vector space V, Improper subspaces are $\{0\}$ and V itself.

Let H be a proper subspace of $\mathbb{R}^2$.
So, dim. H = 1

Let, $\{(x_0, y_0)\}$ is a basis of H.

Let, $(x, y) \in H \Rightarrow (x, y) = \alpha (x_0, y_0)$

$$\therefore x = \alpha x_0 \ \& \ y = \alpha y_0 .$$

$$\Rightarrow \frac{x}{x_0} = \frac{y}{y_0} = \alpha .$$

$$\Rightarrow y = \left(\frac{y_0}{x_0}\right) x .$$

Q③ Let H and K are subspaces of vector space $V(F)$.

Define $H+K = \{ h+k : h \in H \text{ and } k \in K \}$

Prove that :

(i) $H+K$ is a subspace of V.

(ii) If $H \cap K = \{0\}$; Then $\dim(H+K) = \dim H + \dim K$.

ANS : (i) $u, v \in H+K$

let, $u = h_1 + k_1$
$v = h_2 + k_2$.

$\therefore u+v = (h_1+k_1) + (h_2+k_2)$

$= ((h_1+k_1)+h_2) + k_2$ [Associative prop.]

$= ((h_1) + (k_1+h_2)) + k_2$ ["]

$= (h_1 + (h_2+k_1)) + k_2$ [Commutative prop.]

$= ((h_1+h_2)+k_1) + k_2$ [Associative]

$= (h_1+h_2) + (k_1+k_2)$ ["].

$= h' + k' \in H+K$

(ii) Dimension of null space i.e. $\{0\}$ is 0.

Actual result :-

$\dim(H+K) = \dim H + \dim K - \dim(H \cap K)$.

let $\{v_1, v_2, \ldots, v_n\}$ is a basis of H.
$\{u_1, u_2, \ldots, u_m\}$ is a basis of K.

Prove that $B = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$ is a basis of $H+K$.

Definitely, B spans $H+K$.

let, $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_m u_m = 0$
where not all α_i's and β_j's are zero.

let, $\alpha_1, \alpha_2, \ldots, \alpha_n = h$
$\beta_1, \beta_2, \ldots, \beta_m = k$

$h + k = 0 \Rightarrow h = -k \in K$
$\Rightarrow h \in H \cap K = \{0\}$

Let, $h \neq 0$.

$\therefore h = 0$

So, there is a contradiction.

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \Rightarrow \alpha_i = 0$

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Similarly, $\beta_j = 0$

* $H \cap K$ is a subspace of H & K.

$$H \cap K \subset H$$
$$H \cap K \subset K.$$

let $\dim H \cap K = r$.

let $\{x_1, x_2, \ldots, x_r\}$ be a basis of $H \cap K$

$\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r}\}$ is a basis of H

$\{x_1, x_2, \ldots, x_r, z_1, z_2, \ldots, z_{m-r}\}$ is a basis of K.

So, we have to show that

$\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r}, z_1, z_2, \ldots, z_{m-r}\}$ is a basis of $H+K$.

dimension $= r + n - r + m - r = n + m - r$.

Change of Basis

$\mathbb{R}^2$

Standard Basis $= \left\{ \overset{u_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{u_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$.

$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 7 \\ 8 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. $B_3 = \left\{ \begin{pmatrix} 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$.

$\begin{pmatrix} 3 \\ 4 \end{pmatrix}_{B_3} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. ; $\begin{pmatrix} 3 \\ 4 \end{pmatrix}_{B_1} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Linear transformation

let V and W be two vector spaces over the same field F.
A map $T: V \to W$ is called a linear transformation (operator) if

(i) $T(u+v) = Tu + Tv$

(ii) $T(\alpha u) = \alpha Tu$

Examples

① $f : \mathbb{R} \longrightarrow \mathbb{R}$.

$$f(x) = e^x \; ; \quad x \in \mathbb{R}$$

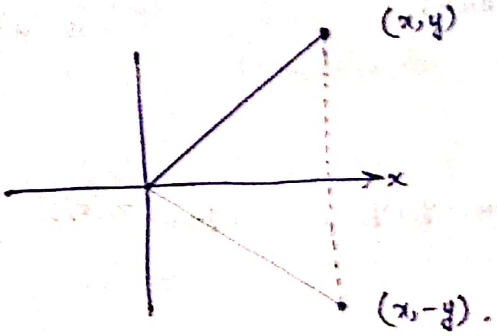
$$f(x+y) = e^{x+y} = e^x \cdot e^y \neq e^x + e^y$$

$$f(e^{\alpha x}) = (e^x)^\alpha \neq \alpha \cdot e^x$$

$\therefore$ This function is not linear.

② $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$



Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$.

$$\therefore f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = f\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -(y_1 + y_2) \end{pmatrix}$$

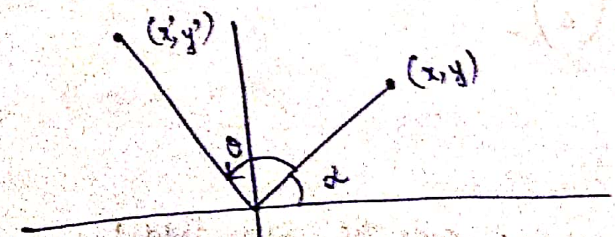
$$= \left(\begin{pmatrix} x_1 \\ -y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix}\right)$$

$$= f\begin{pmatrix} x_1 \\ +y_1 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

$$f\left(\alpha\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \begin{pmatrix} \alpha x \\ -\alpha y \end{pmatrix} = \left(\alpha\begin{pmatrix} x \\ -y \end{pmatrix}\right) = \alpha f\begin{pmatrix} x \\ y \end{pmatrix}.$$

$\therefore$ This function / mapping is linear.

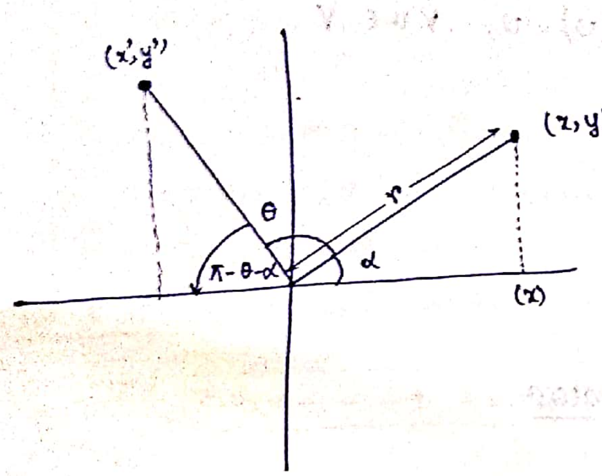
③



$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\Rightarrow f\begin{pmatrix} x \\ y \end{pmatrix} = A_\theta \begin{pmatrix} x \\ y \end{pmatrix}$$; which is linear.

Rotation transformation



$v = (x, y)$

After rotation by θ angle,

$v' = (x', y')$.

$x = r\cos\alpha$

$y = r\sin\alpha$

$x' = r\cos(\theta+\alpha)$

$y' = r\sin(\theta+\alpha)$.

$$\therefore \quad x' = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) = x\cos\theta - y\sin\theta$$

$$y' = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = x\sin\theta + y\cos\theta$$

$$= \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A_\theta} \begin{pmatrix} x \\ y \end{pmatrix} = A_\theta \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A_\theta \begin{pmatrix} x \\ y \end{pmatrix}.$$

- $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\exists$ $m\times n$ order matrix $A_{m\times n}$

Defined by

$T(v) = A_{m\times n} v$; $v \in \mathbb{R}^n$.

$T(u+v) = A_{m\times n}(u+v) = A_{m\times n}u + A_{m\times n}v = Tu + Tv$.

$T(\alpha v) = A_{m\times n}(\alpha v) = \alpha A_{m\times n} v = \alpha Tv$.

Zero transformation :-

$T : V \to W$ is a map such that $T(v) = 0$; $\forall v \in V$.

$T(u+v) = 0 = 0 + 0 = Tu + Tv$.

$T(\alpha u) = 0 = \alpha \cdot 0 = \alpha Tu$.

Zero transformation.

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Identity transformation :-

$I : V \longrightarrow V$ defined as $I(v) = v, \ \forall v \in V$.

- $J : C[0,1] \longrightarrow \mathbb{R}$

$$J(f) = \int_0^1 f(x)\,dx$$

clearly, J is linear transformation.

J is known as Integral operator.

- $D : C'[0,1] \longrightarrow C[0,1]$

$C'[0,1]$ ↳ set of those differentiable continuous functions in $[0,1]$ whose first derivatives are also continuous in $[0,1]$.

$$D(f) = f' = \frac{df}{dx}.$$

Clearly, D is also a linear transformation.

THEOREM :

Let $T : V(F) \longrightarrow W(F)$ (same field) be a linear transformation.

Then

(i) $T(0) = 0_W$

(ii) $T(u-v) = Tu - Tv$

(iii) $T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 T v_1 + \cdots + \alpha_n T v_n$.

$\forall\ u, v, v_1, v_2, \ldots, v_n \in V$ &

$\forall$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Proof

$T(0_V) \in W$

$T(0_V) = T(0_V + 0_V) = T0_V + T0_V$

$T(0_V) - T(0_V) = T0_V + T0_V - T0_V$

$\Rightarrow \ 0_W = T0_V$.

$T(u-v) = T(u + (-1)v)$

$\Rightarrow T(u-v) = T(u) + (-1)Tv$

$= Tu - Tv$

Proof By Mathematical Induction :

$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 Tv_1 + \alpha_2 Tv_2$$

True for K

i.e.

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_K v_K) = \alpha_1 Tv_1 + \alpha_2 Tv_2 + \dots + \alpha_K Tv_K$$

∴ for K+1

$$T(\boxed{\alpha_1 v_1 + \dots + \alpha_K v_K} + \alpha_{K+1} v_{K+1})$$

$$= \underbrace{T(\alpha_1 v_1 + \dots + \alpha_K v_K)} + T(\alpha_{K+1} v_{K+1})$$

$$= \underbrace{\alpha_1 Tv_1 + \alpha_2 Tv_2 + \dots \alpha_K Tv_K} + \alpha_{K+1} Tv_{K+1}$$

Hence, ③ property is proved.

- $T : M_{n \times n} \longrightarrow \mathbb{R}$

$$T(A) = \det A$$

T is not a linear transformation as both conditions fail.

Kernel & Image of a linear transformation :=

Let $T : V \to W$ be a linear transformation.

Then, Kernel of T defined as

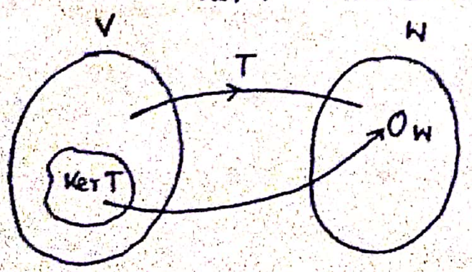
$$\text{Ker}\, T = \{ v \in V \mid Tv = 0_W \}$$

And, Image of T = Im T = $\{ w \in W \mid \exists \text{ some } v \in V \text{ s.t. } Tv = w \}$

Property : let $T : V \to W$ be a linear transformation.

Then, Ker T and Im T are subspaces of V and W respectively.

Proof :



let, $u, v \in \text{Ker}\, T \Rightarrow Tu = 0_W$, $Tv = 0_W$

Now, $T(u+v) = Tu + Tv = 0_W + 0_W = 0_W$

$\Rightarrow u + v \in \text{Ker}\, T$.

let, $u \in \text{Ker}\, T$

$T(\alpha u) = \alpha Tu = \alpha 0_W = 0_W \Rightarrow \alpha u \in \text{Ker}\, T$.

So, $\ker T$ is a subspace of V.

Let, $w_1, w_2 \in \operatorname{Im} T$. $\Rightarrow$ $Tu = w_1$

& $Tv = w_2$, where, [redacted] $u, v \in V$

$\therefore$ $\cancel{T(u+v) = Tu + Tv = w_1 + w_2 \in W}$, (as V [redacted] is a vector space so, if [redacted] $u, v \in V$ then [redacted] $u+v \in V$.

$w_1 + w_2 = Tu + Tv = T(u+v)$

$\therefore$ $w_1 + w_2 \in \operatorname{Im} T$.

Also, $w \in \operatorname{Im} T$

[redacted]

$\alpha w = \alpha Tu = T(\alpha u)$, $u \in V$.

and V is a vector space; so, $\alpha u \in V$.

$\therefore$ $\alpha w \in \operatorname{Im} T$

<u>THEOREM</u> :

Let $T : V \to W$ is a linear map. Then T is injective <u>if and only if</u> $\ker T = \{0_V\}$.

<u>Proof</u> : Suppose T is injective.

Let $x \in \ker T$

$\Rightarrow$ $Tx = 0_W = T(0_V)$

$\Rightarrow$ $x = 0_V$, since T is injective.

$\Rightarrow$ $\ker T = \{0_V\}$.

Suppose $\ker T = \{0_V\}$

Let $T(x) = T(y)$, $x, y \in V$

$\Rightarrow$ $T(x) - T(y) = T(y) - T(y) = 0_W$

$\Rightarrow$ $T(x-y) = 0_W$

$\Rightarrow$ $x - y \in \ker T$

$\Rightarrow$ $x - y = 0_V$

$\Rightarrow$ $x = y$.

- Dimension of Ker T = Nullity T.
- Dimension of Im T = Rank T.

THEOREM : (Rank, Nullity Theorem)

Let V be a finite dimensional vector space of dimension n and $T: V \to W$ is a linear map.

Then,

$$\dim V = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$$

i.e. $$\dim V = \text{Nullity } T + \text{Rank } T.$$

THEOREM :

Let V and W be vector spaces of same dimension. Then, T is injective iff T is surjective.

Proof :

$$\dim V = \underbrace{\text{Nullity } T}_{=0 \text{ as } \operatorname{Ker} T = \{0_V\} \text{ as } T \text{ is injective.}} + \text{Rank } T$$

$$\Rightarrow \underbrace{n = \dim \operatorname{Im} T = \dim W}.$$

$\therefore \operatorname{Im} T = W$

$\Rightarrow T$ is surjective.

Q. Verify that the map $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$ is linear. Find the rank and nullity of T.

Ans :

$u = (x_1, x_2)$
$v = (y_1, y_2)$

$\therefore T(u+v) = T((x_1, x_2) + (y_1, y_2))$

$= T(x_1 + y_1, x_2 + y_2)$

$= ((x_1+y_1)+(x_2+y_2), (x_1+y_1)-(x_2+y_2), x_2+y_2)$.

$= ((x_1+x_2)+(y_1+y_2), (x_1-x_2)+(y_1-y_2), x_2+y_2)$

by commutative property

$= ((x_1+x_2+y_1)+y_2, (x_1-x_2+y_1)-y_2, x_2+y_2)$

$= ((x_1+y_1+x_2)+y_2, (x_1+y_1-x_2)-y_2, x_2+y_2)$

by associative property

$= (x_1+x_2, x_1-x_2, x_2) + (y_1+y_2, y_1-y_2, y_2)$

$= T(x_1, x_2) + T(y_1, y_2)$

$= Tu + Tv$.

$$T(\alpha u) = T(\alpha(x_1, x_2))$$

$$= T(\alpha x_1, \alpha x_2)$$

$$= (\alpha x_1 + \alpha x_2, \alpha x_1 - \alpha x_2, \alpha x_2)$$

$$= \alpha(x_1 + x_2, x_1 - x_2, x_2)$$

$$= \alpha T(x_1, x_2) = \alpha T(u).$$

$\ker T = \{ (x,y) \mid T(x,y) = (0,0,0) \}$

$T(x,y) = (0,0,0)$

$\Rightarrow (x+y,\ x-y,\ y) = (0,0,0)$

$$\left.\begin{aligned} x+y &= 0 \\ x-y &= 0 \\ y &= 0 \end{aligned}\right\} \quad x = y = 0.$$

$\ker T = \{0\}$

Nullity $T = \dim \ker T$

So, by Rank-Nullity theorem,

$2 = \dim \ker T + \dim \operatorname{Im} T$

$\Rightarrow 2 = 0 + \operatorname{Rank} T$,

- Let $T: V \to U$ be a linear map, T is called Singular if there exists a $v \neq 0 \in V$ such that $Tv = 0$. Further $Tv = 0 \Rightarrow v = 0$ then T is called Non-Singular map.

Theorem :-

Let $T: V \to U$ be a linear map. T is non-singular iff image of linearly independent set of vectors is linearly independent.

Proof :

Let T is non-singular and $\{v_1, v_2, v_3, \ldots, v_n\}$ is linearly independent set of vectors of V. We prove $\{Tv_1, Tv_2, \ldots, Tv_n\}$ is linearly independent.

Let $\alpha_1 Tv_1 + \alpha_2 Tv_2 + \alpha_3 Tv_3 + \cdots + \alpha_n Tv_n = 0$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n) = 0$

$\because$ T is linear;

$\because$ T is non-singular

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0.$

$\because \{v_1, v_2, \ldots, v_n\}$ is linearly independent,

So, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

$\therefore \{Tv_1, Tv_2, \ldots, Tv_n\}$ is linearly independent too.

$\{v\}$, $\{Tv\}$

If $v \neq 0$, $\{v\}$ is linearly independent. So, $\{Tv\}$ is linearly independent. Hence $Tv \neq 0$. So, $v \neq 0 \Rightarrow Tv \neq 0$.

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$\therefore Tv = 0 \Rightarrow v = 0.$

$T: V^n \to U$ linear and $\{v_1, v_2, \ldots, v_n\}$ is a basis of V. Then max. linearly independent subset of $\{Tv_1, Tv_2, \ldots, Tv_n\}$ is a basis of $\operatorname{Im} T$.

Pf :
First we need to prove that $\{Tv_1, Tv_2, \ldots, Tv_n\}$ span $\operatorname{Im} T$

Span $\{Tv_1, Tv_2, \ldots, Tv_n\}$

$= \alpha_1 Tv_1 + \alpha_2 Tv_2 + \cdots + \alpha_n Tv_n$

$= T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$.

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$= \text{Span}\{v_1, v_2, \ldots, v_n\}$

$= V.$

Let $u \in \operatorname{Im} T$, $\exists\, v \in V$ such that

$u = Tv$

$= T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$

$= \alpha_1 Tv_1 + \alpha_2 Tv_2 + \cdots + \alpha_n Tv_n$

So, $\{Tv_1, Tv_2, \ldots, Tv_n\}$ spans $\operatorname{Im} T$.

Matrix representation of a linear transformation :

Let $T: V^n \to U^m$ be a linear transformation.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V; and,

$\{u_1, u_2, \ldots, u_m\}$ be a basis of U.

$T(v_j) \in U^m \Rightarrow Tv_j = a_{1j} u_1 + a_{2j} u_2 + \cdots + a_{ij} u_i + \cdots + a_{mj} u_m$

$1 \le i \le m$

$a_{1j}, a_{2j}, \ldots, a_{mj}$ are a unique set of scalars.

$$Tv_j = \sum_{i=1}^{m} a_{ij} u_i$$

$1 \le j \le n$

$\forall\, v_1, v_2, \ldots, v_n$ we get unique mn scalars a_{ij}, $1 \le i \le m$, $1 \le j \le n$.
If we write these unique set of scalars a_{ij} in row and column like as —

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$$A_T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

This matrix is called a matrix of linear transformation from basis of V to basis of U.

Q. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$, using the bases $B_1 = B_2 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\}$. Compute A_T.

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Let $\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$\Rightarrow \alpha - 3\beta = 0$ and $-\alpha + 2\beta = 2$

$$\Rightarrow \left.\begin{array}{l} \alpha = -6 \\ \beta = -2 \end{array}\right]$$

$$T\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3+2 \\ -3-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -5 \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \delta \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$\Rightarrow \gamma - 3\delta = -1$ and $-\gamma + 2\delta = -5$

$\therefore \gamma = 3\delta - 1 = 17$

$\Rightarrow -5 = -(-1+3\delta) + 2\delta$

$\Rightarrow -5 = 1 - \delta$

$\Rightarrow \delta = 6$.

$$\therefore \left.\begin{array}{l} \gamma = 17 \\ \delta = 6. \end{array}\right]$$

So, $A_T = \begin{pmatrix} -6 & 17 \\ -2 & 6 \end{pmatrix}$.

• If $B_1 = B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

and, $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

then, $B_T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$

$$B_T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

$$A_T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 50 \\ 18 \end{pmatrix} \neq \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

$$T : \underset{B_1}{V^m} \longrightarrow \underset{B_2}{U^n}$$

$$(Tv)_{B_2} = A_T (x)_{B_1}.$$

$$T : V^n_{B_1} \longrightarrow U^n_{B_2}$$
$\quad$ (basis) $\quad$ (basis)

$A_{T_{n\times n}} \longleftarrow$ obtained from Standard Basis of U and V.

$$Tx = A_T(x).$$

If B_T is a matrix of T regarding non-std. basis of V and U, Then,

$$(Tx)_{B_2} = A_T(x)_{B_1}..$$

Pf :

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Let $\{e_1, e_2, \cdots, e_n\}$ is std. basis of V.

$$e_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \longrightarrow \text{ith place}$$

Suppose, $Te_1 = w_1$ $\quad$ w_i is m-tuple vector.

$Te_2 = w_2$

$\cdots$

$\cdots$

$Te_i = w_i$

$\cdots$

$Te_n = w_n$.

$$A_T = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}_{m\times n}$$

$$w_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \longleftarrow \text{ith column of } A_T.$$

$$Te_i = w_i$$

$$A_T e_i = \begin{array}{c} \qquad\qquad\qquad \text{ith column} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mi} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} = w_i$$

$$Te_i = A_T e_i \qquad \forall i, 1 \le i \le n$$

$$\Rightarrow Tx = A_T x \qquad \forall x \in \mathbb{R}^n$$

$$\Rightarrow T = A_T$$

Suppose $\exists B_T$ such that, $Tx = B_T x$.

In that case, $\underbrace{A_T x = B_T x}$ $= Tx$.

$$\Rightarrow (A_T - B_T)x = 0$$

$\Rightarrow \underbrace{(A_T - B_T)}_{C} = 0$

$Ce_i = 0$

This means that ith column of C matrix has all its elements $= 0$ $\forall i,\ 1 \le i \le n$

$\therefore$ C is a null matrix

$\therefore$ $A_T = B_T$. ← uniqueness.

Theorem :=

Let V and U be n and m dimensional vector spaces over the same field F respectively. Let $B_1 = \{v_1, v_2, \dots, v_n\}$ and $B_2 = \{u_1, u_2, \dots u_m\}$ be the bases of V and U respectively. Then, there exists a unique matrix A_T for $T : V \longrightarrow U$ such that

$$(Tx)_{B_2} = (A_T(x))_{B_1}.$$

$x \in V$.

Hint :

B_1 is a basis of V.

So, $\exists$ unique set of scalars $c_1, c_2, \dots, c_n$ such that,

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

So,

$$(x)_{B_1} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$A_{T_{m \times n}} (x)_{B_1} = A_T \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}.$$

$$Tx_{B_2} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

$$Tx = d_1 u_1 + d_2 u_2 + \dots + d_m u_m.$$

Change of Basis :-

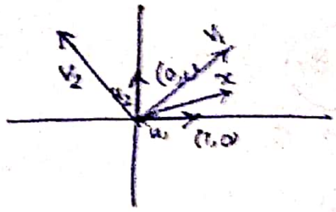
Ex : 1

Let $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha v_1 + \beta v_2 .$$
$$= \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \frac{2}{5} \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{3}{5} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \frac{2}{5} v_1 - \frac{3}{5} v_2$$

Similarly,

$$u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5} v_1 + \frac{1}{5} v_2 .$$



$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$; $B_1 = \left\{ \underset{u_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$

$x = x_1 u_1 + x_2 u_2 .$

$B_2 = \{ v_1 , v_2 \}$

$x = c_1 v_1 + c_2 v_2 . \longrightarrow (\mathrm{I})$

We need to determine the value of c_1, c_2.

Now, $x = x_1 u_1 + x_2 u_2$

$$= x_1 \left(\frac{2}{5} v_1 - \frac{3}{5} v_2 \right) + x_2 \left(\frac{1}{5} v_1 + \frac{1}{5} v_2 \right)$$
$$= \left(\frac{2x_1}{5} + \frac{x_2}{5} \right) v_1 + \left(\frac{-3x_1}{5} + \frac{x_2}{5} \right) v_2 \longrightarrow (\mathrm{II})$$

So, comparing (I) and (II) :

$$c_1 = \frac{2x_1}{5} + \frac{x_2}{5}$$
$$c_2 = -\frac{3}{5} x_1 + \frac{1}{5} x_2 .$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

Transition Matrix from ~~matrix~~ B_1 to B_2 and we have, $(x)_{B_2} = A (x)_{B_1}$

Now, if $x = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

then
$$\begin{pmatrix} 2/5 & 1/5 \\ -3/5 & 1/5 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -13/5 \end{pmatrix}$$

• The $n \times n$ matrix A whose columns are given by $u_j = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{ij}v_i + \cdots + a_{nj}v_n$

or, $(u_j)_{B_2} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$

is called the ~~transformation~~ transition matrix from basis B_1 to basis B_2.
That is,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ (u_1)_{B_2} & (u_2)_{B_2} & (u_n)_{B_2} \end{array}$$

Theorem : Let B_1 and B_2 be the bases for a vector space V. Let A be the transition matrix from B_1 to B_2. Then for every $x \in V$,

$$(x)_{B_2} = A(x)_{B_1}.$$

Theorem : If A is a transition matrix from B_1 to B_2; then A^{-1} is the transition matrix from B_2 to B_1.

Pf :

$(x)_{B_2} = A(x)_{B_1}$

Suppose C is the transition matrix from B_2 to B_1.

$\therefore (x)_{B_1} = C(x)_{B_2}$

$\Rightarrow (x)_{B_2} = AC(x)_{B_2}$.

Prove that $AC = I$.

Rank of a Matrix

Maximum number of linearly independent rows (columns) of matrix is called Rank of a Matrix.

→ Rank of a null matrix is zero.
→ Rank of a non-zero matrix ≥ 1.
→ If A is a $m \times n$ matrix, then rank A $\leq \min\{m, n\}$.
→ Rank of identity matrix of order n is "n", i.e. rank $I_n = n$.

Echelon form of a matrix :-

1. A non-zero row (if any) precede the zero rows.
2. The number of zeros preceding the first non-zero entry in a row increases row by row until zero row remains.

e.g.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank $I_3 = 3$

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 1 \\ 0 & 2 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank A = 4

[Rank = No. of non-zero rows].
After conversion into Echelon form.

Elementary operations :-

1. Interchange any two rows (or, columns) i.e. $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$)
2. Any row (column) multiplied by some non-zero scalar say $k \neq 0$.
$R_i \to kR_i$
$(C_i \to kC_i)$
3. $R_i \to R_i + kR_j$ $(C_i \to C_i + kC_j)$
$k \neq 0$.

Equivalent matrix :

Two matrices A and B of same order are said to be equivalent if one can be obtained by other by applying elementary operations. Symbolically, we write $A \sim B$.

Elementary matrix :

A matrix obtained by applying any one elementary operation on identity matrix is called Elementary matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 6R_2} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Theorem :

Each elementary row (column) operations on a $m \times n$ matrix A can be effected by pre-multiplying (post-multiplying) A by the corresponding elementary matrix of order m (n).

e.g :

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 5 & 3 \end{bmatrix}_{2\times 3} \xrightarrow{R_1 \leftrightarrow R_2} \underset{B}{\begin{bmatrix} 0 & 5 & 3 \\ 1 & 2 & 6 \end{bmatrix}} \xrightarrow{C_2 \leftrightarrow C_3} \underset{C}{\begin{bmatrix} 0 & 3 & 5 \\ 1 & 6 & 2 \end{bmatrix}}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Corresponding elementary matrix for row operation = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\therefore \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 3 \\ 1 & 2 & 6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = B.$$

For column operation ; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Corresponding elementary matrix = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\therefore B \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C.$$

So, $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C.$$

Theorem :

If a n-square matrix A is reduced to the identity matrix I_n by applying a finite number of elementary row operations, then matrix A is non-singular or invertible. Then A^{-1} can be obtained by applying the same sequence of elementary row operations on I_n which were applied on A to reduce it to I_n.

II :
E_r ---------- $E_2 E_1 A = I_n$

$\Rightarrow E_r$ ---------- $E_2 E_1 A A^{-1} = I_n A^{-1}$

$\Rightarrow E_r$ ---------- $E_2 E_1 I_n = A^{-1}$

e.g :

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \qquad A^{-1} = \frac{1}{-1}\begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right]$$

$R_1 \to \frac{1}{2} R_1$

$$\left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 5R_1$

$$\left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & -1/2 & -5/2 & 1 \end{array}\right]$$

$R_2 \to -2R_2$

$$\left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

$R_1 \to R_1 - 3/2 \, R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -7 & 3 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

$\frac{1}{2} - \frac{15}{2} = -\frac{14}{2} = -7$

$0 - 3/\cancel{2}(-\cancel{2}) = 3$

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→ Every elementary matrices are non-singular.

Pf: Let I_n be an identity matrix of order n.
E be a elementary matrix.

$$EE = I \longrightarrow \text{non-singular.}$$

[Product of non-singular matrices is non-singular].

A is non-singular.
B is any matrix.

$\boxed{\text{rank}(AB) = \text{rank } B}$ (if matrix multiplication is possible).

Normal Form of a Matrix :-

By finite number of elementary transformation, every non-zero matrix of order $m \times n$ can be reduced to one of the following forms :

(i) $\left[\begin{array}{c|c} I_r & O_{r\times(n-r)} \\ \hline O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{array}\right]$ (ii) $\left[\begin{array}{c} I_r \\ \hline O \end{array}\right]$ (iii) $\left[\begin{array}{c|c} I_r & O \end{array}\right]$ (iv) $[I_r]$ ↳ possible only when given matrix is non-singular.

where, I_r is the identity matrix of order 'r'.

These four forms are called Normal forms of a matrix.

Theorem :-

A $m\times n$ matrix 'A' is of rank 'r' iff there exists an m sq. non-singular matrix P and an n sq. non-singular matrix Q, such that :

$$PAQ = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$$

if, $PAQ = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

$\Rightarrow \text{rank}(PAQ) = \text{rank}\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

$\Rightarrow \text{rank } AQ = r$

$\Rightarrow \text{rank } A = r$

[Pf. of converse].

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$$P_{m\times m} \, A_{m\times n} \, Q_{n\times n} = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

non-singular

$$A_{m\times n} = I_{m\times m} \, A_{m\times n} \, I_{n\times n}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \qquad P A_{m\times n} Q.$$

Let F be a field and $x_1, x_2, \ldots\ldots, x_n$ are unknown scalars.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (I)$$

where, $a_{ij} \in F$ $(i = 1(1)m$ and $j = 1(1)n)$ & $b_i \in F$, $i = 1(1)m$

$$AX = B$$

where, $A = [a_{ij}]$ i & j $\quad B = (b_1, b_2, \ldots\ldots, b_m) \in F^m$

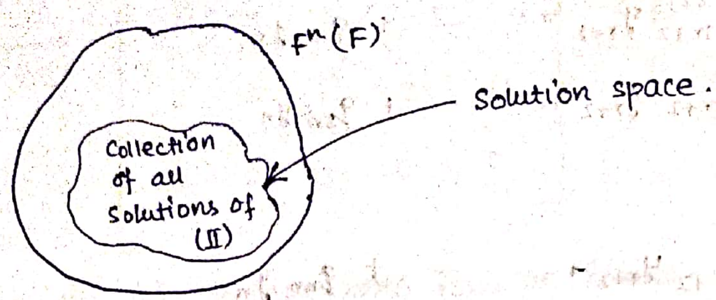
$X = (x_1, x_2, \ldots\ldots, x_n) \in F^n$

$F^n(F)$

- If $B = 0$, then, $AX = 0$ —(II) is called homogeneous linear system of equations.
- System of equations (I) is called consistent if there exist a solution of (I).
- Homogeneous linear system of equations are always consistent.

→ If x and y be two solutions of (II). Then $\alpha x + y$ is also a solution of (II).

$$\begin{aligned} &A(\alpha x + y) \\ &= \alpha Ax + Ay \\ &= \alpha \cdot 0 + 0 \\ &= 0. \end{aligned}$$



Theorem :

The number of linearly independent solutions of the homogeneous linear system of equations $AX = 0$ is $(n-r)$, where r is the rank of $m \times n$ order matrix A.

Pf :

Since A has rank r

$\Rightarrow \exists$ non-singular matrices P and Q of order m and n respectively such that

$$PAQ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right].$$

$$AX = O_{m\times 1}$$

$$PAX = PO = O$$

let $X = QY$; $\underline{Y \in F^n}$

i.e. $Y = (y_1, y_2, \ldots, y^n) \in F^n$.

$\because Q$ is non singular

$\therefore Y = Q^{-1}X$.

for any value of X, we get unique value of Y and vice-versa.

So, $\underline{PAQ}Y = 0$

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] Y = 0$$

This is possible only when first r components of Y are zero i.e. $y_1 = y_2 = 0 = \ldots = y_r$; and $(n-r)$ components of Y i.e. $y_{r+1}, y_{r+2}, \ldots, y_n$ are arbitrary.

Let $Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & & & \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$

Then, $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y_{r+1} \\ \vdots \\ y_n \end{bmatrix}$

$x_1 = q_{1r+1}\, y_{r+1} + q_{1r+2}\, y_{r+2} + \cdots\cdots + q_{1n}\, y_n$

$x_2 = q_{2r+1}\, y_{r+1} + q_{2r+2}\, y_{r+2} + \cdots\cdots + q_{2n}\, y_n$

$\vdots$

$x_n = q_{nr+1}\, y_{r+1} + q_{nr+2}\, y_{r+2} + \cdots\cdots + q_{nn}\, y_n$.

$\because y_{r+1}, \ldots\ldots, y_n$ are arbitrary

So, if we take $y_{r+1} = 1$ & $y_{r+2} = 0 = \cdots\cdots = y_n$

one solution will be

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} q_{1r+1} \\ q_{2r+1} \\ \vdots \\ q_{nr+1} \end{pmatrix}$$

$(r+1)$th column of Q.

Linearly independent solution of given system of equations.

Ex :

$A_{n\times n} X = 0$, $|A| \neq 0$] No linearly independent solution.

$A_{n\times n} X = 0$, $|A| = 0$, rank $A = r < n$, $\therefore (n-r) > 0$.]

Q: Find the dimension and a basis of the solution space of the system of homogeneous linear equations

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$$
$$x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 0$$
$$3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 = 0$$

Sol^n:

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix},$$

$\Big\downarrow R_3 \to R_3 - 2R_2$

*

pivot element

(first non-zero element of a row)

$$* \longrightarrow \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the no. of non-zero rows in Echelon form is 2; so rank of coefficient matrix A is 2.

So, the number of linearly independent solution of given system of equations is $5-2 = 3$.

From the echelon form, our system of equations is reduced to

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - x_5 = 0$$

$\because$ x_1 & x_3 are pivot ; so we take x_2, x_4, x_5 arbitrary.

x_1	-2	5	-7
x_2	1	0	0
x_3	0	-2	2
x_4	0	1	0
x_5	0	0	1

The dimension of solution space of given system of equations is 3, and a basis is

$$\{(-2,1,0,0,0),\ (5,0,-2,1,0),\ (-7,0,2,0,1)\}$$

Q. Find all the solutions of the given system of equations :

$$x_1 + 2x_2 - x_3 = 2$$

$$2x_1 + 3x_2 + 5x_3 = 5$$

$$-x_1 - 3x_2 + 8x_3 = -1 .$$

NOTE : General method to check whether the given system of lin. equ^{n}s $Ax = b$ is consistent or not :

$[A \mid b]$ if rank A = rank$[A|b]$: system is consistent

if rank $A \neq$ rank $[A|b]$: system is not consistent.

Theorem : If x_1 and x_2 be solutions of non-homogeneous linear system of equations $Ax = b$, then their difference i.e. $x_1 - x_2$ is a solution of the associated homogeneous linear system of equations $Ax = 0$.

Corollary : If y is any particular solution of $Ax = b$ and x is a solution of $Ax = b$, then there is a solution h to the associated homogeneous linear system of eq^{n}s $Ax = 0$ such that

$$x = y + h.$$

Previous Q :

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = b$$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 3 & 5 & 5 \\ -1 & -3 & 8 & -1 \end{array}\right]$$

by $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 7 & 1 \\ 0 & -1 & 7 & 1 \end{array}\right]$$

by $R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

here Rank A = Rank $[A|b] = 2$

$\Rightarrow$ System is consistent.

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Pf of Th :

let $Ax_1 = b$ & $Ax_2 = b$

$\therefore Ax_1 - Ax_2 = 0$

$\Rightarrow A(x_1 - x_2) = 0$.

Prev Q

$\therefore$ from echelon form of A :

$x_1 + 2x_2 - x_3 = 2$

$-x_2 + 7x_3 = 1$.

for y_p, let $x_3 = 0$

$\therefore x_2 = -1$, $x_1 = 4$

$\therefore y_p = (4, -1, 0)$

For h,

$x_1 + 2x_2 - x_3 = 0$

$-x_2 + 7x_3 = 0$.

let x_3 is free variable.

$\rightarrow x_2 = 7x_3$.

$\rightarrow x_1 + 14x_3 - x_3 = 0$

$\Rightarrow x_1 = -13x_3$

$\therefore h = (-13x_3, 7x_3, x_3)$

$\therefore h = (-13x_3, 7x_3, x_3)$

$\therefore$ All solutions of given system of equn.s are :

$= (4, -1, 0) + x_3(-13, 7, 1)$ $\quad x_3 \in \mathbb{R}$

Q: Determine K such that the following system of linear equations is consistent and hence obtain its solutions :

$$2x + y - z = 12$$
$$x - y - 2z = -3$$
$$3y + 3z = k$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 12 \\ 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & k \end{array}\right]$$

by $R_2 \rightarrow R_2 - \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 12 \\ 0 & -3/2 & -3/2 & -9 \\ 0 & 3 & 3 & k \end{array}\right]$$

by $R_3 \longrightarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 12 \\ 0 & -3/2 & -3/2 & -9 \\ 0 & 0 & 0 & k-18 \end{array}\right]$$

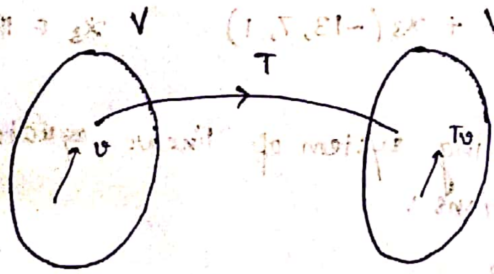
$\therefore$ for the system to be consistent,

$$k - 18 = 0$$
$$\Rightarrow k = 18$$

Eigen Values and Eigen Vectors :

(or) **Proper Value & Proper Vector:**

(or) **characteristic Value & characteristic Vector :**



$T : V \longrightarrow V$

linear transformation

$\boxed{\begin{array}{l} \text{if } v \parallel Tv \\ (\text{if } v \neq 0) \\ \& \ \exists\ \lambda, \text{ st} \\ \underline{Tv = \lambda v} \end{array}}$

Let A be 'n' square matrix with real component (may be complex). The number λ (real or complex) is called Eigen value of A if there exists a non-zero vector $v \neq 0 \in \mathbb{C}^n$, such that :

$$Av = \lambda v$$

here, v is called the eigen vector of A corresponding to the eigen value λ.

Identity transformation

$I : V \longrightarrow V$

$Iv = v = 1.v$

$v \in \mathbb{C}^n$

$\because Av = \lambda v = \lambda I_n v$

$\Rightarrow (A - \lambda I_n)\, v = 0$ ——— (1)

$\because v \neq 0$

for non-trivial solution of equ^n. (1)

$$|A - \lambda I_n| = 0$$

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$\Rightarrow \quad p(\lambda) = (-1)^n \lambda^n + K_1 \lambda^{n-1} + \text{------------} + K_n = 0.$

$p(\lambda)$ is called the "<u>Characteristic polynomial in λ of matrix A</u>" and $p(\lambda)=0$ is called the "<u>characteristic equation of matrix A</u>".

<u>Q:</u> Find the eigen values and corresponding eigen vectors of the following matrix

$$A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$$

<u>Procedure</u>:

① $\det(A - \lambda I) = 0$

② Find the roots of $p(\lambda) = |A - \lambda I| = 0$

③ Solve the $(A - \lambda I)v = 0$ for different λ.

$$\therefore \quad |A - \lambda I| = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -(3-\lambda)(1+\lambda) + 5 = 0$$

$$\Rightarrow \quad -3 - 3\lambda + \lambda + \lambda^2 + 5 = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda + 2 = 0$$

roots are $1 \pm i$

$\lambda_1 = 1+i$ $\quad \lambda_2 = 1-i$

<u>Corresponding to $\lambda_1 = 1+i$</u>

$$\left(A - (1+i)I\right)v = 0$$

$$\left[\begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} - \begin{pmatrix} 1+i & 0 \\ 0 & 1+i \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 - (2+i)x_2 = 0$$

$$(2-i)x_1 - 5x_2 = 0$$

by $R_2 \to R_2 - (2-i)R_1$

$$x_1 - (2+i)x_2 = 0$$

$$0 - 0 = 0$$

One Linear Independent solution is :

$$\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$E_{\lambda = 1+i} = \left\{\begin{pmatrix} 2+i \\ 1 \end{pmatrix}\right\} \qquad E_{1-i} = \left\{\begin{pmatrix} 2-i \\ 1 \end{pmatrix}\right\}$$

$$\underset{D}{\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}} = \underset{C^{-1}}{\begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix}^{-1}} \underset{A}{\begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}} \underset{C}{\begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix}}$$

Q. Compute the eigen values and eigen vectors of $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

Theorem : λ is an eigen value of a square matrix A iff $|A - \lambda I| = 0$

Theorem : Eigen vectors corresponding to distinct eigen values are linearly independent.

i.e. if $v_1, v_2, \ldots, v_n$ are eigen vectors corresponding to distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

Pf : Let $\lambda_1 \neq \lambda_2$

$$\begin{matrix} \lambda_1 & \neq & \lambda_2 \\ \downarrow & & \downarrow \\ v_1 & & v_2 \end{matrix}$$

$Av_1 = \lambda_1 v_1$, and, $Av_2 = \lambda_2 v_2$.

$c_1 v_1 + c_2 v_2 = 0$ —①

$A(c_1 v_1 + c_2 v_2) = A0 = 0$

$\Rightarrow c_1 A v_1 + c_2 A v_2 = 0$

$\Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$ —②

From ② $- \lambda_1 \times$①

$c_2 \lambda_2 v_2 - c_2 \lambda_1 v_2 = 0$

$\Rightarrow c_2(\lambda_2 - \lambda_1) v_2 = 0.$

$\because v_2 \neq 0$ and $\lambda_1 \neq \lambda_2$

$\Rightarrow c_2 = 0$

$\therefore$ By equation ① $\Rightarrow c_1 v_1 + 0 = 0$

$\Rightarrow c_1 = 0$ $[\because v_1 \neq 0]$.

Suppose

$\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Now we prove,

$\{v_1, v_2, \ldots, v_{k+1}\}$ is linearly independent.

Take,

$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k + c_{k+1} v_{k+1} = 0$ —③

$\Rightarrow A(c_1 v_1 + \ldots + c_k v_k + c_{k+1} v_{k+1}) = A0 = 0$

$\Rightarrow c_1 A v_1 + \ldots + c_k A v_k + c_{k+1} A v_{k+1} = 0$

$\Rightarrow c_1 \lambda_1 v_1 + \ldots + c_k \lambda_k v_k + c_{k+1} \lambda_{k+1} v_{k+1} = 0$ —④

From ④ $- \lambda_{k+1} \times$③

$\underline{c_1(\lambda_1 - \lambda_{k+1}) v_1} + \ldots + \underline{c_k(\lambda_k - \lambda_{k+1}) v_k} = 0$

as $\{v_1, v_2, \ldots, v_k\}$ are supposed to be LI, and, $v_1, \ldots, v_k \neq 0$

$\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_{k+1}$

$\because c_1 = c_2 = \ldots = c_k = 0$ —⑤

From ⑤ and ③ & as $v_{k+1} \neq 0$,

So we get $c_{k+1} = 0$.

Cayley - Hamilton Theorem :-

"Every square matrix satisfies its characteristic equation."

i.e. if A is n square matrix and $(-1)^n \lambda^n + k_1 \lambda^{n-1} + \ldots + k_n = 0$ is a characteristic equation of A; then

$$(-1)^n A^n + k_1 A^{n-1} + \ldots + k_n I_n = 0.$$

Further if A is non-singular, then

$$A^{-1} = -\frac{1}{k_n}\left[(-1)^n A^{n-1} + \ldots + k_{n-1} I_n\right]$$

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Example :

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda) - 2 = 0$$

$$\Rightarrow 4 - \lambda - 4\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 2 = 0$$

$$\therefore \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}^2 - 5 \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 5 & 18 \end{bmatrix}$$

$$\therefore A^2 - 5A + 2I_2 = 0$$

$\Rightarrow A^{-1} = -\frac{1}{2}$ [[redacted] $A - 5I_n$]

Q. Find the Eigen values and Eigen vectors for this matrix : $\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

Ans : $A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$.

$\therefore \quad A - \lambda I = \begin{pmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{pmatrix}$

$\therefore \quad |A - \lambda I| = 0$

$\Rightarrow \quad (4-\lambda)^2 - 0 = 0.$

$\Rightarrow \quad \lambda = 4.$

$\therefore$ Corresponding to the eigen value $\lambda = 4$, eigen vector is let $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\therefore \quad (A - 4I)\, v = 0$

$\Rightarrow \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

$\left.\begin{array}{l} x_2 = 0 \\ x_1 \text{ is arbitrary.} \end{array}\right\}$ So, $E_4 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$\therefore \quad A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

→ If, $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

Here also eigen value $\lambda = 4$.

$\therefore \quad (A - 4I)\, v = 0$

$\Rightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

[Eigen space for eigen value 4 for this A is whole xy-plane.]

This equation is satisfied by any vector of $\mathbb{R}^2$.

$\therefore \quad E_4 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\therefore \quad A\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}.$

Algebraic multiplicity of an eigen value λ = The number of times λ repeated.
(AM)

Geometric multiplicity of an eigen value λ = The number of linearly independent eigen vectors corresponding to the eigen value λ.
(GM)

$$\boxed{GM \leq AM}$$

Similar matrices :

Let A and B are two square matrices of same order is said to be similar if there exists a non-singular matrix C such that

$$B = C^{-1} A C$$

- Similar matrices have the same eigen values and hence the same eigen vectors. (not confirmed)

Pf : A and B are similar $\Rightarrow B = C^{-1} A C$.
Then we have to show that $|B - \lambda I| = |A - \lambda I|$.

$$\therefore \; |B - \lambda I| = |C^{-1} A C - \lambda I|$$

$$= |C^{-1} A C - C^{-1}(\lambda I) C|$$

$$= |C^{-1}(A - \lambda I) C|$$

$$= |C^{-1}|\,|A - \lambda I|\,|C|$$

$$= |A - \lambda I|\,|C^{-1}|\,|C|$$

$$= |A - \lambda I|\,|C^{-1} C|$$

$$= |A - \lambda I|\,|I_n|$$

$$= |A - \lambda I|.$$

Cayley - Hamilton Theorem :

Every square matrix satisfies its characteristic equn. i.e. if A is a 'n' square matrix and its characteristic equ.n is

$$|A-\lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + \text{------} + k_n = 0.$$

Then

$$(-1)^n A^n + k_1 A^{n-1} + \text{------} + k_n I_n = 0$$

Pf :

$$P = \operatorname{adj}(A-\lambda I)$$

$\because$ each element of P is a polynomial in λ of degree $\leq (n-1)$

Therefore, we can split P into different matrices that have elements of same power λ.

$$\text{so, } P = P_0 \lambda^{n-1} + P_1 \lambda^{n-2} + \text{----} + P_{n-2} \lambda + P_{n-1}$$

$$\because \quad (A-\lambda I) P = |A-\lambda I| \, I_n$$

$$\Rightarrow (A-\lambda I)(P_0 \lambda^{n-1} + P_1 \lambda^{n-2} + \text{----} + P_{n-2}\lambda + P_{n-1}) = [(-1)^n \lambda^n + k_1 \lambda^{n-1} + \text{-----} + k_{n-1}\lambda + k_n] I_n$$

Comparing the coefficients, we get :

$$-P_0 = (-1)^n I_n \quad \times A^n$$
$$AP_0 - P_1 = k_1 I_n \quad \times A^{n-1}$$
$$AP_1 - P_2 = k_2 I_n \quad \times A^{n-2}$$
$$\text{------}$$
$$AP_{n-2} - P_{n-1} = k_{n-1} I_n \quad \times A$$
$$AP_{n-1} = k_n \quad \times I_n$$

(Add)

$$O = (-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \text{-----} + k_{n-1} A + k_n I_n$$

Proved.

Inner Product Space

$\mathbb{R}^3$

$u = (a_1, a_2, a_3) \in \mathbb{R}^3$
& $v = (b_1, b_2, b_3) \in \mathbb{R}^3$

Dot product $u \cdot v = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$|u| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

$f: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined as

$f(u, v) = u \cdot v$

Definition :=

Let V be a vector space over a field F ($\mathbb{R}$ or $\mathbb{C}$). A map $\bullet : V \times V \longrightarrow F$ is called an Inner Product if the following conditions hold :

[image of (u,v) under '$\cdot$' denoted as $\langle u, v\rangle$ i.e. $\bullet(u,v) = \langle u, v\rangle$]

(I) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

(II) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ (complex conjugate of $\langle u, v\rangle$)

(III) $\langle u, u\rangle \geq 0$ or, $\langle u, u\rangle = 0$ iff $u = 0$.

This condition (I) is known as 'Linear in first co-ordinate'.

Note :

$$\langle u, \alpha v + \beta w\rangle = \bar{\alpha}\langle u, v\rangle + \bar{\beta}\langle u, w\rangle$$

$$\overline{\langle \alpha v + \beta w, u\rangle} = \overline{\alpha\langle v, u\rangle + \beta\langle w, u\rangle} = \bar{\alpha}\,\overline{\langle v, u\rangle} + \bar{\beta}\,\overline{\langle w, u\rangle}$$

- A vector space with an Inner Product defined is called 'Inner Product Space'.

§ An Inner Product Space over a Real field is called Euclidean space.

§ An Inner Product Space over a complex field is called Unitary Space.

Ex ①

$V = \mathbb{C}^n \quad F = \mathbb{C}$

$u = (z_1, z_2, \ldots, z_n) \qquad z_i \in \mathbb{C}$

$v = (w_1, w_2, \ldots, w_n) \qquad w_i \in \mathbb{C}$

$\langle u, v\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n \longrightarrow$ $\boxed{\text{Dot product in complex space.}}$

Is $\langle u, v\rangle$ an Inner Product ?

Ans : $\langle u, u\rangle = z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n$

$= |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$

≥ 0

(condition III).

$\langle u, v\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$

$\langle v, u\rangle = w_1\bar{z}_1 + w_2\bar{z}_2 + \cdots + w_n\bar{z}_n$

$\therefore \overline{\langle v, u\rangle} = \overline{w_1\bar{z}_1 + w_2\bar{z}_2 + \cdots + w_n\bar{z}_n}$

$= \bar{w}_1 z_1 + \bar{w}_2 z_2 + \cdots + \bar{w}_n z_n$

$= z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$

$= \langle u, v\rangle$ (condition II).

Take, $w = (d_1, d_2, \ldots d_n) \quad d_i \in \mathbb{C}$

$\langle \alpha u + \beta v, w\rangle = \langle (\alpha z_1 + \beta w_1, \alpha z_2 + \beta w_2, \ldots, \alpha z_n + \beta w_n), (d_1, d_2, \ldots, d_n)\rangle$

$= (\alpha z_1 + \beta w_1)\bar{d}_1 + (\alpha z_2 + \beta w_2)\bar{d}_2 + \cdots + (\alpha z_n + \beta w_n)\bar{d}_n$

Ex ②

Let V is set of all real valued continuous functions on $[0, 1]$.

$\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$.

Let V be an inner product space. If $\langle u,v\rangle = 0$, $\forall\ v \in V$, then, $u = 0$.

Pf: If $v = u$

$\langle u,u\rangle = 0$

$\Rightarrow u = 0$ (by condition III).

• **Norm or Length of a vector in an Inner Product Space 'V' :-**

The norm or length of a vector v in an Inner Product Space V denoted as $\|v\|$ is defined as

$$\|v\| = \sqrt{\langle v,v\rangle} \quad \text{(non-negative real number).}$$

$$\left\|\frac{u}{\|u\|}\right\|^2 = \left\langle \frac{u}{\|u\|}, \frac{u}{\|u\|}\right\rangle = \frac{1}{\|u\|^2}\langle u,u\rangle = \frac{\|u\|^2}{\|u\|^2} = 1$$

We have,

$\frac{u}{\|u\|} \longrightarrow$ Unit vector / Normalized vector.

Properties :-

① $\|v\| \geq 0$, or, $\|v\| = 0$ iff $v = 0$

② $\|kv\| = |k|\,\|v\|$ (k is any scalar) $\longrightarrow$ $\|kv\|^2 = \langle kv, kv\rangle$

$= k\bar{k}\langle v,v\rangle$

$= |k|^2\langle v,v\rangle$

$= |k|^2\|v\|^2$.

③ $\|u+v\| \leq \|u\| + \|v\|$

Cauchy-Schwarz Inequality:

For any two vectors u and v of an Inner Product Space V,

$$|\langle u,v\rangle| \leq \|u\|\,\|v\|,$$

where u and v are linearly independent.

•
$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \|u\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|v\|^2 \\
&= \|u\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} + \|v\|^2 \\
&= \|u\|^2 + 2\,\mathrm{Re}\langle u,v\rangle + \|v\|^2 \\
&\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2 \\
&\leq \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2
\end{aligned}$$

$\leq (\|u\| + \|v\|)^2$

Angle b/w two vectors in an Euclidean Space :-

Angle b/w two vectors (u, v) in an Euclidean space is :

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\| \, \|v\|}$$

$$|\cos\theta| = \left| \frac{\langle u, v\rangle}{\|u\| \, \|v\|} \right| \leq 1 .$$

$\theta = \pi/2$

$\Rightarrow \langle u, v\rangle = 0$

$u \perp v$ if $\langle u, v\rangle = 0$

Distance b/w u & v is

$\|u - v\|$.

i) $\|u-v\| \geq 0$ & $\|u-v\| = 0$ iff $u = v$.

ii) $\|u-v\| = \|v-u\|$

iii) $\|u-v\| \leq \|u-w\| + \|w-v\|$

Orthogonal and Orthonormal Set of an Inner Product Space V :-

A set $S = \{v_1, v_2, \ldots, v_n\}$ of vectors in an IPS V is called Orthogonal if $\langle v_i, v_j\rangle = 0 \ \forall \ i \neq j$.

If further, $\langle v_i, v_i\rangle = 1$,

Then, S is called Orthonormal set.

e.g. : $R^3(R)$

(1)

$$\{ \underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)} \}$$

$\{e_1, e_2, e_3\}$ is an Orthonormal set.

Q Let V be an Inner Product Space of real valued continuous functions defined in $[-\pi, \pi]$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

$S = \{ \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots \}$ $\longrightarrow$ [Orthogonal but not Orthonormal Set]

$$\underset{(m \neq n)}{\langle \cos mt, \cos nt \rangle} = \int_{-\pi}^{\pi} \cos mt \cos nt \, dt = 0$$

$$\langle \cos mt, \sin nt \rangle = \int_{-\pi}^{\pi} \cos mt \sin nt \, dt = 0.$$

$$\underset{(m \neq n)}{\langle \sin mt, \sin nt \rangle} = \int_{-\pi}^{\pi} \sin mt \cdot \sin nt \, dt = 0$$

§: Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal set of an Inner Product Space V and v is any vector of V, then the vector

$$w = v - \langle v, v_1 \rangle v_1 - \cdots - \langle v, v_i \rangle v_i - \cdots - \langle v, v_n \rangle v_n$$

is ~~the~~ orthogonal to each v_i.

<u>Pf</u>: $\langle w, v_i \rangle = \langle v - \langle v, v_1 \rangle v_1 - \cdots - \langle v, v_i \rangle v_i - \cdots - \langle v, v_n \rangle v_n, v_i \rangle$

$= \langle v, v_i \rangle$ [redacted] $\langle v, v_i \rangle \langle v_i, v_i \rangle$ $=1$

$= 0$

§: Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of non-zero vectors of IPS V, then S is linearly Independent.

<u>Pf</u>:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

$$\langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_i \rangle = \langle 0, v_i \rangle$$

$$\alpha_i \langle v_i, v_i \rangle = 0$$

$$v_i \neq 0 \Rightarrow \underline{\alpha_i = 0}.$$

Gram - Schmidt Orthogonalization Process :-

" Every finite dimensional Inner product space V has an ~~orthogonal~~ orthonormal basis".

Pf : Let V is a finite dimensional IPS of dimension 'n' and $\{v_1, v_2, \dots, v_n\}$ is a basis.

$\{u_1, u_2, \dots, u_n\}$

$$u_1 = \frac{v_1}{\|v_1\|}$$

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$w_2 \neq 0$ [otherwise this contradicts to be $\{v_1, v_2\}$ linearly independent]

$$u_2 = \frac{w_2}{\|w_2\|}.$$

Proceeding in this way

let $w_{i+1} = v_{i+1} - \langle v_{i+1}, u_1 \rangle u_1 - \dots \langle v_{i+1}, u_i \rangle u_i$

Hence, w_{i+1} is orthogonal to each u_i $\quad i = 1(1)i$

$$\therefore u_{i+1} = \frac{w_{i+1}}{\|w_{i+1}\|}$$

Proceeding in this way we construct an orthonormal basis $\{u_1, u_2, \dots, u_n\}$.

Q : Transform the following basis into orthonormal basis.

$\{v_1 = (1,1,1), v_2 = (0,1,1), v_3 = (0,0,1)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{\langle (1,1,1),(1,1,1)\rangle}} = \frac{(1,1,1)}{\sqrt{(1+1+1)}}$$

$$= \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right).$$

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1 = (0,1,1) - \langle (0,1,1) \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right) \rangle \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)$$

$$= (0,1,1) - \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

$$\therefore u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)}{\sqrt{\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9}}} = \frac{\left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)}{\sqrt{\tfrac{6}{9}}} = \left(-\tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right)$$

and, $w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$

$$= (0,0,1) - \langle (0,0,1)\ (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \rangle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$- \langle (0,0,1)\ (-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}) \rangle \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\therefore u_3 = \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + \frac{1}{4}}} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

$\therefore$ The required orthonormal basis is :

$$\left\{ u_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),\ u_2 = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),\ u_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$

- If Q is an Orthogonal matrix, then,

$$QQ^t = I.$$

Q ① If u and v are vectors in IPS, then prove that

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

② If u and v are two vectors in real inner product space V, then $u \perp v$ iff $\|u-v\|^2 = \|u\|^2 + \|v\|^2$.

28.2.19

$\langle AB \rangle = tr(AB^t)$

$S \subset V$

$S^{\perp} = \{ v \in V \mid \langle v, u \rangle = 0 \ \forall u \in S \}$

$W \subset V$

$$\left. \begin{array}{l} V = W + W^{\perp} \\ \& \ W \cap W^{\perp} = \{0\} \end{array} \right\} V = W \oplus W^{\perp}$$

FOURIER SERIES

- $f(x) = f(x+T)$, $\forall x$

  T is a positive no.

  T : period of the function.

- $f(x) = f(-x)$: even $\rightarrow$ symmetric about y-axis.
- $f(-x) = -f(x)$: odd $\rightarrow$ symmetric about origin.

$f(x) \qquad c < x < c+2\pi$

$$f(x) = \frac{1}{2} a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots\ldots$$

$$c < x < c+2\pi.$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)}$$

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos(nx)\,dx.$$

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin(nx)\,dx.$$

Case-1:

$c = -\pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx \quad \text{, and,} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$$

If $f(x)$ is even

then, $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx .$$

If $f(x)$ is odd

then, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

and, $a_0 = a_n = 0$.

Q. Express $f(x) = |x|$, $-\pi < x < \pi$ as Fourier series. Hence show that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

Soln: Since $f(x) = |x|$ is even,

Hence, $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \times \frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\left(x\,\frac{\sin(nx)}{n}\right)_0^{\pi} - \int_0^{\pi} 1.\,\frac{\sin(nx)}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{n}\cdot\left(-\frac{\cos(nx)}{n}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\left(\frac{(-1)^n - 1}{1}\right)\right]$$

$$= \frac{2}{\pi}\,\frac{((-1)^n - 1)}{n^2}.$$

$$= \begin{cases} -\frac{4}{\pi n^2}, & n = \text{odd} \\ 0, & n = \text{even}. \end{cases}$$

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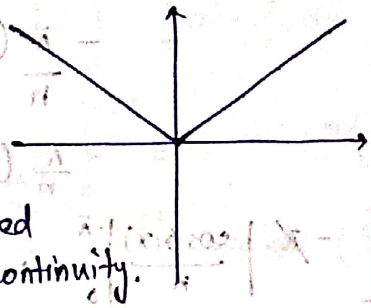
$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\Rightarrow |x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

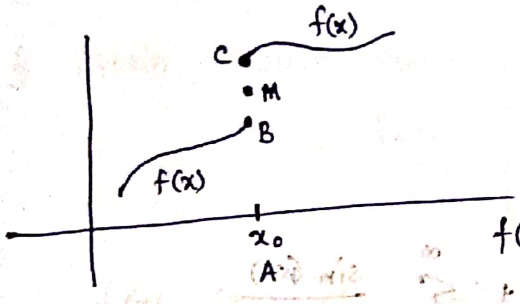
Put $x = 0$

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$



- A discontinuous function can also be expressed in Fourier series only if it has a finite discontinuity.



$x = x_0$ is a finite discontinuity.

$$f(x_0) = \frac{\lim (f(x_0^-)) + \lim f(x_0^+)}{2}$$

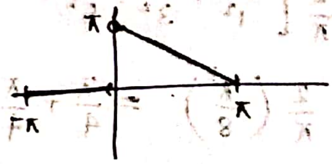
$$= \frac{AB + AC}{2} = AM$$

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f(x)\,dx + \int_{x_0}^{c+2\pi} f(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f(x)\cos(nx)\,dx + \int_{x_0}^{c+2\pi} f(x)\cos(nx)\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f(x)\sin(nx)\,dx + \int_{x_0}^{c+2\pi} f(x)\sin(nx)\,dx\right].$$

Q. Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right] = \frac{1}{\pi}\left[\int_0^{\pi} (\pi - x)\,dx\right] = \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \int_0^{\pi} f(x)\cos(nx)\,dx\right] = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi \sin(nx)}{n}\right)_0^{\pi} - \left(\frac{(-1)^n - 1}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1 - (-1)^n}{n^2}\right].$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \int_0^{\pi} f(x)\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin(nx)\,dx.$$

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Now, $\int_0^{\pi} x \sin(nx)\,dx = \left[x \cdot \left(-\frac{\cos(nx)}{n}\right)\right]_0^{\pi} - \int_0^{\pi} 1 \cdot (-)\frac{\cos nx}{n}\,dx$

$= \left[-\frac{\pi}{n}\left((-1)^n\right) - 0\right] + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}$

$= -\frac{\pi (-1)^n}{n} + 0$

$= -\frac{\pi}{n}(-1)^n.$

$b_n = \frac{1}{\cancel{\pi}}\left[-\cancel{\pi}\left|\frac{\cos(nx)}{n}\right|_0^{\pi} + \frac{\cancel{\pi}}{n}(-1)^n\right]$

$= \left[-\frac{1}{n}\left((-1)^n - 1\right) + \frac{(-1)^n}{n}\right]$

$= \frac{1}{n}.$

$\therefore\ f(x) = \frac{1}{2}\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(\frac{1-(-1)^n}{n^2}\right)\cos(nx) + \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$

$\Rightarrow f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$

Also, $f(\pi) = \frac{\pi}{4} + \frac{2}{\pi}\left[-1 - \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} - \cdots\right]$

and, $f(0) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] + 0$

$= \frac{\pi}{4} + \frac{2}{\pi}\left(\frac{\pi^2}{8}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

$f(0) = \frac{f(0^+) + f(0^-)}{2} = \frac{\pi + 0}{2} = \pi/2$ (verified)

- $f(x)$ be defined in $[c, c+2l]$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$

$[c, c+2l] \longrightarrow [d, d+2\pi]$

$\frac{x}{l} = \frac{z}{\pi}$

$\Rightarrow x = \frac{lz}{\pi}$

$f(x) = f\left(\frac{lz}{\pi}\right) = F(z).$

when,

$x = c$

$$z = \frac{\pi c}{l} = d$$

$x = c + 2l$

$$z = \frac{\pi(c+2l)}{l} = d + 2\pi$$

$$= \frac{\pi c}{l} + 2\pi .$$

$$a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\,dz$$

$$a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos(nz)\,dz .$$

$$b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\sin(nz)\,dz .$$

$$a_0 = \frac{1}{\pi}\int_c^{c+2l} f(x)\cdot\frac{\pi}{l}\,dx = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx .$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx .$$

Q. Obtain Fourier series, for the function :

$$f(x) = \begin{cases} \pi x , & 0 \le x \le 1 \\ \pi(2-x) , & 1 \le x \le 2 . \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$a_0 = \frac{1}{l}\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx\right].$$

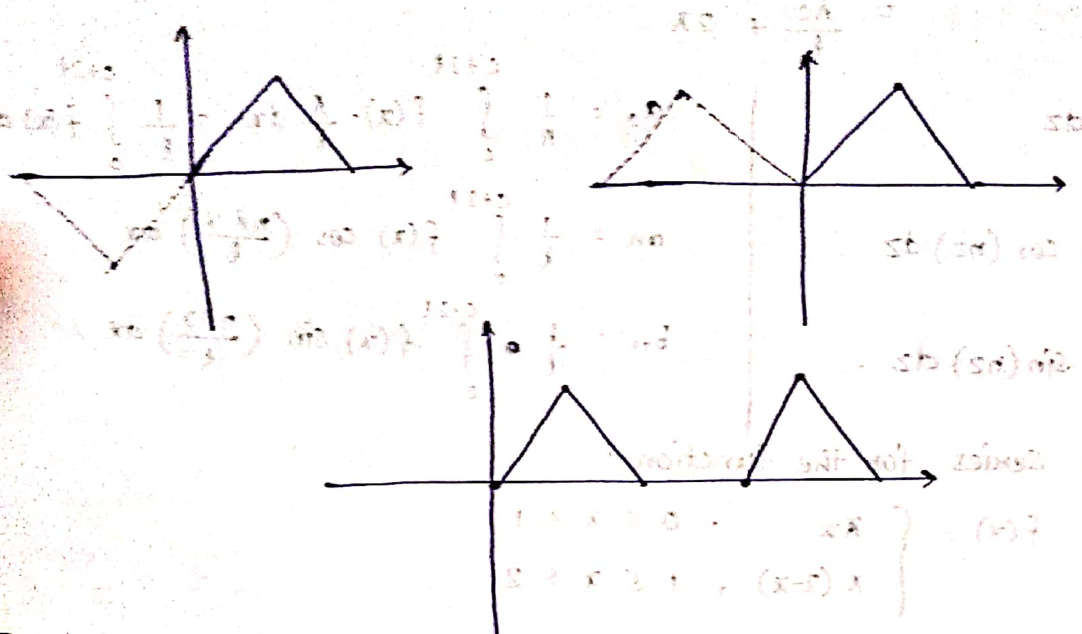
$$a_n = \frac{1}{l}\left[\int_0^1 f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$b_n = \frac{1}{l}\left[\int_0^1 f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_1^2 f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right].$$

HALF RANGE SERIES

$f(x) \longrightarrow (0, l)$

$(-l, 0)$



Q: Expand $f(x) = \pi x - x^2$ in a half range sine series in the interval upto first 3 terms.

$$f(x) = \sum b_n \sin(nx)\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi x - x^2) \cdot \sin(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin(nx)\, dx$$

$$= \frac{2}{\pi}\left[\pi \int_0^{\pi} x \sin(nx)\, dx - \int_0^{\pi} x^2 \sin(nx)\, dx\right]$$

$$= \frac{2}{\pi}\left[\pi\left(\frac{-\pi(-1)^n}{n}\right) - \int_0^{\pi} x^2 \sin(nx)\, dx\right]$$

Now,

$$\int_0^{\pi} x^2 \sin(nx)\, dx = \left[-\frac{x^2(\cos(nx))}{n}\right]_0^{\pi} + \int_0^{\pi} 2x \cdot \frac{\cos(nx)}{n}\, dx$$

$$= -\frac{\pi^2(-1)^n}{n} + \frac{2}{n}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}\, dx\right]$$

$$= -\frac{\pi^2(-1)^n}{n} + \frac{2}{n}\left(0 + \frac{\cos nx}{n^2}\Big|_0^{\pi}\right)$$

$$= -\frac{(-1)^n \pi^2}{n} + \frac{2}{n^3}\left((-1)^n - 1\right)$$

$$\therefore b_n = \frac{2}{\pi}\left[\pi\left(\cancel{\frac{-\pi(-1)^n}{n}}\right) + \cancel{\frac{(-1)^n \pi^2}{n}} - \frac{2}{n^3}\left((-1)^n - 1\right)\right]$$

$$= \frac{-4}{\pi n^3}\left((-1)^n - 1\right)$$

$$= \begin{cases} \dfrac{8}{\pi n^3}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Complex Form or Exponential form of Fourier Series :-

Let $f(x)$ be defined in $c < x < c+2l$ has a F.S. as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\sin\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i}.$$

$$\cos\left(\frac{n\pi x}{l}\right) = \frac{e^{i\left(\frac{n\pi x}{l}\right)} + e^{-i\left(\frac{n\pi x}{l}\right)}}{2}.$$

$$f(x) = \frac{a_0}{2} + \sum a_n\left(\frac{e^{i\left(\frac{n\pi x}{l}\right)} + e^{-i\left(\frac{n\pi x}{l}\right)}}{2}\right) + \sum b_n\left(\frac{e^{i\left(\frac{n\pi x}{l}\right)} - e^{-i\left(\frac{n\pi x}{l}\right)}}{2i}\right)$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{l}} + \sum\left(\frac{a_n + ib_n}{2}\right)e^{-\frac{in\pi x}{l}}$$

Let $C_0 = \frac{a_0}{2}$.

$C_n = \frac{a_n - ib_n}{2}$

$C_{-n} = \frac{a_n + ib_n}{2}$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$\int_c^{c+2l} f(x)\, e^{-in\pi x/l}\, dx = \int_c^{c+2l} C_n e^{\frac{in\pi x}{l}} e^{-\frac{in\pi x}{l}}\, dx$$

$$= 2C_n l.$$

Q. Express $f(x) = e^{-|x|}$, $-2 < x < 2$ in complex form of Fourier series.

Ans:- $l = 2$

$-2 < x < -2 + 2\times(2)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2l} \int_{-2}^{2} f(x) e^{-\frac{in\pi x}{2}} dx$$

$$= \frac{1}{2l} \left[\int_{-2}^{0} e^{x} \cdot e^{-\frac{in\pi x}{2}} dx + \int_{0}^{2} e^{-x} e^{-\frac{in\pi x}{2}} dx \right]$$

$$= \frac{1}{2\times 2} \left[\int_{-2}^{0} e^{\left(1-\frac{in\pi}{2}\right)x} dx + \int_{0}^{2} e^{-\left(1+\frac{in\pi}{2}\right)x} dx \right]$$

$$= \frac{1}{4} \left(\left[\frac{e^{(1-in\pi/2)x}}{1-in\pi/2} \right]_{-2}^{0} + \left[\frac{e^{-(1+n\pi/2)}}{-(1+in\pi/2)} \right]_{0}^{2} \right)$$

$$c_n = \frac{2}{4+n^2\pi^2} \left[1 - (-1)^n e^{-2} \right]$$

Q. Find the complex form of the Fourier series $f(x) = \cos(ax)$ in $-\pi < x < \pi$.

$$\left[\text{Ans :- } c_n = (-1)^n \frac{a \sin(a\pi)}{\pi (a^2 - n^2)} \right].$$

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Approximation by Trigonometric Polynomials
Square Error.

Let $f(x)$ be a periodic function of period 2π.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \sin(nx)$$

$$F(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{N} \alpha_n \cos nx + \sum_{n=1}^{N} \beta_n \sin nx .$$

$$E = \int_{-\pi}^{\pi} (f(x) - F(x))^2 dx = \int_{-\pi}^{\pi} f^2(x)\, dx - \int_{-\pi}^{\pi} 2f(x)F(x)\, dx + \int_{-\pi}^{\pi} F^2(x)\, dx$$

Square Error

$$\Rightarrow E = \int_{-\pi}^{\pi} f^2(x)\, dx - 2\pi \left[\frac{2\alpha_0 a_0}{4} + \sum_{n=1}^{N} (\alpha_n a_n + \beta_n b_n)\right] + \pi \left[\frac{2\alpha_0^2}{4} + \sum_{n=1}^{N} (\alpha_n^2 + \beta_n^2)\right]$$

if $\alpha_0 = a_0$, $\alpha_n = a_n$, $\beta_n = b_n$

$$E^* = \int_{-\pi}^{\pi} f^2(x)\, dx - \pi \left[\frac{2a_0^2}{4} + \sum_{n=1}^{N} (a_n^2 + b_n^2)\right]$$

$$E - E^* = \pi \left\{ \frac{2}{4} (\alpha_0 - a_0)^2 + \sum_{n=1}^{N} \left[(\alpha_n - a_n)^2 + (\beta_n - b_n)^2 \right] \right\}$$

$$E - E^* \geqslant 0$$

$$\Rightarrow E \geqslant E^*$$

So,

$$\boxed{\pi \left[\frac{2a_0^2}{4} + \sum_{n=1}^{N} (a_n^2 + b_n^2)\right] \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x))^2 dx} \Rightarrow \text{Bessel's Inequality.}$$

The above inequality becomes "Parseval's equality" when $N \to \infty$.

$$\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{4} + \lim_{N \to \infty} \frac{1}{2} \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$$

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If the F.S. of $f(x)$ over an interval $(c, c+2l)$ is given as :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

then :

$$\left[\frac{1}{2l}\int_{c}^{c+2l} (f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

Pf :
$$\int_c^{c+2l} (f(x))^2 dx = \frac{a_0}{2}\int_c^{c+2l} f(x)\,dx + \sum \int_c^{c+2l} a_n f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \sum \int_c^{c+2l} b_n f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{l\,a_0^2}{2} + l\left(\sum a_n^2 + \sum b_n^2\right)$$

Q. Find F.S. of x^2 in $[-\pi, \pi]$. Use Parseval's identity to prove that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$

Ans :- $x^2 = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$

$a_0 = \dfrac{2\pi^2}{3}$, $a_n = \dfrac{4(-1)^n}{n^2}$, $b_n = 0$.

Parseval's identity :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2)^2 dx = \frac{(2\pi^2/3)^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\Rightarrow \frac{1}{2\pi}\times 2\int_0^{\pi} x^4 dx = \frac{\pi^4}{9} + 8\sum\frac{1}{n^4}$$

$$\Rightarrow \frac{\pi^4}{5} - \frac{\pi^4}{9} = 8\sum\frac{1}{n^4} \Rightarrow \overset{2}{\cancel{8}}\sum\frac{1}{n^4} = \frac{\cancel{4}\pi^4}{45}$$

$$\Rightarrow \sum\frac{1}{n^4} = \frac{\pi^4}{90}.$$ Proved.

Q. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

Find the function $f(x)$ in the form

$$F(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{N}\alpha_n\cos nx + \sum_{n=1}^{N}\beta_n \sin nx$$

for which the total square error is minimum. Also compute the minimum square error for $N = 1, 3, \ldots$ and what is the smallest N such that $E^* \leq 0.2$?

Complex Analysis

A complex no. is a no. that can be written in an ordered pair.

$z = a + ib \qquad z = (a, b)$

- $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2,\ b_1 + b_2)$
- $(a_1, b_1) \times (a_2, b_2) = (a_1 a_2 - b_1 b_2,\ a_1 b_2 + a_2 b_1)$.

$(a, 0) \simeq a$

$(a, 0) \times (b, 0) = (ab, 0) \simeq ab$

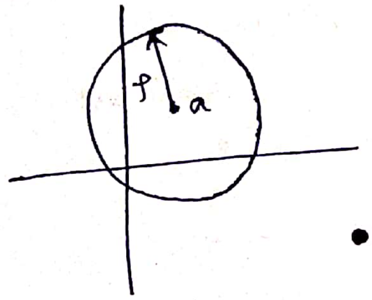
$(0, 1) \times (0, 1) = (-1, 0) \simeq -1$

$\underset{i}{(0,1)} \times \underset{i}{(0,1)}$

So, $\boxed{i^2 = -1}$

$$\begin{aligned} z &= (a, b) \\ &= (a, 0) + (0, b) \\ &= (a, 0) + (b, 0) \times (0, 1) \\ &= a + bi \end{aligned}$$

→ Complex functions are said to be analytic if higher order derivatives exist. Analyticity and Differentiability are different.



$|z - a| < \rho$ [collection of all points inside the circle].

(open circular disk (neighbourhood of point a).

$\rho_1 < |z - a| < \rho_2$ [Annulus region].

- Let S be the collection of complex nos. S is called open if every point of S has a neighbourhood that consist only points of S.

Closed Set : A set S is closed if its complement is an open set.

$|z - a| \leq \rho$

Boundary Point : Set of points such that every neighbourhood about it consists some points of the set 'S' and its complement.

Bounded set : If there exist a circle of sufficient radius inside which the set lies.

Connected set : A set is connected if any two points of set can be joined by finitely many segments of lines consisting of only points of S.

- An Open connected set is called "domain".

S : set of complex numbers.

- Any open set S is called Domain if any two points of it can be joined by a broken line of finitely many segments of lines that contains only points S.
- Region
- $f(z)$

$f(z) = w$

$f(z) = u + iv$

$z = x + iy$

$\because$ w depends on z, and, z depends on x and y

$\Rightarrow$ w depends on x and y.

$\Rightarrow$ u & v depends on x and y.

- $f(z) = u(x, y) + iv(x, y)$

$f(z) = u + iv$.

e.g. : $f(z) = z^2$

$= x^2 - y^2 + i\,2xy$.

Limit :-

$f(x)$

At $x = x_0$,

$\lim_{x \to x_0} f(x) = l \Rightarrow |f(x) - l| < \epsilon$

$\exists\, \delta$ such that $0 < |x - x_0| < \delta$

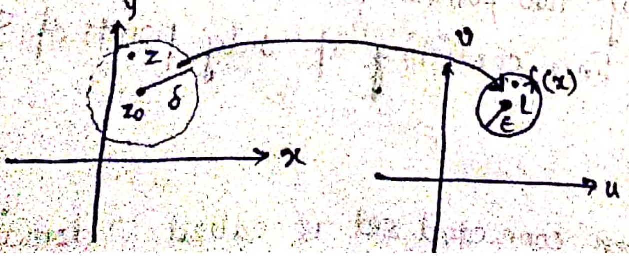
E.g. ① $\lim \dfrac{2xy^2}{x^2 + y^4}$ $(x = my^2)$

② $\lim \dfrac{x^3 + y^3}{x - y}$ $(y = x - mx^3)$.

Limit for complex-valued functions :

A complex valued function $f(z)$ is said to have limit 'l' where z approaches to z_0 (function $f(z)$ may not be defined at $z = z_0$) if for every positive number ϵ (however small but not zero) there exists a positive number δ; such that

$|f(z) - l| < \epsilon$ when $0 < |z - z_0| < \delta$;



Ex $f(z) = i\frac{z}{2}$

$$\lim_{z\to 1} f(z) = \frac{i}{2}.$$

$$\left| f(z) - i/2 \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \left| \frac{z-1}{2} \right| < \epsilon$$

$$\therefore \; |z-1| < 2\epsilon = \delta.$$

Continuous functions :-

A function $f(z)$ is said to be continuous at $z=z_0$ if $f(z)$ is defined at $z=z_0$ and

$$\lim_{z\to z_0} f(z) = f(z_0)$$

A fn $f(z)$ is continuous in a Domain D if $f(z)$ is continuous at each point of the domain.

Differentiability :-

A function $f(z)$ is said to be differentiable at $z=z_0$ if the limit

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

exists.

Let $z_0 + \Delta z = z$

Then,
$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

E.g : ① $f(z) = z^2$

$f'(z) = 2z$.

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(2z+\Delta z)\Delta z}{\Delta z}$$

$$= 2z.$$

(2) $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{[(\cancel{x}+\Delta x) - i(\cancel{y}+\Delta y)] - (\cancel{x} - i\cancel{y})}{\Delta x + i\Delta y}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

(I) Along x-axis, $\lim = 1$

(II) Along y-axis, $\lim = -1$.

$\therefore$ Limit doesnot exist

$\therefore$ $f(z) = \bar{z}$ is not differentiable.

(3) $f(z) = |z|^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\overline{z+\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z\bar{z}} + z\overline{\Delta z} + \Delta z \bar{z} + \Delta z \overline{\Delta z} - \cancel{z\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$$

Let $f(z) = |z|^2$ is differentiable, then limit exists whatever $\Delta z \to 0$.

(I) $\Delta z \to 0$ along the horizontal from $(\Delta x, 0) \to (0,0)$

$\overline{\Delta z} = \Delta z$

$\therefore f'(z) = z + \bar{z}$

(II) $\Delta z \to 0$ vertically ~~along~~ from $(0, \Delta y) \to (0,0)$

$\overline{\Delta z} = -\Delta z$

$\therefore f'(z) = \bar{z} - z$.

~~(iii) $\Delta z \to 0$~~

$\because$ Limit is unique

so, $z+\bar{z} = \bar{z}-z$

$\Rightarrow$ $\underline{z=0}$

$\therefore$ $f(z)=|z|^2$ is differentiable only at the origin.

$f(z)=|z|^2$ is diff. at origin but not analytic at origin.

Analyticity :-

→ A function $f(z)$ is said to be Analytic in a domain D if the function is differentiable at each point of the domain D.

→ A function $f(z)$ is said to be Analytic at point $z=z_0$ if $f(z)$ is differentiable at each point of a neighbourhood of $z=z_0$.

Cauchy-Riemann Equations :-

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at point $z=z_0$. Then partial derivatives of u and v exist at $z=z_0(x_0,y_0)$ and satisfy

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}$$

&

$$\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = -\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}$$

Pf :

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)}\frac{[u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y)] - [u(x_0,y_0)+iv(x_0,y_0)]}{\Delta x + i\Delta y}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)}\frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)}{\Delta x + i\Delta y} + i\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)}{\Delta x + i\Delta y}$$

Case-I

$\Delta z \to 0$ horizontally

from $(\Delta x, 0) \to (0,0)$

$$f'(z_0) = \lim_{\Delta x\to 0}\frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\lim_{\Delta x\to 0}\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

$$= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} \quad \text{--- (1)}$$

Case - II

Let $\Delta z \to 0$ vertically from $(0, \Delta y) \to (0,0)$

i.e.

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}$$

$$= -i \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} \quad \text{——— (2)}$$

Since limit is unique,

so by (1) and (2)

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} = -i \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}$$

Q :

Show that the function defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

& $f(0) = 0$

is continuous, and the C-R equations are satisfied at the origin. yet $f'(0)$ does not exist.

Ans :

$$f(z) = \underbrace{\frac{x^3-y^3}{x^2+y^2}}_{u(x,y)} + i \underbrace{\frac{x^3+y^3}{x^2+y^2}}_{v(x,y)}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{u(h,0) - 0}{h} = \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = 1.$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{v(0, 0+k) - v(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{k^3/k - 0}{k} = 1$$

Similarly, it can be shown, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -1$. $\therefore$ The f^n satisfies CR equn s at $(0,0)$.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{(x,y) \to (0,0)} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$$

$y = mx$

$$= \lim_{x \to 0} \frac{\cancel{x^3}[(1+i) - m^3(1-i)]}{\cancel{x^2}(1+m^2) \cdot \cancel{x}(1+im)}$$

$\Rightarrow f'(0)$ depends on 'm', so, $f(z)$ not differentiable at origin.

Analytic function :-

C-R equations are necessary condition for a funs. to be analytic.

Sufficient condition for a function to be analytic

Let $f(z) = u(x,y) + iv(x,y)$ be defined in a neighbourhood of $z_0 = x_0 + y_0$ and have first order partial derivatives of u and v at $z_0 = x_0 + y_0$, and satisfy Cauchy - Riemann equations at $z_0 = x_0 + y_0$. Then $f(x)$ is analytic at $z_0 = x_0 + y_0$.

Cauchy - Riemann Equ^ns in polar form :-

$z = re^{i\theta}$

$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

be analytic in domain D, then

C-R equations :

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \text{and,} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

& $\boxed{f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}$

Pf : $f(z) = u(x,y) + iv(x,y)$ is analytic [given]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{——— (1)}$$

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$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \quad \text{——— ②}$$

Similarly, we have,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \quad \text{——— ③}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) \quad \text{——— ④}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \left(-\frac{\partial u}{\partial y}\right)(-\sin\theta) + \frac{\partial u}{\partial x}(\cos\theta) \quad \text{——— ⑤}$$

So, from ① and ⑤

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

Similarly, from equ^n ② and ④

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \left(\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}\right) + i\left(\frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right) + i\left(\frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right)$$

$$= \left(\frac{\partial u}{\partial r}\cos\theta + (-\cancel{r})\frac{\partial v}{\partial r}\left(-\frac{\sin\theta}{\cancel{r}}\right)\right) + i\left(\frac{\partial v}{\partial r}\cos\theta + \cancel{r}\frac{\partial u}{\partial r}\left(-\frac{\sin\theta}{\cancel{r}}\right)\right)$$

$$= \left(\frac{\partial u}{\partial r} + \frac{i\partial v}{\partial r}\right)\cos\theta - i\left(i\frac{\partial v}{\partial r} + \frac{\partial u}{\partial r}\right)\sin\theta$$

$$\Rightarrow \boxed{f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}$$

Q: $f(z) = \frac{1}{z}$[redacted] Find $f'(z)$. $[z \neq 0]$.

Ans: $z = re^{i\theta}$

$$f(z) = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}.$$

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$u(r,\theta) = \frac{\cos\theta}{r}$ $\quad v(r,\theta) = \frac{-\sin\theta}{r}$

$\frac{\partial u}{\partial r} = \frac{-\cos\theta}{r^2}$; and, $\frac{\partial v}{\partial r} = \frac{\sin\theta}{r^2}$

$\therefore f'(z) = (\cos\theta - i\sin\theta)\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right)$

$= \frac{i}{r^2}(\cos\theta - i\sin\theta)(\sin\theta + i\cos\theta)$

$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + \frac{i\sin\theta}{r^2}\right)$

$= -\frac{e^{-i\theta}}{r^2}(\cos\theta - i\sin\theta)$

$= -\frac{e^{-i\theta}\cdot e^{-i\theta}}{r^2} = -\frac{1}{re^{i\theta}\cdot re^{i\theta}} = -\frac{1}{z^2}$

- Every analytic function $f(z)$ is free from $\bar{z}$.
  i.e. $\frac{\partial f(z)}{\partial \bar{z}} = 0$.

Pf :

$z = x+iy$

$\bar{z} = x-iy$

$\frac{z+\bar{z}}{2} = x$; $\frac{z-\bar{z}}{2i} = y$

$$\frac{\partial f(z)}{\partial \bar{z}} = \frac{\partial f(z)}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f(z)}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\frac{\partial x}{\partial \bar{z}} + \left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]\frac{\partial y}{\partial \bar{z}}$$

- let $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D and $f'(z) = 0$. Then, $f(z) =$ constant throughout domain D.

Pf : $f'(z) = 0$

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$

$u_x = 0 = u_y$

$v_x = 0 = v_y$

$u_x = u_y = 0$

and $v_x = v_y = 0$.

$\therefore$ Directional derivatives along any other direction $= 0$.

$\therefore \left.\begin{array}{l} u = \text{const.} \\ v = \text{const.}\end{array}\right\} z = \text{const.}$

Harmonic function

If $\phi(x,y)$ is continuous and have first and second order partial derivatives continuous in D and satisfy Laplac[redacted] equ^n

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Then, $\phi(x,y)$ is called Harmonic function.

$f(z) = u(x,y) + i\,v(x,y)$ is analytic.

Theorem : If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in D, then $u(x,y)$ and $v(x,y)$ are Harmonic in D.

Assumption : Let first and second derivatives of u and v are continuous.

$\because$ $f(z)$ is analytic.

So,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—①} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—②}$$

Partial differentiating equ^n ① wrt x —

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \quad \text{——(*)}$$

P.d. equ^n ② wrt y — $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}$ ——(**)

$f(z) = u(x,y) + iv(x,y)$

- If two functions $u(x,y)$ and $v(x,y)$ are Harmonic in domain D and they satisfy C-R equations in D, then $v(x,y)$ is said to be harmonic conjugate of $u(x,y)$.
- If v is a harmonic conjugate of u then $(-u)$ is a harmonic conjugate of v.

Theorem : $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D iff v is harmonic conjugate of u.

E.g ① $f(z) = \underset{u}{x} + i\underset{v}{y}$

$$u_{xx} + u_{yy} = 0 \qquad u_x = 1 = v_y$$
$$u_y = 0 = v_x.$$

② $f(z) = \underset{u}{y} + i\underset{v}{x}$

$$\frac{\partial y}{\partial x} = \frac{\partial x}{\partial y} = 0$$

But, $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$ $(1 \neq -1)$.

Pf: $\Rightarrow$ Obvious by defn.

$\Leftarrow$ v is harmonic conjugate of u implies u & v are harmonic and they satisfy C-R equns. Since u & v harmonic, so their first order partial derivatives of u & v are continuous. By sufficient condition for a func. to be analytic, $f(z) = u + iv$ is analytic.

Q. $u(x,y) = y^3 - 3x^2y$.

Find harmonic conjugate of u and construct an analytic function $f(z) = u + iv$.

Soln:

$$u_x = -6xy$$
$$u_{xx} = -6y \qquad \therefore u_{xx} + u_{yy} = -6y + 6y = 0$$
$$u_y = 3y^2 - 3x^2$$
$$u_{yy} = 6y \qquad \Rightarrow u \text{ is Harmonic.}$$

By C-R equations, ① $\frac{\partial u}{\partial x} = \frac{\partial v(x,y)}{\partial y}$

② $\frac{\partial v(x,y)}{\partial y} = -6xy$.

$$\Rightarrow v(x,y) = -\int_{x=\text{const.}} 6xy\, dy + \phi(x)$$

$$= -6x \cdot \frac{y^2}{2} + \phi(x)$$

$$\Rightarrow v(x,y) = -3xy^2 + \phi(x).$$

② $\frac{\partial v(x,y)}{\partial x} = -\frac{\partial u}{\partial y} = -(3y^2 - 3x^2).$

$$\Rightarrow \frac{\partial}{\partial x}(-3xy^2 + \phi(x)) = -3y^2 + 3x^2$$

$$\Rightarrow \cancel{-3y^2} + \phi'(x) = \cancel{-3y^2} + 3x^2$$

$$\Rightarrow \phi'(x) = 3x^2$$

$$\Rightarrow \phi(x) = x^3 + c$$

$$\therefore \boxed{v(x,y) = -3xy^2 + x^3 + c}$$

$v_x = -3y^2 + 3x^2$ | $v_y = -6xy$

$v_{xx} = 6x$ | $v_{yy} = -6x$

$\Rightarrow v_{xx} + v_{yy} = 6x - 6x = 0.$

So, $\boxed{f(z) = y^3 - 3x^2y + i(x^3 - 3xy^2 + c)}$

Milne's Thompson Method :

Let $f(z) = u(x,y) + iv(x,y)$

$z = x + iy$

$\bar{z} = x - iy$

$$x = \frac{z + \bar{z}}{2}$$

$$y = \frac{z - \bar{z}}{2i}$$

$$\therefore f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Let $z = \bar{z}$

$$f(z) = u(z,0) + iv(z,0)$$

$$f'(z) = u_x + iv_x$$

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Q: $u(x,y) = y^3 - 3x^2y$.

$f'(z) = u_x + i v_x$

By C-R equations, $v_x = -u_y$.

$\therefore f'(z) = u_x - i u_y$

$\Rightarrow f'(z) = -6xy - i(3y^2 - 3x^2)$.

Let $\left.\begin{matrix} x = z \\ y = 0 \end{matrix}\right\}$ $\therefore f'(z) = -i(-3z^2)$

$\Rightarrow \boxed{f(z) = i(z^3 + c)}$

Terminology

Analytic — Regular [Old name]

Analytic — Holomorphic [Modern name]

Q: ① $u - v = (x-y)(x^2 + 4xy + y^2)$
& $f(z) = u + iv$ is an analytic funs.
find $f(z)$ is terms of z.

② If $f(z)$ is a regular function of z, show that :

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$

- If $f(z) = u(x,y) + i v(x,y)$ is analytic, then $u(x,y) = c_1$ and $v(x,y) = c_2$ represents orthogonal family of curves.

Pf :

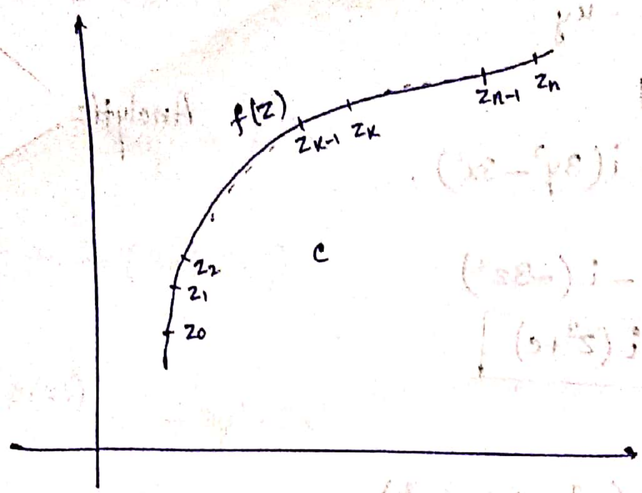
$$du = u_x + u_y \underbrace{\left(\frac{du}{dx}\right)}_{= m_1} = 0$$

$$dv = v_x + v_y \underbrace{\left(\frac{dv}{dx}\right)}_{= m_2} = 0$$

$\therefore m_1 = -\dfrac{u_x}{u_y}$, and, $m_2 = -\dfrac{v_x}{v_y}$.

$$\therefore m_1 m_2 = +\frac{u_x}{\underset{\substack{\| \\ -v_x}}{u_y}} \times \frac{v_x}{\underset{\substack{\| \\ u_x}}{v_y}} = \frac{\cancel{u_x}}{-\cancel{v_x}} \times \frac{\cancel{v_x}}{\cancel{u_x}} = -1.$$

Complex Integration



$z_1, z_2, \text{------}, z_{n-1}$.

Let ξ_k points lie on the arc joining the points from z_{k-1} to z_k $[k = 1(1)n]$

$\therefore$ They form the sum

$$f(z) = f(\xi_1)(z_1 - z_0) + f(\xi_2)(z_2 - z_1) + \text{---} + f(\xi_k)(z_k - z_{k-1}) + \text{------} + f(\xi_n)(z_n - z_{n-1})$$

$$= \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$

$$\therefore f(z) = \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k .$$

If we increase the number of partitions of the curve such that the largest chord length $\Delta z_k \to 0$

$$\lim_{n\to\infty} f(z) = \underline{\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k}$$

$$\downarrow$$

$$\int_{z_0}^{z_n} f(z)\,dz \quad \text{or} \quad \int_c f(z)\,dz .$$

- $\int_C f(z)\,dz$

if $x = x(t)$ $\quad t_1 \le t \le t_2$

and, $y = y(t)$.

Then, $\displaystyle\int_C f(z)\,dz = \int_{t_1}^{t_2} f(z(t))\,\frac{dz}{dt}\cdot dt$

E.g: $\displaystyle\int_C \frac{1}{z}\,dz$ $\qquad$ $c : |z| = 1$

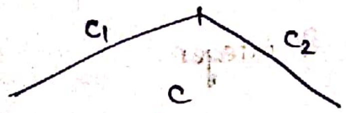
$z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$\therefore dz = e^{i\theta}\cdot i\,d\theta$

$$\therefore \int_C \frac{1}{z}\,dz = \int_0^{2\pi}\left(\frac{1}{e^{i\theta}}\right)\cdot e^{i\theta}\cdot i\,d\theta$$

$$= \int_0^{2\pi} i\,d\theta = 2\pi i.$$

Properties of Complex Integration :-



1. $\displaystyle\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

2. $\displaystyle\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz.$

3. $\displaystyle\int_C K\,f(z)\,dz = K\int_C f(z)\,dz$

K → scalar.

ML inequality :

$$\left| \int_c f(z)\, dz \right| \leq Ml$$

where, $|f(z)| \leqslant M \quad \forall z$ lies on c

and, l is the length of the curve.

Pf :

$$f(z) = \sum_{K=1}^{n} f(\xi_K) \Delta z_K$$

$$|f(z)| = \left| \sum_{K=1}^{n} f(\xi_K) \Delta z_K \right| \leqslant \sum_{K=1}^{n} |f(\xi_K)| |\Delta z_K|$$

$$\leqslant M \sum_{K=1}^{n} |\Delta z_K| .$$

$$\lim_{n\to\infty} |f(z)| \leq \lim_{n\to\infty} M \sum_{K=1}^{n} |\Delta z_K|$$

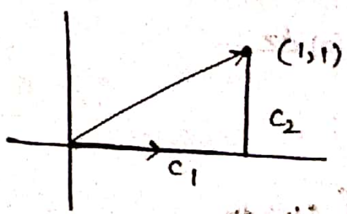
$$\Rightarrow \left| \int_c f(z)\, dz \right| \leqslant Ml .$$

Q: Prove that

$$\int_c (z-z_0)^m \, dz = \begin{cases} 2\pi i , & m = -1 \\ 0 , & m \neq -1 \text{ \& integer.} \end{cases}$$

$c : |z-z_0| = \rho$

Q. Evaluate $\int_c z^2 dz$, $c = c_1 + c_2$



$x = x(t)$

$y = y(t)$

$t_1 \leq t \leq t_2$

[Polynomial functions are analytic everywhere]

$$\int_c f(z)\, dz = \int_{t_1}^{t_2} f(z(t)) \left(\frac{dz}{dt}\right) dt .$$

C_1

$z = x+iy = t$ | $x = t$, $y = 0$

$dz = dt$

C_2

$z = x+iy = 1+it$

$dz = i\,dt$

$$\int_{C_2} z^2 dz = \int_0^1 (1+it)^2\, i\, dt = \int_0^1 (1-t^2+2it)\, i\,dt$$
$$= \left(1-\frac{1}{3}+i\right)i$$
$$= \left(\frac{2}{3}+i\right)i = \frac{2i}{3}-1$$

$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz$$
$$= \int_0^1 t^2\, dt + \int_0^1 (1+it)^2\, i\,dt$$
$$= \frac{1}{3} + \frac{2i}{3} - 1 = \frac{2i}{3} - \frac{2}{3}$$

C_3 :

$x = t$, $y = t$

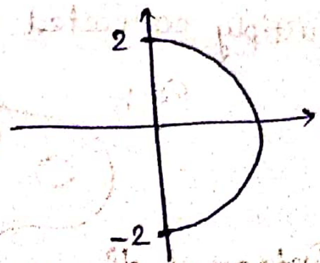
$z = t+it$

$dz = dt + i\,dt$

$$\int_{C_3} z^2 dz = \int_0^1 (t+it)^2 (1+i)\, dt$$
$$= (1+i)^3 \int_0^1 t^2 dt$$
$$= (1-i+3i-3)\frac{1}{3} = \left(\frac{-2+2i}{3}\right)$$

Q : $\int_C \bar{z}\, dz$ $\quad c : |z| = 2$ right half

not an entire/analytic function

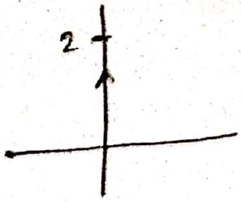


$z = 2e^{i\theta}$, $-\pi/2 \le \theta \le \pi/2$

$dz = 2ie^{i\theta}\, d\theta$

$$\int_C \bar{z}\, dz = \int_{-\pi/2}^{\pi/2} 2e^{-i\theta} \cdot 2ie^{i\theta}\, d\theta = 4i \int_{-\pi/2}^{\pi/2} d\theta = 4i\left(\pi/2+\pi/2\right)$$
$$= 4\pi i.$$

Q. $\int_c \bar{z}\,dz$ c :



$$\int_0^2 -iy(i\,dy)$$

$$= \int_{-2}^{2} y\,dy = 0$$

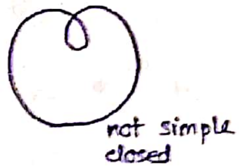
Since $\bar{z}$ is not an entire function

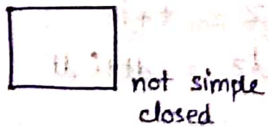
$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$$

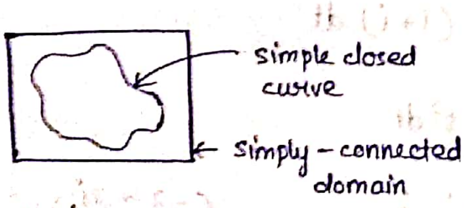
where, $C_3 = C_1 + C_2$.

Simply- Connected Domain :

A domain is called simply connected domain if every simple closed curve in D encloses only pts. of D.



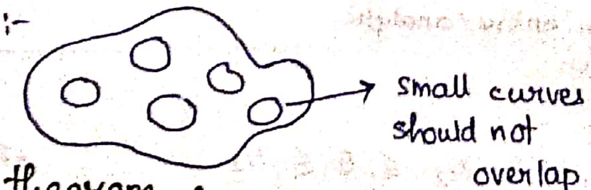




Multiply connected domain :

A domain which is not simply connected is called Multiply connected domain.

Eg :-

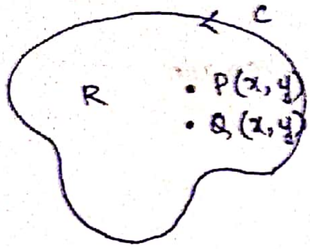


Cauchy Integral theorem :-

If $f(z)$ is $\boxed{\text{analytic}}$ sufficient in a $\boxed{\text{simply connected domain D}}$ necessary cond^n, then for every simple closed curve c that lies in D :

$$\int_c f(z)\,dz = 0$$

Proof: Green's theorem in a plane



$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy = 0+0$$

$$= 0.$$

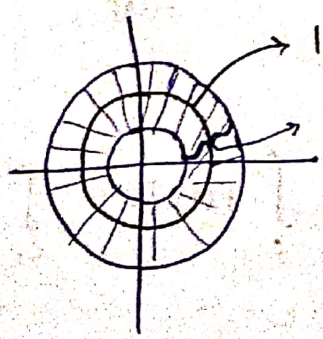
Remarks :-

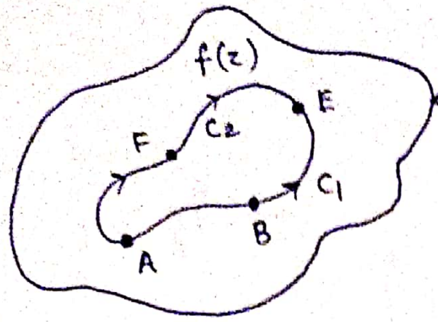
1. $\int_C \frac{1}{z^2}\,dz = 0$ $\quad c: |z| = 1$

2. $\int_C \frac{1}{z}\,dz = 2\pi i$ $\quad c: |z| = 1$
   $\frac{1}{2} < z < \frac{3}{2}$

In ex. 1 $\int \frac{1}{z^2}\,dz = 0$, even though $\frac{1}{z^2}$ is not analytic at 0.

In ex. 2 $\int_C \frac{1}{z}\,dz \neq 0$ since $\frac{1}{z}$ can be analytic only in an annular domain which is a doubly-connected domain.





$$\int_{ABEFA} f(z)\,dz = 0$$

$$\int_{ABE} f(z)\,dz + \int_{EFA} f(z)\,dz = 0$$

$$\int_{ABE} f(z)\,dz = -\int_{EFA} f(z)\,dz$$

$$= \int_{AFE} f(z)\,dz \,.$$

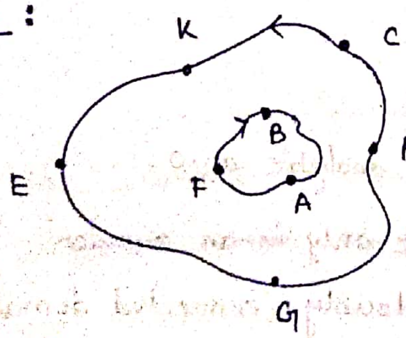
Cauchy Extension Theorem :-

Let $f(z)$ is analytic within and on a bounded domain D bounded by closed curves C and C_1 where C_1 lies inside C.

Then,
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz \,.$$

Here, C and C_1 are traversed in positive sense in their respective interiors.

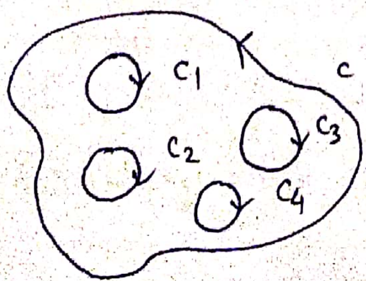
Pf :



$$\int_{EFBAFEGHKE} f(z)\,dz = \cancel{\int_{EF} f(z)\,dz} + \int_{FBAF} f(z)\,dz + \cancel{\int_{FE} f(z)\,dz} + \int_{EGHKE} f(z)\,dz$$

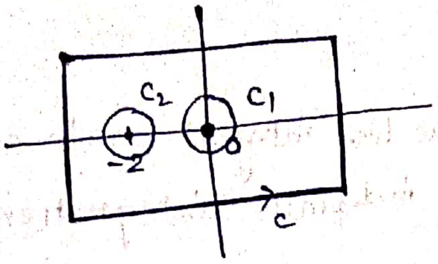
$$\int_{EGHKE} f(z)\,dz = -\int_{FBAF} f(z)\,dz = \int_{FABF} f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz$$

Q: $\oint_C \frac{1}{z(z+2)}\,dz$ C: Rectangle containing points 0 and −2.



$$\oint_C \frac{dz}{z(z+2)} = \frac{1}{2}\left[\oint_{C_1}\left(\frac{1}{z}-\frac{1}{z+2}\right)dz + \oint_{C_2}\left(\frac{1}{z}-\frac{1}{z+2}\right)dz\right]$$

$$= \frac{1}{2}\left[\oint_{C_1}\frac{dz}{z} - \oint_{C_1}\frac{dz}{z+2} + \oint_{C_2}\frac{1}{z}dz - \oint_{C_2}\frac{dz}{z+2}\right]$$

$$= \frac{1}{2}\left[2\pi i - 0 + 0 - 2\pi i\right]$$

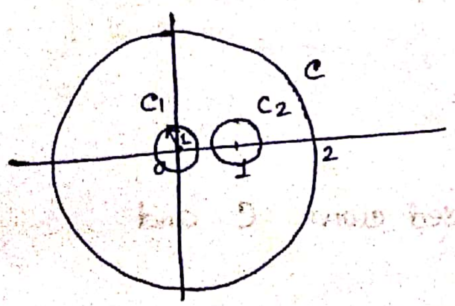
$$= 0.$$

Q: Evaluate $\oint \frac{2z-1}{z(z-1)}\,dz$, $|z|=2$ by Cauchy-Integral theorem.

Ans :- $I = \oint \frac{2z-1}{z(z-1)}\,dz$

$$= \oint\left(\frac{1}{z}+\frac{1}{z-1}\right)dz$$

$$= \oint\frac{dz}{z} + \oint\frac{dz}{z-1}$$



By Cauchy-extension theorem:

$$I = \oint_{C_1}\frac{2z-1}{z(z-1)}dz + \oint_{C_2}\frac{2z-1}{z(z-1)}dz$$

$$\Rightarrow I = \oint_{C_1}\left(\frac{1}{z}+\frac{1}{z-1}\right)dz + \oint_{C_2}\left(\frac{1}{z}+\frac{1}{z-1}\right)dz$$

$$= 2\pi i + 0 + 0 + 2\pi i$$

$$= 4\pi i.$$

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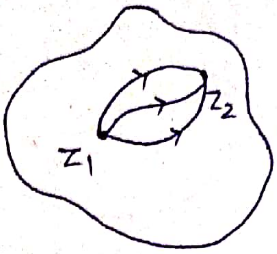
If $f(z)$ is analytic in a simply connected domain D. Then $\int_{z_1}^{z_2} f(z)\,dz$ is independent of the path in D joining any 2 points in D.



Further, the line integral can be evaluated using the indefinite integration method.

Theorem :- If $f(z)$ is analytic in a simply connected domain D and z be points in D, then,

i> $F(z) = \int_a^z f(u)\,du$ is analytic in D.

ii> $F'(z) = f(z)$

$F(z)$ is called antiderivative of $f(z)$.

Proof :

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)$$

$$= \frac{\int_a^{z+\Delta z} f(u)\,du - \int_a^{z} f(u)\,du}{\Delta z} - f(z)$$

$$= \int_z^{z+\Delta z} \frac{f(u)\,du}{\Delta z} - f(z)$$

$$= \frac{1}{\Delta z}\left[\int_z^{z+\Delta z} (f(u) - f(z))\,du\right].$$

Cauchy-Integral Formula :

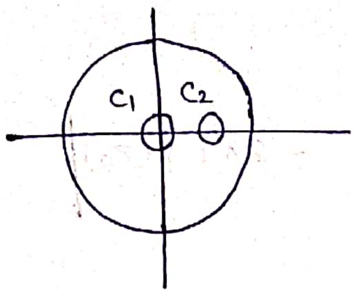
Let $f(z)$ be analytic within and on a simple closed curve C and z_0 be any point inside C. Then,

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\,dz.$$

$$f^{n}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

nth derivative

Q: $\oint_C \frac{2z-1}{z(z-1)}\,dz$ $\quad$ $C: |z| = 2$



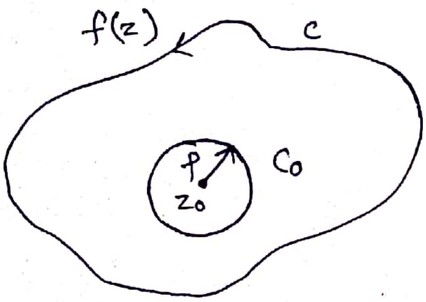
$$I = \oint_{C_1} \frac{2z-1}{z(z-1)}\,dz + \oint_{C_2} \frac{2z-1}{z(z-1)}\,dz$$

$$= \oint_{C_1} \frac{(2z-1)/(z-1)}{z}\,dz + \oint_{C_2} \frac{(2z-1)/z}{(z-1)}\,dz$$

$$= \oint_{C_1} \frac{f(z)}{z}\,dz + \oint \frac{g(z)}{(z-1)}\,dz$$

$$= 2\pi i\, f(0) + 2\pi i\, g(1)$$

$$= 2\pi i (1) + 2\pi i (1) = 4\pi i.$$



$f(z)$ is analytic in the region bounded by the 2 simple closed curves c and C_0.

Pf :

$$\int_c \frac{f(z)}{z-z_0}\,dz = \int_{C_0} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_{C_0} \left[\frac{f(z)-f(z_0)}{z-z_0}\right] dz + \int_{C_0} \frac{f(z_0)}{z-z_0}\,dz$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz + 2\pi i f(z_0)$$

$$\int_c \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$\left|\int_c \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| = \left|\int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \int_{C_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,dz$$

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- A function is said to be continuous at a point if $\forall \epsilon$ (however small but not zero) $\exists \delta$ such that $|f(z) - f(z_0)| < \epsilon$ where $|z-z_0| < \delta$

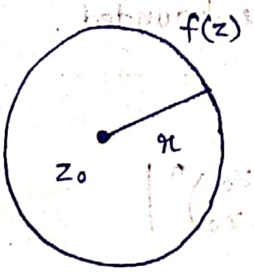
Let $|z-z_0| = \rho < \delta$

$$\left| \int_c \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| \leq \frac{\epsilon}{\rho} (2\pi \rho) = 2\pi \epsilon$$

$$\Rightarrow \left| \int_c \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| \leq 0$$

$$\Rightarrow \int_c \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

Power Series :



$$|f^n(z_0)| \leq \frac{M_n}{r^n}$$

Leibnitz theorem :- Every polynomial of degree n has at least one root in the complex plane.

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n.$$

Every analytic function can be represented as a polynomial function.

Polynomial functions are always analytic.

1. $\sum z^n = 1 + z + z^2 + z^3 + \dots$

Converges absolutely when $|z| < 1$.

2. $\sum \frac{z^n}{n!} = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \frac{z^4}{\lfloor 4} + \dots$

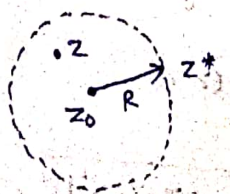
$$\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{z^{n+1}}{z^n} \frac{\lfloor n}{\lfloor n+1}\right| = \lim_{n\to\infty} \frac{z}{n+1} = 0.$$

Absolutely convergent for $z \in R$.

3. $\sum_{n=0}^{\infty} n z^n$: Divergent

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^2$$

- The series converges at $z = z_0$. If the series also converges at another point z^*, then the series converges at all points lying in the circle centred at z_0 and of radius $|z^* - z_0|$.



R = radius of convergence.

Theorem : If the power series converges at a point $z = z^*$, it converges absolutely for every z for which $|z - z_0| < |z^* - z|$ i.e. for each z within the circle through z^* about z.

Then, $\lim_{n\to\infty} a_n(z^*-z_0)^n = 0$

This implies that for $z=z^*$ the terms of series are bounded.

$|a_n(z^*-z_0)^n| < M$ for every $M = 0,1,2,\ldots$

$$|a_n(z-z_0)^n| = \left| a_n(z^*-z_0)^n \left(\frac{z-z_0}{z^*-z_0}\right)^n \right|$$

$$< M \left| \frac{(z_0-z_0)}{(z^*-z_0)} \right|^n.$$

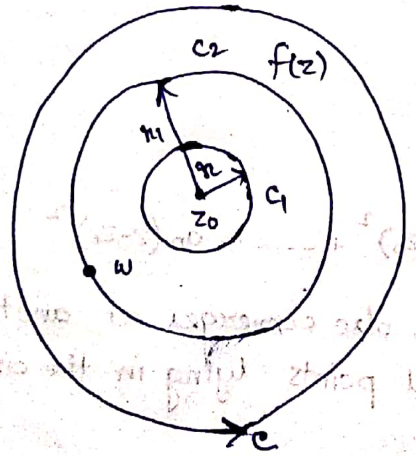
$$\therefore \sum_{n\to\infty} |a_n(z-z_0)^n| < M \sum \left| \frac{z-z_0}{z^*-z_0} \right|^n.$$

Taylor's theorem :-

Let $f(z)$ be analytic within a circle C with centre z_0 and say radius R, and z be an arbitrary point inside C, then, $f(z)$ has representation :

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{\lfloor 2}(z-z_0)^2 + \frac{f'''(z_0)}{\lfloor 3}(z-z$$

$$= f(z_0) + \sum_{n=1}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$$



$|z-z_0| < r$

$r < r_1$

By Cauchy integral formula :

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw$$

$$\frac{1}{w-z} = \frac{1}{w-z_0-(z-z_0)}$$

$$= \frac{1}{(w-z_0)\left[1-\left(\frac{z-z_0}{w-z_0}\right)\right]}$$

$$\Rightarrow \frac{1}{w-z} = \frac{1}{w-z_0}\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\cdots+\frac{(z-z_0)^n}{(w-z_0)^n}+\cdots\right]$$

$$\Rightarrow \frac{1}{w-z} = \left[\frac{1}{w-z_0} + \left(\frac{z-z_0}{(w-z_0)^2}\right) + \left(\frac{(z-z_0)^2}{(w-z_0)^3}\right) + \cdots + \frac{(z-z_0)^{n-1}}{(w-z_0)^n} + \frac{(z-z_0)^n}{(w-z_0)^{n+1}} + \cdots\right] \longrightarrow (I)$$

$(I) \times \frac{f(w)}{2\pi i}$ and integrate each term along curve C_2.

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)\,dw}{w-z_0} + \frac{(z-z_0)}{2\pi i}\int\frac{f(w)\,dw}{(w-z_0)^2} + \frac{(z-z_0)^2}{2\pi i}\int\frac{f(w)\,dw}{(w-z_0)^3} + \cdots$$

$$\cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int\frac{f(w)\,dw}{(w-z_0)^n} + R_n(z).$$

$$R_n(z) = \frac{(z-z_0)^n}{2\pi i}\oint\frac{f(w)\,dw}{(w-z_0)(w-z)}$$

$$f^n(z_0) = \frac{n!}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

$$\lim_{n\to\infty} R_n(z) = 0.$$

- When $z_0 = 0$; $f(z) = f(0) + \sum_{n=1}^{\infty}\frac{f^n(0)\cdot z^n}{\lfloor n}$ is called Maclaurin series.

Q: $f(z) = e^z$

$$f(z) = f(0) + \sum \frac{f^n(0)}{n!}\cdot z^n$$

$f^n(z) = e^z$

$f^n(0) = 1$

$\therefore f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \quad (|z| < \infty)$

Q: $f(z) = \sin z$

$f^{2n}(0) = 0$

$f^{(2n+1)}(0) = (-1)^n$

$$f(z) = f(0) + \sum_{n=0}^{\infty}\frac{f^{2n+1}(0)\cdot z^{2n+1}}{\lfloor(2n+1)}$$

$$= f(0) + \sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^{2n+1}}{\lfloor(2n+1)}$$

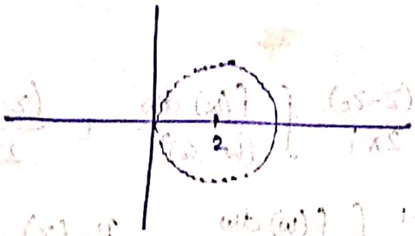
$$\therefore f(z) = \cancelto{0}{f(0)} + \sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^{2n+1}}{(2n+1)!}$$

$$= z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \frac{z^7}{\lfloor 7} + \cdots$$

Q: Show that :

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n \frac{\lfloor n+1}{n!}\left(\frac{z-2}{2}\right)^n \; ; \quad n = 0, 1, 2$$

for $|z-2| < 2$



$$\frac{1}{z^2} = f(2) + \sum_{n=1}^{\infty} \frac{f^n(2)\,(z-2)^n}{n!}$$

$$f'(z) = \frac{-2}{z^3}$$

$$f''(z) = \frac{(-2)(-3)}{z^4}$$

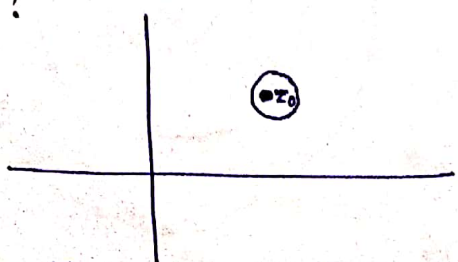
$$f^n(z) = \frac{(-1)^n\,(n+1)!}{z^{n+2}}$$

$$\therefore \frac{1}{z^2} = f(2) + \sum_{n=1}^{\infty} \frac{(-1)^n \lfloor n+1}{n!\,.\,2^{n+2}}\,.\,(z-2)^n$$

$$= \frac{1}{2^2} + \frac{1}{2^2}\sum_{n=1}^{\infty} (-1)^n \frac{\lfloor n+1}{n!}\left(\frac{z-2}{2}\right)^n$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n \frac{\lfloor n+1}{n!}\left(\frac{z-2}{2}\right)^n$$

Q: If $f(z)$ is analytic at pt. $z = z_0$; is Taylor expansion possible at $z = z_0$?



Ans :- yes. There exists a circle however small it may be such that $f(z)$ in that circle is analytic. Hence $f(z)$ can be expanded as a Taylor series in that domain.

Q: $f(z) = \frac{1+2z^2}{z^3+z^5}$. Expand $f(z)$.

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$$f(z) = \frac{1+2z^2}{z^3(1+z^2)} = \frac{2(z^2+1)-1}{z^3(1+z^2)}$$

$$= \frac{2}{z^3} - \frac{1}{z^3}(1-z^2+z^4-z^6+\cdots)$$

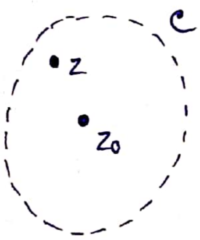
$$= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots$$

$$= \frac{1}{z^3} + \frac{1}{z} - (z - z^3 + z^5 - \cdots)$$

↓

for this to be convergent $|z|<1$.

but for the $f(z)$ to be convergent $0<|z|<1$; since 0 is a singular pt. of $f(z)$ where it is not analytic.



$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^n(z_0)}{\lfloor n} (z-z_0)^n.$$

In Taylor series expansion, all terms have positive powers.

Singularities are those pts. in the domain of a function where it fails to be analytic.

Isolated singular point :- There exist a neighbourhood about a singular point in which the function is analytic everywhere.

Non-isolated singular point :- There doesnot exist any neighbourhood about a singular point where function is analytic everywhere.

Eg :- $f(z) = \dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$

$f(z)$ is not analytic at $z = 0, \frac{1}{n}$ $[n = \pm 1, \pm 2, \pm 3, \ldots]$

Here 0 is a non-isolated singular point and all others are isolated singular pts.

Solution of a Differential Equation :=

1> General solution
2> Particular solution
3> Singular solution.

E.g: ① $\frac{d^2y}{dx^2} + y = 0$

$y(x) = c_1 \sin x$
$y(x) = c_2 \cos x$.

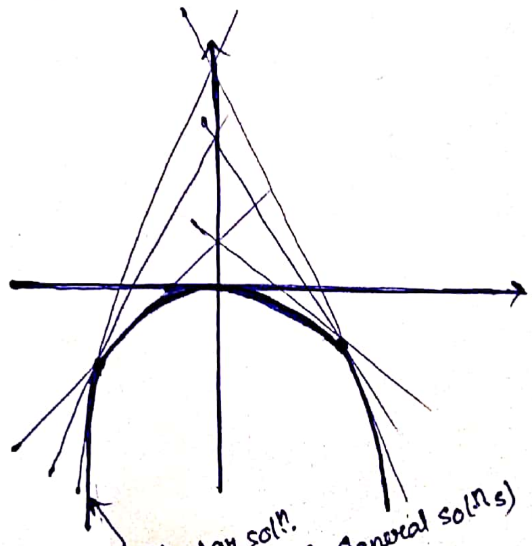
$y(x) = c_1 \sin x + c_2 \cos x \longrightarrow$ General solution.

If we put $c_1 = 1$, $c_2 = 0$ } $y(x) = \sin x \longrightarrow$ Particular solution.

② $(y')^2 + xy' - y = 0$

$y = cx + c^2 \longrightarrow$ General solution

$4y + x^2 = 0$
$\Rightarrow y = -\frac{x^2}{4} \longrightarrow$ singular solution



Singular solⁿ.
(envelope of General solⁿs)

Exact Differential Equation :-

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Necessary condⁿ. for this D. Equⁿ. to be exact is: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

A differential equation is said to be exact if it can be obtained by directly differentiating its primitive without any operations like multiplication, elimination or reductive etc.

$f(x,y) = c$

$df = M\,dx + N\,dy = 0$

$$\Rightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M\,dx + N\,dy = 0$$

i.e. we have to find a f^n $f(x,y)$ such that :

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y).$$

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\Rightarrow f(x,y) = \int_{y=\text{const.}} M(x,y)\,dx + g(y).$$

$$N(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y)$$

$$\Rightarrow g'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]$$

$$\Rightarrow g(y) = \int \left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) dy \quad \text{———} \quad (*)$$

Integrand in equ^n (*) is only f^n of y.

So, $\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\int M\,dx\right] = 0$.

$$\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\int M\,dx\right] = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x\,\partial y}\int M\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y\,\partial x}\int M\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

$$= 0.$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial M}{\partial y}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x}.$$

$$\because \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Q:

$$e^y\,dx + (xe^y + 2y)\,dy = 0$$

$M(x,y) = e^y$ $\qquad \frac{\partial M}{\partial y} = M_y = e^y.$

$N(x,y) = xe^y + 2y,$ $\qquad \frac{\partial N}{\partial x} = N_x = e^y.$

$\because M_y = N_x$ $\therefore$ The given differential equ^n is exact.

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$\therefore$ The solⁿ is :

$$\int_{y=\text{const.}} M\,dx + \int\left(N - \frac{\partial}{\partial y}\int M.dx\right) dy = \text{const.}$$

$$\Rightarrow \int_{y=\text{const.}} e^y\,dx + \int\left(xe^y + 2y - \frac{\partial}{\partial y}\int e^y dx\right) dy = \text{Const.}$$

$$\Rightarrow xe^y + \int\left(\cancel{xe^y} + 2y - \cancel{xe^y}\right) dy = \text{const.} = c$$

$$\Rightarrow \boxed{xe^y + y^2 = c}$$

Note :-

Solution of a exact differential equⁿ of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

is :

$$\int M\,dx + \int\left(\text{Terms of } N \text{ which are free from } x\right) dy = \text{constant}.$$

Q :

$$dx = \frac{y}{1-x^2y^2}\,dx + \frac{x}{1-x^2y^2}\,dy$$

$$\Rightarrow dx = \frac{y\,dx + x\,dy}{1-x^2y^2}$$

$$\Rightarrow dx - (x^2y^2)\,dx = y\,dx + x\,dy$$

$$\Rightarrow (1 - x^2y^2 - y)\,dx + (-x)\,dy = 0.$$

$M = 1 - x^2y^2 - y \Rightarrow M_y = 0 - x^2 \cdot 2y - 1 = -2x^2y - 1$

$N = -x \Rightarrow N_x = -1.$

- Suppose $M dx + N dy = 0$ (1) is non-exact and has a soln.

Let $\mu(x,y)$ is a fn of x and y. $\mu(x,y)$ is called an Integrating factor of (1) if $\mu M dx + \mu N dy = 0$ is exact.

So,

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\Rightarrow \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}.$$

Case - I :

Suppose $\mu(x,y)$ is a function of x only :

$$\mu(x) \cdot \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu(x) \cdot \frac{\partial N}{\partial x}.$$

$$\Rightarrow \mu(x)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{d\mu}{dx} N$$

$$\Rightarrow \frac{d\mu}{\mu(x)} = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] dx.$$

$$\Rightarrow \log(\mu(x)) = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx$$

$$\Rightarrow \mu(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx}.$$

→ when $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a func. of only x say $g(x)$; then, I.F. is $e^{\int g(x) dx}$.

→ Similarly, when $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a func. of y only say $g(y)$; then, I.F. is $e^{\int g(y) dy}$.

Q: Find the orthogonal trajectories of $x^2+y^2 = 2cx$.

$$2x + 2yy' = 2c$$

$$\Rightarrow x + yy' = c.$$

$$\therefore x^2 + y^2 = 2(x+yy')x$$

$$= 2x^2 + 2xy\,y'.$$

$$\Rightarrow y^2 - x^2 = 2xy\left(\frac{dy}{dx}\right).$$

$$\Rightarrow y^2 - x^2 = 2xy\left(-\frac{dx}{dy}\right).$$

$$\Rightarrow (y^2 - x^2)\,dy + 2xy\,dx = 0$$

$$\Rightarrow \underbrace{2xy}_{M}\,dx + \underbrace{(y^2 - x^2)}_{N}\,dy = 0$$

$M_y = 2x$ $\quad$ $M_y \neq N_x$

$N_x = -2x$ $\quad$ So, Diff. Equn. is non-exact.

$$N_x - M_y = -2x - 2x$$
$$= -4x.$$

$$\frac{1}{M}(N_x - M_y) = \frac{1}{2xy}(-4x) = -\frac{2}{y}.$$

$$\therefore \text{I.F} = e^{\int -\frac{2}{y}\,dy} = \frac{1}{y^2}.$$

$$\therefore \frac{2xy}{y^2}\,dx + \left(\frac{y^2 - x^2}{y^2}\right)dy = 0$$

$$\Rightarrow \int \frac{2x}{y}\,dx +$$

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