Linear Algebra – Distance, Hyperplanes and Halfspaces, Eigenvalues, Eigenvectors

Connections between eigenvectors, column space and null space

- We know that eigenvalues can be complex numbers even for real matrices
- When eigenvalues become complex, eigenvectors also become complex
- However, if the matrix is symmetric, then the eigenvalues are always real
- As a result, eigenvectors of symmetric matrices are also real
- Further, there will always be n linearly independent eigenvectors for symmetric matrices

$|A - \lambda I| = 0$

$P_n(\lambda) = 0$

$Ax = \lambda x$

$A = A^T$

$A \quad \lambda_1, \lambda_2, \ldots, \lambda_n$

$\vartheta_1, \vartheta_2, \ldots, \vartheta_n$

$P_n(\lambda) = (\lambda - \lambda_1)^2 P_{n-2}(\lambda)$

$\lambda_1, \lambda_1, \lambda_3 \ldots \lambda_n$

$\vartheta_1 \quad \vartheta_2$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \lambda = 1 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Connections between eigenvectors, column space and null space

- Symmetric matrices have a very important role in data sciences
- In fact symmetric matrices of the form $A^T A$ or AA^T are often encountered
- Eigenvalues of matrices of the form $A^T A$ or AA^T while being real are also non-negative
- As discussed for general symmetric matrices, there will be n linearly independent eigenvectors for matrices of this form also
- What is the connection between the eigenvectors and the column space and null space of a (symmetric) matrix ?

$(A^TA)^T = A^T(A^T)^T = A^TA$

AA^T

$0 \rightarrow$

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Connections between eigenvectors, column space and null space

$$\mathbf{A}v = \lambda v$$

- What happens when the eigenvalues become zero?

$$\mathbf{A}v = \mathbf{0}$$

$AB = 0$ ✓

$v \Rightarrow \lambda = 0$ is a nullspace vector

- The eigenvectors corresponding to zero eigenvalues are in the null space of the matrix
- Conversely, if the eigenvalue corresponding to an eigenvector is not zero then that eigenvector cannot be in the null space

$Av = 0$ and get non-trivial v if A is full rank

Connections between eigenvectors, column space and null space

- Let us assume that there are r eigenvectors corresponding to zero eigenvalue
- This means that the null space dimension is r
- From rank-nullity theorem (discussed before), we know that the column rank should be $\mathrm{n} - r$
- That is $\mathrm{n} - r$ independent vectors are enough to represent all the vectors in the columns of the matrix (column space)
- What could be a basis for this column space or what could be the $\mathrm{n} - r$ independent vectors?

$A_{n \times n}$ is Symmetric
r zero eigenvalues
$n-r$ non-zero eigenvalues
$\to r$ eigenvectors
rank + nullity = n
nullity = r
rank = $n-r$

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Connections between eigenvectors, column space and null space

- Notice that there are $\mathrm{n} - r$ eigenvectors which are not in the null space
- We know that these are independent
- We also know that these vectors are a linear combination of all the column vectors – that is they are in the column space

$$\boldsymbol{Av} = \boldsymbol{\lambda v}$$

- Further, we know that the dimension of the column space is $\mathrm{n} - r$ (rank-nullity theorem)
- This implies that the eigenvectors corresponding to the non-zero eigenvalues form a basis for the column space

A is Symmetric
$\to n$ LI eigenvect

$A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \lambda v$

$v_1 A_1 + v_2 A_2 + \cdots + v_n A_n = \lambda v$

$v = \left(\frac{v_1}{\lambda}\right) A_1 + \cdots + \left(\frac{v_n}{\lambda}\right) A_n$

$c_1 A_1 + \cdots + c_n A_n$

$A_1, \ldots, A_n$

Example

- Consider the following A matrix

$$A = \begin{bmatrix} 0.36 & 0.48 & 0 \\ 0.48 & 0.64 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$A^T = A$

$|A - \lambda I| = 0$

$P_3(\lambda) = 0$

$Ax = \lambda x$

- Notice that this is a symmetric matrix
- The eigenvalues for this matrix are

$$\lambda = (0,1,2)$$

- The eigenvectors corresponding to these eigenvalues are

3-1

n-1

$$\boldsymbol{v}_1 = \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$A v_1$

Example

- From our prior understanding, the eigenvector corresponding to the zero eigenvalue will be in the null space

eigenvectors to zero eigvalue

- We check that

$$A\boldsymbol{v}_1 = \begin{bmatrix} 0.36 & 0.48 & 0 \\ 0.48 & 0.64 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Interestingly, in the initial lectures, it was identified that the null space vector identifies a relationship between the variables
- Hence, the eigenvector corresponding to the zero eigenvalue can be used to identify the relationships among variables

Example

- Let us now check if the other two eigenvectors shown below span the column space

$$v_2 = \begin{bmatrix}0.6\\0.8\\0\end{bmatrix}, v_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

A_1, A_2, A_3 are Lin. Combinat. of λ_2 and λ_3

- This is demonstrated as below

$$A \quad \begin{bmatrix}0.36\\0.48\\0\end{bmatrix} = 6 * \begin{bmatrix}0.6\\0.8\\0\end{bmatrix}$$

$$A_2 \quad \begin{bmatrix}0.48\\0.64\\0\end{bmatrix} = 8 * \begin{bmatrix}0.6\\0.8\\0\end{bmatrix}$$

$$A_3 \quad \begin{bmatrix}0\\0\\2\end{bmatrix} = 2 * \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Summary

- $Ax = \lambda x$ — Symmetric matrices have real eigenvalues ✓
- $Ax = \lambda x$ — Symmetric matrices also have n linearly independent eigenvectors
- $Ax = 0$ — Eigenvectors corresponding to zero eigenvalues span the null space
- $Ax = \lambda x$ — Eigenvectors corresponding to non-zero eigenvalues span the column space for symmetric matrices