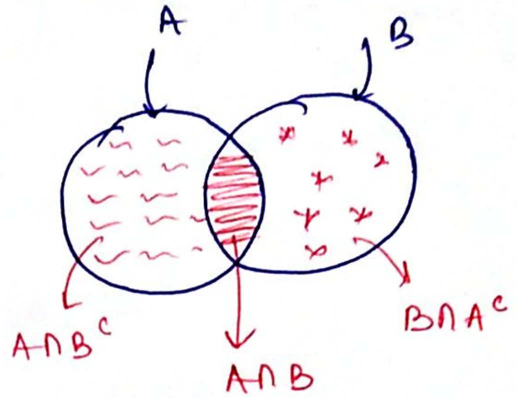
Important results :-

If A and B are any two events connected to a random experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Proof : The events $A \cap B^c$, $A^c \cap B$ and $A \cap B$ are pairwise mutually exclusive events.

$$\therefore \quad A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$$

$$\Rightarrow P(A \cup B) = P\left[(A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)\right]$$

$$= P(A \cap B^c) + P(A \cap B) + P(A^c \cap B), \text{ By axiom 3.} \quad \text{—— (1)}$$

Now $A = (A \cap B^c) \cup (A \cap B)$

$$\Rightarrow P(A) = P\left[(A \cap B^c) \cup (A \cap B)\right]$$

$$= P(A \cap B^c) + P(A \cap B), \text{ by axiom 3.} \quad \text{—— (2).}$$

Similarly,

$$P(B) = P\left[(A^c \cap B) \cup (A \cap B)\right]$$

$$= P(A^c \cap B) + P(A \cap B), \text{ By axiom 3.} \quad \text{—— (3).}$$

Now substituting (2) & (3) in (1).

$$P(A \cup B) = P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)} + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B).$$

Hence proved

Lecture 3 : P(1).

Boole's Inequality:-

If $A_1, A_2, \dots, A_n$ are any n events connected to a random experiment E, then

$$\cancel{P\left(\sum_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)} \qquad P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \quad \text{—(1)}$$

Proof: Let A_1 and A_2 be any two events connected to the r.e. E.

Then $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$\Rightarrow P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$, as $P(A_1 \cap A_2) \geq 0$ by axiom 1.

Let us now consider that the inequality (1) true for $n = m$, where m is a positive integer ≥ 2.

i.e., $P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i)$

Let us now consider $m+1$ events $A_1, A_2, \dots, A_m, A_{m+1}$ connected to E.

Now by hypothesis

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i)$$

$$\therefore P\left(\bigcup_{i=1}^{m} A_i \cup A_{m+1}\right) \leq P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1})$$

$$\leq \sum_{i=1}^{m} P(A_i) + P(A_{m+1})$$

$$= \sum_{i=1}^{m+1} P(A_i).$$

Hence, inequality (1) is true for $n = m+1$.

Hence, proved by the principle of mathematical induction.

Conditional Probability :-

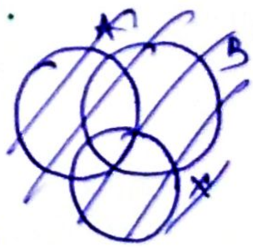
Let A and B be any two events connected to a given random experiment E. The conditional probability of the event A on hypothesis that the event B has occurred, denoted by $P(A|B)$ and is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) \neq 0$.

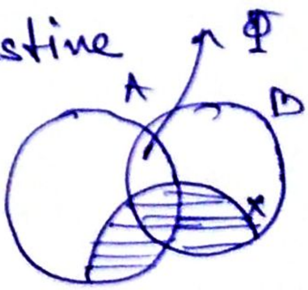
Theorem: Let A and B be two mutually exclusive and exhaustive events connected to a random experiment E. Let X be another event connected to E.

Then $P(X) = P(X|A)P(A) + P(X|B)P(B)$.



Proof: $A \cup B = S$, S is the sample space connected to E.

$A \cap B = \Phi$, as A and B are mutually exclusive and exhaustive given.



$$X = (X \cap A) \cup (X \cap B)$$

$$\Rightarrow P(X) = P[(X \cap A) \cup (X \cap B)]$$

$$= P(X \cap A) + P(X \cap B)$$

$$= P(X|A)P(A) + P(X|B)P(B).$$

proved.

$\because A \cap B = \phi$

$X \cap (A \cap B) = (X \cap A) \cap (X \cap$

$= X \cap \phi = \phi$

$\Rightarrow X \cap A$ & $X \cap B$ are also mutually exclusive.

<u>Theorem</u>: Let A_i's $(i \in I)$, I is the indexed set be pairwise mutually exclusive events, one of which certainly occurs, i.e., A_i's forms an mutually exhaustive set of events. Let B be another event connected to the same r.e., then

$$P(B) = \sum_{i \in I} P(B|A_i)\, P(A_i).$$

provided $P(B|A_i)$ & $P(A_i)$ are defined $\forall i \in I$.

Bayes' Theorem :

Let A and B be two mutually exclusive and exhaustive events connected to the random experiment E.

Let X be another event connected to E.

Let $P(X|A)$ and $P(X|B)$ are known and $P(X) \neq 0$.

then $$P(A|X) = \frac{P(X|A)P(A)}{P(X|A)P(A) + P(X|B)P(B)}$$

Similarly,
$$P(B|X) = \frac{P(X|B)P(B)}{P(X|B)P(B) + P(X|A)P(A)}.$$

Proof: Let S be the event space associated with the r.e. E.

Then $X = X \cap S = X \cap (A \cup B) = (X \cap A) \cup (X \cap B)$

$\Rightarrow P(X) = P[(X \cap A) \cup (X \cap B)]$

$= P(X \cap A) + P(X \cap B)$

$= P(X|A)P(A) + P(X|B)P(B)$

$\because A \cap B = \phi$, given

$X \cap (A \cap B) = X \cap \phi = \phi$

$\Rightarrow (X \cap A) \cap (X \cap B) = \phi$

$\Rightarrow X \cap A$ & $X \cap B$ are mutually exclusive

$$\therefore P(A|X) = \frac{P(A \cap X)}{P(X)}$$

$$= \frac{P(X|A)P(A)}{P(X|A)P(A) + P(X|B)P(B)}.$$

Proved.

Bayes' theorem

Let A_i's, $i \in I$, I being the index set be the pairwise mutually exclusive and exhaustive events connected to a random experiment E.

Let X be an arbitrary event connected to E, s.t. $P(X) \neq 0$ and $P(X|A_i)$ are known $\forall i \in I$.

Then $$P(A_i|X) = \frac{P(X|A_i)\,P(A_i)}{\sum_{j \in I} P(X|A_j)\,P(A_j)} \quad \forall i \in I.$$

Proof: $\because X \subseteq S \Rightarrow X \cap S = X$.

$$\therefore X = X \cap S = X \cap \left(\bigcup_{i \in I} A_i\right)$$

$$= \bigcup_{i \in I} (X \cap A_i)$$

$i \neq j \in I$.

$X \cap (A_i \cap A_j) = X \cap \phi = \phi$

Now $(X \cap A_i) \cap (X \cap A_j) = \phi$ $\forall i \neq j \in I$.

$\Rightarrow X \cap A_i$ & $X \cap A_j$ are pairwise mutually exclusive events.

$$\therefore P(X) = P\left[\bigcup_{j \in I} (X \cap A_j)\right]$$

$$= \sum_{j \in I} P(X \cap A_j)$$

$$= \sum_{j \in I} P(X|A_j)\,P(A_j)$$

$$\therefore P(A_i|X) = \frac{P(X \cap A_i)}{P(X)}$$

$$= \frac{P(X|A_i)\,P(A_i)}{\sum_{j \in I} P(X|A_j)\,P(A_j)} \quad \forall i \in I.$$

Proved.

Lecture 3: P(6).