Linear Algebra – Distance, Hyperplanes and Halfspaces, Eigenvalues, Eigenvectors

Linear Algebra

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Connections between eigenvectors, column space and null space

- We know that eigenvalues can be complex numbers even for real matrices
- When eigenvalues become complex, eigenvectors also become complex
- However, if the matrix is symmetric, then the eigenvalues are always real
- As a result, eigenvectors of symmetric matrices are also real
- Further, there will always be n linearly independent eigenvectors for symmetric matrices

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Linear Algebra

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Connections between eigenvectors, column space and null space

- · Symmetric matrices have a very important role in data sciences
- In fact symmetric matrices of the form A^TA or AA^T are often encountered
- Eigenvalues of matrices of the form $A^T A$ or AA^T while being real are also non-negative
- As discussed for general symmetric matrices, there will be n linearly independent eigenvectors for matrices of this form also
- What is the connection between the eigenvectors and the column space and null space of a (symmetric) matrix?



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Connections between eigenvectors, column space and null space

 $A v = \lambda v$

• What happens when the eigenvalues become zero?

 The eigenvectors corresponding to zero eigenvalues are in the null space of the matrix

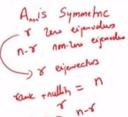
• Conversely, if the eigenvalue corresponding to an eigenvector is not zero then that eigenvector cannot be in the null space

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Connections between eigenvectors, column space and null space

- Let us assume that there are r eigenvectors corresponding to zero eigenvalue
- This means that the null space dimension is r
- From rank-nullity theorem (discussed before), we know that the column rank should be n-r
- That is n r independent vectors are enough to represent all the vectors in the columns of the matrix (column space)
- What could be a basis for this column space or what could be the n − r independent vectors?





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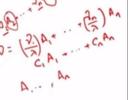
Connections between eigenvectors, column space and null space

- Notice that there are n r eigenvectors which are not in the null space
- We know that these are independent
- We also know that these vectors are a linear combination of all the column vectors – that is they are in the column space

$$Av = \lambda v$$

- Further, we know that the dimension of the column space is n - r (rank-nullity theorem)
- This implies that the eigenvectors corresponding to the nonzero eigenvalues form a basis for the column space

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Example

· Consider the following A matrix

$$A = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

- Notice that this is a symmetric matrix
- · The eigenvalues for this matrix are

$$\lambda = (0,1,2)$$

The eigenvectors corresponding to these eigenvalues are

$$v_{1} = \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix} v_{2} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}, v_{3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



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Example

- From our prior understanding, the eigenvector corresponding to the zero eigenvalue will be in the null space
- · We check that

$$Av_1 = \begin{bmatrix} 0.36 & 0.48 & 0 \\ 0.48 & 0.64 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Interestingly, in the initial lectures, it was identified that the null space vector identifies a relationship between the variables
- Hence, the eigenvector corresponding to the zero eigenvalue can be used to identify the relationships among variables





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Example

· Let us now check if the other two eigenvectors shown below span the column space

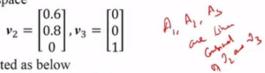
$$v_2 = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is demonstrated as below

trated as below
$$\begin{bmatrix} 0.36 \\ 0.48 \\ 0 \end{bmatrix} = 6 * \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.48 \\ 0.64 \\ 0 \end{bmatrix} = 8 * \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2 * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$





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Summary

· Symmetric matrices have real eigenvalues

· Symmetric matrices also have n linearly independent eigenvectors

· Eigenvectors corresponding to zero eigenvalues span the null space

· Eigenvectors corresponding to non-zero eigenvalues span the column space for symmetric matrices