Two Dimensional Random Variable

Discrete (Joint Distribution of Two Variables).

Let X and Y be two random variables associated with the random experiment E and E' respectively. Let S be the sample space corresponding to the r.e and S' be that of E'.

Let $n(S) = n$ & $n(S') = m$.

Then $\exists$ nm pairs of values (x_i, y_j) ; $i = 1, \ldots, n$, $j = 1, \ldots, m$

Let p_{ij} be the probability assigned to (x_i, y_j)

i.e., $p_{ij} = \text{Prob}(X = x_i, Y = y_j) = P(X = x_i, Y = y_j)$

$i = 1, 2, \ldots, n$

$j = 1, 2, \ldots, m$.

All possible values of (X, Y) and the corresponding probabilities p_{ij} can be shown in joint distribution as follows:

X \ Y	y_1	y_2	y_3	$\cdots$	y_j	$\cdots$	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	$\cdots$	p_{1j}	$\cdots$	p_{1m}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	p_{23}	$\cdots$	p_{2j}	$\cdots$	p_{2m}	$p_{2\cdot}$
x_3	p_{31}	p_{32}	p_{33}	$\cdots$	p_{3j}	$\cdots$	p_{3m}	$p_{3\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	p_{i3}	$\cdots$	p_{ij}	$\cdots$	p_{im}	$p_{i\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}	$\cdots$	p_{nj}	$\cdots$	p_{nm}	$p_{n\cdot}$
Total	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	$\cdots$	$p_{\cdot j}$	$\cdots$	$p_{\cdot m}$	1

As p_{ij} $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ represents the joint pmf of the joint random variable (X, Y), therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} = 1.$$

Marginal probability distribution:

The prob. dist. of the random variable X is given by

$$p_X(x_i) = P(X=x_i) \quad , i=1,2,\dots,n$$

$$= P(X=x_i \cap Y=y_1) + P(X=x_i \cap Y=y_2) + \dots + P(X=x_i \cap Y=y_m)$$

$$= p_{i1} + p_{i2} + \dots + p_{ij} + \dots + p_{im}$$

$$= \sum_{j=1}^{m} p_{ij} = p_{i.}$$

$$\therefore \boxed{p_X(x_i) = P(X=x_i) = p_{i.} \quad ; i=1,2,\dots,n}$$

The prob. dist. of the random variable Y is given by

$$p_Y(y_j) = P(Y=y_j) \quad ; \quad j=1,2,\dots,m$$

$$= P(X=x_1 \cap Y=y_j) + P(X=x_2 \cap Y=y_j) + \dots + P(X=x_i \cap Y=y_j) + \dots + P(X=x_n \cap Y=y_j)$$

$$= p_{1j} + p_{2j} + \dots + p_{ij} + \dots + p_{nj}$$

$$= \sum_{i=1}^{n} p_{ij} = p_{.j} \quad ,$$

$$\therefore \boxed{p_Y(y_j) = P(Y=y_j) = p_{.j} \quad ; \quad j=1,2,\dots,m}$$

Hence, $\sum_{i=1}^{n} p_{i.} = 1$ & $\sum_{j=1}^{m} p_{.j} = 1.$

Conditional Distribution

The conditional distribution of $X = x_i$ given $Y = y_j$ is given by

$$P(X = x_i \mid Y = y_j) = \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}$$

$$\boxed{p_{X|Y}(x_i|y_j) = \frac{p_{ij}}{p_{\cdot j}}} \text{, provided } p_{\cdot j} \neq 0.$$

The conditional probability distribution of $Y = y_j$ given $X = x_i$ is given by

$$P(Y = y_j \mid X = x_i) = \frac{P(X = x_i \cap Y = y_j)}{P(X = x_i)}$$

$$\boxed{p_{Y|X}(y_j|x_i) = \frac{p_{ij}}{p_{i\cdot}}}$$

Table: Conditional dist. of X given $Y = y_3$ (say)

x:	x_1	x_2	$\cdots$	x_i	$\cdots$	x_n	Total
prob:	$\frac{p_{13}}{p_{\cdot 3}}$	$\frac{p_{23}}{p_{\cdot 3}}$	$\cdots$	$\frac{p_{i3}}{p_{\cdot 3}}$	$\cdots$	$\frac{p_{n3}}{p_{\cdot 3}}$	1

Table: Conditional dist. of Y given $X = x_1$ (say)

y:	y_1	y_2	$\cdots$	y_j	$\cdots$	y_m	Total
prob:	$\frac{p_{11}}{p_{1\cdot}}$	$\frac{p_{12}}{p_{1\cdot}}$	$\cdots$	$\frac{p_{1j}}{p_{1\cdot}}$	$\cdots$	$\frac{p_{1m}}{p_{1\cdot}}$	1
	$P(Y=y_1 \mid X=x_1)$	$P(Y=y_2 \mid X=x_1)$		$P(Y=y_j \mid X=x_1)$		$P(Y=y_m \mid X=x_1)$.	

Ex. 1 E : Tossing a coin, E′ : Rolling a die, Let X be the r.v associated with E and Y be the r.v associated with E′.

$S = \{H, T\}$, $S' = \{1,2,3,4,5,6\}$

$H \to 1$, $T \to 0$

E″ = Total no. on the faces, $S'' = \{1,2,3,4,5,6,7\}$

Let Z be the r.v associated with E″

$\therefore$

$Z = 1 \to (0,1) \longrightarrow \{X=0, Y=1\}$

$Z = 2 \to (0,2), (1,1) \to \{X=0, Y=2\} \cup \{X=1, Y=1\}$

$Z = 3 \to (0,3), (1,2) \to \{X=0, Y=3\} \cup \{X=1, Y=2\}$

$Z = 4 \to (0,4), (1,3) \to \{X=0, Y=4\} \cup \{X=1, Y=3\}$

$Z = 5 \to (0,5), (1,4) \to \{X=0, Y=5\} \cup \{X=1, Y=4\}$

$Z = 6 \to (0,6), (1,5) \to \{X=0, Y=6\} \cup \{X=1, Y=5\}$

$Z = 7 \to \cancel{(0,7)}, (1,6) \to \{X=0, Y=7\} \cup \{X=1, Y=6\}$

Then $P\{Z=1\} = \frac{1}{12}$, $P\{Z=i\} = \frac{2}{12} = \frac{1}{6}$, $i = 2,3,4,5,6$

$P\{Z=7\} = \frac{1}{12}$.

$P_{ij} = P(X=i, Y=j) = \frac{1}{12}$, $i = 0,1$; $j = 1,2,3,4,5,6$.

$i \backslash j$	1	2	3	4	5	6	$P(X=i)$ → Marginal prob.
0	P_{01}	P_{02}	P_{03}	P_{04}	P_{05}	P_{06}	$\sum_{j=1}^{6} P_{0j} = P_{0\cdot} = P(X=0)$
1	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	$\sum_{j=1}^{6} P_{1j} = P_{1\cdot} = P(X=1)$
$P(Y=j)$	$\sum_{i=0}^{1} P_{i1} = P_{\cdot 1} = P(Y=1)$						

$P(Y=j)$ → Marginal prob.

Example : The following table gives the joint distribution of x and y :

$x \backslash y$	2	3	7	Total
1	0.10	0.25	0.05	0.40
3	0.30	0.15	0.15	0.60
Total	0.40	0.40	0.20	1

(i) How many pairs of values of X and Y are possible? Write them down and show the corresponding probabilities.

(ii) Show the marginal dist.s of X and Y

(iii) Show the conditional dist. of X given $Y=2$.

(iv) Show the conditional dist. of Y given $X=3$.

(v) Find the probabilities $P(X<Y)$, $P(2X+Y \geq 9)$.

Soln. (i)

$(X=i, Y=j)$:	(1,2)	(1,3)	(1,7)	(3,2)	(3,3)	(3,7)	Total
p_{ij} :	$p_{12}=0.10$	$p_{13}=0.25$	$p_{17}=0.05$	$p_{32}=0.30$	$p_{33}=0.15$	$p_{37}=0.15$	1

(ii)

$P(X=i)=p_{i.}$	0.40	0.60	1
$X=i$	1	3	Total

$Y=j$	2	3	7	Total
$P(Y=j)=p_{.j}$	0.4	0.4	0.2	1

(iii)

$X=i \mid Y=2$	$(X=1 \mid Y=2)$	$(X=3 \mid Y=2)$
$p_{X\mid Y}(i\mid 2)$	$p_{X\mid Y}(1\mid 2)=\frac{p_{12}}{p_{.2}}=\frac{0.10}{0.4}=0.25$	$p_{X\mid Y}(3\mid 2)=\frac{p_{32}}{p_{.2}}=\frac{0.30}{0.4}=0.75$

(iv) Conditional distribution of Y (given X = 3)

$Y=j$	2	3	7	Total
$p_{Y\|X}(j\|3)$	$p_{Y\|X}(2\|3) = \frac{p_{32}}{p_{3.}} = \frac{0.30}{0.60} = 0.50$	$p_{Y\|X}(3\|3) = \frac{p_{33}}{p_{3.}} = \frac{0.15}{0.60} = 0.25$	$p_{Y\|X}(7\|3) = \frac{p_{37}}{p_{3.}} = \frac{0.15}{0.60} = 0.25$	1

(v) $P(X<Y) = P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=7)$
$+ P(X=3, Y=7)$
$= 0.10 + 0.25 + 0.05 + 0.15$
$= 0.55.$

$P(2X+Y \geq 9) = \cancel{P(X=1, Y=2) + P(X=1, Y=3) +}$
$P(X=1, Y=7) + P(X=3, Y=3) + P(X=3, Y=7)$
$= 0.05 + 0.15 + 0.15 = 0.35.$

Ans.

Independent Variables

Two dimensional random variables X and Y are said to be independent if

$P(X=x_i, Y=y_j) = P(X=x_i)\,P(Y=y_j)$

i.e., $p_{ij} = p_{i.} \times p_{.j}$, $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$

Example:

X \ Y	0	2	4	Total
2	0.06	0.15	0.09	0.30 → $p_{2.}$
5	0.14	0.35	0.21	0.70 → $p_{5.}$
Total	0.20 ↓ $p_{.0}$	0.50 ↓ $p_{.2}$	0.30 ↓ $p_{.4}$	1

Here, $p_{20} = p_{.0} \times p_{2.}$; $p_{24} = p_{.4} \times p_{2.}$

$p_{22} = p_{.2} \times p_{2.}$; $p_{24} = p_{.4} \times p_{2.}$; $p_{54} = p_{5.} \times p_{.4}$.

$p_{50} = p_{5.} \times p_{.0}$; $p_{52} = p_{5.} \times p_{.2}$;

Joint Cumulative Distribution Function

The joint cumulative distribution function of two random variables X and Y is given by

$$F_{X,Y}(x,y) = \text{Prob}\{-\infty < X \le x, -\infty < Y \le y\}$$

$$= \text{Prob}(X \le x, Y \le y)$$

$$-\infty < x < \infty, \quad -\infty < y < \infty.$$

To be a bona fied CDF, a fn. must satisfy the following properties, similar to those found in the CDF of a single r.v.

- $0 \le F_{X,Y}(x,y) \le 1. \quad -\infty < x < \infty, \quad -\infty < y < \infty.$
- $F_{X,Y}(x,-\infty) = F_{X,Y}(-\infty,y) = 0$

  $F_{X,Y}(-\infty,-\infty) = 0, \quad F_{X,Y}(+\infty,+\infty) = 1.$
- $F_{X,Y}(x,y)$ must be a non-decreasing fn. of both x and y, i.e.,

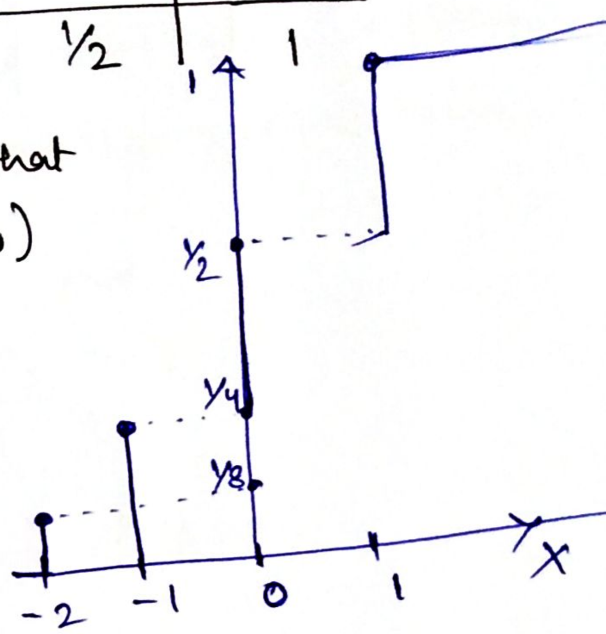
  $$F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2) \text{ if } x_1 \le x_2 \text{ \& } y_1 \le y_2.$$
- The marginal cdf can be derived from the joint cdf as follows

  $$\lim_{y\to\infty} F_{X,Y}(x,y) = F_X(x)$$

  $$\lim_{x\to\infty} F_{X,Y}(x,y) = F_Y(y).$$

Example:- The CDF of a single discrete r.v. X is given in the tabular form as follows

x	$-2 < x$	$-2 \le x < 1$	$-1 \le x < 0$	$0 \le x < 1$	$1 \le x$
$F_X(x)$	0	$1/8$	$1/4$	$1/2$	1



From the table it is understood that

$$P(X < 1.0) = \frac{1}{2} = P(X < 0.5) = P(X \le 0)$$

Similarly, the joint cdf of two discrete r.vs X and Y can also be represented in the tabular form as follows

	$-2 < x$	$-2 \le x < -1$	$-1 \le x < 0$	$0 \le x < 1$	$x \ge 1$.	$F_Y(y)$
$y \ge 2$	0	$1/8$	$1/4$	$1/2$	1	1
$0 \le y < 2$	0	$3/32$	$3/16$	$3/8$	$3/4$	$3/4$
$-2 \le y < 0$	0	$1/32$	$1/16$	$1/8$	$1/4$	$1/4$
$y < -2$	0	0	0	0	0	0
$F_X(x)$	0	$1/8$	$1/4$	$1/2$	1	

From table it is seen ~~tat~~ that

$$P(X < 1, Y < 0) = \frac{1}{8} = P(X \le 0.5, Y < 0)$$
$$= P(X \le 0.0, Y < 0).$$

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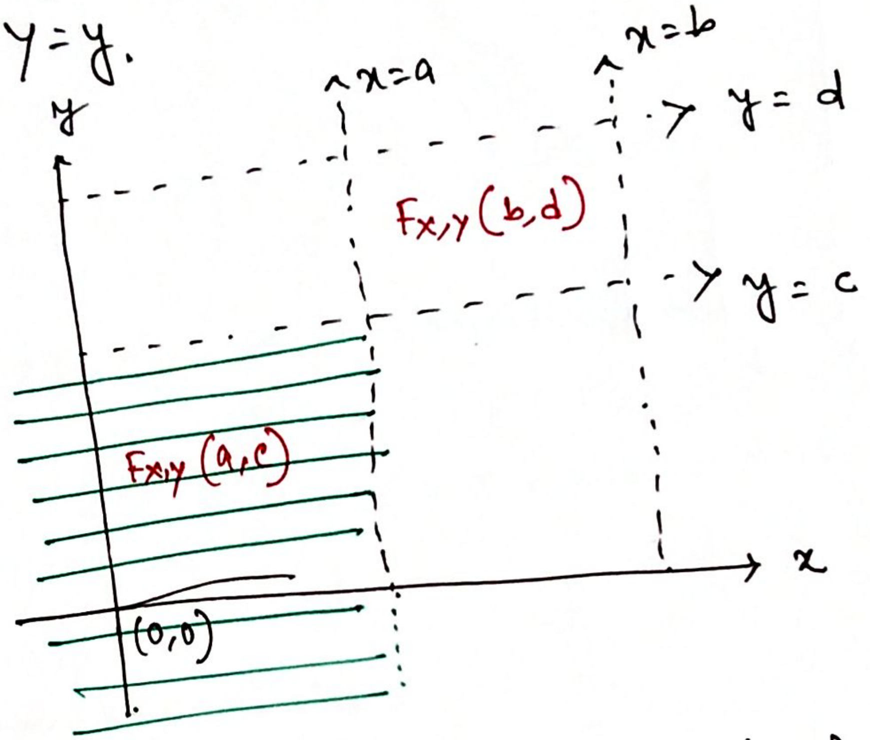
The value of the cdf of two r.v. X and Y at a pt. (x, y) is the prob. of the event that contains all the outcomes that are mapped into the infinite area that lies to the left of a vertical line through $X = x$ and below a horizontal line through $Y = y$.



$F_{X,Y}(a,c) = P(X \le a, Y \le c)$

$F_{X,Y}(b,d) = P(X \le b, Y \le d)$.

\# $P(a < X \le b, Y \le c) = F_{X,Y}(b,c) - F_{X,Y}(a,c)$

\# $P(X \le a, c < Y \le d) = F_{X,Y}(a,d) - F_{X,Y}(a,c)$

\# $P(a < X \le b, c < Y \le d) = F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c) + F_{X,Y}(a,c)$.

clearly seen from the figure.

ABanerji.