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LINEAR ALGEBRA

BINARY OPERATION

denoted by '*'.

$$* : S \times S \longrightarrow S$$

$\mathbb{Z}$ — Set of all integers.

If $a, b \in \mathbb{Z}$
then $a+b \in \mathbb{Z}$
'+' is well - defined binary operation on set of $\mathbb{Z}$.

Examples.

$(\mathbb{Z}, \times)$ well - defined.
$(\mathbb{N}, -)$ — Not a binary operation.

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$(S, +)$ — well - defined.

$A_i \subseteq U$
$S : \{A_1, A_2, \ldots\ldots A_n\}$

$S : P(A)$, A can be any set
$\cup \rightarrow$ union
$\cap \rightarrow$ Intersection] standard binary Operation on a Power Set

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(For any e.g.s, go for matrices)

Properties

→ $*$ is commutative, if
$\forall a, b \in S,$
$a * b = b * a$

→ $*$ is associativety
$a, b, c \in S,$
$a * (b * c) = (a * b) * c$

Matrix multiplication is associative but not commutative.

→ Distributivity of 'o' over '$*$'.

Left distributive:
$a \circ (b * c) = (a \circ b) * (a \circ c)$

Right distributive:

$(b * c) \circ a = (b \circ a) * (c \circ a)$

If commutativity holds, then left dist. = Right dist.

$a+bc = (a+b)(a+c)$

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GROUP :-

A non-empty set G, with a binary operation $*$, is called a group, if

Steps :

(i) $a, b \in G$, $a*b \in G$ [$*$ is closed]

(ii) $a, b, c \in G$, $a*(b*c) = (a*b)*c$

(iii) Existence of Identity (unique)

$\underbrace{\exists!}_{\text{There exists a unique element}}\ e \in G$, s.t $a*e = a$

$e*a = a$

(iv) Existence of Inverse

for every $a \in G$, $\exists : b \in G$

s.t. $a*b = e$

$b*a = e$

b is inverse of a , a^{-1}.

Example

$(\mathbb{R}, +)$

$e = '0'$

$\forall a$, $-a$ is the inverse.

Proof. $a + (-a) = 0$

If the steps hold , $(G, *)$ is a group

$(\mathbb{R} - \{0\}, \times) \rightarrow$ a group.

$(\mathbb{Z}, \times) \rightarrow$ Not a group.

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$\mathbb{R} \sim \{0\} = \mathbb{R}^*$

$(\mathbb{R}-\{0\}, \times) \rightarrow$ multiplicative group on reals

$\{\mathbb{R}^*, 0\}$

<u>**MODULO GROUP**</u> (additive)

$$Z_n = \{0, 1, 2, \ldots, n-1\}$$

$*$: $a*b$: $(a+b) \pmod n$ ← remainder when $(a+b)$ is divided by n.

eg. $Z_5 : \{0, 1, 2, 3, 4\}$

$$2*3 = 0$$
$$4*2 = 1$$

$*$	a_1	$\ldots$	a_n
a_1			
$\vdots$			
a_n			

Group TABLE

* Associativity holds.

Identity element : 0

Inverse $a^{-1} = (n-a)$

$(x \circ y) \circ z$ 6

$x * y$ ~~x z mor~~

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$a \pmod n = a \pmod n$

$[0] = \{0, \pm n, \pm 2n, \dots\}$

Modulo Group is an equivalence relation.

So

$$\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$$

MODULO GROUP (Multiplicative)

$$\mathbb{Z}_p^* = \{1, \dots, p-1\}$$

$(\mathbb{Z}_p^*, \circ)$, p - prime

$$x \circ y = x \times y \pmod p$$

$p \rightarrow$ Prime No.

{ If we take composite no. we can have its factors so rem. becomes 0. }

Reason

$\mathbb{Z}_n = \{1, 2, \dots, n-1\}$

$n = m_1 \times m_2$, $m_1 < n$, $m_2 < n$

$m_1 * m_2 = 0 \notin \mathbb{Z}_n$

- Closure property holds.
- Associativity holds.
- Identity $e = 1$
- Inverse :

e.g. $\mathbb{Z}_7^* = \{1, 2, \dots, 6\}$

6 is inverse of itself

5 is the inverse of 3.

$2^{-1} = 4$

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Abelian Group (Commutative Group)

$(G, *)$ $*$ is commutative

$\mathbb{Q}$ — Set of all rational numbers.

$(\mathbb{Q}, +)$ — Abelian Group.

$(\mathbb{Q}^*, \times)$ — Abelian Group. $\{\mathbb{Q}^* :- \mathbb{Q} - \{0\}\}$

Sub-Group

A Group $(G, *)$

A subset H of G, if it forms a group by itself wrt $*$, then it is called a subgroup.

e.g. $G = (\mathbb{R}, +)$

$H_1 = (\mathbb{Z}, +)$ is a subgroup.

$H_2 = (\mathbb{Q}, +)$

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; a, b, c, d \in \mathbb{R} \right\}$$

$(G, +)$ $e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$G_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ s.t. } ad - bc \neq 0 \right\}$$

$$H_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \pm 1 \right\}$$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \right.$$

FIELDS

$F = \mathbb{R}$ or $\mathbb{C}$

$+$ & $\cdot$

Field is a set with two binary operations satisfying :-

wrt $+, \cdot$ F is closed.

(i) $(F, +)$ is associative –

(ii) There exists an identity element w.r.t. +, additive identity, 0

(i.e.) $a + e = a, \ \forall a \in F$

(iii) There exists an additive inverse for every element in F

$\forall a \in F, \exists a^{-1} \in F$

$a + a^{-1} = e$

(iv) Addition is commutative.

(v) Multiplication is associative.

(vi) Multiplication is commutative.

(vii) $\exists$ a unique non-zero element, $1 \in F$.

s.t. $x.1 = x = 1.x, \ x \in F$

(viii) For each non-zero $x \in F$, $\exists y \in F$.

$x \cdot y = 1 = y \cdot x$; $y \rightarrow$ multiplicative inverse

(ix) multiplication distributes over addition.

$\forall\ x, y, z \in F$, $x\cdot(y+z) = x\cdot y + x\cdot z$ left distributive

$(F, +, \cdot, 0, 1)$ is a field.

$(y+z)\cdot x = y\cdot x + z\cdot x$ Right distributive

SUBFIELD :-

A subset H of F, if it is a field in itself, then H is a subfield of F.

e.g. $\mathbb{R}$ is a subfield of $\mathbb{C}$.
$\mathbb{Q}$ is " " of $\mathbb{R}$ and $\mathbb{C}$.
$\{-1, 0, 1\}$ " " " of $\mathbb{R}$

VECTOR SPACE :-

$V(\mathbb{F}) \rightarrow$ Vector space V over the field $\mathbb{F}$.

A vector space consists of

(1) a field F of scalars. $(F, +, \cdot, 0, 1)$
(2) a set V of objects called vectors.
(iii) a rule on vectors called vector addition associates a vector for each pair of vectors.

(can be any binary operation)

For $x, y \in V$, $x+y \in V$
s.t.

(i) addition is commutative $x+y = y+x$
(ii) addition is associative. $(x+y)+z = x+(y+z)$
(iii) $\exists$ a unique vector

'0' called zero vector, s.t.
$x+0=x$, $\forall x \in V$

(iv) For every $x \in V$, $\exists -x \in V$ s.t.
$x+(-x)=0$

{ Vectors form an Abelian Group w.r.t '+' }

(H) A rule that connects the field F & V by scalar multiplication.
$\alpha \in F$, $x \in V$.
$\alpha . x \in V$.

Following these rules, V is called a vector Space over the field F.

⇒ Scalar Multiplication Satisfies ⇒

$1.x = x$
~~1.~~ $\cancel{1.V = V}$ $\in V$ — $\{1 \in F\}$

~~2.~~ $\cancel{(\alpha_1 \alpha_2).V = \alpha_1(\alpha_2 V)}$

1. $1.x = x$, for every $x \in V$. $1 \in F$

2. $(\alpha_1 . \alpha_2)x = \alpha_1(\alpha_2 x)$

3. $\alpha.(x+y) = \alpha.x + \alpha.y$

4. $(\alpha_1+\alpha_2).x = \alpha_1 x + \alpha_2 x$
$\forall \alpha_1, \alpha_2 \in F$, $x \in V$.

Then V is a vector space over the field F.

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e.g. $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$
$= \{(x,y) : x, y \in \mathbb{R}\} = \mathbb{R}^2$

$\mathbb{C}$ is a vector space over the field $\mathbb{R}$.

$\mathbb{C}^n = \{(z_1, z_2, z_3, \ldots, z_n) : z_k = x_k + iy_k ;$
$x_k, y_k \in \mathbb{R}\}$

$\mathbb{C}^n$ as a vector space over $\mathbb{R}$. ✓
$\mathbb{R}^{2n}$

$\mathbb{R}^n$ over the field $\mathbb{R}$. ✓
$\mathbb{C}^n$ as a vector space over the field $\mathbb{C}$. ✓

$M_{m \times n}(\mathbb{F})$ = Set of all $m \times n$ matrices over the field $\mathbb{F}$.

$$= \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mn} \end{pmatrix}, \quad a_{mn} \in \mathbb{F}$$

e.g. $A = [a_{ij}]$ $B = [b_{ij}]$

$A+B = [a_{ij} + b_{ij}]$

$c \in \mathbb{F} \Rightarrow cA = [ca_{ij}]$,
$\forall\, i,j \quad 1 \le j \le m,$
$1 \le j \le n.$

S – set, $\mathbb{F}$ – field.

$f : S \to \mathbb{F}$

$V = \{$ Set of all functions $f : S \to \mathbb{F} \}$

V is a V.S. over the field $\mathbb{F}$.

Defining, $f, g \in V$, $(f+g)(s) = f(s) + g(s)$

$(cf)(s) = c f(s)$

then we will verify properties.

$0(s) = 0 \quad \forall \; s \in \mathbb{F}$.

Additive Identity.

Additive Inverse :

Suppose $g \in V$,

s.t. $(f+g)(s) = 0$

$g(s) = -f(s), \quad \forall \; s \in S$.

Let $\mathbb{F}$ be the field and let V be the set of all functions f of the form:

$f : \mathbb{F} \to \mathbb{F}$.

$$f(x) = C_0 + C_1 x + \cdots + C_n x^n, \quad x \in \mathbb{F}$$

$C_0, C_1, \ldots, C_n \in \mathbb{F}$ (fixed scalars in $\mathbb{F}$).

A function of this type is called polynomial function

$V = \{$ All polynomial functions on $\mathbb{F} \}$.

V over the field $\mathbb{F}$ is a Vector Space.

Linear Combination of Vectors :-

let V be a vector space over field $\mathbb{F}$.

$\{v_1, v_2, \text{------}, v_n\}$ be a set of vectors from V.

$v \in V$,

$$v = c_1 v_1 + c_2 v_2 + \text{-----} + c_n v_n, \rightarrow (1)$$

where $c_1, c_2, \text{---}, c_n \in \mathbb{F}$.

→ v is said to be a linear combination of $\{v_1, v_2, \text{------}, v_n\}$.

→ If there exists scalars $\{c_1, c_2, \text{---} c_n\}$ s.t. eq. (1) is true.

$\mathbb{F} = R$, $V = \mathbb{R}^3$.

$v_1 = (1, 0, -1)$, $v_2 = (0, 1, 1)$, $v_3 = (1, 1, 1)$

$v = (2, 4, -1)$

$v = c_1 v_1 + c_2 v_2 + c_3 v_3$

$2 = 1 \cdot c_1 + 0 + c_3$ equating compoments.

$c_1 + c_3 = 2$ $\quad$ $c_2 + c_3 = 4$ $\quad$ $-c_1 + c_2 + c_3 = -1$

$-c_1 + 4 = -1$

$c_3 = -3$ $\quad$ $c_2 = 7$ $\quad$ $c_1 = 5$

$(5, 7, -3)$

$v = 5v_1 + 7v_2 - 3v_3$

$x=5$ $\quad f(x)+g(x)$

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1. $(-2,0,3) = c_1(1,3,0) + c_2(2,4,-1)$

$-2 = c_1 + 2c_2$ $\quad 0 = 3c_1 + 4c_2$

$c_1 = -2-2c_2$ $\quad 0 = -6 - 6c_2 + 4c_2$

$c_1 = -2+6$ $\quad 6 = -2c_2$

$c_1 = 4$ $\quad c_2 = -3$

$3 = -c_2$

$$\boxed{c_1 = 4 \;,\; c_2 = -3}$$

2. $(3,4,1) = c_1(1,-2,1) + c_2(-2,-1,1)$

$3 = c_1 - 2c_2$ $\quad 4 = -2c_1 - c_2$

$c_1 = 3+2c_2$ $\quad 4 = -6-4c_2-c_2$

$c_1 = -1$ $\quad 10 = -5c_2$

$c_2 = -2$

$1 = c_1 + c_2$

$-3 \neq 1$

Subspace

A subset W of a vector space V over the field $\mathbb{F}$ is said to be a subspace, if with the same vector additions & scalar multiplication on V, W itself forms a Subspace.

A subset W is a subspace of V if it satisfies the four properties :-

(i) $x+y \in W$, if $\forall\ x, y \in W$ [W is closed under vector addition]

(ii) $\alpha x \in W$, if $\forall\ \alpha \in \mathbb{F}$, $\forall\ x \in W$ [W is closed under scalar multiplication]

(iii) $0 \in W$ [W has zero vector]

(iv) Each vector in W has its inverse in W.

THEOREM 1:

① A non-empty subset W of V is a subspace of V, iff for each pair of vectors $x, y \in W$ and $\forall\ \alpha \in \mathbb{F}$, the vector $\alpha x + y \in W$ ②

① if and only if ②

if ① then ②

& if ② then ①

Proof from 1st ② two properties — put values of α, x, y accordingly to prove 4 properties.

$\alpha = 1$
$x+y \in W$

$\alpha = -1$, $y = x$
$0 \in W$

$y = 0$,
$\alpha x \in W$

$y = 0$, $\alpha = -1$
$-x \in W$

e.g. ① V is subspace of V.

② $\{0\}$ is a subspace of V.

③ $V = \mathbb{F}^n$ $(= \mathbb{C}^n$ or $\mathbb{R}^n)$

$= \{(x_1, x_2, \ldots x_n) : x_i \in \mathbb{F}\}$.

$W = \{(x_1, x_2, \ldots x_n) : x_1 = 0,\ x_i \in \mathbb{F},\ i \neq 1\}$

$\alpha x + y \in W$

So it is a subspace.

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Page $\alpha(1+x_2)+\alpha(1+y_2)$

$\alpha(1+\alpha x_2+1+y_2$

$W_2 = \{(x_1, x_2, \ldots, x_n) : x_1 = 1+x_2, \; x_2, x_3, \ldots, x_n \in \mathbb{F}\}$

Not a vector space

$0 \notin W$.

also closure property doesn't holds.

④ W = Set of Poly. Functions is a subspace of V

V = Set of all functions from $\mathbb{F}$ to $\mathbb{F}$.

⑤ V = Space of $n \times n$ matrices over $\mathbb{F}$.

W_1 = Set of all symmetric matrices over $\mathbb{F}$.

W is a subspace of V.

Hermitian Matrix over the complex $\mathbb{F}$.

$A_{ij} = \overline{A_{ji}}, \; \forall \; i,j$

$$\begin{bmatrix} x & y+iz \\ u+iv & 0 \end{bmatrix} \Rightarrow u+iv = \overline{(y+iz)}$$

$$\Rightarrow \begin{bmatrix} x & y+iz \\ y-iz & 0 \end{bmatrix}$$

$\rightarrow$ diagonal have real entry

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W_2 = Set of all $n \times n$ Hermitian matrices over $\mathbb{C}$.

Not a subspace

$\because A \in W_2 \quad iA \notin W_2$

⑥ W_2 = Set of all $n \times n$ Hermitian matrices over $\mathbb{F}$.

W_2 is a Subspace of V over $\mathbb{R}$.

⑦ Homogeneous linear equation

$$AX = 0$$

V = set of all solutions of equation over $\mathbb{F}$.

$$A_{m \times n} ; \; X_{n \times 1} \in \mathbb{F}.$$

$$\{x_1, x_2, \ldots\ldots x_n\}$$

$$A(x_1 + x_2) = Ax_1 + Ax_2 \quad \{\text{As } x_1 \;\&\; x_2 \text{ are also } \text{sol}^n\}$$
$$= 0$$

$$A(\alpha x_1) = \alpha(Ax_1) = 0.$$

forms a vector space

and subspace of $\mathbb{F}_n$ over the field $\mathbb{F}$.

THEOREM 2.

Suppose W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.

If some elements are common in W_1 and W_2, then their intersection will also have addition, inverse, etc. as both of them will carry these elements.

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$V, \mathbb{F}$

Let S be a set of vectors from V.

Intersection of all the subspaces of V containing S is called the subspace spanned by S.

$S = \{v_1, v_2, \ldots, v_n\}$

W is a subspace spanned by $\{v_1, v_2, \ldots v_n\}$

Theorem:-

The subspace spanned by a non-empty subset S of V, is the linear combination of the vectors in S.

$$W = \{c_1v_1 + c_2v_2 + \ldots + c_nv_n : c_1, c_2, \ldots c_n \in \mathbb{F}, v_i \in S\}$$

e.g. Suppose $V = \mathbb{F}^5$.

F be a subfield of $\mathbb{C}$.

$v_1 = (1,2,0,3,0)$

$v_2 = (0,0,1,4,0)$

$v_3 = (0,0,0,0,1)$

$$W = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in \mathbb{F},\ x_2 = 2x_1,\ x_4 = 3x_1 + 4x_3$$
$$c_1v_1 + c_2v_2 + c_3v_3\}$$

$$= (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$$

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Let $S_1, S_2, \ldots, S_k$ be subsets of V.

The sum of the form
$$v_1 + v_2 + \cdots + v_k, \quad v_i \in S_i$$
called the sum of subsets $S_1, S_2, \ldots S_k$.
denoted by $S_1 + S_2 + \cdots + S_k$.

Let $W_1, W_2, W_3, \ldots, W_k$ be the subspaces spanned by $S_1, \ldots, S_k$.
$W = W_1 + W_2 + \cdots + W_k$ is a subspace of V containing each of $W_1, W_2, \ldots, W_k$.

Proof

$$\alpha x + y \in W$$

let $x, y \in W$

$x_1 = c_1 s_1^1 + c_2 s_2^1 + \cdots c_k s_k^1$

$$x = \underbrace{c_1 x_1}_{\in W_1} + \underbrace{c_2 x_2}_{\in W_2} + \cdots + \underbrace{c_k x_k}_{\in W_k}$$

$$y = d_1 y_1 + d_2 y_2 + \cdots + d_k y_k$$

$x_2 = c_1^2 s_1^2$

$$\alpha x + y = \underbrace{\alpha c_1 x_1 + d_1 y_1} + \alpha c_2 x_2 + d_2 y_2 + \cdots + \alpha c_k x_k + d_k y_k$$

$$= z_1 + z_2 + \cdots + z_k, \quad z_i \in W_i$$

$\mathbb{F} = \mathbb{C}, \quad V = M_{2\times2}(\mathbb{F})$

W_1: Subspace containing all matrices of the form $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$, $x, y, z \in \mathbb{F}$.

W_2: $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, $x, y \in \mathbb{F}$

$$V = W_1 + W_2.$$

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$\Rightarrow$ If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

$$W_1 \cap W_2 := \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbb{F} \right\}$$

It is also subspace

V = Space of polynomial functions over $\mathbb{F}$.

$S = \{f_0, f_1, \ldots\ldots f_n, \ldots\ldots\}$

$f_n(x) = x^n, \quad n = 0, 1, 2, \ldots\ldots$ $\forall x \in \mathbb{F}$.

$S = \{1, x, x^2, \ldots\ldots\}$

V = Subspace spanned by S.

$\Rightarrow$ $\{v_1, v_2, \ldots\ldots, v_k\}$, ~~$v_k \neq 0$~~

If $\exists$ $c_1, c_2, \ldots\ldots, c_k \in \mathbb{F}$ s.t. not all $c_i = 0$

$c_1 v_1 + c_2 v_2 + \ldots\ldots + c_k v_k = 0$

then $\{v_1, v_2, \ldots\ldots v_k$ is said to be linearly dependent$\}$

If $c_i = 0 \quad \forall \; i = 1, 2, \ldots\ldots, k$,
then the vectors are linearly independent.

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$\rightarrow$ $V = \mathbb{F}^3$, $\mathbb{F} \subseteq \mathbb{C}$.

$v_1 = (3, 0, -3)$
$v_2 = (-1, 1, 2)$
$v_3 = (4, 2, 2)$
$v_4 = (2, 1, 1)$

To verify these are linearly dependent or not.

$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_3 v_4 = 0$

$(3c_1 - c_2 + 4c_3 + 2c_4,\ c_2 + 2c_3 + c_4,\ -3c_1 + 2c_2 + 3c_3 + c_4) = 0$

$3c_1 - c_2 + 4c_3 + 2c_4 = 0$ $\quad$ $c_2 + 2c_3 + c_4 = 0 \times 2$

$2c_2 + 4c_3 + 2c_4 = 0$

$-3c_1 + 2c_2 + 3c_3 + c_4 = 0$

$3c_1 = 3c_2$ $\quad$ $c_1 = 0$

$c_1 = c_2$ $\quad$ $c_2 = 0$

$c_3 = 1$

$c_4 = -2$

So vectors are linearly dependent.

$\rightarrow$ $e_1 = (1, 0, 0)$
$e_2 = (0, 1, 0)$
$e_3 = (0, 0, 1)$

$\rightarrow$ linearly independent set.

(Standard linearly independent set).

Standard Basis.

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1. Any set which contains linearly dependent set of vectors, then ~~the~~ given ~~set~~ is also Linearly dependent.

2. Any subset of a Linearly Independent set is also L.I.

3. Any set which contains zero vector is L.D.

$c_1 \cdot (0) = 0$

4. A set of vectors is L.I. iff each finite subset of S is linearly independent (i.e.) iff for distinct vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ of S,

$$c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n = 0 \Rightarrow c_i = 0$$

BASIS

A Basis for V is a linearly independent set of vectors of V that spans V.

$$V = \text{span}\{S\}$$

dimension of $V = |S|$

$\dim(V) = |S| \rightarrow$ no. of vectors in S.

If S is finite, then S is said to be finite dimensional.

$V = \mathbb{R}$ over the field $\mathbb{R}$	$V = \mathbb{R}^2$
$S = \{1\}$	$S = \{(1,0), (0,1)\}$
single dimension	two dimension

$V = \mathbb{F}^n$, over the field $\mathbb{F}$.

$\epsilon_1 = (1, 0, \ldots, 0)$

$\epsilon_2 = (0, 1, \ldots, 0)$ Standard Basis of $\mathbb{R}^n$.

$\vdots$

$\epsilon_n = (0, 0, \ldots, 1)$

$(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$, $x_i \in \mathbb{F}$

$(x_1, x_2, \ldots, x_n) = x_1\epsilon_1 + x_2\epsilon_2 + \ldots + x_n\epsilon_n$

$\rightarrow$ $\mathbb{F} \subseteq \mathbb{C}$.

$P(\mathbb{F})$ = Space of polynomial functions.

Standard basis for poly. functions

$f_0 = 1$

$f_1 = x$ $\{1, x, x^2, \ldots, x^n, \ldots\}$

$f_2 = x^2$

$c_0 f_0 + c_1 f_1 = 0$

$c_0 1 + c_1 x = 0$

linearly independent

$P_n(\mathbb{F})$: Space of polynomials of degree $\leq n$.

$\{1, x, \ldots, x^n\}$

$\dim(P_n) = n+1$

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Theorem:-

Let V be a vector space.
Let $\{v_1, v_2, \ldots, v_m\}$ spans V.
Then any ~~set~~ independent set of V is finite and contains not more than m vectors.

Corollary:- Any two bases of a v.s. V, contains equal number of vectors.
(dimension)

Definition:- The dimension of v.s. V is equal to the no. of elements in its basis.

e.g.

→ $M_{2\times2}(\mathbb{F})$

$$E^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad E^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \in M_{2\times2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= aE^{11} + bE^{12} + cE^{21} + dE^{22}$$

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Q. → $V = \mathbb{R}^3$ $\alpha_1 = (1,0,-1)$ $\alpha_2 = (1,2,1)$ $\alpha_3 = (0,-3,2)$

$\mathbb{B} = \{\alpha_1, \alpha_2, \alpha_3\}$?
(Basis)

I. $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

$(c_1 + c_2,\ 2c_2 - 3c_3,\ -c_1 + c_2 + 2c_3) = 0$

$c_1 + c_2 = 0$ $c_2 = 0$ $c_3 = 0$

$c_1 = -c_2$

$c_1 = 0$

So vectors are linearly independent.

II $v \in \mathbb{R}^3$
$\parallel$
(v_1, v_2, v_3)

$(v_1, v_2, v_3) = d_1\alpha_1 + d_2\alpha_2 + d_3\alpha_3$

$= (d_1 + d_2,\ 2d_2 - 3d_3,\ -d_1 + d_2 + 2d_3)$

Since $\mathbb{R}^3$ would have basis of 3 vectors and this set contains 3 linearly independent vectors. So it can act as Basis.

Corollary:- let V be a finite dimensional vector space & $n = \dim(V)$.

(1) Any subset of V which contains more than n-vectors then the subset is linearly dependent.

(2) No subset of V with less than n-vectors can span V.

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Does $\alpha_1 = (1,0,-1)$
$\alpha_2 = (1,2,1)$
$\alpha_3 = (2,2,0)$
$\alpha_4 = (2,4,2)$ spans V?

No $(0,-3,2)$ is missing.

<u>Theorem</u>:- Let S be a linearly independent set of vectors from V. Suppose β is a vector not belonging to the span(S). Then

$$S' = S \cup \{\beta\}$$

is a linearly independent set.

→ Let W be a subspace of a finite dimensional vector space V.
Every linearly independent set of W is finite and is part of a basis for W.

{Result also holds if $W = V$}

<u>Theorm</u>

→ If W is a proper subspace of finite dim. space V, then basis of W is finite and $\dim W < \dim V$

→ <u>Theorem</u>
Let W_1, W_2 are subspaces of V (finite dim.)
$W_1 + W_2$ is finite dimensional.

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

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Proof:- Let $\{\alpha_1, \alpha_2, ----, \alpha_k\}$ is a basis of $W_1 \cap W_2$.

$\{\beta_1, \beta_2, ----, \beta_l, \alpha_1, \alpha_2, ----, \alpha_k\}$ be the basis of W_1.
$\{\gamma_1, \gamma_2, ----, \gamma_m, \alpha_1, \alpha_2, ----, \alpha_k\}$ be the basis of W_2.

Basis of $W_1 + W_2$.

$\{\alpha_1, \alpha_2, \alpha_3, ----, \alpha_k, \beta_1, \beta_2, ---, \beta_l, \gamma_1, \gamma_2, ---, \gamma_m\}$

$$\dim(W_1 + W_2) = k + l + m.$$

SYSTEM OF LINEAR EQUATIONS :-

$x_1, x_2, ----, x_n$ — n-unknowns
$a_{11}, a_{12}, ----, a_{mn}$ — mn - scalars (coefficient
$b_1, b_2, ----, b_m$ — scalars

$$a_{11}x_1 + a_{12}x_2 + ---- + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ---- + a_{mn}x_n = b_m$$

$$2x_1 - x_2 + 3x_3 + 2x_4 = 0$$
$$x_1 + 4x_2 - x_4 = 0$$
$$2x_1 + 6x_2 - x_3 + 5x_4 = 0$$

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$AX = 0$

$A_{3\times4}$

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. e_1: R_i, $c \in \mathbb{F}$, $R_i \longrightarrow cR_i$

e_2: R_i, R_j, $c \in \mathbb{F}$, $R_i \longrightarrow R_i + cR_j$

e_3: Interchange :- $R_i \longleftrightarrow R_j$

Row - reduced matrix :-

If B is a matrix that can be obtained from A by ~~services~~ series of elementary row op. , then B is said to be row - equivalent to B.

$$e(e_1(R_i)) = R_i$$
$$e_1(e(R_i)) = R_i$$

"Row - equivalent"
equivalence relation

reflexive - A is row eq. to A
Symmetric - A " " " to B
& B " " " to A
Transitive -

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \longrightarrow \begin{array}{l} R_2 \to 2R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 9 & -3 & -4 \\ 0 & 7 & -4 & 3 \end{bmatrix} \quad \begin{array}{l} R_1 \to 9R_1 + R_2 \\ R_3 \to 9R_3 - 7R_2 \end{array}$$

$$\begin{bmatrix} 18 & 0 & 24 & 14 \\ 0 & 9 & -3 & -4 \\ 0 & 0 & -15 & 55 \end{bmatrix} \quad \begin{array}{l} R_1 \to 15R_1 + 24R_3 \\ R_2 \to 5R_2 - R_3 \end{array}$$

$$\begin{bmatrix} 270 & 0 & 0 & 1530 \\ 0 & 45 & 0 & -75 \\ 0 & 0 & -15 & 55 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & -\frac{11}{3} \end{bmatrix}$$

Row-Reduced Echelon Matrix

$$\left(\frac{11}{3}c, \frac{-17}{3}c, \frac{5}{3}c, c\right)$$

Non-zero Rows are kept first in row-reduced echelon matrix.

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An $m \times n$ matrix A is row-reduced echlon

(i) if A is row-reduced.
(ii) Every row of A which has all its entries zero occurs below every row which has non-zero entry.
(iii) If rows $1, 2, ---r$ are the non-zero rows of A and if the leading non-zero entry of row i occurs in column K_i, $i = 1, 2, ---r$ then
$K_1 < K_2 < ---- K_r$

e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem:
If a $m \times n$ matrix A and $m < n$, then the homogeneous system $Ax = 0$ has non-trivial soln.

Theorem:-
If A is an $n \times n$ matrix, then A is row-equivalent to $I_{n \times n}$ matrix iff the $Ax = 0$ has only trivial solution.

Rank of matrix $= r$
(no. of rows with non-zero entries in row-reduced echlon)

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1. Every non-zero row has a leading non-zero entry to be $= 1$.
2. The column containing the leading non-zero entry will have all other entries to be equal to zero.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ is not row-reduced.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$ is row reduced.

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is row-reduced.

$AX = 0$
$RX = 0$
# of non-zero rows

$$\begin{bmatrix} \begin{bmatrix} I_r \end{bmatrix} & \underbrace{\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix}}_{r \times n-r} \\ 0 & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

x_1 to x_r are dependent on x_{r+1} to x_n.

$$x_1 \quad \text{— —} \quad + b_{1_{r+1}} x_{r+1} \quad \text{— — —} \quad x_n = 0$$

$$x_r \quad \text{— — —} \quad + b_{r_{n+1}} x_{r+1} \quad \text{— —} \quad b_{r_n} x_n = 0$$

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For Non-Homogeneous System :-

$$A_{m\times n}\, X_{n\times 1} = Y_{m\times 1}$$

$$A' = [A|Y]_{m\times(n+1)}$$

row reduction

$$R' = [R|Y']$$

$$x_1 - x_2 + 2x_3 = 1$$
$$2x_1 + 2x_3 = 1$$
$$x_1 - x_2 + 4x_3 = 2$$

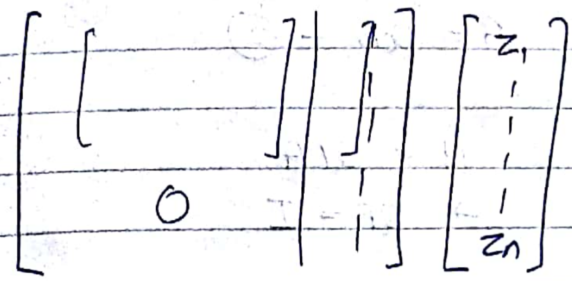
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -1 & 4 & 2 \end{array}\right] \quad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 2 & 1 \end{array}\right] \quad R_1 \to 2R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 2 & 1 \end{array}\right] \quad R_1 \to R_1 - R_3,\ R_2 \to R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$\Rightarrow \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = \frac{1}{2}$



$z_i = 0 \quad ; \quad i > r$

$E \rightarrow$ Elementary matrix
(Matrix obtained after performing 1 row operation on identity matrix.)

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$

~~for A to be a~~
A is a square matrix

$R = \underbrace{E_n(\text{------} E_2(E_1 A))}_{}{}_{n \times n}$

$R = PA$

$P^{-1} R = A$

$[A \mid I_n]_{n \times 2n}$ $\quad B = PA$

$[I_n \mid B]$

B is the inverse of A.

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$B = PA$ — ①
$A = QB$ — ②

$A = QPA$
$\Rightarrow QP = I$

$B=PA$
$QB=A$
$QPA=A$
$QP=I$

Theorem :-

If $A_{n\times n}$ is a matrix, the following are equivalent :-

(i) A is invertible.

(ii) A is row equivalent to an Identity matrix I_n.

(iii) A is a product of elementary matrices.

Corollary :-

If A is $n\times n$ matrix and if a sequence of row operations reduce A to I_n, then the same sequence of operations when performed on I_n, gives A^{-1}.

$$\begin{array}{r} 310 \\ -180 \\ 24 \\ \hline 116 \end{array}$$

Theorem : For any $n\times n$ matrix A, the following are equivalent.

(i) A is invertible.

(ii) The system $AX=0$ has only trivial solⁿ $X=0$

(iii) The system $AX=Y$ has a solution X for each $Y_{n\times 1}$.

$$X = A^{-1}Y$$

$B = PA$

$QB = A$

$QPA = A$

$QP = I$

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$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$\frac{2}{3} - \frac{1}{2}$

$\frac{4-3}{6}$

$\frac{1}{2} - \frac{1}{3}$

$\frac{3-2}{6}$

$\frac{3}{4} - \frac{1}{2}$

$\frac{3-2}{4}$

$\frac{3}{5} - \frac{1}{3}$

$\frac{9-5}{15}$

$[A|I]$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 1 & 0 \\ 1/3 & 1/4 & 1/5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to 2R_2 - R_1$

$R_3 \to 3R_3 - R_1$

$\frac{2}{3} - 1$

$\frac{16}{15} - 1$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & 0 & 0 \\ 0 & 1/6 & 1/6 & -1 & 2 & 0 \\ 0 & 1/4 & 4/15 & -1 & 0 & 3 \end{array}\right]$$

$R_1 \to 2R_1 - 6R_2$

$R_3 \to 4R_3 - 6R_2$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1/3 & 8 & -12 & 0 \\ 0 & 1/6 & 1/6 & -1 & 2 & 0 \\ 0 & 0 & 1/15 & 2 & -12 & 12 \end{array}\right]$$

$R_1 \to 3R_1 + 15R_3$

$R_2 \to 6R_2 - 15R_3$

$\begin{array}{r} 3\,10 \\ -180 \\ 24 \\ \hline 116 \end{array}$

$$\left[\begin{array}{ccc|ccc} \cancel{6} & 0 & 0 & -116 & -36 & 180 \\ 0 & 1 & 0 & 144 & & \\ 0 & 0 & 1/15 & & & \end{array}\right]$$

$R_1 \to R_1/6$

$$\left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 54 & -216 & 180 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1/15 & 2 & -12 & 12 \end{array}\right]$$

$R_3 \to 15R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

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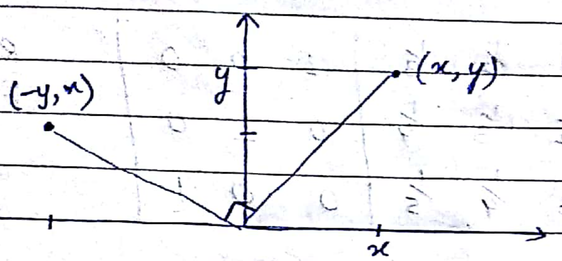
LINEAR TRANSFORMATIONS

$A_{2\times 2}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \underset{\in \mathbb{R}^2}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\in \mathbb{R}^2}{\begin{bmatrix} -y \\ x \end{bmatrix}}$$



$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

origin remains fixed in all cases.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

$\downarrow$

becomes mirror image of (x,y) wrt $y = x$.

$v = (4,0) \Rightarrow$ applying above matrix
$= (2,-2) + (2,2)$ $\downarrow$
$(0,4)$

1. $Av_1 + Av_2 = Av$
2. $A0 = 0$
3. $A(cv) = cAv$

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V, W be two vector space over $\mathbb{F}$.

$T: V \rightarrow W$

T is <u>linear transformation</u> if

(i) $T(v_1 + v_2) = T(v_1) + T(v_2)$; $\forall v_1, v_2 \in V$

(ii) $T(cv) = cT(v)$, $\forall v \in V$; $\forall c \in \mathbb{F}$

<u>Properties Of Linear Transformation</u> :

1. T is linear iff $T(0_V) = 0_W$
2. T is linear iff $T(cv_1 + v_2) = cT(v_1) + T(v_2)$ $\forall v_1, v_2 \in V$, $\forall c \in \mathbb{F}$.
3. If T is linear then $T(x-y) = T(x) - T(y)$
4. for any vectors $v_1, v_2, \ldots v_n \in V$, and $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{F}$

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n)$$

$\Rightarrow$ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

$$T(x_1, x_2) = (2x_1 + x_2, x_1)$$

1. $T(x_1, x_2) + T(y_1, y_2) = (2x_1 + x_2, x_1) + (2y_1 + y_2, y_1)$

$$= (2(x_1 + y_1) + x_2 + y_2, x_1 + y_1)$$

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$T((x_1+x_2),(y_1+y_2)) = (2(x_1+x_2)+y_1+y_2,\ x_1+x_2)$

2. $T(c(x_1,x_2)) = T(cx_1, cx_2)$
$= (2cx_1 + cx_2,\ cx_1)$

$cT(x_1,x_2) = c(2x_1+x_2,\ x_1)$
$= (2cx_1+cx_2,\ cx_1)$

So T is a linear transformation

* we can also verify using
$T(cv_1+v_2) = cT(v_1) + T(v_2)$

$\Rightarrow$ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ map called
$T(a_1,a_2) = (a_1, 0)$ {- Projection on x-axis}

$\Rightarrow$ Define $T: M_{2\times2}(\mathbb{F}) \rightarrow M_{2\times2}(\mathbb{F})$

$T(A) = A^T$
T is linear.

$\Rightarrow$ $T: P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$

$f(x) \in P_n(\mathbb{R})$

$T(f(x)) = f'(x)$

~~$T(f+g) = f'(x) + g'(x)$~~

$$T(c_1 f + c_2 g) = c_1 f'(x) + c_2 g'(x)$$

$$= c_1 T(f) + c_2 T(g)$$

$$T : P_n(\mathbb{R}) \longrightarrow P_{n+1}(\mathbb{R})$$

$x \in [a,b] \in \mathbb{R}$

$$T(f) = \int_a^b f(t) \cdot dt$$

T is linear.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T(x,y) = (x+1, y)$$

is it linear?

no.

$$T(X+Y) \neq T(X) + T(Y)$$

Some Special Transformations :-

→ $T : V \rightarrow V$

$T(v) = v, \ \forall v \in V$

I_V — Identity Transformation.

→ $O : V \rightarrow W$

$O(v) = 0, \ \forall v \in V.$

O_V is called zero transformation.

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Nullspace of T | Kernel of T

$T: V \to W$

$Ker(T)$ or $N(T) = \{x \in V \mid T(x) = 0\} \subset V$

Set of all vectors in V which after transformation becomes 0.

e.g. $T(a_1, a_2) = (a_1, 0)$

Nullspace

$N(T) = \{(0, a_2) : a_2 \in \mathbb{R}\}$

e.g. $T(x,y) = (-y, x)$

$N(T) = \{(0,0)\}$

Range of T :-

$\text{Range}(T)$ or $\underbrace{Im(T)}_{\text{Image of } T}$ or $R(T)$

$= \{w \in W : \exists\, v \in V \; T(v) = w\}$

$= \{T(v) : v \in V\} \subset W$

→ $I_V : V \to V$

$I_V(v) = v$

$N(I_V) = \{0\}$

$R(I_V) = V$

→ T_0 :

$O_V : V \to W$

$T_0(v) = 0$

$N(T_0) = V$

$R(T_0) = \{0\}$

→ $T : \mathbb{R}^3 \to \mathbb{R}^2$

$T(x,y,z) = (x-y, 2z)$

$N(T) = \{(a,a,0) : a \in \mathbb{R}\}$

$R(T) = \mathbb{R}^2$.

<u>Proof</u>: $(a_1, a_2) \in \mathbb{R}^2$, $a_1 = x-y$ $a_2 = 2z$

<u>Theorem</u>:- Let V & W be a Vector Space over $\mathbb{F}$.
Let $T : V \to W$ be linear.
Then $N(T)$ & $R(T)$ is a subspace of V & W respectively.

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Proof :- $x, y \in N(T) \Rightarrow T(x) = T(y) = 0$

$\alpha, \beta \in \mathbb{F}$,

To show that

$\alpha x + \beta y \in N(T)$

$T(\alpha x + \beta y) = \alpha T(x) + \beta (Ty)$

$= 0$

$T(0_V) = 0_W$

$0_W \in R(T)$

Let $x, y \in R(T) \Rightarrow \exists : v_1, v_2 \in V$

$T(v_1) = x$

$T(v_2) = y$

To show

$\alpha x + \beta y \in R(T)$

$v = \alpha v_1 + \beta v_2$

$T(v) = T(\alpha v_1 + \beta v_2)$

$= \alpha T(v_1) + \beta T(v_2)$

$= \alpha x + \beta y \in R(T)$

$\Rightarrow$ **Theorem:**

Let V & W be V.S. over $\mathbb{F}$

& Let $T : V \to W$

Let $\beta = \{\beta_1, \beta_2, \ldots\ldots \beta_n\}$ be a basis for V, then

$R(T) = \text{span}\,(T(\beta))$

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Ordered Basis

$$R(T) = \text{span}\{T(\beta_1), T(\beta_2), \text{------} T(\beta_n)\}$$

Proof

To show 2 sets to be equal.
$A \subseteq B$ & $B \subseteq A \Rightarrow A = B$.

$$\text{span}\{T(\beta_1), \text{------} T(\beta_n)\} \subseteq R(T) \quad \text{—} (1)$$

since $T(\beta_i) \in R(T) \ \forall i$

(i.e) $\sum_{i=1}^{n} \alpha_i T(\beta_i) \in R(T)$

Let $w \in R(T)$ whether $w = \sum \alpha_i T(\beta_i)$
$\exists v \in V$ for some $\alpha_i \in F$
$\exists T(v) = w$

$v = \alpha_1\beta_1 + \alpha_2\beta_2 + \text{------} + \alpha_n\beta_n$ for some $\alpha_1 \in \mathbb{R}$.

$$w = T(v) = T(\alpha_1\beta_1 + \text{------} + \alpha_n\beta_n)$$
$$= \alpha_1 T(\beta_1) + \text{------} + \alpha_n T(\beta_n)$$

$$R(T) \subseteq \text{span}(T(B)) \quad \text{—} (2)$$

from (1) & (2)
$$R(T) = \text{span}(T(B))$$

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E.g. $T: P_2(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$

$$T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$\beta = \{1, x, x^2\}$$

$f(x) = 1, \forall x \in \mathbb{R}.$ $\quad T(1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$f(x) = x, \forall x \in \mathbb{R}$ $\quad T(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

$f(x) = x^2, \forall x \in \mathbb{R}$ $\quad T(x^2) = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$

$$\text{span}(T(\beta)) = \text{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= R(T)$$

$\boxed{\text{Dim}(R(T)) = \text{Rank of } T}$ $= r(T)$

$\boxed{\text{Nullity} = \dim(\text{Ker } T)}$

$\text{rank}(T) = 2$

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$f(1) = f(2) \qquad f(0) = 0$

$$f(x) = ax^2 + bx$$

$$a+b = 4a+2b$$

$$-b = 3a$$

$$f(x) = x^2 - 3x$$

$$N(T) = \text{span}\{x^2 - 3x\}$$

$$\dim(N(T)) = 1$$

Dimension Theorem :- (Rank – Nullity Theorem)

Let $T: V \to W$ be a linear transformation. If V is finite dim.

then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Proof :

$$\dim(V) = n$$

$$\beta = \{\beta_1, \beta_2, \text{----}, \beta_n\}$$

$$\text{Nullity}(T) = r$$

Let $\{\beta_1, \beta_2, \text{----}, \beta_r\}$ be a basis of $N(T)$.

$$T(\beta_i) = 0 \quad \forall \ i = 1, 2, \text{---} r$$

$$\{T(\beta_{r+1}), T(\beta_{r+2}), \text{———} T(\beta_n)\} = R(T)$$

$$\Rightarrow \text{rank}(T) = n - r$$

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Theorem:

$T: V \to W$ is linear

T is one-to-one iff $N(T) = \{0\}$

$(\Rightarrow)$ T is one-one $\Rightarrow N(T) = \{0\}$

$(\Leftarrow)$ $N(T) = \{0\} \Rightarrow T$ is one-to-one.

Suppose $T(x) = T(y)$, $x \neq y$

$T(x-y) = 0$

$x - y = 0$

$\Rightarrow x = y$

V, W are finite dimensional with equal dimensions

$T: V \to W$

The following are equivalent

(i) T is one-to-one.

(ii) T is onto.

(iii) $\dim(V) = \text{rank}(T) = \dim(W)$

$\Rightarrow$ $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (x+y, y)$

$N(T) = \{0\}$.

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For V, W having equal dimensions.

Suppose $\{v_1, v_2, \ldots v_n\}$ be a basis of V.
and $\{w_1, w_2, \ldots w_n\}$ be a basis of W.

There exists exactly one linear transformation $T: V \to W$ such that

$$T(v_i) = w_i, \quad \forall\ i = 1, 2, \ldots n$$

$$x \in V \Rightarrow x = a_1 v_1 + \ldots + a_n v_n = \Sigma a_i v_i, \quad a_i \in \mathbb{F}$$

Define $T: V \to W$

$$T(x) = \Sigma a_i w_i$$

T is linear?

$$u = \Sigma b_i v_i, \quad y = \Sigma c_i v_i$$

$$u+y = \Sigma (b_i + c_i) v_i$$
$$T(u+y) = \Sigma (b_i + c_i) w_i$$

$$T(u+y) = \Sigma (b_i + c_i) w_i = \Sigma b_i w_i + \Sigma c_i w_i$$
$$= T(u) + T(y)$$

$T(v_1)$

$$v_1 = 1 \cdot v_1 + 0 v_2$$
$$T(v_1) = 1 \cdot w_1$$
$$\vdots$$
$$T(v_i) = 1 w_i$$

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To prove for uniqueness :-

Suppose $\mathcal{F}: U \to V \to W$ $\quad U(v_i) = w_i, \; i = 1, 2, \ldots$

$$v \in V, \; U(v) = U\left(\sum a_i v_i\right) = \sum a_i U(v_i)$$
$$= \sum a_i w_i$$
$$= T(V)$$

$\Rightarrow$ $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$T(1,1) = (1,0,2)$$
$$T(2,3) = (1,-1,4)$$

$T(8,11)$

$$(8,11) = 2(1,1) + 3(2,3)$$
$$T(8,11) = 2(1,0,2) + 3(1,-1,4)$$

$\Rightarrow$ $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\beta = \{(1,0), (1,1)\}$$

$$T(1,0) = (1,4)$$
$$T(1,1) = (2,5)$$

Is T one-one ? $\quad$ Find $T(2,3)$

$$\cancel{(0,0) = c_1(1,0) + c_2(1,1)}$$

$c_1 + c_2 = 0$ $\quad$ $\cancel{T(0,0) = 0(1,4) + 0(2,5)}$

$c_2 = 0$ $\quad$ $\cancel{= 0}$

$c_1 + c_2 = 2$ $\quad$ $(2,3) = c_1(1,0) + c_2(1,1)$

$c_2 = 3$ $\quad$ $T(2,3) = -1(1,4) + 3(2,5)$

$c_1 = -1$ $\quad$ $= (5,11)$

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$T(x,y) = (0,0)$
$\Rightarrow x = y = 0.$
So T is one-one.

$(x,y) = c_1(1,0) + c_2(1,1)$
$c_1 + c_2 = x$
$c_2 = y.$
$c_1 = x - y$

$T(x,y) = (x-y)(1,4) + y(2,5)$
$= (x+2y, -4y+5y)$
$= (x+2y, y)$
$x+2y = 0 \qquad y = 0$
$\Rightarrow x = y = 0.$

$\{v_1, v_2, \ldots\ldots v_n\}$ — Ordered basis of V.

$x \in V$: $x = c_1v_1 + c_2v_2 + \ldots + c_nv_n$

$$= \begin{bmatrix} c_1 & c_2 & c_3 & \ldots & c_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$$

$$[x]_c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

coordinate vector wrt ordered basis.

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$\Rightarrow$ $T: V \to W$

V, $\{v_1, \ldots, v_n\}$ ordered basis
$\dim(V) = n$

W, $\{w_1, \ldots, w_m\}$
$\dim(W) = m$

$T(v_1) = a_{11} w_1 + a_{12} w_2 + \ldots + a_{1m} w_m$

$\vdots$

$T(v_n) = a_{n1} w_1 + a_{n2} w_2 + \ldots + a_{nm} w_m$

$$[T]_{v_*}^{w_*} = A = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & & a_{nm} \end{bmatrix}$$

$\downarrow$

matrix ~~to~~ representation for linear transformation.

e.g.

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x,y) = (x+3y, 0, 2x-4y)$

$\beta = \{e_1, e_2\}$ where $e_1 = (1,0)$, $e_2 = (0,1)$

$\beta' = \{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$T(1,0) = (1,0,2)$
$= 1(e_1) + 0 \cdot e_2 + 2e_3$

$T(0,1) = (3, 0, -4)$

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$$= 3e_1 + 0e_2 - 4e_3$$

$$[T]_{\beta}^{\beta'} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$$

If we take $\beta_W = \{e_3, e_2, e_1\}$ $\{\beta' \neq \beta_W\}$

$$T(1,0) = 2e_3 + 0e_2 + 1e_1$$
$$T(0,1) = -4e_3 + 0e_2 + 3e_1$$

$$[T]_{\beta}^{\beta'} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$$

e.g.

$$T: \mathbb{P}_3(\mathbb{R}) \longrightarrow \mathbb{P}_2(\mathbb{R})$$

$$T(f(x)) = f'(x)$$

$$\beta = \{1, x, x^2, x^3\}$$
$$\beta' = \{\underset{e_1}{1}, \underset{e_2}{x}, \underset{e_3}{x^2}\}$$

$$T(1) = 0 = 0e_1 + 0e_2 + 0e_3$$
$$T(x) = 1 = 1e_1$$
$$T(x^2) = 2x = 2e_2$$
$$T(x^3) = 3x^2 = 3e_3$$

$$[T]_{\beta}^{\beta'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

V, W

$\Rightarrow$ $T, U : V \to W$

Define, $S_v = (T+U)v = T(v) + U(v)$
$(cT)(v) = c\,T(v)$

To check for this to be linear

$$(aT + bU)(v) = aT(v) + bU(v)$$

$$\{T;\ T : V \to W\} = \mathcal{L}(V, W) \text{ is a vector space.}$$

Theorem :

$$T, U \in \mathcal{L}(V, W)$$

$$[T+U]_{\beta}^{\beta'} = [T]_{\beta}^{\beta'} + [U]_{\beta}^{\beta'}$$

$$[cT]_{\beta}^{\beta'} = c\,[T]_{\beta}^{\beta'}$$

e.g.

$T : \mathbb{R}^2 \to \mathbb{R}^3$.

$$T(x,y) = (x+3y,\ 0,\ 2x-4y)$$
$$U(x,y) = (x-y,\ 2x,\ 3x+2y)$$

$$[T]_{\beta}^{\beta'} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix} \qquad [U]_{\beta}^{\beta'} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$S = \{0\}$
$Dim(S) = 0$

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$(T+U)(x,y) = (2x+2y,\ 2x,\ 5x-2y)$

$$[T+U]_{\beta}^{\beta'} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$$

$\Rightarrow \quad \mathcal{L}(V) = \{T : V \to V\}$

we can have 2 diff. basis for same set

$$\boxed{\text{If we take same basis } \quad [T]_{\beta}^{\beta'} = I_n.}$$

Composition

$T : V \to W$
$U : W \to Z$

$U \circ T : V \to Z$

$T \circ U(v) =$ not defined.

$U \circ T(v) = U[T(v)]$

$U \circ T$ is linear

$U \circ T(av_1 + bv_2) = a\, U \circ T(v_1) + b\, U \circ T(v_2)$

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$$U_2, T, U_1 \in \mathcal{L}(V)$$

$$T \cdot U(v) = T(U(v))$$

Properties -

1. $T(U_1 + U_2) = T \circ U_1 + T \circ U_2$
2. $(U_1 + U_2) \cdot T = U_1 \circ T + U_2 \circ T$
3. $T \circ (U_1 \circ U_2) = (T \circ U_1) \circ U_2$
4. $a(T \circ U) = (aT) \circ U = T \circ (aU)$

→ Matrix Representation for Composition :

$$T \circ U : V \to V$$

$$[T \circ U]_\beta^{\beta'} = [T]_\beta^{\beta'} \, [U]_\beta^{\beta'}$$

$$T : P_3 \to P_2$$

$$T(f(x)) = f'(x)$$

$$U : P_2 \to P_3$$

$$U(f(x)) = \int_0^x f(t) \cdot dt$$

$$U \circ T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$$

$$(U \cdot T)(f(x)) = f(x) = U(f'(x)) = f(x)$$
$$= I(f(x))$$

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$$[T]_{\beta}^{\beta'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$[U]_{\beta}^{\beta'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$[U \circ T]_{\beta}^{\beta'} = [U][T]_{\beta}^{\beta'} = I_4$$

Theorem →

Let V, W, Z be vector Space over $\mathbb{F}$.
Let α, β, γ be their Ordered basis,

$$T: V \to W \quad \text{and} \quad U: W \to Z$$
$$UT: V \to Z$$

$$[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$$

Corollary →

$$T: V \to V \quad \text{and} \quad U: V \to V.$$

$$[UT]_{\alpha} = [U]_{\alpha} \cdot [T]_{\alpha}$$

U)

$= f(x)$

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α: Ordered basis

$v \in V \quad [v]_\alpha$ — coordinate vector

let $T: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$: $\beta = \langle 1, x, x^2 \rangle$

$\alpha = (1, x, x^2, x^3)$

$T(f(x)) = f'(x)$ $\qquad = \langle 1, x, x^2, x^3 \rangle$

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$P(x) = 2 - 4x + 3x^2 + x^3$$

$$[P(x)]_\alpha = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}$$

$$T(P(x)) = -4 + 6x + 3x^2$$

$$[T(P(x))]_\beta = \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix}$$

$$[T]_\alpha^\beta \cdot [P(x)]_\alpha = \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix} = [T(P(x))]_\beta$$

$T: V \to W$ with $\alpha \& \beta$ — ordered basis

$$[T(v)]_\beta = [T]_\alpha^\beta \cdot [v]_\alpha$$

$V = \mathbb{R}^2$

$\beta = \{(1,1), (1,-1)\}$

$\beta' = \{(2,4), (3,1)\}$

$(2,4) = 3(1,1) - 1(1,-1)$
$(3,1) = 2(1,1) + 1(1,-1)$

$$Q = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

$Q_{\beta}^{\beta'}$

$$[v]_\beta = Q[v]_{\beta'}$$

$[v]_\beta = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\quad [v]_{\beta'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad (2,4) = 1(2,4) + 0(3,1)$

$$Q[v]_{\beta'} = [v]_\beta$$

$v = (5,5)$

$[v]_{\beta'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad [v]_\beta = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

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$\beta = \{\alpha_1, \alpha_2, \text{------} \alpha_n\}$

$\beta' = \{\alpha_1', \alpha_2', \text{------} \alpha_n'\}$

$\alpha_j' = \sum_{i=1}^{n} P_{ij}\, \alpha_i$, $\exists$: a matrix $P_{n\times n}$ with unique scalars.

$$[v]_{\beta'} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

$v = x_1'\alpha_1' + \text{---} + x_n'\alpha_n'$

$v = \sum_{j=1}^{n} x_j' \cdot \alpha_j'$

$$v = \sum_{j=1}^{n} x_j' \left(\sum_{i=1}^{n} P_{ij} \cdot \alpha_i \right)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} (P_{ij} \cdot x_j')\, \alpha_i$$

$$x_i = \sum_{j=1}^{n} P_{ij}\, x_j'$$

$$X = PX'$$

$$X' = P^{-1}X$$

$$[v]_{\beta} = P\,[v]_{\beta'}$$

$$[v]_{\beta'} = P^{-1}\,[v]_{\beta}$$

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Proof If $x=0$ only possible thing $x'=0$

So it is one-one.

$\Rightarrow$ P should be invertible.

or

$x = Px'$ $x' = Qx$ $\Rightarrow$ $PQ = I$ So P is invertible.

Eg. $W \subseteq \mathbb{R}^4$.

$$W = \text{span}\{\underset{\alpha_1}{(1,2,2,1)}, \underset{\alpha_2}{(0,2,0,1)}, \underset{\alpha_3}{(-2,0,-4,3)}\}$$

1. Show that they form a basis.

* 2. $\beta = (b_1, b_2, b_3, b_4) \in W$

$[\beta]_\alpha = ?$

3. $\alpha_1' = (1,0,2,0)$

$\alpha_2' = (0,2,0,1)$ Show that α' is a basis of W.

$\alpha_3' = (0,0,0,3)$ $X = PX'$.

1. $$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 5 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{3}R_3$

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$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2 \quad R_3 \to \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$X = PX'$

$$P = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $T : V \to V$

$T^{-1} : V \to V$

T^{-1} is invertible if $TT^{-1} = I = T^{-1}T$

$\Rightarrow$ $T : V \to W$

$U : W \to V$

$TU = I_W$

$UT = I_V$

$T : V \to W$

T is invertible iff

(i) T is one-one $\Rightarrow T\alpha = T\beta \Rightarrow \alpha = \beta$.

(ii) T is onto $\Rightarrow R(T) = W$.

T is non-singular, if $T(v) = 0 \Rightarrow v = 0$.

T is one-to-one iff T is non-singular

$\Rightarrow$ $T : V \to W$

$\{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ is a linearly independent set

* # T is non-singular iff every linearly independent set in V is mapped onto a linearly independent set in W.

$T: \mathbb{R}^2 \to \mathbb{R}^2$.

$T(x,y) = (x+y, x)$

<u>T is non-singular</u>

$T(x,y) = 0$

$\Rightarrow x+y = 0$

$x = 0$

$\Rightarrow x = y = 0$

<u>T is onto:</u>

$(z_1, z_2) \in \mathbb{R}^2$

$T(x,y) = (z_1, z_2)$

$x+y = z_1$

$x = z_2$

$T(z_2, z_1 - z_2) = (z_1, z_2)$

$T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (y, x-y)$

<u>Theorem:-</u>

$V, W \to \dim V = \dim W$

If $T: V \to W$ is a linear transformation.

The following are equivalent.

(i) T is invertible.

(ii) T is non-singular.

(iii) If $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is basis of V, then $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is a basis of W.

(iv) There is some basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of V such that $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is a basis.

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$T: V \to V$

Two ordered basis $[T]_\beta$

$\underbrace{\beta \quad \beta'}_{P}$

$$[T]_{\beta'} = P^{-1} \cdot [T]_\beta \cdot P$$

then $[T]_\beta$ or $[T]_{\beta'}$ are similar.

If $\exists\, P$

$A = PBP^{-1}$

A & B are similar

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$\beta = \{(1,0), (0,1)\}$

$\beta' = \{(1,1), (2,1)\}$

$$[T]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$(1,1) = 1e_1 + 1e_2$

$(2,1) = 2e_1 + e_2$

whenever $T: V \to V$

T is an operator.

$T(e_1) = \sum \alpha_{ij} e_i = 1e_1 + 0 = e_1$

$T(e_2) = 0$

$T(x,y) = (x,0)$

$[T]_{\beta'} = P^{-1} [T]_{\beta} P$

$= \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$

$T(e_1') = (1,0) = -e_1' + e_2'$

$T(e_2') = (2,0) = -2e_1' + 2e_2'$

Characteristic Value :-

Vector Space V

(operator) $T: V \to V$

A characteristic value of T is a scalar $c \in F$. such that there is a $v \in V$, $v \neq 0$

$T(v) = cv$

$c \to$ eigen value

$v \to$ eigen vector / characteristic vector

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$\Rightarrow$ $T(v) = cv$

$$(T - cI)v = 0$$

As $v \neq 0$

$$\Rightarrow \det(T - cI) = 0$$

$A \in \mathbb{F}^{n \times n}$, $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

A scalar $c \in \mathbb{F}$ is called an eigen value if $v \in \mathbb{F}^n$, s.t.

$$Av = cv$$

$c \rightarrow$ eigen value or spectral value or characteristic value.

$v \rightarrow$ eigen vector.

$$(A - cI)v = 0$$

$\det(A - cI) = 0 \Rightarrow$ for solving value of c.

e.g. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

For eigen values

$$\det(A - cI) = \begin{vmatrix} -c & -1 \\ 1 & -c \end{vmatrix}$$

$$= c^2 + 1$$

$$c^2+1=0$$
$$c = \pm i$$

we are talking about $\mathbb{R}$.
So eigen values does not exist.

e.g. $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

$$\det(A-cI) = \begin{bmatrix} 3-c & 2 & -1 \\ 2 & 2-c & -1 \\ 2 & 2 & -c \end{bmatrix}$$

$$= \begin{vmatrix} 1-c & 0 & -1+c \\ 0 & -c & -1+c \\ 2 & 2 & -c \end{vmatrix}$$

$$= (1-c)(c^2+2-2c) + 2(-c+c^2)$$

$$= c^2+2-2c-c^3-2c+2c^2-2c+2c^2$$

$$= -c^3+5c^2-6c+2$$

$$0 = \underline{(1-c)(c^2-4c+2)}$$

$$c = 1,\ 2\pm\sqrt{2}$$

characteristic polynomial
(of n degree for nXn matrix)

we get n eigen values
(but not all distinct)

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$\Rightarrow$ $T: V \to V$ on finite dim. V
and let c be a scalar

The following are equivalent:
(i) c is an eigen value of T.
(ii) $T - cI$ is singular (not invertible)
(iii) $\det(T - cI) = 0$

$A_{n \times n}$

$$f(c) = \det(A - cI) = a_0 + a_1 c + \cdots + a_n c^n$$

<u>Cayley – Hamilton Theorem</u> :-

Let $A \in \mathbb{F}^{n \times n}$
Every $n \times n$ matrix satisfies its characteristic polynomial.

$$f(A) = 0$$

$$f(c) = -c^3 + 5c^2 - 6c + 2$$

$$A^{-1} f(A) = -A^2 + 5A - 6I + 2A^{-1}$$

$$2A^{-1} = A^2 - 5A + 6I$$

(another way to find inverse)

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⇒ <u>For finding eigen vector :-</u>

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

$$\det(A - cI) = 0$$

$$\begin{pmatrix} 1-c & 1 \\ 4 & 1-c \end{pmatrix} = 0$$

$$(1-c)^2 = 4$$

$1-c = 2$ $\qquad\qquad$ $1-c = -2$

$c = -1$ $\qquad\qquad$ $c = 3$

Let $x = (x_1, x_2) \in \mathbb{R}^2$

<u>$c=3$</u>

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 2x_1 = x_2$$

$$x = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\left\{ t\begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in \mathbb{F} \right\}$ → Eigen space corresponding to eigen value $c = 3$.

$\underline{c = -1}$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 = -x_2$$

$\left\{ t \begin{pmatrix} 1 \\ -2 \end{pmatrix} : t \in \mathbb{F} \right\} \to$ eigen space

$\boxed{\text{Eigen vectors} \Rightarrow \text{Ker}(A - cI)}$

e.g. $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

$$\det(A - cI) = \begin{vmatrix} 4-c & 2 & -2 \\ 2 & 5-c & 0 \\ -2 & 0 & 3-c \end{vmatrix}$$

$$= -2(10 - 2c) + (3-c)\left[(4-c)(5-c) - 4\right]$$

$$= -4(5-c) + (3-c)(4-c)(5-c) - 4(3-c)$$

$$= -20 + 4c + (3-c)\left[20 - 9c + c^2 - 4\right]$$

$$= -20 + 4c + (3-c)(c^2 - 9c + 16)$$

Algebraic multiplicity :→ No. of identical values in eigen values for each eigen value

Geometric multiplicity :→ dimension of eigen space

$= -20 + 4c + 3c^2 - 27c + 48 - c^3 + 9c^2 - 16c$

$= -c^3 + 12c^2 - 39c + 28$

$c = 1, 7, 4$

eigen vectors :-

⇒ A & B are similar matrices if there exists an invertible matrix P such that

$$A = P^{-1} B P$$

⇒ A matrix $A \in \mathbb{F}^{n \times n}$ is said to be diagnonalizable if A is similar to a diagonal matrix.

\# A be an $n \times n$ matrix

$L_A : \mathbb{F}^n \to \mathbb{F}^n$, $x \in \mathbb{F}^n$

n- tupple

$L_A(x) = Ax \to$ (column vector)

\- Left multiplication operation

⇒ $T : V \to V$

T - linear transformation ; β - ordered basis

If $[T]_\beta$ is a diagonal matrix then T is diagonalizable.

A is diagonalizable if L_A is diagonalizable.

Theorem :- An operator T on a finite dimensional space V is diagonalizable iff there exists an ordered basis β for V consisting of eigen vectors of T. Furthermore, If T is diagonalisable $\beta = \{v_1, v_2, \dots v_n\}$ is an ordered basis of eigen vectors of T and $D = [T]_\beta$ then D is diagonal and D_{jj} is the eigen value of T corresponding to vector v_j, $j = 1, 2, \dots n$.

as eigen vectors are L.I.

e.g. $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

$$L_A(x) = Ax$$

$$\lambda_1 = -1 \longrightarrow v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 3 \longrightarrow v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

$$Q^{-1}AQ = \frac{1}{\cancel{4}}\begin{pmatrix} \cancel{1} & 3 \\ \cancel{7} & \cancel{9} \end{pmatrix}\begin{pmatrix} 1 & \cancel{1} \\ -2 & \cancel{2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \cancel{\frac{1}{4}}\begin{pmatrix} \cancel{4} & \cancel{4} \\ \cancel{16} & \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

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eg.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\det(A - cI) = \begin{bmatrix} -c & 1 & 0 & 1 \\ 1 & -c & 1 & 0 \\ 0 & 1 & -c & 1 \\ 1 & 0 & 1 & -c \end{bmatrix}$$

$$\begin{array}{cccc} + & - & + & - \end{array}$$

$$= \begin{vmatrix} -c & 0 & c & 0 \\ 0 & -c & 0 & c \\ 0 & 1 & -c & 1 \\ 1 & 0 & 1 & -c \end{vmatrix}$$

$$= -c \begin{vmatrix} -c & 0 & c \\ 1 & -c & 1 \\ 0 & 1 & -c \end{vmatrix} + c \begin{vmatrix} 0 & -c & c \\ 0 & 1 & 1 \\ 1 & 0 & -c \end{vmatrix}$$

$$= -c\left(-c(c^2-1) + c(1)\right) + c\left(c(-1) + c(-1)\right)$$

$$= -c\left(-c^3 + 2c\right) + c\left(-2c\right)$$

$$= +c^4 - 4c^2$$

$$c^4 - 4c^2 = 0$$

$$c^2(c^2-4) = 0$$

$$c = 0, \pm 2$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$Ax = 0$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Rank = 2
dim(V) = 4
Nullity = 2
Nullity = dim(eigen space)

in the process to find eigen vectors,
Rank - Nullity Theorem holds.

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$$\begin{pmatrix} x_2 + x_4 \\ x_1 + x_3 \\ x_2 + x_4 \\ x_1 + x_3 \end{pmatrix} = 0$$

$x_2 = -x_4$, $x_1 = -x_3$

eigen vectors = Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ dim.(eigen space) = 2

$(A+2I)x = 0$ $\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$

matrix rank = 3

$\therefore$ dim(eigen space) = 1 $\begin{pmatrix} 2x_1 + x_2 + x_4 \\ x_1 + 2x_2 + x_3 \\ x_2 + 2x_3 + x_4 \\ x_1 + x_3 + 2x_4 \end{pmatrix} = 0$

~~$x_2 + x_1 + x_4 - x_3$~~
~~$3x_1 + 3x_2 + x_3 + x_4 = 0$~~

~~A+2I~~
$(A-2I)x = 0$

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

eigen space corresponding to $\lambda = 0$ span $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

" " " $\lambda = 2$ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

" " " $\lambda = -2$ " $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$