Fundamentals of optimization

Multivariate optimization

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

$$st$$

$$3x_1 + 2x_2 - 12 = 0$$

$-\nabla f = \lambda \nabla h$

$h(x) = 0$

$-\nabla f = -\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -4x_1 \\ -8x_2 \end{pmatrix}$

$\begin{pmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-\frac{\partial f}{\partial x_1} = \lambda \frac{\partial h}{\partial x_1}$

$-4x_1 = 3\lambda$

$-8x_2 = 2\lambda$

First order condition

$$\begin{aligned} -4x_1 &= 3\lambda \\ -8x_2 &= 2\lambda \\ (3x_1 + 2x_2 - 12) &= 0 \end{aligned} \quad \xrightarrow{\text{solving}} \quad \begin{bmatrix} x_1^* \\ x_2^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 3.27 \\ 1.09 \\ -4.36 \end{bmatrix}$$

MULTIVARIATE OPTIMIZATION WITH INEQUALITY CONSTRAINTS

Fundamentals of optimization

Multivariate optimization with constraints

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

st

$$3x_1 + 2x_2 \le 12$$

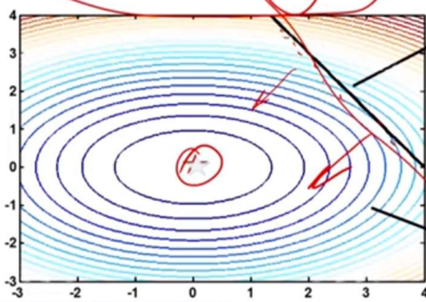
$3x_1 + 2x_2 - 12 \le 0$

$n^T x + b \le 0$

$3x_1 + 2x_2 = 12$

Constrained minimum

Unconstrained minimum



Unconstrained minimum is the same as constrained minimum

feasible region

Fundamentals of optimization

Multivariate optimization with constraints

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

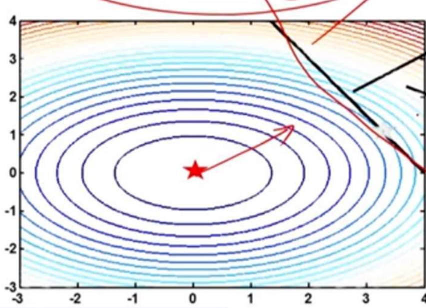
st

$$3x_1 + 2x_2 \ge 12$$

$3x_1 + 2x_2 = 12$

Constrained minimum

Unconstrained minimum



feasible region

Where does the optimum of the constrained function lie?

General formulation

Multivariate optimization

$$\min_{\bar{x}} \; f(\bar{x})$$

st

$$h_i(\bar{x}) = \bar{0}, \; i = 1, .. \; m$$
$$g_j(\bar{x}) \le \bar{0}, \; j = 1, 2 \,.. \; l$$

$g_i(x) \geq 0$

$-g_i(x) \leq 0$

Necessary condition for $\overline{x^*}$ to be the minimizer

KKT conditions has to be satisfied

Sufficient condition

$\nabla^2 L(\overline{x^*})$ has to be positive definite

Data science for Engineers

Summary – KKT conditions

Multivariate optimization

When both equality and inequality constraints are present, at the optimum we have

KKT (Karush-Kuhn-Tucker) conditions

$\nabla f(\overline{x^*}) + \sum_{i=1}^{l} [\nabla h_i(\overline{x^*})] \lambda_i^* + \sum_{j=1}^{m} [\nabla g_j(\overline{x^*})] \mu_j^* = 0$	Gradient of the "Lagrangian function" at x^* $L(\overline{x^*}, \lambda^*, \mu^*) = f(\overline{x^*}) + \sum_{i=1}^{l} \lambda_i h(\overline{x^*}) + \sum_{j=1}^{m} \mu_j g_j(\overline{x^*})$
$h_i(x^*) = 0, i = 1 \ldots l$	Ensures that the optimum satisfies equality constraints
$\lambda_i \epsilon R, i = 1 \ldots l$	
$g_j(\overline{x^*}) \le 0, j = 1 \ldots m$	Ensures that the optimum is in the feasible region
$\mu_j^*(g_j(\overline{x^*}))$=0	Complementary slackness
$\mu_j^* \ge 0, j = 1 \ldots m$	➢ No possibility of improvement near the active constraints

Summary – KKT conditions

Multivariate optimization

- In general it is difficult to use the KKT conditions to solve for the optimum of an inequality constrained problem (than for a problem with equality constraints only) because we do not know *a priori* which constraints are active at the optimum.
- Makes this a combinatorial problem
- KKT conditions are used to verify that a point we have reached is a candidate optimal solution.
- Given a point, it is easy to check which constraints are binding.

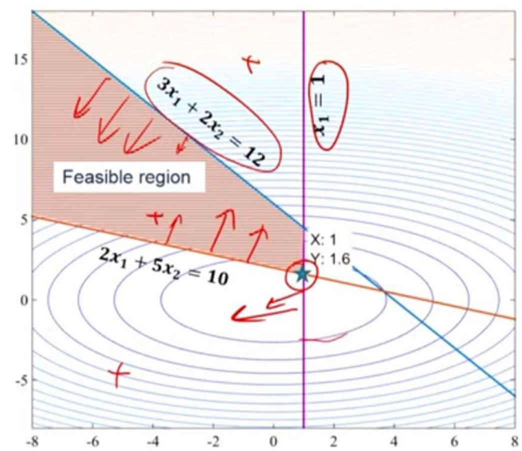
Fundamentals of optimization

Multivariate optimization-quadratic programming

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

st

$$3x_1 + 2x_2 \le 12$$
$$2x_1 + 5x_2 \ge 10$$
$$x_1 \le 1$$



Fundamentals of optimization

Multivariate optimization-quadratic programming

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

st

$3x_1 + 2x_2 \leq 12 \Rightarrow (a)$ μ_1 A

$2x_1 + 5x_2 \geq 10 \Rightarrow (b)$

$x_1 \leq 1 \Rightarrow (c)$

$10 - 2x_1 - 5x_2 \leq 0$

- Lagrangian

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3) = 2x_1^2 + 4x_2^2 + \mu_1(3x_1 + 2x_2 - 12) + \mu_2(10 - 2x_1 - 5x_2) + \mu_3(x_1 - 1)$$

- First order KKT conditions

$$4x_1 + 3\mu_1 - 2\mu_2 + \mu_3 = 0$$
$$8x_2 + 2\mu_1 - 5\mu_2 = 0$$

$$\mu_1(3x_1 + 2x_2 - 12) = 0$$
$$\mu_2(10 - 2x_1 - 5x_2) = 0$$
$$\mu_3(x_3 - 1) = 0$$
$$\mu_i \geq 0$$

3 eqns μ_1, μ_2, μ_3

$3x_1 + 2x_2 - 12 \leq 0$

$10 - 2x_1 - 5x_2 \leq 0$

$3x_1 + 2x_2 - 12 = 0$

Fundamentals of optimization

Multivariate optimization-quadratic programming

Sl.no	Active (A) /Inactive (I) constraints (a)	(b)	(c)	Solution (x, μ)	Possible optima (Y/N)	Remark
1	A	A	A	Infeasible	N	Equations do not have a valid solution.
2	A	A	I	$x = [3.6364 \quad 0.5455]$ $\mu = [-5.2 \quad -1.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied, $\mu_1 < 0$, $\mu_2 < 0$
3	A	I	A	$x = [1 \quad 4.5]$ $\mu = [-18 \quad 0 \quad 50]$	N	$\mu_1 < 0$
4	I	A	A	$x = [1 \quad 1.6]$ $\mu = [0 \quad 2.56 \quad 1.12]$	Y	All constraints and KKT conditions satisfied
5	A	I	I	$x = [3.27 \quad 1.09]$ $\mu = [-4.36 \quad 0 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
6	I	A	I	$x = [1.21 \quad 1.51]$ $\mu = [0 \quad 2.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
7	I	I	A	$x = [1 \quad 0]$ $\mu = [0 \quad 0 \quad -4]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied
8	I	I	I	$x = [0 \quad 0]$ $\mu = [0 \quad 0 \quad 0]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied

Fundamentals of optimization