

Special Continuous Probability Distributions

NORMAL DISTRIBUTION (Gaussian distribution)

The normal distribution was first discovered in 1733 by English mathematician De-Moivre, who obtained this continuous dist as a limiting case of the binomial dist. and applied it to problems arising in the game of chance. It was also known ~~as~~ to Laplace, no later than 1774 but through a historical error it was credited to Gauss, who first made reference to it in the beginning of 19th century (1809), as the dist. of error in Astronomy.

Definition: A r.v X is said to have a normal distribution with parameter μ (called 'mean') and σ^2 (called 'variance') if its p.d.f. is given by the probability law:

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} ; -\infty < x < \infty, -\infty < \mu < \infty$$
$$\sigma > 0.$$

#1. $X \sim N(\mu, \sigma^2)$.

#2. If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X - \mu}{\sigma}$ is a standard normal variate with $E(Z) = 0$ and $\text{var}(Z) = 1$ and we write $Z \sim N(0, 1)$.

#3. The p.d.f of standard normal variate Z is given by:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} ; -\infty < z < \infty.$$

and the corresponding dist. fn., denoted by $\Phi(z)$ is given by:

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^z \phi(u) du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-u^2/2} du.$$

Result 1. $\Phi(-z) = 1 - \Phi(z)$, $z > 0$

~~Result 2.~~ $\Phi(z) = P(Z \leq z) = P(Z \geq -z) = 1 - P(Z \leq -z)$
Proof. $= 1 - \Phi(-z).$

✓ Result 2. $P(a \leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

where $X \sim N(\mu, \sigma^2)$.

Proof: $P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$
 $= P\left(Z \leq \frac{b-\mu}{\sigma}\right) - P\left(Z \leq \frac{a-\mu}{\sigma}\right)$
 $= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$

Normal distribution as a Limiting form of Binomial dist.

Normal dist. is another limiting form of the binomial dist. under the following conditions:

- (i) n , the no. of trials is indefinitely large, i.e., $n \rightarrow \infty$; and
- (ii) neither p nor q is very small.

Normal dist. can also be obtained as a limiting case of Poisson dist. with parameter $\lambda \rightarrow \infty$.

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Normal distribution / Gaussian distribution

$$X \sim N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}; \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0.$$

Verification

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \sigma du$$

$$\text{let } \frac{x-\mu}{\sigma} = u$$

$$dx = \sigma du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$$\text{let } \frac{u^2}{2} = t$$

$$= \frac{1 \times 2}{\sqrt{2\pi}} \int_0^{\infty} e^{-u^2/2} du \quad (\text{as } e^{-u^2/2} \text{ is an even fn.})$$

$$\text{let } \frac{u^2}{2} = t$$

$$\Rightarrow \frac{2u du}{2} = dt$$

$$\Rightarrow du = \frac{dt}{\sqrt{2t}}$$

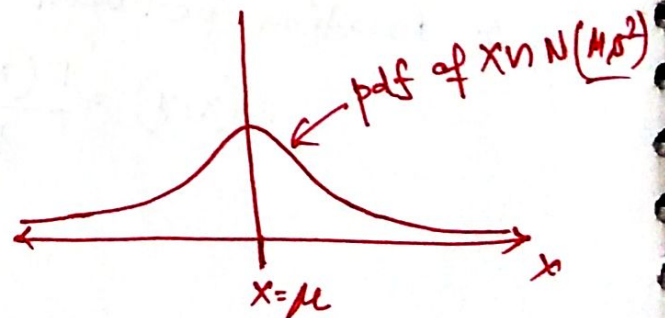
$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{-1/2} dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1.$$

Hence proved.



symmetric about the point $x = \mu$.

MGF of $X \sim N(\mu, \sigma^2)$

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{t(\mu+\sigma u)} e^{-\frac{u^2}{2}} \sigma du.$$

Let $\frac{x-\mu}{\sigma} = u$
 $\Rightarrow dx = \sigma du$
 $x = \mu + \sigma u$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma u} e^{-\frac{u^2}{2}} du$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2 - 2t\sigma u)} du$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2 - 2t\sigma u + t^2\sigma^2 - t^2\sigma^2)} du$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u - t\sigma)^2} \cdot e^{\frac{t^2\sigma^2}{2}} du$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u - t\sigma)^2} du.$$

Let $u - t\sigma = z$
 $\Rightarrow du = dz$

$$= \frac{2\pi e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz.$$

Let $\frac{z^2}{2} = y$
 $\Rightarrow z dz = dy$
 $\Rightarrow dz = \frac{dy}{\sqrt{2y}}$

$$= \frac{2\pi e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_0^{\infty} e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{2\pi e^{t\mu + \frac{t^2\sigma^2}{2}}}{2\sqrt{\pi}} \int_0^{\infty} e^{-y} y^{-\frac{1}{2}} dy$$

$$= \frac{\pi e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{\pi}} \int_0^{\infty} e^{-y} y^{\frac{1}{2}-1} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{t\mu + \frac{t^2\sigma^2}{2}} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} e^{t\mu + \frac{t^2\sigma^2}{2}} \cdot \sqrt{\pi}$$

$$M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

Answer

$$M_X(t) = e^{\mu t + \frac{t^2 \sigma^2}{2}}$$

$$\begin{aligned} \frac{d}{dt} M_X(t) &= \left(\mu + \frac{2t\sigma^2}{2} \right) e^{\mu t + \frac{t^2 \sigma^2}{2}} \\ &= (\mu + t\sigma^2) e^{\mu t + \frac{t^2 \sigma^2}{2}} \end{aligned}$$

$$\therefore E(X) = \left. \frac{d}{dt} M_X(t) \right|_{t=0} = \mu \Rightarrow \boxed{E(X) = \mu}$$

$$\frac{d^2}{dt^2} M_X(t) = \sigma^2 e^{\mu t + \frac{t^2 \sigma^2}{2}} + (\mu + t\sigma^2)(\mu + t\sigma^2) e^{\mu t + \frac{t^2 \sigma^2}{2}}$$

$$\Rightarrow \left. \frac{d^2}{dt^2} M_X(t) \right|_{t=0} = \sigma^2 + \mu^2 = E(X^2)$$

$$\begin{aligned} \therefore \text{Var}(X) &= E(X^2) - \{E(X)\}^2 \\ &= \sigma^2 + \mu^2 - \mu^2 = \sigma^2 \end{aligned}$$

$$\therefore \boxed{\text{Var}(X) = \sigma^2}$$

$$\text{Let } X \sim N(\mu, \sigma^2)$$

$$\text{Let } Z = \frac{X - \mu}{\sigma}$$

$$\text{Then } Z \sim N(0, 1)$$

$$\text{Hence, } f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

$$M_Z(t) = E(e^{zt}) = \int_{-\infty}^{\infty} e^{zt} f_Z(z) dz$$

$$= \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2zt)} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2zt + t^2 - t^2)} dz.$$

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$$M_Z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} e^{tz/2} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$$= 2 \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-u^2/2} du$$

$$= \sqrt{\frac{2}{\pi}} e^{t^2/2} \int_0^{\infty} e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{1}{\sqrt{\pi}} e^{t^2/2} \int_0^{\infty} e^{-y} y^{-1/2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{t^2/2} \Gamma(1/2) = \frac{1}{\sqrt{\pi}} e^{t^2/2} \cdot \sqrt{\pi}$$

$$\boxed{M_Z(t) = e^{t^2/2}}$$

$$\frac{d}{dt} M_Z(t) = t e^{t^2/2} \Rightarrow E(Z) = \left. \frac{d}{dt} M_Z(t) \right|_{t=0} = 0$$

$$\therefore \boxed{E(Z) = 0}$$

$$\frac{d^2}{dt^2} M_Z(t) = e^{t^2/2} + t^2 e^{t^2/2} \Rightarrow E(Z^2) = \left. \frac{d^2}{dt^2} M_Z(t) \right|_{t=0} = 1$$

$$\therefore \text{Var}(Z) = E(Z^2) - \{E(Z)\}^2 = 1 - 0 = 1$$

$$\therefore \boxed{\text{Var}(Z) = 1}$$

$$\text{let } \frac{(z-t)^2}{2} = u$$

$$\Rightarrow z-t = u$$

$$\text{let } \frac{u^2}{2} = y$$

$$\Rightarrow u du = dy$$

$$\Rightarrow du = \frac{dy}{\sqrt{2y}}$$

Asaneji
Lecture 13 P(1)

Q. Let independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$

$i=1, 2, \dots, n$.

$$M_{X_i}(t) = e^{\mu_i t + \frac{t^2 \sigma_i^2}{2}}, \quad i=1, 2, \dots, n$$

$$Y = \sum_{i=1}^n a_i X_i$$

$$\text{Then } M_Y(t) = \prod_{i=1}^n M_{a_i X_i}(t) = \prod_{i=1}^n M_{X_i}(a_i t)$$

$$= e^{\sum_{i=1}^n \mu_i a_i t + \frac{a_i^2 t^2 \sigma_i^2}{2}}$$

$$= e^{(\sum_{i=1}^n \mu_i a_i) t + \frac{t^2}{2} (\sum_{i=1}^n a_i^2 \sigma_i^2)}$$

$$\Rightarrow Y \sim N\left(\sum_{i=1}^n \mu_i a_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right).$$

Area property (Normal Probability Integral)

Let $X \sim N(\mu, \sigma^2)$.

$$\text{Then } P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\text{Let } z = \frac{x-\mu}{\sigma} \Rightarrow dz = \frac{dx}{\sigma}$$

$$\therefore P(\mu < X < x_1) = \int_0^{z_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma dz, \text{ where } z_1 = \frac{x_1 - \mu}{\sigma}$$

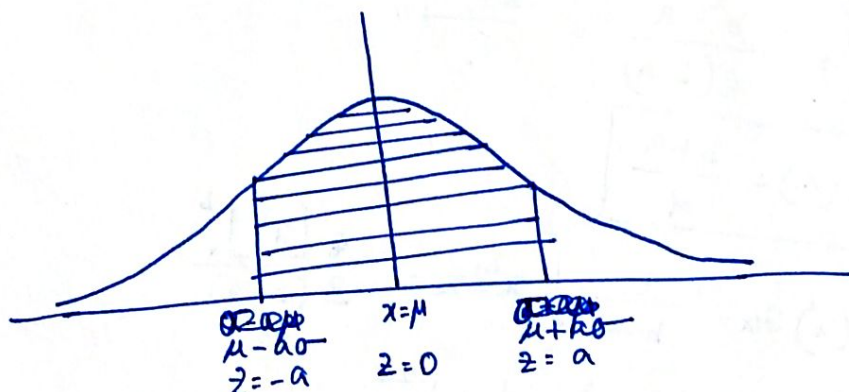
$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{z^2}{2}} dz$$

$$= P(0 < Z < z_1).$$

where $Z \sim N(0, 1)$. $\therefore z_1 = \frac{x_1 - \mu}{\sigma}$

$\int_0^{z_1} \phi(z) dz$ is the area under standard normal curve between the ordinate $z=0$ and $z=z_1$. Given in normal table

$$Z = \frac{X - \mu}{\sigma}$$



$$P(\mu - a\sigma < X < \mu + a\sigma) = 2P(\mu < X < \mu + a\sigma) \quad \text{due to symmetry}$$

$$\begin{aligned} P(\mu - a\sigma < X < \mu + a\sigma) &= 2P(\mu < X < \mu + a\sigma) \\ &= 2P\left(\frac{\mu - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + a\sigma - \mu}{\sigma}\right) \\ &= 2P(0 < Z < a). \end{aligned}$$

Characteristics of the normal distribution and normal probability curve :-

Let $X \sim N(\mu, \sigma^2)$.

then its pdf is given by $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$-\infty < x < \infty$$

$$-\infty < \mu < \infty, \sigma > 0.$$

The normal curve has the following properties

- (i) The curve is bell shaped and symmetric about the line $x = \mu$.
- (ii) Mean, median and mode of the distribution coincide at $x = \mu$.

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

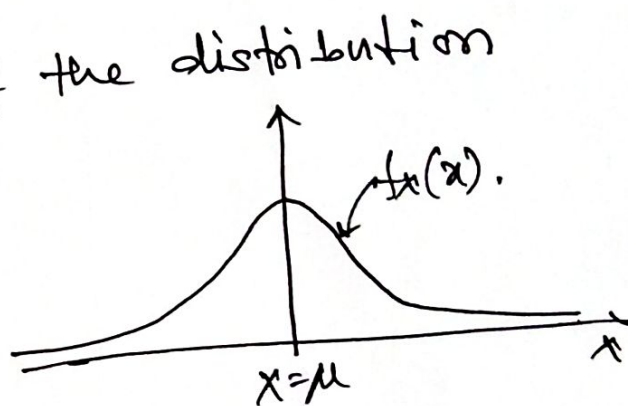
$$= \frac{\frac{b-\mu}{\sigma}}{\frac{a-\mu}{\sigma}} \int \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$\text{let } \frac{x-\mu}{\sigma} = u.$$

$$= \int_{\frac{(a-\mu)/\sigma}{\sigma}}^{\frac{(b-\mu)/\sigma}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du - \int_{-\infty}^{\frac{(b-\mu)/\sigma}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

where $\Phi \rightarrow$ cdf of standard normal variate



Example:- Let X be a normally distributed r.v. with mean 12 and standard deviation 4.

Find (a) (i) $P(X \geq 20)$, (ii) $P(X \leq 20)$, (iii) $P(0 \leq X \leq 12)$

(b) Find x' such that $P(X > x') = 0.24$.

(c) Find x_0' and x_1' s.t. $P(x_0' < X < x_1') = 0.50$

and $P(X > x_1') = 0.24$.

Given that $f_1(t) = \int_{-\infty}^t \phi(z) dz$ and $f_2(t) = \int_0^t \phi(z) dz$.

$f_1(2) = 0.97725$, $f_2(2) = 0.4772$, $f_2(3) = 0.4987$,

$f_1(3) = 0.9987$, $f_2(0.71) = 0.26$, $f_2(0.67) = 0.25$

Soln. $X \sim N(12, 4^2)$, i.e., $X \sim N(12, 16)$. $\therefore \mu = 12, \sigma = 4$.

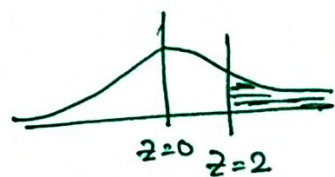
(a)(i) $P(X \geq 20) = P\left(\frac{X-\mu}{\sigma} \geq \frac{20-\mu}{\sigma}\right)$

$= P(Z \geq 2) =$

$= 0.5 - f_2(2) \quad [\text{OR}, 1 - f_1(2)]$

$= 0.5 - 0.4772 \quad [\text{OR}, 1 - 0.97725]$

$= 0.0228 \quad [\text{OR}, 0.0228]$



(ii) $P(X \leq 20) = 1 - P(X \geq 20) = 1 - 0.0228 = 0.9772$

OR $P(X \leq 20) = P\left(\frac{X-\mu}{\sigma} \leq \frac{20-\mu}{\sigma}\right) = P(Z \leq 2) = f_1(2) \quad (\text{OR } f_2(2))$

(iii) $P(0 \leq X \leq 12) = P\left(\frac{0-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{12-\mu}{\sigma}\right)$

$= P(-3 \leq Z \leq 0)$

$= P(0 \leq Z \leq 3) \quad [\text{due to symmetry}]$

$= f_2(3) = 0.4987$

$$(b) P(X > x'_1) = 0.24$$

$$\Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{x'_1-\mu}{\sigma}\right) = 0.24$$

$$\Rightarrow P(Z > z_1) = 0.24, \text{ where } z_1 = \frac{x'_1-\mu}{\sigma}$$

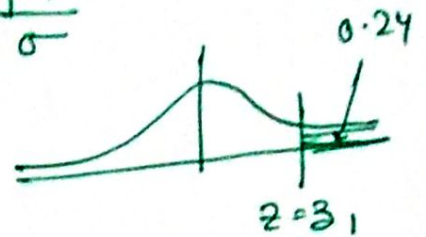
$$\Rightarrow 0.5 - P(0 < Z < z_1) = 0.24$$

$$\Rightarrow P(0 < Z < z_1) = 0.26 = f_2(0.71) \quad [\text{Given}]$$

$$\Rightarrow z_1 = 0.71$$

$$\Rightarrow \frac{x'_1 - 12}{4} = 0.71$$

$$\Rightarrow x'_1 = 0.71 \times 4 + 12 = 14.84$$



$$(c) P(x'_0 < X < x'_1) = 0.50 \quad \& \quad P(X > x'_1) = 0.25$$

$$\Rightarrow P\left(\frac{x'_0-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{x'_1-\mu}{\sigma}\right) = 0.50 \quad \Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{x'_1-\mu}{\sigma}\right) = 0.25$$

$$\Rightarrow P(z_1 < Z < z_2) = 0.50 \quad \text{--- (I)} \quad \Rightarrow P(Z > z_2) = 0.25 \quad \text{--- (II)}$$

From fig it is clear that $z_1 = -z_2$.

Now from (II)

$$\Rightarrow 0.5 - P(0 \leq Z \leq z_2) = 0.25$$

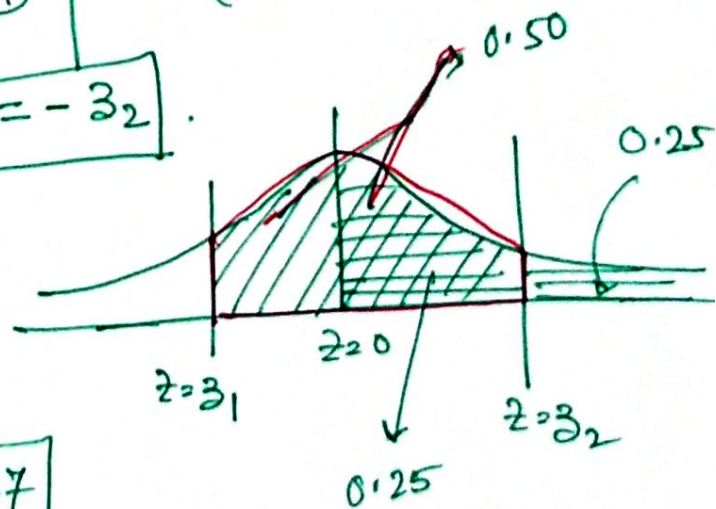
$$\Rightarrow P(0 \leq Z \leq z_2) = 0.25$$

$$\Rightarrow z_2 = 0.67 \quad \Rightarrow z_1 = -0.67$$

$$\text{Hence, } x'_1 = \sigma z_2 + \mu = 14.68$$

$$\& \quad x'_0 = \sigma z_1 + \mu = 9.32$$

Ans.



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