# Compound or Joint Experiment

Let E and E′ be two random experiment.
Let S and S′ be the respective event spaces.
Let us now consider the random experiment E″ which represents the occurence of E first and the E′.
Then E″ is called the compound random experiment of E and E′.

**Example :-**

Let E ⟶ tossing an unbiased coin
E′ ⟶ throwing an unbiased die.

Then $S = \{H, T\}$, $S' = \{1,2,3,4,5,6\}$

Let S″ be the sample space corresponding to the r.e. E″.

Then $S'' = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

\# Let us assume that S and S′ are mostly enumerable.
Then two r.e. E and E′ are said to be independent if for any simple event $(u_i, v_j)$ connected to E″.

$$P(u_i, v_j) = P(u_i)\,P(v_j)$$

**Theorem :-** Let A and B are two events connected to the random experiment E and E′, respectively.
If E and E′ are independent, then

$$P(A, B) = P(A)\,P(B).$$

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## Independent trials :-

Let E be a given random experiment with sample space S. Let E be repeated $n$ times, where $n$ is the positive integer, then resulting experiment gives a compound experiment $E_n$ with sample space

$S^n = S \times S \times \cdots \times S$ ($n$ factors).

This compound experiment $E_n$ results in $n$ trials of E.

These $n$ trials are said to independent if for any simple event $(x_1, x_2, \ldots, x_n)$ connected to $E_n$,

$$P(x_1, x_2, \ldots, x_n) = P(x_1) P(x_2) \cdots P(x_n).$$

## Bernoulli trials :-

Let E be a random experiment with the sample space S. Let S contains two distinct outcomes called 'success' (denoted by 's') and 'failure' (denoted by 'f'). If E be repeated $n$ times under identical conditions, then we get $n$ independent trials of E. These trials are called Bernoulli sequence of trials if the probability of 's' (or 'f') remains constant in each trial. of E.

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Example :- If an unbiased coin is tossed n times under identical condition we get a Bernoulli sequence in which the probability of getting a 'head' (i.e., success) in each trial is $\frac{1}{2}$ and that of getting a 'tail' (i.e., failure) is also $1/2$.

## Bernoulli distribution

Let us consider a Bernoulli sequence of trials. Let $X$ be the random variable corresponding to the Bernoulli sequence. s.t.

$$X = 0 \;;\; \text{when 'f' occurs}$$
$$\;\;= 1 \;;\; \text{when 's' occurs.}$$

Let $p$ be the probability of 's' and $1-p = q$ be the probability of 'f'.

Then $P(X=0) = 1-p = q$

$P(X=1) = p$

Hence, $\boxed{P(X=x) = p^x (1-p)^{1-x} \;;\; x = 0, 1}$

$X \rightarrow$ said to follow Bernoulli distribution with pmf $P(X=x) = p^x q^{1-x} \;;\; x = 0, 1.$

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$$E(X) = \sum_x x\,p(x) = 0\cdot(1-p) + 1\cdot p = p$$

$$E(X^2) = \sum_x x^2 p(x) = 0^2\cdot(1-p) + 1^2\cdot p = p$$

$$\therefore Var(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p) = pq.$$

The MGF of $X$ is given by

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} p(x)$$

$$= e^{t\cdot 0}\cdot q + e^{t\cdot 1}\cdot p = q + pe^t.$$

$$\therefore \boxed{M_X(t) = q + pe^t}$$

$$\frac{d}{dt} M_X(t) = pe^t \Rightarrow \frac{d}{dt} M_X(t)\Big|_{t=0} = p$$

$$\Rightarrow \boxed{E(X) = p}$$

$$\frac{d^2}{dt^2} M_X(t) = pe^t \Rightarrow \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = p.$$

$$\therefore \boxed{Var(X) = pq}$$

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# BINOMIAL DISTRIBUTION

Discovered by James Bernoulli (1654 – 1705) in the year 1700.

# Let us consider a sequence of $n$ independent Bernoulli trials.

# Let us repeat the Bernoulli trial $n$ times.

# Let us consider a set of $n$ independent Bernoulli trials.

$n \rightarrow$ finite no.

$p \rightarrow$ prob. of 'success' in each trial.

$q = 1-p \rightarrow$ prob. of 'failure' in each trial

$p$ and $q$ are remain constant in each trials.

Let us try to find out the probability of getting $r$ success in $n$ independent trial.

SSSFFSSFFFF SFSSF .... FFFFS

Here $S \rightarrow$ success

$F \rightarrow$ failure

$$P(\text{SSSFFSSFFFFSFSSF} \ldots \text{FFFFS})$$

$$= P(S)P(S)P(S)P(F)P(F) \ldots P(S)$$

$$= \underbrace{P(S)P(S) \ldots P(S)}_{r \text{ factors}} \; \underbrace{P(F)P(F) \ldots P(F)}_{(n-r) \text{ factor}}$$

$$= p^r q^{n-r}.$$

Now $r$ success in $n$ trials can occur in ${}^nC_r$ ways.

$\therefore$ Probability of getting $r$ success in $n$ independent sequence of Bernoulli trials is given by

$$P(X=r) = {}^nC_r \, p^r q^{n-r} \quad ; \; r = 0,1,2,\ldots,n. \quad q = 1-p$$

$$= 0 \quad ; \text{ otherwise}$$

Then $X$ is said to follow Binomial distribution with pmf $P(X=r) = {}^nC_r\, p^r q^{n-r}$ ; $r = 0, 1, \ldots, n$
$q = 1-p$

$= 0$ ; otherwise

$X \to$ Binomial variate

$n$ & $p \to$ parameters of Binomial distribution

$X \sim B(n, p)$.

<u>MGF</u>

$$M_X(t) = E(e^{tX})$$
$$= \sum_{r=0}^{\infty} e^{tr}\, {}^nC_r\, p^r q^r$$
$$= \sum_{r=0}^{\infty} {}^nC_r (pe^t)^r q^r$$
$$\boxed{M_X(t) = (pe^t + q)^n}$$

$$\frac{d}{dt} M_X(t) = n(pe^t+q)^{n-1} pe^t.$$

Now $\left.\frac{d}{dt} M_X(t)\right]_{t=0} = n(p+q)^{n-1} p = np.$

$\therefore \boxed{E(X) = np}$

$$\frac{d^2}{dt^2} M_X(t) = npe^t (pe^t+q)^{n-1} + n(n-1)(pe^t+q)^{n-2} p^2 e^{2t}$$

$\therefore \left.\frac{d}{dt^2} M_X(t)\right]_{t=0} = np + n(n-1)p^2 = \cancel{np(\ldots)}$

$\therefore E(X^2) = \cancel{np}\ np + n(n-1)p^2$

$\therefore Var(X) = E(X^2) - \{E(X)\}^2 = \cancel{n^2p - n^2p^2 = n^2p(1-p)}$

$= \cancel{n^2pq}.\ np + n(n-1)p^2 - n^2p^2$

$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2} = \boxed{npq = Var(X)}$

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Let $X_1 \sim B(n_1, p_1)$ & $X_2 \sim B(n_2, p_2)$

$X_1$ and $X_2$ are two independent random variables.

Then what is the distribution of $X_1 + X_2$

As $X_1 \sim B(n_1, p_1) \Rightarrow M_{X_1}(t) = (p_1 e^t + q_1)^{n_1}$

$X_2 \sim B(n_2, p_2) \Rightarrow M_{X_2}(t) = (p_2 e^t + q_2)^{n_2}$

Now $M_{X_1+X_2}(t) = (p_1 e^t + q_1)^{n_1} (p_2 e^t + q_2)^{n_2}$

Let $p_1 = p_2 = p$

& $q_1 = q_2 = q = 1-p$.

Then $M_{X_1+X_2}(t) = (pe^t + q)^{n_1+n_2}$

$\Rightarrow X_1 + X_2 \sim B(n_1+n_2, p)$.

Let $X \sim B(n, p)$. Then what is the distribution of $Y = n - X$.

As $X \sim B(n, p)$ then $M_X(t) = (pe^t + q)^n$

$\therefore M_Y(t) = E(e^{tY}) = E(e^{t(n-X)}) = E(e^{tn} \cdot e^{-tX})$

$= e^{tn} E(e^{-tX}) = e^{tn} M_X(-t)$

$= e^{tn} (pe^{-t} + q)^n = (p + qe^t)^n$

$\Rightarrow Y \sim B(n, q)$.

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Characteristic function

$$\phi_X(t) = E(e^{itX})$$

$$= \cancel{\sum e^{itx}} \sum_{r=0}^{n} e^{itr}\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} {}^nC_r\, (e^{it}p)^r\, q^{n-r}$$

$$= (pe^{it} + q)^n$$

Probability Generating function:

$$P(z) = \sum_{r=0}^{n} P(X=r)\, z^r = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r} z^r$$

$$= \sum_{r=0}^{n} {}^nC_r (pz)^r\, q^{n-r}$$

$$= \cancel{(pz)}\, (pz+q)^n.$$

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