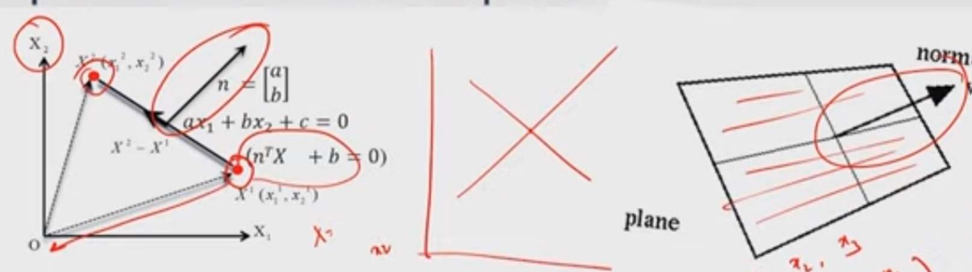
Linear Algebra – Distance, Hyperplanes and Halfspaces, Eigenvalues, Eigenvectors

Representation of line and plane



$3-1=2$
$3-2=1$
$3-3=0$

$a^T b = 0$

x_1, x_2, x_3

$n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$n_1x_1 + n_2x_2 + n_3x_3 + b = 0$
$n^T x + b = 0$

Point X¹ lies on the line $\Rightarrow n^T X^1 + b = 0 - (1)$
Point X² lies on the line $\Rightarrow n^T X^2 + b = 0 - (2)$
Subtracting (2) from (1)
$n^T (X^2 - X^1) = 0.$
Thus n is perpendicular to $(X^2 - X^1)$

Projections

- We can define the projection ($\hat{X}$) of a vector (X) onto a lower dimension (two dimensions in the picture) mathematically as

$$\hat{X} = c_1 v_1 + c_2 v_2$$

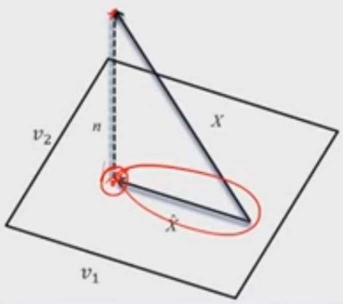
v_1, v_2

- Using vector addition

$$X = c_1 v_1 + c_2 v_2 + n$$

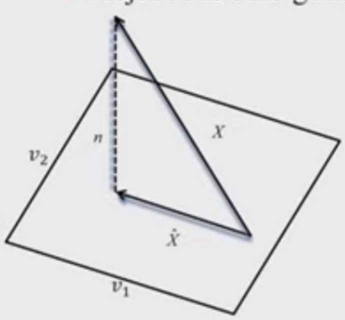
$n^T v_1 = v_1^T n = 0$

$n^T v_2 = v_2^T n = 0$



Projections

- Projections onto general orthogonal directions (two dimensions in this case)



$$v_1^T n = 0$$
$$v_1^T (X - (c_1 v_1 + c_2 v_2)) = 0$$
$$v_1^T X - c_1 v_1^T v_1 = 0$$
$$\hat{X} = \frac{v_1^T X}{v_1{}^T v_1} v_1 + \frac{v_2^T X}{v_2{}^T v_2} v_2$$

$-c_2 v_1^T v_2$

$v_2^T n = 0$

$c_1 = \frac{v_1^T x}{v_1^T v_1}$ $\quad c_2 = \frac{v_2^T x}{v_2^T v_2}$

Projections: Example

$$X = [1 \;\; 2 \;\; 3]^T$$

- Projecting this vector onto the space spanned by the vectors $v_1 = [1 \; -1 \; -2]^T$ and $v_2 = [2 \;\; 0 \;\; 1]^T$
- Thus, finding the projection onto the plane defined by v_1 and v_2 is

$$\hat{X} = \frac{\overbrace{[1 \;\; 2 \;\; 3]}^{X^T}\overbrace{\begin{bmatrix}1\\-1\\-2\end{bmatrix}}^{v_1}}{6}\underbrace{\begin{bmatrix}1\\-1\\-2\end{bmatrix}}_{v_1} + \frac{\overbrace{[1 \;\; 2 \;\; 3]}^{X^T}\begin{bmatrix}2\\0\\1\end{bmatrix}}{5}\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$[1 \; -1 \; -2]\begin{bmatrix}2\\0\\1\end{bmatrix} = 2-2=0$$

Projections: Example

$$\hat{X} = \frac{-7}{6}\begin{bmatrix}1\\-1\\-2\end{bmatrix} + 1\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$\hat{X} = \begin{bmatrix}5/6\\7/6\\20/6\end{bmatrix} \leftarrow \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Projection -Generalization

$X \in R^n$ $v_1, \dots v_k \in R^n$

- Projections onto general directions
- Consider the problem of projection of X onto a space spanned k linearly independent vectors

$$\hat{X} = \sum_{j=1}^{k} c_j v_j$$

$c_1 v_1 + c_2 v_2 + \dots + c_k v_k$

$$\hat{X} = [v_1 \quad \cdots \quad v_k]_{n\times k} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}_{k\times 1}$$

$$\hat{X} = Vc$$

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$$\hat{X} = \sum_{j=1}^{k} c_j v_j$$

$$\hat{X} = [v_1 \quad \cdots \quad v_k] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

$$\hat{X} = Vc$$

$= c_1 v_1 + c_2 v_2 \cdots + c_k v_k$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\times 1 + 1\times 1 \\ 0\times 1 + 1\times 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \left[1\times\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Projection -Generalization

$X = \hat{X} + n$

$n = X - \hat{X}$

- Using orthogonality idea

$$V^T(X - \widehat{X}) = V^T(X - Vc) = 0$$

$$\begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

$$V^T X - V^T V c = 0$$

$$c = (V^T V)^{-1} V^T X$$

$$\hat{X} = V(V^T V)^{-1} V^T X$$