POISSON DISTRIBUTION

Poisson distribution is a limiting case of the binomial distribution under following conditions:

(i) n, the no. of Bernoulli trials is ~~identic~~ indefinitely large, i.e., $n \to \infty$

(ii) p, the constant probability of success for each trial is indefinitely small, i.e., $p \to 0$

(iii) $np = \lambda$, (say) finite.

Thus. $p = \frac{\lambda}{n}$ and $q = 1 - p = 1 - \frac{\lambda}{n}$.

Then the probability of x success in n independent repeatation of Bernoulli trial is

$$b(x; n, p) = {}^{n}C_x p^x q^{n-x} \quad ; x = 0, 1, 2, \ldots, n.$$

for $x \neq 0$ ~~Now~~

$$\lim {}^{n}C_x p^x q^{n-x} = \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n.(n-1).\ldots.(n-x)(n-x-1)\ldots 1}{x!\,(n-x)!} \; \frac{\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{n.(n-1)\ldots(n-x+1)\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}}$$

$$= \frac{1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{\left(1-\frac{\lambda}{n}\right)^x} \; \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^n$$

Now using the result $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n/a} = e$ if $a \neq 0$.

we get, $$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{(-n/\lambda)}\right)^{-n/\lambda}\right]^{-\lambda}$$

$$= e^{-\lambda}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) = 1.$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x = 1.$$

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$\therefore \lim_{n\to\infty} b(x; n, p) = \lim_{n\to\infty} {}^{n}C_x\, p^x q^{n-x}$

$$= \frac{\lambda^x}{x!} e^{-\lambda}, \quad x \neq 0.$$

Let $x = 0$

$$\lim_{n\to\infty} b(x; n, p) = \lim_{n\to\infty} {}^{n}C_0\, p^0 q^{n-0}$$

$$= \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{(-n/\lambda)}\right)^{-n/\lambda}\right]^{-\lambda}$$

$$= e^{-\lambda}$$

$$= \frac{e^{-\lambda}\lambda^0}{0!}$$

$$\therefore \lim_{n\to\infty} b(x; n, p) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Moment Generating Function:

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$$

$$\boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$

$$\frac{d}{dt} M_X(t) = \lambda e^t e^{\lambda(e^t-1)}$$

$$\therefore \frac{d}{dt} M_X(t)\Big|_{t=0} = \boxed{\lambda = E(X)}$$

$$\frac{d^2}{dt^2} M_X(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda e^t \cdot \lambda e^t e^{\lambda(e^t-1)}$$

$$\therefore \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \boxed{\lambda + \lambda^2 = E(X^2)}$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2$$
$$= \lambda + \lambda^2 - \lambda^2 = \lambda.$$
$$\therefore \boxed{\text{Var}(X) = \lambda}$$

Characteristic function of the Poisson distribution

$\because X \sim P(\lambda).$

$$\phi_X(t) = E(e^{itX}) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{it}\lambda)^x}{x!}$$
$$= e^{-\lambda} \cdot e^{e^{it}\lambda} = e^{\lambda(e^{it}-1)}.$$
$$\therefore \boxed{\phi_X(t) = e^{\lambda(e^{it}-1)}}$$

If $X_i \sim P(\lambda_i)$; $i = 1, 2, \ldots, n$; where each X_i's are independent random variables. Then $\sum_{i=1}^{n} X_i \sim P\left(\sum_{i=1}^{n} \lambda_i\right)$.

Proof: $M_{X_i}(t) = e^{\lambda_i(e^t-1)}$; $i = 1, 2, \ldots, n$

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$$
$$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \cdots e^{\lambda_n(e^t-1)}$$
$$= e^{\left(\sum_{i=1}^{n}\lambda_i\right)(e^t-1)}$$

$$\Rightarrow \sum_{i=1}^{n} X_i \sim P\left(\sum_{i=1}^{n} \lambda_i\right).$$

The difference of two independent Poisson variates is not a Poisson variate.

Let $X_1 \sim P(\lambda_1)$ and $X_2 \sim P(\lambda_2)$

$\therefore M_{X_1}(t) = e^{\lambda_1(e^t-1)}$

$M_{X_2}(t) = e^{\lambda_2(e^t-1)}$

Now $M_{X_1-X_2}(t) = M_{X_1+(-X_2)}(t) = M_{X_1}(t)\, M_{-X_2}(t)$

$= M_{X_1}(t)\, M_{X_2}(-t)$

$= e^{\lambda_1(e^t-1)}\, e^{\lambda_2(e^{-t}-1)}$.

$\Rightarrow X_1 - X_2$ is not a Poisson variate.

Probability Generating Function:

$$P(z) = \sum_{x=0}^{\infty} z^x P(X=x) = \sum_{x=0}^{\infty} z^x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda z)^x}{x!} = e^{-\lambda} e^{\lambda z}$$

$$= e^{\lambda(z-1)}.$$

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Geometric DISTRIBUTION

Let us suppose that we have a sequence of independent Bernoulli trials s.t. the probability of 'success' in each trial is 'p' and remains ~~saim~~ same.

Then the probability that there are x failure preceding the first success is given by $q^x p$ where $q = 1-p$.

Let X be the random variable denoting the no. of failure preceding the first success in sequence of independent Bernoulli trials.

Then

$$p(x) = P(X=x) = \begin{cases} q^x p & ; x = 0,1,2,\dots, \quad 0 \le p \le 1, \quad q = 1-p. \\ 0 & ; \text{otherwise} \end{cases}$$

Now $$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} q^x p = p(1+q+q^2+\dots)$$

$$= \frac{p}{1-q} = \frac{p}{p} = 1.$$

$\Rightarrow p(x) = q^x p, \; x = 0,1,2,\dots$ is a pmf.

Since $q^x p$; $x = 0,1,2,3,\dots$ are the terms of a geometric progression series, $(X, p(x))$ is said to be a geometric distribution.

* G.P series: $a, ar, ar^2, \dots, ar^n, \dots, r \neq 0$.

Example (i) $2, 2^2, 2^3, \dots$

(ii) $3, 3^2, 3^3, \dots$

Moment-Generating function

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} q^x p = p \sum_{x=0}^{\infty} (e^t q)^x$$

$$= \frac{p}{1 - e^t q} = p(1 - qe^t)^{-1}.$$

$$E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = -p(1-qe^t)^{-2} \cdot (-qe^t)\Big|_{t=0}$$

$$= pqe^t(1-qe^t)^{-2}\Big]_{t=0}$$

$$= pq(1-q)^{-2} = \frac{q}{p}.$$

$$\therefore \boxed{E(X) = \frac{q}{p}}$$

$$\frac{d^2}{dt^2} M_X(t) = pqe^t(1-qe^t)^{-2} - 2pqe^t(1-qe^t)^{-3}(-qe^t)$$

$$\therefore \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = pq(1-q)^{-2} + 2pq^2(1-q)^{-3}$$

$$= \frac{pq}{p^2} + \frac{2pq^2}{p^3}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big]_{t=0}$$

$$= \frac{q}{p} + 2\frac{q^2}{p^2}$$

$$\therefore Var(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{q}{p} + 2\frac{q^2}{p^2} - \frac{q^2}{p^2}$$

$$= \frac{q}{p} + \frac{q^2}{p^2} = \frac{q}{p}\left(1 + \frac{q}{p}\right) = \frac{q}{p}\frac{(p+q)}{p}$$

$$\boxed{Var(X) = \frac{q}{p^2}.}$$

Q. Let X_1 and X_2 are two independent random variables having geometric distribution $q^k p$; $k=0,1,\ldots$ Show that the conditional distribution of X_1 given X_1+X_2 is uniform.

$$P(X_1=r \mid (X_1+X_2=n)) = \frac{P(X_1=r \cap X_1+X_2=n)}{P(X_1+X_2=n)}$$

$$= \frac{P(X_1=r \cap X_2=n-r)}{\sum_{i=0}^{n} P(X_1=i \cap X_2=n-i)}$$

$$= \frac{P(X_1=r)P(X_2=n-r)}{\sum_{i=0}^{n} P(X_1=i)P(X_2=n-i)}$$

$$= \frac{pq^r \cdot pq^{n-r}}{\sum_{i=0}^{n} pq^i \cdot pq^{n-i}}$$

$$= \frac{\cancel{p^2} q^n}{\cancel{p^2} \sum_{i=0}^{n} q^n} = \frac{q^n}{(n+1)q^n} = \frac{1}{n+1}.$$

$$\therefore P(X_1=r \mid X_1+X_2=n) = \frac{1}{n+1} \; ; \; r=0,1,2,\ldots,n.$$

Hence proved.

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The Negative Binomial Distribution

Let us consider a sequence of Bernoulli trials with probability of 'success' p and that of failure $1-p=q$.

Let X be a random variable which defines the number of failures before a specified number of successes (say, r) occurs.

Then X is said to follow a negative Binomial distribution with parameter r & p.

The PMF (pmf) of X is given by

$$P(X=x) = \binom{x+r-1}{x}(1-p)^r q^x \; ; \; x = 0,1,2,\ldots \quad 0<p<1.$$

Example :- In rolling a fair die, let us suppose that getting 6 is a success and getting any other face is a failure. Let X represent the no. of failure before getting 5th success. Then X is said to follow a negative Binomial distribution with parameters $r=5$, $p=\frac{1}{6}$.

$$MGF \equiv M_X(t) = \left(\frac{1-p}{1-pe^t}\right)^r, \quad t < -\log p$$

$$CF \equiv \phi_X(t) = \left(\frac{1-p}{1-pe^{it}}\right)^r \quad \forall\, t \in \mathbb{R}$$

$$PGF \equiv G_X(z) = \left(\frac{1-p}{1-pz}\right)^r \quad \forall\, |z| < \frac{1}{p}.$$

$$E(X) = \frac{pr}{1-p}$$

$$Var(X) = \frac{pr}{(1-p)^2}.$$

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Discrete uniform distribution :-

Let X be a r.v with state space $0, 1, 2, \cdots, n$ and pmf.

$$P(X=x) = \frac{1}{n+1} \ \forall \ x = 0, 1, 2, \cdots, n$$

Then X is said to follow uniform distribution.

$$E(X) = \sum_{i=0}^{n} x_i p_i = \frac{1}{n+1} \sum_{i=1}^{n} i = \frac{1}{(n+1)} \cdot \frac{n(n+1)}{2} = \frac{n}{2}.$$

$$E(X^2) = \sum_{i=0}^{n} x_i^2 p_i = \frac{1}{n+1} \sum_{i=1}^{n} i^2 = \frac{1}{\cancel{(n+1)}} \cdot \frac{n\cancel{(n+1)}(2n+1)}{6}$$

$$= \frac{n(2n+1)}{6}$$

$$Var(X) = \frac{n(2n+1)}{6} - \frac{n^2}{4} = \frac{n}{2}\left[\frac{2n+1}{3} - \frac{n}{2}\right]$$

$$= \frac{n}{2} \times \frac{4n+2-3n}{6} = \frac{n(n+2)}{12}$$

$$M_X(t) = E(e^{tX}) = \sum_{i=0}^{n} e^{tx_i} p_i = \frac{1}{n+1} \sum_{i=0}^{n} e^{ti}$$

$$= \frac{1}{n+1}\left[1 + e^{t} + e^{2t} + \cdots + e^{nt}\right]$$

$$= \frac{1}{n+1} \frac{1 - e^{(n+1)t}}{(1 - e^{t})}$$

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Degenerate Random Variable

Let X be a discrete r.v. with pmf as

$$P(X=x) \begin{cases} = 1, & x = c \\ = 0, & \text{otherwise} \end{cases} \Rightarrow$$ The r.v X is degenerate at the pt 'c'.

$E(X) = c$

$E(X^2) = c^2$

$Var(X) = 0.$

Then X is said to be a degenerate r.v. and is characterised by $Var(X) = 0$.

$MGF = M_X(t) = e^{ct}.$

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