Functions of a Random Variable

Let X be a r.v that assigns the value x to an outcome. Let $g(X)$ be the fn. of X, then $Y = g(X)$ is a r.v. and assigns the value $g(x)$ to that outcome.

The r.v. $Y = g(X)$ is said to be a <u>derived random variable</u>.

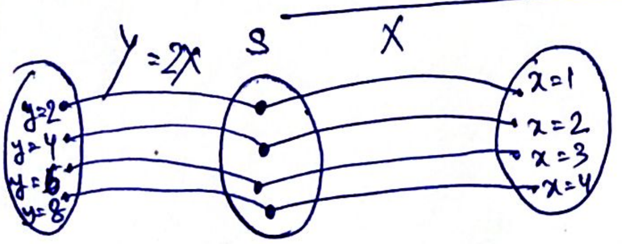
Let X be a discrete r.v. with pmf $P_X(x)$. Let $Y = g(X)$ be some fn. of X. We want to find the pmf of Y. Let us consider the following example.

<u>Example 1:</u>- Let X be a discrete r.v. with pmf given by

$$P_X(x) = \begin{cases} 1/10 \; ; & x = 1 \\ 2/10 \; ; & x = 2 \\ 3/10 \; ; & x = 3 \\ 4/10 \; ; & x = 4 \\ 0 \; ; & \text{otherwise} \end{cases}$$

Let us consider the r.v. $Y = 2X$.

<u>To find pmf. of Y</u>



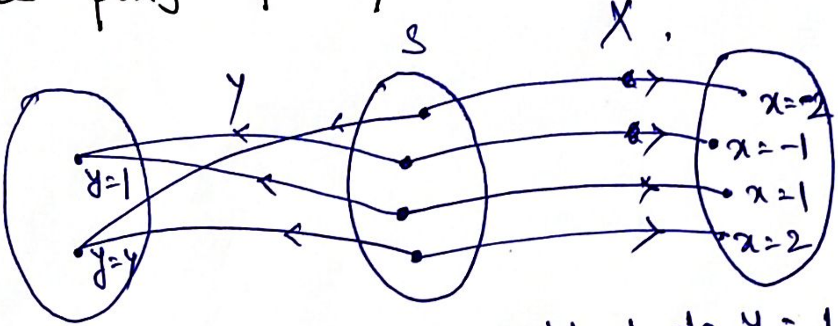
Hence

$$P_Y(y) = \text{Prob}(Y = g(x)) = \text{Prob}(X = x) = P_X(x).$$

Example 2: Let X be a discrete r.v. with pmf. as

$$p_X(x) = \begin{cases} 1/10 \ ; & x=-2 \\ 2/10 \ ; & x=-1 \\ 3/10 \ ; & x=1 \\ 4/10 \ ; & x=2 \\ 0 \ ; & \text{otherwise} \end{cases}$$

Find the pmf for $Y = X^2$.



In this case each outcome mapped to $Y=1$ or 4.

$\therefore \ Y=1 \iff X = 1$ or -1

$Y=4 \iff X = 2$ or -2.

$$\therefore \ p_Y(y) = \begin{cases} \frac{2}{10} + \frac{3}{10} = \frac{5}{10} = 1/2 \ , & y=1. \\ \frac{1}{10} + \frac{4}{10} = \frac{5}{10} = 1/2 \ , & y=4 \\ 0 & , \text{otherwise} \end{cases}$$

Hence, we may conclude that pmf of $Y = g(X)$ is equal to the pmf of X, if $g(x_1) \neq g(x_2)$, when $x_1 \neq x_2$. Otherwise, the pmf for Y is obtained as

$$p_Y(y) = \sum_{x:\, g(x)=y} p_X(x)$$

Example 3.

Let X be a discrete r.v. that is defined on the integers in the interval $[-3,4]$. Let the pmf of X is given as follows

$$p_X(x) = \begin{cases} 0.05, & x \in \{-3,4\} \\ 0.10, & x \in \{-2,3\} \\ 0.15, & x \in \{-1,2\} \\ 0.20, & x \in \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$

Find the pmf for $Y = X^2 - |X|$.

Soln. The possible values of the r.v. X are

$\{-3,-2,-1,0,1,2,3,4\}$.

$\therefore$ The possible values for the r.v Y are

$\{6,2,0,0,0,2,6,12\} \equiv \{0,2,6,12\}$

$$\therefore p_Y(y) = \begin{cases} 0.15 + 0.20 + 0.20 = 0.55 \; ; & y = 0 \rightarrow x = -1,0,1 \\ 0.10 + 0.15 = 0.25 \; ; & y = 2 \rightarrow x = -2,2 \\ 0.05 + 0.10 = 0.15 \; ; & y = 6 \rightarrow x = -3,3 \\ 0.05 = 0.05 \; ; & y = 12 \rightarrow x = 4 \\ = 0 \; ; & \text{otherwise} \end{cases}$$

Mathematical Expectation

Let X be a discrete r.v. with prob. mass fn.

$P[X=x_i] = p_i \ \forall i \in I$ OR $P[X=x] = p_X(x) \ \forall x$.

Then the expected value of X or expectation of X, is given by

$$\mu = E(X) = \sum_{i \in I} x_i p_i \quad \Big| \quad E(X) = \sum_{\forall x} x\, p_X(x) \quad \Big| \quad E(X) = \sum_{i=1}^{n} i\, p_i\ ;\ p_i = P[X=i],\ i=1,2,\cdots n.$$

Expected value of $g(X)$, a fn. of the r.v. X is given by $E[g(X)] = \sum_{i \in I} g(x_i)\, p_i$

Let $g(X) = X^2$

Then $E[g(X)] = E[X^2] = \sum_{i \in I} x_i^2 p_i$.

Example: Let us consider the example 2 of Lecture 5 P(2). and find $E(X^2)$ & $E(Y)$.

Soln. $E[Y] = \sum_{\forall y} y\, p_Y(y) = 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = \frac{1}{2} + 2 = 2\frac{1}{2}$

OR

$$E[X^2] = \sum_{x} x^2 p_X(x) = 4 \cdot \frac{1}{10} + 1 \cdot \frac{2}{10} + 1 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10}$$

$$= 4 \cdot \frac{5}{10} + 1 \cdot \frac{5}{10} = 2\frac{1}{2}$$

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n-th moment / n-th raw moment / rth moment about origin of a r.v. X is given by

$$\mu_n' = E[X^n] = \sum_{i \in I} x_i^n p_i$$

nth moment about a pt $X=A$ is given by

$$E[(X-A)^n] = \sum_{i \in I} (x_i - A)^n p_i$$

nth central moment $\mu_n = E[(X-E(X))^n]$

$$= \sum_{i \in I} (x_i - E(x_i))^n p_i$$

The second central moment is called the variance of the r.v X and is written as

$$\sigma_X^2 \equiv Var[X] \equiv E[(X-E(X))^2] = \sum_{i \in I} (x_i - E(X))^2 p_i$$

A frequently used formula for variance is

$$Var[X] = E[X^2] - (E[X])^2.$$

Since $E[X^2]$ is the mean-square value of X, the variance is equal to the mean-square value of X minus the square of the mean value of X.

The variance can never be negative and characterizes the dispersion of the r.v. X about its mean value.

The square root of variance is called the standard deviation (s.d.) of the r.v. X and is denoted by σ_X.

The s.d. (σ_X) yields a number whose units are the same as those of the r.v. X. and hence provides a clear picture of the dispersion of the r.v. about its mean

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The coefficient of variance of the r.v. X is given by

$$\boxed{C_X = \frac{\sigma_X}{E(X)} = \frac{\sigma_X}{\mu}.}$$

It is a dimensionless measure of the variability of the r.v. X.

Properties :-

1. $E[c] = c$, $c \to$ constant.

Proof: Let X be a r.v. with pmf. $P[X=x_i] = p_i \quad \forall i \in I$

Let $x_i = c \;\; \forall i$

Then $E[c] = \sum_{i\in I} c\, p_i = c \sum_{i\in I} p_i = c.$

2. $E[X+Y] = E[X] + E[Y]$
3. $E[aX+bY] = aE[X] + bE[Y]$; $a, b \in \mathbb{R}$.
4. $E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$; $a_i \in \mathbb{R}$.
5. If X and Y are independent r.v's. then $E[XY] = E[X]E[Y]$.

Proofs of Properties 2–5 will be discussed after introducing 2-D r.v's.

6. $Var(1) = 0$
7. $Var[X] = E[X^2] - [E(X)]^2$

Proof:- $Var[X] = E(X-\mu)^2 = E(X^2 - 2\mu X + \mu^2)$

$= E(X^2) - 2\mu E(X) + E(\mu^2)$

$= E(X^2) - 2\mu.\mu + \mu^2 = E(X^2) - \mu^2$

Proved.

8. $Var(cX) = c^2 Var(X)$.

Proof: $Var(cX) = E(cX)^2 - [E(cX)]^2$

$$= E[c^2X^2] - [c E(X)]^2$$

$$= c^2 E(X^2) - c^2 [E(X)]^2$$

$$= c^2 [E(X^2) - \{E(X)\}^2]$$

$$= c^2 Var(X).$$

Proved.

9. If X and Y are independent r.v's. then

$$Var[X+Y] = Var(X) + Var(Y)$$

10. $Var(aX+bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2cov(X,Y)$.

11. If X and Y are uncorrelated r.v's, then

$$Var(X+Y) = Var(X-Y).$$

Proofs of 9-11 will be provided after introducing 2D-rvs.