Linear Algebra – Distance, Hyperplanes and Halfspaces, Eigenvalues, Eigenvectors

Hyperplanes

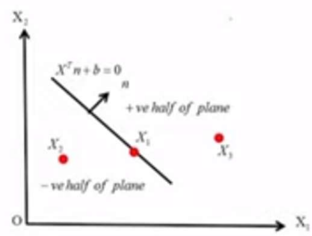
- Geometrically, hyperplane is a geometric entity whose dimension is one less than that of its ambient space.
- For instance, the hyperplanes for a 3D space are 2D planes and hyperplanes for a 2D space are 1D lines and so on.
- The hyperplane is usually described by an equation as follows

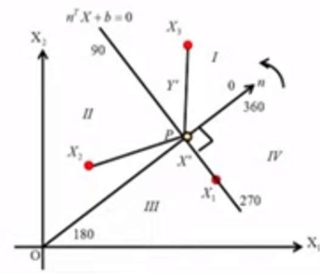
$$X^T n + b = 0$$

Halfspace

- We can observe that the equation can be evaluated for the two halfspaces
- It can be seen that

$$X^T n + b = 0 \;\forall\; X \in line$$
$$X^T n + b > 0 \;\forall\; X \in subspace\ in\ the\ n\ direction\ (X_3)$$
$$X^T n + b < 0 \;\forall\; X \in subspace\ in\ the\ -n\ direction\ (X_2)$$





Data science for Engineers

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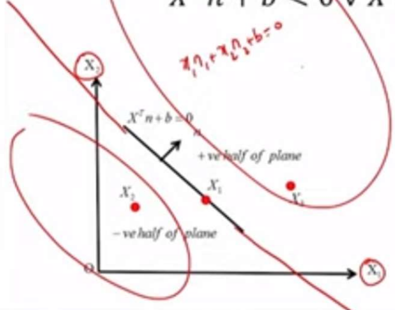

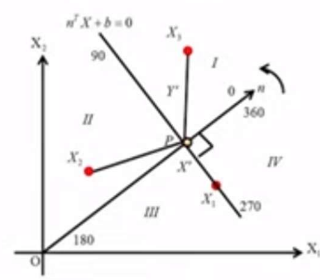

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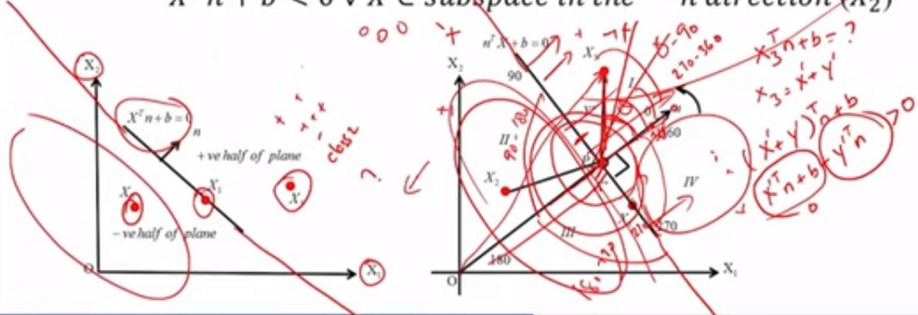
$a^T b = |a||b|\cos\theta$

$X_1^T n + b < 0$



Hyperplanes and halfspaces: Example

- Let us consider a 2D geometry with $n = \begin{bmatrix}1\\3\end{bmatrix}$ and $b = 4$

$$X = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$X^T n + b = 0$$

$$[x_1\ x_2]\begin{bmatrix}1\\3\end{bmatrix} + 4 = 0$$

$$x_1 + 3x_2 + 4 = 0$$

(−1, −1)
(1, −1)
(1, −2)

lin: $x_1 + 3x_2 + 4 = 0$ (−1, −1)
+ve hs: $1 - 3 + 4 = 2 > 0$ (1, −1)
−ve hs: $1 - 6 + 4 = -1 < 0$ (1, −2)

- The hyperplane is the equation of a line
- The halfspaces corresponding to this hyperplane are

$$x_1 + 3x_2 + 4 > 0 : \text{Positive halfspace}$$

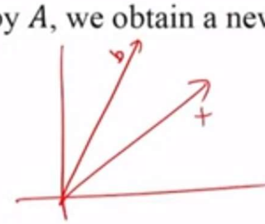
$$x_1 + 3x_2 + 4 < 0 : \text{Negative halfspace}$$

Eigenvalues and eigenvectors

- We have previously seen linear equations of the form $Ax = b$

$A_{n \times n} x_{n \times 1} = b_{n \times 1}$

- What is the geometrical interpretation of this equation?
- We can make an interpretation as follows
  - When vector x is operated on by A, we obtain a new vector b with a different orientation



Ax

Eigenvalues and eigenvectors

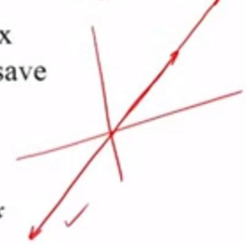
- Operator representation

$$x \longrightarrow \boxed{A} \longrightarrow Ax = b$$

- The newly obtained b vector represents a new orientation. So we ask the following question
- Are there directions for a matrix A such that when the matrix operates on these directions they maintain their orientation save for multiplication by a scalar (positive or negative)?
- That is

$$x \longrightarrow \boxed{A} \longrightarrow Ax = b = \lambda x$$

$A\underline{x} = \lambda \underline{x}$

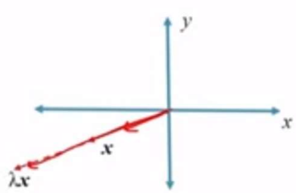


Eigenvalues and eigenvectors

- The mathematical formulation of our question is

$$Ax = \lambda x$$

- The constant λ $(positive)$ represents the amount of stretch or shrinkage the attributes x go through in the x direction
- The solutions (x) are known as eigenvectors and their corresponding λ are eigenvalues



Eigenvalues and eigenvectors

- We can find the eigenvalues as follows

$$Ax = \lambda x \quad A(n \times n); x(n \times 1)$$
$$Ax - \lambda I x = 0$$
$$(A - \lambda I)x = 0$$

- Thus the eigenvalues of the equation can be identified using

$$|A - \lambda I| = 0$$

- Substituting the eigenvalues in the original equation will help us find solutions for the eigenvector x

$x = 0$ (trivial solution)

x is in the null space of $(A - \lambda I)$

(rank) + nullity = n

Eigenvalues and eigenvectors: Examples

- Consider the following example with the given A matrix

$$A = \begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$
$$|A - \lambda I| = \left|\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} 8-\lambda & 7 \\ 2 & 3-\lambda \end{bmatrix}\right| = 0$$
$$(8-\lambda)(3-\lambda) - 14 = 0$$
$$\lambda^2 - 11\lambda + 10 = 0$$
$$\lambda = (10,1)$$

$\lambda_1 = 10$
$\lambda_2 = 1$

R Code

```
A = matrix(c(8,7,2,3), 2, 2, byrow=TRUE)
ev =eigen(A)
values = ev$values
```

Console output

```
> values
[1] 10 1
```

- Thus we identify two eigenvalues and now we proceed to find the corresponding eigenvectors

Eigenvalues and eigenvectors: Examples

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\lambda = 1$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 7x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$8x_1 + 7x_2 = x_1$

$7x_1 + 7x_2 = 0 \Rightarrow x_1 + x_2 = 0$ ✓

$2x_1 + 3x_2 = x_2$

$2x_1 + 2x_2 = 0 \rightarrow x_1 + x_2 = 0$

- Thus the eigenvector (unit) corresponding to $\lambda = 1$is

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \quad \begin{bmatrix} k \\ -k \end{bmatrix}$$

$\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2}$

$\sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$

Eigenvalues and eigenvectors: Examples

- $\lambda = 10$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 7x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

$$7x_2 = 2x_1$$ ✓

- Thus the eigenvector (unit) corresponding to $\lambda = 10$

$$X = \begin{bmatrix} \frac{7}{\sqrt{53}} \\ \frac{2}{\sqrt{53}} \end{bmatrix} \quad \begin{bmatrix} k \\ \frac{2k}{7} \end{bmatrix}$$

R Code

```
A = matrix(c(8,7,2,3), 2, 2, byrow=TRUE)
ev =eigen(A)
vectors <- ev$vectors
> vectors
          [,1]       [,2]
[1,] 0.9615239 -0.7071068
[2,] 0.2747211  0.7071068
```

$\sqrt{\frac{49}{53} + \frac{4}{53}}$

$= \sqrt{1} = 1$

Summary

- $Ax = b$ — Geometric interpretation
- $Ax = \lambda x$ — Eigenvalue-eigenvector equation
- λ — N eigenvalues from $|A - \lambda I| = 0$
- x — Eigenvectors, generally expressed in unit vector form