Moment Generating fn. (MGF) $M_X(t)$

The moment generating fn. of a r.v. X is denoted by $M_X(t)$ and is defined for real values of t, as

$$M_X(t) = E(e^{tX}).$$

It exists if the above expectation is finite for all t in an open interval aroun zero, i.e., $-c < t < c$, for some positive constant c. The range of values of t for which $M_X(t)$ exists is called the region of convergence.

If X is a discrete r.v. with pmf $p_X(x)$ then

$$M_X(t) = \sum_{\forall x} e^{tx} p_X(x)$$

similarly, if X is a continuous r.v with pdf $f_X(x)$

then $$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx.$$

Example: Let X be a discrete r.v. with pmf as

$$p_X(x) = \begin{cases} 1/n, & x = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

find the MGF of X.

Soln. $M_X(t) = \sum_{\forall x} e^{tx} p_X(x) = \sum_{x=1}^{n} \frac{e^{tx}}{n} = \frac{1}{n}(e^{t} + e^{2t} + \cdots + e^{nt})$

$$= \frac{1}{n}\left(\frac{e^{t} - e^{(n+1)t}}{1-e^{t}}\right)$$

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$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^rX^r}{r!} + \cdots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' \; , \quad \mu_r' \rightarrow r\text{th moment about origin.}$$

Note :- the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r'

Since $M_X(t)$ generates all the moments of the r.v. X, it is known as the moment generating fn. (mgf) of X.

Now $$\frac{d}{dt} M_X(t)\Big|_{t=0} = \left[E(X) + tE(X^2) + \cdots + \frac{t^{r-1}}{(r-1)!}E(X^r) + \cdots\right]_{t=0}$$

$$= E(X) = \mu_1' \rightarrow \text{1st moment about origin}$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \left[\mu_2' + t\mu_3' + \cdots + \frac{t^{r-2}}{(r-2)!}\mu_r' + \cdots\right]_{t=0}$$

$$= \mu_2' \longrightarrow \text{2nd moment about origin.}$$

$\vdots$

$$\frac{d^r}{dt^r} M_X(t)\Big|_{t=0} = \left[\mu_r' + t\mu_{r+1}' + \cdots\right]_{t=0}$$

$$= \mu_r' \rightarrow r\text{th moment about origin.}$$

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The MGF of X about a point $X=A$.

$$M_X(t)\ (\text{about } X=A) = E\left(e^{t(X-A)}\right)$$

$$= E\left[1+t(X-A)+\frac{t^2}{2!}(X-A)^2+\cdots+\frac{t^r}{r!}(X-A)^r+\cdots\right]$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X-A)^r$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r$$

where $\mu_r = E(X-A)^r \rightarrow$ rth moment of X about the pt $X=A$.

Properties of MGF :-

Prop. 1. $M_{cX}(t) = M_X(ct)\ \forall\ c\in\mathbb{R}$

Proof :-

$$M_{cX}(t) = E\left(e^{t(cX)}\right) = E\left(e^{(ct)X}\right) = M_X(ct)$$

Proved.

Prop. 2. If X_i; $i=1,2,\ldots,n$ are n independent r.v.s then mgf of $Y=\sum_{i=1}^{n}X_i$ is given by $M_Y(t)=\prod_{i=1}^{n}M_{X_i}(t)$.

Proof :-

$$M_Y(t) = E\left(e^{t\sum_{i=1}^{n}X_i}\right) = E\left(e^{tX_1}\cdot e^{tX_2}\cdots e^{tX_n}\right)$$

$$= E\left(e^{tX_1}\right)E\left(e^{tX_2}\right)\cdots E\left(e^{tX_n}\right)$$

$$= M_{X_1}(t)\,M_{X_2}(t)\cdots M_{X_n}(t)$$

$$= \prod_{i=1}^{n}M_{X_i}(t)$$

Proved

Prop. 3. Let X be a random variable

Let $U = \frac{X-a}{h}$ where a and h are constants.

Then $M_U(t) = e^{-at/h} M_X(t/h)$

<u>Proof</u> :- $M_U(t) = E(e^{tU})$

$$= E\left(e^{t\left(\frac{X-a}{h}\right)}\right) = E\left(e^{\frac{tX-ta}{h}}\right)$$

$$= E\left(e^{\frac{tX}{h}} \cdot e^{-\frac{at}{h}}\right)$$

$$= e^{-\frac{at}{h}} E\left(e^{\frac{tX}{h}}\right)$$

$$= e^{-at/h} M_X(t/h).$$

<u>Proved</u>.

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Let X be a cont. r.v.

The mgf of X is closely related to the Laplace transform. The Laplace transform of a real valued fn. $h(t), t \geq 0$ is given by

$$h^*(s) = \int_0^\infty e^{-sx} h(x)\,dx. \quad \text{—— (1)}$$

s may be a complex no. or a real no.

However,

$$MGF \equiv M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \quad , t \in \mathbb{R}$$

t is always a real no.

In (1) $h(x)$ need not to be a density fn.

Let X be a cont. r.v. taking the values in $(0, \infty)$.

Then

$$\mathcal{L}_X(s) = h^*(s) = M_X(-s) = \int_0^\infty e^{-sx} h(x)\,dx$$

where $h(x)$ is the pdf of X.

In this case

$$E(X^r) = (-1)^r \frac{d^r}{ds^r} \mathcal{L}_X(s) \Big|_{s=0} \quad , r = 1, 2, \ldots$$

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Lecture 7 (5).

Uniqueness Theorem of MGF

The moment generating fn. of a distribution, if exist, uniquely determines the distribution. This implies that corresponding to a given probability distribution, there is only one mgf (provided it exists) and corresponding to a given mgf, there is only one prob. distribution.

Hence, $M_X(t) = M_Y(t) \Leftrightarrow$ the r.vs X and Y are uniquely / identically distributed.

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Probability Generating Function for Discrete Random Variable :-

Let X be a discrete r.v. with pmf.

$$p_i = \text{Prob}\{X=i\}, \quad i = 0,1,2,\ldots$$

Then the probability generating fn. of X is given by

$$G_X(z) = \sum_{i=0}^{\infty} p_i z^i = p_0 + p_1 z + p_2 z^2 + \cdots + p_i z^i + \cdots$$

$G_X(z)$ is also called the z-transform of X.

It can be easily proved that $G_X(z)$ converge for any complex number z for which $|z|<1$.

$$|G_X(z)| = \left|\sum_{i=0}^{\infty} p_i z^i\right| = \sum_{i=0}^{\infty} |p_i||z^i| \quad \text{for } |z|<1.$$

$$\leq \sum_{i=0}^{\infty} p_i = 1$$

Example: Let R be a random variable which denotes the number of heads in three tosses of a fair coin. Find the pgf of R.

Soln.

$$G_R(z) = \sum_{i=0}^{3} p_i z^i = z^0 \frac{1}{8} + z^1 \frac{3}{8} + z^2 \frac{3}{8} + z^3 \frac{1}{8}$$

$$= \frac{1}{8}(1 + 3z + 3z^2 + z^3)$$

Now for $z=1$

$$G_R(1) = \frac{1}{8}(1+3+3+1) = 1.$$

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Characteristic fn.

$\phi_X(t)$

The characteristic function of a r.v X is defined by

$$\phi_X(t) = E(e^{itX}) = \sum_{\forall j} e^{itx_j} p_j$$

Note:- Here X is considered to be a discrete r.v with pmf $P(X=j) = p_j \ \forall j$

and $i = \sqrt{-1}$.

Now

$$|\phi_X(t)| = \left| \sum_j e^{itx_j} p_j \right|$$

$$\leq \sum_j |e^{itx_j}||p_j|$$

$$= \sum_j |p_j| \ ; \text{ as } |e^{itx_j}| = 1.$$

$$= 1.$$

$\Rightarrow |\phi_X(t)| \leq 1.$

$\Rightarrow$ The characteristic fn. of the r.v X always exist.

Let X be a cont. r.v. with pdf $f_X(x)$, $-\infty < x < \infty$.

Then

$$|\phi_X(t)| = \left| \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx \right| \leq \int_{-\infty}^{\infty} |e^{itx}||f_X(x)|\, dx$$

$$= \int_{-\infty}^{\infty} |f_X(x)|\, dx \quad \text{as } |e^{itx}| = 1.$$

$$= 1. \quad \because f_X(x) \text{ is a pdf.}$$

Hence proved $|\phi_X(t)| \leq 1$.

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Now

$$\phi_X(t) = E(e^{itX})$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r'$$

where $\mu_r' = E(X^r) \rightarrow$ rth moment of X about origin.

$$\frac{d}{dt}\phi_X(t) = \sum_{r=1}^{\infty} \frac{(it)^{r-1}}{(r-1)!} \mu_r'$$

$$\Rightarrow \frac{d}{dt}\phi_X(t)\Big|_{t=0} = \mu_1'$$

$$\frac{d^2}{dt^2}\phi_X(t) = \sum_{r=2}^{\infty} \frac{(it)^{r-2}}{(r-2)!} \mu_r'$$

$$\frac{d^2}{dt^2}\phi_X(t)\Big|_{t=0} = \mu_2'$$

$$\vdots$$

$$\frac{d^r}{dt^r}\phi_X(t)\Big|_{t=0} = \mu_r'.$$

Properties.

Prop. 1. $\phi_X(0) = 1$.

$$\phi_X(0) = \sum_x e^{i.0.x} p_X(x) = \sum_x p_X(x) = 1.$$

Prop. 2. $|\phi_X(t)| \leq 1$.

Prop 3. $\phi_{cX}(t) = \phi_X(ct) \ \forall c \in \mathbb{R}$.

Prop 4. If X_i's are independent r.vs then

$$\phi_{\sum_i X_i}(t) = \prod_i \phi_{X_i}(t).$$

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Prop 5. Let $U = \dfrac{X-a}{h}$, a and h are constants, then

$$\phi_U(t) = e^{-iat/h}\,\phi_X(t/h).$$

Theorem :- Uniqueness theorem of characteristic fn.

Characteristic fn. uniquely determines the dist., i.e., a necessary and sufficient condition for two distribution with pmf/pdf $p_1(\cdot)$ & $p_2(\cdot)$ to be identical is that the characteristic fn. $\phi_1(t)$ and $\phi_2(t)$ are identical.

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